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Acoustic black hole: the Sagnac effect, the geodesics and the bounding values of the parameters

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Abstract Sagnac interference experiment is theoretically analyzed in the curved spacetime of the Rotating Acoustic Black hole metric. The Zero and the infinite Sagnac delay has been analyzed. The geodesic motion in the metric is discussed very briefly to derive the formula for the Sagnac delay. For the first time, the values of the two constant parameters related to the metric of the acoustic black hole have been found to be restricted within certain limit by the use of the formula for the Sagnac delay. The equation for finding the sonic horizon has also been deduced.

1 Introduction

A Black hole in Einstein's theory of gravitation or general relativity (GR) is essentially an extremely contracted state of a massive body when all the mass of it has been concentrated into its Schwarzschild radius creating such a curvature in nearby region that even light cannot escape a predefined surface called event horizon. Black holes (BH) are *bodies caught in an inexorable gravitational collapse* [1] and remain out of direct observation because no emission comes out of it. When a BH forms a binary system with a main-sequence stars or a protostar, mass from the later is accreted into the BH with a very high velocity. The trajectory of this accreting mass form a predictable geometry and emits high energy photons. The indirect experimental evidences of several properties of BH involve spectral and timing analyses of the data of these high energy photons (mostly in UV and X-ray region) collected through several satellite missions (NICER, NuSTAR, CHANDRA, XMM-Newton, Suzaku and several others) as both parts of the spectrum are absorbed in the earth's atmospheric region [2]. Apart from the event horizon, a region of spacetime called ergo-region [3] exists for a rotating BH. By projecting a particle to it, energy can be extracted from such a BH through a process named after Penrose who first showed this. The wave analogue of Penrose process [3] is called superradiance first proposed by Misner [4]. Although with the application of cutting-edge technologies involving several countries and availability of high power computational facilities, X-ray/UV astronomy gives several indirect but measurable properties of BHs in binary systems, most characteristic properties remain unrealizable. Very recently, however, event horizon telescope reported picture of the event horizon of a BH shot directly creating a breakthrough in the experimental measurement of terrestrial BHs. This created the first ever possibilities of observing phenomena related to an isolated BH which was so far unattainable.

In 1981, Unruh [5] has proposed that certain aspects of BH in GR is analogous to some characteristics of the flow of liquid flowing in supersonic speed. As a moving fluid drags sound wave along with its flows, supersonic flow traps sound waves – much like trapping of electromagnatic waves in geometric BHs. Hence this system is known as sonic or acoustic black hole and was created in the laboratory in 2009 with Bose-Einstein condensate [6]. A system with draining bathtub-like feature with rotating inviscid and barotropic fluid, consequently, resembles a rotating BH and is called rotating acoustic black hole (ABH). Matt Visser [7,8] proposed the metric of the ABH and discussed its properties analogous to any geometric BH in fair details. Few effects shown by terrestrial BHs were observed for this type of *analogous BH* in the laboratory like Hawking radiation [9–13]. A phenomenon related to superradiance from BH, termed superresonance for ABH, first discussed by Basak and Majumdar [14,15] was also experimentally observed [16]. Another remarkable feature of ABH is the use of its concept in engineering and structural applications as vibration damping and

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its control and the absorption of sound in gases and liquids and control of noise in structural design [17,18].

In the present paper we will discuss an effect known as Sagnac effect after G. Sagnac [19] who was known to perform it first (although later evidences showed that it had been performed several times earlier, the name stuck; an account from the historical perspective may be found in Refs. [20–22]) with a hope that the experiment can be performed in ABH. The original set-up of Sagnac effect was an interference-type experiment on board a rotating disc-like platform with visible light where the two beams in phase were made to travel a round trip before meeting again at the position of the beam-splitter to produce an interference fringe. The rotation of the disc induces the required path difference and thus phase difference occurs between the corotating and counter rotating beams. After Sagnac's experiment it has been repeated several times. Historically, this experiment has been cited to nullify the validity of special relativity (SR). Although later it was found to be quite consistent with SR, this experiment indeed poses a counter intuitive challenge to it. The related conceptual issues drew the attention of the community particularly in the last three decades.

Sagnac effect is interesting in many ways. On the theoretical side, it poses both conceptual and interpretational problems to relativistic physics as it is an experiment where the optical effect is created by rotational motion or a uniformly accelerating motion with, however, no dynamical effect of acceleration involved. SR now has to stand face to face to the challenge of its formulation in accelerating frame. The formulation for Sagnac effect has also been used to propose resolution to some longstanding relativistic paradoxes [21–24] and to make ground to handle acceleration (rotation) in SR [25]. Resolution to an important conceptual issue related to the precise reason behind the origin of Sagnac effect has been recently proposed by Bhadra, Ghosh and Raychaudhuri [26].

The experimental arena has also been alive with several arrangements to measure the Sagnac effect more and more precisely and with other waves than light. Although the ingenuine design of the experiment has been first conceived to measure the velocity of earth with respect to the pre-relativistic aether (and also earth's rotation), the later experimentalists have found that the experiment is worth performing in its own right. One of the reasons is that the correct measurement of the phaseshift will give the precise dependence of γ-factor on the velocity of the frame thus offering experimental verification of the SR. One of the most important application of this effect is to make global positioning system (GPS) more precise with Sagnac Correction [27] where the earth itself acts as the rotating platform. This correction is also necessary in LISA-type interferrometers. Sagnac correction due to the earth's rotation has to be taken in to account by Hafele and Keating [28,29] in their famous experiment of 1971. General relativistic correction had not been included into the final calculation for the obvious reason that the experiment preceded the theoretical analysis.

All the theoretical calculations stated above had been done in the background of flat spacetime of SR. In 1975, Ashtekar and Magnon [30,31] first discussed Sagnac effect in curved spacetime using the sophisticated language of differential geometry. A nice simplified treatment was offered by Tartaglia [32] for the Kerr metric following the metrical treatment offered by Logunov [33] (and by Malykin [34] in SR). Logunov–Tartaglia treatment has become quite standard and followed by many [22,35,36].

In the present discourse, we analyse, using standard method outlined in Refs. [22,30,32–34,36] of finding Sagnac delay (SD), in the spacetime field of the ABH with the hope that the experimentalists will find this experiment performable in the laboratory. We, however do not propose any design of the possible experiment as such nor do we perform any simulation. Indeed, no possibility for experimental measurement of correction terms due to curvature of spacetime for geometric BH can be foreseen in near future because true verification the correction terms without any weak field approximation needs us to perform the experiment in the field of a massive source like a terrestial BH. The only possibility is that an experiment with ABH in laboratory can test the correction term. One of our motivations to do this work is precisely this anticipation. Moreover, the issue of existence of two formulas for the SD [22,23,37] has yet to be settled by achieving the precision to (at least) second order or more of v/c. An experiment with ABH is performable in laboratory unlike, obviously, that for a general (geometrical) black hole (GBH). Sagnac effect offers a simple way to discuss the nature of orbits near event horizon [22,36] which will be shown for this case of ABH too here. Geodesic motion in ABH is also discussed very briefly to suit our purpose. Moreover, we have also deduced, taking cue from an earlier work, an equation for the sonic horizon. The remarkable fact is that from the expressions of the SD, the values of the parameters A and B are found to be restricted within some limits. To our knowledge, this is reported for the first time.

The paper is organized as follows. In the next subsection 1.1 we brief about the Sagnac delay in SR and related topics to set the stage for our present treatise. We very briefly discuss the standard results of SD and comment over it. In Sect. 2 we present our main result of the Sagnac delay term for the ABH. In the following two Sects. 2.1 and 2.2, we discuss the appearance of zero and infinite Sagnac delay and their significances. Section 3 is, in a sense, out of our main sequence of development where we briefly discuss the geodesic equations of ABH spacetime although no rigorous analysis is provided. In the last section we presented our argument to restrict the values of the constants present in ABH, namely A and B, using the expressions obtained for SD and comment over it.

1.1 Sagnac delay

Although the experimentally observed quantity in Sagnac experiment is the interference fringe shift formed at the screen like any interferometer, it has now become customary to write the theoretical formula in terms of the Sagnac delay (SD) which is the difference between the round-trip times of the counter-rotating and corotating beams while they meet at the beamsplitter (screen) again after the round-trip. The special relativistic result is given by

$$\Delta t = \frac{4\pi\omega r^2}{c^2}\gamma, \tag{1}$$

where Δt and ω are the SD and the angular velocity of the platform respectively. r is the radius of the circular path of the beams, c and γ being the velocity of light in vacuum and the relativistic gamma-factor.

The appearance of γ in the formula created a debate because that did not appear in the original Sagnac's formula and was perhaps first pointed out by Selleri [37]. There was no scope for any empirical way to settle the issue that needs to measure to a precision of v^2/c^2 as $v << c$. The issue was discussed at length by Ghosal et al. in relation to Ehrenfest paradox [23]. Another formula that is widely used while discussing Sagnac-type experiments is given by [23,38]

$$\Delta t = \frac{2Lv}{c^2}, \tag{2}$$

where $v = \omega r$ and L is the length of the path. This formula forces the Sagnac formula out from the area dependence [20,39] and rightly shows its dependence of the velocity of the platform. Indeed it was shown proposing a gedanken experiment resembling Sagnac effect which uses a suitable experimental arrangement – called linear Sagnac effect [23,40], that SD does not depend on the area enclosed by the tracks of the circumnavigating rays (an experiment performed by Wang et. el. corroborates this). However, recently Bhadra, Ghose and Raychaudhuri have showed by proposing a gedanken experiment that the origin of the Sagnac effect has to be attributed to the asymmetrically placed observers with respect to the journeys of light [26] settling the age-old problem of explanation of the effect on-board the rotating platform.

2 Sagnac effect in a rotating acoustic black hole metric

In this section we investigate the Sagnac delay around a rotating accoustic black hole which is analogous to $(2+1)D$ shrinking fluid vortex. We employ the general relativistic treatment of Sagnac effect as mentioned earlier. Such a treatment is possible once we have a metric defined on the related spacetime – which uniquely describes the spacetime. Earlier we have seen such an approach of treating superradiance (superresonance) simply using the metric of the ABH [14,15]. The metric of ABH is given by (see, for example Visser [7,8])

$$\begin{aligned} ds^2 = &-\left(1-\frac{A^2+B^2}{c_s^2r^2}\right)dt^2 + \left(1-\frac{A^2}{c_s^2r^2}\right)^{-1}dr^2 \\ &-\frac{2B}{c_s}\,d\phi\,dt + r^2\,d\phi^2, \end{aligned} \tag{3}$$

where A, B are constants and c_s is the velocity of phonon in that fluid medium. Note that $A = 0$ means no radial flow [7]. The BH may be the vortex in a superfluid HeII with sink at the centre.

We allow the signals to move a round trip in counter rotating directions with suitable arrangements and then to meet again. The radius remaining constant ($r = 0$), this condition has the usual meaning $ds = 0$. If Ω is the angular velocity of the signal (the sonic wave), then $\phi = \Omega t$. This gives two solutions for Ω given by

$$\Omega_\pm = \frac{B \pm \sqrt{c_s^2r^2 - A^2}}{c_s r^2}. \tag{4}$$

For the observer on board the platform (the vortex) rotating with an angular velocity $\omega_0 t$, we write $\phi_0 = \omega_0 t$. We also note that $d\phi$ is the difference of the two values of ϕ_0. Therefore

$$\begin{aligned} d\phi &= \phi_{0+} - \phi_{0-} \\ &= \frac{4\pi\omega_0^2c_s^2r^4 - 4\pi\omega_0c_sr^2\left(c_s^2r^2-A^2\right)^{1/2}}{B^2 - \left[\left(c_s^2r^2-A^2\right)^{1/2} - \omega_0c_sr^2\right]^2}. \end{aligned} \tag{5}$$

Thus the SD is given by

$$\delta\tau = \frac{d\phi}{\omega_0}\left[-\left(1-\frac{A^2+B^2}{c_s^2r^2}\right) - \frac{2B}{c_s}\Omega + r^2\Omega^2\right]. \tag{6}$$

When the angular velocity of sonic wave is equal to the angular velocity of the observer ($\Omega = \omega_0$), we obtain

$$\delta\tau = \frac{4\pi r\left[\left(c_s^2r^2-A^2\right)^{1/2} - \omega_0c_sr^2\right]\left[\omega_0^2c_s^2r^4 - 2B\omega_0c_sr^2 - c_s^2r^2 + (A^2+B^2)\right]^{1/2}}{\omega_0^2c_s^2r^4 - 2\omega_0c_sr^2\left(c_s^2r^2-A^2\right)^{1/2} + c_s^2r^2 - (A^2+B^2)}. \tag{7}$$

A stationary observer is an observer who does not see the metric change in his motion. The related Killing vector is denoted by $u^\mu = u^l\,(1, 0, 0, \omega)$ where $\omega = \mathrm{d}\phi/\,\mathrm{d}t$ is the angular velocity of the observer. The equation governing this quantity in case of circular geodesic motion is

$$\omega^2 r^2 - 4\frac{B}{c_s}\omega - \left(1 - \frac{A^2 + B^2}{c_s^2 r^2}\right) = 0, \tag{8}$$

giving

$$\omega_\pm = \frac{2B \pm \left(3B^2 - A^2 + c_s^2 r^2\right)^{1/2}}{c_s r^2}. \tag{9}$$

Putting this value of ω in place of ω_0 in the expression of SD,we obtain

$$\begin{aligned}\delta\tau_\pm = 4\pi r &\left[\left(c_s^2 r^2 - A^2\right)^{1/2} - 2B \pm \left(3B^2 - A^2 + c_s^2 r^2\right)^{1/2}\right]\\ &\times B^{1/2}\left[4B - 2\left(3B^2 - A^2 + c_s^2 r^2\right)^{1/2}\right]^{1/2}\\ &\times\Big[c_s^8 r^8 - 4(A^2 + B^2)c_s^6 r^6 + (6A^4 c_s^4 + 10A^2 B^2 c_s^4\\ &-66B^4 c_s^4)r^4 - (12A^4 B^2 c_s^2 - 132A^2 B^4 c_s^2 - 135B^6 c_s^2\\ &+4A^6 c_s^2)r^2 + A^8 + 62A^4 B^4 + 9B^8\\ &+4A^6 B^2 + 180A^2 B^6\Big]^{-1}.\end{aligned} \tag{10}$$

This delay depends on the constants A and B and the phonon velocity in the medium. There are two values of the delay depending upon the value of ω and denotes whether the orbit is direct or retrograde. Note that the expression restricts the value of the constants A and B depending on the radius and the phonon by demanding the real values of the delay $\delta\tau_\pm$ for obvious reason. We will discuss the issue later in the appropriate place.

Note that the ABH metric (3) turns into a $(2+1)D$ Minkowskian metric if

$$A = B = 0.$$

This essentially means that

$$\vec{\mathrm{v}} = \frac{A}{r}\hat{r} + \frac{B}{r}\hat{\phi} = 0 \tag{11}$$

where $\vec{\mathrm{v}}$ is the velocity profile [14]. Hence, ABH ceases to exist for vanishing A and B. Also, as mentioned earlier, there is no radial flow in the fluid for $A = 0$. Consequently, there should not be any special relativistic limit for analogue gravity as such. This is also evident from the way the metric is usually established [8]. Moreover, it is known that black hole type situation will not be created unless $A < 0$.

In GR, any metric reduces to the Minkowskian one when the mass function appearing in the metric vanishes. Consequently any general relativistic formula immediately reduces to the special relativistic one at this limit.

Interestingly, the expression for the Sagnac delay in ABH (Eq. (10)) vanishes once A and B are set to zero and there remains no SR counterpart whatsoever. This corroborates the fact stated above that for this case the flow ceases to exist, and hence the blackhole too. Moreover, in the general relativistic case, it is always possible to expand the formula to obtain a pure SR term (or, the original Sagnac formula (1)) and a series containing the mass term. This separation, obviously, means nothing in the present context. Hence, the original Sangac formula (classical or special relativistic) is unattainable from the present expression under any circumstance.

2.1 Zero Sagnac delay

Sagnac delay is the difference between the round-trip times of the counter and co-rotating light beams which creates the interferences fringe shift. If the delay is zero, both beams reach the observer simultaneously. In flat spacetime, this essentially means that the observer does not rotate at all. Even for non-rotating spacetime such as Schwarzschild one, Sagnac delay can never be zero unless the beam-splitter stops rotating [21,22]. But in curved spacetime it essentially means that the observer is *non-rotating relative to the local space-time geometry* – the observer is locally non-rotating which is termed as *dragging of inertial frame* [41]. The observer considers both the directions perfectly equivalent [36]. The expressions for radius can be obtained by solving the vanishing numerator of the SD (10) algebraically for r. We find that the SD vanishes at the radii

$$r_+ = \frac{1}{c_s}\left(A^2 + \frac{B^2}{16}\right)^{\frac{1}{2}}, \tag{12}$$

and

$$r_{++} = \frac{1}{c_s}\left(A^2 + B^2\right)^{1/2}. \tag{13}$$

We plot in Fig. 1 the vanishing Sagnac delay against the radius r where it occurs for some chosen value of the parameters. The values are so chosen as to be consistent with their domain of definition discussed later in Sect. 5 and the values do not have any particular significance but to show the functional characteristics.

2.2 Infinite Sagnac delay

The SD will be infinitely large when the denominator of the expression of $\delta\tau$ in Eq. (10) is zero. This is an eighth degree polynomial for r. The zeros of this function gives the radii where the SD is infinite. Infinite SD signifies that the rotational speed of the fluid vortex is equal to the speed of the phonon. Exact solutions will be complicated although we do not venture to find it as it is not that necessary for our purpose. Rather, we draw a representative curve with set of values of the parameters. The red dots represent the two values of the radii of the orbits where the Sagnac delay would be infinite

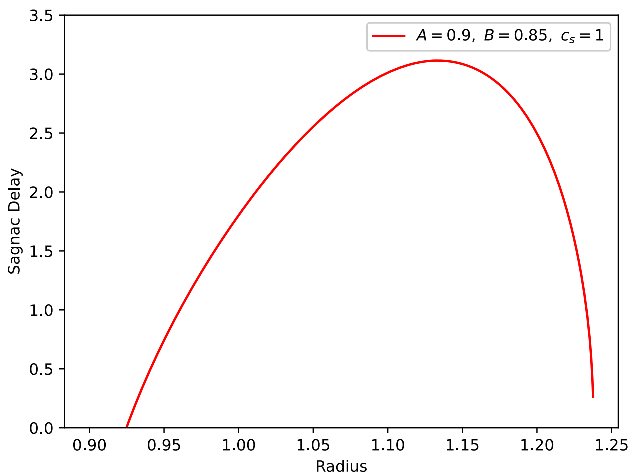

Fig. 1 Zero Sagnac delay

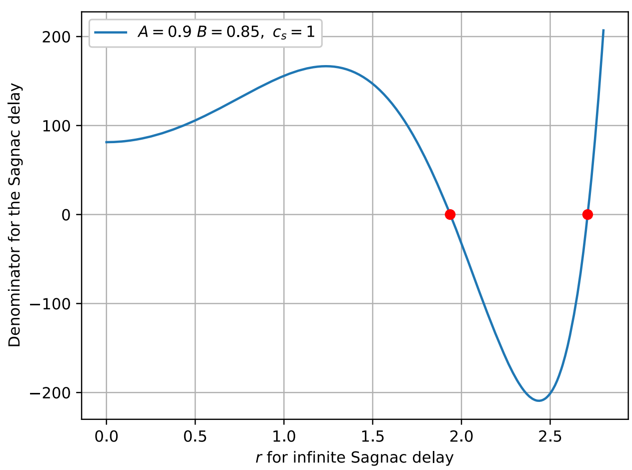

Fig. 2 Infinite Sagnac delay

for the chosen values of the parameters. Other values are either imaginary or negative hence they are not shown here (Fig. 2).

3 Geodesic equation

In this section, we discuss the geodesic equations of acoustic spacetime at length. We follow the standard method of Lagrangian [42]. The related Lagrangian in this case is

$$\mathscr{L} = -\frac{1}{2}\left(1 - \frac{A^2+B^2}{c_s^2 r^2}\right)\dot{t}^2 + \frac{1}{2}\left(1 - \frac{A^2}{c_s^2 r^2}\right)^{-1}\dot{r}^2 - \frac{2B}{c_s}\dot{\phi}\dot{t} + r^2\dot{\phi}^2, \tag{14}$$

where by the overhead dot, as per convention, we denote a differentiation with respect to some affine parameter λ along the geodesic. The corresponding canonical momenta are given by

$$\begin{aligned} p_t &= \frac{\partial\mathscr{L}}{\partial\dot{t}} = -\left(1 - \frac{A^2+B^2}{c_s^2 r^2}\right)\dot{t} - \frac{B}{c_s}\dot{\phi},\\ p_r &= \frac{\partial\mathscr{L}}{\partial\dot{r}} = \left(1 - \frac{A^2}{c_s^2 r^2}\right)^{-1}\dot{r},\\ p_\phi &= \frac{\partial\mathscr{L}}{\partial\dot{\phi}} = -\frac{B}{c_s}\dot{t} + r^2\dot{\phi}. \end{aligned} \tag{15}$$

The equality of the related Hamiltonian to the Lagrangian $\mathscr{H} = p_\mu \dot{x}^\mu - \mathscr{L} = \mathscr{L}$ has the usual and naïve interpretation that the problem does not have any potential term. The constants of motion related to the timelike Killing vector ∂_t and ∂_ϕ are given by

$$\begin{aligned} E &= -\left(1 - \frac{A^2+B^2}{c_s^2 r^2}\right)\dot{t} - \frac{B}{c_s}\dot{\phi},\\ J &= -\frac{B}{c_s}\dot{t} + r^2\dot{\phi}. \end{aligned} \tag{16}$$

E is a constant identified with energy at infinity. We thus find

$$\dot{\phi} = \frac{(EB - Jc_s)\, c_s r^2 + J(A^2+B^2)}{r^2\left(A^2 - c_s^2 r^2\right)}, \tag{17}$$

and

$$\dot{t} = \frac{Ec_s^2 r^2 + JBc_s}{A^2 - c_s^2 r^2}. \tag{18}$$

The radial timelike geodesic for $J = 0$ is given by

$$\dot{r} = \frac{\left[c_s^2\left(E^2-1\right)r^2 + A^2\right]^{1/2}}{c_s r}. \tag{19}$$

From Eqs. (17) and (19) the expression of the orbits is found to be given by

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\left(A^2 - c_s^2 r^2\right)}{EBc_s^2 r}\left[c_s^2\left(E^2-1\right)r^2 + A^2\right]^{1/2}, \tag{20}$$

or,

$$\phi = \int \frac{EBc_s^2 r}{\left(A^2 - c_s^2 r^2\right)}\left[\left(E^2-1\right)c_s^2 r^2 + A^2\right]^{-1/2}\mathrm{d}r. \tag{21}$$

The integral is hyperbolic in nature and the solution is given by

$$r = \frac{A}{c\sqrt{E^2-1}}\left[E^2\tanh^2\left(\frac{B}{A}\phi + \xi\right) - 1\right]^{1/2}. \tag{22}$$

The quantity ξ arises as the integration constant and acts like a phase variable. Again a representative polar plot is given here. As was done earlier, the chosen values of the constants have no particular significance except for their restricted values (Fig. 3).

4 Sonic horizon

The event horizon of a sonic black hole is that boundary where the speed of the fluid crosses the sound velocity barrier – from supersonic to subsonic. Its position is indicated by the

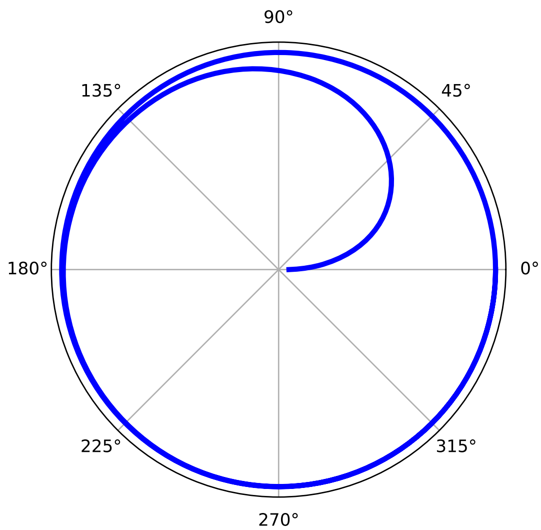

Fig. 3 Plot of orbit, $r = r(\phi)$, with $A = 0.9$, $B = 0.7$, $c_s = 1$, $E = 1.5$ and $\xi = 10$

overlapping zone of the velocities of the fluid and the spatial distribution curve of the sonic velocities as shown by Yang et al. [43]. The fluid velocity is given by the expression

$$\vec{v}(\vec{r}, t) = \frac{\dot{\rho}(t)}{\rho(t)}\vec{r}, \tag{23}$$

with $\rho(t)$ being the system distribution width parameter and $\rho_0(t)$ being the initial distribution width. Again $\rho(t) = (A_0 \sin \omega t + B_0)^{1/2}$ where

$$A_0 = \left[\left(\frac{\rho_0^2}{2} + \frac{\hbar^2}{2m^2\Omega^2\rho_0^2}\right)^2 - \frac{C_2\hbar^2}{2C_1m^2\Omega^2}\right]^{1/2},$$
$$B_0 = \left(\frac{\rho_0^2}{2} + \frac{C_2\hbar^2}{2C_1m^2\Omega^2\rho_0^2}\right),$$

which dictate that the system distribution width shows an oscillatory nature between the maximum value $(B_0 + A_0)^{\frac{1}{2}}$ and the minimum value $(B_0 - A_0)^{\frac{1}{2}}$ with the frequency ω. The phonon velocity is given by the formula

$$c_s(\vec{r}, t) = \left(\frac{g_0}{m}\right)^{\frac{1}{2}} \frac{C_0}{\rho(t_0)^{\frac{3}{2}}} \exp\left[\frac{-r^2}{2\rho(t)^2}\right], \tag{24}$$

where g_0 is the nonlinear interaction term, The boundary location of the sonic horizon radius r_s is determined by the solution of the following equation

$$[v(r, t) - c_s(r, t)]_{r=r_s} = 0. \tag{25}$$

This relation gives an equation for the determination of sonic radius r_s

$$r^2 + 2\rho(t)^2 \ln(r) + 2\rho(t)^2 \ln\left[\left(\frac{m}{g_0}\right)^{\frac{1}{2}} \frac{\dot{\rho}(t)}{\rho(t)} \frac{\rho_0^{\frac{3}{2}}}{C_0}\right] = 0. \tag{26}$$

This equation is highly nonlinear and can be solved numerically with the choice of appropriate values of the parameters obtained from experimental considerations.

5 Discussion

In this paper we have discussed the general relativistic Sagnac effect in the acoustic blackhole using standard methods outlined by Malykin [34] for flat spacetime, Ashtekar and Magnon [30] in curved spacetime, simplified by Tartaglia [32] and followed by several authors. Raychaudhuri [22,36] showed that Sagnac effect can be used as a tool to understand several features of spacetime, flat or curved. Sagnac effect in (2 + 1)D was first discussed by Raychaudhuri [36] in BTZ spacetime. The discussion of the effect in any given metric reveals many interesting features of the spacetime and the present case is no different. The metric of ABH (3) has two arbitrary constants A and B and no particular information of their domain is available in the literature. With the analysis presented here we can throw some light over it.

Whatever this paper presents is essentially a theoretical analysis without mentioning anything about the possible experimental ingradients. The reason is that the present authors are unaware of the related experiments, nor they consider themselves capable enough to do so. The calculation presented here is from the perspective of laboratory frame. Phonons are to be generated by creating suitable perturbations in the rotating fluid to observe the interference pattern created from the perspective of the laboratory.

Although we repeatedly used the word *beamsplitter* following the standard description (arrangement) of the optical phenomenon, in ABH there is no necessity of the beamsplitter. Indeed, no direct analogy can be drawn with the optical phenomenon for the reason that here the career of the signals itself creats the blackhole. Rather, the phonon is to be created by perturbation into the fluid itself. The resulting phonons will rotate in both the directions naturally as it do in any normal fluid.

It is evident from the brief treatise on Sagnac effect that the Sagnac delay (Eq. (10)) should obviously be real. Hence we find

$$r^2 > \frac{A^2}{c_s^2}. \tag{27}$$

Moreover, imposing the same consideration we must have

$$B^2 > \frac{1}{2} B \left(3B^2 - A^2 + c_s^2 r^2\right)^{1/2}. \tag{28}$$

Depending on whether B is greater or less than 1, the inequality sign will change its direction upon squaring. If we consider $B > 1$, we obtain

$$B^2 > \frac{1}{4} \left(3B^2 - A^2 + c_s^2 r^2\right)$$

which yields $A^2 + B^2 > c_s^2 r^2$. This is unacceptable as it reverses the signature of the time coordinate in the metric. The equality is also excluded due to the metric signature. The only option left, ($B < 1$), reverses the direction of the inequality sign and gives

$$A^2 + B^2 < c_s^2 r^2 \tag{29}$$

which is consistent with the signature of the metric. Also, we must have $B > 0$ due to the metric signature. So we conclude that

$$0 < B < 1. \tag{30}$$

The null geodesic at fixed radius is given by

$$-\left(1 - \frac{A^2 + B^2}{c_s^2 r^2}\right) dt^2 - \frac{2B}{c_s} d\phi\, dt + r^2 d\phi^2 = 0. \tag{31}$$

The velocity profile of any circle around the the sink (taken at $r = 0$), gives the change of phase of the wavefunction of the fluid along that circle to be $d\phi = 2\pi\kappa B$ [14,15] where κ is the proportionality constant which depends on the microscopic properties of the fluid. Hence we obtain

$$dt = \frac{\left[\frac{16B^4\kappa^2\pi^2}{c_s^2} + 16B^2\kappa^2\pi^2 r^2\left(1 - \frac{A^2+B^2}{c_s^2 r^2}\right)\right]^{1/2} - \frac{4B^2\kappa\pi}{c_s}}{2\left(1 - \frac{A^2+B^2}{c_s^2 r^2}\right)}, \tag{32}$$

or

$$dt = 2\kappa\pi B c_s r^2 \frac{-B + r\left(c_s^2 - A^2\right)^{1/2}}{c_s^2 r^2 - A^2 - B^2}. \tag{33}$$

The value of κ can thus be obtained inducing the value of $d\phi$ in Eq. (5).

$$\kappa = \frac{d\phi}{2\pi B} = \frac{1}{2\pi B} \frac{4\pi\omega_0^2 c_s^2 r^4 - 4\pi\omega_0 c_s r^2\left(c_s^2 r^2 - A^2\right)^{1/2}}{B^2 - \left[\left(c_s^2 r^2 - A^2\right)^{1/2} - \omega_0 c_s r^2\right]^2}. \tag{34}$$

Now, from Eq. (33) we obtain

$$A < c_s, \tag{35}$$

which means that the value of A must be less than the velocity of phonon in that medium. We exclude the equality due to Eq. (17) through Eq. (20) as it ensures a discontinuity. Equations (30) and (35) give the domain of definition of the parameters A and B. Note that these quantities A and B are related to the velocity profile of the acoustic blackhole.

The ergosphere of the ABH is formed at [8]

$$r_{\text{ergo}} = \frac{\sqrt{A^2 + B^2}}{c_s}. \tag{36}$$

The limits of the values of A and B also restricts the values of the radius of the ergosphere. This also restricts the value of the horizon radius. This corroborates the fact that acoustic black hole occurs when $A < 0$ [8]. These domains of definition of the constants also impose restrictions on the velocity profile of the fluid

$$\vec{v} = \frac{A}{r}\hat{r} + \frac{B}{r}\hat{\phi},$$

at a given radius.

ABH is a state of fluid which simulates blackhole-like features that has been created in the laboratory and the emission of Hawking radiation and superradiance have been experimentally verified. We calculated the Sagnac delay for ABH and analysed the zero Sagnac delay and infinite Sagnac delay for the spacetime metric graphically. Infinite Sagnac delay was first discussed by Raychaudhuri [21,22]. There and in Ref. [36] it revealed some interesting facts. We expect that this remarkable effect can be experimentally observed in ABH in near future. The astonishing prediction of Hawking radiation, though proposed way back in 1975 still remained unattainable in the astrophysical scales due to the limitations of our observational skills [9], has been observed for ABH created in the laboratory [10]. Suitable procedure for such a setup may be suggested and designed by competent hands. The experimental verification of this effect can logically be extended to other spacetime metrics too and expected to reveal the nature of the spacetime considered.

Now, the pertinent question that arises here is what we expect from such a study of Sagnac effect. As we know, the measurement of the delay (or the related fringe shift) in the gravitational field is important for the GPS as well as in the defense studies application. Although it is true that analogue gravity shares a little with the Einstein's geometric theory of gravity, the result found here, if be experimentally confirmed, will speak for the already obtained results from GR since an equivalent mathematical treatment is followed in both cases. The infinite or zero Sagnac delay are two possible interesting observations which probably may not be performable in geometric cases at present and in near future. But these speak of some interesting characteristics of the spacetime concerned. If this can be experimentally tested in analogue gravity, we may be assured of that the same conclusions drawn using the similar method for geometric cases are correct and hence acceptable.

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