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Double Copy Relations in Massive Theories

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Abstract

The studies of scattering amplitudes during recent years has uncovered many new features of various theories that are invisible from the action point of view. One of the most interesting features is the relation between gauge theories and gravity theories, known as double copy. The most famous example is the relation between pure Yang-Mills theory and axion-dilaton gravity, however over the years many more examples of this relation have been found. It is interesting to understand what is special about these theories that can be related by double copy. Most of the known examples of such theories have massless force carriers so a natural question is can double copy be extended to massive theories? The work presented in this thesis suggests that such massive theories must be strongly constrained since a naive attempt to double copy a generic massive gauge theory leads to unphysical scattering amplitudes. In 4d the only known examples are Kaluza-Klein theories of 5d massless gauge theories. However in 3d the situation is different. The work of this thesis suggests that topologically massive gauge and gravity theories are related by double copy. We observed this relation between scattering amplitudes (for 3,4 and 5 point tree level amplitudes) as well as between classical solutions. Also, we found a 3d equivalent of Weyl double copy relating the Cotton tensor to a product of two gauge theory field strength tensors.

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Copyright Declaration

List of publications

During the PhD project the following papers have been published:

[1] L. Alberte, C. de Rham, A. Momeni, J. Rumbutis and A. J. Tolley, *EFT of Interacting Spin-2 Fields*, *JHEP* **01** (2020) 131 [1910.05285].

[2] L. Alberte, C. de Rham, A. Momeni, J. Rumbutis and A. J. Tolley, *Positivity Constraints on Interacting Spin-2 Fields*, *JHEP* **03** (2020) 097 [1910.11799].

[3] L. Alberte, C. de Rham, A. Momeni, J. Rumbutis and A. J. Tolley, *Positivity Constraints on Interacting Pseudo-Linear Spin-2 Fields*, *JHEP* **07** (2020) 121 [1912.10018].

[4] A. Momeni, J. Rumbutis and A. J. Tolley, *Massive Gravity from Double Copy*, *JHEP* **12** (2020) 030 [2004.07853].

[5] A. Momeni, J. Rumbutis and A. J. Tolley, *Kaluza-Klein from colour-kinematics duality for massive fields*, *JHEP* **08** (2021) 081 [2012.09711].

[6] M. C. González, A. Momeni and J. Rumbutis, *Massive double copy in three spacetime dimensions*, *JHEP* **08** (2021) 116 [2107.00611].

[7] M. C. González, A. Momeni and J. Rumbutis, *Massive double copy in the high-energy limit*, *JHEP* **04** (2022) 094 [2112.08401].

[8] M. C. González, A. Momeni and J. Rumbutis, *Cotton Double Copy for Gravitational Waves*, 2202.10476.

Statement of Originality

The work presented in this thesis thesis is based on [4, 5, 6, 7, 8]. Contributions from other works have been appropriately referenced.

Conventions

Scattering Kinematics: We consider all particles to be ingoing, so that their four-momenta satisfy $\sum_i p_i = 0$. In 4 particle scattering we define the Mandelstam variables to be

$$s = -(p_1 + p_2)^2,\ t = -(p_1 + p_3)^2,\ u = -(p_1 + p_4)^2, \tag{1}$$

which are linearly dependent by $s + t + u = \sum 4m_i^2$. Similarly Mandelstam variables for higher point amplitudes are defined as $s_{ij} = -(p_i + p_j)^2$.

Metric Signature: The signature of the metric used in this work is mostly plus: $(-, +, +, .., +)$.

Scattering Amplitude: Our definition of the n-particle scattering amplitude, A_n, is the following:

$$\langle f|\hat{S} - \hat{1}|i\rangle = (2\pi)^d \delta^{(d)} \left(\sum_{i=1}^{n} p_i \right) A_n(\{p_i\}), \tag{2}$$

where $|i\rangle$ is the initial state, $|f\rangle$ is the final state, $\hat{S}$ is the S-matrix operator and p_i are the momenta of the particles.

Lie Algebra Generators of the Gauge Group: We use the following conventions for the generators, T_a:

$$\mathrm{Tr}[T_a T_b] = \delta_{ab}. \tag{3}$$

These are related to the usual generators (for example in [1]) as $T_a = \sqrt{2} t_a$. We define the structure constants, f_{abc} as

$$[T_a, T_b] = i f_{abc} T_c, \tag{4}$$

which again are larger by a factor of $\sqrt{2}$ than the structure constants in [1]. In terms of these f_{abc} the field strength tensor $F^a_{\mu\nu}$ is written as:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \frac{1}{\sqrt{2}} f_{abc} A^b_\mu A^c_\nu. \tag{5}$$

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Chapter 1

Introduction

During recent years, the study of scattering amplitudes has uncovered a number of new mathematical structures. One of the most surprising features discovered is a highly nontrivial relation between gauge theory and gravity scattering amplitudes, known as double copy. The fact that these theories are related can be quite surprising since perturbative calculations in general relativity seem to be much more complicated than those in gauge theories, at least by using Feynman diagrams. In fact the perturbative expansion of the Einstein-Hilbert term produces an infinite number of two derivative interaction terms, whereas Yang Mills action contains only a finite number of such terms. On the other hand one can anticipate the existence of such relations between gauge theory of gravity if they both can be embedded into a more fundamental unifying theory such as string theory. The most studied and understood example of double copy is the relation between (super) Yang-Mills and (super) gravity theories (in particular there has been a lot of studies of amplitude relations up to five loops between super Yang-Mills and $\mathcal{N} > 4$ supergravity theories [2, 3, 4, 5, 6, 7, 8]), however by now there is a large number of other theories related by double copy, including extensions with massive matter fields as well and many more. At present it is not entirely clear what makes these theories special so that they can be related by double copy (some work on this question has been undertaken in [9]). In particular it is not obvious if double copy could be extended to theories with only massive fields. Extension to the massive case is interesting for multiple reasons. First of all, massive gravity is a very interesting subject on its own since it is far less understood compared to general relativity due to more complicated nature of its interactions. A better analytic understanding of classical solutions and the Vainshtein mechanism would be desirable for this theory to be used as a model of cosmic acceleration. If a double copy construction for ghost-free massive gravity existed, it could simplify various calculations just like the standard double copy is used for gravitational wave calculations in general relativity and may lead to new insights in the non-linear interactions. This was the initial motivation for looking at a massive double copy in [10] which will be outlined below. A careful treatment of higher point amplitudes lead us to discover a problem with spurious poles[1] that one generally encounters in trying to double copy a generic massive theory. An example of a theory in which this problem occurs is massive Yang-Mills, whose naive double copy as discussed below is observed to exhibit the spurious poles explicitly in the 5 point amplitude [10]. The search for a way of avoiding these spurious poles has revealed that in four spacetime dimensions the only known examples of massive gauge theories that can be double copied are Kaluza-Klein theories associated with massless gauge theory in five dimensions [11]. Since we know that these theories can be double copied in five dimensions, the spurious pole cancellation in Kaluza-Klein theory works the same way as in the massless case. However in three spacetime dimensions there is another way of avoiding these unphysical singularities, and an example of that is topologically massive theories that can be related by double copy [12]. In the work described in this thesis we have checked the amplitude relation between topologically massive theories only up to 5 point level but our further investigations of classical double copy [13] suggested that there might be all loop and all multiplicity relations between the amplitudes of the two theories.

In the rest of this introduction chapter, I summarize the standard massless BCJ and classical double copy formalism. Since a large part of this thesis will be about extending the double copy formalism from massless to massive case, it is first necessary to understand the massless case. I then introduce the relevant massive gauge and gravity theories which will play a significant role in the remaining discussion.

[1]Spurious poles are poles in the scattering amplitude that are not associated with the exchange of physical states.

1.1 BCJ Double Copy

The Bern-Carrasco Johansson (BCJ) double-copy [2, 6] is a relation between the scattering amplitudes of gauge and gravity theories. There are many different examples of this relation, but the original and the most famous one is between (super) Yang-Mills and (super) gravity [6]. It has been proven to hold for all n point tree level amplitudes and it is conjectured to hold at loop level as well [14, 15, 16, 17]. This particular example of double copy can be understood from a string theory point view by considering relations between open and closed string amplitudes, known as KLT relations [18], and by looking at the low energy effective field theories of these two string theories. However double copy was found to be more general, there are known examples of double copy relations between two non-gravitational theories for example the non-linear sigma models and DBI and special Galileon [19, 20, 21, 22, 23, 24, 25, 26]. Also, there are many examples of relations between super Yang-Mills and supergravity theories [27]. Recently double copy has been extended for gauge theories with massive matter fields [28, 29]. The double copy procedure starts with writing the n point tree level gauge theory amplitude in the following form[2]

$$A_n = g^{n-2} \sum_i \frac{c_i n_i}{\prod_{\alpha_i} D_{\alpha_i}}, \tag{1.1}$$

where c_i are colour factors, n_i are kinematic factors containing products of momenta and polarizations and D_α are propagator factors. The sum i runs over distinct Feynman graphs with only (trivalent) cubic vertices. The kinematic factors are not unique since the colour factors are related by Jacobi-type identities, $c_i + c_j + c_k = 0$, which causes ambiguity in defining n's. To do double copy we need to fix this freedom by choosing n's satisfying colour-kinematics (CK) duality which states that the n's have to satisfy the same algebraic relations as the c's:

$$c_i + c_j + c_k = 0, \to n_i + n_j + n_k = 0. \tag{1.2}$$

[2]Note that we are using the amplitude definition given in (2). All of our scattering amplitudes are given as the momentum space delta function stripped amplitudes of $\langle \{k_f\} | \hat{S} - \hat{1} | \{k_i\} \rangle$. *i.e.* we forgo the introduction of an i as in $\hat{S} = \hat{1} + i\hat{T}$.

Then the colour factors can be replaced by kinematic factors, $c_i \to \bar{n}_i$ (these can be kinematic factors of a different gauge theory) and the coupling constant to gravitational coupling constant, $g \to \kappa/2$, where $\kappa = \sqrt{32\pi G}$ to get the double copy amplitude:

$$M_n = i\left(\frac{\kappa}{2}\right)^{n-2}\sum_i \frac{\bar{n}_i n_i}{\prod_{\alpha_i} D_{\alpha_i}}. \tag{1.3}$$

If we go in the opposite direction and replace n's by c's we obtained so called zeroth copy amplitude:

$$A_n^0 = g^{n-2}\sum_i \frac{c_i \bar{c}_i}{\prod_{\alpha_i} D_{\alpha_i}}, \tag{1.4}$$

corresponding to the amplitude of a bi-adjoint scalar theory, (the scalar field has indices of two gauge group Lie algebras $\phi^{aa'}$).

1.2 Classical Double Copy

Double copy relations go beyond scattering amplitudes. Many different classical solutions in gauge and gravity theories have been found to be related by the so called classical double copy. There are several different formulations of classical double copy, the most common one is Kerr-Schild double copy. A metric in Kerr-Schild coordinates is of the following form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa k_\mu k_\nu \phi \ , \tag{1.5}$$

where $\bar{g}_{\mu\nu}$ is the background metric and k^μ is a vector tangent to a null geodesic with the respect to both $\bar{g}$ and g:

$$\bar{g}_{\mu\nu} k^\mu k^\nu = 0, \ k^\mu \bar{\nabla}_\mu k_\nu = 0, \tag{1.6}$$

where $\bar{\nabla}$ is the background covariant derivative and the indices are raised/lowered with $\bar{g}$. In these coordinates the Ricci tensor is linear in ϕ [30]. Similarly the Kerr-Schild form of the gauge field is:

$$A^{a\ \mu} = c^a A^\mu = c^a k^\mu \phi \ , \tag{1.7}$$

where c^a is some constant colour factor. Again, this ansatz linearises the field strength tensor, $F^a_{\mu\nu}$, and the Yang-Mills equations of motion. Therefore, in both gravity and gauge theory the equations of motion are linear relations between ϕ and the sources. For example, when $\bar{g} = \eta$ and we are in the vacuum, both Yang-Mills and gravity equations of motion are solved if ϕ satisfies $\Box\phi = 0$ *i.e.* the equation of motion of a scalar. This can be interpreted as the zeroth copy - bi-adjoint scalar field given by

$$\Phi^{aa'} = c^a (c')^{a'} \phi. \tag{1.8}$$

Therefore we see that the classical Kerr-Schild double copy prescription is replacing c^a by k_μ in order to relate zeroth ($\Phi^{aa'} = c^a (c')^{a'} \phi$), single ($A^a_\mu = c^a k_\mu \phi$) and double copy ($h_{\mu\nu} = \phi k_\mu k_\nu$) solutions. In general when sources are present one needs to relate them in order to find double copy relations. Note that the graviton field in Kerr-Schild coordinates is traceless and symmetric so the axion and dilaton fields that appear in BCJ double copy are not present here which means that Kerr-Schild double copy is a relation between gauge theory and pure Einstein gravity. As an example consider Schwarzschild solution in Kerr-Schild coordinates:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\kappa^2 M}{8\pi r} k_\mu k_\nu, \tag{1.9}$$

where $k^\mu = \left(1, \frac{x^i}{r}\right)$ and $r^2 = x^i x_i$. The graviton field (or double copy) in this case is $h_{\mu\nu} = \frac{\kappa M}{8\pi r} k_\mu k_\nu$. Then the single copy is obtained by replacing $\frac{\kappa}{2} k_\mu M \to g c^a$

$$A^a_\mu = \frac{g c^a k_\mu}{4\pi r}. \tag{1.10}$$

This solves Yang-Mills equations which are Maxwell equations in this case (since the Kerr-Schild ansatz linearises the equation of motion):

$$\partial_\mu F^{a\mu\nu} = J^{a\nu}, \tag{1.11}$$

where $J^{a\nu}$ is a static point source located at the origin:

$$J^{a\nu} = -gc^a v^\nu \delta^3(\mathbf{x}), \tag{1.12}$$

where $v^\mu = (1,0,0,0)$. There are many other examples of Kerr-Schild double copy including Kerr black holes, black branes, plane waves, shock waves [30] and (A)dS spaces [31].

A related but slightly different approach is to relate the Weyl curvature tensor to the field strength tensor directly. This is known as Weyl double copy. At linearised order the Weyl tensor can be related to the square of the YM field strength

$$W^{\text{lin.}}_{\mu\nu\rho\lambda} = \frac{1}{2}\frac{F^{\text{lin.}}_{\mu\nu}F^{\text{lin.}}_{\rho\lambda}}{e^{ip\cdot x}} \ . \tag{1.13}$$

This relationship is known to extend beyond linearised case for exact solutions of Petrov Type D and Type N spacetimes when written in terms of spinors [32, 33]. Recently it has been attempted to explained it from a twistor space perspective [34, 35, 36].

There is a wide range of applications of double copy. For example, double copy is used for UV considerations of effective field theories [37, 38, 39, 40], efficient gravitational wave calculations [41, 42, 43, 44, 45, 46] and relations between classical solutions in different theories [47, 30, 31, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66]. The double copy has been studied for scattering amplitudes [67, 68] and correlation functions [69, 70, 71, 72] around more general backgrounds.

1.3 Massive Yang-Mills

Now we review the theories that will play a role in understanding massive double copy. The first theory which will be the starting point in attempting to do massive double copy is massive Yang-Mills. It can be thought of as the low energy effective action of Yang-Mills - Higgs theory, obtained by integrating out the Higgs field. We consider the case of gauge symmetry breaking which gives all gauge bosons the same mass, m. The cut off scale of the resulting effective field theory (EFT) is then the Higgs mass. In unitary gauge the leading (in 1/cutoff) terms in the effective Lagrangian are the following:

$$\mathcal{L}_{mYM} = -\frac{1}{4}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2}m^2\mathrm{tr}(A_\mu A^\mu), \tag{1.14}$$

where g is the coupling constant. This theory is not renormalizable and should be understood as an EFT which contains an infinite number of interactions suppressed by the cutoff scale. For example, we could add a quartic interaction $\mathrm{tr}(A_\mu A^\mu)^2$ to (1.14). To better understand the structure of this effective field theory we can reintroduce Stückelberg fields (Goldstone modes) by the following replacement

$$A_\mu \to \frac{\sqrt{2}i}{g}V(x)^{-1}D_\mu V(x) \tag{1.15}$$

where $D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}}A_\mu$ is the gauge covariant derivative and $V(x) = \exp\left[\frac{i}{\sqrt{2}\Lambda}T^a\phi^a(x)\right]$ where $\phi^a(x)$ are the Stückelberg fields, so that the Lagrangian with the restored gauge symmetry is

$$\mathcal{L}_{mYM} = -\frac{1}{4}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \Lambda^2\mathrm{tr}(D_\mu V D^\mu V^{-1}), \tag{1.16}$$

where $\Lambda = m/g$. It is manifestly gauge invariant under $D_\mu \to U(x)^{-1}D_\mu U(x)$ under which $V(x)$ transform as $V(x) \to U(x)^{-1}V(x)$ where $U(x) = \exp\left[\frac{i}{\sqrt{2}\Lambda}T^a\xi^a(x)\right]$ and $\xi^a(x)$ is the gauge transformation parameter. The unitary gauge Lagrangian, (1.14), is recovered by fixing the gauge $\phi^a = 0$.

The highest cutoff scale of the resulting EFT is $\Lambda = m/g$ (Goldstone mode decay constant). Additional operators in the effective action could lower the cutoff, but we assume that is not the case so Λ is the controlling scale. This allows to take the following decoupling limit: $g \to 0$, $m \to 0$ for fixed Λ. This results in a theory containing a free massless spin-1 field and an interacting non-linear sigma model

$$\mathcal{L}_{DL} = \lim_{g\to 0,\Lambda\text{ fixed}}\mathcal{L}_{mYM} = -\frac{1}{4}\sum_a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \Lambda^2\mathrm{tr}(\partial_\mu V\partial^\mu V^{-1}). \tag{1.17}$$

This illustrates the 'Goldstone equivalence theorem' which states that the leading interactions for the helicity-0 modes of the massive spin-1 states are determined by the effective field theory for the Goldsone modes described by (1.17). Note that adding additional unitary gauge interactions, such as $\mathrm{tr}(A_\mu A^\mu)^2$, would result in the addition of further irrelevant operators to the nonlinear sigma model Lagrangian in the decoupling limit, which have been considered in the double copy context for example in [37, 38, 39, 40].

The reason why we choose unitary gauge is that it is well suited for the tree amplitude calculations. This is because the expressions for off-shell vertices in massive and massless Yang-Mills are the same, so the only difference in calculating amplitudes is the massive propagator:

$$\frac{-i\hat{\eta}_{\mu\nu}}{p^2+m^2}\,, \tag{1.18}$$

where $\hat{\eta}_{\mu\nu} = \eta_{\mu\nu} + p_\mu p_\nu/m^2$. Similarly to the massless case [6] we express the tree level n-point scattering amplitude of massive Yang-Mills as:

$$A_n = g^{n-2}\sum_i \frac{c_i n_i}{\prod_{\alpha_i}(-p_{\alpha_i}^2 - m^2)}, \tag{1.19}$$

where c_i are colour factors which are products of the structure constants of the gauge group, n_i are the kinematic factors, i labels distinct Feynman diagrams and α_i labels internal propagators in the Feynman diagram. This is the same expression for the amplitude as in massless case but the massless propagators $p_{\alpha_i}^2$ are replaced now by massive $p_{\alpha_i}^2 + m^2$. However, the expressions of kinematic factors, n_i, in terms of products of polarizations and momenta are not the same as in the massless case since they contain information about the massive polarization states in the massive propagator $\hat{\eta}_{\mu\nu}$, and also, the external momenta now satisfy $p_i^2 = -m^2$ instead of $p_i^2 = 0$. Therefore, it is not guaranteed that the colour-kinematics duality still holds.

1.4 dRGT Massive Gravity

If massive Yang-Mills, described in the previous section, can be double copied it should give a massive gravity theory. A natural candidate (for massive graviton interactions in the double copy) is dRGT theory of massive gravity [73], since it is the highest cutoff effective field theory for interacting massive spin-2 fields just like massive Yang-Mills is the highest cutoff effective field theory for massive spin-1 fields. In this theory the diffeomorphism symmetry is broken by introducing the non-dynamical reference metric, $f_{\mu\nu}$, in the action making the graviton massive. Generic interactions in massive gravity action would lead to a theory that breaks down at the scale $\Lambda_5 = (m^4 M_{Pl})^{1/5}$ due to appearance of Boulware-Deser ghost [74]. However, a specific tuning of the interaction terms was found in [73, 75] which removes the ghost and raises the cutoff scale to $\Lambda_3 = (m^2 M_{Pl})^{1/3}$ where the perturbative unitarity breaks down. This theory is known as de Rham, Gabadadze and Tolley (dRGT) massive gravity.

Massive gravity is interesting from both a theoretical and a phenomenological point of view. It is a candidate effective theory for gravity at large scales because it can explain the late-time accelerated expansion of the universe without the tuning of the cosmological constant, *i.e.* it can provide a solution to the cosmological constant problem. There are two different ideas of how this can be achieved. The first idea is to allow the cosmological constant to be large but make gravity weaker at large distances, relevant to cosmology. Then the effect of large cosmological constant on the acceleration of the universe can be much smaller than that in general relativity. This is known as degravitation and the corresponding massive gravity solutions are known as screening solutions. The second idea is called self-acceleration. It turns out that in massive gravity it is possible to have zero cosmological constant (perhaps due to some symmetry or mechanism forcing the vacuum energy to be zero) and still find accelerating cosmological solutions with the Hubble scale of the order of the graviton mass. This acceleration can be thought as being sourced by a condensate of massive gravitons. These two mechanisms are opposite of each other but both screening and self-accelerating solutions are expected to be present in massive gravity. For more details see [76, 77].

There are other important differences between massive and massless gravity due to different number of degrees of freedom. For example, even in the massless limit the Newtonian potential in massive gravity is different due to the helicity-0 graviton exchange between interacting objects. This effect is known as vDVZ discontinuity [78, 79] which seems to be in contradiction with solar system observations. However there is a region in which this helicity-0 contribution is suppressed as first noted by Vainshtein [80]. The linear theory in this region breaks down and the non-linearities become important which recover the result of general relativity. This will be explained in a bit more detail after the description of the decoupling limit.

In unitary gauge the action of dRGT massive gravity can be written in terms of the following variable [73]

$$\mathcal{K}^\mu_\nu(f,g) = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu_\nu \,, \tag{1.20}$$

where $g_{\mu\nu}$ is the dynamical metric and $f_{\mu\nu}$ is the non-dynamical metric that breaks diffeomorphism invariance. This variable $\mathcal{K}^\mu_\nu$ has a strange square root metric structure, which is required in order to avoid having a ghost [73]. We will see in section 2.1.5 that this square root metric structure has a straightforward Λ_3 decoupling limit. The 4d Lagrangian in unitary gauge is [77]

$$\mathcal{L}_{mGR} = \frac{M_{\rm Pl}^2}{2}\sqrt{-g}R + \frac{m^2 M_{\rm Pl}^2}{4}\sqrt{-g}\sum_{n=0}^{4}\kappa_n \mathcal{U}_n\left[\mathcal{K}\right], \tag{1.21}$$

where [3]

$$\mathcal{U}_n = \varepsilon_{\mu_1\ldots\mu_n\ldots\mu_4}\varepsilon^{\nu_1\ldots\nu_n\ldots\nu_4}\mathcal{K}^{\mu_1}_{\nu_1}\ldots\mathcal{K}^{\mu_n}_{\nu_n}\delta^{\mu_{n+1}}_{\nu_{n+1}}\ldots\delta^{\mu_4}_{\nu_4}. \tag{1.22}$$

When we expend the theory around a flat background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ the κ_0 term gives rise to a cosmological constant term and κ_1 to a term linear in $h_{\mu\nu}$ *i.e.* a tadpole so if we want flat space to be the solution of the vacuum equations of motion we set $\kappa_0 = \kappa_1 = 0$. The κ_2 coefficient will appear in the kinetic Fierz-Pauli term in the action so κ_2 can be fixed by canonical normalization of $h_{\mu\nu}$. This corresponds to setting $\kappa_2 = 1$. Explicitly the terms in the

[3] We use Euclidean conventions so that for flat spacetime $\varepsilon_{0123} = \varepsilon^{0123} = 1$, i.e. in the Lorentzian $\varepsilon_{\mu\nu\alpha\beta} = -\eta_{\mu\mu'}\eta_{\nu\nu'}\eta_{\alpha\alpha'}\eta_{\beta\beta'}\varepsilon^{\mu'\nu'\alpha'\beta'}$. As long as we are clear that we use one of them with all indices up and the other with all indices down together with $\varepsilon_{i_1\ldots i_k i_{k+1}\ldots i_d}\varepsilon^{i_1\ldots i_k j_{k+1}\ldots j_d} = k!\,\delta^{j_{k+1}\ldots j_d}_{i_{k+1}\ldots i_d}$ with the generalized Kronecker delta expressed as a determinant of a matrix built out of δ's.

potential are

$$\mathcal{U}_2(\mathcal{K}) = 2\left([\mathcal{K}]^2 - [\mathcal{K}^2]\right) , \tag{1.23}$$

$$\mathcal{U}_3(\mathcal{K}) = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3] , \tag{1.24}$$

$$\mathcal{U}_4(\mathcal{K}) = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4] , \tag{1.25}$$

where the squared brackets denote the trace. The free parameters are $\kappa_3\,, \kappa_4$ as well as the graviton mass m^2. We will set $f_{\mu\nu}$ to be Minkowski metric, $\eta_{\mu\nu}$ for scattering amplitude calculations in Minkowski spacetime. It can be shown that the terms in (1.21) are the only allowed interactions for which the equations of motion for all 5 propagating degrees of freedom are second order [81].

If we consider this theory as an effective field theory its cutoff is $\Lambda_3 = (m^2 M_{Pl})^{1/3}$. This is the highest possible cutoff for Lorentz invariant interacting massive spin-2 theories [73, 75] in 4 dimensions. From this point of view the action (1.21) should be thought of as the leading terms in the expansion in $1/\Lambda_3$ and the sub leading terms should be of the form

$$\Delta\mathcal{L} = \Lambda_3^4 \sqrt{-g}\, F[g_{\mu\nu}, K_{\mu\nu}, \frac{\nabla_\mu}{\Lambda_3}, M_{\rm Pl} R_{\mu\nu\rho\sigma}] \,. \tag{1.26}$$

where F contains the sum of all diffeomorphism invariant scalar operators[4] with dimensionless Wilson coefficients.

Similarly to massive Yang-Mills we can restore the diffeomorphism symmetry with Stückelberg fields. Since the symmetry is broken by the non-dynamical metric $f_{\mu\nu}$ we can restore it by making $f_{\mu\nu}$ transform as a tensor. This is achieved by the following replacement:

$$f_{\mu\nu} \to \partial_\mu \Phi^A \partial_\nu \Phi^B \eta_{AB} \,, \tag{1.27}$$

where the Latin indices A, B have the same range as Lorentz indices. If we shift the Stückelberg fields as $\Phi^A(x) = x^A + \pi^A(x)$, then $\pi^A(x)$ is the analogue of the $\phi^a(x)$ in $V(x)$ (1.16), and the unitary gauge is $\pi^A(x) = 0$. The full derivation of Λ_3 decoupling limit can be found in [82].

[4] All breaking of diffeomorphism invariance is captured by the tensor $\mathcal{K}_{\mu\nu}$, so if $\mathcal{K}_{\mu\nu}$ was transforming as a tensor under diffeomorphism all terms in the Lagrangian would be diffeomorphism invariant.

Unlike the massive Yang-Mills case where in the decoupling limit the helicity-1 is free and the only interactions are the helicity-0 self interactions, in massive gravity decoupling limit the helicity-1 - helicity-0 interactions are still present as well as mixing between helicity-2 and helicity-0 modes. However the helicity-1 modes appear quadratically so it consistent to truncate the theory by switching them off, since they will not be sourced by the other helicity modes. Then the decoupling limit action for helicity-0 and helicity-2 modes, without helicity-0 and helicity-2 kinetic mixing terms (which are removed by a local field redefinition of h) is [77]

$$\mathcal{L}_{DL} = \lim_{m\to 0, M_{Pl}\to\infty, \Lambda_3 \text{ fixed}} \mathcal{L}_{mGR} = \frac{1}{4} h^{\mu\nu}\mathcal{E}h_{\mu\nu} + \sum_{n=2}^{5} c_n \mathcal{L}^{(n)}_{Gal}[\pi] + b h_{\mu\nu} X^{\mu\nu}(\Pi) \tag{1.28}$$

where

$$h^{\mu\nu}\mathcal{E}h_{\mu\nu} = -\varepsilon^{\nu_1\nu_2\nu_3\nu_4}\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\delta^{\mu_1}_{\nu_1} h^{\mu_2}_{\nu_2}\partial_{\nu_3}\partial^{\mu_3} h^{\mu_4}_{\nu_4}, \tag{1.29}$$

$\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi$, b is a constant, X is defined as

$$X^{\mu\nu}(\Pi) = \varepsilon^{\mu}_{\mu_1\mu_2\mu_3}\varepsilon^{\nu\nu_1\nu_2\nu_3}\Pi^{\mu_1}_{\nu_1}\Pi^{\mu_2}_{\nu_2}\Pi^{\mu_3}_{\nu_3} \tag{1.30}$$

and $\mathcal{L}^{(n)}_{Gal}$ are the Galileon terms

$$\mathcal{L}^{(n)}_{Gal}[\pi] = \pi\mathcal{U}_{n-1}(\Pi). \tag{1.31}$$

There is still a coupling between helicity-2 and helicity-0 given by the last term in (1.28). It cannot be removed by a local field redefinition only by a non-local one of the form $h_{\mu\nu} \to h_{\mu\nu} + 2bD_{\mu\nu\rho\sigma}X^{\rho\sigma}$, where $D = \mathcal{E}^{-1}$ is the graviton propagator. Therefore, we see that the decoupling limit of massive gravity in terms of local operators is a scalar theory with self interactions given by the Galileon terms (1.31) (known as Galileon theory) coupled to a massless spin-2 field.

From the decoupling limit we can see the origin of the vDVZ discontinuity [78, 79]. If we have external matter minimally coupled to massive gravity through the energy-stress tensor $T_{\mu\nu}$ then in the decoupling limit we have two additional terms:

$$\frac{1}{M_{Pl}} h_{\mu\nu} T^{\mu\nu} + \frac{1}{M_{Pl}} \pi T^{\mu}_{\mu}. \tag{1.32}$$

We see that for the sources where T^{μ}_{μ} is non-zero we get an additional contribution to the force between them given by the exchange of π at linear level. As mentioned before this gives rise to vDVZ discontinuity since the massless limit of massive gravity does not agree with general relativity. For example the Newtonian potential in this limit is $-\frac{4GM}{3R}$ rather than $-\frac{GM}{R}$ as in linearized general relativity. However, the distance scale where the linearized theory breaks down and non-linearities become important in massive gravity is different from that of general relativity in which this distance scale for spherically symmetric solutions is the Schwarzschild radius $r_S = 2GM$. Instead, in Λ_3 massive gravity this distance is the Vainshtein radius $r_V = \left(\frac{GM}{m^2}\right)^{1/3}$. Inside the region $r \ll r_V$ we cannot trust the linearized theory and it turns out non-linear interactions of π make its contribution to the Newtonian potential suppressed relative to the $h_{\mu\nu}$ exchange [83] so the scalar "fifth" force is screened inside this region. For astrophysical sources and small graviton mass r_V can be much larger compared to r_S so the solar systems tests (which would be in the region r_V) should be probing the non-linear regime of massive gravity theory. The full theory of massive gravity is quite complicated [73] so it is hard to obtained exact classical solutions and check if they exhibit the Vainshtein screening mechanism. However it is believed that these features of massive gravity are captured by the Galileon theory in which the screening mechanism has been studied mostly numerically due to difficulties in doing analytic calculations in non-linear regime [84, 85, 86, 87]. Thus we see why the double copy construction of massive gravity could be beneficial - it might simplify the analytic calculations (just like it simplifies graviational wave calculations in general relativity) allowing us to better understand the Vainshtein screening mechanism. This would of course require non-perturbative construction of massive gravity solutions (since the perturbative methods used in general relativity such as post-Newtonian or post-Minkowskian expansion would not help to probe the non-linear dynamics) perhaps given by classical double copy. Also, a double copy construction of massive gravity might give a simpler way of calculating gravitational waveforms in this theory that could be tested experimentally.

1.5 Topologically Massive Gauge and Gravity Theories in 3d

As will be described in the second chapter of this thesis, there are interesting examples of double copy relations between topologically massive gauge and gravity theories in three spacetime dimensions. This section contains a short introduction of these theories. We start with topologically massive Yang-Mills theory (TMYM) which propagates one spin-1 degree of freedom and is given by a standard Yang-Mills term supplemented with a Chern-Simons term. The TMYM action is

$$S_{TMYM} = \int d^3x \frac{1}{g^2}\left(-\frac{1}{4}\text{tr}\left[F^{\mu\nu}F_{\mu\nu}\right] + \epsilon_{\mu\nu\rho}\frac{m}{2}\text{tr}\left[A^\mu\partial^\nu A^\rho + \frac{2}{3}A^\mu A^\nu A^\rho\right]\right), \tag{1.33}$$

where m is the mass of the gauge field and g the coupling strength. The mass term is proportional to the Chern-Simons level. Under gauge transformations, $A_\mu \to U^{-1}A_\mu U + U^{-1}\partial_\mu U$, the TMYM is not fully invariant, instead it changes by two terms:

$$S_{TMYM} \to S_{TMYM} + \delta S_1 + \delta S_2 \tag{1.34}$$

where

$$\delta S_1 \propto \int d^3x \epsilon^{\mu\nu\alpha}\text{tr}[\partial_\mu\left(A_\alpha\partial_\nu U U^{-1}\right)] \tag{1.35}$$

and

$$\delta S_2 \propto \int d^3x \epsilon^{\mu\nu\alpha}\text{tr}[\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\alpha U U^{-1}]. \tag{1.36}$$

The first term, δS_1, is a total derivative and it vanishes if we only consider gauge transformations such that $U(x)$ goes to the identity at infinity. The second term, δS_2, does not vanish in the non-Abelian case. Instead, it is equal to $8\pi^2\frac{m}{g^2}w(U)$, where $w(U)$ is the winding number of the gauge transformation which is an integer. Since the partition function should be invariant under gauge transformations, $Z = \int DAe^{iS} = \int DAe^{iS+i8\pi^2\frac{m}{g^2}w(U)} = Ze^{i8\pi^2\frac{m}{g^2}w(U)}$, $8\pi^2\frac{m}{g^2}w(U)$

must be an integer multiple of 2π so the mass of the gauge field must be quantized:

$$4\pi\frac{m}{g^2} = \text{integer}. \tag{1.37}$$

Pure Yang-Mills theory in 3d describes a single massless spin-1 degree of freedom, while pure Chern-Simons theory is topological, with no propagating degrees of freedom. However, when combined together into topologically massive Yang-Mills theory they describe a singe massive spin-1 degree of freedom [88]. A massive spin-1 field in a Lorentz invariant (including discrete transformations such as parity) 3d theory would have 2 degrees of freedom, but this theory breaks parity due to the parity odd Chern-Simons term. Unlike 4d massive Yang-Mills theory, topologically massive Yang-Mills is renormalizable (in fact it is UV finite or super-renormalizable) [88].

Pure Chern-Simons theory itself has some interesting properties even though it does not have propagating degrees of freedom. The expectation value of a Wilson loop calculated along some closed path in this theory can be related to the knot invariants of that path [89]. Also, it has a connection to the 4d topological theta term $S_\theta = \int_{M_4} \mathrm{tr}[F \wedge F]$ which is a total derivative $S_\theta = \int_{M_4} dK$, where $K = \mathrm{tr}[(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)]$. The 3d Chern-Simons term can be obtained by considering S_θ on a manifold M_4 with a boundary $\partial M_4 = M_3$ and using Stokes' theorem: $\int_{M_4} dK = \int_{\partial M_4} K = \int_{M_3} \mathrm{tr}[A \wedge dA + \frac{2}{3}A \wedge A \wedge A]$ [88].

To see what effect Chern-Simons term has on the force mediated by A_μ, we can look at the Abelian case, *i.e.* topologically massive electrodynamics (TME). The equation of motion with the source J^ν is

$$\partial_\mu F^{\mu\nu} + \frac{m}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = J^\nu. \tag{1.38}$$

This is solved by

$$A^\mu = \frac{1}{-\Box + m^2}\left(\eta^{\mu\nu} + \frac{1}{m}\epsilon^{\mu\nu\alpha}\partial_\alpha\right)J_\nu + \text{gauge dependent term}. \tag{1.39}$$

This looks like the standard massive vector field but the source is "twisted": $J'^\mu = J^\mu - \frac{m}{\Box}\epsilon^{\mu\alpha\beta}\partial_\alpha J_\beta$. This twisting transformation makes non-spinning sources create field profiles as if they were spinning. In particular if $J^0 = e\delta^2(\mathbf{x})$ (describing point static charge) and $J_i =$

$g\epsilon_{ij}\partial_j\delta^2(\mathbf{x})$ (describing dipole current) then the electric and magnetic fields created by this source are

$$E_i = F^0_i = (e+gm)\partial_i Y(r),\ B = \epsilon^{ij}F_{ij} = (e+gm)\nabla^2 Y(r), \tag{1.40}$$

where $Y(r) = \frac{1}{2\pi}K_0(mr)$. $K_0(x)$ is Bessel function which behaves as $\ln(x)$ at the origin and $e^{-x}/\sqrt{x}$ at infinity, so the fields have a finite range due to the mediator being massive. Also, we see that the electric field is not proportional to e and the magnetic field is not proportional to g as in the case of normal electromagnetism. Instead they are both proportional to the combination $e+mg$ which means that there is a magnetic field even when $g=0$ and an electric field even when $e=0$. Therefore, we see that the topological massive vector field sees the source J as if it has a different classical spin. Also, for the specific choice of $e+gm=0$ the fields are zero so it is possible to have charge and current configurations that do not create electromagnetic fields.

Topologically massive gravity (TMG) action is given by

$$S_{TMG} = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\left(-R - \frac{1}{2m}\epsilon^{\mu\nu\rho}\left(\Gamma^\alpha_{\mu\sigma}\partial_\nu\Gamma^\sigma_{\alpha\rho} + \frac{2}{3}\Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\beta}\Gamma^\beta_{\rho\alpha}\right)\right)\ . \tag{1.41}$$

Similarly to TMYM it is an addition of a gravitational Chern-Simons term to the Einsten-Hilbert term. However, the sign of the Einstein-Hilbert term is opposite to the conventional one; this is required so that the sign of the kinetic term of the physical spin-2 mode is correct and the theory is ghost free. Like in TMYM case the change under gauge (diffeomorphism in this case) transformation gives a term related to the topology of the gauge transformation. It is easier to see this using the first order formalism, where we write TMG action in terms of dreibein, e^a_μ, and the spin connection, $\omega^a_{b\mu}$, one-forms

$$S_{TMG} = \frac{1}{\kappa^2}\int\left(-e^a\wedge d\omega_a - \epsilon_{abc}e^a\wedge\omega^b\wedge\omega^c - \frac{1}{2m}\left(\omega^a\wedge d\omega_a + \frac{2}{3}\epsilon_{abc}\omega^a\wedge\omega^b\wedge\omega^c\right)\right), \tag{1.42}$$

where $\omega^a_\mu = \frac{1}{2}\epsilon^{abc}\omega_{bc\mu}$. The latin dreibein indices can be transformed with a local Lorentz transformation $\Lambda^{a'}_a$, which leaves the Einstein-Hilbert term invariant but might change the gravitational Chern-Simons term by a total derivative. Comparing this with (1.33) we see that the gravitational Chern-Simons term has a similar structure to the gauge theory Chern-Simons term. In fact under local Lorentz transformations, $\delta e^a = \epsilon^a_{bc} e^b \lambda^c$, ω^a_μ transforms just like a gauge field: $\delta\omega^a = d\lambda^a + \epsilon^{abc}\omega_b\lambda_c$ [90]. Therefore, the change in the action under these local Lorentz transformations will be also given by (1.36), however, in this case the gauge group (of local Lorentz transformations) is non-compact $SO(2,1)$ and the homotopy of its maximal compact subgroup, $SO(2)$, is trivial [91], so the additional term vanishes and the action is fully gauge invariant. Therefore, no quantization of the mass is required.

Similarly to TMYM, this theory describes a single massive degree of freedom but now it is spin-2 [88]. Again, a massive spin-2 particle in a Lorentz invariant theory in 3d should have 2 degrees of freedom but the gravitational Chern-Simons term breaks parity. The Einstein-Hilbert term alone in 3d gives no propagating degrees of freedom, that is why its negative sign does not introduce ghosts. There is a similar construction of the gravitational Chern-Simons term as in gauge theory case. This is done by considering the 4d Hirzebruch-Pontryagin density, $\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R_{\mu\nu\rho\sigma}R^{\rho\sigma}_{\alpha\beta}$, (which is a total derivative $\partial_\mu X^\mu$ for some X_μ) and analogously to gauge theory case relating its integral over 4d manifold to an integral over the boundary gives the 3d gravitational Chern-Simons term [88].

To see how matter coupled to topological massive gravity interact again we consider the linearised equation of motion (in de Donder gauge),

$$\partial^2 h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial^2 h + \frac{1}{m}\partial_\rho \epsilon^{\rho\lambda}{}_{(\mu}\partial^2 h_{\nu)\lambda} = \kappa T_{\mu\nu} \ . \tag{1.43}$$

Similar to the TME case, it is possible to rewrite this as

$$\left(-\partial^2 + m^2\right) h_{\mu\nu} = \kappa\frac{m^2}{\partial^2}\left[T_{\mu\nu} - \eta_{\mu\nu}T - \frac{1}{m}\partial_\rho\epsilon^{\rho\lambda}{}_{(\mu}T_{\nu)\lambda} + \frac{1}{2m^2}\left(\eta_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu\right)T\right] \ . \tag{1.44}$$

The right hand side of this equation again is "twisted" transformation of the stress-energy tensor, $T_{\mu\nu}$. The solution sourced by a stationary spinning particle at the origin, given by $T^0_0 = -M\delta^2(\mathbf{r})$, $T^i_0 = -\frac{1}{2}\sigma\epsilon^{ij}\partial_j\delta^2(\mathbf{r})$, $T_{ij} = 0$ is the following [92]:

$$h_{0i} = \frac{\kappa}{m}(M + m\sigma)\epsilon^{ij}\partial_j(C(r) - Y(r)) \tag{1.45}$$

$$h_{ij} = (\kappa(M + m\sigma)Y(r) - 2\kappa MC(r))\,\delta_{ij} \tag{1.46}$$

$$h_{00} = \kappa(M + m\sigma)Y(r), \tag{1.47}$$

where $C(x) = -\frac{1}{2\pi}\ln(x)$ and $Y(r) = \frac{1}{2\pi}K_0(mr)$ as before. h_{0i} gives a gravito-magnetic field and is responsible for the frame dragging effect. In general relativity it would be zero for non-rotating sources ($\sigma = 0$) but here it is non zero. Also, h_{00} gives a Newtonian potential and we see that it is not just proportional to M but instead to $M + m\sigma$. This means that the equivalence principle between gravitational and inertial mass is no longer true in this theory. Note that $M + m\sigma$ can be positive or negative so it is possible in this theory to have repulsive gravity as well as the case $M + m\sigma = 0$ where gravitational field vanishes in linearised theory. It is easy to notice from this short summary that these topologically massive gauge and gravity theories share a lot of similarities; both can be obtained from 4d topological terms, both break parity and describe a single massive degree of freedom and see the sources as if they were "twisted". As we shall see later there is strong evidence that these theories are related by double copy to all orders.

1.6 Structure of the Thesis

The rest of the thesis is structured as following:

1. In the second chapter our proposed massive extension of BCJ double copy is introduced, which was first discussed in [10], and the origin of unphysical poles is explained, following an example of massive theories in which the double copy procedure gives these spurious poles. An example of a theory avoiding the spurious poles is given, namely Kaluza-Klein theory, and it is explained how the attempts of deforming this theory in order to find new massive theories compatible with double copy fail. The massive double copy in 3d is described where massive theories that are not Kaluza-Klein theories can avoid spurious poles in the double copy amplitudes and an example of this relation is given between topologically massive gauge and gravity theories.

2. In the third chapter this relation between topologically massive gauge and gravity theories is explored further by considering eikonal high energy scattering as well as classical solutions. This includes a newly found Weyl double copy analog for 3d - Cotton double copy that has been observed in type N solutions [93].

Chapter 2

Massive BCJ Double Copy

In this chapter I will introduce a proposal for a double copy for massive theories, first given in [10]. This chapter is organised as follows: first BCJ double copy in matrix notation will be introduced, from which it will be easy to see spurious poles in the double copy of a generic massive gauge theory. Then I discuss an example of a theory that has this problem - massive Yang-Mills and an example of a theory which avoids spurious poles - Kaluza Klein theory. Then I discuss a proposal [12] for massive double copy relation in three spacetime dimensions which avoids spurious poles at least at 5 point level.

2.0.1 BCJ Double Copy in Matrix Notation

This section contains a review of the BCJ double copy formalism for scattering amplitudes with matrix notation, as introduced in [11, 12], which is useful to understand the issues with spurious poles that arise when trying to square amplitudes including massive gauge fields. In a generic gauge theory the n-point tree level amplitude, A_n, can be written as

$$A_n = g^{n-2} c^T D^{-1} n \ , \tag{2.1}$$

where g is the coupling, c is the vector of colour factors, D is the diagonal matrix with elements given by products of inverse propagators and n is the column vector of kinematic numerators.

The Jacobi identities and CK duality[1] in matrix form can be written as

$$Mc = 0 \;\rightarrow\; Mn = 0 \;, \tag{2.2}$$

for a matrix M with entries ± 1. Once a representation satisfying CK duality is found, the corresponding amplitude in the gravitational theory, M_n, is given as

$$M_n = i\left(\frac{\kappa}{2}\right)^{n-2} n^T D^{-1} n \;. \tag{2.3}$$

The kinematic factors directly calculated from Feynman diagrams may not satisfy Jacobi relations, therefore they must be shifted as

$$n \rightarrow n + \Delta n \;, \tag{2.4}$$

such that the amplitude in (2.1) is unchanged. This can be achieved by setting

$$D^{-1}\Delta n = M^T v \;, \tag{2.5}$$

where v is a vector to be determined. For a specific theory there could be some other form of the shift that leaves the amplitude unchanged, however for a generic theory, where we only know the Jacobi identities between colour factors, this is the most general shift. Such shifts are usually referred to as generalized gauge transformations since in the massless case they can be obtained by a gauge transformation and a field redefinition of the gauge field. In order to satisfy the CK duality, the shifted n must obey the following equation

$$M(n + \Delta n) = 0 \;, \tag{2.6}$$

[1]Note that unlike in massless case where CK duality is required for gauge invariance of the amplitudes, in the massive case with no gauge symmetry it is not clear if CK has to be imposed. In fact it is possible that some other representation of kinematic numerators is more physical for doing massive double copy due to some other reasons. We leave this point for future work and assume that the rules for massive double copy are the same as in the massless case even though this might be too restrictive.

which combined with (2.5) gives

$$MDM^T v = -Mn \ . \tag{2.7}$$

The number of non-zero rows of M will be equal to the number of Jacobi identities, N_j. Therefore, MDM^T is block diagonal with a $N_j \times N_j$ symmetric block matrix A and all other elements equal to zero. Mn will have at most N_j non-zero elements so we can write it as

$$Mn = (U, 0, ..., 0) \ . \tag{2.8}$$

Note that the vector U measures the violation of the CK algebra. We can see that in order to find the shifts we need to find v, for which we need to invert the matrix A:

$$v = -(A^{-1}U, 0, .., 0) \ . \tag{2.9}$$

It may be that A does not have full rank as it happens in pure Yang-Mills theory. In that case, we can still invert A in the subspace orthogonal to all null eigenvectors of A if U is in that subspace,

$$U \cdot \text{null}(A) = 0, \tag{2.10}$$

i.e. if there are certain relations between kinematic factors known as BCJ relations.
If we now substitute the shifted n back into (2.15) we obtain the following gravity amplitude:

$$\begin{aligned} -i\left(\frac{\kappa}{2}\right)^{-(n-2)} M_n &= (n+\Delta n)^T D^{-1} (n+\Delta n) \\ &= (n+\Delta n)^T D^{-1} n + (n+\Delta n)^T M^T v \\ &= n^T D^{-1} n + \Delta n^T D^{-1} n \ , \end{aligned} \tag{2.11}$$

where going from the first to the second line we expanded the expression and used (2.5), and going from the second to the third line we used (2.6) to set the last term to zero. Now we can replace Δn using (2.5) and (2.9) to obtain

$$-i\left(\frac{\kappa}{2}\right)^{-(n-2)} M_n = n^T D^{-1} n + v^T M n = n^T D^{-1} n - U^T A^{-1} U \ . \tag{2.12}$$

Again, in the case when A does not have full rank, A^{-1} and U must be in the subspace orthogonal to the null vectors.

2.0.2 Spurious Poles

Note that the poles of M_n come from kinematic configurations for which either D or A becomes singular. Since D contains the physical poles in the gauge theory amplitude, spurious poles could only arise from A. In particular, since, $A^{-1} = \frac{1}{\det(A)}$(cofactor matrix) we see that the double copy amplitude has the factor of $\det(A)$ in the denominator. In general this factor is not equal to a product of physical poles and therefore it gives spurious poles.

Now let us consider the 4-point amplitude with all massive states with mass m in the adjoint representation. We have a single Jacobi identity, $c_s + c_t + c_u = 0$, so M has only one non zero row equal to $(1, 1, 1)$ and $D = \mathrm{diag}(s - m^2, t - m^2, u - m^2)$. In this case $Mn = (n_s + n_t + n_u, 0, 0)$, so $U = n_s + n_t + n_u$ and A is 1×1 matrix $A = s - m^2 + t - m^2 + u - m^2 = m^2$. Therefore, using (2.12) the double copy of the massive 4-point amplitude is

$$-i\left(\frac{\kappa}{2}\right)^{-2} M_4 = \frac{n_s^2}{s - m^2} + \frac{n_t^2}{t - m^2} + \frac{n_u^2}{u - m^2} - \frac{(n_s + n_t + n_u)^2}{m^2} \ . \tag{2.13}$$

This shows that no spurious poles arise for the 4-point amplitudes. The 5-point case is more complicated and generically gives rise to spurious poles. An example of a massive gauge theory that has this problem this will be described in the next section.

2.1 Massive Yang-Mills and dRGT Massive Gravity

The most natural starting point for attempting to do massive double copy is to try square the massive Yang-Mills theory which is obtained by adding a mass term to Yang-Mills theory *i.e.* it is a low energy effective field theory of Yang-Mills coupled to Higgs field which spontaneously breaks the gauge symmetry in a way that all of the gauge bosons acquire the same mass. We expect the double copy of this theory to contain a massive spin-2 field with self interactions described by ghost-free dRGT massive gravity. These massive effective field theories are described in 1.3 and 1.4. Some evidence that they could be related can be seen from their decoupling limits described in the previous sections. The decoupling limit of massive Yang-Mills theory is a non-linear sigma model while that of dRGT massive gravity is a Galileon-like theory [94, 75, 73, 82] and it is known that the special Galileon theory can be obtained as a double copy of a non-linear sigma model [20, 95, 21, 25, 26]. Also the possibility of this relation was proposed before for example in [96].

While at 3 and 4 point level the double copy of this theory matches the amplitudes of a massive gravity theory at higher points the relation fails due to appearance of spurious poles as was first found by [97]. Despite this problem even up to 4pt level this massive double copy has some interesting features. For example, the double copy procedure does not commute with taking decoupling (high energy) limit. These features will be described in this section.

2.1.1 Degrees of Freedom

First we look at the spectrum of the double copy theory. The asymptotic states in the gravitational theory are identified with the tensor products of gauge theory asymptotic states, ignoring their colour indices. For example, if we double copy pure Yang-Mills theory we get the following states:

$$A_\mu \otimes A_\nu = h_{\mu\nu} \oplus B_{\mu\nu} \oplus \phi. \tag{2.14}$$

This is obtained by decomposing the tensor product of two vector (spin 1) representations of Lorentz group into irreducible representations. $h_{\mu\nu}$ is the graviton, $B_{\mu\nu}$ is a 2-form field and ϕ is a massless scalar field known as dilaton. In the case of four dimensions and massless fields $B_{\mu\nu}$ is dual to a pseudo-scalar, known as axion. In terms of the number of degrees of freedom it is $2 \times 2 = 2 + 1^* + 1$.

In the case of massive Yang-Mills, we still have (2.14) but all of the fields are massive: $h_{\mu\nu}$ is a massive spin-2 field, $B_{\mu\nu}$ is a massive 2-form field which in four dimensions is dual to a massive spin-1 field and ϕ is a massive scalar. In terms of the number degrees of freedom now we have $3 \times 3 = 5 + 3 + 1$. Instead of working with the $B_{\mu\nu}$ field we work with its dual massive vector field, A_μ. We can see that just by the looking at the double copy of the free theory there already is an interesting difference between the spectrum of massless and massive double copy: the B field in the massive case is spin-1 rather than a spin-0 field like in the massless case.

2.1.2 Double Copy Construction of Scattering Amplitudes

Now that we know the spectrum on the massive gravity side we proceed to scattering amplitudes. We assume that, just as in the massless case, the representation of the massive Yang-Mills amplitude in (2.1) must satisfy the colour-kinematics duality [2], which states that when we have three colour factors, c_i, c_j and c_k related by the Jacobi identity, $c_i + c_j + c_k = 0$, we need the corresponding kinematic factors to obey the same algebraic relation: $n_i + n_j + n_k = 0$. In the massless case it has been conjectured [2] such choice of kinematic factors by choosing a gauge and performing field redefinitions is always possible. We will see later that in the massive case that is generally not true if we want to keep locality. However the four point case is special and the kinematic factors calculated directly from (1.14) using Feynman rules satisfy the colour-kinematics duality.

In the usual double copy procedure, once the correct representation for (2.1) is chosen, the colour factors can be replaced with kinematic factors in order to obtain an amplitude of a gravitational theory [6]. We follow the same procedure and conjecture that the following expression gives an amplitude in a massive gravity theory:

$$M_n = i\left(\frac{\kappa}{2}\right)^{n-2}\sum_i \frac{n_i\tilde{n}_i}{\prod_{\alpha_i}(-p_{\alpha_i}^2 - m^2)}, \tag{2.15}$$

where $\tilde{n}_i$ are the kinematic factors of the second massive Yang-Mills. The products of Yang-Mills polarization tensors in n_i and $\tilde{n}_i$, ϵ_μ and $\tilde{\epsilon}_\nu$ respectively, are decomposed into polarization tensors of the fields in the gravitational theory. This corresponds to decomposition of a tensor product of two vector representations of the little group (for massive particles in 4d it is $SO(3)$) into irreducible representations. Schematically this is done as follows:

$$\epsilon_\mu^{((j}\tilde{\epsilon}^{\kappa))} \to \epsilon_{\mu\nu}^{(h)jk} \tag{2.16}$$

$$\epsilon_\mu^{[j}\tilde{\epsilon}_\nu^{k]} \to \epsilon_{\mu\nu}^{(B)jk}, \tag{2.17}$$

$$\epsilon_\mu^j\tilde{\epsilon}^\kappa\delta_{jk} \propto \epsilon_{\mu\nu}^{(\phi)}. \tag{2.18}$$

where j, k are little group indices, $(())$ denotes the symmetric traceless part corresponding to the graviton polarization, $\epsilon^{(h)}$, and the antisymmetric part denoted as $[]$ corresponds to the spin-1 polarization in terms of the B field, $\epsilon^{(B)}$. However instead of working with the massive $B_{\mu\nu}$ field in this paper, we construct the action in terms of the vector field A_μ which is dual to $B_{\mu\nu}$. The dualization procedure is explained in Appendix D. We define the map between B field polarization tensor and A polarization vector to be:

$$\epsilon_{\mu\nu}^{(B)} = \frac{i}{\sqrt{2}m}\varepsilon_{\mu\nu\rho\sigma}p^\rho\epsilon^{(A)\sigma}, \tag{2.19}$$

where p^σ is the four-momentum of the external state and the factor of $\sqrt{2}$ is required for the correct normalization. The trace part of the tensor product, given in (2.18), is the polarization tensor corresponding to the scalar, ϕ. As we show in Appendix C from explicit calculation in helicity basis we find it to be

$$\epsilon_{\mu\nu}^{(\phi)} = \frac{1}{\sqrt{3}}\left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right), \tag{2.20}$$

which up to a sign could equally have been fixed by the requirement that it is a tracefull, transverse and normalized.

2.1.3 Three-point Amplitude

We start with the simplest case - 3 pt amplitudes. For massive Yang-Mills the 3 pt amplitude is exactly same as that of massless Yang-Mills:

$$A_3(1^a, 2^b, 3^c) = \sqrt{2} g f_{abc}(-\epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot p_1 + \epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot p_1 - \epsilon_1 \cdot p_2\, \epsilon_2 \cdot \epsilon_3). \tag{2.21}$$

The difference is that now the on-shell momenta satisfy $p_i^2 = -m^2$ and there are 3 possible polarization states. Our conventions for these are given in Appendix B. The three-point amplitude in dRGT massive gravity is as follows:

$$\begin{aligned} M_3 =& i\kappa\Big((\epsilon_1^{\mu\nu}\epsilon_{3\mu\nu}\epsilon_{2\alpha\beta}p_1^\alpha p_1^\beta + 2\epsilon_{1\mu\nu}\epsilon_2^{\mu\alpha}\epsilon_{3\beta}^{\nu}p_{1\alpha}p_{2\beta} + \text{cyclic permutations of 1,2,3}) \\ &+ \frac{3}{2}(1+\kappa_3)\epsilon_1^{\mu\nu}\epsilon_{2\nu\alpha}\epsilon_{3\mu}^{\alpha}m^2\Big), \end{aligned} \tag{2.22}$$

where the coupling constant $\kappa = 2/M_{\rm Pl}$. The first term is already proportional to the square of Yang-Mills three-point colour-stripped amplitude if we write the polarization tensors as products of two spin-1 polarization vectors, $(\epsilon^i)_{\mu\nu} = (\epsilon^i)_\mu(\epsilon^i)_\nu$ [2], M_3. Therefore, in order for double copy to work we need to choose κ_3 such that the second term vanishes, *i.e.* $\kappa_3 = -1$. We see that already at cubic level the double copy construction picks a particular one parameter (κ_4) subset of theories from 2-parameter family of massive gravity theories.

Since on the gravity side, in addition to massive spin 2 we have massive spin 0 and spin 1 states, we have to construct the three point amplitudes for all possible scattering processes between them. By applying (2.15) to three-point amplitudes explicitly we obtain the following relation:

$$M_3 = i\frac{\kappa}{2}A_3\tilde{A}_3, . \tag{2.23}$$

where the 3 point amplitudes have their structure constants, f_{abc}, stripped off. By substituting (2.21) and (2.16), (2.19) and (2.20) we get the following three-point vertices in a gravitational theory:

[2] Note that only polarization tensors for helicity ± 2 can be written as $(\epsilon^i)_{\mu\nu} = (\epsilon^i)_\mu(\epsilon^i)_\nu$, for helicities $\pm 1, 0$ we need to sum over the products of different helicities weighted by Clebsch–Gordan coefficients $\epsilon^\lambda_{\mu\nu} = \sum_{\lambda'\lambda''} C^\lambda_{\lambda'\lambda''}\epsilon^{\lambda'}_\mu\epsilon^{\lambda''}_\nu$.

$$M_{AAh} = i\frac{\kappa}{2}\left(\frac{3}{2}m^2\epsilon_1^\mu\epsilon_2^\nu\epsilon_{3\mu\nu} - p_1^\alpha p_2^\beta\epsilon_1^\mu\epsilon_2^\mu\epsilon_{3\alpha\beta} + p_1^\mu p_2^\nu\epsilon_1^\nu\epsilon_2^\alpha\epsilon_{3\mu\alpha} + p_1^\mu p_2^\nu\epsilon_1^\alpha\epsilon_{2\mu}\epsilon_{3\nu\alpha}\right) \tag{2.24}$$

$$M_{AA\phi} = -i\frac{\kappa}{8\sqrt{3}}\left(15m^2\epsilon_1^\mu\epsilon_{2\mu} + 2p_{1\nu}p_{2\mu}\epsilon_1^\mu\epsilon_2^\nu\right) \tag{2.25}$$

$$M_{\phi hh} = -i\frac{\sqrt{3}\kappa}{4}m^2\epsilon_{2\mu\nu}\epsilon_3^{\mu\nu} \tag{2.26}$$

$$M_{\phi\phi h} = -i\frac{3\kappa}{4}p_{1\mu}p_{2\nu}\epsilon_3^{\mu\nu} \tag{2.27}$$

$$M_{\phi\phi\phi} = -i\frac{11\sqrt{3}}{16}\kappa m^2 \tag{2.28}$$

$$M_{hhh} = i\kappa\left((\epsilon_1^{\mu\nu}\epsilon_{3\mu\nu}\epsilon_{2\alpha\beta}p_1^\alpha p_1^\beta + 2\epsilon_{1\mu\nu}\epsilon_2^{\mu\alpha}\epsilon_{3\beta}^\nu p_{1\alpha}p_{2\beta} + \text{cyclic permutations of 1,2,3})\right) \tag{2.29}$$

As mentioned before, M_{hhh} matches three graviton amplitude of massive gravity if we choose $\kappa_3 = -1$ (or $c_3 = 1/4$ using the parametrization of the theory as in [75, 98]). The M_{AAh} and $M_{\phi\phi h}$ amplitudes are different from those obtained from vector and scalar kinetic terms minimally coupled to gravity (for example a minimally coupled scalar would give $M_{\phi\phi h} = -i\kappa\epsilon_{3\mu\nu}p_1^\mu p_2^\nu$. This is expected, since we know theories containing massive spin-2 field do not have diffeomorphism symmetry, and we allow couplings between our fields and the reference metric which in this case is the Minkowski metric. In this way we evade the usual equivalence principle requirements for a massless spin-2 particle. As already mentioned we see that M_{hhh} matches the 3 point amplitude of massive gravity with $\kappa_3 = -1$.

2.1.4 Four-point Amplitude

At 4pt level we express the four-point amplitude of massive Yang-Mills in the form given in Eq. (2.1) by defining the colour factors to be:

$$c_s = f_{abe}f_{cde} \tag{2.30}$$

$$c_t = f_{cae}f_{bde} \tag{2.31}$$

$$c_u = f_{bce}f_{ade}. \tag{2.32}$$

so that

$$A_4(1^a, 2^b, 3^c, 4^d) = g^2 \left(\frac{c_s n_s}{s - m^2} + \frac{c_t n_t}{t - m^2} + \frac{c_u n_u}{u - m^2} \right) , \tag{2.33}$$

where the kinematic factors are

$$\begin{aligned} n_s = & -\frac{i}{2}(\epsilon_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3 p_1 \cdot \epsilon_4 + 4\epsilon_2 \cdot \epsilon_4 p_1 \cdot \epsilon_3 p_2 \cdot \epsilon_1 - 2\epsilon_2 \cdot \epsilon_3 p_1 \cdot \epsilon_4 p_2 \cdot \epsilon_1 + 3\epsilon_1 \cdot \epsilon_2 p_1 \cdot \epsilon_4 p_2 \cdot \epsilon_3 \\ & + 4\epsilon_2 \cdot \epsilon_4 p_2 \cdot \epsilon_1 p_2 \cdot \epsilon_3 - 4\epsilon_1 \cdot \epsilon_4 p_1 \cdot \epsilon_2 (p_1 \cdot \epsilon_3 + p_2 \cdot \epsilon_3) - 3\epsilon_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3 p_2 \cdot \epsilon_4 \\ & - 2\epsilon_2 \cdot \epsilon_3 p_2 \cdot \epsilon_1 p_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3 p_2 \cdot \epsilon_4 + 4\epsilon_3 \cdot \epsilon_4 p_1 \cdot \epsilon_2 p_3 \cdot \epsilon_1 - 4\epsilon_3 \cdot \epsilon_4 p_2 \cdot \epsilon_1 p_3 \cdot \epsilon_2 \\ & + 2\epsilon_1 \cdot \epsilon_3 p_1 \cdot \epsilon_2 (p_1 \cdot \epsilon_4 + p_2 \cdot \epsilon_4 - p_3 \cdot \epsilon_4) + \epsilon_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3 p_3 \cdot \epsilon_4 + 2\epsilon_2 \cdot \epsilon_3 p_2 \cdot \epsilon_1 p_3 \cdot \epsilon_4 \\ & - \epsilon_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3 p_3 \cdot \epsilon_4 + \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 (m^2 - s) + \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 (-m^2 + s) + \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 t \\ & - \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 u), \end{aligned} \tag{2.34}$$

$$\begin{aligned} n_t = & \frac{i}{2}(\epsilon_1 \cdot \epsilon_3 p_1 \cdot \epsilon_2 p_1 \cdot \epsilon_4 + 4\epsilon_2 \cdot \epsilon_4 p_1 \cdot \epsilon_3 p_2 \cdot \epsilon_1 + \epsilon_1 \cdot \epsilon_3 p_1 \cdot \epsilon_2 p_2 \cdot \epsilon_4 + 4\epsilon_3 \cdot \epsilon_4 p_1 \cdot \epsilon_2 p_3 \cdot \epsilon_1 \\ & - 2\epsilon_2 \cdot \epsilon_3 p_1 \cdot \epsilon_4 p_3 \cdot \epsilon_1 - 4\epsilon_2 \cdot \epsilon_4 p_2 \cdot \epsilon_3 p_3 \cdot \epsilon_1 + 2\epsilon_2 \cdot \epsilon_3 p_2 \cdot \epsilon_4 p_3 \cdot \epsilon_1 + 3\epsilon_1 \cdot \epsilon_3 p_1 \cdot \epsilon_4 p_3 \cdot \epsilon_2 \\ & - \epsilon_1 \cdot \epsilon_3 p_2 \cdot \epsilon_4 p_3 \cdot \epsilon_2 + 4\epsilon_3 \cdot \epsilon_4 p_3 \cdot \epsilon_1 p_3 \cdot \epsilon_2 - 4\epsilon_1 \cdot \epsilon_4 p_1 \cdot \epsilon_3 (p_1 \cdot \epsilon_2 + p_3 \cdot \epsilon_2) \\ & - 3\epsilon_1 \cdot \epsilon_3 p_1 \cdot \epsilon_2 p_3 \cdot \epsilon_4 - 2\epsilon_2 \cdot \epsilon_3 p_3 \cdot \epsilon_1 p_3 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_3 p_3 \cdot \epsilon_2 p_3 \cdot \epsilon_4 + 2\epsilon_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3 (p_1 \cdot \epsilon_4 \\ & - p_2 \cdot \epsilon_4 + p_3 \cdot \epsilon_4) + \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 s + \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 (m^2 - t) + \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 (-m^2 + t) \\ & - \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 u), \end{aligned} \tag{2.35}$$

$$\begin{aligned} n_u = & -\frac{i}{2}(4\epsilon_1 \cdot \epsilon_4 p_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3 - 4\epsilon_2 \cdot \epsilon_4 p_2 \cdot \epsilon_1 p_2 \cdot \epsilon_3 - 4\epsilon_2 \cdot \epsilon_4 p_2 \cdot \epsilon_3 p_3 \cdot \epsilon_1 + 4\epsilon_2 \cdot \epsilon_3 p_2 \cdot \epsilon_4 p_3 \cdot \epsilon_1 \\ & - 4\epsilon_1 \cdot \epsilon_4 p_1 \cdot \epsilon_3 p_3 \cdot \epsilon_2 + 4\epsilon_3 \cdot \epsilon_4 p_2 \cdot \epsilon_1 p_3 \cdot \epsilon_2 + 4\epsilon_3 \cdot \epsilon_4 p_3 \cdot \epsilon_1 p_3 \cdot \epsilon_2 - 4\epsilon_2 \cdot \epsilon_3 p_2 \cdot \epsilon_1 p_3 \cdot \epsilon_4 \\ & + 4\epsilon_1 \cdot \epsilon_2 p_2 \cdot \epsilon_3 (p_2 \cdot \epsilon_4 + p_3 \cdot \epsilon_4) - 4\epsilon_1 \cdot \epsilon_3 p_3 \cdot \epsilon_2 (p_2 \cdot \epsilon_4 + p_3 \cdot \epsilon_4) + \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 s - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 t \\ & + \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 (m^2 - u) + \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 (-m^2 + u)), \end{aligned} \tag{2.36}$$

where the Mandelstam variables are defined as standard:

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 + p_3)^2, \quad u = -(p_1 + p_4)^2, \tag{2.37}$$

with all momenta incoming. These expressions for kinematic factors are very similar to those obtained from massless Yang-Mills theory but there are two differences: the relation between Mandelstam variables is now $s + t + u = 4m^2$, rather than $s + t + u = 0$, and the locations of the poles now are at $s, t, u = m^2$. Because of that the terms coming from quartic Yang-Mills vertex now have to be multiplied by $s - m^2$, $t - m^2$ and $u - m^2$ in order to recast the amplitude into the form (2.1).

In general, kinematic factors of a given scattering amplitude are not unique. They are not invariant under field redefinitions. However in massless Yang-Mills theory for any choice of kinematic factors of four-point amplitude, the colour-kinematics duality, $c_s + c_t + c_u = 0 \rightarrow n_s + n_t + n_u = 0$, is satisfied [99]. In our case of massive Yang-Mills theory, it is not immediately clear whether this is still true. However, explicit calculation shows that our colour and kinematic factors (directly calculated from usual Feynman rules) in (2.34),(2.35) and (2.36) still obey $n_s + n_t + n_u \propto p_4 \cdot \epsilon_4 = 0$ and $c_s + c_t + c_u = 0$. The fact that this still holds for the massive theory can be understood by noticing that the only difference between massive and massless kinematic factors is coming from the terms proportional to m^2 in (2.34), (2.35) and (2.36) (in fact we do not need to use the relation between s, t and u here). It is easy to see that these six terms add to zero, therefore the value of $n_s + n_t + n_u$ is the same for massless and massive theory and colour-kinematics duality for four-point amplitude still holds in the massive case.

We start with $hh \rightarrow hh$ amplitude which is calculated using (2.34), (2.35), (2.36), (2.15) and (2.16). By comparing it with $hh \rightarrow hh$ amplitude calculated using dRGT massive gravity action, M_4^{mGr}, we find the following:

$$M_4 = M_4^{\text{mGr}} - i\frac{3}{16}\kappa^2 m^4 \left(\frac{\epsilon_{1\mu\nu}\epsilon_2^{\mu\nu}\epsilon_{3\alpha\beta}\epsilon_4^{\alpha\beta}}{s - m^2} + \frac{\epsilon_{1\mu\nu}\epsilon_3^{\mu\nu}\epsilon_{2\alpha\beta}\epsilon_4^{\alpha\beta}}{t - m^2} + \frac{\epsilon_{1\mu\nu}\epsilon_4^{\mu\nu}\epsilon_{3\alpha\beta}\epsilon_2^{\alpha\beta}}{u - m^2} \right), \tag{2.38}$$

with the free coefficients in the massive gravity action chosen to be $\kappa_3 = -1$ and $\kappa_4 = \frac{7}{24}$ ($c_3 = \frac{1}{4}$ and $d_5 = -\frac{7}{192}$ using the parametrization of [75]). The second term on the right hand side of (2.38) corresponds to a scalar exchange with three-point vertex given in (2.26).

Having fixed the spin-2 interactions, we then construct the scattering amplitudes for all other 2-2 scattering processes (for example $h\phi \to AA$) from the double copy prescription, and make an ansatz for the action which gives these amplitudes. A couple of general features emerge. We find that all 3 and 4 point amplitudes containing odd numbers of A are zero as one would expect since A is a vector. Furthermore we find that none of the amplitudes scale with energy more that E^6 at high energies. Since all of them have $\kappa^2 = 4/M_{\rm Pl}^2$ in front (can be seen from (2.15)), the lowest scale appearing in the resulting theory to this order is $\Lambda_3 = (M_{\rm Pl} m^2)^{1/3}$, the well-known highest possible scale for a Lorentz invariant theory of massive gravity.

As already stated, from (2.29) and (2.38) we see that the self interactions of h up to quartic order in h can be described by dRGT massive gravity action. Anticipating that the n-point scattering amplitudes are controlled by the scale Λ_3 to all orders, it is natural to write the interactions for all the fields in the dRGT form, taking particular care to choose combinations which are natural from the point of view of the decoupling limit effective theory, namely those that automatically lead to Λ_3 interactions to all orders. This process is somewhat labourious, and we quote only our final form for the action which is

$$
\begin{aligned}
S = \int d^4x \sqrt{-g} \Bigg(& \frac{2}{\kappa^2} R[g] + \frac{m^2}{\kappa^2} \sum_{n=2}^{4} \kappa_n \mathcal{U}_n [\mathcal{K}] \\
& - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \\
& - \frac{1}{2} \mathcal{K}^{\mu\nu} F_{\nu\alpha} F_\mu{}^\alpha + \frac{1}{8} \mathcal{K}^\mu_\mu F_{\nu\alpha} F^{\nu\alpha} - \frac{1}{4} \nabla_\mu \phi \nabla_\nu \phi \left(\mathcal{K}^{\mu\nu} - g^{\mu\nu} \mathcal{K}^\alpha_\alpha \right) - \frac{\sqrt{3}}{2} \frac{m^2}{\kappa} \phi \left(\mathcal{K}^{\mu\nu} \mathcal{K}_{\mu\nu} - \mathcal{K}^\mu_\mu \mathcal{K}^\nu_\nu \right) \\
& + \frac{1}{24\sqrt{3}} \frac{\kappa}{m^2} \phi \left([\Phi]^2 - [\Phi^2] \right) + \frac{-3}{8\sqrt{3}} \kappa m^2 \phi^3 - \frac{\kappa}{\sqrt{3}} m^2 A^\mu A_\mu \phi - \frac{\kappa}{16\sqrt{3}} F^{\mu\nu} F_{\mu\nu} \phi \\
& + \text{quartic contact terms} \Bigg),
\end{aligned}
\tag{2.39}
$$

where $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ is the dynamical metric, $f_{\mu\nu} = \eta_{\mu\nu}$ is the reference metric, $\mathcal{K}^{\mu}_{\nu} = \delta^{\mu}_{\nu} - (\sqrt{g^{-1}f})^{\mu}_{\nu}$, $\Phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$ and the crucial contact terms which fix the form of the 2-2 scattering amplitude are given in Appendix A. The indices are raised/lowered with g. The self interactions of the scalar, ϕ, contain galileon interactions (the cubic term in (2.39) and the quartic one in (A.1)), ϕ^3 term and two additional two and four derivative contact terms to this order. The action has been intentionally written in a manner which is diffeomorphism invariant in terms of $\mathcal{K}$. The reference metric η that breaks diffeomorphism invariances only enters through $\mathcal{K}$, and in this sense $\mathcal{K}$ is a 'spurion' field for the breaking of diffeomorphisms.

Since the S-matrix is invariant under field redefinitions, the cubic ϕ interactions are ambiguous since we may for example use field redefinitions to trade the cubic Galileon term for a potential ϕ^3 and vice versa without changing the on-shell vertex. A similar story holds for the ϕK^2 and $(\nabla\phi)^2 K$ terms. However changing the off-shell structure in this way also changes the form of the quartic interactions. Anticipating that the decoupling limit is a Galileon-like theory (which is implicit in the Λ_3 scale), we have intentionally chosen to put the cubic interactions in a form for which the quartic interactions are also manifestly Galileon-like. In other words the desire to have a Galileon-like decoupling limit theory gives us guidance in writing the nonlinear off-shell structure of the theory that goes beyond what is immediately inferred from the on-shell scattering amplitudes, even though the diffeomorphism symmetry is broken by the mass term. That is the decoupling limit for the Stückelberg fields/Goldstone modes gives us an indication of the best way to structure the interacting Lagrangian and this explains many of our choices of interactions in (2.39) and Appendix A. Although we have not calculated beyond four-point level, the implicit nonlinearly realized diffeomorphism symmetry present in the Stückelberg formulation fixes a set of interactions at all orders as is familiar in effective theories with broken symmetries.

2.1.5 Λ_3 Decoupling Limit

Having successfully constructed the interaction Lagrangian for the double copy effective theory, at least to quartic order, it is useful to understand its decoupling limit. This will give us insight into the interactions that arise beyond 2-2 scattering, and the overall structure of the effective theory, but it will also allow us to understand better the connection between the massive Yang-Mills decoupling limit and that for the double copy massive gravity theory. First of it is instructive to understand what happens to the kinematic factors in the decoupling limit. We will show that taking the decoupling limit and performing the double copy procedure do not commute. The origin of this non-commutativity is that there are terms needed in the kinematic factors to satisfy colour-kinematics duality that are singular in the decoupling limit but nevertheless cancel out of the gauge theory amplitudes. However when we construct the gravity amplitudes by squaring these kinematic factors, they no longer cancel and give additional non-zero contributions that are finite in the decoupling limit. To be precise, the kinematic factors which satisfy colour-kinematics duality $n_s + n_t + n_u = 0$ take the form

$$n_s = \frac{s-m^2}{m^3}\Sigma(s,t,u) + \frac{1}{m^2}\hat{n}_s, \quad n_t = \frac{t-m^2}{m^3}\Sigma(s,t,u) + \frac{1}{m^2}\hat{n}_t, \quad n_u = \frac{u-m^2}{m^3}\Sigma(s,t,u) + \frac{1}{m^2}\hat{n}_u\,, \tag{2.40}$$

where $\Sigma(s,t,u)$ (triple crossing symmetric) and $\hat{n}_i$ are finite as $m \to 0$. Here Σ arises in a manner similar to the generalized gauge transformations in the massless case, a fact which is crucial to understanding why its contribution is finite. The explicit expressions for Σ and $\hat{n}_i$ are given in Eqs. (F.12), (F.13), (F.14) and (F.15). Since in the massive case $s+t+u = 4m^2$ we have $\hat{n}_s + \hat{n}_t + \hat{n}_u = -m\Sigma$ and so in the limit $m \to 0$, $\hat{n}_i$ by themselves satisfy colour-kinematics duality. The $1/m^3$ behaviour in n_i comes from helicity $0,0,0,\pm1$ interactions since the polarization tensor for a massive helicity-0 gluon scales as $1/m$ but that for helicity-1 is finite as $m \to 0$. The term Σ cancels out of the gauge theory amplitudes

$$A_4^{\rm mYM} = g^2\left(\frac{c_s n_s}{s-m^2} + \frac{c_t n_t}{t-m^2} + \frac{c_u n_u}{u-m^2}\right) = \frac{1}{\Lambda^2}\left(\frac{c_s \hat{n}_s}{s-m^2} + \frac{c_t \hat{n}_t}{t-m^2} + \frac{c_u \hat{n}_u}{u-m^2}\right), \tag{2.41}$$

by virtue of the colour relation $c_s + c_t + c_u = 0$, demonstrating the natural decoupling limit scaling.

By contrast, when we square to construct the gravity amplitudes, Σ survives as a contact term. For instance the naive leading $1/m^6$ term enters in the gravity amplitudes in the combination

$$\frac{1}{M_{\rm Pl}^2}\left(\frac{n_s n_s'}{s-m^2}+\frac{n_t n_t'}{t-m^2}+\frac{n_u n_u'}{u-m^2}\right) \sim \frac{\Sigma\Sigma'}{M_{\rm Pl}^2 m^6}\left((s-m^2)+(t-m^2)+(u-m^2)\right)+\ldots \sim \frac{\Sigma\Sigma'}{\Lambda_3^6}+\ldots\,, \tag{2.42}$$

and hence it contributes at the Λ_3 scale. Specifically this will show up as a non-zero spin-2, helicity $0,0,0,\pm2$ interaction. Similarly the naive $1/m^5$ term is suppressed by virtue of the kinematic relation $\hat{n}_s + \hat{n}_t + \hat{n}_u = -m\Sigma$ and we have in full as an exact statement

$$\frac{1}{M_{\rm Pl}^2}\left(\frac{n_s n_s'}{s-m^2}+\frac{n_t n_t'}{t-m^2}+\frac{n_u n_u'}{u-m^2}\right) = \frac{-\Sigma\Sigma'}{\Lambda_3^6}+\frac{1}{\Lambda_3^6}\left(\frac{\hat{n}_s \hat{n}_s'}{s-m^2}+\frac{\hat{n}_t \hat{n}_t'}{t-m^2}+\frac{\hat{n}_u \hat{n}_u'}{u-m^2}\right)\,. \tag{2.43}$$

Since Σ does not contribute to the gauge theory amplitudes, first taking the decoupling limit of them (giving a non-linear sigma model) and performing the double copy procedure (giving a special Galileon) will lead to a different result in which the $\frac{\Sigma\Sigma'}{\Lambda_3^6}$ term is absent[3]. The kinematic factors inferred from the decoupling limit $\hat{n}_i(m=0)$ will necessarily be finite in the decoupling limit, and these do not correspond to the decoupling limit of the above kinematic factors (2.40) which are singular. Indeed in the decoupling limit, the gauge theory kinematic factors come purely from helicity-0 gluons by the Goldstone equivalence theorem. However, if we first double copy and then take the decoupling limit then the helicity ±1 modes survive and we end up with a different theory. The expressions of the amplitudes in the decoupling limit are given in Appendix F.

Now we proceed to the the decoupling limit of the action of the double copy theory. We have

[3]It is of course technically true that if we only compute amplitudes in which the spin-1 helicity-1 polarizations are set to zero, then $\Sigma = \Sigma' = 0$ and we will recover the special Galileon amplitudes in the decoupling limit. But this is an inconsistent procedure from the point of view of the gravity theory, and has no relation to the massive gravity theory whose decoupling limit is a special Galileon. There may however exist an extension of the recipe along the lines discussed in [100, 101, 46, 41] which allows for a consistent removal of additional degrees of freedom.

intentionally written the interacting Lagrangian (2.39) in as covariant form as possible, so that the decoupling limit is easily derived. Following the standard recipe (see for example [77] for a review), after denoting the reference metric from which $\mathcal{K}^{\mu}{}_{\nu}$ is constructed by

$$f_{\mu\nu} = \partial_\mu \Phi^A \partial_\nu \Phi^B \eta_{AB} \,, \tag{2.44}$$

we further decompose

$$\Phi^A = x^A - \frac{1}{m M_{\rm Pl}} V^A - \frac{1}{\Lambda_3^3} \eta^{AB} \partial_B \pi \,. \tag{2.45}$$

so that we may identify V^A as the helicity-1 and π as the helicity-0 modes of the spin-2 particle. Further for the massive spin-1 state A_μ we replace it by

$$A_\mu \to A_\mu + \frac{1}{m} \partial_\mu \chi \,, \tag{2.46}$$

where χ is the original Stückelberg scalar, the helicity-0 state of the spin-1. The normalizations, which are standard, are chosen so that all the additional Stückelberg fields have a finite (and non-zero) kinetic term in the decoupling limit. The metric may be denoted $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Remembering that $\kappa = 2/M_{\rm Pl}$, the decoupling limit is defined by $m \to 0$, $\kappa \to 0$ in such a way that $\Lambda_3^3 = m^2 M_{\rm Pl}$ is kept finite. The Lagrangian has been written in a judicious way to ensure that no term diverges in this limit.

Crucially, we have

$$\lim_{m \to 0, \Lambda_3 {\rm fixed}} \mathcal{K}_{\mu\nu} = \frac{\Pi_{\mu\nu}}{\Lambda_3^3} := \frac{\partial_\mu \partial_\nu \pi}{\Lambda_3^3} \,, \tag{2.47}$$

which explains the emergence of the Galileon symmetry for π in the decoupling limit, since $\Pi_{\mu\nu}$ is invariant under $\pi \to \pi + c + v_\mu x^\mu$, and our choice of $\mathcal{K}$ as the building block. Hence for all terms in the Lagrangian for which the coefficients are finite in the Λ_3 limit, it is sufficient to replace $K_{\mu\nu}$ by $\Pi_{\mu\nu}$ and the metric $g_{\mu\nu}$ by $\eta_{\mu\nu}$. The decoupling limit Lagrangian is found to be (keeping track only of those terms which contribute to quartic order)

$$\mathcal{L}_{DL} = \frac{1}{2} h^{\mu\nu} \mathcal{E} h_{\mu\nu} + h_{\mu\nu} X^{\mu\nu} - \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} (\partial \chi)^2 + \mathcal{L}_{A,V}$$

$$
-\frac{1}{4\Lambda_3^3}\partial_\mu\phi\partial_\nu\phi\left(\Pi^{\mu\nu}-\eta^{\mu\nu}[\Pi]\right)-\frac{\sqrt{3}}{4}\frac{1}{\Lambda_3^3}\phi\left(\Pi^{\mu\nu}\Pi_{\mu\nu}-\Pi^\mu_\mu\Pi^\nu_\nu\right)
$$
$$
+\frac{1}{12\sqrt{3}}\frac{1}{\Lambda_3^3}\phi\left([\Phi]^2-[\Phi^2]\right)+\frac{11}{864}\frac{1}{\Lambda_3^6}\phi\left([\Phi]^3-3[\Phi][\Phi^2]+2[\Phi^3]\right)+\frac{7}{48\Lambda_3^6}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\nu'\alpha'\beta}\Phi^\mu_{\mu'}\,\Pi^\nu_{\nu'}\Pi^\alpha_{\alpha'}\phi
$$
$$
+\frac{11}{8\sqrt{3}}\frac{1}{\Lambda_3^6}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\nu'\alpha'\beta}\Pi^\mu_{\mu'}\Phi^\nu_{\nu'}\Phi^\alpha_{\alpha'}\phi-\frac{11}{24\sqrt{3}}\frac{1}{\Lambda_3^6}\phi\left([\Pi]^3-3[\Pi][\Pi^2]+2[\Pi^3]\right)\,, \tag{2.48}
$$

where all indices are raised and lowered with $\eta_{\mu\nu}$. We have separated out the spin-2 and spin-1 helicity-1 contributions which even in the case of standard massive gravity is particularly complicated [82], and they are schematically

$$
\mathcal{L}_{A,V}=-\frac{1}{4}F_{\mu\nu}K^{\mu\nu\alpha\beta}F_{\alpha\beta}-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{K}^{\mu\nu\alpha\beta}\mathcal{F}_{\alpha\beta} \tag{2.49}
$$

where $\mathcal{F}_{\mu\nu}=\partial_\mu V_\nu-\partial_\nu V_\mu$, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and the kinetic term coefficients $K^{\mu\nu\alpha\beta}$ and $\mathcal{K}^{\mu\nu\alpha\beta}$ are tensors constructed from $\Pi_{\mu\nu}/\Lambda_3^3$ and $\Phi_{\mu\nu}/\Lambda_3^3$. Since V_μ and A_μ are not sourced, classically it is consistent to set them to zero. They would of course contribute in loop processes.

The tensor $X_{\mu\nu}$, which is characteristic of the massive gravity decoupling limit, needs to be identically conserved to ensure that that $h_{\mu\nu}$ preserves spin-2 gauge invariance (linear diffeomorphisms) $h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$. This is the decoupling limit remnant of full diffeomorphism invariance. Explicitly its form is

$$
X^{aA}=\varepsilon^{abcd}\varepsilon_{ABCD}\left[\frac{1}{2}\delta^B_b\delta^C_c\Pi^D_d-\frac{1}{4\Lambda_3^3}\delta^B_b\Pi^C_c\Pi^D_d+\frac{1}{24\Lambda_3^6}\Pi^B_b\Pi^C_c\Pi^D_d+\frac{1}{24\Lambda_3^6}\Phi^B_b\Phi^C_c\Pi^D_d\right.
$$
$$
\left.-\frac{1}{72\sqrt{3}\Lambda_3^6}\Phi^B_b\Phi^C_c\Phi^D_d-\frac{1}{8\sqrt{3}\Lambda_3^6}\Phi^B_b\Pi^C_c\Pi^D_d\right] \tag{2.50}
$$

The tensor (2.50) is indeed identically conserved by virtue of the double ε structure. The full decoupling limit action (2.48) is invariant under two separate Galileon symmetries $\pi\to\pi+v_\mu x^\mu$, $\phi\to\phi+u_\mu x^\mu$ and thus describes a bi-Galileon theory [102] coupled to a massless spin-2 field. Indeed it may be put in a more manifest bi-Galileon form by performing a 'demixing' transformation that removes the mixed $h\pi$ and $h\pi\pi$ terms, namely

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \frac{1}{2}\pi\delta_{\mu\nu} - \frac{1}{4\Lambda_3^3}\pi\Pi_{\mu\nu}\,. \tag{2.51}$$

We may make use of the fact that up to total derivatives

$$\frac{1}{2}h^{\mu\nu}\mathcal{E}h_{\mu\nu} = -\frac{1}{2}\varepsilon^{abcd}\varepsilon_{ABCD}\delta_a^A h_b^B \partial_c\partial^C h_d^D \tag{2.52}$$

The resulting Lagrangian then takes the form

$$\mathcal{L}_{DL} = \frac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}\tilde{h}_{\mu\nu} - \frac{3}{4}(\partial\pi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\chi)^2 + \mathcal{L}^{\rm int}_{\text{bi-Galileon}}(\phi,\pi) + \frac{1}{24\Lambda_3^6}\varepsilon^{abcd}\varepsilon_{ABCD}\tilde{h}_a^A\tilde{\Pi}_b^B\tilde{\Pi}_c^C\tilde{\Pi}_d^D + \mathcal{L}_{A,V}\,, \tag{2.53}$$

where $\tilde{\Pi}_{ab} = \partial_a\partial_b\tilde{\pi}$ and $\tilde{\pi} = \pi - \frac{1}{\sqrt{3}}\phi$. The term $\mathcal{L}^{\rm int}_{\text{bi-Galileon}}$ contains standard cubic and quartic[4] bi-Galileon interactions:

$$\mathcal{L}^{\rm int}_{\text{bi-Galileon}} = a_0\pi(\varepsilon\varepsilon\delta^2\Pi^2) + \phi\sum_{n=1}^{3} a_n(\varepsilon\varepsilon\delta^2\Phi^{n-1}\Pi^{3-n}) + b_0\pi(\varepsilon\varepsilon\delta\Pi^3) + \phi\sum_{n=1}^{4} b_n(\varepsilon\varepsilon\delta\Phi^{n-1}\Pi^{4-n})\,, \tag{2.54}$$

where we have used the shorthand $\epsilon\epsilon XYZW = \varepsilon^{abcd}\varepsilon_{ABCD}X_a^AY_b^BZ_c^CW_d^D$ and the coefficients are given by $(a_0,a_1,a_2,a_3) = (-\frac{1}{8}, \frac{\sqrt{3}}{8}, -\frac{1}{8}, \frac{1}{24\sqrt{3}})$ and $(b_0,b_1,b_2,b_3,b_4) = (\frac{5}{96}, -\frac{25\sqrt{3}}{144}, \frac{1}{6}, \frac{197\sqrt{3}}{432}, \frac{11}{864})$.

The quartic interactions of the form $\tilde{h}\tilde{\Pi}^3$ cannot be removed with a local field redefinition, as is well known from the standard massive gravity case mentioned in the introduction. This is as it should be since it is precisely these interactions that describe the nonzero helicity $0,0,0,\pm 2$ amplitudes that arise from the $\Sigma\Sigma'$ contact term in the decoupling limit, as described in equation (2.43) and implicit in the full answer (E.3) and explicit in (E.4). Indeed the combination $\tilde{\pi}$ is exactly the combination which identifies the diagoanlized parts of π and ϕ that correspond to the spin-1 helicity-0 polarization tensor squared $\epsilon_0^{\mu}\epsilon_0^{\nu}$[5].

[4]Strictly speaking there are also quintic interactions, however since we have only fixed the Lagrangian by reproducing the $2-2$ scattering amplitude, we cannot take seriously the inferred coefficients of the quintic interactions.

[5]To see this, note that at leading order in the decoupling limit $K_{\mu\nu} \sim \frac{1}{\Lambda_3^3}\partial_\mu\partial_\nu\pi \sim -\sqrt{3/2}\frac{1}{M_{\rm Pl}}\epsilon^{\lambda=0}_{\mu\nu}\pi$. Since in unitary gauge $K_{\mu\nu} = \frac{1}{M_{\rm Pl}}h_{\mu\nu} + \mathcal{O}(h^2)$, the canonically normalizalized unitary gauge helicity-zero mode is in

As noted in the introduction, since the decoupling limit of massive Yang-Mills is a nonlinear sigma model and the double copy of the latter is the special Galileon, we might have expected the massive gravity theory to be that corresponding to a special Galileon. Interestingly however, this was never possible since the decoupling limit of dRGT massive gravity never gives rise to a special Galileon. This is easily seen by the manner in which the Galileon interactions arise from mixing with $h_{\mu\nu}$. The decoupling limit of dRGT massive gravity for general κ_3 and κ_4 is (ignoring helicity-1 contributions)

$$\mathcal{L}_{DL} = -\frac{1}{2}\varepsilon^{abcd}\varepsilon_{ABCD}\delta_a^A h_b^B \partial_c \partial^C h_d^D + h_{\mu\nu}X^{\mu\nu} , \tag{2.55}$$

where

$$X_{\mu\nu} = \varepsilon^{abcd}\varepsilon_{ABCD}\left[\frac{1}{2}\delta_b^B\delta_c^C\Pi_d^D + \frac{1}{4\Lambda_3^3}(2+3\kappa_3)\delta_b^B\Pi_c^C\Pi_d^D + \frac{1}{4\Lambda_3^6}(4\kappa_4+\kappa_3)\Pi_b^B\Pi_c^C\Pi_d^D\right] . \tag{2.56}$$

Since the special Galileon in four dimensions is a pure quartic Galileon, we need that after performing the demixing

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \frac{1}{2}\pi\eta_{\mu\nu} + \frac{1}{4\Lambda_3^3}(2+3\kappa_3)\pi\Pi_{\mu\nu} , \tag{2.57}$$

there is no cubic Galileon term. This requires $(2+3\kappa_3) = 0$ which does not correspond to the value obtained from double copy. Even with this choice, we then have

$$\begin{aligned}\mathcal{L}_{DL} &= -\frac{1}{2}\varepsilon^{abcd}\varepsilon_{ABCD}\delta_a^A\tilde{h}_b^B\partial_c\partial^C\tilde{h}_d^D + \frac{1}{8\Lambda_3^6}\varepsilon^{abcd}\varepsilon_{ABCD}(4\kappa_4+\kappa_3)\pi\delta_a^A\Pi_b^B\Pi_c^C\Pi_d^D \\ &+\frac{1}{4\Lambda_3^6}(4\kappa_4+\kappa_3)\varepsilon^{abcd}\varepsilon_{ABCD}\tilde{h}_a^A\Pi_b^B\Pi_c^C\Pi_d^D ,\end{aligned} \tag{2.58}$$

and so we only have a non-vanishing quartic Galileon term when there is also a non-zero $h\pi\pi\pi$ interaction which cannot itself be removed with a field redefinition since it contributes to the $\pm 2, 0, 0, 0$ scattering amplitude. Furthermore higher order n-point amplitudes will receive con-

effect $-\sqrt{3/2}\pi$, whence the combination arising in (E.5) is $\frac{2}{\sqrt{6}}(-\sqrt{3/2}\pi + \frac{1}{\sqrt{2}}\phi) = -\pi + \frac{1}{\sqrt{3}\phi} = -\tilde{\pi}$.

tributions from intermediate graviton exchange which do not arise in the pure quartic Galileon theory. Hence the special Galileon does not strictly speaking arise in standard massive gravity in any form. Thus we see that taking decoupling limit and doing double copy do not commute, since the double copy of decoupling limit of massive Yang-Mills gives special Galileon and the decoupling limit of double copy theory gives a different Galileon-like theory.

2.1.6 5pt Amplitudes and Spurious Poles

In this section, we will analyze the 5-point amplitude which we will show has spurious poles. We can write the 5-point amplitude as

$$A_5 = g^3 \sum_{i=1}^{15} \frac{c_i n_i}{s_i - m^2} \ , \tag{2.59}$$

where the 15 colour factors of the adjoint representation fields are:

$$\begin{aligned}
c_1 &\equiv f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5} \,, \quad & c_2 &\equiv f^{a_2 a_3 b} f^{b a_4 c} f^{c a_5 a_1} \,, \quad & c_3 &\equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2} \,, \\
c_4 &\equiv f^{a_4 a_5 b} f^{b a_1 c} f^{c a_2 a_3} \,, \quad & c_5 &\equiv f^{a_5 a_1 b} f^{b a_2 c} f^{c a_3 a_4} \,, \quad & c_6 &\equiv f^{a_1 a_4 b} f^{b a_3 c} f^{c a_2 a_5} \,, \\
c_7 &\equiv f^{a_3 a_2 b} f^{b a_5 c} f^{c a_1 a_4} \,, \quad & c_8 &\equiv f^{a_2 a_5 b} f^{b a_1 c} f^{c a_4 a_3} \,, \quad & c_9 &\equiv f^{a_1 a_3 b} f^{b a_4 c} f^{c a_2 a_5} \,, \\
c_{10} &\equiv f^{a_4 a_2 b} f^{b a_5 c} f^{c a_1 a_3} \,, \quad & c_{11} &\equiv f^{a_5 a_1 b} f^{b a_3 c} f^{c a_4 a_2} \,, \quad & c_{12} &\equiv f^{a_1 a_2 b} f^{b a_4 c} f^{c a_3 a_5} \,, \\
c_{13} &\equiv f^{a_3 a_5 b} f^{b a_1 c} f^{c a_2 a_4} \,, \quad & c_{14} &\equiv f^{a_1 a_4 b} f^{b a_2 c} f^{c a_3 a_5} \,, \quad & c_{15} &\equiv f^{a_1 a_3 b} f^{b a_2 c} f^{c a_4 a_5} \,.
\end{aligned} \tag{2.60}$$

There are 9 independent Jacobi identities that can be written in the form $Mc = 0$, where $c = (c_1, .., c_{15})$, and with the matrix M given by

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.61}$$

Meanwhile, the matrix of propagators, D, is given by

$$\begin{aligned} D =&\text{diag}\{D_{12}D_{45}, D_{15}D_{23}, D_{12}D_{34}, D_{23}D_{45}, D_{15}D_{34}, D_{14}D_{25}, D_{14}D_{23}, D_{25}D_{34}, \\ &D_{13}D_{25}, D_{13}D_{24}, D_{15}D_{24}, D_{12}D_{35}, D_{24}D_{35}, D_{14}D_{35}, D_{13}D_{45}\} \ , \end{aligned} \tag{2.62}$$

where $D_{ij} = -(p_i + p_j)^2 - m^2 = s_{ij} - m^2$. Using momentum conservation we find that there are only 5 independent Mandelstam invariants; here, we choose them to be s_{12}, s_{13}, s_{14}, s_{23} and s_{24}.

In order to find the shift of kinematic numerators, Δn, we need to build and invert the 9×9 matrix A defined in (2.7). Explicit calculation of the determinant of A at 5-points gives:

$$\det(A) = m^8 (\prod_{i<j} D_{ij}) P(s_{kl}, m) \ , \tag{2.63}$$

where $P(s_{kl}, m)$ is a polynomial of the Mandelstam invariants and the mass, given as:

$$
\begin{aligned}
P(s_{kl}, m) = & 320m^8 + 36m^6(9s_{12} + 4(s_{13} + s_{14} + s_{23} + s_{24})) \\
& + m^4 \big(117s_{12}^2 + 108s_{12}(s_{13} + s_{14} + s_{23} + s_{24}) + 4 \left(s_{13}(13s_{14} + 4s_{23} + 17s_{24}) \right.\\
& \qquad \left. +4s_{13}^2 + 4s_{14}^2 + 17s_{14}s_{23} + 4s_{14}s_{24} + 4s_{23}^2 + 13s_{23}s_{24} + 4s_{24}^2\right)\big) \\
& + 2m^2 \big(9s_{12}^3 + 13s_{12}^2(s_{13} + s_{14} + s_{23} + s_{24}) + s_{12} \left(s_{13}(10s_{14} + 6s_{23} + 17s_{24}) \right.\\
& \qquad \left. +4s_{13}^2 + 4s_{14}^2 + s_{14}(17s_{23} + 6s_{24}) + 2(2s_{23} + s_{24})(s_{23} + 2s_{24})\right) \\
& \qquad +2 \big(s_{13}^2(s_{14} + 2s_{24}) + s_{13} \left(s_{14}^2 + s_{14}(s_{23} + s_{24}) + s_{24}(s_{23} + 2s_{24})\right) \\
& \qquad +s_{23} \left(s_{24}(s_{14} + s_{23}) + 2s_{14}(s_{14} + s_{23}) + s_{24}^2\right)\big)\big) \\
& + 2s_{24} \left(s_{23} \left(s_{12}^2 + s_{12}(s_{13} + s_{14}) - s_{13}s_{14}\right) + s_{12}(s_{12} + s_{13})(s_{12} + s_{13} + s_{14})\right) \\
& + \left(s_{12}(s_{12} + s_{13} + s_{14}) + s_{23}(s_{12} + s_{14})\right)^2 + s_{24}^2(s_{12} + s_{13})^2 \ ,
\end{aligned} \tag{2.64}
$$

with $\prod_{i<j} D_{ij}$ the product of all 10 physical poles. The complicated polynomial, $P(s_{kl}, m)$, cannot be expressed as a product of physical poles and it appears in the denominator of A^{-1} of double copy answer in (2.12), therefore it seems that by double copying a generic theory of adjoint fields, all of the same mass gives an unphysical amplitude. Also, by explicit calculations we found the residue of this spurious pole being non-zero. Therefore, the relation between massive Yang-Mills and massive gravity fails at higher points. One of the goals of the rest of this thesis and ongoing work is to find a way around this issue with the spurious poles.

2.2 Kaluza-Klein Theories

In the previous section we saw that the double copy of a generic massive gauge theory introduces spurious poles at 5pt and higher multiplicity amplitudes preventing an interpretation of the resulting double copy as a local field theory. It is natural to ask if there are any massive gauge theories avoiding this problem. This question was first analysed in [97] where certain conditions on the mass spectrum of the theory were derived, that are necessary to avoid the spurious poles.

These spectral conditions were derived by requiring the massive propagators to satisfy the same algebraic relations as the massless ones, ensuring the rank of the massive KLT matrix (or A matrix in (2.9) in our language) to be the same as in the massless case, where spurious poles do not appear. However, massive Yang-Mills theory does not satisfy those conditions and it remains unclear whether the spurious poles can be removed by adding new irrelevant operators or new fields to the massive Yang-Mills action. For example, as suggested in [103] the spurious poles could signal a presence of a new massive state that has to be included in the spectrum.

On the other hand, there is at least one known theory of interacting massive spin-1 states which satisfies these spectral conditions: Kaluza-Klein (KK) gauge theory. It is known that for 5 dimensional (5d) theories on 4d Minkowski $\times S^1$ which are compatible with colour-kinematics duality, the double copy procedure works after we compactify the theory to four dimensions (4d). The condition for the absence of spurious poles in 4d is essentially automatically satisfied by virtue of the kinematics and mass spectrum implied from 5d, together with the charge conservation inherited from 5d compactified translation invariance. Thus as an example, pure 5d Yang-Mills compactified on an S^1 automatically gives a 4d theory of interacting massive spin-1 states for which colour-kinematics duality is respected, and for which there is a known double copy.

In this section we explore whether there are other consistent effective field theories for interacting massive spin-1 states, which still respect colour-kinematics duality, for which there is a possibility of developing a double-copy. Since the conditions necessary to remove the spurious poles are highly non-trivial, and it is not straightforward to satisfy them, we will rather fix the spectrum and charge assignments of our interacting effective field theory to be identical to that of 5d Yang-Mills compactified on an S^1. This allows us to avoid the automatic appearance of spurious poles, and still gives us considerable freedom in choosing local interactions. In particular, we do not require that our 4d theory respects the full compactified 5d gauge invariance, rather only that it preserves 4d gauge invariance, together with a global $U(1)$ symmetry which is the remnant of 5d translations. There are then a huge number of distinct operators which can be included in the EFT which are consistent with the symmetries and are unconstrained by other low energy criteria.[6] Our goal then is to explore this huge space of effective field theories to establish whether there are other possible cases which admit colour-kinematics duality.

Remarkably we find that if we demand the naive extension of CK duality for massive states proposed in [10, 97] to an array of scattering amplitudes with different external states, the huge freedom in our EFT coupling constant space is reduced, and to the order we have calculated, namely up to 5pt amplitudes and order $1/\Lambda^4$ in an EFT expansion, the unique solution which allows for CK duality is the Kaluza-Klein compactification of the 5d theory

$$\mathcal{L}_{5d} = \left(\frac{-1}{4} \mathrm{tr}(F^2) + \frac{G_{5d}}{\Lambda^2} \mathrm{tr}(F^3) - \frac{9G_{5d}^2}{16\Lambda^4} \mathrm{tr}([F,F]^2) \right) + \mathcal{O}(\Lambda^{-6}) \tag{2.65}$$

which in the uncompactified 5d limit is known to admit a double copy. In this sense, and to the order we have calculated, we are in effect able to 'derive' Kaluza-Klein theory as the only consistent EFT which admits colour-kinematics duality for a given spectrum of states and charges. Whilst this result is very powerful, it is also disappointing in terms of limiting the search for other examples of interacting massive theories which may admit a double copy.
Our results stop short of being a proof of the inevitability of Kaluza-Klein given our chosen spectrum of states as we have not considered all possible EFT operators that arise up to the calculated EFT order $1/\Lambda^4$, nor have we computed 6pt or higher amplitudes. At each order a multitude of new contact terms can be included in the EFT expansion. Nevertheless, it becomes quite readily apparent that as the order of amplitude is increased, given the increasing number of constraints required in order to satisfy colour-kinematics, it becomes increasingly hard to find any remaining freedom beyond that from the compactification of the known CK compliant higher derivative operators.

We begin this section with a review of the conditions necessary to remove spurious poles and

[6] We do expect these couplings to be constrained by other assumptions for example by positivity bounds on the scattering of massive spin-1 states similar to those considered in [104, 105].

a brief derivation of the BCJ relations for massive theories, with some mild assumption on the exchanged particles. In section 2.2.2 we specify the large class of effective field theories we consider and begin the process of putting constraints on the multiple different coupling constants by demanding that the 4pt amplitudes respect the BCJ relations. Whilst this process leaves some freedom, this remaining freedom is removed by considering BCJ relations for the 5pt function as we do in section 2.2.9. In section 2.2.16 we briefly consider some alternative possible EFT operators.

2.2.1 Spectral conditions and massive BCJ relations

As described in section 2.0.1 in a generic theory, the matrix A in (2.12) will be invertible and the inverse of A will have elements with poles at locations which are some complicated expressions of masses in the theory giving spurious poles in the double copy amplitude. Specifically the location of the spurious poles will be determined by $\det A = 0$ and since $\det A$ is in general a complicated function of kinematic invariants and masses, the poles will typically be uncorrelated with those demanded by locality and unitarity. Hence while it is possible to construct kinematic factors which respect CK duality, the resulting gravitational theory defined by (2.15) cannot be interpreted as a local field theory. This is for example the situation that arises for the proposed double copy of massive Yang-Mills as discussed in the previous section.

One way of 'solving' this problem would be look what happens in the massless case in which double copy does not give unphysical poles. Setting the mass to zero in (2.62) and calculating A (which is the 9×9 non-zero block of MDM^T) reveals that A is singular, i.e. has rank 5 instead of 9 as in the massive case. Now, as mentioned in 2.0.1 the CK duality can only be satisfied if the vector U is in the subspace orthogonal to null vectors of A and then A can be inverted in this subspace:

$$U \cdot \text{null}(A) = 0 \tag{2.66}$$

Since there are 4 null vectors we have 4 conditions that components of U have to satisfy. These components are linear in kinematic numerators and can also be expressed in terms of partial amplitudes, in terms of which these relations are known as the BCJ relations [2]. These relations are satisfied for Yang-Mills so to satisfy CK we need to invert A in 5 dimensional subspace in which the expression of A^{-1} is much simpler and only contains physical poles. This mechanism of spurious pole cancellations is similar at higher points.

Now returning to the massive case, if we restrict to massive theories where only one mass is exchanged mass per channel and all the fields are in the adjoint representation, we see that we can copy this spurious poles cancelling mechanism from the massless case to the massive one by carefully choosing the spectrum of the theory. In particular the spurious poles are guaranteed to cancel if the components of D satisfy the same algebraic relations as in the massless case and massive analogs of BCJ relations are satisfied. Then the rank of A will be the same as in massless case and the expression for A^{-1} in the subspace orthogonal to null vectors will have a similar form without spurious poles. This idea was first proposed in [97] where the requirement for D to satisfy the same relations as in massless case was called spectral conditions. The analysis there was done in KLT language which is equivalent to our analysis in BCJ language [11].

We illustrate these spectral conditions in the simplest case of 4pt amplitude (even though as showed in 2.0.1 there are no spurious poles in the case of single mass at 4pt). In 4pt pure Yang-Mills amplitude we have one Jacobi identity, $c_s + c_t + c_u = 0$ (so M has only one non-zero row, $\{1, 1, 1\}$) and $D = \text{diag}\{s, t, u\}$ which obeys $\text{Tr}D = s + t + u = 0$. A is then 1×1 matrix A equal to $\text{Tr}D = 0$ so its rank is zero. Now if we have a massive gauge theory and we scatter m_1, m_2, m_3, m_4 with $D = \text{diag}\{s - m_{12}^2, t - m_{13}^2, u - m_{14}^2\}$, imposing the same condition as in the massless case $A = \text{Tr}D = 0$, we get

$$s + t + u - m_{12}^2 - m_{13}^2 - m_{14}^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 - m_{12}^2 - m_{13}^2 - m_{14}^2 = 0, \tag{2.67}$$

which is the 4pt spectral condition of [97]. Since A has one null one-component vector, (2.66) gives

$$n_s + n_t + n_u = 0, \tag{2.68}$$

which is just the Jacobi identity for kinematic factors. However, now this is a constraint on n's since the value of this equation cannot be shifted by shifting n's (since the matrix A is zero) so for example the n's directly calculated from Feynman diagrams must satisfy CK straight away. There is a BCFW recursion proof of BCJ relations [106, 107, 108] so in theories for which BCFW recursion works, like pure YM, the lower point BCJ relations (and spectral conditions) imply the higher point ones. As mentioned before just from 4pt considerations alone, it is not necessary to impose (2.67), since no spurious poles appear in the 4pt amplitude.

At five points for a gauge theory with all of the fields in the adjoint representation of the gauge group the colour factors are given in (2.60) and the matrix M of 9 independent Jacobi identities given by (2.61). Again we assume that there is a single mass exchange for each colour factor but now the masses in the D matrix (2.62) are different $D_{ij} = -(p_i + p_j)^2 - m_{ij}^2 = s_{ij} - m_{ij}^2$. To impose the spectral conditions we required the 9×9 block matrix in (2.7), A, to be of rank 5, such that there are 4 BCJ relations between the n's just like in the massless case. By imposing the five 5pt spectral conditions of [97]

$$\begin{aligned}
m_{15}^2 &= 2m_1^2 - m_{12}^2 - m_{13}^2 - m_{14}^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2 \\
m_{25}^2 &= m_1^2 - m_{12}^2 + 2m_2^2 - m_{23}^2 - m_{24}^2 + m_3^2 + m_4^2 + m_5^2 \\
m_{34}^2 &= 2m_1^2 - m_{12}^2 - m_{13}^2 - m_{14}^2 + 2m_2^2 - m_{23}^2 - m_{24}^2 + 2m_3^2 + 2m_4^2 + m_5^2 \\
m_{35}^2 &= -m_1^2 + m_{12}^2 + m_{14}^2 - m_2^2 + m_{24}^2 - m_4^2 \\
m_{45}^2 &= -m_1^2 + m_{12}^2 + m_{13}^2 - m_2^2 + m_{23}^2 - m_3^2 \, ,
\end{aligned} \tag{2.69}$$

one can check that the matrix A, which is given explicitly in Appendix G, has rank 5 as in the massless case. The conditions (2.69) can also be derived by demanding that that 4pt spectral condition is satisfied on every 4-point amplitude that appears in factorization channels for internal propagators going on-shell as shown in Fig. 2.8. Since A has rank 5 it has four null-eigenvectors given in (G.2). Now to satisfy CK we need the kinematic factors to satisfy the 5pt BCJ relations. Usually they are written in terms of partial amplitudes (for example in

[97]) but we write them in terms of kinematic factors:

$$\begin{aligned}&D_{12}^2(-(D_{15}D_{34}n_4+D_{23}D_{45}n_5+D_{34}D_{45}n_2))+D_{12}(D_{15}(D_{23}(D_{14}n_8+D_{34}(-n_1+n_{13}+n_6)\\&-D_{45}n_3)-D_{24}D_{34}n_4+D_{25}D_{34}n_7)+D_{23}(D_{14}D_{25}n_5-D_{24}n_5(D_{35}+D_{45})\\&+D_{34}D_{35}n_{11})+D_{34}n_2(D_{14}D_{25}-D_{24}D_{45}))-D_{15}D_{23}D_{24}(D_{34}(n_1-n_{12})+n_3(D_{35}+D_{45}))\\&=0,\end{aligned} \tag{2.70}$$

$$\begin{aligned}&D_{14}(D_{15}D_{34}(D_{12}n_4-D_{25}n_7)-D_{23}(-D_{12}D_{45}n_5+D_{24}D_{25}n_5+D_{34}D_{35}n_{11})+D_{12}D_{34}D_{45}n_2\\&+D_{15}D_{23}(-D_{24}n_8+D_{34}(n_1-n_{13}-n_6)+D_{45}n_3)-D_{24}D_{34}n_2(D_{25}+D_{35}))\\&+D_{14}^2(-(D_{15}D_{23}n_8+D_{23}D_{25}n_5+D_{25}D_{34}n_2))+D_{15}D_{24}D_{34}(D_{23}(n_{14}-n_6)-n_7(D_{25}+D_{35}))\\&=0,\end{aligned} \tag{2.71}$$

$$\begin{aligned}&-D_{15}(D_{12}D_{34}n_4(D_{24}+D_{45})+D_{13}D_{24}D_{34}n_4+D_{23}D_{45}(-D_{14}n_8+D_{34}(n_1+n_{10}-n_6)+D_{45}n_3)\\&+D_{23}D_{24}(D_{34}(n_1-n_{15})+D_{45}n_3)-D_{25}D_{34}D_{45}n_7)-D_{45}((D_{23}n_5+D_{34}n_2)(D_{12}(D_{24}+D_{45})\\&-D_{14}D_{25})+D_{13}D_{34}(D_{23}n_{11}+D_{24}n_2))=0,\end{aligned} \tag{2.72}$$

$$\begin{aligned}&D_{15}(D_{12}D_{25}D_{34}n_4-D_{23}D_{24}(n_8(D_{13}+D_{14})+D_{34}(n_6-n_9))+D_{23}D_{25}(-D_{14}n_8\\&+D_{34}(n_1+n_{10}-n_6)+D_{45}n_3)-D_{25}D_{34}n_7(D_{24}+D_{25}))-D_{25}((D_{23}n_5+D_{34}n_2)\\&(D_{14}(D_{24}+D_{25})-D_{12}D_{45})+D_{13}D_{23}(D_{24}n_5-D_{34}n_{11}))=0.\end{aligned} \tag{2.73}$$

Of course, these are the same BCJ relations as in the massless case, with the replacement $s_{ij} \to D_{ij}$ in [2]. In other words this mechanism of spurious pole cancellation in massive double copy is just considering massive theories in which all algebraic relations between partial amplitudes and propagators are the same as in massless theories, compatible with the colour-kinematics duality. It is straightforward to generalize this procedure to n-pt amplitudes by requiring the rank of A to be reduced by the number of BCJ relations at that order. The key idea of having a local double copy is to require the reduced rank matrix to admit an inverse without spurious poles.

Now having established the spectral conditions the questions is what theories satisfy it? As mentioned before one known class of examples is Kaluza-Klein theories obtained by compatifying 5d massless theories obeying the CK. This is because the massive propagators in the Kaluza-Klein theory, D_{ij}, satisfy the same algebraic relations as the massless propagator in 5d since dimensional reduction is just a rewriting a theory in terms of different variables. If the 5d kinematic factors satisfy the massless BCJ relations, then the kinematic factors of the compactified theory should obey massive BCJ relations as well.

Furthermore the spectral conditions (2.67) and (2.69) follow automatically from 5d momentum conservation which in 4d terms can be interpreted as charge conservation for a global $U(1)$ charge. It remains unclear whether there are other nontrivial solutions of these spectral conditions other than that given by Kaluza-Klein theories. Given this, in what follows we shall follow the opposite approach. We shall construct effective theories with the same spectrum and global $U(1)$ charge conservation properties as Kaluza-Klein, so that the spectral conditions (2.67) and (2.69) are automatically satisfied. We then ask what freedom there exists in the form of their interactions, which will necessarily alter the kinematic factors n_i, such that they still satisfy the BCJ relations (2.70)–(2.73). According to our procedure, as long as the latter are satisfied, it is possible to solve for v, hence for Δn, and hence determine new local kinematic factors $n + \Delta n$ which do respect CK duality, from which a double copy theory may be constructed as described in section 2.0.1.

2.2.2 KK inspired action

We consider a 4d effective field theory of interacting massless scalar and massive and massless vectors fields with the same spectrum as Kaluza-Klein theory. In particular, the massive states will be charged under a global $U(1)$ symmetry, the remnant of 5d translation symmetry, and the mass will be proportional to the charge, with the infinite spectrum of charges integer spaced. The massive vectors will further transform in the adjoint representation of some gauge group G. We shall express the action for the massive states in 4d unitary gauge[7], *i.e.* we will not introduce any Stückelberg fields which arise naturally from compactification from the higher dimensional gauge symmetry [109], meaning that the quadratic part of the action for the massive states is a complex gauged Proca theory. The remaining 4d gauge symmetry, the gauge freedom of the 4d massless gluon, is however made manifest.

In the EFT context, there is still a huge freedom in the choice of interactions between the states, even given the assumed 4d gauge symmetry and global $U(1)$ symmetry. In order to make calculational progress we will restrict to what remains a very large set of possible interactions. This set is chosen by the requirement that all the terms in the 4d action do indeed arise from compactification of 5d pure Yang-Mills together with the following additional operators:

$$\frac{1}{\Lambda^2}\text{tr}(F^3) \tag{2.74}$$

and

$$\frac{-9}{16\Lambda^4}\text{tr}([F_{\mu\nu}, F_{\alpha\beta}][F^{\mu\nu}, F^{\alpha\beta}]). \tag{2.75}$$

We chose these higher order operators because it was shown in [110] that they are compatible with colour-kinematics duality. These particular operators are among those appearing in the low energy EFT of open bosonic and super string theory at $\mathcal{O}(\alpha'^2)$. However both of these EFTs also contain some operators that are are incompatible with CK duality which is based on cubic graphs with colour factors made of structure constants. For example the colour factor of $\text{tr}(F^{\mu}_{\nu}F^{\nu}_{\rho}F^{\rho}F^{\sigma}_{\mu})$ is $\text{tr}(T^aT^bT^cT^d)$, which contains the totally symmetric colour structure, d^{abcd}, that cannot be expressed purely in terms of antisymmetric structure constants, f^{abc}. However,

[7]This is unitary gauge for the gauge symmetries which are broken by the mass for the spin-1 states and has nothing to do with the unbroken 4d gauge symmetry G. In other words, if we did not choose any gauge in 5d theory, reducing it to 4d would give Stückelberg fields such that the 4d action would have a non-linearly realised full 5d gauge symmetry. By choosing unitary gauge we set all of Stückelberg fields to zero.

	coefficient	Interactions		coefficient	Interactions
$\mathcal{L}_{AAA}$	g_{ijk}	$DAAA$	$\mathcal{L}^{F^4}_{AAAA1}$	c_{ijkl}	$(DA)^4$
$\mathcal{L}_{AA\phi}$	g'_{ijs}	$AA\phi$	$\mathcal{L}^{F^4}_{AAAA2}$	C_{ijkl}	$m^2(DA)^2A^2$
$\mathcal{L}_{AAAA}$	g_{ijkl}	$AAAA$	$\mathcal{L}^{F^4}_{AAA\phi1}$	c_{ijks}	$m(DA)^2A\phi$
$\mathcal{L}_{AA\phi\phi}$	g_{ijss}	$AA\phi\phi$	$\mathcal{L}^{F^4}_{AAA\phi2}$	C_{ijks}	$m^3A^3D\phi$
$\mathcal{L}_{AAA^0}$	g_i	AAF^0	$\mathcal{L}^{F^4}_{AA\phi\phi1}$	c_{ijss}	$(DA)^2(D\phi)^2$
$\mathcal{L}^{F^3}_{AAA^0}$	G_i	$DADAF^0$	$\mathcal{L}^{F^4}_{AA\phi\phi2}$	$c^{(2)}_{ijss}$	$(mA)^2(D\phi)^2$
$\mathcal{L}^{F^3}_{AAA1}$	G_{ijk}	$(DA)^3$	$\mathcal{L}^{F^4}_{AA\phi\phi3}$	$c^{(3)}_{ijss}$	$(mA)^2(D\phi)^2$
$\mathcal{L}^{F^3}_{AAA2}$	$\hat{G}_{ijk}$	m_im_jDAAA	$\mathcal{L}^{F^4}_{AA\phi\phi3}$	$c^{(3)}_{ijss}$	$(mA)^2(D\phi)^2$
$\mathcal{L}^{F^3}_{AA\phi}$	G'_{ijs}	$AD\phi DA$	$\mathcal{L}^{F^4}_{\phi\phi\phi\phi}$	$c_{\phi4}$	$(D\phi)^4$
$\mathcal{L}^{F^3}_{A\phi\phi}$	G_{0ss}	$D\phi D\phi F^0$	$\mathcal{L}^{F^4}_{AAAAA1}$	c_{ijklm}	$(DA)^3A^2$
$\mathcal{L}^{F^3}_{AAAA1}$	G_{ijkl}	$DADAAA$	$\mathcal{L}^{F^4}_{AAAAA2}$	C_{ijklm}	$m^2(A)^4DA$
$\mathcal{L}^{F^3}_{AAAA2}$	$\hat{G}_{ijkl}$	m_im_jAAAA	$\mathcal{L}^{F^4}_{\phi AAAA1}$	c_{ijkls}	$m(A)^3DAD\phi$
$\mathcal{L}^{F^3}_{AA\phi\phi1}$	G_{ijss}	$AAD\phi D\phi$	$\mathcal{L}^{F^4}_{\phi AAAA2}$	C_{ijklm}	$m(DA)^2(A)^2\phi$
$\mathcal{L}^{F^3}_{AA\phi\phi2}$	$\hat{G}_{ijss}$	$A\phi DAD\phi$	$\mathcal{L}^{F^4}_{\phi AAAA3}$	$\hat{C}_{ijkls}$	$(m)^3(A)^4\phi$
$\mathcal{L}^{F^3}_{AAA\phi1}$	$\hat{G}_{ijks}$	$mAAAD\phi$	$\mathcal{L}^{F^4}_{\phi\phi AAA1}$	c_{ijkss}	$DA(A)^2(D\phi)^2$
$\mathcal{L}^{F^3}_{AAA\phi2}$	G_{ijks}	$mAADA\phi$	$\mathcal{L}^{F^4}_{\phi\phi AAA2}$	C_{ijkss}	$A(DA)^2(D\phi)^2$
$\mathcal{L}^{F^3}_{AAAAA}$	G_{ijklm}	$DAAAAA$	$\mathcal{L}^{F^4}_{\phi\phi AAA3}$	$\hat{C}^{(3)}_{ijkss}$	$m^2(A)^3D\phi\phi$
$\mathcal{L}^{F^3}_{\phi AAAA}$	G_{ijkls}	$mA\phi AAA$	$\mathcal{L}^{F^4}_{\phi\phi AAA4}$	$\hat{C}^{(4)}_{ijkss}$	$m^2(A)^3D\phi\phi$
$\mathcal{L}^{F^3}_{\phi\phi AAA1}$	G_{ijkss}	$D\phi\phi AAA$	$\mathcal{L}^{F^4}_{\phi\phi AAA5}$	$\hat{C}^{(5)}_{ijkss}$	$m^2(A)^3D\phi\phi$
$\mathcal{L}^{F^3}_{\phi\phi AAA2}$	$\hat{G}_{ijkss}$	$DA\phi A\phi A$	$\mathcal{L}^{F^4}_{\phi\phi\phi AA1}$	c_{ijsss}	$mA^2D\phi D\phi\phi$
$\mathcal{L}^{F^4}_{\phi\phi\phi AA2}$	$c^{(2)}_{ijsss}$	$mA^2(D\phi)^2\phi$	$\mathcal{L}^{F^4}_{\phi\phi\phi AA3}$	$c^{(3)}_{ijsss}$	$mA^2(D\phi)^2\phi$

Table 2.1: Coefficients of the interactions.

to cancel such colour structures it is possible to take a certain linear combination of open bosonic (bs) and superstring (ss) amplitudes at the order $\mathcal{O}(\alpha'^2)$, $A^{\alpha'^2}_{bs} - A^{\alpha'^2}_{ss}$ [110]. This particular combination is the one corresponding to the operators in (2.74) and (2.75). Since KLT relations are linear in amplitudes the linear combination of two amplitudes is guaranteed to satisfy them, so the reason why these particular operators satisfy CK can be understood as a low energy limit of KLT relations between a linear combination of the ampltitudes in two theories satisfying them, that cancels the colour structures which are not made of f^{abc}.

Crucially though we allow each of the coefficients of the various invariant terms in the action to be arbitrary, and in principle to be different for different interacting massive states. In doing so, the 4d action loses any further remnant information of the underlying 5d gauge symmetry. Thus while KK theory lies as a special point in our chosen class of EFTs, the class as a whole is still huge.

Specifically then, we have an action with an uncharged massless scalar field ϕ and multiple charged massive vector fields A^i transforming as matter fields in adjoint representation of some non-Abelian group G for which A^0 is the gauge connection. To order $1/\Lambda^4$ the Lagrangian is given as:

$$\mathcal{L} = \mathcal{L}^{F^2} + \mathcal{L}^{F^3} + \mathcal{L}^{F^4}, \tag{2.76}$$

where $\mathcal{L}^{F^2}$, $\mathcal{L}^{F^3}$ and $\mathcal{L}^{F^4}$ contain the operators appearing in the compactification of $\text{tr}(F^2)$, $\text{tr}(F^3)$ and $\text{tr}([F_{\mu\nu}, F_{\alpha\beta}][F^{\mu\nu}, F^{\alpha\beta}])$ respectively that contribute to up to 5pt amplitudes. They are as follows:

$$\begin{aligned}\mathcal{L}^{F^2} \quad &= \text{tr}\left(-\tfrac{1}{2}D_\mu\phi D^\mu\phi - \tfrac{1}{4}F^0_{\mu\nu}F^{0\mu\nu} - \tfrac{1}{2}\textstyle\sum_{i\in\mathbb{Z}_{\neq 0}}\tfrac{1}{2}|D_\mu A^i_\nu - D_\nu A^i_\mu|^2 - 2g_i A^{i\mu}A^{-i\nu}F^0_{\mu\nu} + m_i^2|A^i_\mu|^2\right) \\ &\quad + \mathcal{L}_{AAA} + \mathcal{L}_{AA\phi} + \mathcal{L}_{AA\phi\phi} + \mathcal{L}_{AAAA}\,,\end{aligned} \tag{2.77}$$

$$\begin{aligned}\mathcal{L}^{F^3} = \quad & \frac{1}{\Lambda^2}\text{tr}\left(GF^0_{\mu\nu}F^{0\nu\rho}F^{0\mu}_{\rho} + \sum_{i\in\mathbb{Z}_{\neq 0}}\left(3G_i D_\mu A^{i\nu}D_\nu A^{-i\rho}F^{\mu}_{0\rho}\right)\right) \\ & + \mathcal{L}^{F^3}_{AAA1} + \mathcal{L}^{F^3}_{AAA2} + \mathcal{L}^{F^3}_{AA\phi} + \mathcal{L}^{F^3}_{A\phi\phi} + \mathcal{L}^{F^3}_{AAAA1} + \mathcal{L}^{F^3}_{AAAA2} + \mathcal{L}^{F^3}_{AA\phi\phi 1} + \mathcal{L}^{F^3}_{AA\phi\phi 2} + \mathcal{L}^{F^3}_{AAA\phi 1} \\ & + \mathcal{L}^{F^3}_{AAA\phi 2} + \mathcal{L}^{F^3}_{AAAAA} + \mathcal{L}^{F^3}_{\phi AAAA} + \mathcal{L}^{F^3}_{\phi\phi AAA1} + \mathcal{L}^{F^3}_{\phi\phi AAA2}\,,\end{aligned} \tag{2.78}$$

$$\begin{aligned}\mathcal{L}^{F^4} \quad &= \mathcal{L}^{F^4}_{AAAA1} + \mathcal{L}^{F^4}_{AAAA2} + \mathcal{L}^{F^4}_{AAA\phi 1} + \mathcal{L}^{F^4}_{AAA\phi 2} + \mathcal{L}^{F^4}_{AA\phi\phi 1} + \mathcal{L}^{F^4}_{AA\phi\phi 2} + \mathcal{L}^{F^4}_{AA\phi\phi 3} + \mathcal{L}^{F^4}_{\phi\phi\phi\phi} \\ &\quad + \mathcal{L}^{F^4}_{AAAAA1} + \mathcal{L}^{F^4}_{AAAAA2} + \mathcal{L}^{F^4}_{\phi AAAA1} + \mathcal{L}^{F^4}_{\phi AAAA2} + \mathcal{L}^{F^4}_{\phi AAAA3} + \mathcal{L}^{F^4}_{\phi\phi AAA1} + \mathcal{L}^{F^4}_{\phi\phi AAA2} \\ &\quad + \mathcal{L}^{F^4}_{\phi\phi AAA3} + \mathcal{L}^{F^4}_{\phi\phi\phi AA1} + \mathcal{L}^{F^4}_{\phi\phi\phi AA2},\end{aligned} \tag{2.79}$$

where,

$$D_\mu = \partial_\mu + igA^0_\mu\,, \qquad F^{0a}_{\mu\nu} = \partial_\mu A^{0a}_\nu - \partial_\nu A^{0a}_\mu + \frac{g}{\sqrt{2}}f^{abc}A^{0b}_\mu A^{0c}_\nu\,, \tag{2.80}$$

and schematically, the interacting terms with the relevant coefficients are in table 2.1. The exact terms are given in the Appendix G.1.

We write couplings only of distinct operators coming from the compactification of the 5d Yang-

Mills. The couplings of identical operators coming from different 5d terms are combined. For example, we get the interactions of the form

$$f^{abe} f^{cde} \sum_{i,j,k,l\in\mathbb{Z}_{\neq 0}} \left(A^{ia}_{[\mu} A^{jb}_{\nu]} A^{kc[\mu} A^{ld\nu]} \right), \tag{2.81}$$

from the compactification of $\mathcal{L}^{F^2}$ as well as $\mathcal{L}^{F^4}$, which has $\frac{1}{\Lambda^4}$. We combine the couplings into a single one, g_{ijkl}. Hence, the coupling g_{ijkl} is dependent on the scale Λ. Note that in KK theory $g_{ijkl} = g^2 + \frac{18 m_1 m_2 m_3 m_4}{\Lambda^4} G^2$.

Demanding the global $U(1)$ symmetry imposes charge conservation at each vertex as in KK theory

$$\sum_{I=1}^{n} m_{i_I} = 0 \tag{2.82}$$

for every non-zero couplings where $I = 1, ..., n$ labels the legs of the vertex. Note that m_i is really labelling the charge of $A^{i\mu}$ so we allow negative values $m_{-i} = -m_i$ with the understanding that the mass is $|m_i|$. The condition (2.82) ensures that the spectral conditions are satisfied [97]. Since we have considered the EFT expansion of the action only up to $1/\Lambda^4$ order, it is only consistent to calculate scattering amplitudes up to this order, which is precisely what we will do in Section 2.2.3 and Section 2.2.9 for the 4pt and 5pt functions respectively.

2.2.3 4-point amplitudes

In this section we constrain the couplings of the Lagrangian by calculating 2-2 scattering amplitudes up to $1/\Lambda^4$ in the EFT expansion (for self consistency since we only add irrelevant operators up to $1/\Lambda^4$ in our action) and requiring the numerators to satisfy the 4pt colour-kinematic duality, $n_s + n_t + n_u = 0$. We consider the scattering processes where the couplings are not fixed by 4d gauge invariance, for example, we do not consider $A^0 A^0 \to A^0 A^0$.

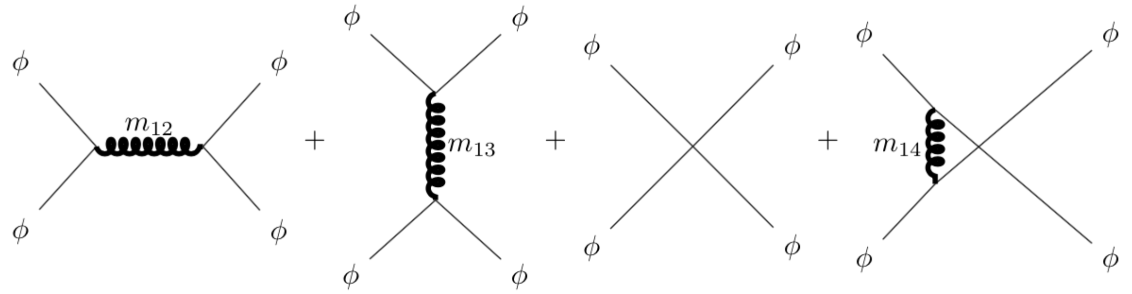

Figure 2.1: Feynman diagrams for the $\phi\phi \to \phi\phi$ process. Note that the bold curly line represents a gluon and $m_{ij} = 0$.

2.2.4 $\phi\phi \to \phi\phi$

As the simplest example, let us first consider all the external states to be the massless scalars. By charge conservation, any exchanged state must be massless, and the only possibility for our chosen theory is the massless gluon. The Feynman diagrams for this process are in Fig. 2.1 and the four point amplitude is found to be:

$$\begin{aligned} i\mathcal{A}_{\phi\phi\phi\phi} = &\frac{i}{2\Lambda^4 st(s+t)}\Bigg(c_u st(s-t)\big(-18c_{\phi 4}(s+t)^2 + 3G_{0ss}(s+t)\left(\sqrt{2}g\Lambda^2 + 6G_{0ss}(s+t)\right) \\ &+g\Lambda^2\left(g\Lambda^2 + 3\sqrt{2}G_{0ss}(s+t)\right)\big) - (s+t)\Bigg(c_s t(s+2t)\big(-18c_{\phi 4}s^2 + g^2\Lambda^4 - 6\sqrt{2}gG_{0ss}\Lambda^2 s \\ &+18G_{0ss}^2 s^2\big) - c_t s(2s+t)\left(-18c_{\phi 4}t^2 + g^2\Lambda^4 - 6\sqrt{2}gG_{0ss}\Lambda^2 t + 18G_{0ss}^2 t^2\right)\Bigg)\Bigg) \end{aligned} \tag{2.83}$$

The BCJ relation gives the following condition

$$-\frac{i\left(c_{\phi 4} - G_{0ss}^2\right)\left(3s^2t + 2s^3 - 3st^2 - 2t^3\right)}{\Lambda^4} = 0,$$

which is satisfied if

$$c_{\phi 4} = G_{0ss}^2. \tag{2.84}$$

This simple example already illustrates the power of demanding CK duality. From a low energy EFT point of view, there is no reason for the Wilson coefficient for the quartic $(D\phi)^4$ operator to be associated with that for the cubic $D\phi D\phi F^0$ operator. Demanding CK duality enforces the relation 'quartic coupling= cubic coupling squared'. Note that in this particular amplitude, no massive states are involved. We shall find that this general idea translates into constraints

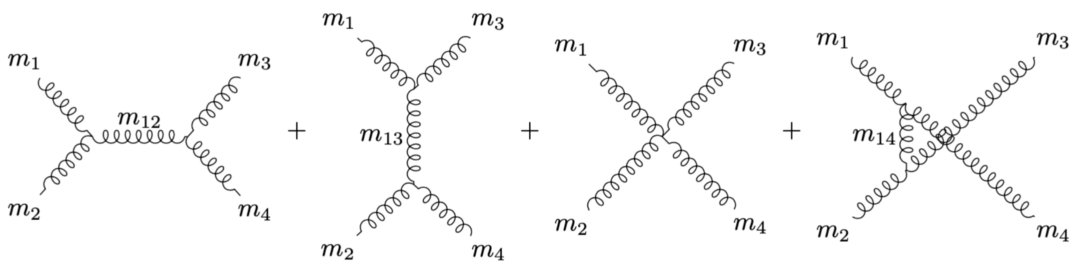

Figure 2.2: Feynman diagrams of the $AA \to AA$ process for the general case. The curly lines represents massive spin-1 fields.

on nearly all the quartic operators from demanding CK duality. In the remainder, we will not include the explicit expression for the scattering amplitudes and BCJ relations as they are overly complicated, but shall give only the implied conclusion for the constraints on the coupling constants.

2.2.5 General condition

We first calculate the 2-2 scattering amplitude of spin-1 fields with masses m_I, with $I = 1, 2, 3, 4$ labelling the amplitude legs, such that $\sum_{I=1}^{4} m_{i_I} = 0$, but for which charge conservation forbids the exchange of massless states. To avoid cumbersome notation we shall replace the charge label i for the massive state with the leg label I so that m_{i_I} is denoted m_I. Similarly a coupling $G_{i_I i_J}$ is replaced with G_{IJ}. Furthermore, for cubic interactions between 3 charged states, for which the charge of the 3rd state is fixed by charge conservation, we shall drop the 3rd label. So for example $G_{i_1 i_2 i_3}$ for which $m_3 = -m_1 - m_2$ may be written in short hand as G_{12}. The interacting terms $\mathcal{L}_{AAA}$, $\mathcal{L}_{AAAA}$ and $\mathcal{L}_{AA\phi}$ lead to the following amplitude (see Fig. 2.2):

$$
\begin{aligned}
i\mathcal{A}_4 \propto & \left(V_{g_{12}}^{AAA} + V_{G_{12}}^{AAA1} + V_{\hat{G}_{12}}^{AAA2}\right) \frac{i}{s - m_{12}^2} \left(V_{g_{34}}^{AAA} + V_{G_{34}}^{AAA1} + V_{\hat{G}_{34}}^{AAA2}\right) \\
& + \left(V_{g_{13}}^{AAA} + V_{G_{13}}^{AAA1} + V_{\hat{G}_{13}}^{AAA2}\right) \frac{i}{t - m_{13}^2} \left(V_{g_{24}}^{AAA} + V_{G_{24}}^{AAA1} + V_{\hat{G}_{24}}^{AAA2}\right) \\
& + \left(V_{g_{14}}^{AAA} + V_{G_{14}}^{AAA1} + V_{\hat{G}_{14}}^{AAA2}\right) \frac{i}{u - m_{14}^2} \left(V_{g_{23}}^{AAA} + V_{G_{23}}^{AAA1} + V_{\hat{G}_{23}}^{AAA2}\right) \\
& + \left(V_{g_{1234}}^{AAAA} + V_{G_{1234}}^{AAAA1} + V_{\hat{G}_{1234}}^{AAAA2} + V_{c_{1234}}^{AAAA1} + V_{C_{1234}}^{AAAA2}\right),
\end{aligned} \tag{2.85}
$$

where $V^{ABC(D)}_{g_{ijk(l)}}$ represents the three or four point vertex with the relevant coupling, $g_{ijk(l)}$ or $G_{ijk(l)}$. The coupling g_{ij} is a simplified notation for g_{ijk} with $m_I + m_J + m_K = 0$. Here we assume that the couplings are symmetric in all of their indices, for example $g_{jik} = g_{jik} = g_{kji}$ so that the $A_iA_jA_k$ vertex without $1/\Lambda^{2n}$ corrections has the same structure as the pure Yang-Mills three point vertex. Later we will explore what happens if we do not assume that. By finding the numerators of (2.85) and imposing the colour-kinematic duality, $n_s + n_t + n_u = 0$, we find the following constraints on the couplings:

$$\begin{aligned} &g_{1234} - \frac{18 m_1 m_2 m_3 m_4}{\Lambda^4} c_{1234} = g_{12}g_{34} = g_{13}g_{24} = g_{14}g_{23} = \frac{G^2_{1234}}{c_{1234}}, \\ &G_{1234} = G_{12}g_{34} = G_{13}g_{24} = G_{14}g_{23} = g_{12}G_{34} = g_{13}G_{24} = g_{14}G_{23}, \\ &G_{ij} = \hat{G}_{ij}, \quad G_{1234} = \hat{G}_{1234}, \quad c_{1234} = C_{1234}, \\ &c_{1234} = G_{12}G_{34} = G_{13}G_{24} = G_{14}G_{23}\,. \end{aligned} \tag{2.86}$$

As mentioned in section 2.2.2, g_{1234} can be scale dependent as it is the combination of two terms coming from the compactification of $\mathcal{L}^{F^2}$ and $\mathcal{L}^{F^4}$. Therefore the coefficient of $1/\Lambda^4$ in the equation above, c_{1234}, does not have to be zero.

2.2.6 $m_I + m_J = 0$

Further focusing on the case of 2-2 scattering amplitude of spin-1 fields with masses m_I, $I = 1, 2, 3, 4$, such that $m_I + m_J = 0$ means that the exchange diagrams can now be mediated by the massless states, A^0 and ϕ. First we consider the case where two pairs of masses add up to zero then the second case where four pairs of masses add up to zero.

First case: Without loss of generality, we consider $m_1 + m_2 = 0$, which implies $m_3 + m_4 = 0$. Moreover, we assume that none of the masses are individually zero. These conditions lead to an exchange of gluon and scalar in the s-channel and exchange of massive spin-1 field in the t-channel and the u-channel. Hence, from the interacting terms $\mathcal{L}_{AAA}$, $\mathcal{L}_{AAAA}$ and $\mathcal{L}_{AA\phi}$ we get the following amplitude (see Fig. 2.3): :

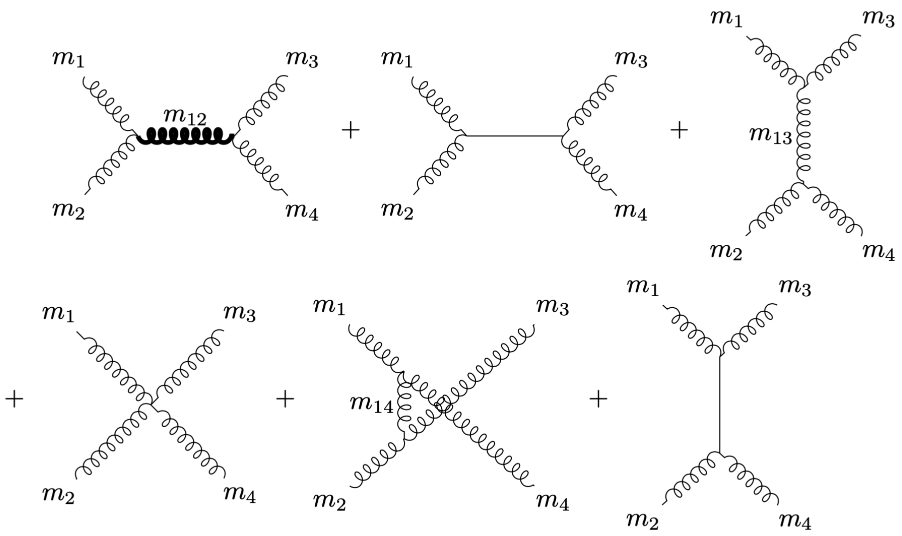

Figure 2.3: Feynman diagrams of the $AA \to AA$ process for the first case of $m_I + m_J = 0$, i.e. $m_1 + m_2 = 0$ and $m_3 + m_4 = 0$. The bold curly line represents a gluon, where $m_{12} = 0$ and the straight line a scalar field. The last diagram, where we have a scalar exchange in the t-channel contributes to the second case of $m_I + m_J = 0$ where $m_1 = -m_2 = m_3 = -m_4$.

$$
\begin{aligned}
i\mathcal{A}_4 \propto & \left(V_g^{AAA_0} + V_{g_i}^{AAA_0} + V_{G_i}^{AAA_0} + V_{\hat{G}_i}^{AAA_0}\right) \frac{i}{s} \left(V_g^{AAA_0} + V_{g_i}^{AAA_0} + V_{G_i}^{AAA_0} + V_{\hat{G}_i}^{AAA_0}\right) \\
& + \left(V_{g'_{12s}}^{AA\phi} + V_{G'_{12s}}^{AA\phi}\right) \frac{i}{s} \left(V_{g'_{34s}}^{AA\phi} + V_{G'_{34s}}^{AA\phi}\right) \\
& + \left(V_{g_{13}}^{AAA} + V_{G_{13}}^{AAA1} + V_{\hat{G}_{13}}^{AAA2}\right) \frac{i}{t - m_{13}^2} \left(V_{g_{-1-3}}^{AAA} + V_{G_{-1-3}}^{AAA1} + V_{\hat{G}_{-1-3}}^{AAA2}\right) \\
& + \left(V_{g_{1-3}}^{AAA} + V_{G_{1-3}}^{AAA1} + V_{\hat{G}_{1-3}}^{AAA2}\right) \frac{i}{u - m_{14}^2} \left(V_{g_{-13}}^{AAA} + V_{G_{-13}}^{AAA1} + V_{\hat{G}_{-13}}^{AAA2}\right) \\
& + \left(V_{g_{1-13-3}}^{AAAA} + V_{G_{1-13-3}}^{AAAA1} + V_{\hat{G}_{1-13-3}}^{AAAA2} + V_{c_{1-13-3}}^{AAAA1} + V_{C_{1-13-3}}^{AAAA2}\right).
\end{aligned} \tag{2.87}
$$

By finding the numerators of (2.87) and imposing the colour-kinematic duality, $n_s + n_t + n_u = 0$, we find the following constraints on the couplings:

$$
\begin{aligned}
& g_{1-13-3} - \frac{18 m_1^2 m_3^2}{\Lambda^4} c_{1-13-3} = g^2 = g_{13} g_{-1-3} = g_{1-3} g_{-13}, \\
& G_{ij} = \hat{G}_{ij}, \quad g_i = g - G_i \frac{3\sqrt{2} m_i^2}{\Lambda^2}, \quad g'_{1-1s} g'_{3-3s} = g^2 m_1 m_3, \\
& \hat{G}_{1-13-3} = G_{1-13-3} = G_3 g = G_1 g = G_{13} g_{-1-3} = g_{13} G_{-1-3} = G_{1-3} g_{-13} = g_{1-3} G_{-13} = \frac{g'_{1-1s} G'_{3-3s}}{m_1 m_3}, \\
& c_{1-13-3} = C_{1-13-3} = G_1 G_3 = G_{1-3} G_{-13} = G_{13} G_{-1-3}.
\end{aligned} \tag{2.88}
$$

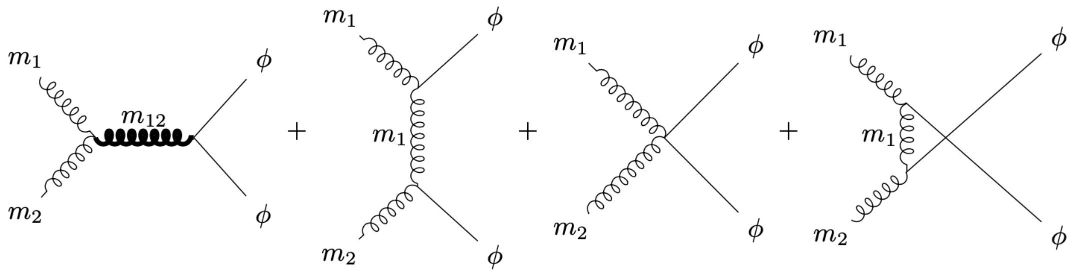

Figure 2.4: Feynman diagrams of the $AA \to \phi\phi$ process for $m_I + m_J = 0$ case. The bold curly line represents a gluon, where $m_{12} = 0$ and the straight line a scalar field.

When $m_4 = m_3 = 0$ we find that

$$G_i = G, \quad c_{1-100} = C_{1-100} = G^2 \,. \tag{2.89}$$

Second case: Now we consider four pairs of masses to be zero, for example, $m_1 + m_2 = m_1 + m_3 = m_3 + m_4 = m_2 + m_4 = 0$ and $m_1 \neq 0$. These conditions lead to an exchange of gluon and scalar in the s and t channels and exchange of massive spin-1 field in the u-channel. Hence, from the interacting terms $\mathcal{L}_{AAA}$, $\mathcal{L}_{AAAA}$ and $\mathcal{L}_{AA\phi}$ we get the following amplitude (see Fig. 2.3):

$$\begin{aligned}
i\mathcal{A}_4 \propto & \left(V_g^{AAA_0} + V_{g_i}^{AAA_0} + V_{G_i}^{AAA_0}\right) \frac{i}{s} \left(V_g^{AAA_0} + V_{g_i}^{AAA_0} + V_{G_i}^{AAA_0}\right) \\
& + \left(V_{g'_{1-1s}}^{AA\phi} + V_{G'_{1-1s}}^{AA\phi}\right) \frac{i}{s} \left(V_{g'_{-11s}}^{AA\phi} + V_{G'_{-11s}}^{AA\phi}\right) \\
& + \left(V_{g_{1-1}}^{AAA} + V_{G_{1-1}}^{AAA1} + V_{\hat{G}_{1-1}}^{AAA2}\right) \frac{i}{t} \left(V_{g_{-11}}^{AAA} + V_{G_{-11}}^{AAA1} + V_{\hat{G}_{-11}}^{AAA2}\right) \\
& + \left(V_{g'_{1-1s}}^{AA\phi} + V_{G'_{1-1s}}^{AA\phi}\right) \frac{i}{t} \left(V_{g'_{-11s}}^{AA\phi} + V_{G'_{-11s}}^{AA\phi}\right) \\
& + \left(V_{g_{11}}^{AAA} + V_{G_{11}}^{AAA1} + V_{\hat{G}_{11}}^{AAA2}\right) \frac{i}{u - m_{11}^2} \left(V_{g_{-1-1}}^{AAA} + V_{G_{-1-1}}^{AAA1} + V_{\hat{G}_{-1-1}}^{AAA2}\right) \\
& + \left(V_{g_{1-1-11}}^{AAAA} + V_{G_{1-1-11}}^{AAAA1} + V_{\hat{G}_{1-1-11}}^{AAAA2} + V_{c_{1-1-11}}^{AAAA1} + V_{C_{1-1-11}}^{AAAA2}\right) .
\end{aligned} \tag{2.90}$$

By finding the numerators of (2.90) and imposing the colour-kinematic duality, $n_s+n_t+n_u=0$, we find the following constraints on the couplings [8]:

$$
\begin{aligned}
&g_{1-1-11}-\frac{18m_1^4}{\Lambda^4}c_{1-1-11}=g^2=g_{11}g_{-1-1}, \quad g_i=g-G_i\frac{3\sqrt{2}m_i^2}{\Lambda^2}, \quad g'_{1-1s}g'_{-11s}=g^2m_1^2,\\
&\hat{G}_{1-1-11}=G_{1-1-11}=G_{-1}g=G_1g=G_{11}g_{-1-1}=g_{11}G_{-1-1}=\frac{g'_{1-1s}G'_{-11s}}{m_1^2},\\
&c_{1-1-11}=C_{1-1-11}=G_1G_{-1}=G_{1-1}G_{-1-1}.
\end{aligned} \tag{2.91}
$$

2.2.7 $AA\to\phi\phi$

Now we consider the case where $m_3=m_4=0$ and we calculate the $AA\to\phi\phi$ amplitude. A gluon is exchanged in the s-channel and a massive spin-1 field in the t and u-channel. From the terms $\mathcal{L}_{AAA}$, $\mathcal{L}_{AA\phi\phi}$, $\mathcal{L}_{AA\phi}$ and $\mathcal{L}_{\phi\phi A}$ we have (see Fig. 2.4):

$$
\begin{aligned}
i\mathcal{A}_4 \propto\, &\left(V_g^{AAA_0}+V_{g_i}^{AAA_0}+V_{G_i}^{AAA_0}\right)\frac{i}{s}\left(V_g^{A_0\phi\phi}+V_{G_{0ss}}^{A_0\phi\phi}\right)\\
&+\left(V_{g'_{1-1s}}^{AA\phi}+V_{G'_{1-1s}}^{AA\phi}\right)\frac{i}{t-m_1^2}\left(V_{g'_{1-1s}}^{AA\phi}+V_{G'_{1-1s}}^{AA\phi}\right)\\
&+\left(V_{g'_{1-1s}}^{AA\phi}+V_{G'_{1-1s}}^{AA\phi}\right)\frac{i}{u-m_1^2}\left(V_{g'_{1-1s}}^{AA\phi}+V_{G'_{1-1s}}^{AA\phi}\right)\\
&+\left(V_{g_{1-100}}^{AA\phi\phi}+V_{G_{1-1ss}}^{AA\phi\phi1}+V_{\hat{G}_{1-1ss}}^{AA\phi\phi2}+V_{c_{1-1ss}}^{AA\phi\phi1}+V_{c^{(2)}_{1-1ss}}^{AA\phi\phi2}+V_{c^{(3)}_{1-1ss}}^{AA\phi\phi3}\right).
\end{aligned} \tag{2.92}
$$

Colour-kinematics duality puts the following constraints on the couplings:

$$
\begin{aligned}
&g_{i-iss}=g^2, \quad G_i=\frac{2g'^2_{i-is}-m_1^2g(g+g_i)}{3\sqrt{2}gm_1^2}\frac{\Lambda^2}{m_1^2}, \quad c_{i-iss}=G_{0ss}G_i, \quad c^{(2)}_{i-iss}=\frac{G'^2_{i-is}}{m_1^2},\\
&c^{(3)}_{i-iss}=G_{0ss}G_i+\frac{\sqrt{2}\Lambda^2}{6m_1^2}g\left(G_{0ss}-G_i\right), \quad \hat{G}_{i-iss}=gG_i,\\
&G_{i-iss}=-gG_i+\frac{2g'^2_{i-is}G_{0ss}}{gm_1^2}-\frac{3\sqrt{2}G'^2_{i-is}}{\Lambda^2}.
\end{aligned} \tag{2.93}
$$

---

[8] We found another set of solution where the quartic operators $G_{1-1-11}=\hat{G}_{1-1-11}=0$ but $G_{14}\neq\hat{G}_{14}$ which is incompatible with the result obtained from the general case.

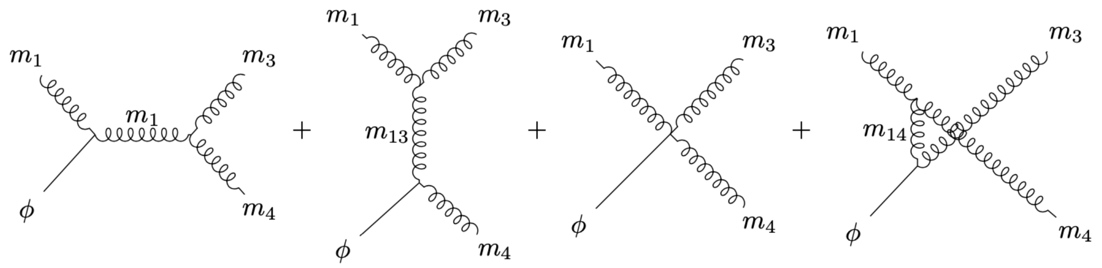

Figure 2.5: Feynman diagrams of the $A\phi \to AA$ process for $m_I = 0$ case.

2.2.8 $m_I = 0$

In this section, we calculate 2-2 scattering amplitude of one massless external state and three massive spin-1 fields satisfying $m_I + m_J + m_K = 0$. We consider the massless state to be a scalar with $m_2 = 0$ (note that the case where the massless state is a gluon reproduces the same results as in the general case and we will give the constraints at the end of this section for completeness). The amplitude for this process is the following (see Fig. 2.5):

$$
\begin{aligned}
i\mathcal{A}_4 \propto & \left(V^{AA\phi}_{g'_{1-1s}} + V^{AA\phi}_{G'_{1-1s}}\right) \frac{i}{s - m_1^2} \left(V^{AAA}_{g_{34}} + V^{AAA1}_{G_{34}} + V^{AAA2}_{\hat{G}_{34}}\right) \\
& + \left(V^{AAA}_{g_{13}} + V^{AAA1}_{G_{13}} + V^{AAA2}_{\hat{G}_{13}}\right) \frac{i}{t - m_{13}^2} \left(V^{AA\phi}_{g'_{4-4}s} + V^{AA\phi}_{G'_{4-4}s}\right) \\
& + \left(V^{AAA}_{g_{14}} + V^{AAA1}_{G_{14}} + V^{AAA2}_{\hat{G}_{14}}\right) \frac{i}{u - m_{14}^2} \left(V^{AA\phi}_{g'_{3-3s}} + V^{AA\phi}_{G'_{3-3s}}\right) \\
& + \left(V^{AAA\phi 1}_{\hat{G}_{134s}} + V^{AAA\phi 2}_{G_{134s}} + V^{AAA\phi 1}_{c_{134s}} + V^{AAA\phi 2}_{C_{134s}}\right),
\end{aligned}
\tag{2.94}
$$

Note that in this case we have $m_1 + m_3 + m_4 = 0$, so:

$$
\begin{aligned}
g_{34(-3-4)} = g_{341}, \quad g_{13(-1-3)} = g_{134}, \quad g_{14(-1-4)} = g_{143}, \\
G_{34(-3-4)} = G_{341}, \quad G_{13(-1-3)} = G_{134}, \quad G_{14(-1-4)} = G_{143},
\end{aligned}
\tag{2.95}
$$

where we assume that g_{ijk} and G_{ijk} are fully symmetric in their indices and $(\pm I \pm J)$ represents $\pm m_I \pm m_J$. Hence, in our simplified notation we have $g_{34} = g_{13} = g_{14}$ and $G_{34} = G_{13} = G_{14}$ in this case. Imposing the colour-kinematic duality on $A\phi \to AA$, we get the following constraints:

$$
\begin{aligned}
&\frac{g'_{4-4s}}{g'_{3-3s}} = \frac{m_4}{m_3}, \quad \frac{g'_{1-1s}}{g'_{3-3s}} = \frac{m_1}{m_3}, \quad G_{34} = \hat{G}_{34} \\
&\frac{G_{13}g'_{4-4}}{m_4} = \frac{G_{14}g'_{3-3}}{m_3} = \frac{g_{13}G'_{4-4s}}{m_4} = \frac{g_{14}G'_{3-3s}}{m_3} = \hat{G}_{134s} = G_{134s}, \\
&c_{134s} = C_{134s} = \frac{G_{1-1s}G_{34}}{m_1} = \frac{G_{4-4s}G_{13}}{m_4} = \frac{G_{3-3s}G_{14}}{m_3}.
\end{aligned} \tag{2.96}
$$

The conditions on couplings where one external state ($m_2 = 0$) is a gluon can be derived from (2.86) by setting $g_{12} = g_{24} = g_{23} = g$ and $g_{1234} = g^2$ which leads to $g_i = g - G_i \frac{3\sqrt{2}m_i^2}{\Lambda^2}$ and $g_{34}g = g^2$ so $g_{ij} = g$ for arbitrary i, j, and $G_1 g = G_{13} g$ which, when combined with (2.86) and (2.89), implies $G_i = G_{ij} = \hat{G}_{ij} = G$, i.e. we fix all cubic couplings except for G_{0ss}. The two quartic $AAAA^0$ couplings coming from $\mathcal{L}^{F^3}_{AAA1}$ and $\mathcal{L}^{F^3}_{AAA2}$ in (G.4) are both equal to $G_{34}g$ by gauge invariance so BCJ relation does not impose an additional constraint on them.

Summary of results: Combining all of the constraints obtained from different cases and different processes we obtain the results summarized in table 2.2. These match the values of couplings of the 4d KK theory of 5d Yang-Mills with coupling $\sqrt{2\pi R}g$ plus $\frac{\sqrt{2\pi R}G}{\Lambda^2}tr(F^3) - \frac{9\pi RG^2}{8\Lambda^4}tr([F_{\mu\nu}, F_{\alpha\beta}][F^{\mu\nu}, F^{\alpha\beta}])$ operators, where R is the radius of the S^1, if $G_{0ss} = G$, but this is not fixed from 4pt processes. The reason for this is that $\mathcal{L}^{F^3}_{A\phi\phi}$ vertex in (G.4) is zero when A is on-shell. This means that the only 4pt processes that can constrain G_{0ss} are $\phi\phi \to \phi\phi$ and $AA \to \phi\phi$ but as we saw in the previous sections there is a freedom between G_{0ss} and quartic Λ^4 coefficients in both of these processes. Therefore, this coupling can only be fixed by 5pt scattering.

2.2.9 5-point amplitudes

In this section we consider different 5pt scattering amplitudes of external massive and massless fields and find the CK constraints on contact couplings imposed by the BCJ relations. Since we know from the previous section that all 3pt and 4pt couplings, except for G_{0ss}, are fixed to be that of KK theory, the only remaining freedom at 5pt is that from the as yet undetermined G_{0ss},

	coefficient	CK constrained value		coefficient	CK constrained value
$\mathcal{L}_{AAA}$	g_{ijk}	g	$\mathcal{L}^{F^3}_{AA\phi\phi1}$	G_{ijss}	$g(2G_{0ss}-G)-\frac{3\sqrt{2}m_i^2G^2}{\Lambda^2}$
$\mathcal{L}_{AA\phi}$	g'_{ijs}	$m_i g$	$\mathcal{L}^{F^3}_{AA\phi\phi2}$	$\hat{G}_{ijss}$	gG
$\mathcal{L}_{AAAA}$	g_{ijkl}	$g^2+\frac{18m_im_jm_km_l}{\Lambda^4}G^2$	$\mathcal{L}^{F^3}_{AAA\phi1}$	$\hat{G}_{ijks}$	gG
$\mathcal{L}_{AA\phi\phi}$	g_{ijss}	g^2	$\mathcal{L}^{F^3}_{AAA\phi2}$	G_{ijks}	gG
$\mathcal{L}^{F^4}_{AAF^0}$	g_i	$g_i=g-G\frac{3\sqrt{2}m_i^2}{\Lambda^2}$	$\mathcal{L}^{F^4}_{AAAA1}$	c_{ijkl}	G^2
$\mathcal{L}^{F^3}_{AAF^0}$	G_i	$G_i=G$	$\mathcal{L}^{F^4}_{AAAA2}$	C_{ijkl}	G^2
$\mathcal{L}^{F^3}_{AAA1}$	G_{ijk}	G	$\mathcal{L}^{F^4}_{AAA\phi1}$	c_{ijks}	G^2
$\mathcal{L}^{F^3}_{AAA2}$	$\hat{G}_{ijk}$	G	$\mathcal{L}^{F^4}_{AAA\phi2}$	C_{ijks}	G^2
$\mathcal{L}^{F^3}_{AA\phi}$	G'_{ijs}	m_iG	$\mathcal{L}^{F^4}_{AA\phi\phi1}$	c_{ijss}	$G_{0ss}G$
$\mathcal{L}^{F^3}_{A\phi\phi}$	G_{0ss}	not constrained	$\mathcal{L}^{F^4}_{AA\phi\phi2}$	$c^{(2)}_{ijss}$	G^2
$\mathcal{L}^{F^3}_{AAAA1}$	G_{ijkl}	gG	$\mathcal{L}^{F^4}_{AA\phi\phi3}$	$c^{(3)}_{ijss}$	$G_{0ss}G+\frac{\sqrt{2}\Lambda^2}{6m_i^2}g\,(G_{0ss}-G)$
$\mathcal{L}^{F^3}_{AAAA2}$	$\hat{G}_{ijkl}$	gG	$\mathcal{L}^{F^4}_{\phi\phi\phi\phi}$	$c_{\phi4}$	G^2_{0ss}

Table 2.2: Coefficients of the interactions constrained by the demands of colour-kinematics duality.

and the additional $1/\Lambda^2$ and $1/\Lambda^4$ suppressed quintic operators. We focus on the cases where we have multiple quintic contact terms whose coefficients are not fixed by gauge invariance. For instance, the A^0AAAA operators are clearly fixed by determining the $AAAA$ interactions by gauge invariance. As we will show below the remaining undetermined cubic coupling G_{0ss} is fixed by considering CK duality for $\phi\phi AAA$ scattering. In addition we shall find that the quintic contact terms are fixed in terms of a single coupling constant. This result is perhaps not so surprising since there are more BCJ relations to be satisfied, more independent contractions of polarizations and momenta, yet fewer overall free coefficients for quintic contact terms.

2.2.10 General case

We consider the 5-point scattering amplitude of five external spin-1 fields of mass m_I, $I = 1,2,3,4,5$ such that $m_1+m_2+m_3+m_4+m_5=0$ and there are no I,J such that $m_I+m_J=0$. Only massive spin-1 fields are exchanged in the 25 diagrams contributing to the amplitude (note that we have 15 kinematic factors as we can absorb the quartic contact terms into cubic diagrams). The two types of diagrams are shown in Fig. 2.6 and we obtained all 25 diagrams by relabelling the external states of these two. We then calculate the kinematic numerators and 5pt BCJ relations given in (2.70) - (2.73) and require them to be zero which gives us the constraints

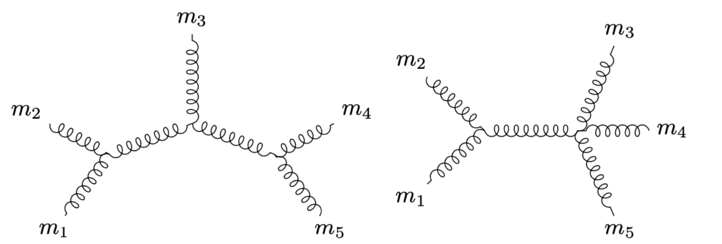

Figure 2.6: Five point diagrams for the general case.

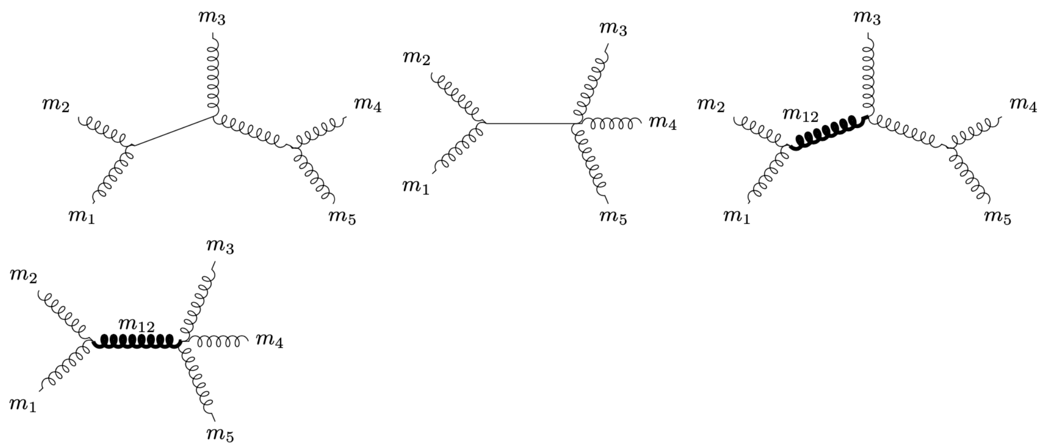

Figure 2.7: Five point diagrams for the general case when $m_1 + m_2 = 0$. The bold curly line represents a massless gluon, where $m_{12} = 0$ and the straight line a massless scalar.

on quintic contact term coefficients. We use Table 2.2 for the cubic and quartic couplings and find that all the three coefficients, G_{ijklm}, c_{ijklm} and C_{ijklm} are fixed by $(\epsilon_1 \cdot \epsilon_2)(\epsilon_4 \cdot \epsilon_5)(\epsilon_3 \cdot p_1)$ term in (2.70) to be of the same values as in KK theory:

$$G_{ijklm} = g^2 G, \;\; c_{ijklm} = C_{ijklm} = gG^2. \tag{2.97}$$

2.2.11 $m_1 + m_2 = 0$

Next we consider the 5-point scattering amplitude of five external spin-1 field such that $m_1 + m_2 = 0$. In this case, 6 out of the 30 diagrams have a massless particle exchange, a gluon and a scalar. These diagrams are shown in Fig. 2.7. In exactly the same way as in previous case, (2.70) alone forces the quintic couplings to be as in (2.97).

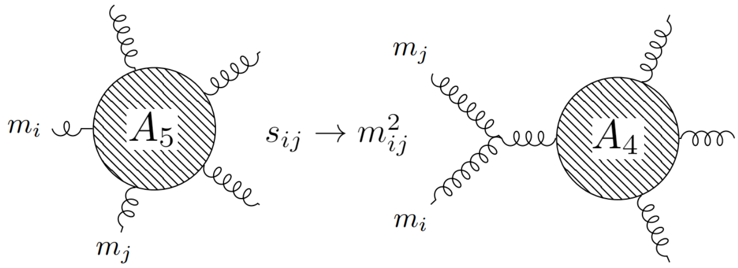

Figure 2.8: Factorization limits of the 5-point amplitude. By only imposing the 4-point BCJ relations on A_4 for all possible factorizations, the 5-point BCJ can be satisfied

2.2.12 $AAAAA$ without Λ^{-2n} operators

Let us temporarily take a step back, and consider the $AAAAA$ amplitude at leading order in the EFT expansion, Λ^0. We find that imposing the constraints from $AA \to AA$ scattering is enough for the BCJ relation to hold at 5-point, at least to this order in the EFT expansion. This can be seen by taking the factorization limits of the 5-point amplitude and imposing BCJ relations on the sub 4-point amplitudes (see Fig. 2.8). To show our procedure of checking the BCJ at 5-point, consider the amplitude for this process as:

$$\mathcal{A}_5 = g_{24}g_{15}g_{3(2+4)}\frac{(...)}{D_{15}D_{24}} + g_{24}g_{35}g_{1(2+4)}\frac{(...)}{D_{24}D_{35}} + g_{24}g_{135(2+4)}\frac{(...)}{D_{24}} + \text{other contributions} \quad (2.98)$$

where (...) represents the contraction of polarizations, momenta and the colour factors. From (2.86), we express the quartic coupling $g_{135(24)}$ as a product of two cubic diagrams:

$$\begin{aligned} g_{135(24)} &= g_{35}g_{1(3+5)}, \\ &= g_{15}g_{3(1+5)}, \\ &= g_{13}g_{5(1+3)}. \end{aligned} \quad (2.99)$$

We also express the 9 remaining contact couplings in terms of products of cubic ones. We combine all 30 equalities (3 equalities per contact coupling), simultaneously solve them and we get 24 constraints. Once these constraints imposed on the kinematic factors, the 5-point BCJ relation is satisfied. We conclude that for our theory, the BCJ relations at 5-point are satisfied and can be obtained from the coupling conditions derived at 4-point. This is consistent with

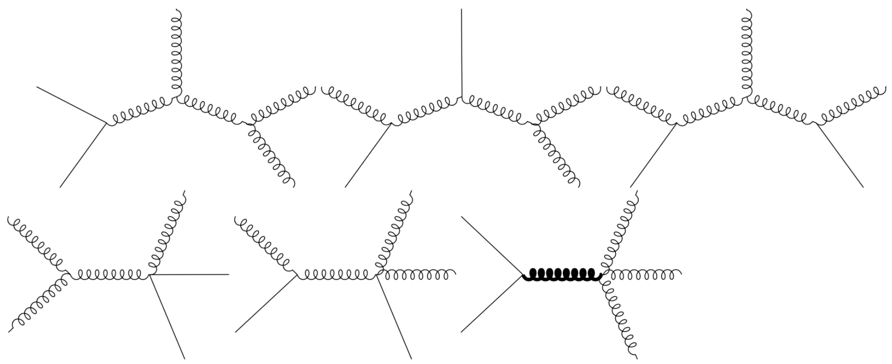

Figure 2.9: The six types of diagrams for $\phi\,\phi AAA$ process. The bold curly line represents a gluon and the straight line a scalar field.

the fact that BCJ relation can be proved recursively using BCFW recursion for theories in which amplitudes can be constructed that way [106]. Of course this will not be true if we add $1/\Lambda^{2n}$ corrections because the 5pt contact term couplings there are independent from 3 and 4pt couplings.

2.2.13 $\phi\,\phi AAA$

Now we consider two massless scalars and three massive vectors with the diagrams shown in Fig. 2.9. By inspection of the $(\epsilon_3 \cdot p_1)(\epsilon_5 \cdot p_2)(\epsilon_4 \cdot p_2)$, $(\epsilon_3 \cdot \epsilon_4)(\epsilon_5 \cdot p_2)$ and $(\epsilon_5 \cdot \epsilon_4)(\epsilon_3 \cdot p_2)$ terms in (2.70) and (2.73) we are able to finally fix $G_{0ss} = G$ (which fixes all 4pt couplings as in (2.2)) and the four quintic couplings to be of their KK values:

$$G_{ijkss} = \hat{G}_{ijkss} = g^2 G,\; c_{ijkss} = C_{ijkss} = \hat{C}^{(3)}_{ijkss} = \hat{C}^{(4)}_{ijkss} = \hat{C}^{(5)}_{ijkss} = gG^2. \tag{2.100}$$

2.2.14 $\phi AAAA$

Now we consider the first state to be a scalar, and the remaining states to be massive vectors. Let us first assume there are no pairs of vectors for which $m_I + m_J \neq 0$. Then the four types of diagrams are shown in Fig. 2.10. Inspecting the $(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$ and $(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot p_1)(\epsilon_5 \cdot p_2)$

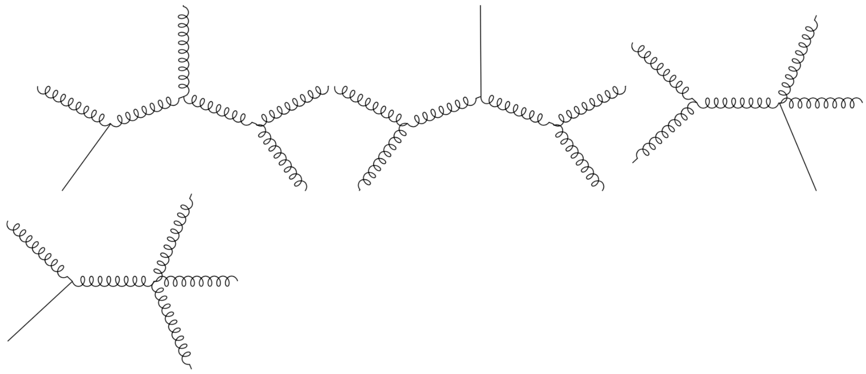

Figure 2.10: The four types of diagrams for $\phi AAAA$ process.The straight line is a scalar field.

terms in (2.70) and (2.72) fix the four quintic couplings to be of their KK values:

$$G_{ijkls} = \hat{G}_{ijkls} = g^2 G,\ c_{ijkls} = C_{ijkls} = \hat{C}_{ijkls} = gG^2. \tag{2.101}$$

From this we can consider two special cases where either a single or a double pair of massive vectors satisfy $m_I + m_J \neq 0$.

$m_I + m_J = 0$

Case 1: Consider again one scalar and four vectors but only one pair of masses add up to zero, for example, we consider $m_2 + m_3 = m_4 + m_5 = 0$. There are more diagrams now which are shown in Fig. 2.11. We use the constraint $G_{0ss} = G$ obtained from $\phi\,\phi AAA$. Now inspecting the $(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$ and $(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot p_2)(\epsilon_5 \cdot p_3)$ terms in (2.71) fixes the four quintic couplings to be of their KK values given in (2.101).

Case 2: We consider once again one scalar and four vectors but their masses now satisfy $m_2 = -m_3 = -m_4 = m_5$. The diagrams still look like the ones in Fig. 2.11 but now we have more channels with scalar or massless gluon exchanges. Just as before $(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$ and $(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot p_2)(\epsilon_5 \cdot p_3)$ terms in (2.71) fix the four quintic couplings to be of their KK values given in (2.101).

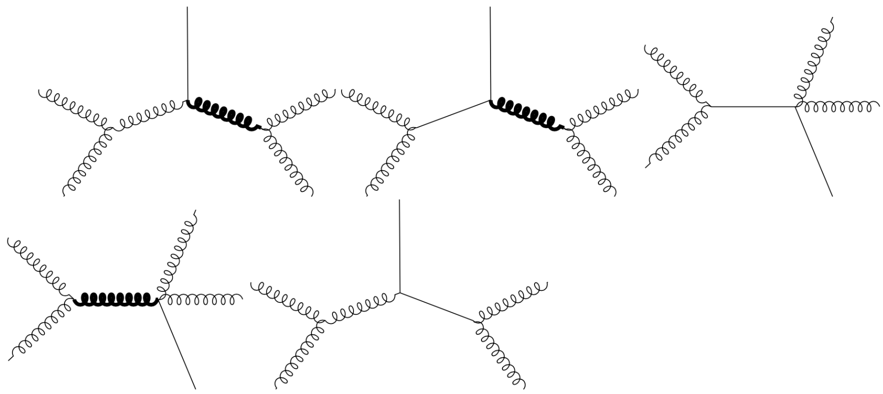

Figure 2.11: The five types of diagrams for $\phi AAAA$ process when $m_i + m_j = 0$. The bold curly line represents a gluon and the straight line a scalar field.

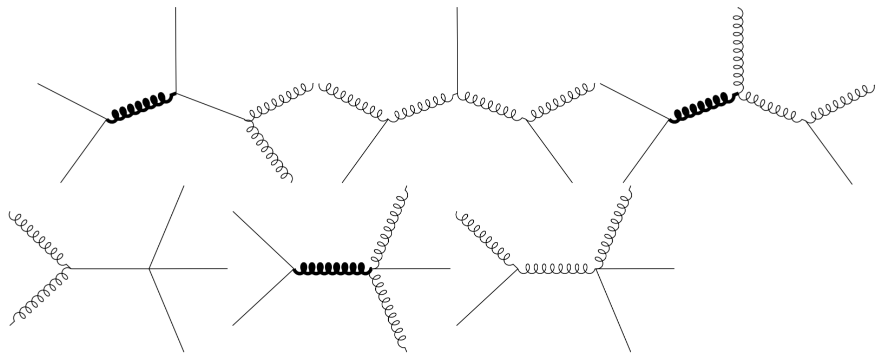

Figure 2.12: The six types of diagrams for $\phi\,\phi\,\phi AA$ process. The bold curly line represents a gluon and the straight line a scalar field.

2.2.15 $\phi\,\phi\,\phi AA$

Finally we consider three scalars and two vectors with the diagrams shown in Fig. 2.12. We now make use of the constraint $G_{0ss} = G$ obtained from $\phi\,\phi AAA$. In this case we only have the $1/\Lambda^4$ quintic operators and by inspecting $\epsilon_4 \cdot \epsilon_5$ and $(\epsilon_5 \cdot p_2)(\epsilon_4 \cdot p_1)$ terms in (2.70) we fix their coefficients to be:

$$c_{ijsss} = c^{(2)}_{ijsss} = c^{(3)}_{ijsss} = gG^2. \tag{2.102}$$

2.2.16 Non-symmetric couplings

In the previous sections we assumed the couplings g_{ijk}, g_{ijkl} and G_{ijkl} to be fully symmetric in all of the indices. This is of course what we obtain from KK reduction, however it is obviously not the most general possibility. In this section we will briefly consider more general couplings that are not symmetric. Now the cubic and quartic AAA and $AAAA$ vertices (without $1/\Lambda^{2n}$ corrections) are not of the same form as Yang-Mills vertices. For example, the Λ^0 terms contributing to three point vertex $A_1A_2A_3$ are:

$$\frac{1}{\sqrt{2}} f^{abc} \left(g_{123}\partial_{[\mu}A^{1a}_{\nu]}A^{2b\mu}A^{3c\nu} + g_{231}\partial_{[\mu}A^{2b}_{\nu]}A^{3c\mu}A^{1a\nu} + g_{312}\partial_{[\mu}A^{3c}_{\nu]}A^{1a\mu}A^{2b\nu} \right), \tag{2.103}$$

which gives a different Feynman rule than 3pt Yang-Mills vertex. For example, the on shell vertex coming from this term is

$$\begin{aligned} A_3(1^a, 2^b, 3^c) \propto f_{abc}\Big(&- (g_{123} + g_{231})\epsilon_1 \cdot \epsilon_2 \; \epsilon_3 \cdot p_1 + (g_{123} + g_{312})\epsilon_1 \cdot \epsilon_3 \; \epsilon_2 \cdot p_1 \\ &- (g_{312} + g_{231})\epsilon_1 \cdot p_2 \; \epsilon_2 \cdot \epsilon_3 \Big), \end{aligned} \tag{2.104}$$

which is of different structure than Yang-Mills 3pt vertex,

$$A_3(1^a, 2^b, 3^c) \propto f_{abc} \left(-\epsilon_1 \cdot \epsilon_2 \; \epsilon_3 \cdot p_1 + \epsilon_1 \cdot \epsilon_3 \; \epsilon_2 \cdot p_1 - \epsilon_1 \cdot p_2 \; \epsilon_2 \cdot \epsilon_3 \right). \tag{2.105}$$

Similarly we may consider non-symmetric $A_1A_2A_3A_4$ couplings:

$$\begin{aligned} \mathcal{L}_{AAAA} = {} & f^{abe} f^{cde} (g_{1234} A^{1a} \cdot A^{3c} A^{2b} \cdot A^{4d} + g_{1243} A^{1a} \cdot A^{4d} A^{2b} \cdot A^{3c}) \\ &+ f^{ace} f^{bde} (g_{1324} A^{1a} \cdot A^{2b} A^{3c} \cdot A^{4d} + g_{1342} A^{1a} \cdot A^{4d} A^{3c} \cdot A^{2b}) \\ &+ f^{ade} f^{bce} (g_{1423} A^{1a} \cdot A^{3c} A^{2b} \cdot A^{4d} + g_{1432} A^{1a} \cdot A^{2b} A^{3c} \cdot A^{4d}), \end{aligned} \tag{2.106}$$

$$\begin{aligned} \mathcal{L}^{F^3}_{AAAA} = f^{abe} f^{cde} \Big(& G_{1234} D_{[\mu} A^a_{1\nu]} D^{[\nu} A^{b\rho]}_2 A^c_{3\rho} A^{\mu d}_4 + G_{1324} D_{[\mu} A^a_{1\nu]} D^{[\nu} A^{b\rho]}_3 A^c_{2\rho} A^{\mu d}_4 \\ &+ G_{1423} D_{[\mu} A^a_{1\nu]} D^{[\nu} A^{b\rho]}_4 A^c_{2\rho} A^{\mu d}_3 + G_{2413} D_{[\mu} A^a_{2\nu]} D^{[\nu} A^{b\rho]}_4 A^c_{1\rho} A^{\mu d}_3 \\ &+ G_{2314} D_{[\mu} A^a_{2\nu]} D^{[\nu} 3^{b\rho]}_4 A^c_{1\rho} A^{\mu d}_4 + G_{3412} D_{[\mu} A^a_{3\nu]} D^{[\nu} 3^{b\rho]}_4 A^c_{1\rho} A^{\mu d}_2 \Big), \end{aligned} \tag{2.107}$$

where any other operators obtain by permuting $1, 2, 3, 4$ labels of one of these operators does not give a new operator. Note that if we allow for non-symmetric couplings we do not need to write $\mathcal{L}^{F^3}_{AAAA2}$ term from (G.4) because these operators are already included in (2.106). For simplicity we only considered $A_1A_2 \to A_3A_4$ scattering amplitude with these non-symmetric couplings up to $1/\Lambda^2$ order. We found that by imposing the BCJ relation we do not get a new solution, *i.e.* all the couplings must be symmetric as before.

2.2.17 Combining all results

We see that imposing BCJ relations for all 4 and 5 point amplitudes and combining all of the constrains fixes all 3, 4 and 5 point couplings to be of the same value as in the Kaluza-Klein theory obtained from compactification of the 5d Lagrangian $\left(\frac{-1}{4}\mathrm{tr}(F^2) + \frac{G_{5d}}{\Lambda^2}\mathrm{tr}(F^3) - \frac{9G_{5d}^2}{16\Lambda^4}\mathrm{tr}([F,F]^2)\right)$ on an S^1. Our results may be interpreted as the statement that Kaluza-Klein theory may be derived from the requirement that a specific spectrum of states which automatically satisfy the spectral conditions, admit a local double copy. While the fact that Kaluza-Klein theory is a consistent solution is itself not surprising, one might of thought that allowing for higher derivative/irrelevant operators in the EFT expansion would give us more freedom, and could be used to relax some of the constraints of the BCJ relations. To the order that we have calculated this situation does not arise.

It remains the case that there could still be some freedom in higher point amplitudes or at higher orders in the EFT expansion. At a given order in the EFT expansion, there are in general many other operators in the adjoint representation that could be included, in particular non-symmetric couplings of the type considered in section 2.2.16, as well as operators which do not appear in the Kaluza-Klein compactification of 5d Yang-Mills plus its higher derivative terms. Another possibility is to add multiple fields in different representations in 5d and to consider their compactification. We leave it to future works to explore or otherwise exclude these possibilities.

Given our results, it would be extremely helpful to have other nontrivial solutions of the spectral conditions. However, ultimately it may be necessary to relax the naive rules of the double copy to account for interacting theories with massive states, or similarly to account for higher order operators in an EFT expansion. By trying to mirror the massless double copy procedure as closely as possible, we have also forced ourselves into working with massive theories that are related to massless ones (in this case massless ones in higher dimensions), and to a large extent this explains the limitation of our results.

2.3 Avoiding Spurious Poles in 3D

Having seen that it is hard to find a massive theory compatible with colour kinematics duality in 4D that is not a Kaluza-Klein theory we now move to 3D. Upon closer inspection the polynomial appearing in the denominator of 5pt double copy amplitude given in (2.64) has a special structure, it can be expressed as:

$$P(s_{kl}, m) = 16 \det(p_i \cdot p_j) \ , \quad i, j < 5 \ , \tag{2.108}$$

where $\det(p_i \cdot p_j)$ is the Gram determinant of the momenta of 4 out of the 5 external states.

Note that $P(s_{kl}, m) \neq 0$, only if the spacetime dimension is larger than 3 because we cannot have 4 independent vectors in less than four dimensions. Therefore it is zero in our 3D case and A has a null eigenvector, which corresponds to a BCJ relation. First, we need to check if U is orthogonal to that null vector, that is, if the BCJ relation is true. If that is the case, we can invert A in the subspace orthogonal to the null vector. Otherwise, we cannot satisfy colour-kinematics duality. In 3D, the null vector, e_0, turns out to have a very simple form:

$$e_0 = \begin{bmatrix} \epsilon(1,2,3) \\ -\epsilon(1,2,4) - \epsilon(1,3,4) + \epsilon(2,3,4) \\ \epsilon(1,2,3) + \epsilon(1,2,4) \\ -\epsilon(1,3,4) \\ \epsilon(1,2,4) - \epsilon(2,3,4) \\ \epsilon(1,2,3) + \epsilon(1,2,4) - \epsilon(2,3,4) \\ -\epsilon(1,2,4) - \epsilon(1,3,4) \\ \epsilon(2,3,4) \\ \epsilon(1,2,3) + \epsilon(1,2,4) + \epsilon(1,3,4) \end{bmatrix} . \tag{2.109}$$

In the result above, we have expressed the Mandelstam variables in terms of the products of the 3D Levi-Civita tensor and momenta, $\epsilon(i,j,k) = \epsilon_{\mu\nu\sigma} p_i^\mu p_j^\nu p_k^\sigma$, as explained in the Appendix H.2. As mentioned before the vector U must satisfy

$$U \cdot e_0 = 0 \ , \tag{2.110}$$

in order to be able to satisfy the colour-kinematics duality.

For generic kinematics, all 9 components of the null vector are non-zero so we can use the freedom of adding the null eigenvector to v in (2.9) to eliminate v's ninth component[9]. In other words, we can restrict ourselves to an 8-dimensional subspace to invert the 8×8 submatrix of A. This 8×8 matrix, however, still has a complicated polynomial in its determinant:

$$\det A_{8x8} = -2m^6 \left(\prod_{i<j} D_{ij} \right) P_1(s_{kl}, m) \ , \tag{2.111}$$

[9] We pick the ninth component as a specific example, but we can alternatively choose any other component since the final answer does not depend on which component we choose.

with $P_1(s_{kl}, m)$ given by

$$P_1(s_{kl}, m) = 4(\epsilon(1,2,3) + \epsilon(1,2,4) + \epsilon(1,3,4))^2 \ , \tag{2.112}$$

where as before we have used the special 3D kinematics from Appendix H.2 to simplify this expression. At first sight, it appears like we are in trouble again, this polynomial seems to give spurious poles in the double copy of the 5-point amplitude for a generic 3D theory. However we will show that the amplitude does not have spurious poles if (2.110) is satisfied.

One way of seeing the cancellation of the spurious poles is by considering the pseudo-inverse of the matrix A:

$$(A + \varepsilon I)^{-1} = \frac{1}{\det(A + \varepsilon I)} C \ , \tag{2.113}$$

where C is the cofactor matrix of $(A + \varepsilon I)$. Explicit calculation of these quantities gives the following:

$$\det(A + \varepsilon I) = -\varepsilon \, 8m^6 \left(\prod_{i<j} D_{ij} \right) e_0 \cdot e_0 + O(\varepsilon^2) \ , \tag{2.114}$$

and

$$C_{ij} = -8m^6 \left(\prod_{i<j} D_{ij} \right) (e_0)_i (e_0)_j + \varepsilon \Big(w_i (e_0)_j + w_j (e_0)_i + K_{ij} e_0 \cdot e_0 \Big) + O(\varepsilon^2) \ , \tag{2.115}$$

with w_i a vector and K_{ij} a matrix. We can see that in the limit $\varepsilon \to 0$, $U^T(A + \varepsilon I)^{-1} U = \frac{U^T C U}{\det(A+\varepsilon I)}$ is finite if U is orthogonal to e_0, that is, if the BCJ relation in (2.110) is satisfied. Moreover the factor of $e_0 \cdot e_0$ in the denominator cancels out since only K_{ij} contributes to the double copy amplitude. This contribution can be expressed as:

$$\begin{aligned} \lim_{\varepsilon \to 0} U^T (A + \varepsilon I)^{-1} U &= - \frac{U^T K U}{8m^6 \left(\prod_{i<j} D_{ij} \right)} \\ &= \sum_{i=1}^{9} \left(\frac{u_i^2}{m^2 D_i} \right) + \frac{(u_1 - u_2 + u_3 - u_4 + u_5 + u_6 - u_7 - u_8 + u_9)^2}{m^2 D_{35}} \\ &\quad + \frac{1}{8m^2 \epsilon(1,3,4) \epsilon(2,3,4)} \sum_{i=1}^{6} q_i U \cdot e_i \end{aligned} \tag{2.116}$$

where q_i is a linear combination of the components of U, e_i are 9-component vectors linear in U components and polynomial in Mandelstam variables. Above, we used shorthand notation for the propagators $D_i = \{D_{34}, D_{12}, D_{45}, D_{23}, D_{15}, D_{14}, D_{25}, D_{13}, D_{24}\}$.

Let us first look at the terms in the second line of (2.116). They contain only physical poles and, as we will see in the following, are consistent with factorization of the scattering amplitude. When D_i goes on shell, (2.116) goes to $u_i^2/(m^2 D_i)$ where u_i is the violation of the 4pt BCJ relation associated to this factorization channel. The second term in the second line of (2.116) corresponds to the violation of the 10th Jacobi identity which is not independent from the 9 Jacobi identities encoded in $U = Mn$. Nevertheless, it is needed to have the correct factorization in the $D_{35} \to 0$ limit. As an explicit example, we now illustrate the factorization of the 5pt double copy amplitude, M_5, when $D_{12} \to 0$. In this limit the $n^T D^{-1} n$ term in the 5-point double copy (2.12) goes to

$$\frac{1}{D_{12}} \left(\frac{n_1^2}{D_{45}} + \frac{n_3^2}{D_{34}} + \frac{n_{12}^2}{D_{35}} \right), \tag{2.117}$$

which is easy to see since D matrix is diagonal and given in (2.62). Meanwhile, from (2.116) we find that the $U^T A^{-1} U$ term goes to

$$\frac{1}{D_{12}} \left(\frac{u_2^2}{m^2} \right) = \frac{1}{D_{12}} \left(\frac{(-n_1 + n_3 + n_{12})^2}{m^2} \right), \tag{2.118}$$

since U is a linear combination of the kinematic factors, $U = Mn$ (in particular $u_2 = -n_1 + n_3 + n_{12}$). Using the fact that the 5pt gauge theory amplitude, A_5, factorises into the 3-point amplitude $A_3(12I)$ and the 4-point amplitude $A_4(I345)$, where I is an intermediate state., we can write the following expression for the kinematic factors in this limit:

$$n_1 = -n_s A_3(12I), \quad n_3 = n_t A_3(12I), \quad n_{12} = n_u A_3(12I), \tag{2.119}$$

where n_s, n_t and n_u are the kinematic factors of the 4pt amplitude $A_4(I345)$ (if we identify $s_{45} = s$, $s_{34} = t$, $s_{35} = u$ and $c_1 = -f^{a_1 a_2 b} c_s$, $c_3 = f^{a_1 a_2 b} c_t$, $c_{12} = f^{a_1 a_2 b} c_u$). Substituting (2.117), (2.118) and (2.119) into (2.12) we get the following expression for the 5pt double copy amplitude in $D_{12} \to 0$ limit:

$$M_5 \to \frac{A_3(12I)^2}{D_{12}} \left(\frac{n_s^2}{s-m^2} + \frac{n_t^2}{t-m^2} + \frac{n_u^2}{u-m^2} - \frac{(n_s+n_t+n_u)^2}{m^2} \right) = \frac{M_3(12I)M_4(I345)}{D_{12}}, \tag{2.120}$$

where we used the 3pt double copy relation $M_3(12I) = A_3(12I)^2$. This shows that the 5pt double copy amplitude correctly factorizes into the 3-point and 4-point double copy amplitudes. This argument can be repeated for all other factorization channels.

Now we analyze the third line of (2.116) which appears to have unphysical poles. However, we will show that the residues of these poles are zero if the BCJ relation in (2.110) is satisfied. To show this, we note that we can write the third line of (2.116) with different expressions for $\{q_i, e_i\}$ which correspond to different ways of splitting the answer into q_i and e_i. In Appendix H.1, we give two explicit expression for $\{q_i, e_i\}$. We have checked numerically that one of these expressions has the property that, for kinematics when $\epsilon(1,3,4) = 0$, all $e_i || e_0$. Similarly, for the other expression, all $e_i || e_0$ when $\epsilon(2,3,4) = 0$. Therefore, if $U \cdot e_0 = 0$, then $U \cdot e_i = 0$ on the residues of these unphysical poles; so these residues are zero. We conclude that the condition in (2.110) is sufficient for the double copy to give a physical amplitude at 5-points in 3D with all fields with the same mass and in the adjoint representation. Note that this is quite different from the usual requirement of having 4 BCJ relations like in massless Yang-Mills. Here one relation is enough. Now, an interesting question arises: which theories in three spacetime dimensions satisfy (2.110)? To tackle this question, in the following sections we will introduce the topologically massive theories. First, we will analyze the 3, 4, and 5-point amplitudes of TMYM and how to write them in terms of kinematic numerators that satisfy the colour-kinematics duality. Afterwards, we will look at the TMG case and show how this corresponds to the double copy of TMYM.

2.4 Topologically Massive Yang-Mills

The explicit expression of TMYM action in our conventions is

$$S_{TMYM} = \int d^3x \Bigg(-\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} + \epsilon_{\mu\nu\rho} \frac{m}{12} \left(6A^{a\mu} \partial^\nu A^\rho_a + g\sqrt{2} f_{abc} A^{a\mu} A^{b\nu} A^{c\rho} \right) \Bigg) \ , \qquad (2.121)$$

where m is the mass of the gauge field and g the coupling strength. The equations of motion can be easily obtain from (1.33) and read

$$D_\mu F^{\mu v} + \frac{m}{2} \epsilon^{v\alpha\beta} F_{\alpha\beta} = 0 \ , \qquad (2.122)$$

where $D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}} A_\mu$, $F_{\mu\nu} = F^a_{\mu\nu} T^a$, with $F^a_{\mu\nu}$ the Yang-Mills field strength and T^a the generators of the gauge group. In the following, we choose to work in Lorenz gauge where $\partial_\mu A^\mu = 0$, and $A^\mu = A^{\mu\ a} T^a$. It is easy to see that plane waves of the form $A^\mu = \varepsilon^\mu e^{ip\cdot x}$ are solutions to the linearised equations of motion as long as the polarisation vectors, ε, satisfy

$$\varepsilon^a_\mu + \frac{i}{m} \epsilon_{\mu\nu\rho} p^\nu \varepsilon^a_\rho = 0 \ . \qquad (2.123)$$

This equation constrains the allowed polarisations of the topologically massive gauge field. Note that we will be denoting the polarisations with ε, to distinguish them from the Levi-Civita symbol denoted with ϵ. Another important ingredient for our scattering amplitudes calculation is the propagator of the gauge field. In an arbitrary gauge the colour stripped propagator is

$$D_{\mu\nu}[\alpha] = \frac{-i}{p^2 + m^2} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{im}{p^2} \epsilon_{\mu\nu\sigma} p^\sigma \right) - \frac{i\alpha}{p^4} p_\mu p_\nu \ . \qquad (2.124)$$

We will work in Landau gauge where $\alpha = 0$, hence,

$$D_{\mu\nu} = \frac{-i}{p^2 + m^2} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{im}{p^2} \epsilon_{\mu\nu\sigma} p^\sigma \right) \ . \qquad (2.125)$$

When we look at the gravitational case, we will see how the graviton propagator can arise as the "square" of this one. Given their simplicity, we also present here the three-point and four-point off-shell vertices:

$$V_3^{\mu\nu\rho} = \frac{ig}{\sqrt{2}} f^{a_1a_2a_3} \left(m\epsilon_{\mu\nu\rho} + i\eta^{\mu\nu}(p_1^\rho - p_2^\rho) + i\eta^{\mu\rho}(p_3^\nu - p_1^\nu) + i\eta^{\nu\rho}(p_2^\mu - p_3^\mu) \right) , \tag{2.126}$$

$$V_4^{\mu\nu\rho\sigma} = \frac{ig^2}{2} \left((c_s - c_t)\eta^{\mu\sigma}\eta^{\nu\rho} + (c_u - c_s)\eta^{\mu\rho}\eta^{\nu\sigma} + (c_t - c_u)\eta^{\mu\nu}\eta^{\rho\sigma} \right) , \tag{2.127}$$

where

$$c_s = f^{a_1a_2b} f^{ba_3a_4} , \quad c_u = f^{a_2a_3b} f^{ba_1a_4} , \quad c_t = f^{a_3a_1b} f^{ba_2a_4} . \tag{2.128}$$

In the following we construct the three, four, and five-point amplitudes of TMYM and show how to shift the kinematic numerator so that they satisfy the colour-kinematics duality.

2.4.1 TMYM Scattering Amplitudes

3-point Amplitude The three-point on-shell amplitude is:

$$A_3 = g \left(\sqrt{2}(ee_{13}pe_{12} - ee_{23}pe_{21} + ee_{12}pe_{23}) + \frac{im}{\sqrt{2}} \epsilon^{\mu\nu\rho} \varepsilon_{1\mu}\varepsilon_{2\nu}\varepsilon_{3\rho} \right) , \tag{2.129}$$

where we have defined $ee_{ij} \equiv \varepsilon_{i\mu}\varepsilon_j^\mu$ and $pe_{ij} \equiv p_{i\mu}\varepsilon_j^\mu$. Using the equation of motion for the polarisation vectors, $\varepsilon_1, \varepsilon_2, \varepsilon_3$, as given in (2.123), one can express the first term as follows :

$$(ee_{13}pe_{12} - ee_{23}pe_{21} + ee_{12}pe_{23}) = -\frac{3im}{2} \epsilon^{\mu\nu\rho} \varepsilon_{1\mu}\varepsilon_{2\nu}\varepsilon_{3\rho} . \tag{2.130}$$

Hence, the 3-point amplitude of topological massive Yang-Mills can be written as:

$$A_3 = -ig\sqrt{2} m \epsilon^{\mu\nu\rho} \varepsilon_{1\mu}\varepsilon_{2\nu}\varepsilon_{3\rho} . \tag{2.131}$$

4-point Amplitude The 4-point TMYM amplitude can be expressed as follows:

$$A_4 = g^2 \left(\frac{c_s n_s}{s - m^2} + \frac{c_t n_t}{t - m^2} + \frac{c_u n_u}{u - m^2} \right) , \tag{2.132}$$

where the colour factor are given in (2.128) and the kinematic factors are computed using Feynman rules. It is possible to simplify these factors by using the reconstruction methods explained in [111, 112]. Doing this we find the simpler expressions

$$\begin{aligned}
n_s &= \frac{-i}{32s}\Bigg(ee_{12}ee_{34}(16m^4 - 12s^2 + 36m^2t - 35st - 11t^2) \\
&\quad - ee_{13}ee_{24}(16m^4 + 72m^2s + 23s^2 + 11(4m^2+s)t) \\
&\quad + ee_{14}ee_{23}(-160m^4 + s(12s+11t) + 4m^2(40s+11t))\Bigg) , \\
n_t &= \frac{i}{8t}\Bigg(- ee_{12}ee_{34}(4m^4 + 29m^2t + 3t^2) - ee_{13}ee_{24}(4m^2 - 2s - t)(m^2 + 3t) \\
&\quad + ee_{14}ee_{23}(4m^4 + 29m^2t + 3t^2)\Bigg) , \\
n_u &= \frac{-i}{32u}\Bigg(ee_{12}ee_{34}(672m^4 - 424m^2t + (s+t)(12s+65t)) \\
&\quad - ee_{13}ee_{24}(672m^4 - 424m^2t - (41s - 12t)(s+t)) \\
&\quad + ee_{14}ee_{23}(-848m^4 + 12s^2 + 53st - 12t^2 + 8m^2(20s+33t))\Bigg) .
\end{aligned} \tag{2.133}$$

These kinematics factors do not satisfy automatically the colour-kinematics duality, i.e. $n_s + n_t + n_u \neq 0$. Since we are interested in finding the double copy of TMYM, we need to shift the numerators such that the colour-kinematics duality is satisfied. The new kinematic factors read

$$\hat{n}_s = n_s - \frac{(n_s+n_t+n_u)(s-m^2)}{(m^2)}, \quad \hat{n}_t = n_t - \frac{(n_s+n_t+n_u)(t-m^2)}{(m^2)}, \quad \hat{n}_u = n_u - \frac{(n_s+n_t+n_u)(u-m^2)}{(m^2)} , \tag{2.134}$$

which indeed satisfy the CK duality

$$\hat{n}_s + \hat{n}_t + \hat{n}_u = \left(\frac{n_s+n_t+n_u}{m^2}\right)\left(4m^2 - (s+t+u)\right) = 0 , \tag{2.135}$$

given that $s + t + u = 4m^2$.

5-point Amplitude The TMYM 5-point amplitude can be written as in (2.59). Just as in the previous case, the kinematic factors calculated directly from the Feynman rules do not satisfy the CK algebra. Their explicit expressions are complicated so we do not show them here, but they can be found in the ancillary Mathematica file, FivePointKinematicFactors.m, included in the submission of [12]. The shifted numerators that satisfy the CK duality can be found as explained in Section 2.3. The 9 component vector U was constructed using (2.8), and we used the 8×8 submatrix of A, A_{8x8} for constructing the shifted kinematic factors. In the construction of A_{8x8}, we can eliminate any arbitrary nth column and nth row since the final result does not depend on this.

2.5 Topologically Massive Gravity and the Double Copy

The action for TMG is

$$S_{TMG} = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \left(-R - \frac{1}{2m} \epsilon^{\mu\nu\rho} \left(\Gamma^{\alpha}_{\mu\sigma} \partial_\nu \Gamma^{\sigma}_{\alpha\rho} + \frac{2}{3} \Gamma^{\alpha}_{\mu\sigma} \Gamma^{\sigma}_{\nu\beta} \Gamma^{\beta}_{\rho\alpha} \right) \right) . \tag{2.136}$$

Note that the sign of the Einstein-Hilbert term is the opposite to the conventional one; this is required so that the physical spin-2 mode is not ghostly as mentioned in the introduction. The equations of motion are given by

$$G^{\mu\nu} + \frac{1}{m} C^{\mu\nu} = 0 , \tag{2.137}$$

where $G^{\nu\mu} \equiv R^{\nu\mu} - \frac{1}{2} R g^{\nu\mu}$ is the Einstein tensor and $C^{\mu\nu} \equiv \varepsilon^{\mu\alpha\beta} \nabla_\alpha \left(R^\nu_\beta - \frac{1}{4} g^\nu_\beta R \right)$ the Cotton tensor which is the 3D analogue of the Weyl tensor.

We now proceed to analyze some elements that are required for the scattering amplitude computations, and how these elements themselves can be constructed as a double copy of the analogue Yang-Mills object. First, we obtain the linearised equations of motion by expanding around flat space as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. We will work in de Donder gauge where $\partial_\mu h^{\mu\nu} - \frac{1}{2} \partial^\nu h = 0$.

As in the Yang-Mills case, the plane wave solution $h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ip\cdot x}$ is a solution of the linearised equations of motion when

$$\left(\delta^\mu_\alpha \delta^\nu_\beta + \frac{i}{2m}(\epsilon^\nu_{\ \rho\sigma} p^\rho \delta^\sigma_\alpha \delta^\mu_\beta + \epsilon^\mu_{\ \rho\sigma} p^\rho \delta^\sigma_\alpha \delta^\nu_\beta)\right)\varepsilon^{\alpha\beta} = 0 \ . \tag{2.138}$$

This again restricts the allowed polarisations of the massive graviton. It is interesting to notice that we can already see a double copy relation at this level. The on-shell polarisation tensors of TMG can be written as the square of the on-shell polarisation vector of TMYM

$$\varepsilon_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu \ . \tag{2.139}$$

If ε_μ satisfies (2.123) then the polarisation tensor defined above will satisfy (2.138). When we write our scattering amplitudes below, we will be using this relation and writing them in terms of the polarisation vectors ε_μ.

It is also instructive to look at the propagator of topologically massive gravitons. In an arbitrary gauge this reads

$$\mathcal{D}^{\mu\nu\rho\sigma} = D^{\mu\nu\rho\sigma} + \alpha\frac{i}{4q^2}\left(- 4\eta^{\mu(\sigma}\eta^{\rho)\nu} + \left(2\eta^{\nu(\sigma} P_1^{\rho)\mu} + \nu\sigma \leftrightarrow \mu\rho\right)\right) , \tag{2.140}$$

with

$$\begin{aligned} D^{\mu\nu\rho\sigma} = \frac{i}{p^2+m^2}\Bigg(&- 3\eta^{\mu\nu}\eta^{\rho\sigma} + \left(\eta^{\rho\sigma}P_1^{\mu\nu} + \mu\nu \leftrightarrow \rho\sigma\right) + 2P_1^{\mu(\sigma}P_1^{\rho)\nu} - \left(P_2^{\mu(\rho}P_1^{\sigma)\nu} + P_2^{\nu(\rho}P_1^{\mu)\sigma}\right) \\ &- \frac{m^2}{p^2}\left(2\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\nu(\sigma}P_1^{\rho)\mu} + \frac{2}{p^2}\eta^{\mu(\sigma}p^{\rho)}p^\nu\right) - \frac{1}{p^4}p^\mu p^\nu p^\rho p^\sigma\Bigg) , \end{aligned} \tag{2.141}$$

where $X^{a(b}X^{c)d} = \frac{1}{2}(X^{ab}X^{cd} + X^{ac}X^{bd})$ and

$$P_1^{ab} = \eta^{ab} - \frac{p^a p^b}{p^2}, \quad P_2^{ab} = P_1^{ab} - \frac{im}{p^2}\epsilon^{abc}p_c \ . \tag{2.142}$$

As before, we will work in de Donder gauge where $\alpha = 0$; hence $\mathcal{D}^{abcd} = D^{abcd}$. A double copy between the propagators of TMYM and TMG has been proposed in [113]. It was shown that they can be related as

$$D_{\mu\nu\rho\sigma} = \left(p^2 + m^2\right) D_{\rho(\mu} D_{\nu)\sigma} \ , \tag{2.143}$$

which matches the TMG case if we ignore terms that vanish when they are contracted to conserved currents. For the gravitational case, we do not show the explicit expression of the off-shell vertices since this is quite involved and gives no insights to our discussion. We will now go ahead and compute the three, four, and five -point amplitudes of TMG and show how they correspond to the double copy of TMYM.

2.5.1 TMG Scattering Amplitudes

3-point Amplitude The 3-point amplitude of TMG can be simplified by using the 3-point relations in Appendix H.2, which arise from the polarisation vector equations of motion. After using these relations, the 3-point TMG amplitude can be written as follows:

$$M_3 = 2iee_{12}ee_{13}ee_{23}m^2\kappa \ , \tag{2.144}$$

(where $ee_{ij} = \varepsilon_{i\mu}\varepsilon_j^{\mu}$ and $pe_{ij} = p_{i\mu}\varepsilon_j^{\mu}$ as before) which is equivalent to:

$$M_3 = -i\left(\epsilon^{\mu\nu\rho}\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{3\rho}\right)^2 m^2\kappa \ . \tag{2.145}$$

From this, we can see that the three-point double copy relation

$$M_3 = i\frac{\kappa}{2}A_3A_3 \ , \tag{2.146}$$

is satisfied with A_3 given by (2.131) and M_3 by (2.145). We conclude that at 3-points, the double copy of TMYM is TMG.

4-point Amplitude We compute the 4-point amplitude directly from the Feynman rules and, given its length, only show our result in the Appendix H.3. Here, we focus on understanding if it corresponds to the TMYM double copy. At 4-points, the double copy relation is the following:

$$M_4 = i\left(\frac{\kappa}{2}\right)^2\left(\frac{\hat{n}_s^2}{s-m^2}+\frac{\hat{n}_t^2}{t-m^2}+\frac{\hat{n}_u^2}{u-m^2}\right)\,, \tag{2.147}$$

where $\hat{n}$ satisfies the colour-kinematics duality. To find the TMYM double copy, we plug-in the kinematic numerators from (2.134). The analytic expressions obtained for the TMYM double copy are highly involved and complicated to simplify. In this case, we can simplify them by using the Breit coordinate system. It is well known that it is advantageous to use this coordinate system to investigate the analytic properties of the amplitude (see e.g.[114]). For elastic scattering processes, the momenta in the Breit coordinate system are defined as

$$\begin{aligned} p_1^\mu &= (\sqrt{\vec{p}^2+m^2},\vec{p})\,, \qquad p_2^\mu = (E,-\vec{p}+\lambda\vec{e})\,,\\ p_3^\mu &= (\sqrt{\vec{p}^2+m^2},-\vec{p})\,, \quad p_4^\mu = (E,\vec{p}+\lambda\vec{e})\,, \end{aligned} \tag{2.148}$$

where $\vec{e}\cdot\vec{p} = 0$, $|\vec{e}|= 1$, and the arrow denotes 2-dimensional spatial vectors. Note that all momenta are incoming. We choose the following directions, $\vec{p} = (p, 0)$ and $\vec{e} = (0, 1)$. In terms of Mandelstam variables we have,

$$t = -4p^2, \quad E = \sqrt{p^2+m^2+\lambda^2} = \frac{s-2m^2+t/2}{\sqrt{4m^2-t}}, \tag{2.149}$$

and the external polarisations are obtained from (H.13). The explicit expressions for the shifted numerators in this coordinate system are given in (H.11). Once we simplify the double copy of TMYM using these relations, we find that it indeed corresponds to the 4-point amplitude in (H.10). Therefore, we conclude that at 4-point TMG is the double copy of TMYM.

Another way to check the double copy of TMYM that does not require a specific coordinate system is to use the simplified kinematic numerators in (2.133). However, it is still is complicated to see that TMG is the double copy of TMYM. This can only be seen once we relate the product of polarisations and Mandelstam variables as shown in (H.5). These relations are satisfied on-shell and were obtained by using random on-shell momenta and polarisation vectors. Our numerical method used to obtain these random kinematics is explained in Appendix H.4. Therefore, we can conclude that once (H.5) is imposed, the 4-pt TMG is the double copy of TMYM.

5-point Amplitude At 5-points, we have shown that the double copy in 3D has no spurious poles as long as (2.110) is satisfied. We have verified numerically that the TMYM vector U satisfies this equation for multiple random on-shell kinematics. Thus, there are no spurious poles in the double copy of the TMYM 5-point amplitude. We obtained the 5-point TMYM double copy using (2.12) and compared it with the 5-point amplitude of TMG computed from Feynman rules. Because of the complexity of the analytic expression of the TMG 5-point amplitude, we only compared the values of both amplitudes evaluated on random kinematic configurations. Some examples of these values are given in Table H.1. We found that they agree exactly, further confirming the absence of spurious poles in the double copy of the 5-point TMYM amplitude.

2.6 Discussion

In this section we explored BCJ double copy of the amplitudes in massive theories. While the first attempt to double copy massive Yang-Mills failed to give a physical amplitude at five and higher point level, it revealed that the spectrum of a massive gauge theory that is compatible with double copy is strongly constrained. Then we analyzed the constraints that arise on a large class of low energy effective theories for a tower of interacting massive spin-1 states coupled to a massless gluon and scalar, by demanding that they respect colour-kinematic duality, a necessary precursor to a double copy, without introducing any spurious poles. The mechanism of spurious pole cancellation was the same as in massless case as suggested in [115].

Since in general, the spectral conditions of [115] are difficult to solve, we have made the expedient choice that the spectrum of states should be identical to that that arises in Kaluza-Klein theories, specifically the standard compactification from five dimensions on an S^1, together with demanding the preservation of the associated global $U(1)$. This ensures that spectral conditions are identically satisfied. There is nevertheless a huge class of effective field theories which satisfy these requirements. Thus it is only necessary to impose the BCJ relations in order to ensure colour-kinematics duality is kept intact. Our analysis shows that that at least to quintic order in the Lagrangian, and up to order $1/\Lambda^4$ in the effective field theory expansion, the unique theory within our class which respects the colour-kinematics duality is the theory obtained from compactification of the 5d Lagrangian

$$\left(\frac{-1}{4}\text{tr}(F^2) + \frac{G_{5d}}{\Lambda^2}\text{tr}(F^3) - \frac{9G_{5d}^2}{16\Lambda^4}\text{tr}([F,F]^2)\right) \tag{2.150}$$

on an S^1. The latter is of course known to admit a local double copy, including the higher derivative operators [110], which can be understood as a particular combination of operators appearing in open bosonic string and superstring low energy effective action, that contains only the colour factors built of the structure constants (in order for BCJ double copy formalism to work). Interestingly, while 4pt processes alone could not fix all of cubic and quartic couplings, we found that by combining them with 5pt processes we fixed all the cubic, quartic and quintic interactions. This showed that despite a very large freedom that we have in our action with arbitrary couplings, the BCJ relations eliminate all of it. Therefore, it seems that if we restrict ourself to theories where the spurious poles cancel in the same way as in massless theories we are only left with Kaluza-Klein theories and in order to find new massive double copy examples we need a different mechanism of spurious pole cancellation.

Such a new mechanism has been found in 3d (at least for 5 point tree level amplitudes). The special kinematics arising in a three-dimensional spacetime allow us to construct a well-defined massive double copy that does not require a tower of massive states. Here, we have shown how the spurious poles that generically appear in 5-point amplitudes can be avoided with a single BCJ relation. This BCJ relation was written in terms of the kinematic numerators, or more precisely, in terms of the breaking of the CK algebra for the kinematic factors, i.e., the vector U. It is possible to rewrite this relation in terms of partial amplitudes; nevertheless, the expression for the BCJ relation is largely involved as seen in (H.14). An explicit example of a theory that satisfies such BCJ relation was found to be Topologically Massive Yang-Mills and its double copy is well defined and corresponds to Topologically Massive Gravity. The expressions for these scattering amplitudes become quite involved at higher points, and required the use of numerical methods to verify our results. It is quite likely that there are better variables in which the double copy relation becomes cleaner. For example, it would interesting to see if using spinor-helicity variables, similar to [113], can lead to more compact expressions that allow to prove the 5-point amplitude double copy analytically.

The fact that topologically massive theories satisfy a double copy relation makes us ponder how do these theories fit in the larger web of relations for scattering amplitudes [21]. Examples of scalar effective field theories in this web have been shown to satisfy a double copy relation which is inherited from that of YM and gravity. It is interesting to note that a common feature of these theories is that they exhibit conformal invariance at a given spacetime dimension [9, 116]. In our case, while the gravitational Chern-Simons term is conformally invariant, the whole action including the Einstein-Hilbert term is not. Similarly, TMYM does not have conformal invariance. It is not clear if there is a similar feature or an obvious way of relating these theories to the broader web of amplitude relations, but it would be interesting to explore that possibility. Similarly, it is compelling to understand how does the different versions of the classical double copy work for topologically massive theories? This will be explored in the second part of this thesis.

Besides TMYM, it is interesting to understand if other 3D theories can also have a well-defined double copy, and if these theories need to satisfy the BCJ relation in (2.110). To understand this, we can explore the simple case of massive Yang-Mills in 3D. By following the procedure in Section 2.3, we can analyze the 5-point massive Yang-Mills amplitude. As in 4D, the local numerator factors calculated directly from the Feynman rules do not satisfy the colour-kinematics duality, i.e., (2.2) is not satisfied. In order to satisfy CK duality, we need to perform the shifts (2.4) and solve (2.7) to find v. However, if we consider the amplitude with external polarisations that satisfy the TMYM equations of motion, that is, (2.123), the local kinematics of massive Yang-Mills calculated directly from the Feynman rules satisfy the colour-kinematics duality and do not require any shifts. In that case, the double copy will only have physical poles. This shows that the polarisations of TMYM play a special role in giving rise to a physical double copy. It would be interesting to understand if these are the only polarisations that remove the spurious poles in the massive Yang-Mills double copy, or if a generalized procedure should be found in order to construct a double copy for more general polarisations.

So far, the massive double copy has only been explored for tree-level processes. It will be intriguing to understand how this generalizes to loop order. A simpler task towards this goal consists of understanding higher corrections in the eikonal limit. At tree-level, we can already see an interesting structure arising. The 4-point amplitudes of the TMYM and TMG in the eikonal limit, $s \to \infty$ and $t \ll m^2$, read

$$A^{TYM} = g^2 \frac{\frac{ms}{\sqrt{t}} c_t}{-m^2} \,, \quad iM^{TMG} = \frac{-\kappa^2}{2} \frac{s^2}{t} = 2 \left(\frac{\kappa}{2}\right)^2 \frac{\left(\frac{ms}{\sqrt{t}}\right)^2}{-m^2} \,. \tag{2.151}$$

We can immediately observe a double copy relation arising, given that in this limit the propagator is $t - m^2 \to -m^2$. This is not the standard relation, instead it has an extra factor of 2. In fact, it is straightforward to understand this extra factor. In the 4-point double copy, (2.13), the kinematic factor n_t dominates in the eikonal limit which gives rise to $iM = 2\left(\frac{\kappa}{2}\right)^2 \frac{n_t^2}{m^2}$. Given this new feature arising already at tree-level, we would like to understand how this procedure works at higher orders in the eikonal limit. This will be explored in the second part of the thesis, where we look at these theories coupled to matter fields and study eikonal scattering and classical double copy.

Chapter 3

Massive Classical Double Copy in 3d

3.1 Introduction

In this chapter we will take a step forward in understanding the topologically massive double copy involving matter fields. First, we introduce the topologically massive theories including a minimal coupling to matter fields in Section 3.2. We take a look at the 2-2 scattering of scalars through a massive mediator and find that the double copy requires an extra contact term interaction between the scalars. This extra term becomes subdominant in the eikonal limit in which the sources are highly energetic and their stress-energy tensor becomes traceless, leading to the standard double copy relation as suggested in [117]. Given this, we will explore the eikonal limit in more detail in Section 3.3. We take advantage of the fact that both abelian and non-abelian objects can double copy to the same gravitational object [118] and look at the linearized TMYM case, that is, Topologically Massive Electrodynamics (TME). We prove that the TMG and TME amplitudes exponentiate in the eikonal limit, but a simple double copy relation as in the massless 4d case does not arise. Instead, we show that information beyond the eikonal limit is required to construct the correct massive double copy. Nevertheless, we can construct a simple double copy for the phase shift. To further understand the double copy relation of topologically massive theories in the high-energy limit, we take a look at the classical solutions generated by a highly energetic particle in Section 3.4. We show that a coordinate space Kerr-Schild double copy can be obtained for wave solutions when taking into account a special set of boundary conditions. In the process, we show how the choice of $i\epsilon$ prescription for obtaining the phase shift is related to the boundary conditions of the topologically massive field. Lastly, we in Section 3.5 by discussing the 3d equivalent of Weyl double, which is a way of directly relating the Cotton curvature tensor to the field strength tensor. This relationship holds in curved backgrounds for wave solutions. We give an explicit proof for Type N spacetimes and show few explicit examples.

3.2 Topologically Massive Theories with Matter Couplings

In this section we briefly review the actions of Topologically Massive Yang-Mills (TMYM) and Topologically Massive Gravity (TMG) theory with a minimal coupling to matter. The action of TMYM with a source is,

$$S_{TMYM} = \int d^3x \Bigg(-\frac{1}{4}F^{a\mu\nu}F_{a\mu\nu} + \epsilon_{\mu\nu\rho}\frac{m}{12}\left(6A^{a\mu}\partial^{\nu}A^{\rho}_{a} + g\sqrt{2}f_{abc}A^{a\mu}A^{b\nu}A^{c\rho}\right) + \frac{g}{\sqrt{2}}A^{\mu a}J_{\mu a}\Bigg), \tag{3.1}$$

where m is the mass of the gauge field and g the coupling strength. The equations of motion can be easily obtain from (3.1) and read

$$D_\mu F^{\mu\nu} + \frac{m}{2}\varepsilon^{\nu\rho\gamma}F_{\rho\gamma} = \frac{g}{\sqrt{2}}J^{\nu}, \tag{3.2}$$

where $D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}}A_\mu$, $F_{\mu\nu} = F^a_{\mu\nu}T^a$, with $F^a_{\mu\nu}$ the Yang-Mills field strength and T^a the generators of the gauge group. A large simplification occurs when we consider an ansatz for the gauge field of the form $A^{\mu\ a} = c^a A^\mu$ such that the equations of motion become linear and read

$$\partial_\mu F^{\mu\nu} + \frac{m}{2}\varepsilon^{\nu\rho\gamma}F_{\rho\gamma} = \frac{g}{\sqrt{2}}J^{\nu}, \tag{3.3}$$

where $F^{\mu\nu}$ is the Maxwell field strength since we have linearized the theory.

On the gravitational side, we use the conventions $\kappa^2 = 16\pi G$ and $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Therefore, the action of TMG is,

$$S_{TMG} = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \left(-R - \frac{1}{2m} \epsilon^{\mu\nu\rho} \left(\Gamma^{\alpha}_{\mu\sigma} \partial_\nu \Gamma^{\sigma}_{\alpha\rho} + \frac{2}{3} \Gamma^{\alpha}_{\mu\sigma} \Gamma^{\sigma}_{\nu\beta} \Gamma^{\beta}_{\rho\alpha} \right) + \mathcal{L}_{Matter} \right) , \tag{3.4}$$

and the equations of motion are,

$$G_{\mu\nu} + C_{\mu\nu}/m = -\kappa^2 T_{\mu\nu}/2 , \tag{3.5}$$

where $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$ is the Einstein tensor and

$$C^{\mu\nu} = \epsilon^{\mu\alpha\beta} \nabla_\alpha (R^\nu_\beta - \frac{1}{4} g^\nu_\beta R) \tag{3.6}$$

is the Cotton tensor. Despite the fact that the action is third order in derivatives, the theory is ghost free [88], due to a $\epsilon^{\mu\nu\rho}$ structure of the gravitational Chern-Simons term which makes the theory to be second order in time derivatives. The equations of motion largely simplify if we consider a Kerr-Schild ansatz for the graviton field $h_{\mu\nu} = \phi k_\mu k_\nu$ for which the equation of motion is linear.

3.2.1 2-2 Scattering of Matter

In this subsection, we look at the scattering of minimally coupled massive scalars through a topologically massive mediator and analyze their double copy relation [1]. We take the mass of the scalars to be that of the topologically massive mediators [2]. We write the tree level 2-2 scattering amplitude of scalars in the adjoint representation coupled to TMYM as

$$A_4 = g^2 \sum_{i=1}^{3} \frac{c_i n_i}{s_i - m^2} , \tag{3.7}$$

[1] In all the scattering amplitude calculations presented here, we work in Lorenz gauge for TMYM and in de Donder gauge for TMG.

[2] When the mass of the scalar, m, is not the same as the mass of the mediator, M, the double copy of A_4 can be written as $M_4^{DC} = M_4 + \frac{P(s,t,u)}{stu}$, where P is a polynomial. This $\frac{P(s,t,u)}{stu}$ term has massless poles and their residues are proportional to $m^2 - M^2$. If we require that M_4^{DC} only has contributions from the exchange of a massive mediator and contact terms, we have to set $M = m$.

where the kinematic factors are given by

$$\begin{aligned} n_s &= \frac{i}{2}(u-t) - 2m\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{s}, \\ n_t &= \frac{i}{2}(s-u) - 2m\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{t}, \\ n_u &= \frac{i}{2}(t-s) - 2m\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{u}, \end{aligned} \tag{3.8}$$

where $s = -(p_1+p_2)^2$, $t = -(p_1+p_3)^2$ and $u = -(p_1+p_4)^2$. Here, the coupling to TMYM is given by Eq. (3.1) with $J^{\mu\ a} = f^{abc}\partial^\mu\phi_b\phi_c$. It looks like there are massless poles in (3.8) however, the residues of these poles are zero. This is generic feature of the amplitudes in topologically massive theories due to $1/p^2$ terms in the propagators (2.125) and (2.141) that do not correspond to a pole of a physical particle exchange. It is likely that there is another choice of kinematic variables in which these massless poles are not present.

Similarly, the minimally coupled scalar scattering amplitude in TMG is given as

$$\begin{aligned} M_4 = &\Bigg(\frac{8\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho m\left(4m^2-2s-t\right) - 32im^4 s + 8im^2\left(s^2+st+t^2\right) + it^3}{t\left(m^2-t\right)} + \\ &\frac{8\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho m\left(-4m^2+s+2t\right) + i\left(-32m^4 t + 8m^2\left(s^2+st+t^2\right)+s^3\right)}{s\left(m^2-s\right)} - \\ &\frac{i\left(-8i\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho m(s-t) + 192m^6 - 112m^4(s+t) + 4m^2\left(5s^2+8st+5t^2\right) - (s+t)^3\right)}{\left(-4m^2+s+t\right)\left(-3m^2+s+t\right)} \Bigg)\frac{\kappa^2}{16}, \end{aligned} \tag{3.9}$$

where we have used $s+t+u = 4m^2$ to express u in terms of s and t. The double copy of (3.7), M_4^{DC}, differs from (3.9) by

$$M_4 - M_4^{DC} = -im^2\kappa^2, \tag{3.10}$$

which means that we can match them by adding a contact term, $-\frac{\kappa^2 m^2}{4!}\phi^4$, in the action of TMG with a minimally coupled scalar. This non-trivial realization of the double copy reduces to the trivial case when taking the high-energy (large s and small t) limit. In such a limit, the contact term contribution becomes subdominant since the scattering through the topologically massive graviton grows as s^2. In the rest of this section we explore in detail the double copy in the eikonal limit and leave the analysis of the double copy with more general matter for future studies.

3.3 Double Copy in the Eikonal Limit

The high-energy limit of scattering processes has been largely studied due to its connections to classical backgrounds, which was first explored in [119]. Recently, the focus on the eikonal limit[3] has increased given the ability of obtaining classical observables that describe the inspiral phase of the coalescence of compact binaries from the phase shift [120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133]. In this limit, it has been shown that a simple double copy relation arises in 4 dimensions [47, 134, 135]. Since the eikonal amplitude includes information at all loop orders, a double copy relation for topologically massive eikonal amplitudes will be the first hint for an all orders double copy.

We proceed to analyse in detail the topologically massive double copy in the eikonal limit where we expect it to hold without requiring extra interactions on the gravitational side. We consider the 2-2 scattering of external scalar fields with the following kinematics,

$$p_1^\mu = \left(\frac{1}{2p^v}\left(\frac{q^{\,2}}{4}+m^2\right), p^v, \frac{q}{2}\right) , \qquad p_3^\mu = \left(\frac{-1}{2p^v}\left(\frac{q^{\,2}}{4}+m^2\right), -p^v, \frac{q}{2}\right) , \tag{3.11}$$

$$p_2^\mu = \left(p^u, \frac{1}{2p^u}\left(\frac{q^{\,2}}{4}+m^2\right), -\frac{q}{2}\right) , \qquad p_4^\mu = \left(-p^u, \frac{-1}{2p^u}\left(\frac{q^{\,2}}{4}+m^2\right), \frac{-q}{2}\right) . \tag{3.12}$$

These momenta are on-shell, $p_1^2 = p_2^2 = p_3^2 = p_4^2 = -m^2$, and satisfy the momentum conservation condition $p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0$. Here we work in lightcone coordinates (u, v, x^1),

$$u = \frac{1}{\sqrt{2}}\left(x^0 - x^1\right) , \quad v = \frac{1}{\sqrt{2}}\left(x^0 + x^1\right) . \tag{3.13}$$

[3]Note that our definition of eikonal limit is the limit when the center of mass energy is larger than any other scale. There is a different definition of eikonal limit when the center of mass energy is much larger than the momentum transfer but smaller than the mass of interacting objects.

The independent Mandelstam invariants are

$$s = -(p_1 + p_2)^2 = \frac{(4m^2 + 8p_v p_u + q^2)^2}{32 p_u p_v} \tag{3.14}$$

$$t = -(p_1 + p_3)^2 = -q^{\ 2} \,. \tag{3.15}$$

In the eikonal limit, the momenta p^v and p^u are much larger than q and m and hence $s \approx -u >> t$ and $s >> m^2$.

In the following, we compute the eikonal amplitude to all orders for TMG and TME. We focus on the Abelian case for simplicity since we expect that in the eikonal limit both Abelian and non-Abelian cases double copy to the same gravitational solution, as in both cases the same diagrams contribute to the eikonal scattering amplitude. Also, we know that eikonal amplitudes are related to classical shock wave solutions, which are solutions to both Abelian and non-Abelian theories.

3.3.1 Eikonal resummation in TMG

In 4d, it has been shown that the ladder and cross-ladder diagrams for massive particles of arbitrary spin, which are expected to dominate in the eikonal limit, re-sum in impact parameter space [136]. The eikonal $2-2$ amplitude to all loop orders is given by

$$i\mathcal{M}^{\text{eik}}\left(s,t\right) = 2s \int \mathrm{d}^{D-2}\vec{b} e^{i\vec{q}\cdot\vec{b}} \left(e^{i\delta(s,\vec{b})} - 1\right) \ , \tag{3.16}$$

where the leading order in t/s and m^2/s eikonal phase reads

$$\delta(s,\vec{b}) = \frac{1}{2s} \int \frac{\mathrm{d}^{D-2}\vec{q}}{(2\pi)^{D-2}} e^{-i\vec{q}\cdot\vec{b}} M_{\text{tree}}(s, t = -(\vec{q})^2) \ , \tag{3.17}$$

with $M_{\text{tree}}(s, t = -(\vec{q})^2)$ the 4-point, tree level scattering amplitude given by the t-channel graph in the eikonal limit. Furthermore, this phase can be expressed in terms of the square of 3-point amplitudes by applying a BCFW-like shift. We prove the eikonal resummation for topologically massive theories in Appendix I. For TMG M_{tree} is given as

$$M_{\text{tree}}(s, t = -(q^y)^2) = \frac{-i\kappa^2 s^2 m}{2(q^y)^2(q^y + im)}. \tag{3.18}$$

To compute the phase shift explicitly, we see that we need to regulate the following divergent integral,

$$\delta = \frac{-i\kappa^2 sm}{4} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{q^2(q+im)} e^{-ibq} , \tag{3.19}$$

that is, we need to choose some $i\epsilon$ prescription for integrating around the pole at $q = 0$. We will see later that this freedom corresponds to the freedom in choosing boundary conditions of the shockwave solution corresponding to this scattering process and we will make the choice best suitable for double copy. Following [137] we shift $q \to q - i\epsilon$ and close the integration contour in the lower half plane when $b > 0$ and in the upper half plane when $b < 0$. This way the contour at infinity goes to zero and we can evaluate (3.19) as a contour integral. It picks the residue of the pole at $q = -im$ when $b > 0$, and the residue of the pole at $q = 0$ when $b < 0$. Therefore we can write

$$\delta = \frac{\kappa^2 sm}{4} \left(-\text{Res}_{q=-im} \left(\frac{1}{q^2(q+im)} e^{-ibq} \right) \theta(b) + \text{Res}_{q=0} \left(\frac{1}{q^2(q+im)} e^{-ibq} \right) \theta(-b) \right). \tag{3.20}$$

Evaluating the residues we get that the phase shift is given by

$$\delta = \frac{\kappa^2 s}{4m} \left(e^{-mb}\theta(b) + (1 - mb)\theta(-b) \right) . \tag{3.21}$$

Then the eikonal amplitude reads

$$i\mathcal{M}_{\text{eik}} = 2s \int_{-\infty}^{\infty} db e^{-ibq} \left(\exp\left(\frac{i\kappa^2 s}{4m} \left(e^{-mb}\theta(b) + (1 - mb)\theta(-b) \right) \right) - 1 \right) . \tag{3.22}$$

Note that this could be explicitly evaluated in terms of incomplete gamma functions as in [138]. To understand the connection between the boundary conditions and regularization of the integral, we can see what happens if we choose a different $i\epsilon$ prescription for the $q = 0$ pole, for example if we shift it as $q \to q + i\epsilon$. Then phase shift changes by

$$\frac{\kappa^2 sm}{4}\left(-\text{Res}_{q=0}\left(\frac{1}{q^2(q+im)}e^{-ibq}\right)\right) = \frac{\kappa^2 s}{4m}\left(1 - mb\right). \tag{3.23}$$

We will see in Section 3.4.1 that this corresponds to homogeneous solution of TMG equations of motion that can only be fixed by boundary conditions.

3.3.2 Eikonal resummation in TME

The sum of all loop diagrams for TMYM is complicated due to the different colour factors arising at each loop order. Since we are interested in shock wave solutions which are also solutions of the linearised theory here we will consider the eikonal amplitude in topological massive electrodynamics (TME) of two scalars of charge Q. From now on we slightly change the notation by absorbing the $1/\sqrt{2}$ factor into Q. In other words, the covariant derivative acting on the scalar is now $D\phi = (d - igQA)\phi$. The calculation of the TME eikonal amplitude is given in Appendix I and the expressions are very similar to the TMG case:

$$i\mathcal{A}_{\text{eik}} = 2s\int_{-\infty}^{\infty} dbe^{-ibq}\left(e^{i\delta} - 1\right), \tag{3.24}$$

where the phase shift reads

$$\delta = \frac{1}{2s}\int_{-\infty}^{\infty}\frac{dq^y}{2\pi}A_{\text{tree}}(s, t = -(q^y)^2)e^{-ibq^y}, \tag{3.25}$$

and A_{tree} is given as

$$A_{\text{tree}}(s, t = -(q^y)^2) = \frac{2sg^2Q^2}{q^y(q^y + im)}. \tag{3.26}$$

Evaluating this explicitly and choosing the same contour of integration as in the TMG case gives

$$\delta = -i\frac{g^2Q^2}{2}2\left(-\text{Res}_{q=-im}\left(\frac{1}{q(q+im)}e^{-ibq}\right)\theta(b) + \text{Res}_{q=0}\left(\frac{1}{q(q+im)}e^{-ibq}\right)\theta(-b)\right), \tag{3.27}$$

which finally leads to a compact expression for the TME phase shift

$$\delta = -\frac{g^2 Q^2}{m}\left(e^{-mb}\theta(b) + \theta(-b)\right) \ . \tag{3.28}$$

3.3.3 Double Copy of Eikonal Amplitudes and Phase Shift

After showing that the exponentiation in the eikonal limit is a feature of TMG and TME amplitudes just like in the gravity and Yang-Mills case in 4d, we would like to understand if a simple double copy relation arises in this limit just like in 4d [47]. To do so, it is useful to write the $n-1$ loop diagrams of the TME eikonal amplitude as

$$\frac{i\mathcal{A}_{n-1}}{(\sqrt{2}g)^{2n}} = \frac{i}{n!}\left(\frac{1}{2s}\right)^{n-1}\int_{-\infty}^{\infty} dbe^{-ibq}\left(\int_{-\infty}^{\infty}\frac{dq^y}{2\pi}\frac{sQ^2(q^y - im)}{q^y((q^y)^2+m^2)}e^{-ibq^y}\right)^n . \tag{3.29}$$

comparing this with ,

$$\frac{i\mathcal{A}_{n-1}}{(\sqrt{2}g)^{2n}} \sim \int \frac{cN}{D} , \tag{3.30}$$

where c is the colour factor, N is the kinematic factor and D is the product of all propagators (which also includes the factor of $(2s)^{n-1}$ which comes from propagators), we can identify $c = Q^{2n}$, $N = (s(1 - im/q^y))^n$ and $D = (2s)^{n-1}((q^y)^2 + m^2)^n$. Following the prescription of leaving propagators untouched and exchanging colour (in this case electric charge Q) for kinematics, we can now find the double copy by considering the replacement $Q^2 \to s(1-im/q^y)$ which leads to

$$\frac{i\mathcal{M}^{\text{D.C.}}_{n-1}}{(i\kappa/2)^{2n}} = \frac{1}{n!}\left(\frac{1}{2s}\right)^{n-1}\int_{-\infty}^{\infty} dbe^{-ibq}\left(\int_{-\infty}^{\infty}\frac{dq^y}{2\pi}\frac{s^2(q^y - im)}{(q^y)^2(q^y+im)}e^{-ibq^y}\right)^n . \tag{3.31}$$

When comparing to the TMG result in Eq. (I.12), Eq. (I.13) and Eq. (I.14), we can see that there is a mismatch in the amplitudes. Naively, this could be interpreted as requiring new degrees of freedom on the double copy side. Nevertheless, we will show that this is not the case, and instead it is just an artifact of the massive double copy.

We now proceed to understand the origin of the mismatch between the double copy and TMG eikonal amplitude by looking at the tree level result in detail. We start by looking at the eikonal limit of the kinematic factors of the four-point scalar amplitude in topological massive Yang-Mills:

$$n_s = n_u = -i\frac{s}{2}, \quad n_t = s\left(i \pm \frac{m}{\sqrt{-t}}\right), \tag{3.32}$$

where the $\pm$ sign comes from $\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho = \pm\frac{1}{2}\sqrt{stu}$. We see that in the Yang-Mills amplitude the t channel dominates since the s and u channels are suppressed by $1/s$:

$$A_{\phi 4} \to g^2 \frac{c_t n_t}{t - m^2}. \tag{3.33}$$

However, when constructing the massive double copy we have a new term proportional to $(n_s + n_t + n_u)^2$ coming from requiring the CK duality. In this term all channels contribute equally:

$$n_s + n_t + n_u = \pm \frac{ms}{\sqrt{-t}}. \tag{3.34}$$

Therefore, the double copy of this amplitude is not simply proportional to n_t^2:

$$\begin{aligned} -i\left(\frac{\kappa}{2}\right)^{-2} M_4 &= \frac{n_s^2}{s-m^2} + \frac{n_t^2}{t-m^2} + \frac{n_u^2}{u-m^2} - \frac{(n_s+n_t+n_u)^2}{m^2} \to \frac{n_t^2}{t-m^2} - \frac{(n_s+n_t+n_u)^2}{m^2} \\ &= -\frac{2ms^2(m \pm i\sqrt{-t})}{t(t-m^2)}\ , \end{aligned} \tag{3.35}$$

which correctly reproduces the TMG eikonal amplitude. This tells us that to correctly double copy the scattering amplitude in the eikonal limit we require information beyond the eikonal limit. Alternatively, one could further require that $|t| \ll m^2$ in which case a simple double copy relation arises if we take $Q^2 \to \frac{ms}{\sqrt{-t}}$ and note that in this limit the propagators are given by $-m^2$ [12]. Nevertheless, restricting to the large mass limit would lead to an incorrect computation of the phase shift as can be seen from the previous sections.

Note that despite this issue at the level of the scattering amplitudes, a double copy for the phase shift will arise in the same way as in the 4d Yang-Mills and gravity case. To see this, one should note that given our choice of boundary conditions, the phase shift is only physical for $y > 0$. On this side of the shock wave, the phase shift scales as expected for a scattering through a massive mediator of spin J, that is, $\delta \sim s^{J-1}e^{-mb}$. Thus we see that

$$\frac{\delta^{\text{TME}}}{g^2} = \frac{Q^2}{m}e^{-mb} \quad \xrightarrow[Q^2 \to s]{} \quad \frac{\delta^{\text{TMG}}}{(\kappa/2)^2} = \frac{s}{m}e^{-mb} \ . \tag{3.36}$$

3.4 Double Copy of Classical Solutions

In this section, we will relate the eikonal amplitudes computed above to classical field profiles for the graviton and the gauge field. We do so by interpreting the 4-point scalar amplitudes as the scattering of a scalar off a shock wave background, which in turn is generated by a point-particle with large momentum (the second scalar). Since it is possible to write the gravitational shock wave in Kerr-Schild coordinates, we will explore if we can construct a classical double copy for such solutions. For the standard massless Yang-Mills and Gravity cases, the double copy of shock waves has been explored in various contexts [47, 139, 118, 140].

We proceed by looking at the Kerr-Schild double copy, single copy and zeroth copy ansatze and understanding the equations of motion that they satisfy. Given a metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa k_\mu k_\nu \phi \ , \tag{3.37}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and k^μ is null and geodetic, the single copy is given by

$$A^{a\ \mu} = c^a A^\mu = c^a k^\mu \phi \ . \tag{3.38}$$

To understand if this ansatz gives a solution to TMYM we look at the trace reversed TMG equations with one upper and one lower index

$$R^\mu_\nu + \frac{1}{m}C^\mu_\nu = -\frac{\kappa^2}{2}(T^\mu_\nu - T g^\mu_\nu) \ . \tag{3.39}$$

Contracting this equation with a Killing vector V^μ one finds

$$\nabla_\lambda F^{\lambda\mu} + \frac{1}{m}\epsilon^{\mu\alpha\beta}\nabla_\alpha\nabla^\lambda F_{\lambda\beta} + \frac{V^\nu}{V^\lambda k_\lambda}\left(X^\mu_\nu + Y^\mu_\nu\right) = \frac{\kappa}{2}J^\mu \ , \tag{3.40}$$

$$J^\mu \equiv \frac{2V^\nu}{V^\rho k_\rho}\left(T^\mu_\nu - \delta^\mu_\nu T - \frac{1}{2m}\epsilon^\mu_{\ \alpha\nu}\nabla^\alpha T\right) \ , \tag{3.41}$$

where ∇ is the covariant derivative of η and

$$X^\mu_\nu \equiv -\bar{\nabla}_\nu\left[A^\mu\left(\bar{\nabla}_\lambda k^\lambda + \frac{k^\lambda\bar{\nabla}_\lambda\phi}{\phi}\right)\right] \ , \tag{3.42}$$

$$Y^\mu_\nu \equiv F^{\rho\mu}\bar{\nabla}_\rho k_\nu - \bar{\nabla}_\rho\left(A^\rho\bar{\nabla}^\mu k_\nu - A^\mu\bar{\nabla}_\rho k_\nu\right) \ . \tag{3.43}$$

We see that the equation of motion in TMG is third order in derivatives. This suggests that in order to solve it we will need more boundary conditions compared to general relativity where the equations of motion are second order. We will see later that in fact the boundary conditions play important role in the classical double copy relations between shockwaves in TMYM and TMG; the relations are only manifest for a specific choice of boundary conditions. This is different from the usual massless case where classical double copy relations are apparent at the level of equations of motion, without imposing any boundary conditions.

The equation for the single copy largely simplifies when we consider wave solutions. In such case, the source either vanishes or corresponds to a particle sourcing a shock wave so that the trace of the stress energy tensor vanishes. Furthermore, we can work with lightcone coordinates such that

$$\eta_{\mu\nu}dx^\mu dx^\nu = -2du dv + dy^2 \ . \tag{3.44}$$

Meanwhile, the Kerr-Schild and Killing vector are given by

$$k_\mu dx^\mu = -du \ , \quad V_\mu dx^\mu = dv \ , \quad k\cdot V = 1 \tag{3.45}$$

The single copy equation of motion now reads

$$\nabla_\lambda F^{\lambda\mu} + \frac{1}{m}\epsilon^{\mu\alpha\beta}\nabla_\alpha\nabla^\lambda F_{\lambda\beta} = gJ^\mu = 2gV^\nu T^\mu_\nu \ , \tag{3.46}$$

where we have taken $\kappa/2 \to g$. We note that the single copy does not automatically satisfy the linearized equation of motion of TMYM unless the covariant derivatives pull out factors of the mass and give

$$\varepsilon^{\mu\rho\gamma} k_\gamma \nabla_\rho \left(\frac{\nabla^2 \phi}{m^2} \right) = \varepsilon^{\mu\rho\gamma} k_\gamma \nabla_\rho \phi \ . \tag{3.47}$$

This is satisfied as long as the zeroth copy, $\phi^{a\tilde{a}} = c^a c^{\tilde{a}} \phi$, satisfies the linearized massive biadjoint scalar equation of motion for a vacuum solution or away of a localized source. To see that this is a consistent requirement, we obtain the zeroth copy eom by contracting Eq. (3.46) with the Killing vector V and find

$$\nabla^2 \phi + \frac{m\epsilon^{\mu\lambda\rho} V_\mu (\nabla_\lambda \phi) k_\rho}{k \cdot V} + k \cdot Z = g \frac{J \cdot V}{k \cdot V} \equiv j \ , \tag{3.48}$$

where

$$Z^\nu \equiv (V^\rho k_\rho) \, \bar{\nabla}_\mu \left(\phi \bar{\nabla}^{[\mu} k^{\nu]} - k^\mu \bar{\nabla}_\nu \phi \right) + m\epsilon^{\mu\lambda\rho} V_\mu (\nabla_\lambda k_\rho) \phi \ . \tag{3.49}$$

Considering again the case of wave solutions, we find that the zeroth copy satisfies the following equation of motion

$$\nabla^2 \phi + m\epsilon^{\mu\lambda\rho} V_\mu (\nabla_\lambda \phi) k_\rho = j = 2 V^\nu V^\mu T^\mu_\nu \ . \tag{3.50}$$

Requiring consistency of the double copy restricts the zeroth copy to satisfy $\partial_y \phi = -m\phi$. Thus, the Kerr-Schild double copy for TMG waves fixes the zeroth copy to satisfy

$$\phi = A e^{-my} \ , \tag{3.51}$$

where A is a constant. It is trivial to see that plane waves will satisfy the double copy relation. Hence, in the following we analyze in detail the more involved case of shock wave solutions.

3.4.1 Shock Waves

Shock wave solutions are closely related to scattering amplitudes in the eikonal limit. A probe particle moving in a shock wave background will experience a time delay which can also be computed by considering the 2 to 2 scattering in the eikonal limit of such particle with the massless particle generating the shock wave. Understanding the double copy of shock wave solutions could give a hint of an all order double copy relation. Here, we will analyze in detail how to construct a double copy for these classical solutions. In TMG, these solutions have been previously studied in [138, 137, 141] where an important feature is highlighted, the need to choose boundary conditions to fully fix the metric. In the following, we construct the TMG, TME and biadjoint scalar shock waves by choosing a special case of boundary conditions that makes the double copy relation explicit.

We start by constructing the shock wave solution in TMG for a source

$$T^{\mu\nu} = E\delta(u)\delta(y)\delta^{\mu}_{v}\delta^{\nu}_{v} \ , \tag{3.52}$$

with energy E. The metric can be written in lightcone Kerr-Schild coordinates with the Kerr-Schild scalar given by

$$\phi = \delta(u)g(y) \ , \tag{3.53}$$

where g satisfies

$$g'''(y) + mg''(y) = \kappa E m\delta(y) \ . \tag{3.54}$$

The TMG shock wave is not fixed by requiring asymptotic flatness as in the GR case. Since flatness only requires $g''(y) = 0$, given a solution g_1 of Eq. (3.54), $g_2 = g_1 + c(u)G$ is an asymptotically flat shock wave as long as $G'' = 0$. So one could ask if there are certain boundary conditions that allow for a double copy relation in coordinate space. Since we would like to connect our classical solution to the eikonal amplitudes, we will choose our boundary conditions such that they are consistent with the phase shift calculation in the previous section.

The 2-2 amplitude in the eikonal limit can be reproduced by considering the propagation of a point particle in the shock wave background. Following [138][4], we change the coordinate v to

$$v \to v + \frac{\kappa}{2}\theta(u)g(y) \tag{3.55}$$

which changes $dv \to dv + \frac{\kappa}{2}\left(\delta(u)g(y)du + \theta(u)d(g(y))\right)$ so the metric is now

$$ds^2 = -2dudv + dy^2 - \kappa\,\theta(u)\,du\,d(g(y))\;. \tag{3.56}$$

For $u < 0$ the metric if Minkowski in u, v, y coordinates but for $u > 0$ it is Minkowski in $u, v + \frac{\kappa}{2}g(y), y$ coordinates. Therefore, we can write the wavefunction of the incoming particle with momentum p (in $u < 0$ region) as

$$\psi_{in} = \frac{1}{(2\pi)^{3/2}}e^{ip\cdot x}\;, \tag{3.57}$$

while for outgoing particle of momentum p' (in $u > 0$ region) it is

$$\psi_{out} = \frac{1}{(2\pi)^{3/2}}e^{ip'\cdot x + \frac{1}{2}ip'_v\kappa g(y)}\;. \tag{3.58}$$

The scattering amplitude M_{eik} defined as

$$\delta(p_v - p'_v)\delta(p_u - p'_u)M^{\text{p.p.}}_{eik}(q_y = p'_y - p_y) = \int d^3x\psi_{in}(x)\psi^*_{out}(x)\;, \tag{3.59}$$

is equal to

$$M^{\text{p.p.}}_{eik} = \int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{i\left(-qy - \frac{1}{2}\kappa p'_v g(y)\right)}, \tag{3.60}$$

where for our kinematics $p' - p = -p_2 - p_4 = q$ and $p_u = E$ so $p'_v = \frac{s}{2E}$. This matches the result in (3.22) if

$$g(y) = -\kappa\frac{E}{m}\left(e^{-my}\theta(y) + (1 - my)\theta(-y)\right), \tag{3.61}$$

[4]Note that there is a minus sign in front of q in (3.60) compared to [138], since the shock wave geometry is sourced by particle 1 and the incoming test particle is particle 2. Therefore $p_y - k_y$ of [138] is equal to $p_2^y + p_4^y = -q$ in our convention.

when taking into account the non-relativistic normalization and conventions:

$$M_{eik}^{\text{p.p.}} = \delta(q) + \frac{\mathcal{M}_{\text{eik}}}{4\pi s} \ . \tag{3.62}$$

We can see that this choice gives boundary conditions such that in one side ($y > 0$) of the shock wave the metric is Cartesian, i.e., $\lim_{y\to\infty} h_{\mu\nu} = 0$, while for $y < 0$ it is flat, even if it is in non-Cartesian coordinates. A different choice of boundary conditions would correspond to adding some homogeneous solution of (3.54). Now we can connect this freedom of choosing boundary conditions to the freedom in regulating the divergent integral in the eikonal scattering amplitude calculation given in Section 3.3.1. We can easily see that changing the phase shift by (3.23) corresponds to adding a homogeneous solution of (3.54), since a function linear in y is a homogeneous solution of (3.54).

We now proceed to compute the shock wave for linearized TMYM, that is, TME in a similar manner. Consider a source $J^\mu = Q\delta(u)\delta(y)\delta^\mu_v$ and an ansatz for the shock-wave solution in TMYM of the form

$$A^a = -c^a \delta(u) f(y) du \ . \tag{3.63}$$

Plugging this in the TMYM eom gives

$$f''(y) + mf'(y) = -gQ\delta(y) \ . \tag{3.64}$$

As in the gravitational case, the shock wave is not fully determined by requiring that the field strength vanishes at infinity. In this case, given a solution f_1 of Eq. (3.64), $f_2 = f_1 + c(u)F$ is also a shock wave with asymptotically vanishing field strength as long as $f' = 0$. This leaves us with the freedom of imposing stronger boundary conditions on the gauge field to fully fix it. We will proceed as in the gravitational case and fix this boundary condition by looking at the eikonal scattering amplitudes. We consider the scattering amplitude for the propagation of a point particle in the shock wave gauge background. Similar to the TMG case, we first perform a gauge transformation on A

$$A \to A + d(\theta(u)f(y)) = \theta(u)f'(y)dy \ . \tag{3.65}$$

The wavefunction of a point particle with charge Q moving in an electromagnetic field, satisfying $D^\mu D_\mu \psi = 0$, can be written as

$$\psi = e^{igQ\int^x A_\mu dx^\mu + ip\cdot x} \ , \tag{3.66}$$

where the integral can be taken over any path that ends at x. We choose the path so that it starts in the $u < 0$ region. The wavefunction of the incoming particle with momentum p (in the $u < 0$ region) is then

$$\psi_{in} = \frac{1}{(2\pi)^{3/2}} e^{ip\cdot x} \ , \tag{3.67}$$

while for the outgoing particle with momentum p' (in the $u > 0$ region) is

$$\psi_{out} = \frac{1}{(2\pi)^{3/2}} e^{igQ\int^y f'(y')dy' + ip'\cdot x} = \frac{1}{(2\pi)^{3/2}} e^{igQf(y) + \text{const.} + ip'\cdot x} \ . \tag{3.68}$$

Additionally, we choose the path such that the constant of integration is zero. Then by (3.59) the point-particle scattering amplitude, $A^{\text{p.p.}}_{eik}$, is equal to

$$A^{\text{p.p.}}_{eik} = \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{-iqy - igQf(y)}. \tag{3.69}$$

Matching this to the eikonal amplitude in Eq. (3.28) and (I.20), and taking into account the non-relativistic normalization of the point particle amplitude in Eq. (3.62) we find that

$$f(y) = \frac{gQ}{m}\left(e^{-my}\theta(y) + \theta(-y)\right) \ . \tag{3.70}$$

This choice corresponds to boundary conditions in which the field strength is zero for $y < 0$ and on the other side of the shock wave we have $\lim_{y\to\infty} A^\mu = 0$.

Lastly, we look at the zeroth copy, $\phi^{a\tilde{a}} = c^a c^{\tilde{a}} S$, shock wave which is a solution of the linearized bi-adjoint scalar equations of motion:

$$(\nabla^2 - m^2)S = -\lambda\delta(u)\delta(y) \ . \tag{3.71}$$

The scalar field shock wave solution is

$$S = \frac{\lambda}{2m}\left(e^{-my}\theta(y) + e^{my}\theta(-y)\right)\delta(u) \ . \tag{3.72}$$

Note that for the scalar case, there is no analog of having the curvature, or field strength vanishing, or an extra freedom in the solution from choosing boundary conditions. In fact, in this case the field approaches 0 at both $y = \pm\infty$.

Now, we can proceed to construct the double copy of the TMYM shock wave to understand if it corresponds to the TMG shock wave. We can immediately see that this construction is highly dependent on our choice of boundary conditions. If we simply look at the equations of motion, we would naively conclude that the double copy of the TMYM shock wave does not correspond to the TMG shock wave. Instead, it would suggest that the source on the gravitational side is given by $T_{uu} = \frac{E}{m}\delta(u)\partial_y\delta(y)$ with all other components being zero. Nevertheless, one should be careful since we need to choose the appropriate boundary conditions to completely fix the shock wave solutions. Considering the special choice used in the computations above, we can see that the Kerr-Schild double copy holds on the $y > 0$ side of the shock wave. In this side of the shock wave, the condition for the Kerr-Schild zeroth copy, Eq. (3.51), is fulfilled and the double copy relation is satisfied when we consider the replacements:

$$\frac{\kappa}{2} \longleftrightarrow g \longleftrightarrow 1 \ , \quad 2E \longleftrightarrow Q \longleftrightarrow \lambda \ , \tag{3.73}$$

where the factor of 2 is standard in relating the Kerr-Schild sources as seen in Eq. (3.46). On the other hand, the relation does not hold for $y < 0$, but this should not cause alarm, since on that side of the shock wave the spacetime is flat and the field strength vanishes. Hence, the apparent mismatch is simply explained by the choice of boundary conditions on that side of the shock wave which obscures the double copy relation. This conclusion is similar to the time delay computation presented in [137]. Naively, computing the time delay, $\Delta x^- = \delta(s,b)/|p^-|$, using the phase shift in Eq. (3.21) and (3.28) will give a non-zero result on the $y < 0$ which is unphysical since in this side of the shock wave the space is flat (the field strength vanishes).

3.4.2 Gyratons

Now we consider a generalization of the shock wave metric by adding a classical spin to the source. In this subsection, we will construct such solutions for TMG, TMYM, and the biadjoint scalar. In gravitational settings, this type of solutions have been dubbed gyratons and their metric is

$$ds^2 = -2dudv + dy^2 + \kappa\phi(u,y)du^2 + 2\kappa\alpha(u,y)dudy \; . \tag{3.74}$$

The stress tensor is now given by

$$T_{\mu\nu} = \left(Ek_\mu k_\nu + \sigma k_{(\mu}\epsilon_{\nu)}^{\alpha\beta} k_\alpha \partial_\beta\right)\delta(u)\delta(y) \; , \tag{3.75}$$

where E is the energy of the source and σ its spin. Writing $\phi = g(y)\delta(u)$, the TMG equation of motion now gives

$$\partial_y^2\left(g'(y)\delta(u) - 2\partial_u\alpha(u,y)\right) + m\partial_y\left(g'(y)\delta(u) - 2\partial_u\alpha(u,y)\right) = \kappa m\delta(u)\left(E\delta(y) - \sigma\delta'(y)\right) \; . \tag{3.76}$$

We see that outside the sources the equation of motion is similar to that of the shock wave but now the y derivative of ϕ is shifted to $\partial_y\phi = \partial_y\phi - 2\partial_u\alpha$. The metric (3.74) is invariant under the following transformation [142]:

$$v \to v + \kappa\,\lambda(u,y), \quad \alpha \to \alpha - \partial_y\lambda, \quad \phi \to \phi - 2\partial_u\lambda \; . \tag{3.77}$$

We can fix this gauge freedom by imposing

$$\partial_y\alpha = 0 \; . \tag{3.78}$$

Now if we assume α has the same u dependence as $\phi(u,y) = g(y)\delta(u)$, *i.e.* $\alpha(u,y) = \alpha(y)\delta(u)$, then the gauge condition implies that $\alpha(y) = $ constant. Then the solution of (3.76), with the same boundary conditions for ϕ as before, is given as

$$g = \frac{\kappa}{m}(E + m\sigma)e^{-my}\theta(y) + \frac{\kappa}{m}(E + m\sigma - Emy)\theta(-y) \ , \tag{3.79}$$

$$\alpha = 0 \ . \tag{3.80}$$

Here we have chosen $\alpha = 0$ so that the metric is in Cartesian coordinates on the $y > 0$ side of the gyraton. Note that with this choice the metric is in Kerr-Schild coordinates as in the shock waves case.

It is interesting to note that the inclusion of classical spin changed the expression of the shock-wave Kerr-Schild scalar on the physical side ($y > 0$) by shifting the energy as

$$E \to E\left(1 + m\frac{\sigma}{E}\right) \ . \tag{3.81}$$

This type of energy shift was originally found when looking at gravitational anyons in [92]. It is not surprising that the same shift arises for gyratons, since we can think of them as being sourced by highly-boosted anyons. Alternatively, this shift can be obtained by shifting the y coordinate as $y \to y - \sigma/E$ and taking the small σ/E limit. This shift is reminiscent of the spin deformations of 3-point on-shell amplitudes in 3d [143], which in 4d are related to the Newman-Janis shift[144, 145, 146, 147, 148, 149]. We will see in the following that this shift also arises for the TME and biadjoint scalar gyratons.

On the gauge theory side we can consider the following gauge field:

$$A^a = c^a(\varphi(u,y)du + \beta(u,y)dy), \tag{3.82}$$

which gives only one non-vanishing component of field strength $F_{uy} = -\partial_y\varphi + \partial_u\beta$ just like in the shock wave case. Expressing $\varphi = f(y)\delta(u)$, the equation of motion with the spinning source,

$$J_\mu = \left(Qk_\mu + Q'\epsilon_\mu^{\alpha\beta}k_\alpha\partial_\beta\right)\delta(u)\delta(y) \ , \tag{3.83}$$

gives the following:

$$\partial_y\left(f'(y)\delta(u) - \partial_u\beta(u,y)\right) + m\left(f'(y)\delta(u) - \partial_u\beta(u,y)\right) = g\left(Q\delta(u)\delta(y) - Q'\delta(u)\delta'(y)\right) \ . \tag{3.84}$$

We now choose to impose the Lorenz gauge condition which implies

$$\partial_y\beta = 0 \ , \tag{3.85}$$

but we still have some residual freedom from choosing boundary conditions which we fix by picking the same boundary conditions as in the shock waves case, that is, that the field strength vanishes on one side of the shock wave and on the other side the gauge field vanishes asymptotically. With this choice, we get the following solution:

$$f = g\frac{Q + mQ'}{m}\left(\theta(y)e^{-my}\right) + g\frac{Q}{m}\theta(-y), \tag{3.86}$$

$$\beta = 0. \tag{3.87}$$

An important feature of this choice is that the gauge field is null, as required by the Kerr-Schild single copy ansatz.

Finally, we construct the zeroth copy solution for a spinning source. The linearized equation of motion reads

$$(\nabla^2 - m^2)S = -\left(\lambda\delta(u)\delta(y) - \lambda'\delta(u)\delta'(y)\right) \ , \tag{3.88}$$

and its solution is given by

$$S = \frac{1}{2m}\left((\lambda + m\lambda')e^{-my}\theta(y) + (\lambda - m\lambda')e^{my}\theta(-y)\right)\delta(u) \ . \tag{3.89}$$

Consequently, we see that Kerr-Schild double copy works in a similar way as before in the region $y > 0$ where the curvature (field strength) is non-zero, with the replacements now given by

$$\frac{\kappa}{2} \longleftrightarrow g \longleftrightarrow 1 \ , \quad 2(E + m\sigma) \longleftrightarrow Q + mQ' \longleftrightarrow \lambda + m\lambda' \ . \tag{3.90}$$

3.4.3 dRGT Shock Waves

As a special case, we will analyze the massive double copy of shockwaves in $d \geq 4$. Although it is known that the double copy construction fails to reproduce dRGT massive gravity at 5-points due to the appearance of spurious poles, it would be interesting to understand if it is possible that the 4-point double copy holds beyond tree-level. A simple example that that can help us understand this consists of analyzing the classical shock wave solutions since this can be entirely reproduced from looking at the 2-2 eikonal scattering. In dRGT, the shock wave solutions for a stress tensor of the form $T^{\mu\nu} = E\delta(u)\delta(\vec{x}-\vec{x}_0)\delta^\mu_v\delta^\nu_v$ can be written in Kerr-Schild form as

$$\mathrm{d}s^2 = -2\ \mathrm{d}u\mathrm{d}v + \mathrm{d}\vec{x}^2 + \kappa\delta(u)F(\vec{x})\,\mathrm{d}u^2 \ , \quad (-\nabla^2 + m^2)F(\vec{x}) = \kappa E\delta(\vec{x} - \vec{x}_0) \ . \tag{3.91}$$

In fact, this is a solution to the equations of motion for a massive graviton with an arbitrary potential [150], even if such cases include ghosts. The Kerr-Schild vector and scalar are given by

$$k_\mu dx^\mu = -du \ , \quad \phi = \delta(u)F(\vec{x}) \ . \tag{3.92}$$

Thus, the single copy is given by

$$A^{\mu\ a} = -c^a\delta(u)F(\vec{x})\,\delta^\mu_v \ . \tag{3.93}$$

Using this in the massive Yang-Mills equations of motion and considering the replacements in Eq. (3.73), we find

$$(-\nabla^2 + m^2)F\,(\vec{x})\,\delta(u) = gQ\delta(\vec{x} - \vec{x}_0)\ , \tag{3.94}$$

which tells us that this indeed corresponds to a shock wave solution of massive Yang-Mills with a source $J^\mu = Q\delta(u)\delta(\vec{x} - \vec{x}_0)\delta^\mu_v$. One should note that this double copy relation holds for all $d \geq 4$. This simple relation might be a hint that the loop level massive double copy holds at 4-points for massive gravity.

3.5 Cotton Double Copy

Having seen that the Kerr-Schild double copy between shockwave solutions of TMYM and TMG holds on the non trivial side of the shockwave, $y > 0$, where the curvature and field strength are non-zero, raises a question: can we get a cleaner formulation of double copy by directly relating the curvature to the field strength? In 4d this is known as Weyl double which is summarized in the introduction. However in 3d the Weyl tensor is zero so another formulation of double copy is needed. It turns out that the Cotton tensor plays the the Weyl tensor role, therefore we attempt to relate Cotton tensor of TMG to the field strength tensor of TMYM. Since the Weyl double copy is usually formulated in terms of spinors we will use 3d spinor formalism for Cotton double copy.

Since we will be interested in curved backgrounds we write the action of TMYM in a curved spacetime as

$$\begin{aligned} S_{TMYM} = \int d^3x\sqrt{-g}\bigg(&-\frac{1}{4}F^{a\mu\nu}F_{a\mu\nu} + \frac{\mathrm{g}}{\sqrt{2}}A^{\mu a}J_{\mu a} \\ &+\varepsilon_{\mu\nu\rho}\frac{m}{12}\left(6A^{a\mu}\partial^\nu A^\rho_a + \mathrm{g}\sqrt{2}f_{abc}A^{a\mu}A^{b\nu}A^{c\rho}\right)\bigg), \end{aligned} \tag{3.95}$$

where m is the mass of the gauge field, g the coupling strength, and $\varepsilon_{\mu\nu\rho}$ is the Levi-Civita tensor given by $\varepsilon_{\mu\nu\rho} = \sqrt{-g}\epsilon_{\mu\nu\rho}$, with $\epsilon_{\mu\nu\rho}$ the Levi-Civita symbol. As before, we will consider gauge fields and sources of the form $A^{\mu\ a} = c^a A^\mu$ and $J^{\mu\ a} = c^a J^\mu$, with c^a a constant color charge, so that the equations of motion become linear and read

$$\nabla_\mu F^{\mu\nu} + \frac{m}{2}\varepsilon^{\nu\rho\gamma}F_{\rho\gamma} = \frac{\mathrm{g}}{\sqrt{2}}J^\nu, \tag{3.96}$$

where $F_{\mu\nu}$ is the linearised Yang-Mills field strength. As we saw in the previous sections the double copy of TMYM corresponds to TMG theory whose action (including the cosmological constant),

$$\begin{aligned} S_{TMG} =& \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\Bigg(- R + 2\Lambda + \mathcal{L}_{Matter} \\ & - \frac{1}{2m}\varepsilon^{\mu\nu\rho}\left(\Gamma^\alpha_{\mu\sigma}\partial_\nu\Gamma^\sigma_{\alpha\rho} + \frac{2}{3}\Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\beta}\Gamma^\beta_{\rho\alpha}\right)\Bigg) , \end{aligned} \tag{3.97}$$

where $\kappa^2 = 16\pi G$, and Λ is the cosmological constant. The equations of motion read

$$G_{\mu\nu} + C_{\mu\nu}/m = -\kappa^2\frac{T_{\mu\nu}}{2} - \Lambda g_{\mu\nu} , \tag{3.98}$$

where $G^{\mu\nu}$ is the Einstein tensor and $C^{\mu\nu} = \varepsilon^{\mu\alpha\beta}\nabla_\alpha(R^\nu_\beta - \frac{1}{4}g^\nu_\beta R)$ the Cotton tensor.

We proceed to understand whether one can construct an analogue of the Weyl double copy for topologically massive theories. When trying to generalise this to 3d, one immediately hits a roadblock since the Weyl tensor is zero. Instead, we will look at the analogue of the Weyl tensor in 3d which is the Cotton tensor. In 3d, the Cotton tensor is invariant under conformal transformations and thus is zero for conformally flat spacetimes, just like the Weyl tensor for $d > 3$. Since the Cotton tensor appears in the TMG equations of motion, Eq. (3.98), this tells us that we could write it as a *square* of terms in the TMYM equations of motion. By a simple counting of derivatives, we see that an appropriate ansatz is

$$C^{\text{lin.}}_{\mu\nu} = -\frac{1}{4}\frac{\left(\partial^\lambda F^{\text{lin.}}_{\lambda(\nu}\right)\left(\varepsilon_{\mu)\rho\gamma}(F^{\rho\gamma})^{\text{lin.}}\right)}{e^{ip\cdot x}} , \tag{3.99}$$

which is satisfied for plane waves. Note that we can use the TMYM equations of motion, (3.96), to rewrite this relation in a simpler form. Considering only localised sources, outside of the source we have

$$C_{\mu\nu}^{\text{lin.}} = \frac{m}{2} \frac{{}^{\star}F^{\text{lin.}}_{(\mu}{}^{\star}F_{\nu)}{}^{\text{lin.}}}{e^{ip\cdot x}} \ , \tag{3.100}$$

where ${}^{\star}F_{\rho} = \varepsilon_{\mu\nu\rho}F^{\mu\nu}/2$ is the dual field strength. In the following section we will use this relation as motivation for the Cotton double copy.

3.6 3d Spinor formalism for the Cotton double copy

The spinor formalism in 3d has been considered in [151, 152, 153]. It uses the fact that the 3d Lorenz group $SO(2,1)$ is isomorphic to $SL(2,\mathrm{R})/\mathrm{Z}_2$ to rewrite the tangent space Lorentz transformations. This allows us to write a vector in tangent space as

$$v_{\mathrm{a}} = -(\sigma_{\mathrm{a}})_{AB}v^{AB}, \tag{3.101}$$

where the sigma matrices, σ_{a}, form a basis of $SL(2,\mathrm{R})$ that satisfy the Clifford algebra

$$(\sigma_a)_{AB}(\sigma_b)^{B}{}_{C} + (\sigma_b)_{AB}(\sigma_a)^{B}{}_{C} = 2\eta_{ab}\epsilon_{AC}. \tag{3.102}$$

We choose the sigma matrices as

$$\sigma^{A}{}_{B} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) . \tag{3.103}$$

Note that σ_{AB} and σ^{AB} are symmetric.

To move between coordinate space and tangent space we use the frame e^{μ}_{a} that satisfies $\eta_{\mathrm{ab}} = e^{\mu}{}_{\mathrm{a}}e^{\nu}{}_{\mathrm{b}}g_{\mu\nu}$. Thus, we can write a vector in coordinate space as $v^{\mu} = -e^{\mu}{}_{\mathrm{a}}(\sigma^{\mathrm{a}})_{AB}v^{AB}$. The $SL(2,\mathrm{R})$ indices, $A,B = 1,2$, are lowered and raised with the 2d Levi-Civita symbol ϵ_{AB} according to the following conventions $\psi^{A} = \psi_{B}\epsilon^{BA}$, $\psi_{A} = \epsilon_{AB}\psi^{B}$ [152]. In the following, it will be useful to work with a spinor basis given by a dyad (ι, o) that satisfies

$\iota_A\iota^A = o_A o^A = 0$, $\iota_A o^A = -1$. Thus we can write $\epsilon_{AB} = 2\iota_{[A}o_{B]}$.

Now we proceed to expressing the linearised TMYM field strength tensor, $F_{\mu\nu}$, in terms of spinors. Contracting the Lorentz indices with σ matrices gives F_{ABCD}, which must be antisymmetric with respect to exchanging AB to CD. Using this antisymmetry property and that the fact that antisymmetrization with respect two indices A and B must be proportional to ϵ_{AB}, it is easy to see that F_{ABCD} must be of the following form:

$$F_{ABCD} = f_{BD}\epsilon_{AC} + f_{AC}\epsilon_{BD} \ , \tag{3.104}$$

where $f_{AB} = f_{BA}$ can be interpreted as the dual field strength $f^\mu = -\sigma^\mu_{AB} f^{AB} \propto \epsilon^{\mu\alpha\beta}F_{\alpha\beta}$. Then substituting this into the equation of motion (3.96) in vacuum ($J^\nu = 0$) and expressing everything in terms of spinor indices we get

$$\nabla^C_{(A} f_{B)C} = \frac{m}{\sqrt{2}} f_{AB} \ . \tag{3.105}$$

Now we use the Bianchi identity, $dF = 0$, which in terms of spinors is

$$\nabla_{EF}F_{ABCD} + \nabla_{AB}F_{CDEF} + \nabla_{CD}F_{EFAB} = 0. \tag{3.106}$$

Contracting this with ϵ^{EC} and ϵ^{BD} and substituting (3.104) gives $\nabla^C_{[F}\phi_{A]C} = 0$, which implies that symmetrization on the left hand side of (3.105) is redundant. Therefore the equation of motion of f_{AB} can be written as

$$\nabla^C_A f_{BC} = \frac{m}{\sqrt{2}} f_{AB} \ . \tag{3.107}$$

Similarly, we need to write TMG equation of motion in terms of spinor equivalents of curvature tensors. In a vacuum 3d spacetime ($T_{\mu\nu} = 0$), the TMG equations of motion can be written as

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{1}{6}Rg_{\mu\nu} - \frac{1}{m}C_{\mu\nu}, \tag{3.108}$$

since contracting the indices of the equation of motion and using the fact that $C_{\mu\nu}$ is traceless gives $\Lambda = -R/6$. This implies that the Cotton tensor is proportional to the traceless part of the Ricci tensor, $\Phi_{\mu\nu}$, on the solutions of the equations of motion:

$$C_{\mu\nu} = -m\left(R_{\mu\nu} - \frac{R}{3}g_{\mu\nu}\right) = -m\Phi_{\mu\nu}. \tag{3.109}$$

The definition of the Cotton tensor, $C_{\mu\nu} = \varepsilon^{\mu\alpha\beta}\nabla_\alpha(R^\nu_\beta - \frac{1}{4}g^\nu_\beta R)$, together with contracted Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$, which in terms of spinors is

$$\nabla^{AB}\Phi_{ABCD} + \frac{1}{6}\nabla_{CD}R = 0, \tag{3.110}$$

implies that the Cotton spinor is related to the spinor corresponding to the traceless part of Ricci tensor, Φ_{ABCD}, in the following way [152]

$$C_{ABCD} = -\sqrt{2}\nabla^E_{(A}\Phi_{BCD)E}. \tag{3.111}$$

The fact that $C_{\mu\nu}$ is symmetric and trace free implies that C_{ABCD} is totally symmetric. Substituting the spinorial version of (3.109) into this equation gives the following equation of motion for the Cotton spinor:

$$\nabla^E_{(A}C_{BCD)E} = \frac{m}{\sqrt{2}}C_{ABCD}. \tag{3.112}$$

Using the Bianchi identity (3.110) in vacuum where $R = -6\Lambda$ is constant implies that $\nabla^E_{[A}\Phi_{B]CDE} = \nabla^E_{[A}C_{B]CDE} = 0$, therefore the symmetrization of the indices of the left hand side of (3.112) is redundant so the equation of motion for the Cotton spinor can be written as

$$\nabla^{EA}C_{BECD} = \frac{m}{\sqrt{2}}C^A{}_{CDE}\ . \tag{3.113}$$

Motivated by the linear relationship found in Eq. (3.100), we propose that the analogue of the

Weyl double copy between the Cotton and field strength spinors is [5]

$$\boxed{C_{ABCD} = \frac{m}{2}\frac{f_{(AB}f_{CD)}}{S}\ .} \tag{3.114}$$

Below we will proof that this relationship is satisfied for Type N spacetimes with a scalar field S satisfying the massive Klein Gordon equation with a non-minimal coupling in curved spacetimes. Note that (3.114) follows the KLT double copy philosophy. In our case S plays the role of the KLT kernel and can be thought of as a linearised solution of the massive biadjoint scalar when considering the ansatz $S^{ab} = c^a c^b S$; this is commonly referred to as the zeroth copy.

3.7 Algebraic classification of tensors in 2+1 d

Now we briefly review the 2+1 d analog of classification of tensors in 3+1 d. First we need to introduce the concept of principal spinor. A symmetric n-index spinor, $\phi_{AB...E}$ can be written as

$$\phi_{AB...E} = \alpha_{(A}\beta_B...\delta_{E)}, \tag{3.115}$$

where the spinors α, β, ...,δ are called principal spinors of ϕ. This can be seen by contracting $\phi_{AB...E}$ with n factors of an arbitrary spinor $\xi^A = (\xi_1, \xi_2)$, which gives $(\xi_2)^n$ times an n th degree polynomial in ξ_1/ξ_2:

$$\phi_{AB...E}\xi^A\xi^B...\xi^E = (\xi_2)^n\left(\phi_{11...1}\left(\frac{\xi_1}{\xi_2}\right)^n + n\phi_{21...1}\left(\frac{\xi_1}{\xi_2}\right)^{n-1} + ... + \phi_{22...2}\right). \tag{3.116}$$

By the fundamental theorem of algebra this polynomial must be a product of its roots, z_i:

$$(\xi_2)^n\prod_{i=1}^{n}\left(\frac{\xi_1}{\xi_2} - z_i\right) = \prod_{i=1}^{n}\left(\xi_1 - z_i\xi_2\right), \tag{3.117}$$

[5]The mass factor in Eq. (3.114) is a choice of conventions. It could be absorbed in S or we could write the relation for the traceless Ricci spinor since the TMG equations tell us that $C_{ABCD} = -m\Phi_{ABCD}$, where the traceless Ricci tensor is given as $S^{\mu\nu} \equiv R^{\mu\nu} - Rg^{\mu\nu}/3 = \sigma^\mu_{AB}\sigma^\mu_{CD}\Phi_{ABCD}$.

which can be written as

$$\phi_{AB...E}\xi^A\xi^B...\xi^E = \alpha_A\xi^A\beta_B\xi^B...\delta_E\xi^E, \tag{3.118}$$

where, for example, we can choose $\alpha_A = (1, z_1)$, $\beta_A = (1, z_2)$ etc. This gives (3.115) because ξ is arbitrary. Since symmetric traceless tensors correspond to totally symmetric spinors, they can be classified by the multiplicity of the principal spinors.

As was shown in (3.104) a two-form can be written in terms of symmetric spinor f_{AB}. Once a basis for spinors (ι, o) is chosen, f_{AB} can be expanded as

$$f_{AB} = \left(\Phi_0\iota^2 + 2\Phi_1\iota o + \Phi_2 o^2\right)_{(AB)}, \tag{3.119}$$

and its algebraic classification [154] is found in Table 3.1. Different types correspond to different ways of how a symmetric real ($f_{AB} = \widehat{f_{AB}}$ where hat denotes conjugation) spinor can be a product of principal spinors. We see that this classification corresponds to classifying the dual field strength f^μ as spacelike, timelike or null.

Table 3.1: Algebraic classification of two form field in 2+1 d [154].

$f_\mu f^\mu$	**Normalization**	**Principal spinors**
>0	$\Phi_1 = 0, \Phi_2/\Phi_0 < 0$	$f_{AB} = \alpha_{(A}\beta_{B)}$ with $\widehat{\alpha}_A = \alpha_A, \widehat{\beta}_A = \beta_A$
<0	$\Phi_1 = 0, \Phi_2/\Phi_0 > 0$	$f_{AB} = \alpha_{(A}\widehat{\alpha}_{B)}$
$=0$	$\Phi_0 = 0, \Phi_1 = 0, \Phi_2 = \pm 1$	$f_{AB} = \pm\alpha_{(A}\alpha_{B)}$ with $\widehat{\alpha}_A = \alpha_A$

Similarly, as mentioned before, the spinor equivalent of a symmetric tensor and traceless rank-2 tensor is totally symmetric. In such case, one can write the corresponding spinor as

$$C_{ABCD} = \left(\Psi_0\iota^4 + 4\Psi_1\iota^3 o + 6\Psi_2\iota^2 o^2 + 4\Psi_3\iota o^3 + \Psi_4 o^4\right)_{(ABCD)}, \tag{3.120}$$

and the algebraic classification [154] is found in Table 3.2. The curvature scalars Ψ_i are the analogous of the 3+1 d Newman–Penrose (NP) scalars [155]. Like in 4d the symmetric tensors can be classified by the multiplicity of their principal spinors, however in 3d a further classification can be obtained, since when two principal spinors, α and β, are not proportional to each other, α can either be proportional to the conjugate spinor of β, $\widehat{\beta}$, (whose components in

Lorentzian signature are $\widehat{\beta}_A = (\beta_A)^*$) or not. In the rest of this thesis we will focus on type N spacetimes. One can in principle look for double copy relations in type D spacetimes. However, it was shown in [156] that all type D solutions of TMG are equivalent to the biaxially squashed Anti-de Sitter space which is not an asymptotically flat space, therefore it is not obvious what should be the corresponding single copy solution in TMYM and on which background it should be defined.

Table 3.2: Algebraic classification of a rank 2, totally symmetric, traceless tensor in 2+1 d [154]. If we restrict ourselves to real principal spinors, $\alpha_A, \beta_A, \gamma_A, \delta_A$, we have the same Petrov types as in the standard 3+1 classification. Note that the spinor dyad (ι, o) will change under Lorentz transformations, which in turn induces a change on the NP scalars, Ψ_i, so the normalization changes under Lorentz transformations.

Type	**Normalization**	**Principal spinors**
I	$\Psi_1 = \Psi_3 = 0, \Psi_0 = \Psi_4, 3\Psi_2/\Psi_0 < -1$	$\Phi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$ with $\widehat{\alpha} = \alpha, \widehat{\beta} = \beta, \widehat{\gamma} = \gamma, \widehat{\delta} = \delta$
IZ	$\Psi_1 = \Psi_3 = 0, \Psi_0 = -\Psi_4 \neq 0$	$\Phi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\widehat{\gamma}_{D)}$ with $\widehat{\alpha} = \alpha, \widehat{\beta} = \beta$
IZZ	$\Psi_1 = \Psi_3 = 0, \Psi_0 = \Psi_4, 3\Psi_2/\Psi_0 > 1$	$\Phi_{ABCD} = \pm\alpha_{(A}\widehat{\alpha}_B\beta_C\widehat{\beta}_{D)}$
II	$\Psi_1 = \Psi_3 = \Psi_4 = 0, \Psi_2/\Psi_0 < 0$	$\Phi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}$ with $\widehat{\alpha} = \alpha, \widehat{\beta} = \beta, \widehat{\gamma} = \gamma$
IIZ	$\Psi_1 = \Psi_3 = \Psi_4 = 0, \Psi_2/\Psi_0 > 0$	$\Phi_{ABCD} = \pm\alpha_{(A}\alpha_B\beta_C\widehat{\beta}_{D)}$ with $\widehat{\alpha} = \alpha$
D	$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \Psi_2 \neq 0$	$\Phi_{ABCD} = \pm\alpha_{(A}\alpha_B\beta_C\beta_{D)}$ with $\widehat{\alpha} = \alpha, \widehat{\beta} = \beta$
DZ	$\Psi_1 = \Psi_3 = 0, \Psi_0 = \Psi_4 = 3\Psi_2 \neq 0$	$\Phi_{ABCD} = \pm\alpha_{(A}\alpha_B\widehat{\alpha}_C\widehat{\alpha}_{D)}$
III	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0, \Psi_3 = 1$	$\Phi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}$ with $\widehat{\alpha} = \alpha, \widehat{\beta} = \beta$
N	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \Psi_4 = \pm 1$	$\Phi_{ABCD} = \pm\alpha_{(A}\alpha_B\alpha_C\alpha_{D)}$ with $\widehat{\alpha} = \alpha$
O	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0,$	$\Phi_{ABCD} = 0$

3.8 Type N solutions

We will consider Cotton double copy for Type N solutions of TMG in Minkowski and Anti-de Sitter (AdS) backgrounds. For AdS we will work in Poincare coordinates in which the AdS metric is written as

$$ds^2_{AdS} = \frac{L^2}{y^2}\left(-2dudv + dy^2\right) \ , \tag{3.121}$$

where L is the AdS radius.

For Type N solutions, which encode transverse radiation, the Cotton spinor and field strength spinor can be written as

$$C_{ABCD} = \Psi_4 o_A o_B o_C o_D \ , \quad f_{AB} = \Phi_2 o_A o_B \ , \tag{3.122}$$

where ψ_4 and Φ_2 are Newman-Penrose (NP) scalars. In this case, the double copy can simply be expressed as

$$\Psi_4 = \frac{m\Phi_2^2}{2S} \ . \tag{3.123}$$

We will now prove that the Cotton double copy holds for type N spacetimes in curved backgrounds by deriving the equation of motion of the zeroth copy S.

We start by substituting (3.122) into (3.113) and (3.107), and contracting the equations with ι and o to get:

$$o_A \nabla^A{}_C \log \Psi_4 + 4 o_A \iota^B \nabla^A{}_C o_B - \iota_A o^B \nabla^A{}_C o_B = \frac{m}{\sqrt{2}} o_c \ , \tag{3.124}$$

$$o_A \nabla^A{}_C \log \Phi_2 + 2 o_A \iota^B \nabla^A{}_C o_B - \iota_A o^B \nabla^A{}_C o_B = \frac{m}{\sqrt{2}} o_c \ , \tag{3.125}$$

$$o^B o_C \nabla^{CA} o_B = 0 \ . \tag{3.126}$$

From the Cotton double copy in (3.123), together with Eqs. (3.124) and (3.125), we find

$$o_A \nabla^A{}_C \log S - \iota_A o^B \nabla^A{}_C o_B = -\frac{m}{\sqrt{2}} o_c \ . \tag{3.127}$$

To show that S satisfies the Klein-Gordon equation with a non-minimal coupling term first we write $\nabla_\mu \nabla^\mu S$ as

$$-\nabla_{AB} \nabla^{AB} S = -\epsilon_{AC} \nabla^C{}_B \nabla^{AB} S = 2\iota_C o_A \nabla^C{}_B \nabla^{AB} S. \tag{3.128}$$

Then, one can use the Leibniz rule and (3.127) to eliminate the derivatives of S:

$$\begin{aligned} 2\iota_C o_A \nabla^C{}_B \nabla^{AB} S = &-2\iota_C \nabla^C{}_B o_A \nabla^{AB} S \\ &+ 2\iota_C \nabla^C{}_B \left(S \iota_A o^D \nabla^{AB} o_D - \frac{m}{\sqrt{2}} S o^B \right) . \end{aligned} \tag{3.129}$$

Expanding (3.129) and using (3.127) to eliminate ∇S terms we get:

$$
\begin{aligned}
&- \nabla_{AB}\nabla^{AB}S = S\Big(2\iota_C\nabla^C{}_B\iota_A o^D\nabla^{AB}o_D \\
&+ 2\iota_C\iota_A\nabla^C{}_B o^D\nabla^{AB}o_D + 2\iota_D\iota_A\nabla^A{}_C o^D\iota_E o^F\nabla^{EC}o_F \\
&- m\sqrt{2}\left(\iota_C\nabla^C{}_B o^B + \iota_D\iota_A\nabla^A{}_C o^D o^C - \iota_C\iota_E o^F\nabla^{EC}o_F\right) \\
&+ m^2 + 2\iota_C\iota_A o^D\nabla^C{}_B\nabla^{AB}o_D\Big) \ .
\end{aligned}
\tag{3.130}
$$

The first three terms as well as the terms linear in m add up to zero by (3.126). The term with the second derivative of o can be related to curvature spinors by the following relation [152]:

$$
\nabla_{D(A}\nabla_{B)}{}^D o_C = \frac{1}{2}\Phi_{ABCD}o^D + \frac{1}{24}R\left(\epsilon_{AC}o_B + \epsilon_{BC}o_A\right), \tag{3.131}
$$

where Φ_{ABCD} is the spinor equivalent of the traceless Ricci tensor which is proportional C_{ABCD}, see footnote 5. By substituting (3.122), we see that the term proportional to Φ_{ABCD} does not contribute. Therefore we find that

$$
2\iota_C\iota_A o^D\nabla^C{}_B\nabla^{AB}o_D = \frac{1}{6}R \ .
$$

Finally, substituting everything into (3.130) we get

$$
-\nabla_{AB}\nabla^{AB}S = \Box S = \left(m^2 + \frac{1}{6}R\right)S. \tag{3.132}
$$

This proves that the Cotton double copy is satisfied for Type N solutions with the zeroth copy given by a linearized massive bi-adjoint scalar with a non-minimal coupling. Note that we obtained the same non-minimal coupling as in the 4d zeroth copy [56, 55, 34], but in 3d it does not give a conformally invariant equation. We will now show explicit examples of Type N spacetimes where the Cotton double copy holds.

3.8.1 pp-waves

We analyze the double copy relation for plane-fronted waves with parallel propagation (pp-waves). For TMG, any solution that admits a null Killing vector, well-defined through all space, is a pp-wave solution [6] [158]. In flat space the metric of pp-waves can always be written as [156]

$$\mathrm{d}s^2 = \mathrm{d}y^2 - 2\ \mathrm{d}u\ \mathrm{d}v + \mathrm{e}^{-my} f(u)\mathrm{d}u^2, \tag{3.133}$$

where u, v are lightcone coordinates. Now the pp-wave metric in AdS space reads

$$\mathrm{d}s^2 = \mathrm{d}y^2 - 2\mathrm{e}^{2\frac{y}{L}}\mathrm{d}u\ \mathrm{d}v + \mathrm{e}^{\frac{(1-mL)}{L}y} f(u)\mathrm{d}u^2, \tag{3.134}$$

Note that we can obtain the dS solution by taking $L \to iL$. On the TMYM side, we can write the pp-wave solution as

$$A^a = c^a \mathrm{e}^{-my} g(u) du, \tag{3.135}$$

for the Minkowski, AdS, and dS cases. In Table 3.3 we show the NP scalars for the corresponding pp-waves in Minkowski and AdS. One can easily see that the scalar S, which is computed using the Cotton double copy in Eq. (3.123) satisfies

$$(\nabla^2 - m^2 - \frac{R}{6})S(u, y) = (\partial_y^2 - m^2 + \frac{1}{L^2})S = 0\ . \tag{3.136}$$

	Ψ_4	Φ_2	S
Mink.	$\frac{m^3}{4}\mathrm{e}^{-my} f(u)$	$-\frac{m}{\sqrt{2}}\mathrm{e}^{-my} g(u)$	$\frac{g(u)^2}{f(u)}\mathrm{e}^{-my}$
AdS	$\frac{m^3}{4}\mathrm{e}^{-(\frac{3}{L}+m)y} f(u)$	$-\frac{m}{\sqrt{2}}\mathrm{e}^{-(\frac{2}{L}+m)y} g(u)$	$\frac{g(u)^2}{f(u)}\mathrm{e}^{-(\frac{1}{L}-m)y}$

Table 3.3: In this table we show the NP-scalars for the Cotton spinor and the dual field strength spinor. We also show the scalar S constructed from the Cotton double copy in Eq. (3.123).

[6]The nomenclature of pp-waves for the non-zero cosmological constant case can be misleading since the null Killing vector is not covariantly conserved [157].

3.8.2 Shock waves and gyratons

Minkowski

We now consider solutions with a source corresponding to a fast moving particle whose stress tensor is traceless and is given by

$$T_{\mu\nu} = \left(Ek_\mu k_\nu + \sigma k_{(\mu}\epsilon_{\nu)}^{\alpha\beta}k_\alpha\partial_\beta\right)\delta(u)\delta(y) \ , \tag{3.137}$$

where the null vector k^μ is defined as $k_\mu dx^\mu = du$, E is the energy of the source particle and σ is its classical spin. Note that this source can be thought of as a boosted gravitational anyon. If the particle has no classical spin ($\sigma = 0$) then it generates shockwaves; otherwise, the solutions are dubbed gyratons. In flat space, both of these solutions have a metric of the form [7]

$$\mathrm{d}s^2 = \mathrm{d}y^2 - 2\ \mathrm{d}u\ \mathrm{d}v + \kappa F(u,y)\mathrm{d}u^2. \tag{3.138}$$

For these solutions, we have that the only non-zero NP Cotton scalar is

$$\Psi_4 = -\frac{1}{4}\partial_y^3 F(u,y) \ , \tag{3.139}$$

where F satisfies the following equation of motion

$$\partial_y^3 F(u,y) + m\partial_y^2 F(u,y) = \kappa m\delta(u)\left(E\delta(y) - \sigma\delta'(y)\right) \ . \tag{3.140}$$

On the gauge theory side, we will also consider a boosted spinning source whose current is given by

$$J_\mu = \left(Qk_\mu + Q'\epsilon_\mu^{\alpha\beta}k_\alpha\partial_\beta\right)\delta(u)\delta(y) \ , \tag{3.141}$$

[7]Note that the gyraton metric is generically written as $ds^2 = -2dudv + dy^2 + \kappa F(u,y)du^2 + 2\kappa\alpha(u,y)dudy$, where the cross term proportional to $\alpha(u,y)$ allows us to see the rotation explicitly [142]. Here we have chosen a gauge where $\alpha = 0$.

where Q is the electric charge and Q' contributes, together with Q, to the magnetic flux. We consider the following gauge field

$$A^a = c^a G(u,y)du, \tag{3.142}$$

which linearises the TMYM equations of motion and gives only one non-vanishing component of field strength $F_{uy} = -\partial_y G(u,y)$. Hence the only non-zero NP field strength scalar is

$$\Phi_2 = \frac{1}{\sqrt{2}}\partial_y G(u,y) \ , \tag{3.143}$$

where G satisfies

$$\partial_y^2 G(u,y) + m\partial_y G(u,y) = g\delta(u)\left(Q\delta(y) - Q'\delta'(y)\right) \ . \tag{3.144}$$

Then the scalar S in the Cotton double copy, Eq. (3.114), is given as

$$S = -m\frac{(\partial_y G(u,y))^2}{\partial_y^3 F(u,y)}. \tag{3.145}$$

Equations (3.140) and (3.144) imply that outside the sources the following is true:

$$(\nabla^2 - m^2)S(u,y) = 0. \tag{3.146}$$

To see the double copy for an explicit gyraton or shock wave solution, we need to pick boundary conditions for the metric. As realised in [138], we cannot have the same coordinate chart on both sides of the shockwave. In section 3.4.1 we showed that a useful prescription to observe the double copy relation is to a consider boundary conditions where the metric is flat for $y < 0$ and Cartesian for $y > 0$ [137]. Then we can solve (3.140) imposing these boundary conditions:

$$F(u,y) = \frac{\kappa}{m}(E + m\sigma)e^{-my}\delta(u)\theta(y) + \frac{\kappa}{m}(E + m\sigma - Emy)\delta(u)\theta(-y) \ . \tag{3.147}$$

Choosing the analogue boundary condition in TMYM, namely $F_{\mu\nu} = 0$ for $y < 0$ and $\lim_{y\to\infty} A^\mu =$

0, leads to

$$G(u,y) = g\frac{Q + mQ'}{m}\delta(u)\left(\theta(y)e^{-my}\right) + g\frac{Q}{m}\delta(u)\theta(-y). \tag{3.148}$$

On the $y < 0$ side of the gyraton, the double copy is trivial since $\Psi_4 = \Phi_2 = 0$. On the other hand, in the $y > 0$ side the NP scalars are

$$\Phi_2 = -\frac{g(Q + mQ')}{\sqrt{2}}e^{-my}\delta(u), \quad \Psi_4 = \frac{\kappa(E + m\sigma)m^2}{4}e^{-my}\delta(u). \tag{3.149}$$

They are proportional to those of the flat space pp-waves in Table 3.3 times $\delta(u)$. The analog of gyratons in the zeroth copy satisfies:

$$(\partial_y^2 - m^2)S(u,y) = (\lambda + \lambda'\partial_y)\delta(y)\delta(u), \tag{3.150}$$

which is solved by (with the boundary conditions similar as before)

$$S(u,y) = -\frac{(\lambda - \lambda' m)}{2m}e^{-my}\delta(u)\theta(y) - \frac{(\lambda + \lambda' m)}{2m}e^{my}\delta(u)\theta(-y), \tag{3.151}$$

We see that making the replacement $\frac{gm(Q+mQ')^2}{2(\lambda-\lambda' m)} \to \frac{\kappa m^2}{4}(E + m\sigma)$ leads to the double copy relation in Eq. (3.123). Note that the shockwave solutions can be obtained by setting $\sigma = 0$ and $Q' = 0$. We then see that the gyraton NP scalar is obtained from the shockwave one by the shift $E \to E\left(1 + m\frac{\sigma}{E}\right)$ in TMG and $Q \to Q(1 + m\frac{Q'}{Q})$ in TMYM, which arise from spin deformations of on-shell 3-point amplitudes [143] and was originally found for gravitational anyons [92].

Anti-de Sitter space

We proceed to consider gyraton solutions of TMG and TMYM in a three dimensional AdS background (the background metric is defined in (3.121)). Just like gravitational shockwaves in AdS, the gyraton solution can be written in Poincare coordinates as

$$ds^2 = \frac{L^2}{y^2}\left(-2dudv + dy^2 + \delta(u)F(y)du^2\right) \ , \tag{3.152}$$

where F satisfies:

$$\frac{y}{L}F''' + mF'' - m\frac{F'}{y} = \kappa m\left(E\frac{L}{y}\delta(y-y_0) - \sigma\delta'(y-y_0)\right) \ , \tag{3.153}$$

where $y_0 \neq 0$ is the location of the source in the bulk and we will assume $mL > 1$. As before, we need to fix the boundary conditions to find the explicit solution. We choose to have the same boundary conditions as in the Minkowski case in the flat space limit. This is equivalent to imposing Brown-Henneaux boundary conditions and requiring a regular solution in the bulk. The explicit solution with these boundary conditions is

$$\begin{aligned} F(y) = &-\frac{\kappa L^2 mE}{2\left(1-(Lm)^2\right)} \\ &\left[2\left(1+\frac{\sigma}{E}\left((1+mL)/L\right)\right)\left(\frac{y}{y_0}\right)^{1-Lm}\theta\left(y-y_0\right)\right. \\ &+\left((1-Lm)\left(\frac{y}{y_0}\right)^2+\left(1+\frac{2\sigma}{EL}\right)(1+Lm)\right) \\ &\left.\theta\left(-y+y_0\right)\right] . \end{aligned} \tag{3.154}$$

On the non-trivial side of the gyraton solution, $y > y_0$, we have that the only non-zero Cotton NP scalar is

$$\begin{aligned} \Psi_4 = -\frac{1}{2}\frac{y}{L}\delta(u)F'''(y) = \frac{\kappa}{2}E\left(1+\frac{\sigma}{E}\left((1+mL)/L\right)\right) \\ L^2m^2\left(\frac{y}{y_0}\right)^{-1-Lm}\delta(u) \ . \end{aligned} \tag{3.155}$$

On the other hand, the linearised gyraton solution for TMYM in an AdS background is given by

$$A^a = c^a\delta(u)G(y)du, \tag{3.156}$$

where the function G satisfies

$$\begin{aligned}&\frac{y^4}{L^4}\left(G''+\frac{1+Lm}{y}G'\right)\\&=\frac{y^3}{L^3}g\left(Q\delta(y-y_0)-Q'\frac{y}{L}\delta'(y-y0)\right)\ .\end{aligned}\tag{3.157}$$

The explicit gyraton solution with boundary condition analogue to the gravitational case, namely $F_{\mu\nu}=0$ for $y<y_0$ and $\lim_{y\to\infty}A^\mu=0$, is given by

$$\begin{aligned}G(y)=&-\frac{gQ}{m}\\&\left[\left(1+\frac{Q'}{QL}(1+mL)\right)\left(\frac{y}{y_0}\right)^{-Lm}\theta\left(y-y_0\right)\right.\\&\left.+\left(1+\frac{Q'}{QL}\right)\left(\frac{y_0}{L}\right)^3\theta\left(-y+y_0\right)\right]\ .\end{aligned}\tag{3.158}$$

Thus we have that for $y>y_0$ the only non-zero dual field strength NP scalar is

$$\begin{aligned}\Phi_2&=2\delta(u)\frac{y}{L}f'(y)\\&=2gQ(1+\frac{Q'}{Q}((1+mL)/L))\delta(u)\left(\frac{y}{y_0}\right)^{-Lm}\ .\end{aligned}\tag{3.159}$$

Lastly, we consider the linearised biadjoint scalar, $S^{a\tilde{a}}=c^a c^{\tilde{a}}S$, living in an AdS background with a non-minimal coupling as in Eq. (3.132) and sourced by $(\lambda+\lambda'\frac{y}{L}\frac{\partial}{\partial_y})\delta(y-y_0)\delta(u)$ The gyraton solution is now given by

$$\begin{aligned}S=-\frac{\lambda L}{2my_0}&\left[\left(1-m\frac{\lambda'}{\lambda}\right)\left(\frac{y}{y_0}\right)^{1-Lm}\theta\left(y-y_0\right)\right.\\&\left.+\left(1+m\frac{\lambda'}{\lambda}\right)\left(\frac{y}{y_0}\right)^{1+Lm}\theta\left(-y+y_0\right)\right]\delta(u)\ ,\end{aligned}\tag{3.160}$$

where we chose boundary conditions by requiring that the field vanishes deep in the bulk and as we approach the AdS boundary. We note that this solution corresponds to the scalar that arises from Eq. (3.123), which shows that the Cotton double copy is satisfied for AdS shock waves as expected. Again the shockwave solutions can be obtained by setting σ and Q' to zero. In a similar manner to the flat space case, one can consider shifts of the charge and energy to obtain the gyraton double copy from shockwaves. In this case the shifts are given $E \to E\left(1 + \frac{\sigma}{E}\left((1+mL)/L\right)\right)$ for the TMG case and $Q \to Q(1 + \frac{Q'}{Q}\left((1+mL)/L\right))$ in TMYM. In future work, we will explore whether the origin of these shifts can be traced down to spin deformations of three-point correlators.

3.9 Discussion

We have analyzed the high-energy limit of topologically massive theories from two different perspectives. First, by looking at the scattering amplitudes in the eikonal limit; and second, by looking at the shock wave solutions for both a spinless and a spinning source. In the former analysis, we found that to construct the double copy of the eikonal amplitudes, we need information outside of the eikonal limit at tree-level. This is in stark difference with the massless $d \geq 4$ case where a simpler double copy relation arises. In the latter, we obtained a double copy relation which is only manifest for a specific choice of boundary conditions. Along the way, we showed how the eikonal amplitude is related to the classical shockwave solutions and how the choice of $i\epsilon$ prescription required to regulate the phase shift corresponds to the choice of boundary conditions of the topologically massive field. This allowed us to choose the appropriate prescriptions to make the coordinate space double copy clear on the non-trivial (where the curvature and field strength are non-zero) side of the shockwave. This suggested that a cleaner double copy relation might arise when looking at the curvature and field strength, as in the $4d$ Weyl double copy, instead of looking directly at the fields, as in the Kerr-Schild double copy case. We have constructed a double copy relation for topologically massive theories that gives the Cotton spinor as the square of the dual field strength spinor in curved spacetime backgrounds. This generalises the 4d Weyl double copy to 3d spacetimes. We have focused on Type N spacetimes, which correspond to radiative solutions, gave a proof of the Cotton double copy for gravitational waves and showed explicit examples. Other examples that we didn't look at explicitly can be found in [157, 159]. It would be interesting to understand whether the double copy holds for Type D solutions, which describe fields around isolated objects such as black holes. Previous analysis studying the scattering of massive fields, which represent isolated objects, through topologically massive mediators [13, 143, 117] have shown that this is not straightforward, and further investigations should clarify this intriguing case.

Several open questions remain when it comes to fully understanding the massive double copy. Regarding scattering amplitudes, it has not been shown if the double copy relation holds for six and higher-point amplitudes or for loop corrections. In the special case of topologically massive theories, a complete understanding of the situation when including couplings to generic matter is lacking. Some progress has been made in [113, 143, 117] and we have contributed to clarifying the situation in the high-energy limit in this work. Nevertheless, a broader exploration for more generic sources for both classical solutions and scattering amplitudes is still missing.

Chapter 4

Conclusion

4.1 Summary of Thesis Achievements

In this work, we have explored the possibility of extending the double copy procedure to massive gauge and gravity theories. First we checked if massive Yang-Mills theory can be related to dRGT massive gravity theories which can be expected from the decoupling limits of the two theories (namely non-linear sigma model and special Galileon) that are known to be related by double copy. However, while double copy of 3 and 4 point massive Yang-Mills amplitudes gives massive gravity amplitudes the relation fails at higher points, due to the appearance of spurious poles. The decoupling limit argument is found to be incorrect due to an interesting feature of massive double copy - taking the decoupling limit and doing double copy do not commute and the decoupling limit of massive gravity theory obtained from double copy of 4 point massive Yang-Mills amplitudes is not special Galileon. Even though the relation between these particular theories was found to fail it led to further exploration of what massive theories can be double copied without having spurious poles. One obvious example we explored is Kaluza-Klein theories. We have tried to find new massive theories that can be double copied by deforming the couplings in a Kaluza-Klein theory obtained by compatifying a 5d Yang-Mills theory on a circle. However, we were not be able to find any new theories this way, since imposing the necessary conditions for double copy to be physical (spectral conditions and BCJ relations) lead back to the coupling of the original Kaluza-Klein theory. While this result does not prove in general that the only massive theory compatible with double copy in 4d is Kaluza-Klein theory, the vast amount of freedom that is available in choosing different coupling for different fields suggests that massive theories that can be double copied are very strongly constrained.

The situation in 3d was found to be more interesting. We found a new mechanism in 3d of how spurious poles can cancel in 5 point amplitudes which is completely different from how it works in massless and Kaluza-Klein theories (in particular the number and the form of BCJ relations is different). Then we found an example of this mechanism - double copy relation between topological massive gauge and gravity theories. This relation was only explicitly checked for 3, 4 and 5 point amplitudes but the further work on eikonal amplitudes and classical double copy suggests that these topologically massive theories could be related at all orders. This study of classical double copy between these theories revealed some interesting features. First of all the eikonal amplitudes themselves are not directly related since the double copy of the eikonal amplitudes needs information outside of the eikonal limit at tree-level. However, once the classical solutions are obtained from these eikonal amplitudes, the double copy relation is manifest for a specific choice of boundary conditions. By exploring relations between type N solutions we found that there is a cleaner double copy relation directly between the curvature and the field strength tensor. This is a 3d equivalent of Weyl double copy - Cotton double copy. We proved that this relation holds for type N spacetimes and explored several explicit examples.

4.2 Future Work

There are a lot of future directions one can take to get a better understanding of massive double copy. It is still not clear whether the spurious pole problem in double copying massive Yang-Mills could be fixed by adding extra fields or irrelevant operators to the action.

Also it would be interesting to continue looking for new massive gauge theories in 4d that can be double copied. Following our work on Kaluza-Klein theories it would be interesting to see if the freedom in deforming couplings is eliminated at all orders and if more general deformations of Kaluza-Klein theory (adding operators that do not appear in the compactified theory) leads to the same result.

A lot of questions about massive double copy in 3d still remain unanswered. It is not clear if there are any other examples of the new spurious pole cancellation mechanism. As for topological massive theories it would be interesting to explore relations between more general classical solutions, higher point tree and loop amplitudes to understand if these two theories are really related at all orders.

If double copy relation between topologically massive gauge and gravity theories holds at all orders it would be interesting to understand its origin. For example the double copy relaiton between massless Yang-Mills and gravity can be understood from string theory as low energy limit of KLT relations between open and closed string amplitudes. Is there a string theory origin for the relations between these topologically massive theories? On the other hand, recently there has been done a lot of work in understanding classical Weyl double copy relations from twistor space [36, 35]. It would be interesting to check if its 3d equivalent - Cotton double copy could also be explained similarly using mini-twistor space suitable for 3d [160, 161].

The existence of a 3D massive double copy opens a path for a profuse amount of questions as we have discussed above. We hope that this exploration will stimulate future explorations to advance our understanding of the applicability of the double copy.

Appendix A

Contact Terms in the Double Copy of Massive YM

Below are the various contact terms needed in (2.39) to reproduce the desired quartic interactions. All terms are written in a covariant form, with the understanding that they enter the action with a $\sqrt{-g}$ prefactor.

$$\mathcal{L}^{(4)}_{\phi\phi\phi\phi} = \frac{11}{3456}\,\frac{\kappa^2}{m^4}\phi\left([\Phi]^3 - 3[\Phi][\Phi^2] + 2[\Phi^3]\right) + \frac{21}{128}\,\kappa^2\nabla^\mu\phi\nabla_\mu\phi\;\phi\;\phi + \frac{-1}{96}\,\frac{\kappa^2}{m^2}\nabla^\rho\phi\nabla^\sigma\phi\Phi_{\rho\sigma}\;\phi \tag{A.1}$$

where we have used $\Phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$.

$$\mathcal{L}^{(4)}_{AAAA} = \frac{-1}{512}\,\frac{\kappa^2}{m^2}(F^{\mu\nu}F_{\mu\nu})^2 + \frac{3}{256}\,\frac{\kappa^2}{m^2}F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu} \tag{A.2}$$

$$\mathcal{L}^{(4)}_{AAhh} = \frac{-3}{16}\,F_{\mu\nu}F_{\rho\sigma}\mathcal{K}^{\mu\rho}\mathcal{K}^{\nu\sigma} + \frac{-1}{4}\,F^{\mu\nu}F_{\mu\sigma}\mathcal{K}_{\nu\rho}\mathcal{K}^{\rho\sigma} + \frac{-1}{16}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\nu'\alpha'\beta'}F^\mu_{\mu'}F^\nu_{\nu'}\mathcal{K}^\alpha_{\alpha'}\mathcal{K}^\beta_{\beta'} \tag{A.3}$$

$$\begin{aligned}\mathcal{L}^{(4)}_{hh\phi\phi} =&\frac{7}{48}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\nu'\alpha'\beta}\Phi^{\mu}_{\mu'}\,\mathcal{K}^{\nu}_{\nu'}\mathcal{K}^{\alpha}_{\alpha'}\phi + \frac{3}{8}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\nu'\alpha'\beta}\nabla^{\mu}\phi\nabla_{\mu'}\mathcal{K}^{\nu}_{\nu'}\mathcal{K}^{\alpha}_{\alpha'}\phi\\
&+\frac{-17}{48}\,m^2\,\phi\,\phi\,\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} + \frac{1}{24}\,\frac{1}{m^2}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\nu'\alpha'\beta'}\nabla^{\mu}\phi\,\nabla_{\mu'}\phi\,\nabla^{\nu}\mathcal{K}^{\alpha}_{\alpha'}\nabla_{\nu'}\mathcal{K}^{\beta}_{\beta'}\\
&+\frac{1}{12}\,\frac{1}{m^2}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\mu\alpha'\beta'}\nabla^{\rho}\phi\,\nabla^{\nu}\phi\,\nabla_{\mu'}\mathcal{K}^{\alpha}_{\alpha'}\nabla_{[\rho}\mathcal{K}^{\beta}_{\beta']} + \frac{1}{48}\,\frac{1}{m^2}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\nu'\alpha'\beta'}\Phi^{\mu}_{\mu'}\Phi^{\nu}_{\nu'}\mathcal{K}^{\alpha}_{\alpha'}\mathcal{K}^{\beta}_{\beta'}\end{aligned} \tag{A.4}$$

$$\begin{aligned}\mathcal{L}^{(4)}_{h\phi\phi\phi} =&\frac{-1}{144\sqrt{3}}\,\frac{\kappa}{m^4}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\nu'\alpha'\beta'}\mathcal{K}^{\mu}_{\mu'}\Phi^{\nu}_{\nu'}\Phi^{\alpha}_{\alpha'}\Phi^{\beta}_{\beta'} + \frac{-19}{48\sqrt{3}}\,\kappa\,\mathcal{K}^{\mu\nu}\nabla_{\mu}\phi\nabla_{\nu}\phi\,\phi\\
&+\frac{11}{16\sqrt{3}}\,\frac{\kappa}{m^2}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\nu'\alpha'\beta}\mathcal{K}^{\mu}_{\mu'}\Phi^{\nu}_{\nu'}\Phi^{\alpha}_{\alpha'}\phi\end{aligned} \tag{A.5}$$

$$\begin{aligned}\mathcal{L}^{(4)}_{hhh\phi} =&\frac{1}{12\sqrt{3}}\,\frac{1}{\kappa}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu'\nu'\alpha'\beta'}\mathcal{K}^{\mu}_{\mu'}\mathcal{K}^{\nu}_{\nu'}\mathcal{K}^{\alpha}_{\alpha'}\Phi^{\beta}_{\beta'} + \frac{-2}{\sqrt{3}}\,\frac{1}{\kappa}\,\nabla^{[\beta}\mathcal{K}^{\nu]\alpha}\nabla_{[\beta}\mathcal{K}_{\mu]\alpha}\mathcal{K}^{\mu}_{\nu}\phi\\
&+\frac{8}{\sqrt{3}}\frac{1}{\kappa}R^{\rho\sigma}_{\mu\nu}\mathcal{K}^{\mu}_{\rho}\mathcal{K}^{\nu}_{\sigma}\phi + \frac{-11}{12\sqrt{3}}\,\frac{m^2}{\kappa}\phi\left([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]\right)\end{aligned} \tag{A.6}$$

with $\nabla_{[\mu}A_{\nu]\rho} = \frac{1}{2}(\nabla_{\mu}A_{\nu\rho} - \nabla_{\nu}A_{\mu\rho})$

$$\begin{aligned}\mathcal{L}^{(4)}_{AhA\phi} =&\frac{1}{8\sqrt{3}}\,\kappa m^2\,A^{\mu}A^{\nu}\mathcal{K}_{\mu\nu}\phi + \frac{1}{16\sqrt{3}}\,\frac{\kappa}{m^2}\,F^{\mu\nu}F^{\rho\sigma}\mathcal{K}_{\nu\rho}\Phi_{\mu\sigma} + \frac{-1}{16\sqrt{3}}\,\frac{\kappa}{m^2}\,\nabla^{\rho}F^{\mu\nu}\nabla_{\sigma}F_{\mu\nu}\mathcal{K}^{\sigma}_{\nu}\phi\\
&+\frac{-1}{4\sqrt{3}}\,\frac{\kappa}{m^2}\,\nabla^{\nu}F^{\mu\rho}F_{\mu\sigma}\nabla_{\rho}\mathcal{K}^{\sigma}_{\nu}\phi + \frac{-1}{8\sqrt{3}}\,\frac{\kappa}{m^2}\,\nabla^{\rho}F^{\nu}_{\sigma}\nabla_{\rho}F^{\mu\sigma}\mathcal{K}_{\mu\nu}\phi + \frac{-1}{8\sqrt{3}}\,\frac{\kappa}{m^2}\,F_{\mu\nu}F_{\rho\sigma}\nabla^{\mu}\nabla^{\sigma}\mathcal{K}^{\nu\rho}\phi\end{aligned} \tag{A.7}$$

$$\begin{aligned}\mathcal{L}^{(4)}_{A\phi A\phi} =&\frac{1}{384}\,\frac{\kappa^2}{m^4}\,F^{\mu\nu}F_{\mu\nu}\Phi^{\rho\sigma}\Phi_{\rho\sigma} + \frac{1}{64}\,\kappa^2\,F^{\mu\nu}F_{\mu\rho}\phi\phi + \frac{-1}{48}\,\frac{\kappa^2}{m^2}\,F^{\mu\nu}F^{\rho}_{\nu}\nabla_{\rho}\phi\nabla_{\mu}\phi\\
&+\frac{-11}{32}\,\kappa^2 m^2\,A^{\mu}A_{\mu}\phi\phi + \frac{1}{192}\,\kappa^2\,A^{\mu}A^{\nu}\nabla_{\mu}\phi\nabla_{\nu}\phi\\
&+\frac{-1}{192}\,\frac{\kappa^2}{m^4}\,\nabla^{\rho}F_{\mu\nu}\nabla^{\sigma}F^{\mu\nu}\nabla_{\rho}\phi\nabla_{\sigma}\phi + \frac{1}{128}\,\frac{\kappa^2}{m^2}\,\nabla_{\rho}F_{\mu\nu}\nabla^{\rho}F^{\mu\nu}\phi\phi\\
&+\frac{-1}{192}\,\frac{\kappa^2}{m^4}\,\nabla^{\rho}F^{\mu}_{\nu}\nabla_{\mu}F^{\sigma}_{\rho}\nabla^{\nu}\phi\nabla_{\sigma}\phi + \frac{1}{96}\,\frac{\kappa^2}{m^4}\,\nabla^{\rho}F^{\mu\nu}\nabla_{\sigma}F^{\sigma}_{\nu}\nabla_{\rho}\phi\nabla_{\mu}\phi\end{aligned} \tag{A.8}$$

Appendix B

Polarizations in 4d

The four momenta in the centre of mass frame with scattering angle θ and three momenta $p = \frac{1}{2}\sqrt{s - 4m^2}$ is defined as:

$$p^\mu = (\frac{\sqrt{s}}{2}, p\sin\theta, 0, p\cos\theta). \tag{B.1}$$

We define the polarization vectors in the helicity basis as follows:

$$\begin{aligned}
\epsilon^\mu_{\lambda=1} &= \frac{1}{\sqrt{2}}(0, -\cos\theta, -i, \sin\theta)\,, \\
\epsilon^\mu_{\lambda=-1} &= \frac{1}{\sqrt{2}}(0, \cos\theta, -i, -\sin\theta)\,, \\
\epsilon^\mu_{\lambda=0} &= \frac{1}{m}(p, E\sin\theta, 0, E\cos\theta)\,.
\end{aligned} \tag{B.2}$$

where θ is the scattering angle in the centre of mass frame and p the three-momentum. The polarizations clearly satisfy the transverse and completeness relations,$i.e$

$$\begin{aligned}
p_\mu \epsilon^\mu_\lambda &= 0\,, \\
\sum_{\lambda=1}^{3} \epsilon^\mu_\lambda (\epsilon^\nu_\lambda)^* &= \eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\,,
\end{aligned} \tag{B.3}$$

where $(\epsilon^{\mu}_{\lambda})^* = (-1)^{\lambda}\epsilon^{\mu}_{-\lambda}$. The polarization tensors for the spin-2 field with different helicities are constructed from the polarization vectors with appropriate Clebsch-Gordan (CG) coefficients as (we review the construction in detail in C),

$$\begin{aligned}
\epsilon^{\mu\nu}_{\lambda=\pm 2} &= \epsilon^{\mu}_{\pm}\epsilon^{\nu}_{\pm}\,, \\
\epsilon^{\mu\nu}_{\lambda=\pm 1} &= \frac{1}{\sqrt{2}}(\epsilon^{\mu}_{\pm}\epsilon^{\nu}_{0} + \epsilon^{\mu}_{0}\epsilon^{\nu}_{\pm})\,, \\
\epsilon^{\mu\nu}_{\lambda=0} &= \frac{1}{\sqrt{6}}(\epsilon^{\mu}_{+}\epsilon^{\nu}_{-} + \epsilon^{\mu}_{-}\epsilon^{\nu}_{+} + 2\epsilon^{\mu}_{0}\epsilon^{\nu}_{0})\,.
\end{aligned} \tag{B.4}$$

The polarization tensors satisfy the transverse, traceless and completeness relations

$$\begin{aligned}
& p_{\mu}\epsilon^{\mu\nu}_{\lambda} = 0\,, \epsilon^{\mu}_{\mu\lambda} = 0\,, \\
& \sum_{\lambda=-2}^{2} \epsilon^{\mu\nu}_{\lambda}(\epsilon^{\alpha\beta}_{\lambda})^* = \frac{1}{2}\left(G^{\mu\alpha}G^{\nu\beta} + G^{\mu\beta}G^{\nu\alpha} - \frac{2}{3}G^{\mu\nu}G^{\alpha\beta}\right),
\end{aligned} \tag{B.5}$$

where $G^{\mu\alpha} = \eta^{\mu\nu} + \frac{p^{\mu}p^{\nu}}{m^2}$.

Appendix C

Construction of gravity states from massive Yang-Mills

As mentioned in 2.1.1, from the tensor product of two massive spin-1 states we get a massive spin-2, a massive spin-1 and a massive spin-0 on the gravity side. In this section we review how the gravity on-shell states are constructed from such product, *i.e*, $|1, \lambda_1 > \otimes |1, \lambda_2 >$. The polarization tensor of the particle of spin J with helicity λ is given as,

$$\epsilon_{\mu\nu}^{J,\lambda} = \sum_{\lambda'\lambda''} C_{\lambda'\lambda''}^{J,\lambda} \epsilon_{\mu}^{\lambda'} \epsilon_{\nu}^{\lambda''}, \tag{C.1}$$

where $\lambda = \lambda' + \lambda''$. We start from the spin-0 state which is obtained from $|0, \lambda >= |1, \lambda' > \otimes |1, \lambda'' >$, with $\lambda = 0 = \lambda' + \lambda''$. This polarization state is obtained by considering the following:

$$\begin{aligned} \epsilon_{\mu\nu}^{(\phi)} \equiv \epsilon_{\mu\nu}^{0,0} =& \sum_{\lambda'\lambda''} C_{\lambda'\lambda''}^{0,0} \epsilon_{\mu}^{\lambda'} \epsilon_{\nu}^{\lambda''} \\ =& \frac{1}{\sqrt{3}} \left(\epsilon_{\mu}^{0} \epsilon_{\nu}^{0} - \epsilon_{\mu}^{+} \epsilon_{\nu}^{-} - \epsilon_{\mu}^{-} \epsilon_{\nu}^{+} \right) \end{aligned} \tag{C.2}$$

where $C^0_{\lambda'\lambda''}$ are the CG coefficients given in (C.7). By substituting (B.2), we can see that (C.2) can be expressed as:

$$\epsilon^{(\phi)}_{\mu\nu} = \frac{1}{\sqrt{3}}\left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right). \tag{C.3}$$

Hence, the factor of $\frac{1}{\sqrt{3}}$ in (2.20) which follows from the CG coefficient.

The spin-2 state is obtained from $|2,\lambda >= |1,\lambda' > \otimes |1,\lambda'' >$, with $-2 \leq \lambda \leq 2$.

$$\epsilon^{2,\lambda}_{\mu\nu} = \sum_{\lambda'\lambda''} C^{2,\lambda}_{\lambda'\lambda''}\epsilon^{\lambda'}_{\mu}\epsilon^{\lambda''}_{\nu}. \tag{C.4}$$

To give an explicit example, the helicity $\lambda = +2$ is,

$$\begin{aligned}
\epsilon^{2,+2}_{\mu\nu} &= \sum_{\lambda'\lambda''} C^{2,+2}_{\lambda'\lambda''}\epsilon^{\lambda'}_{\mu}\epsilon^{\lambda''}_{\nu}, \\
&= C^{2,+2}_{+1+1}\epsilon^{+1}_{\mu}\epsilon^{+1}_{\nu}, \\
&= 1 \times \epsilon^{+1}_{\mu}\epsilon^{+1}_{\nu}.
\end{aligned} \tag{C.5}$$

In this work we use the polarization states to be a superposition of different helicities and we do not focus on specific choices, for example for the graviton polarisation we have,

$$\epsilon^{(h)}_{\mu\nu} = \sum_{\lambda=-2}^{+2} \alpha_\lambda \epsilon^{2,\lambda}_{\mu\nu}. \tag{C.6}$$

$$\begin{aligned}
\text{Spin-2}: \quad & C^{2,2}_{++} = C^{2,-2}_{--} = 1, \\
& C^{2,1}_{0+} = C^{2,1}_{+0} = C^{2,-1}_{-0} = C^{2,-1}_{0-} = \frac{1}{\sqrt{2}}, \\
& C^{2,0}_{+-} = C^{2,0}_{-+} = \frac{1}{\sqrt{6}}, \quad C^{2,0}_{00} = \sqrt{\frac{2}{3}}, \\
\text{Spin-1}: \quad & C^{1,1}_{+0} = -C^{1,1}_{0+} = C^{1,-1}_{0-} = -C^{1,-1}_{-0} = \frac{1}{\sqrt{2}} \\
& C^{1,0}_{+-} = -C^{1,0}_{-+} = \frac{1}{\sqrt{2}}, \quad C^{1,0}_{00} = 0 \\
\text{Spin-0}: \quad & C^{0,0}_{+-} = C^{0,0}_{-+} = \frac{-1}{\sqrt{3}}, \quad C^{0,0}_{00} = \frac{1}{\sqrt{3}}
\end{aligned} \tag{C.7}$$

Appendix D

Dualization of the massive B field in 4d

We follow the dualization procedure explained in [162]. The Stückelberg action of free massive 2-form field, B, is

$$S = \int -\frac{1}{2} dB \wedge *dB - \frac{1}{2}(mB - d\lambda) \wedge *(mB - d\lambda), \tag{D.1}$$

where λ is a 1-form Stückelberg field which is needed to restore the gauge symmetry which acts on the fields as follows:

$$B \to B + d\Lambda,$$

$$\lambda \to \lambda + m\Lambda.$$

The first step in the dualization procedure is to rewrite the action in terms of field strengths, $H = dB$ and $G = mB - d\lambda$. To do that we need to impose Bianchi identities,

$$dH = 0, \tag{D.2}$$

$$dG - mdB = 0, \tag{D.3}$$

with Lagrange multipliers. We first do it for G:

$$S = \int -\frac{1}{2} dB \wedge *dB - \frac{1}{2} G \wedge *G + A \wedge d(G - mB), \tag{D.4}$$

where A is a 1-form Lagrange multiplier imposing (D.3). By integrating the last term by parts we can find the equation of motion for G to be

$$G = - * dA. \tag{D.5}$$

Substituting this back to the action and integrating by parts the last term we get

$$S = \int -\frac{1}{2} dB \wedge *dB - \frac{1}{2} dA \wedge *dA + A \wedge mdB. \tag{D.6}$$

Now we can replace dB by H and impose (D.2) with a scalar Lagrange multiplier, χ. This gives the following

$$S = \int -\frac{1}{2} H \wedge *H - \frac{1}{2} dA \wedge *dA + A \wedge mH + \chi dH. \tag{D.7}$$

Now again we integrate last term by parts and find the equation of motion for H to be

$$H = - * (mA - d\phi). \tag{D.8}$$

Substituting this back in the (D.7) gives the Stueckelberg action for massive spin-1 field, A, known as Proca action:

$$S = \int -\frac{1}{2} dA \wedge *dA - \frac{1}{2} (mdA - d\chi) \wedge *(mdA - d\chi), \tag{D.9}$$

where χ is now the Stückelberg scalar field. From (D.8) we can see that in unitary gauge, $\chi = 0$, the relation between the B and A fields is $dB = - * mA$ which in coordinate basis can be written as:

$$A_\mu = -\frac{1}{2m} \varepsilon_{\mu\nu\rho\sigma} \nabla^\nu B^{\rho\sigma}. \tag{D.10}$$

This means that the relationship between the polarization vector of A, $\epsilon^{(A)}$, and the polarization tensor of B, $\epsilon^{(B)}$, will be of the form:

$$\epsilon^{(A)}_{\mu} \propto \frac{i}{m} \varepsilon_{\mu\nu\rho\sigma} p^{\nu} \epsilon^{(B)\rho\sigma}, \tag{D.11}$$

where the overall constant can be found by requiring $\epsilon^{(A)}_{\mu}$ to be normalised (*i.e.* consistent with (B.3)). This relation can be inverted by multiplying both sides by the ε tensor and p, which using $p^2 = -m^2$ and imposing normalisation condition gives (2.19).

Appendix E

Double Copy of the 4-Point Scattering Amplitude in the Decoupling Limit

We take the Λ_3 decoupling limit,

$$m \to 0, \quad M_{pl} \to \infty, \quad \text{keeping } \Lambda_3 = (m^2 M_{pl})^{1/3} \text{ fixed,} \tag{E.1}$$

of the full scattering amplitude obtained from double copy with external states arbitrary superpositions of h and ϕ fields defined as: (setting the vectors to zero for simplicitly)

$$\begin{aligned}
\epsilon_{1\mu\nu} &= \alpha_{T1}\epsilon^{2,+2}_{\mu\nu}(p_1) + \alpha_{T2}\epsilon^{2,-2}_{\mu\nu}(p_1) + \alpha_{T3}\epsilon^{2,+1}_{\mu\nu}(p_1) + \alpha_{T4}\epsilon^{2,-1}_{\mu\nu}(p_1) + \alpha_{T5}\epsilon^{2,0}_{\mu\nu}(p_1) + \alpha_S\epsilon^{0,0}_{\mu\nu}(p_1),\\
\epsilon_{2\mu\nu} &= \beta_{T1}\epsilon^{2,+2}_{\mu\nu}(p_2) + \beta_{T2}\epsilon^{2,-2}_{\mu\nu}(p_2) + \beta_{T3}\epsilon^{2,+1}_{\mu\nu}(p_2) + \beta_{T4}\epsilon^{2,-1}_{\mu\nu}(p_2) + \beta_{T5}\epsilon^{2,0}_{\mu\nu}(p_2) + \beta_S\epsilon^{0,0}_{\mu\nu}(p_2),\\
\epsilon_{3\mu\nu} &= \gamma_{T1}\epsilon^{2,+2}_{\mu\nu}(p_3) + \gamma_{T2}\epsilon^{2,-2}_{\mu\nu}(p_3) + \gamma_{T3}\epsilon^{2,+1}_{\mu\nu}(p_3) + \gamma_{T4}\epsilon^{2,-1}_{\mu\nu}(p_3) + \gamma_{T5}\epsilon^{2,0}_{\mu\nu}(p_3) + \gamma_S\epsilon^{0,0}_{\mu\nu}(p_3),\\
\epsilon_{4\mu\nu} &= \sigma_{T1}\epsilon^{2,+2}_{\mu\nu}(p_4) + \sigma_{T2}\epsilon^{2,-2}_{\mu\nu}(p_4) + \sigma_{T3}\epsilon^{2,+1}_{\mu\nu}(p_4) + \sigma_{T4}\epsilon^{2,-1}_{\mu\nu}(p_4) + \sigma_{T5}\epsilon^{2,0}_{\mu\nu}(p_4) + \sigma_S\epsilon^{0,0}_{\mu\nu}(p_4).
\end{aligned} \tag{E.2}$$

This gives the following amplitude:

$$\begin{aligned}
M_4 \to i\frac{stu}{2304}\Big(&6\alpha_{T3}\beta_{T3}\gamma_S\sigma_S - 6\alpha_{T4}\beta_{T3}\gamma_S\sigma_S - 6\alpha_{T3}\beta_{T4}\gamma_S\sigma_S + 6\alpha_{T4}\beta_{T4}\gamma_S\sigma_S + 10\alpha_{T5}\beta_{T5}\gamma_S\sigma_S\\
&- 6\alpha_{T3}\beta_S\gamma_{T3}\sigma_S + 6\alpha_{T4}\beta_S\gamma_{T3}\sigma_S - 6\sqrt{2}\alpha_{T5}\beta_{T3}\gamma_{T3}\sigma_S - 6\sqrt{2}\alpha_{T3}\beta_{T5}\gamma_{T3}\sigma_S + 6\alpha_{T3}\beta_S\gamma_{T4}\sigma_S
\end{aligned}$$

$$
\begin{aligned}
&- 6\alpha_{T4}\beta_S\gamma_{T4}\sigma_S - 6\sqrt{2}\alpha_{T5}\beta_{T4}\gamma_{T4}\sigma_S - 6\sqrt{2}\alpha_{T4}\beta_{T5}\gamma_{T4}\sigma_S + 10\alpha_{T5}\beta_S\gamma_{T5}\sigma_S\\
&- 6\sqrt{2}\alpha_{T4}\beta_{T3}\gamma_{T5}\sigma_S - 6\sqrt{2}\alpha_{T3}\beta_{T4}\gamma_{T5}\sigma_S - 2\sqrt{2}\alpha_{T5}\beta_{T5}\gamma_{T5}\sigma_S + 11\alpha_{T5}\beta_S\gamma_S\sqrt{2}\sigma_S\\
&+ 6\alpha_{T5}\beta_{T4}\gamma_{T3}\sqrt{2}\sigma_S + 6\alpha_{T4}\beta_{T5}\gamma_{T3}\sqrt{2}\sigma_S + 6\alpha_{T5}\beta_{T3}\gamma_{T4}\sqrt{2}\sigma_S + 6\alpha_{T3}\beta_{T5}\gamma_{T4}\sqrt{2}\sigma_S\\
&+ 6\alpha_{T3}\beta_{T3}\gamma_{T5}\sqrt{2}\sigma_S + 6\alpha_{T4}\beta_{T4}\gamma_{T5}\sqrt{2}\sigma_S + 2\alpha_{T2}\beta_S\gamma_S\sqrt{3}\sigma_S + 4\alpha_{T5}\beta_{T5}\gamma_{T1}\sqrt{3}\sigma_S\\
&+ 4\alpha_{T5}\beta_{T5}\gamma_{T2}\sqrt{3}\sigma_S + 4\alpha_{T5}\beta_{T1}\gamma_{T5}\sqrt{3}\sigma_S + 4\alpha_{T5}\beta_{T2}\gamma_{T5}\sqrt{3}\sigma_S + 4\alpha_{T2}\beta_{T5}\gamma_{T5}\sqrt{3}\sigma_S\\
&+ 2\alpha_{T5}\beta_{T1}\gamma_S\sqrt{6}\sigma_S + 2\alpha_{T5}\beta_{T2}\gamma_S\sqrt{6}\sigma_S + 2\alpha_{T2}\beta_{T5}\gamma_S\sqrt{6}\sigma_S + 2\alpha_{T5}\beta_S\gamma_{T1}\sqrt{6}\sigma_S\\
&+ 2\alpha_{T5}\beta_S\gamma_{T2}\sqrt{6}\sigma_S + 2\alpha_{T2}\beta_S\gamma_{T5}\sqrt{6}\sigma_S - 6\alpha_{T3}\beta_S\gamma_S\sigma_{T3} + 6\alpha_{T4}\beta_S\gamma_S\sigma_{T3}\\
&- 6\sqrt{2}\alpha_{T5}\beta_{T3}\gamma_S\sigma_{T3} - 6\sqrt{2}\alpha_{T3}\beta_{T5}\gamma_S\sigma_{T3} + 12\alpha_{T5}\beta_{T5}\gamma_{T3}\sigma_{T3} - 6\sqrt{2}\alpha_{T5}\beta_S\gamma_{T4}\sigma_{T3}\\
&- 12\alpha_{T5}\beta_{T5}\gamma_{T4}\sigma_{T3} - 6\sqrt{2}\alpha_{T3}\beta_S\gamma_{T5}\sigma_{T3} - 12\alpha_{T5}\beta_{T3}\gamma_{T5}\sigma_{T3} + 12\alpha_{T5}\beta_{T4}\gamma_{T5}\sigma_{T3}\\
&- 12\alpha_{T3}\beta_{T5}\gamma_{T5}\sigma_{T3} + 12\alpha_{T4}\beta_{T5}\gamma_{T5}\sigma_{T3} + 6\alpha_{T3}\beta_S\gamma_S\sigma_{T4} - 6\alpha_{T4}\beta_S\gamma_S\sigma_{T4}\\
&- 6\sqrt{2}\alpha_{T5}\beta_{T4}\gamma_S\sigma_{T4} - 6\sqrt{2}\alpha_{T4}\beta_{T5}\gamma_S\sigma_{T4} - 6\sqrt{2}\alpha_{T5}\beta_S\gamma_{T3}\sigma_{T4} - 12\alpha_{T5}\beta_{T5}\gamma_{T3}\sigma_{T4}\\
&+ 12\alpha_{T5}\beta_{T5}\gamma_{T4}\sigma_{T4} - 6\sqrt{2}\alpha_{T4}\beta_S\gamma_{T5}\sigma_{T4} + 12\alpha_{T5}\beta_{T3}\gamma_{T5}\sigma_{T4} - 12\alpha_{T5}\beta_{T4}\gamma_{T5}\sigma_{T4}\\
&+ 12\alpha_{T3}\beta_{T5}\gamma_{T5}\sigma_{T4} - 12\alpha_{T4}\beta_{T5}\gamma_{T5}\sigma_{T4} + 10\alpha_{T5}\beta_S\gamma_S\sigma_{T5} - 6\sqrt{2}\alpha_{T4}\beta_{T3}\gamma_S\sigma_{T5}\\
&- 6\sqrt{2}\alpha_{T3}\beta_{T4}\gamma_S\sigma_{T5} - 2\sqrt{2}\alpha_{T5}\beta_{T5}\gamma_S\sigma_{T5} - 6\sqrt{2}\alpha_{T3}\beta_S\gamma_{T3}\sigma_{T5} - 12\alpha_{T5}\beta_{T3}\gamma_{T3}\sigma_{T5}\\
&+ 12\alpha_{T5}\beta_{T4}\gamma_{T3}\sigma_{T5} - 12\alpha_{T3}\beta_{T5}\gamma_{T3}\sigma_{T5} + 12\alpha_{T4}\beta_{T5}\gamma_{T3}\sigma_{T5} - 6\sqrt{2}\alpha_{T4}\beta_S\gamma_{T4}\sigma_{T5}\\
&+ 12\alpha_{T5}\beta_{T3}\gamma_{T4}\sigma_{T5} - 12\alpha_{T5}\beta_{T4}\gamma_{T4}\sigma_{T5} + 12\alpha_{T3}\beta_{T5}\gamma_{T4}\sigma_{T5} - 12\alpha_{T4}\beta_{T5}\gamma_{T4}\sigma_{T5}\\
&- 2\sqrt{2}\alpha_{T5}\beta_S\gamma_{T5}\sigma_{T5} + 12\alpha_{T3}\beta_{T3}\gamma_{T5}\sigma_{T5} - 12\alpha_{T4}\beta_{T3}\gamma_{T5}\sigma_{T5} - 12\alpha_{T3}\beta_{T4}\gamma_{T5}\sigma_{T5}\\
&+ 12\alpha_{T4}\beta_{T4}\gamma_{T5}\sigma_{T5} - 28\alpha_{T5}\beta_{T5}\gamma_{T5}\sigma_{T5} - \alpha_S\Big(- 11\sqrt{2}\beta_{T5}\gamma_S\sigma_S - 2\sqrt{6}\beta_{T5}\gamma_{T1}\sigma_S\\
&- 2\sqrt{6}\beta_{T5}\gamma_{T2}\sigma_S + 6\beta_{T3}\gamma_{T3}\sigma_S - 6\beta_{T4}\gamma_{T3}\sigma_S - 6\beta_{T3}\gamma_{T4}\sigma_S + 6\beta_{T4}\gamma_{T4}\sigma_S - 2\sqrt{6}\beta_{T2}\gamma_{T5}\sigma_S\\
&- 10\beta_{T5}\gamma_{T5}\sigma_S - 2\beta_{T2}\gamma_S\sqrt{3}\sigma_S - 2\sqrt{6}\beta_{T5}\gamma_S\sigma_{T1} - 4\sqrt{3}\beta_{T5}\gamma_{T5}\sigma_{T1} - 2\sqrt{6}\beta_{T5}\gamma_S\sigma_{T2}\\
&- 4\sqrt{3}\beta_{T5}\gamma_{T5}\sigma_{T2} + 6\beta_{T3}\gamma_S\sigma_{T3} - 6\beta_{T4}\gamma_S\sigma_{T3} - 6\sqrt{2}\beta_{T5}\gamma_{T3}\sigma_{T3} - 6\sqrt{2}\beta_{T4}\gamma_{T5}\sigma_{T3}\\
&- 6\beta_{T3}\gamma_S\sigma_{T4} + 6\beta_{T4}\gamma_S\sigma_{T4} - 6\sqrt{2}\beta_{T5}\gamma_{T4}\sigma_{T4} - 6\sqrt{2}\beta_{T3}\gamma_{T5}\sigma_{T4} - 2\sqrt{6}\beta_{T2}\gamma_S\sigma_{T5}\\
&- 10\beta_{T5}\gamma_S\sigma_{T5} - 4\sqrt{3}\beta_{T5}\gamma_{T1}\sigma_{T5} - 4\sqrt{3}\beta_{T5}\gamma_{T2}\sigma_{T5} - 6\sqrt{2}\beta_{T4}\gamma_{T3}\sigma_{T5} - 6\sqrt{2}\beta_{T3}\gamma_{T4}\sigma_{T5}\\
&- 4\sqrt{3}\beta_{T2}\gamma_{T5}\sigma_{T5} - \beta_S\Big(6\gamma_{T3}\sigma_{T3} - 6\gamma_{T4}\sigma_{T3} - 6\gamma_{T3}\sigma_{T4} + 6\gamma_{T4}\sigma_{T4} + 10\gamma_{T5}\sigma_{T5} - \gamma_S\Big(\\
&- 17\sigma_S - 2\sqrt{3}\sigma_{T1} - 2\sqrt{3}\sigma_{T2} - 11\sqrt{2}\sigma_{T5}\Big) + 11\gamma_{T5}\sigma_S\sqrt{2} + 2\gamma_{T1}\sigma_S\sqrt{3} + 2\gamma_{T2}\sigma_S\sqrt{3}\\
&+ 2\gamma_{T5}\sigma_{T1}\sqrt{6} + 2\gamma_{T5}\sigma_{T2}\sqrt{6} + 2\gamma_{T1}\sigma_{T5}\sqrt{6} + 2\gamma_{T2}\sigma_{T5}\sqrt{6}\Big) + 6\beta_{T5}\gamma_{T4}\sigma_{T3}\sqrt{2}
\end{aligned}
$$

$$+ 6\beta_{T3}\gamma_{T5}\sigma_{T3}\sqrt{2} + 6\beta_{T5}\gamma_{T3}\sigma_{T4}\sqrt{2} + 6\beta_{T4}\gamma_{T5}\sigma_{T4}\sqrt{2} + 6\beta_{T3}\gamma_{T3}\sigma_{T5}\sqrt{2} + 6\beta_{T4}\gamma_{T4}\sigma_{T5}\sqrt{2}$$
$$+ 2\beta_{T5}\gamma_{T5}\sigma_{T5}\sqrt{2} + 2\beta_{T1}\Big(- \gamma_S\sigma_S - \sqrt{2}\gamma_{T5}\sigma_S - \sqrt{2}\gamma_S\sigma_{T5} - 2\gamma_{T5}\sigma_{T5}\Big)\sqrt{3}\Big)$$
$$+ 6\alpha_{T5}\beta_{T4}\gamma_S\sigma_{T3}\sqrt{2} + 6\alpha_{T4}\beta_{T5}\gamma_S\sigma_{T3}\sqrt{2} + 6\alpha_{T5}\beta_S\gamma_{T3}\sigma_{T3}\sqrt{2} + 6\alpha_{T4}\beta_S\gamma_{T5}\sigma_{T3}\sqrt{2}$$
$$+ 6\alpha_{T5}\beta_{T3}\gamma_S\sigma_{T4}\sqrt{2} + 6\alpha_{T3}\beta_{T5}\gamma_S\sigma_{T4}\sqrt{2} + 6\alpha_{T5}\beta_S\gamma_{T4}\sigma_{T4}\sqrt{2} + 6\alpha_{T3}\beta_S\gamma_{T5}\sigma_{T4}\sqrt{2}$$
$$+ 6\alpha_{T3}\beta_{T3}\gamma_S\sigma_{T5}\sqrt{2} + 6\alpha_{T4}\beta_{T4}\gamma_S\sigma_{T5}\sqrt{2} + 6\alpha_{T4}\beta_S\gamma_{T3}\sigma_{T5}\sqrt{2} + 6\alpha_{T3}\beta_S\gamma_{T4}\sigma_{T5}\sqrt{2}$$
$$+ 4\alpha_{T5}\beta_{T5}\gamma_S\sigma_{T1}\sqrt{3} + 4\alpha_{T5}\beta_S\gamma_{T5}\sigma_{T1}\sqrt{3} + 4\alpha_{T5}\beta_{T5}\gamma_S\sigma_{T2}\sqrt{3} + 4\alpha_{T5}\beta_S\gamma_{T5}\sigma_{T2}\sqrt{3}$$
$$+ 4\alpha_{T5}\beta_{T1}\gamma_S\sigma_{T5}\sqrt{3} + 4\alpha_{T5}\beta_{T2}\gamma_S\sigma_{T5}\sqrt{3} + 4\alpha_{T2}\beta_{T5}\gamma_S\sigma_{T5}\sqrt{3} + 4\alpha_{T5}\beta_S\gamma_{T1}\sigma_{T5}\sqrt{3}$$
$$+ 4\alpha_{T5}\beta_S\gamma_{T2}\sigma_{T5}\sqrt{3} + 4\alpha_{T2}\beta_S\gamma_{T5}\sigma_{T5}\sqrt{3} + 2\alpha_{T1}\Big(\beta_{T5}\Big(2\gamma_{T5}\sigma_S + \gamma_S\sqrt{2}\sigma_S + 2\gamma_S\sigma_{T5}$$
$$+ 2\gamma_{T5}\sigma_{T5}\sqrt{2}\Big) - \beta_S\Big(- \gamma_S\sigma_S - \sqrt{2}\gamma_{T5}\sigma_S - \sqrt{2}\gamma_S\sigma_{T5} - 2\gamma_{T5}\sigma_{T5}\Big)\Big)\sqrt{3}$$
$$+ 2\alpha_{T5}\beta_S\gamma_S\sigma_{T1}\sqrt{6} + 4\alpha_{T5}\beta_{T5}\gamma_{T5}\sigma_{T1}\sqrt{6} + 2\alpha_{T5}\beta_S\gamma_S\sigma_{T2}\sqrt{6} + 4\alpha_{T5}\beta_{T5}\gamma_{T5}\sigma_{T2}\sqrt{6}$$
$$+ 2\alpha_{T2}\beta_S\gamma_S\sigma_{T5}\sqrt{6} + 4\alpha_{T5}\beta_{T5}\gamma_{T1}\sigma_{T5}\sqrt{6} + 4\alpha_{T5}\beta_{T5}\gamma_{T2}\sigma_{T5}\sqrt{6} + 4\alpha_{T5}\beta_{T1}\gamma_{T5}\sigma_{T5}\sqrt{6}$$
$$+ 4\alpha_{T5}\beta_{T2}\gamma_{T5}\sigma_{T5}\sqrt{6} + 4\alpha_{T2}\beta_{T5}\gamma_{T5}\sigma_{T5}\sqrt{6}\Big). \tag{E.3}$$

This amplitude simplifies considerable if we focus on scattering processes of the form $+2XXX$. We may easily see that X can only be a scalar mode and this amplitude then takes the form

$$M_4(+2XXX) = \frac{istu}{96\sqrt{6}}(\beta_{T5} + \frac{1}{\sqrt{2}}\beta_S)(\gamma_{T5} + \frac{1}{\sqrt{2}}\gamma_S)(\sigma_{T5} + \frac{1}{\sqrt{2}}\sigma_S)\,. \tag{E.4}$$

The combination $\beta_{T5} + \frac{1}{\sqrt{2}}\beta_S$ is precisely the combination of polarizations that picks out the helicity-0 squared term $\epsilon_0^\mu\epsilon_0^\nu$

$$\beta_{T5}\epsilon^{\mu\nu}_{2,0} + \beta_S\epsilon^{\mu\nu}_{0,0} = \frac{2}{\sqrt{6}}(\beta_{T5} + \frac{1}{\sqrt{2}}\beta_S)\epsilon_0^\mu\epsilon_0^\nu + \frac{1}{\sqrt{6}}(\beta_{T5} - \sqrt{2}\beta_S)(\epsilon_+^\mu\epsilon_-^\nu + \epsilon_-^\mu\epsilon_+^\nu)\,. \tag{E.5}$$

Since the helicity $+2$ mode has polarization tensor $\epsilon_+^\mu\epsilon_+^\nu$ we recognize that $M_4(+2XXX)$ is the double copy of the $+1000$ massive Yang-Mills amplitude and comes specifically from the $\Sigma\Sigma'$ contact term (2.43).

Appendix F

Decoupling limit of massive Yang-Mills amplitude

In this section we derive the decoupling limit of the massive Yang-Mills amplitude which is expected to be the amplitude of NLSM, derive the kinematic factors and double copy it to show that we recover the 4 point amplitude of a special Galileon. We also show that taking the decoupling limit and performing the double copy do not commute. From (2.1), the 4-point amplitudes of massive Yang-Mills is expressed as:

$$A_4^{\text{mYM}} = \frac{m^2}{\Lambda^2}\left(\frac{c_s n_s}{s-m^2} + \frac{c_t n_t}{t-m^2} + \frac{c_u n_u}{u-m^2}\right), \tag{F.1}$$

with the n's given by (2.34), (2.35) and (2.36). By plugging the polarization vectors which are arbitrary superpositions of all helicities given as:

$$\begin{aligned}
\epsilon_{1\mu} &= \alpha_1\epsilon_\mu^{+1}(p_1) + \alpha_2\epsilon_\mu^{-1}(p_1) + \alpha_3\epsilon_\mu^{0}(p_1),\\
\epsilon_{2\mu} &= \beta_1\epsilon_\mu^{+1}(p_2) + \beta_2\epsilon_\mu^{-1}(p_2) + \beta_3\epsilon_\mu^{0}(p_2),\\
\epsilon_{3\mu} &= \gamma_1\epsilon_\mu^{+1}(p_3) + \gamma_2\epsilon_\mu^{-1}(p_3) + \gamma_3\epsilon_\mu^{0}(p_3),\\
\epsilon_{4\mu} &= \sigma_1\epsilon_\mu^{+1}(p_4) + \sigma_2\epsilon_\mu^{-1}(p_4) + \sigma_3\epsilon_\mu^{0}(p_4),
\end{aligned} \tag{F.2}$$

and four momenta into the n's, they can be rearranged in the following form (as mentioned in (2.40)):

$$n_s = \frac{s-m^2}{m^3}\Sigma(s,t,u)+\frac{1}{m^2}\hat{n}_s, \quad n_t = \frac{t-m^2}{m^3}\Sigma(s,t,u)+\frac{1}{m^2}\hat{n}_t, \quad n_u = \frac{u-m^2}{m^3}\Sigma(s,t,u)+\frac{1}{m^2}\hat{n}_u\,, \tag{F.3}$$

with $n_s + n_t + n_u = 0$ and $\hat{n}_s + \hat{n}_t + \hat{n}_u = -m\Sigma$. The explicit expressions for the $\hat{n}$'s and $\Sigma(s,t,u)$ are given in (F.12) and (F.13) (F.14) (F.15). The amplitude can be written as,

$$A_4^{\text{mYM}} = \frac{m^2}{\Lambda^2}\left(\frac{c_s n_s}{s-m^2}+\frac{c_t n_t}{t-m^2}+\frac{c_u n_u}{u-m^2}\right), \tag{F.4}$$

$$= \frac{1}{\Lambda^2}\left(\frac{c_s \hat{n}_s}{s-m^2}+\frac{c_t \hat{n}_t}{t-m^2}+\frac{c_u \hat{n}_u}{u-m^2}\right) + \frac{1}{m\Lambda^2}\Sigma(s,t,u)\left(c_s+c_t+c_u\right), \tag{F.5}$$

and as mentioned in the introduction, the last term which seems at first ill defined in the decoupling limit $m \to 0$, Λ fixed, is zero by virtue of Jacobi identity. Focusing on the non-zero term, the amplitude in the dcoupling limit is as follows:

$$\begin{aligned} A_4^{\text{DL}} &= \lim_{m\to 0,\ \Lambda \text{fixed}} \frac{1}{\Lambda^2} A^{\text{mYM}}, \\ &= -i\frac{1}{12\Lambda^2}\Bigg(c_s(t-u)+c_t(u-s)+c_u(s-t)\Bigg)\alpha_3\beta_3\sigma_3\gamma_3. \end{aligned} \tag{F.6}$$

We see that only helicity-0 polarization states remain interacting in this decoupling limit. The kinematic factors of this amplitude are,

$$n_s = -\frac{is}{12}(t-u), \quad n_t = -\frac{it}{12}(u-s), \quad n_u = -\frac{iu}{12}(s-t). \tag{F.7}$$

Note that in this limit we have $s+t+u=0$ and can see that the colour-kinematics duality is satisfied.

Using the kinematic factors of this amplitude we double copy it and obtain the following:

$$A^{DC} = i\frac{\alpha_3^2\beta_3^2\gamma_3^2\sigma_3^2}{16\Lambda_3^6}stu, \tag{F.8}$$

which is equal to the scattering amplitude of a galileon theory.

It seems that we could have defined the kinematic factors of the full massive Yang-Mills theory without the $1/m^3$ terms in (F.3) since they cancel in the full amplitude. However without them the colour-kinematics duality is not satisfied. This is in contrast to the massless double copy where at four-points any representation of kinematic factors satisfy this duality. If we tried to double copy, *i.e.*

$$\frac{1}{M_{\rm Pl}^2}\left(\frac{n_s n_s'}{s-m^2}+\frac{n_t n_t'}{t-m^2}+\frac{n_u n_u'}{u-m^2}\right) = \frac{-\Sigma\Sigma'}{\Lambda_3^6}+\frac{1}{\Lambda_3^6}\left(\frac{\hat{n}_s\hat{n}_s'}{s-m^2}+\frac{\hat{n}_t\hat{n}_t'}{t-m^2}+\frac{\hat{n}_u\hat{n}_u'}{u-m^2}\right), \tag{F.9}$$

without using $\Sigma(s,t,u)$, we would have obtained a theory whose Λ_3 decoupling limit is the special galileon because only $\hat{n}$ terms could have contributed to the double copy amplitude, *i.e.* we would have obtained

$$A^{DC}_{\hat{n}^2} = \frac{i}{M_{pl}^2}\sum_{i=1}^{3}\frac{\hat{n}_i\hat{n}_i'}{m^4 s_i} = i\frac{\alpha_3^2\beta_3^2\gamma_3^2\sigma_3^2}{16\Lambda_3^6}stu, \tag{F.10}$$

where $i=1,2,3$ labels s,t,u respectively. However, in our case, when we square $\Sigma(s,t,u)$, they sum to a $1/m^4$ contribution to the double copy scattering amplitude

$$A^{DC}_{\Sigma^2(s,t,u)} = i\frac{\left(\alpha_3\beta_3\gamma_3\left(\sigma_2-\sigma_1\right)+\alpha_3\beta_3\left(\gamma_2-\gamma_1\right)\sigma_3+\gamma_3\sigma_3\left(\alpha_3\left(\beta_1-\beta_2\right)+\left(\alpha_1-\alpha_2\right)\beta_3\right)\right)^2}{8\Lambda_3^6}stu, \tag{F.11}$$

which contains helicity ±1 polarizations and the decoupling limit of the resulting theory is not the double copy of the decoupling limit of the massive Yang-Mills, *i.e.* the operations of taking decoupling limit and performing double copy do not commute.

The explicit expressions for $\Sigma(s,t,u)$, $\hat{n}_s$, $\hat{n}_t$, $\hat{n}_u$ are given below:

$$\Sigma(s,t,u) = i\frac{\sqrt{stu}\left(\alpha_3\beta_3\gamma_3\left(\sigma_1-\sigma_2\right)+\alpha_3\beta_3\left(\gamma_1-\gamma_2\right)\sigma_3+\gamma_3\sigma_3\left(\alpha_3\left(\beta_2-\beta_1\right)+\left(\alpha_2-\alpha_1\right)\beta_3\right)\right)}{2\sqrt{2}} \tag{F.12}$$

$$\begin{aligned}
\hat{n}_s = &-\frac{i}{4\Big(4m^2-s\Big)}(16(\alpha_2(-\beta_2\gamma_1\sigma_1+\beta_2\gamma_2\sigma_2-\beta_3\gamma_3\sigma_2+\beta_3\gamma_1\sigma_3)+\alpha_1(\beta_1\gamma_1\sigma_1-\beta_3\gamma_3\sigma_1\\
&-\beta_1\gamma_2\sigma_2+\beta_3\gamma_2\sigma_3)+\alpha_3(\beta_2\gamma_3\sigma_1+\beta_1\gamma_3\sigma_2-\beta_1\gamma_1\sigma_3-\beta_2\gamma_2\sigma_3))m^6-4(4u(\alpha_1\beta_1+\alpha_2\beta_2\\
&-\alpha_3\beta_3)(\gamma_1\sigma_1+\gamma_2\sigma_2-\gamma_3\sigma_3)+t(-\alpha_3(\beta_1-\beta_2)(\gamma_3(\sigma_1-\sigma_2)+(\gamma_1-\gamma_2)\sigma_3)+\alpha_2\beta_3(\gamma_3(\sigma_1\\
&-\sigma_2)+(\gamma_1-\gamma_2)\sigma_3)+4\alpha_3\beta_3(\gamma_1\sigma_1+\gamma_2\sigma_2-\gamma_3\sigma_3)-2\alpha_2\beta_2(3\gamma_1\sigma_1+\gamma_2\sigma_2-2\gamma_3\sigma_3)\\
&+\alpha_1(\beta_3\gamma_3(\sigma_2-\sigma_1)+\beta_3(\gamma_2-\gamma_1)\sigma_3-2\beta_1(\gamma_1\sigma_1+3\gamma_2\sigma_2-2\gamma_3\sigma_3)))+s(\alpha_1(5\beta_1\gamma_1\sigma_1\\
&+9\beta_3\gamma_3\sigma_1-5\beta_1\gamma_2\sigma_2-11\beta_3\gamma_2\sigma_3-16\beta_1\gamma_3\sigma_3)+\alpha_2(-5\beta_2\gamma_1\sigma_1+5\beta_2\gamma_2\sigma_2+9\beta_3\gamma_3\sigma_2\\
&-11\beta_3\gamma_1\sigma_3-16\beta_2\gamma_3\sigma_3)+\alpha_3(-11\beta_2\gamma_3\sigma_1-11\beta_1\gamma_3\sigma_2+9\beta_1\gamma_1\sigma_3+9\beta_2\gamma_2\sigma_3-4\beta_3(4\gamma_1\sigma_1\\
&+4\gamma_2\sigma_2-9\gamma_3\sigma_3))))m^4+2\sqrt{2}\sqrt{stu}(\alpha_2\beta_2(-9\gamma_3\sigma_1+7\gamma_3\sigma_2-9\gamma_1\sigma_3+7\gamma_2\sigma_3)+\alpha_1\beta_1(\\
&-7\gamma_3\sigma_1+9\gamma_3\sigma_2-7\gamma_1\sigma_3+9\gamma_2\sigma_3)+\alpha_2\beta_3(-9\gamma_1\sigma_1-7\gamma_2\sigma_2+12\gamma_3\sigma_3)+\alpha_1\beta_3(7\gamma_1\sigma_1\\
&+9\gamma_2\sigma_2-12\gamma_3\sigma_3)+\alpha_3(\beta_2(-9\gamma_1\sigma_1-7\gamma_2\sigma_2+12\gamma_3\sigma_3)+12\beta_3(\gamma_3\sigma_1-\gamma_3\sigma_2+\gamma_1\sigma_3\\
&-\gamma_2\sigma_3)+\beta_1(7\gamma_1\sigma_1+9\gamma_2\sigma_2-12\gamma_3\sigma_3)))m^3+2s(t(\alpha_3\beta_2(-6\gamma_3\sigma_1+5\gamma_3\sigma_2-6\gamma_1\sigma_3\\
&+5\gamma_2\sigma_3)+\alpha_2\beta_3(-6\gamma_3\sigma_1+5\gamma_3\sigma_2-6\gamma_1\sigma_3+5\gamma_2\sigma_3)-2\alpha_2\beta_2(3\gamma_1\sigma_1-\gamma_2\sigma_2+5\gamma_3\sigma_3)\\
&+\alpha_3\beta_1(5\gamma_3\sigma_1-6\gamma_3\sigma_2+5\gamma_1\sigma_3-6\gamma_2\sigma_3)+\alpha_1\beta_3(5\gamma_3\sigma_1-6\gamma_3\sigma_2+5\gamma_1\sigma_3-6\gamma_2\sigma_3)\\
&+2\alpha_1\beta_1(\gamma_1\sigma_1-3\gamma_2\sigma_2-5\gamma_3\sigma_3)-2\alpha_3\beta_3(5\gamma_1\sigma_1+5\gamma_2\sigma_2-13\gamma_3\sigma_3))+2u(\alpha_3\beta_3(-3\gamma_1\sigma_1\\
&-3\gamma_2\sigma_2+5\gamma_3\sigma_3)+\alpha_1\beta_1(\gamma_1\sigma_1+\gamma_2\sigma_2-3\gamma_3\sigma_3)+\alpha_2\beta_2(\gamma_1\sigma_1+\gamma_2\sigma_2-3\gamma_3\sigma_3))\\
&+s(\alpha_2(\beta_3(9\gamma_3\sigma_2-14\gamma_1\sigma_3)-2\beta_2(\gamma_1\sigma_1-\gamma_2\sigma_2+8\gamma_3\sigma_3))+\alpha_1(\beta_3(9\gamma_3\sigma_1-14\gamma_2\sigma_3)\\
&+2\beta_1(\gamma_1\sigma_1-\gamma_2\sigma_2-8\gamma_3\sigma_3))+\alpha_3(-14\beta_2\gamma_3\sigma_1-14\beta_1\gamma_3\sigma_2+9\beta_1\gamma_1\sigma_3+9\beta_2\gamma_2\sigma_3\\
&-4\beta_3(4\gamma_1\sigma_1+4\gamma_2\sigma_2-11\gamma_3\sigma_3))))m^2-2\sqrt{2}s\sqrt{stu}(\alpha_2\beta_2(\gamma_3(\sigma_2-3\sigma_1)+(\gamma_2-3\gamma_1)\sigma_3)
\end{aligned}$$

$$
\begin{aligned}
&-\alpha_1\beta_1(\gamma_3(\sigma_1-3\sigma_2)+(\gamma_1-3\gamma_2)\sigma_3)-\alpha_2\beta_3(3\gamma_1\sigma_1+\gamma_2\sigma_2-3\gamma_3\sigma_3)+\alpha_1\beta_3(\gamma_1\sigma_1+3\gamma_2\sigma_2\\
&-3\gamma_3\sigma_3)+\alpha_3(3\beta_3(\gamma_3\sigma_1-\gamma_3\sigma_2+\gamma_1\sigma_3-\gamma_2\sigma_3)-\beta_2(3\gamma_1\sigma_1+\gamma_2\sigma_2-3\gamma_3\sigma_3)+\beta_1(\gamma_1\sigma_1\\
&+3\gamma_2\sigma_2-3\gamma_3\sigma_3)))m+4s^3\alpha_3\beta_3\gamma_3\sigma_3+s^2(s(-2\alpha_2\beta_3\gamma_3\sigma_2+4\alpha_2\beta_3\gamma_1\sigma_3+4\alpha_2\beta_2\gamma_3\sigma_3+\alpha_1(\\
&-2\beta_3\gamma_3\sigma_1+4\beta_3\gamma_2\sigma_3+4\beta_1\gamma_3\sigma_3)+\alpha_3(4\beta_2\gamma_3\sigma_1+4\beta_1\gamma_3\sigma_2-2\beta_1\gamma_1\sigma_3-2\beta_2\gamma_2\sigma_3\\
&+\beta_3(4\gamma_1\sigma_1+4\gamma_2\sigma_2-17\gamma_3\sigma_3)))+2(u((\alpha_1\beta_1+\alpha_2\beta_2)\gamma_3\sigma_3+\alpha_3\beta_3(\gamma_1\sigma_1+\gamma_2\sigma_2-4\gamma_3\sigma_3))\\
&+t(3\alpha_1\beta_1\gamma_3\sigma_3+3\alpha_2\beta_2\gamma_3\sigma_3-\alpha_1\beta_3(\gamma_3(\sigma_1-2\sigma_2)+(\gamma_1-2\gamma_2)\sigma_3)+\alpha_2\beta_3(2\gamma_3\sigma_1-\gamma_3\sigma_2\\
&+2\gamma_1\sigma_3-\gamma_2\sigma_3)-\alpha_3(\beta_2(-2\gamma_3\sigma_1+\gamma_3\sigma_2-2\gamma_1\sigma_3+\gamma_2\sigma_3)+\beta_1(\gamma_3\sigma_1-2\gamma_3\sigma_2+\gamma_1\sigma_3\\
&-2\gamma_2\sigma_3)-3\beta_3(\gamma_1\sigma_1+\gamma_2\sigma_2-3\gamma_3\sigma_3))))\Big)
\end{aligned}
\tag{F.13}
$$

$$
\begin{aligned}
\hat{n}_t=\frac{i}{4\left(s-4m^2\right)^2}\Big(&-64(-\alpha_3(\beta_3\gamma_1\sigma_1+4\beta_2\gamma_3\sigma_1+\beta_3\gamma_2\sigma_2+4\beta_1\gamma_3\sigma_2)+\alpha_3(\beta_1\gamma_1+\beta_2\gamma_2\\
&+4\beta_3\gamma_3)\sigma_3+\alpha_1(3\beta_2\gamma_2\sigma_1+\beta_3\gamma_3\sigma_1+5\beta_1\gamma_2\sigma_2-4\beta_3\gamma_2\sigma_3-\beta_1\gamma_3\sigma_3)+\alpha_2(5\beta_2\gamma_1\sigma_1\\
&+3\beta_1\gamma_1\sigma_2+\beta_3\gamma_3\sigma_2-4\beta_3\gamma_1\sigma_3-\beta_2\gamma_3\sigma_3))m^8-16(t(\alpha_3(4\gamma_3(\beta_1\sigma_1+\beta_2\sigma_2)+(3\beta_1\gamma_1\\
&-7\beta_2\gamma_1-7\beta_1\gamma_2+3\beta_2\gamma_2)\sigma_3-4\beta_3(-2\gamma_2\sigma_1+\gamma_2\sigma_2+\gamma_1(\sigma_1-2\sigma_2)+\gamma_3\sigma_3))-\alpha_2(\beta_2(\\
&-16\gamma_1\sigma_1+5\gamma_2\sigma_1+5\gamma_1\sigma_2-2\gamma_2\sigma_2+4\gamma_3\sigma_3)+\beta_3(7\gamma_3\sigma_1-3\gamma_3\sigma_2-4\gamma_2\sigma_3)+\beta_1(5\gamma_1\sigma_1\\
&+6\gamma_1\sigma_2+5\gamma_2\sigma_2-8\gamma_3\sigma_3))+\alpha_1(\beta_3(3\gamma_3\sigma_1-7\gamma_3\sigma_2+4\gamma_1\sigma_3)+\beta_1(2\gamma_1\sigma_1-5\gamma_2\sigma_1-5\gamma_1\sigma_2\\
&+16\gamma_2\sigma_2-4\gamma_3\sigma_3)-\beta_2(5\gamma_1\sigma_1+6\gamma_2\sigma_1+5\gamma_2\sigma_2-8\gamma_3\sigma_3)))-2s(-2\alpha_3(\beta_3\gamma_1\sigma_1+4\beta_2\gamma_3\sigma_1\\
&+\beta_3\gamma_2\sigma_2+4\beta_1\gamma_3\sigma_2)+2\alpha_3(\beta_1\gamma_1+\beta_2\gamma_2+4\beta_3\gamma_3)\sigma_3+\alpha_1(7\beta_2\gamma_2\sigma_1+2\beta_3\gamma_3\sigma_1+9\beta_1\gamma_2\sigma_2\\
&-8\beta_3\gamma_2\sigma_3-2\beta_1\gamma_3\sigma_3)+\alpha_2(9\beta_2\gamma_1\sigma_1+7\beta_1\gamma_1\sigma_2+2\beta_3\gamma_3\sigma_2-8\beta_3\gamma_1\sigma_3-2\beta_2\gamma_3\sigma_3)))m^6\\
&-8\sqrt{2}\sqrt{stu}(\alpha_1(\beta_3\gamma_1\sigma_1-5\beta_3\gamma_2\sigma_1-\beta_1\gamma_3\sigma_1+5\beta_2\gamma_3\sigma_1+4\beta_3\gamma_2\sigma_2+4\beta_1\gamma_3\sigma_2-(\beta_1\gamma_1\\
&-4\beta_1\gamma_2+5\beta_2\gamma_2+5\beta_3\gamma_3)\sigma_3)+\alpha_3(\beta_2(-4\gamma_1\sigma_1+5\gamma_2\sigma_1-\gamma_2\sigma_2+5\gamma_3\sigma_3)+5\beta_3(\gamma_3\sigma_1\\
&-\gamma_3\sigma_2+\gamma_1\sigma_3-\gamma_2\sigma_3)+\beta_1(4\gamma_2\sigma_2+\gamma_1(\sigma_1-5\sigma_2)-5\gamma_3\sigma_3))+\alpha_2(-5\beta_1\gamma_3\sigma_2+5\beta_1\gamma_1\sigma_3\\
&+\beta_2(-4\gamma_3\sigma_1+\gamma_3\sigma_2-4\gamma_1\sigma_3+\gamma_2\sigma_3)+\beta_3(-4\gamma_1\sigma_1+5\gamma_1\sigma_2-\gamma_2\sigma_2+5\gamma_3\sigma_3)))m^5-4((\\
&-5\alpha_3(\beta_3\gamma_1\sigma_1+4\beta_2\gamma_3\sigma_1+\beta_3\gamma_2\sigma_2+4\beta_1\gamma_3\sigma_2)+5\alpha_3(\beta_1\gamma_1+\beta_2\gamma_2+4\beta_3\gamma_3)\sigma_3\\
&+\alpha_1(19\beta_2\gamma_2\sigma_1+5\beta_3\gamma_3\sigma_1+21\beta_1\gamma_2\sigma_2-5(4\beta_3\gamma_2+\beta_1\gamma_3)\sigma_3)+\alpha_2(21\beta_2\gamma_1\sigma_1+19\beta_1\gamma_1\sigma_2
\end{aligned}
$$

$$+5\beta_3\gamma_3\sigma_2 - 5(4\beta_3\gamma_1 + \beta_2\gamma_3)\sigma_3))s^2 + t(-2\alpha_3(\beta_2\gamma_3(15\sigma_1 + \sigma_2) + \beta_1\gamma_3(\sigma_1 + 15\sigma_2)$$

$$+\beta_3(5\gamma_1\sigma_1 + 3\gamma_2\sigma_1 + 3\gamma_1\sigma_2 + 5\gamma_2\sigma_2)) + \alpha_3(13\beta_1\gamma_1 + 5\beta_2\gamma_1 + 5\beta_1\gamma_2 + 13\beta_2\gamma_2$$

$$+52\beta_3\gamma_3)\sigma_3 + \alpha_1(-2\beta_1\gamma_1\sigma_1 + 5\beta_2\gamma_1\sigma_1 + 5\beta_1\gamma_2\sigma_1 + 22\beta_2\gamma_2\sigma_1 + 13\beta_3\gamma_3\sigma_1 + 5\beta_1\gamma_1\sigma_2$$

$$-24\beta_1\gamma_2\sigma_2 + 5\beta_2\gamma_2\sigma_2 + 5\beta_3\gamma_3\sigma_2 - 2(\beta_3\gamma_1 + 15\beta_3\gamma_2 + 5\beta_1\gamma_3 + 3\beta_2\gamma_3)\sigma_3) + \alpha_2(5\beta_1\gamma_1\sigma_1$$

$$-24\beta_2\gamma_1\sigma_1 + 5\beta_2\gamma_2\sigma_1 + 5\beta_3\gamma_3\sigma_1 + 22\beta_1\gamma_1\sigma_2 + 5\beta_2\gamma_1\sigma_2 + 5\beta_1\gamma_2\sigma_2 - 2\beta_2\gamma_2\sigma_2 + 13\beta_3\gamma_3\sigma_2$$

$$-2(15\beta_3\gamma_1 + \beta_3\gamma_2 + 3\beta_1\gamma_3 + 5\beta_2\gamma_3)\sigma_3))s + t^2(8\alpha_3\beta_3(\gamma_1 - \gamma_2)(\sigma_1 - \sigma_2) - 4\alpha_3(\beta_1$$

$$-\beta_2)(\gamma_3(\sigma_1 - \sigma_2) + (\gamma_1 - \gamma_2)\sigma_3) + \alpha_2(\beta_2(-21\gamma_1\sigma_1 + 5\gamma_2\sigma_1 + 5\gamma_1\sigma_2 - 5\gamma_2\sigma_2 + 8\gamma_3\sigma_3)$$

$$+4\beta_3(\gamma_3\sigma_1 - \gamma_3\sigma_2 + \gamma_1\sigma_3 - \gamma_2\sigma_3) + \beta_1(5\gamma_1\sigma_1 + 3\gamma_2\sigma_1 + 3\gamma_1\sigma_2 + 5\gamma_2\sigma_2 - 8\gamma_3\sigma_3))$$

$$+\alpha_1(4\beta_3(-\gamma_3\sigma_1 + \gamma_3\sigma_2 - \gamma_1\sigma_3 + \gamma_2\sigma_3) + \beta_1(5\gamma_2\sigma_1 - 21\gamma_2\sigma_2 + 5\gamma_1(\sigma_2 - \sigma_1) + 8\gamma_3\sigma_3)$$

$$+\beta_2(5\gamma_1\sigma_1 + 3\gamma_2\sigma_1 + 3\gamma_1\sigma_2 + 5\gamma_2\sigma_2 - 8\gamma_3\sigma_3))))m^4 + 2\sqrt{2}\sqrt{stu}(s(\alpha_1(\beta_3\gamma_1\sigma_1 - 5\beta_3\gamma_2\sigma_1$$

$$-\beta_1\gamma_3\sigma_1 + 5\beta_2\gamma_3\sigma_1 + 4\beta_3\gamma_2\sigma_2 + 4\beta_1\gamma_3\sigma_2 - (\beta_1\gamma_1 - 4\beta_1\gamma_2 + 5\beta_2\gamma_2 + 5\beta_3\gamma_3)\sigma_3) + \alpha_3(\beta_2($$

$$-4\gamma_1\sigma_1 + 5\gamma_2\sigma_1 - \gamma_2\sigma_2 + 5\gamma_3\sigma_3) + 5\beta_3(\gamma_3\sigma_1 - \gamma_3\sigma_2 + \gamma_1\sigma_3 - \gamma_2\sigma_3) + \beta_1(4\gamma_2\sigma_2 + \gamma_1(\sigma_1$$

$$-5\sigma_2) - 5\gamma_3\sigma_3)) + \alpha_2(-5\beta_1\gamma_3\sigma_2 + 5\beta_1\gamma_1\sigma_3 + \beta_2(-4\gamma_3\sigma_1 + \gamma_3\sigma_2 - 4\gamma_1\sigma_3 + \gamma_2\sigma_3) + \beta_3($$

$$-4\gamma_1\sigma_1 + 5\gamma_1\sigma_2 - \gamma_2\sigma_2 + 5\gamma_3\sigma_3))) + t(\alpha_3(\beta_2(-17\gamma_1\sigma_1 + 5\gamma_2\sigma_1 + 5\gamma_1\sigma_2 - 9\gamma_2\sigma_2$$

$$+24\gamma_3\sigma_3) + 24\beta_3(\gamma_3\sigma_1 - \gamma_3\sigma_2 + \gamma_1\sigma_3 - \gamma_2\sigma_3) + \beta_1(9\gamma_1\sigma_1 - 5\gamma_2\sigma_1 - 5\gamma_1\sigma_2 + 17\gamma_2\sigma_2$$

$$-24\gamma_3\sigma_3)) + \alpha_2(\beta_2(-17\gamma_3\sigma_1 + 9\gamma_3\sigma_2 - 17\gamma_1\sigma_3 + 9\gamma_2\sigma_3) + \beta_3(-17\gamma_1\sigma_1 + 5\gamma_2\sigma_1 + 5\gamma_1\sigma_2$$

$$-9\gamma_2\sigma_2 + 24\gamma_3\sigma_3) + 5\beta_1(\gamma_3\sigma_1 - \gamma_3\sigma_2 + \gamma_1\sigma_3 - \gamma_2\sigma_3)) + \alpha_1(\beta_1(-9\gamma_3\sigma_1 + 17\gamma_3\sigma_2 - 9\gamma_1\sigma_3$$

$$+17\gamma_2\sigma_3) + 5\beta_2(\gamma_3\sigma_1 - \gamma_3\sigma_2 + \gamma_1\sigma_3 - \gamma_2\sigma_3) + \beta_3(9\gamma_1\sigma_1 - 5\gamma_2\sigma_1 - 5\gamma_1\sigma_2 + 17\gamma_2\sigma_2$$

$$-24\gamma_3\sigma_3))))m^3 + 2s((-\alpha_3(\beta_3\gamma_1\sigma_1 + 4\beta_2\gamma_3\sigma_1 + \beta_3\gamma_2\sigma_2 + 4\beta_1\gamma_3\sigma_2) + \alpha_3(\beta_1\gamma_1 + \beta_2\gamma_2$$

$$+4\beta_3\gamma_3)\sigma_3 + \alpha_1(4\beta_2\gamma_2\sigma_1 + \beta_3\gamma_3\sigma_1 + 4\beta_1\gamma_2\sigma_2 - 4\beta_3\gamma_2\sigma_3 - \beta_1\gamma_3\sigma_3) + \alpha_2(4\beta_2\gamma_1\sigma_1$$

$$+4\beta_1\gamma_1\sigma_2 + \beta_3\gamma_3\sigma_2 - 4\beta_3\gamma_1\sigma_3 - \beta_2\gamma_3\sigma_3))s^2 + t(\alpha_2(-4\beta_2\gamma_1\sigma_1 - \beta_3\gamma_3\sigma_1 + 8\beta_1\gamma_1\sigma_2$$

$$+12\beta_3\gamma_3\sigma_2 + \beta_3(\gamma_2 - 23\gamma_1)\sigma_3 + (\beta_1 - 11\beta_2)\gamma_3\sigma_3) + \alpha_1(-4\beta_1\gamma_2\sigma_2 - 11\beta_1\gamma_3\sigma_3$$

$$+\beta_2(8\gamma_2\sigma_1 + \gamma_3\sigma_3) + \beta_3(12\gamma_3\sigma_1 - \gamma_3\sigma_2 + \gamma_1\sigma_3 - 23\gamma_2\sigma_3)) + \alpha_3(\beta_2(-23\gamma_3\sigma_1 + \gamma_3\sigma_2$$

$$-\gamma_1\sigma_3 + 12\gamma_2\sigma_3) + \beta_3(\gamma_2\sigma_1 - 11\gamma_2\sigma_2 + \gamma_1(\sigma_2 - 11\sigma_1) + 36\gamma_3\sigma_3) + \beta_1(\gamma_3\sigma_1 - 23\gamma_3\sigma_2$$

$$+12\gamma_1\sigma_3 - \gamma_2\sigma_3)))s + t^2(\alpha_1(-8\beta_1\gamma_2\sigma_2 - 9\beta_1\gamma_3\sigma_3 + \beta_2(4\gamma_2\sigma_1 + 4\gamma_1\sigma_2 + \gamma_3\sigma_3)$$

$$+\beta_3(7\gamma_3\sigma_1 - 11\gamma_3\sigma_2 + 7\gamma_1\sigma_3 - 11\gamma_2\sigma_3)) + \alpha_2(\beta_3(-11\gamma_3\sigma_1 + 7\gamma_3\sigma_2 - 11\gamma_1\sigma_3 + 7\gamma_2\sigma_3)$$

$$
\begin{aligned}
&+\beta_1(4\gamma_2\sigma_1+4\gamma_1\sigma_2+\gamma_3\sigma_3)-\beta_2(8\gamma_1\sigma_1+9\gamma_3\sigma_3))+\alpha_3(\beta_2(-11\gamma_3\sigma_1+7\gamma_3\sigma_2-11\gamma_1\sigma_3\\
&+7\gamma_2\sigma_3)+\beta_3(\gamma_2\sigma_1-9\gamma_2\sigma_2+\gamma_1(\sigma_2-9\sigma_1)+32\gamma_3\sigma_3)+\beta_1(7\gamma_3\sigma_1-11\gamma_3\sigma_2+7\gamma_1\sigma_3\\
&-11\gamma_2\sigma_3))))m^2+st\sqrt{2}\sqrt{stu}(2\alpha_1\beta_1(\gamma_3(\sigma_1-3\sigma_2)+(\gamma_1-3\gamma_2)\sigma_3)+\alpha_1\beta_3(-2\gamma_1\sigma_1\\
&-6\gamma_2\sigma_2+7\gamma_3\sigma_3)+\alpha_3(-2\beta_1\gamma_1\sigma_1+6\beta_2\gamma_1\sigma_1-7\beta_3\gamma_3\sigma_1-6\beta_1\gamma_2\sigma_2+2\beta_2\gamma_2\sigma_2+7\beta_3\gamma_3\sigma_2\\
&+7(\beta_3(\gamma_2-\gamma_1)+(\beta_1-\beta_2)\gamma_3)\sigma_3)+2\alpha_2\beta_2(3\gamma_3\sigma_1-\gamma_3\sigma_2+3\gamma_1\sigma_3-\gamma_2\sigma_3)+\alpha_2\beta_3(6\gamma_1\sigma_1\\
&+2\gamma_2\sigma_2-7\gamma_3\sigma_3))m+s^2t(2s(-\beta_3\gamma_3(\alpha_1\sigma_1+\alpha_2\sigma_2)+(2\alpha_2\beta_3\gamma_1+2\alpha_1\beta_3\gamma_2+\alpha_1\beta_1\gamma_3\\
&+\alpha_2\beta_2\gamma_3)\sigma_3+\alpha_3(\beta_3\gamma_1\sigma_1+2\beta_2\gamma_3\sigma_1+\beta_3\gamma_2\sigma_2+2\beta_1\gamma_3\sigma_2-(\beta_1\gamma_1+\beta_2\gamma_2+3\beta_3\gamma_3)\sigma_3))+t(\\
&-2\beta_3\gamma_3(\alpha_2(\sigma_2-2\sigma_1)+\alpha_1(\sigma_1-2\sigma_2))-2\beta_3(\alpha_2(\gamma_2-2\gamma_1)+\alpha_1(\gamma_1-2\gamma_2))\sigma_3+4(\alpha_1\beta_1\\
&+\alpha_2\beta_2)\gamma_3\sigma_3+\alpha_3(-2\gamma_3(\beta_1\sigma_1-2\beta_2\sigma_1-2\beta_1\sigma_2+\beta_2\sigma_2)-2(\beta_1\gamma_1-2\beta_2\gamma_1-2\beta_1\gamma_2\\
&+\beta_2\gamma_2)\sigma_3+\beta_3(4\gamma_1\sigma_1+4\gamma_2\sigma_2-11\gamma_3\sigma_3)))))\Big)
\end{aligned}
\tag{F.14}
$$

$$
\begin{aligned}
\hat{n}_u=&-\frac{i}{4\left(s-4m^2\right)^2}\Big(192(\alpha_3(-\beta_3\gamma_1\sigma_1+\beta_2\gamma_3\sigma_1-\beta_3\gamma_2\sigma_2+\beta_1\gamma_3\sigma_2)+\alpha_1(\beta_1\gamma_1\sigma_1-\beta_2\gamma_2\sigma_1\\
&+\beta_3\gamma_2\sigma_3-\beta_1\gamma_3\sigma_3)+\alpha_2(-\beta_1\gamma_1\sigma_2+\beta_2\gamma_2\sigma_2+\beta_3\gamma_1\sigma_3-\beta_2\gamma_3\sigma_3))m^8-16(2s(\alpha_2(\\
&-7\beta_1\gamma_1\sigma_2+3\beta_2\gamma_2\sigma_2-6\beta_3\gamma_3\sigma_2+13\beta_3\gamma_1\sigma_3)+\alpha_1(3\beta_1\gamma_1\sigma_1-7\beta_2\gamma_2\sigma_1-6\beta_3\gamma_3\sigma_1\\
&+13\beta_3\gamma_2\sigma_3)+\alpha_3(13\beta_2\gamma_3\sigma_1+13\beta_1\gamma_3\sigma_2-6\beta_1\gamma_1\sigma_3-6\beta_2\gamma_2\sigma_3-12\beta_3\gamma_3\sigma_3))+t(\alpha_3(\beta_2(\\
&-\gamma_3\sigma_1+5\gamma_3\sigma_2-8\gamma_1\sigma_3+4\gamma_2\sigma_3)+4\beta_3(-3\gamma_1\sigma_1+2\gamma_2\sigma_1+2\gamma_1\sigma_2-3\gamma_2\sigma_2+\gamma_3\sigma_3)\\
&+\beta_1(5\gamma_3\sigma_1-\gamma_3\sigma_2+4\gamma_1\sigma_3-8\gamma_2\sigma_3))-\alpha_2(\beta_2(-26\gamma_1\sigma_1+5\gamma_2\sigma_1+5\gamma_1\sigma_2-8\gamma_2\sigma_2\\
&+12\gamma_3\sigma_3)+\beta_3(8\gamma_3\sigma_1-4\gamma_3\sigma_2+\gamma_1\sigma_3-5\gamma_2\sigma_3)+\beta_1(5\gamma_1\sigma_1+6\gamma_1\sigma_2+5\gamma_2\sigma_2-8\gamma_3\sigma_3))\\
&-\alpha_1(\beta_3(-4\gamma_3\sigma_1+8\gamma_3\sigma_2-5\gamma_1\sigma_3+\gamma_2\sigma_3)+\beta_1(-8\gamma_1\sigma_1+5\gamma_2\sigma_1+5\gamma_1\sigma_2-26\gamma_2\sigma_2\\
&+12\gamma_3\sigma_3)+\beta_2(5\gamma_1\sigma_1+6\gamma_2\sigma_1+5\gamma_2\sigma_2-8\gamma_3\sigma_3))))m^6-8\sqrt{2}\sqrt{stu}(\alpha_3(\beta_2(-13\gamma_1\sigma_1\\
&+5\gamma_2\sigma_1-8\gamma_2\sigma_2+19\gamma_3\sigma_3)+19\beta_3(\gamma_3\sigma_1-\gamma_3\sigma_2+\gamma_1\sigma_3-\gamma_2\sigma_3)+\beta_1(8\gamma_1\sigma_1-5\gamma_1\sigma_2\\
&+13\gamma_2\sigma_2-19\gamma_3\sigma_3))+\alpha_2(-5\beta_1\gamma_3\sigma_2+5\beta_1\gamma_1\sigma_3+\beta_2(-13\gamma_3\sigma_1+8\gamma_3\sigma_2-13\gamma_1\sigma_3\\
&+8\gamma_2\sigma_3)+\beta_3(-13\gamma_1\sigma_1+5\gamma_1\sigma_2-8\gamma_2\sigma_2+19\gamma_3\sigma_3))+\alpha_1(\beta_1(-8\gamma_3\sigma_1+13\gamma_3\sigma_2-8\gamma_1\sigma_3\\
&+13\gamma_2\sigma_3)+5\beta_2(\gamma_3\sigma_1-\gamma_2\sigma_3)+\beta_3(8\gamma_1\sigma_1-5\gamma_2\sigma_1+13\gamma_2\sigma_2-19\gamma_3\sigma_3)))m^5
\end{aligned}
$$

$$
\begin{aligned}
&+4((\alpha_1(3\beta_1\gamma_1\sigma_1-19\beta_2\gamma_2\sigma_1-32\beta_3\gamma_3\sigma_1+59\beta_3\gamma_2\sigma_3+17\beta_1\gamma_3\sigma_3)+\alpha_2(-19\beta_1\gamma_1\sigma_2\\
&+3\beta_2\gamma_2\sigma_2-32\beta_3\gamma_3\sigma_2+59\beta_3\gamma_1\sigma_3+17\beta_2\gamma_3\sigma_3)+\alpha_3(59\beta_2\gamma_3\sigma_1+59\beta_1\gamma_3\sigma_2-32\beta_1\gamma_1\sigma_3\\
&-32\beta_2\gamma_2\sigma_3+17\beta_3(\gamma_1\sigma_1+\gamma_2\sigma_2-4\gamma_3\sigma_3)))s^2+t(\alpha_2(\beta_2(50\gamma_1\sigma_1-5\gamma_2\sigma_1-5\gamma_1\sigma_2+8\gamma_2\sigma_2\\
&+10\gamma_3\sigma_3)+\beta_3(6\gamma_3\sigma_1-22\gamma_3\sigma_2+41\gamma_1\sigma_3-7\gamma_2\sigma_3)-\beta_1(5\gamma_1\sigma_1+22\gamma_1\sigma_2+5\gamma_2\sigma_2\\
&-6\gamma_3\sigma_3))+\alpha_1(\beta_3(-22\gamma_3\sigma_1+6\gamma_3\sigma_2-7\gamma_1\sigma_3+41\gamma_2\sigma_3)+\beta_1(8\gamma_1\sigma_1-5\gamma_2\sigma_1-5\gamma_1\sigma_2\\
&+50\gamma_2\sigma_2+10\gamma_3\sigma_3)-\beta_2(5\gamma_1\sigma_1+22\gamma_2\sigma_1+5\gamma_2\sigma_2-6\gamma_3\sigma_3))+\alpha_3(\beta_1(-7\gamma_3\sigma_1+41\gamma_3\sigma_2\\
&-22\gamma_1\sigma_3+6\gamma_2\sigma_3)+\beta_2(41\gamma_3\sigma_1-7\gamma_3\sigma_2+6\gamma_1\sigma_3-22\gamma_2\sigma_3)+2\beta_3(5\gamma_1\sigma_1+3\gamma_2\sigma_1+3\gamma_1\sigma_2\\
&+5\gamma_2\sigma_2-38\gamma_3\sigma_3)))s+t^2(4\alpha_3((\beta_1-\beta_2)(\gamma_3(\sigma_1-\sigma_2)+(\gamma_1-\gamma_2)\sigma_3)-2\beta_3(\gamma_1-\gamma_2)(\sigma_1\\
&-\sigma_2))-\alpha_2(\beta_2(-21\gamma_1\sigma_1+5\gamma_2\sigma_1+5\gamma_1\sigma_2-5\gamma_2\sigma_2+8\gamma_3\sigma_3)+4\beta_3(\gamma_3\sigma_1-\gamma_3\sigma_2+\gamma_1\sigma_3\\
&-\gamma_2\sigma_3)+\beta_1(5\gamma_1\sigma_1+3\gamma_2\sigma_1+3\gamma_1\sigma_2+5\gamma_2\sigma_2-8\gamma_3\sigma_3))-\alpha_1(4\beta_3(-\gamma_3\sigma_1+\gamma_3\sigma_2-\gamma_1\sigma_3\\
&+\gamma_2\sigma_3)+\beta_1(-5\gamma_1\sigma_1+5\gamma_2\sigma_1+5\gamma_1\sigma_2-21\gamma_2\sigma_2+8\gamma_3\sigma_3)+\beta_2(5\gamma_1\sigma_1+3\gamma_2\sigma_1+3\gamma_1\sigma_2\\
&+5\gamma_2\sigma_2-8\gamma_3\sigma_3))))m^4+2\sqrt{2}\sqrt{stu}(t(\alpha_3(\beta_2(-17\gamma_1\sigma_1+5\gamma_2\sigma_1+5\gamma_1\sigma_2-9\gamma_2\sigma_2\\
&+24\gamma_3\sigma_3)+24\beta_3(\gamma_3\sigma_1-\gamma_3\sigma_2+\gamma_1\sigma_3-\gamma_2\sigma_3)+\beta_1(9\gamma_1\sigma_1-5\gamma_2\sigma_1-5\gamma_1\sigma_2+17\gamma_2\sigma_2\\
&-24\gamma_3\sigma_3))+\alpha_2(\beta_2(-17\gamma_3\sigma_1+9\gamma_3\sigma_2-17\gamma_1\sigma_3+9\gamma_2\sigma_3)+\beta_3(-17\gamma_1\sigma_1+5\gamma_2\sigma_1\\
&+5\gamma_1\sigma_2-9\gamma_2\sigma_2+24\gamma_3\sigma_3)+5\beta_1(\gamma_3\sigma_1-\gamma_3\sigma_2+\gamma_1\sigma_3-\gamma_2\sigma_3))+\alpha_1(\beta_1(-9\gamma_3\sigma_1+17\gamma_3\sigma_2\\
&-9\gamma_1\sigma_3+17\gamma_2\sigma_3)+5\beta_2(\gamma_3\sigma_1-\gamma_3\sigma_2+\gamma_1\sigma_3-\gamma_2\sigma_3)+\beta_3(9\gamma_1\sigma_1-5\gamma_2\sigma_1-5\gamma_1\sigma_2\\
&+17\gamma_2\sigma_2-24\gamma_3\sigma_3)))+s(\alpha_3(\beta_2(-25\gamma_1\sigma_1+5\gamma_2\sigma_1-12\gamma_2\sigma_2+33\gamma_3\sigma_3)+33\beta_3(\gamma_3\sigma_1\\
&-\gamma_3\sigma_2+\gamma_1\sigma_3-\gamma_2\sigma_3)+\beta_1(12\gamma_1\sigma_1-5\gamma_1\sigma_2+25\gamma_2\sigma_2-33\gamma_3\sigma_3))+\alpha_2(-5\beta_1\gamma_3\sigma_2\\
&+5\beta_1\gamma_1\sigma_3+\beta_2(-25\gamma_3\sigma_1+12\gamma_3\sigma_2-25\gamma_1\sigma_3+12\gamma_2\sigma_3)+\beta_3(-12\gamma_2\sigma_2+5\gamma_1(\sigma_2-5\sigma_1)\\
&+33\gamma_3\sigma_3))+\alpha_1(\beta_1(-12\gamma_3\sigma_1+25\gamma_3\sigma_2-12\gamma_1\sigma_3+25\gamma_2\sigma_3)+5\beta_2(\gamma_3\sigma_1-\gamma_2\sigma_3)\\
&+\beta_3(12\gamma_1\sigma_1-5\gamma_2\sigma_1+25\gamma_2\sigma_2-33\gamma_3\sigma_3))))m^3-2s((\alpha_1(-4\beta_2\gamma_2\sigma_1-14\beta_3\gamma_3\sigma_1\\
&+26\beta_3\gamma_2\sigma_3+11\beta_1\gamma_3\sigma_3)+\alpha_2(-4\beta_1\gamma_1\sigma_2-14\beta_3\gamma_3\sigma_2+26\beta_3\gamma_1\sigma_3+11\beta_2\gamma_3\sigma_3)\\
&+\alpha_3(11\beta_3\gamma_1\sigma_1+26\beta_2\gamma_3\sigma_1+11\beta_3\gamma_2\sigma_2+26\beta_1\gamma_3\sigma_2-14\beta_1\gamma_1\sigma_3-14\beta_2\gamma_2\sigma_3\\
&-32\beta_3\gamma_3\sigma_3))s^2+t(\alpha_2(-\beta_1(8\gamma_1\sigma_2+\gamma_3\sigma_3)+\beta_2(12\gamma_1\sigma_1+23\gamma_3\sigma_3)+\beta_3(15\gamma_3\sigma_1-21\gamma_3\sigma_2\\
&+37\gamma_1\sigma_3-10\gamma_2\sigma_3))+\alpha_1(\beta_3(-21\gamma_3\sigma_1+15\gamma_3\sigma_2-10\gamma_1\sigma_3+37\gamma_2\sigma_3)-\beta_2(8\gamma_2\sigma_1+\gamma_3\sigma_3)\\
&+\beta_1(12\gamma_2\sigma_2+23\gamma_3\sigma_3))+\alpha_3(\beta_1(-10\gamma_3\sigma_1+37\gamma_3\sigma_2-21\gamma_1\sigma_3+15\gamma_2\sigma_3)+\beta_2(37\gamma_3\sigma_1
\end{aligned}
$$

$$
\begin{aligned}
&-10\gamma_3\sigma_2+15\gamma_1\sigma_3-21\gamma_2\sigma_3)+\beta_3(23\gamma_1\sigma_1-\gamma_2\sigma_1-\gamma_1\sigma_2+23\gamma_2\sigma_2-72\gamma_3\sigma_3)))s+t^2(\alpha_2(\\
&-\beta_1(4\gamma_2\sigma_1+4\gamma_1\sigma_2+\gamma_3\sigma_3)+\beta_2(8\gamma_1\sigma_1+9\gamma_3\sigma_3)+\beta_3(11\gamma_3\sigma_1-7\gamma_3\sigma_2+11\gamma_1\sigma_3\\
&-7\gamma_2\sigma_3))-\alpha_1(\beta_2(4\gamma_2\sigma_1+4\gamma_1\sigma_2+\gamma_3\sigma_3)-\beta_1(8\gamma_2\sigma_2+9\gamma_3\sigma_3)+\beta_3(7\gamma_3\sigma_1-11\gamma_3\sigma_2\\
&+7\gamma_1\sigma_3-11\gamma_2\sigma_3))+\alpha_3(\beta_1(-7\gamma_3\sigma_1+11\gamma_3\sigma_2-7\gamma_1\sigma_3+11\gamma_2\sigma_3)+\beta_2(11\gamma_3\sigma_1-7\gamma_3\sigma_2\\
&+11\gamma_1\sigma_3-7\gamma_2\sigma_3)+\beta_3(9\gamma_1\sigma_1-\gamma_2\sigma_1-\gamma_1\sigma_2+9\gamma_2\sigma_2-32\gamma_3\sigma_3))))m^2-\sqrt{2}s(s\\
&+t)\sqrt{stu}(2\alpha_2\beta_2(\gamma_3(\sigma_2-3\sigma_1)+(\gamma_2-3\gamma_1)\sigma_3)-2\alpha_1\beta_1(\gamma_3(\sigma_1-3\sigma_2)+(\gamma_1-3\gamma_2)\sigma_3)\\
&+\alpha_2\beta_3(-6\gamma_1\sigma_1-2\gamma_2\sigma_2+7\gamma_3\sigma_3)+\alpha_1\beta_3(2\gamma_1\sigma_1+6\gamma_2\sigma_2-7\gamma_3\sigma_3)+\alpha_3(\beta_2(-6\gamma_1\sigma_1\\
&-2\gamma_2\sigma_2+7\gamma_3\sigma_3)+7\beta_3(\gamma_3\sigma_1-\gamma_3\sigma_2+\gamma_1\sigma_3-\gamma_2\sigma_3)+\beta_1(2\gamma_1\sigma_1+6\gamma_2\sigma_2-7\gamma_3\sigma_3)))m\\
&+s^2(s+t)(s(2(\alpha_1(-\beta_3\gamma_3\sigma_1+2\beta_3\gamma_2\sigma_3+\beta_1\gamma_3\sigma_3)+\alpha_2(-\beta_3\gamma_3\sigma_2+2\beta_3\gamma_1\sigma_3+\beta_2\gamma_3\sigma_3))\\
&+\alpha_3(4\beta_2\gamma_3\sigma_1+4\beta_1\gamma_3\sigma_2-2\beta_1\gamma_1\sigma_3-2\beta_2\gamma_2\sigma_3+\beta_3(2\gamma_1\sigma_1+2\gamma_2\sigma_2-5\gamma_3\sigma_3)))\\
&+t(4\alpha_1\beta_1\gamma_3\sigma_3+4\alpha_2\beta_2\gamma_3\sigma_3-2\alpha_1\beta_3(\gamma_3(\sigma_1-2\sigma_2)+(\gamma_1-2\gamma_2)\sigma_3)+2\alpha_2\beta_3(2\gamma_3\sigma_1-\gamma_3\sigma_2\\
&+2\gamma_1\sigma_3-\gamma_2\sigma_3)+\alpha_3(\beta_3(4\gamma_1\sigma_1+4\gamma_2\sigma_2-11\gamma_3\sigma_3)-2(\beta_2\gamma_3(\sigma_2-2\sigma_1)+\beta_1\gamma_3(\sigma_1-2\sigma_2)\\
&+\beta_2(\gamma_2-2\gamma_1)\sigma_3+\beta_1(\gamma_1-2\gamma_2)\sigma_3))))\Big)
\end{aligned}
\tag{F.15}
$$

Appendix G

BCJ Relations in 5pt Massive Amplitudes

In 5pt scattering the size of block diagonal matrix A (defined by (2.7)) is 9×9 and it can be expressed as:

$$
A = \begin{pmatrix}
B_1 & D_{12}D_{34} & 0 & 0 & -D_{15}D_{34} & 0 & D_{25}D_{34} & 0 & 0 \\
D_{12}D_{34} & B_2 & D_{12}D_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & D_{12}D_{45} & B_3 & D_{23}D_{45} & 0 & 0 & 0 & D_{13}D_{45} & 0 \\
0 & 0 & D_{23}D_{45} & B_4 & D_{14}D_{23} & 0 & 0 & 0 & \\
-D_{15}D_{34} & 0 & 0 & D_{15}D_{23} & B_5 & 0 & 0 & 0 & -D_{15}D_{24} \\
0 & 0 & 0 & D_{14}D_{23} & 0 & B_6 & D_{14}D_{25} & 0 & 0 \\
D_{25}D_{34} & 0 & 0 & 0 & 0 & D_{14}D_{25} & B_7 & -D_{13}D_{25} & 0 \\
0 & 0 & D_{13}D_{45} & 0 & 0 & 0 & -D_{13}D_{25} & B_8 & D_{13}D_{24} \\
0 & 0 & 0 & 0 & -D_{15}D_{24} & 0 & 0 & D_{13}D_{24} & B_9
\end{pmatrix},
$$

where the B_i's are:

$$\begin{aligned}
B_1 &= D_{12}D_{34} + D_{15}D_{34} + D_{25}D_{34},\\
B_2 &= D_{12}D_{34} + D_{12}D_{35} + D_{12}D_{45},\\
B_3 &= D_{12}D_{45} + D_{13}D_{45} + D_{23}D_{45},\\
B_4 &= D_{14}D_{23} + D_{15}D_{23} + D_{23}D_{45},\\
B_5 &= D_{15}D_{23} + D_{15}D_{24} + D_{15}D_{34},\\
B_6 &= D_{14}D_{23} + D_{14}D_{25} + D_{14}D_{34},\\
B_7 &= D_{13}D_{25} + D_{14}D_{25} + D_{25}D_{34},\\
B_8 &= D_{13}D_{24} + D_{13}D_{25} + D_{13}D_{45},\\
B_9 &= D_{13}D_{24} + D_{15}D_{24} + D_{24}D_{35}.
\end{aligned} \tag{G.1}$$

Imposing the spectral conditions (2.69), reduces the rank of A from 9 to 5, where the following vectors form the basis for the null space of A:

$$\begin{aligned}
u_1^T &= \left(0, \frac{D_{24}}{D_{12}}, -\frac{D_{24}}{D_{12}+D_{13}+D_{23}}, 0, -\frac{D_{24}}{D_{12}+D_{13}+D_{14}}, 0, 0, 0, 1\right),\\
u_2^T &= \left(-\frac{D_{13}}{D_{12}+D_{13}+D_{14}+D_{23}+D_{24}}, 0, -\frac{D_{13}}{D_{12}+D_{13}+D_{23}}, 0, -\frac{D_{13}}{D_{12}+D_{13}+D_{14}}, 0, 0, 1, 0\right),\\
u_3^T &= \left(-\frac{D_{12}+D_{23}+D_{24}}{D_{12}+D_{13}+D_{14}+D_{23}+D_{24}}, 0, -\frac{D_{12}+D_{23}+D_{24}}{D_{12}+D_{13}+D_{23}}, -\frac{-D_{12}-D_{23}-D_{24}}{D_{23}}, 0, 0, 1, 0, 0\right),\\
u_4^T &= \left(\frac{D_{14}}{D_{12}+D_{13}+D_{14}+D_{23}+D_{24}}, \frac{D_{14}}{D_{12}}, 0, -\frac{D_{14}}{D_{23}}, 0, 1, 0, 0, 0\right).
\end{aligned} \tag{G.2}$$

These null vectors determine the BCJ relations via (2.10), expressed here as linear relations on the kinematic factors. To obtain (2.70), (2.71), (2.72) and (2.73) once the spectral condition is applied, we consider linear combinations of u_α^T in (2.10),

$$\sum_\alpha \beta_\alpha u_\alpha^T U = 0, \tag{G.3}$$

the values of β_α for a given BCJ relation is given in the following table:

	β_1	β_2	β_3	β_4
(2.70)	A_1	-A_1	-A_1	0
(2.71)	-A_2	A_2	A_2	$\frac{D_{24}}{D_{14}}A_2$
(2.72)	0	-A_3	-A_3	0
(2.73)	0	A_4	$\frac{D_{12}+D_{23}}{D_{12}+D_{23}+D_{24}}A_4$	0

Table G.1: Values of β_α in (G.3) in order to reproduce the four BCJ relations, where
$A_1 = D_{12}(D_{12} + D_{13} + D_{14})D_{23}(D_{12} + D_{13} + D_{14} + D_{23} + D_{24})$,
$A_2 = D_{14}(D_{12} + D_{13} + D_{14})D_{23}(D_{12} + D_{13} + D_{14} + D_{23} + D_{24})$,
$A_3 = (D_{12} + D_{13} + D_{14})D_{23}(D_{12} + D_{13} + D_{23})(D_{12} + D_{13} + D_{14} + D_{23} + D_{24})$,
$A_4 = -(D_{12} + D_{13} + D_{14})D_{23}(D_{12} + D_{23} + D_{24})(D_{12} + D_{13} + D_{14} + D_{23} + D_{24})$.

G.1 Interacting terms in KK inspired action

Here are the explicit expressions of the terms listed in Table 2.1:

$$
\begin{aligned}
\mathcal{L}_{AAA} &= \frac{1}{\sqrt{2}} f^{abc} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} g_{ijk} \left((D_{[\mu} A^{i}_{\nu]})^{a} A^{jb\mu} A^{kc\nu} \right) , \\
\mathcal{L}_{AA\phi} &= \frac{i}{\sqrt{2}} f^{abc} \sum_{i,j\in\mathbb{Z}_{\neq 0}} g'_{ijs} \left(A^{ia}_{\mu} A^{jb\mu} \phi^{c} \right) , \\
\mathcal{L}_{AAAA} &= \frac{-1}{8} f^{abe} f^{cde} \sum_{i,j,k,l\in\mathbb{Z}_{\neq 0}} g_{ijkl} \left(A^{ia}_{[\mu} A^{jb}_{\nu]} A^{kc[\mu} A^{ld\nu]} \right) , \\
\mathcal{L}_{AA\phi\phi} &= \frac{-1}{4} f^{abe} f^{cde} \sum_{i,j\in\mathbb{Z}_{\neq 0}} g_{ijss} \left(A^{ia}_{\mu} A^{jc\mu} \phi^{b} \phi^{d} \right) , \\
\mathcal{L}^{F^3}_{AAA1} &= \frac{4}{\Lambda^2} f^{abc} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} G_{ijk} \left(D^{[\mu} A^{i\nu]} D_{\nu} A^{j\rho} D_{[\rho} A_{k\mu]} \right) , \\
\mathcal{L}^{F^3}_{AAA2} &= \frac{3}{\Lambda^2} f^{abc} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} \hat{G}_{ijk} \left(m_i m_j A^{ai\mu} A^{bj\nu} D_{[\nu} A^{kc}_{\mu]} \right) , \\
\mathcal{L}^{F^3}_{AA\phi} &= \frac{-6i}{\Lambda^2} f^{abc} \sum_{i,j\in\mathbb{Z}_{\neq 0}} G'_{ijs} \left(A^{ia\mu} D^{\rho} \phi^{b} D_{[\rho} A^{jc}_{\mu]} \right) , \\
\mathcal{L}^{F^3}_{A\phi\phi} &= \frac{3}{2\Lambda^2} f^{abc} G_{0ss} \left(D^{\mu} \phi^{a} D^{\nu} \phi^{b} F^{0c}_{\nu\mu} \right) , \\
\mathcal{L}^{F^3}_{AAAA1} &= \frac{3\sqrt{2}}{\Lambda^2} f^{abe} f^{cde} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} G_{ijkl} \left(D^{[\mu} A^{ia\nu]} D_{[\nu} A^{jb}_{\rho]} A^{kc\rho} A^{dl}_{\mu} \right) , \\
\mathcal{L}^{F^3}_{AAAA2} &= \frac{-3}{2\sqrt{2}\Lambda^2} f^{abe} f^{cde} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} \hat{G}_{ijkl} \left(m_i m_j A^{ia}_{\mu} A^{jb\nu} A^{kc\mu} A^{ld}_{\nu} \right) , \\
\mathcal{L}^{F^3}_{AA\phi\phi1} &= \frac{3}{2\sqrt{2}\Lambda^2} f^{abe} f^{cde} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} G_{ijss} \left(A^{ia}_{\mu} A^{jb}_{\nu} D^{\mu} \phi^{c} D^{\nu} \phi^{d} \right) , \\
\mathcal{L}^{F^3}_{AA\phi\phi2} &= \frac{3\sqrt{2}}{\Lambda^2} f^{abe} f^{cde} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} \hat{G}_{ijss} \left(A^{ia\mu} \phi^{b} D_{[\mu} A^{jc}_{\nu]} D^{\nu} \phi^{d} \right) , \\
\mathcal{L}^{F^3}_{AAA\phi1} &= \frac{-3i}{\sqrt{2}\Lambda^2} f^{abe} f^{cde} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} \hat{G}_{ijks} \left(m_k A^{ia}_{\mu} A^{jb}_{\nu} A^{kc\mu} D^{\nu} \phi^{d} \right) ,
\end{aligned}
\tag{G.4}
$$

$$\mathcal{L}^{F^3}_{AAA\phi 2} = \frac{-3\sqrt{2}i}{\Lambda^2} f^{abe} f^{cde} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} G_{ijks}\left(m_i A^{ja\mu}\phi^b A^{ic\nu} D_{[\mu}A^{kd}_{\nu]}\right),$$
$$\mathcal{L}^{F^3}_{AAAAA} = \frac{3}{2\Lambda^2} f^{a_1bc} f^{ba_2a_3} f^{ca_4a_5} \sum_{i,j,k,l,m\in\mathbb{Z}_{\neq 0}} G_{ijklm}\left(D^{[\mu}A^{ia_1\nu]} A^{ja_2\nu} A^{ka_3\rho} A^{la_4}_{\rho} A^{ma_5\mu}\right),$$
$$\mathcal{L}^{F^3}_{\phi AAAA} = \frac{3i}{2\Lambda^2} f^{a_1bc} f^{ba_2a_3} f^{ca_4a_5} \sum_{i,j,k,l\in\mathbb{Z}_{\neq 0}} G_{ijkls}\left(m_i A^{ia_1\mu}\phi^{a_2} A^{ja_3\nu} A^{ka_4}_{\mu} A^{ma_5}_{\nu}\right), \quad \text{(G.5)}$$
$$\mathcal{L}^{F^3}_{\phi\phi AAA1} = \frac{-3}{2\Lambda^2} f^{a_1bc} f^{ba_2a_3} f^{ca_4a_5} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} G_{ijkss}\left(D^{\mu}\phi^{a_1}\phi^{a_2} A^{ia_3\nu} A^{ja_4}_{\mu} A^{ka_5}_{\nu}\right),$$
$$\mathcal{L}^{F^3}_{\phi\phi AAA2} = \frac{3}{2\Lambda^2} f^{a_1bc} f^{ba_2a_3} f^{ca_4a_5} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} \hat{G}_{ijks}\left(D_{[\mu}A^{ia_1}_{\nu]}\phi^{a_2} A^{ja_3\mu}\phi^{a_4} A^{ka_5\nu}\right),$$

$$\mathcal{L}^{F^4}_{AAAA1} = \frac{-9}{\Lambda^4} f^{abe} f^{cde} \sum_{i,j,k,l\in\mathbb{Z}} c_{ijkl}\left(D^{[\mu}A^{ia\nu]} D_{[\alpha}A^{jb}_{\beta]} D_{\mu}A^{ck}_{\nu} D^{[\alpha}A^{ld\beta]}\right), \quad \text{(G.6)}$$
$$\mathcal{L}^{F^4}_{AAAA2} = \frac{9}{\Lambda^4} f^{abe} f^{cde} \sum_{i,j,k,l\in\mathbb{Z}} C_{ijkl}\left(m_i A^{ia\mu} D_{[\alpha}A^{jb}_{\beta]} m_k A^{ck}_{\mu} D^{[\alpha}A^{ld\beta]}\right),$$
$$\mathcal{L}^{F^4}_{AAA\phi 1} = \frac{18i}{\Lambda^4} f^{abe} f^{cde} \sum_{i,j,k\in\mathbb{Z}} c_{ijks}\left(m_i A^{ia\mu} D_{[\alpha}A^{jb}_{\beta]} D_{\mu}\phi^c D^{[\alpha}A^{ld}_{\beta]}\right),$$
$$\mathcal{L}^{F^4}_{AAA\phi 2} = \frac{-9i}{\Lambda^4} f^{abe} f^{cde} \sum_{i,j,k\in\mathbb{Z}} C_{ijks}\left(m_i A^{ia\mu} m_j A^{jb\nu} D_{\mu}\phi^c m_k A^{ld}_{\nu}\right),$$
$$\mathcal{L}^{F^4}_{AA\phi\phi 1} = \frac{-9}{\Lambda^4} f^{abe} f^{cde} \sum_{i,j\in\mathbb{Z}} c_{ijss}\left(D^{[\mu}A^{ia\nu]} D_{\alpha}\phi^b D_{\mu}A^{ck}_{\nu} D^{\alpha}\phi^d\right),$$
$$\mathcal{L}^{F^4}_{AA\phi\phi 2} = \frac{9}{2\Lambda^4} f^{abe} f^{cde} \sum_{i,j\in\mathbb{Z}} c^{(2)}_{ijss}\left(m_i A^{ia\mu} D_{\nu}\phi^b D_{\mu}\phi^c m_j A^{jd\nu}\right),$$
$$\mathcal{L}^{F^4}_{AA\phi\phi 3} = \frac{9}{2\Lambda^4} f^{abe} f^{cde} \sum_{i,j\in\mathbb{Z}} c^{(3)}_{ijss}\left(m_i A^{ia\mu} D_{\nu}\phi^b D^{\nu}\phi^c m_j A_{jd\mu}\right),$$
$$\mathcal{L}^{F^4}_{\phi\phi\phi\phi} = -\frac{9}{4\Lambda^4} f^{abe} f^{cde}\, c_{\phi 4}\left(D^{\mu}\phi^a D_{\nu}\phi^b D_{\mu}\phi^c D^{\nu}\phi^d\right)$$
$$\mathcal{L}^{F^4}_{AAAAA1} = \frac{-18}{\sqrt{2}\Lambda^4} f^{a_1a_2b} f^{a_3cb} f^{ca_4a_5} \sum_{i,j,k,l,m\in\mathbb{Z}_{\neq 0}} c_{ijklm}\left(D^{[\mu}A^{ia_1\nu]} D_{[\alpha}A^{ja_2}_{\beta]} D_{[\mu}A^{ka_3}_{\nu]} A^{la_4\alpha} A^{ma_5\beta}\right),$$
$$\mathcal{L}^{F^4}_{AAAAA2} = \frac{9}{\sqrt{2}\Lambda^4} f^{a_1a_2b} f^{a_3cb} f^{ca_4a_5} \sum_{i,j,k,l,m\in\mathbb{Z}_{\neq 0}} C_{ijklm}\left(m_i m_k A^{ia_1\mu} D_{[\alpha}A^{ja_2}_{\beta]} A^{ka_3}_{\mu} A^{la_4\alpha} A^{ma_5\beta}\right),$$
$$\mathcal{L}^{F^4}_{\phi AAAA1} = \frac{-9i}{\sqrt{2}\Lambda^4} f^{a_1a_2b} f^{a_3cb} f^{ca_4a_5} \sum_{i,j,k,l\in\mathbb{Z}_{\neq 0}} c_{ijkls}\left(m_j D^{\mu}\phi^{a_1} D_{[\alpha}A^{ia_2}_{\beta]} A^{ja_3}_{\mu} A^{ka_4\alpha} A^{la_5\beta}\right),$$

$$\mathcal{L}^{F^4}_{\phi AAAA2} = \frac{-18i}{\sqrt{2}\Lambda^4} f^{a_1a_2b} f^{a_3cb} f^{ca_4a_5} \sum_{i,j,k,l\in\mathbb{Z}_{\neq 0}} C_{ijkls} \left(m_j D^{[\mu} A^{ia_1\nu]} A^{ja_2}_{\beta} D_{[\mu} A^{ka_3}_{\nu]} \phi^{a_4} A^{la_5\beta} \right),$$

$$\mathcal{L}^{F^4}_{\phi AAAA3} = \frac{-9i}{\sqrt{2}\Lambda^4} f^{a_1a_2b} f^{a_3cb} f^{ca_4a_5} \sum_{i,j,k,l\in\mathbb{Z}_{\neq 0}} \hat{C}_{ijkls} \left(m_i m_j m_k A^{ia_1\mu} A^{ja_2}_{\beta} A^{ka_3}_{\mu} \phi^{a_4} A^{la_5\beta} \right),$$

$$\mathcal{L}^{F^4}_{\phi\phi AAA1} = \frac{9}{\sqrt{2}\Lambda^4} f^{a_1a_2b} f^{a_3cb} f^{ca_4a_5} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} c_{ijkss} \left(D^{\mu} \phi^{a_1} D_{[\alpha} A^{ia_2}_{\beta]} D_{\mu} \phi^{a_3} A^{ka_4\alpha} A^{la_5\beta} \right),$$

$$\mathcal{L}^{F^4}_{\phi\phi AAA2} = \frac{18}{\sqrt{2}\Lambda^4} f^{a_1a_2b} f^{a_3cb} f^{ca_4a_5} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} C_{ijkss} \left(D^{[\mu} A^{ia_1\nu]} D_{\beta} \phi^{a_2} D_{[\mu} A^{ka_3}_{\nu]} \phi^{a_4} A^{la_5\beta} \right),$$

$$\mathcal{L}^{F^4}_{\phi\phi AAA3} = \frac{-9}{\sqrt{2}\Lambda^4} f^{a_1a_2b} f^{a_3cb} f^{ca_4a_5} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} \hat{C}^{(3)}_{ijkss} \left(m_i m_j A^{ia_1\mu} D_{\beta} \phi^{a_2} A^{ja_3}_{\mu} \phi^{a_4} A^{ka_5\beta} \right),$$

$$\mathcal{L}^{F^4}_{\phi\phi AAA4} = \frac{-9}{\sqrt{2}\Lambda^4} f^{a_1a_2b} f^{a_3cb} f^{ca_4a_5} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} \hat{C}^{(4)}_{ijkss} \left(m_i m_j A^{ia_1\mu} D_{\beta} \phi^{a_2} A^{ja_3\beta} \phi^{a_4} A^{ka_5}_{\mu} \right),$$

$$\mathcal{L}^{F^4}_{\phi\phi AAA5} = \frac{-9}{\sqrt{2}\Lambda^4} f^{a_1a_4b} f^{ca_3a_5} f^{a_2bc} \sum_{i,j,k\in\mathbb{Z}_{\neq 0}} \hat{C}^{(5)}_{ijkss} \left(m_j m_k A^{ia_1\mu} D_{\beta} \phi^{a_2} A^{ja_3}_{\mu} \phi^{a_4} A^{ka_5\beta} \right),$$

$$\mathcal{L}^{F^4}_{\phi\phi\phi AA1} = \frac{-9i}{\sqrt{2}\Lambda^4} f^{a_1a_5b} f^{a_2bc} f^{a_3a_4c} \sum_{i,j\in\mathbb{Z}_{\neq 0}} c_{ijsss} \left(m_j A^{ia_1\mu} A^{ja_2\nu} D_{\mu} \phi^{a_3} D_{\nu} \phi^{a_4} \phi^{a_5} \right),$$

$$\mathcal{L}^{F^4}_{\phi\phi\phi AA2} = \frac{-9i}{\sqrt{2}\Lambda^4} f^{a_1a_4b} f^{a_3bc} f^{a_2a_5c} \sum_{i,j\in\mathbb{Z}_{\neq 0}} c^{(2)}_{ijsss} \left(m_i A^{ia_1\mu} A^{ja_2\nu} D_{\mu} \phi^{a_3} D_{\nu} \phi^{a_4} \phi^{a_5} \right),$$

$$\mathcal{L}^{F^4}_{\phi\phi\phi AA3} = \frac{-9}{\sqrt{2}\Lambda^4} f^{a_1a_2b} f^{a_3cb} f^{ca_4a_5} \sum_{i,j\in\mathbb{Z}_{\neq 0}} c^{(3)}_{ijsss} \left(m_i D_{\mu} \phi^{a_1} A^{ia_2}_{\beta} D_{\mu} \phi^{a_3} \phi^{a_4} A^{la_5\beta} \right).$$

Here a, b, c, d, e are the indices of the adjoint representation of the gauge group and various numerical factors are chosen such that in the case of 4d KK reduction of 5d Yang-Mills with coupling g plus $\frac{G}{\Lambda^2}\mathrm{tr}(F^3)$ and $\frac{-9G^2}{16\Lambda^2}\mathrm{tr}([F,F]^2)$ operators all couplings appear without any numerical factors, *i.e.* they are expressed as products of g and G.

Appendix H

Double Copy in 3d

H.1 Explicit Expressions of q_i and e_i

In this appendix we give the explicit expressions for q_i and e_i which were introduced in (2.116). Since we can split this expression in different ways, we have the freedom to choose different expressions for q_i and e_i. Here we show two different cases that are useful to understand the residues at the spurious poles. One possible choice for q_i and e_i is

$$\begin{aligned}
q_1 &= -u_2 - u_4 - u_7 + u_9 \ , \\
q_2 &= 0 \ , \\
q_3 &= -u_2 - u_4 - u_7 + u_9 \ , \\
q_4 &= u_1 - u_2 + u_3 - u_4 + u_6 - u_7 + 2u_9 \ , \\
q_5 &= u_2 + u_3 + 2u_4 + u_5 + u_6 + u_7 - u_9 \ , \\
q_6 &= u_3 + u_4 + u_5 + u_6 \ ,
\end{aligned}$$

$$e_1=\frac{1}{2}\begin{bmatrix} -m^2\left(15s_{12}+17s_{13}+4s_{14}+10s_{23}+6s_{24}\right)+46m^4+s_{12}^2+2s_{13}^2+s_{14}s_{23}+s_{13}s_{24}+s_{12}\left(4s_{13}+s_{14}+s_{23}+s_{24}\right)\\ -m^2\left(-7s_{12}-11s_{13}-14s_{14}-7s_{23}-3s_{24}\right)-31m^4-2s_{13}^2+s_{12}\left(-2s_{13}-3s_{14}\right)-4s_{13}s_{14}-2s_{13}s_{23}-3s_{14}s_{23}+s_{13}s_{24}-2s_{14}s_{24}\\ -m^2\left(8s_{12}+13s_{13}-5s_{14}+4s_{23}+4s_{24}\right)+32m^4+2s_{13}^2-2s_{14}^2+s_{12}\left(3s_{13}+s_{14}\right)\\ 0\\ -m^2\left(7s_{12}+11s_{13}+14s_{14}+7s_{23}+3s_{24}\right)+31m^4+2s_{13}^2+4s_{13}s_{14}+s_{12}\left(2s_{13}+3s_{14}\right)+2s_{13}s_{23}+3s_{14}s_{23}-s_{13}s_{24}+2s_{14}s_{24}\\ -m^2\left(22s_{12}+28s_{13}+18s_{14}+17s_{23}+9s_{24}\right)+77m^4+s_{12}^2+4s_{13}^2+4s_{13}s_{14}+2s_{13}s_{23}+4s_{14}s_{23}+2s_{14}s_{24}+s_{12}\left(6s_{13}+4s_{14}+s_{23}+s_{24}\right)\\ -m^2\left(7s_{12}+4s_{13}+9s_{14}+6s_{23}+2s_{24}\right)+14m^4+s_{12}^2+2s_{14}^2+s_{14}s_{23}+s_{13}s_{24}+s_{12}\left(s_{13}+s_{23}+s_{24}\right)\\ -m^2\left(-14s_{12}-15s_{13}-23s_{14}-13s_{23}-5s_{24}\right)-45m^4-s_{12}^2-2s_{13}^2-2s_{14}^2-4s_{13}s_{14}-2s_{13}s_{23}-4s_{14}s_{23}+s_{12}\left(-3s_{13}-3s_{14}-s_{23}-s_{24}\right)-2s_{14}s_{24}\\ -m^2\left(8s_{12}+13s_{13}-5s_{14}+4s_{23}+4s_{24}\right)+32m^4+2s_{13}^2-2s_{14}^2+s_{12}\left(3s_{13}+s_{14}\right) \end{bmatrix},$$

$$e_2=\begin{bmatrix} m^2\left(2s_{12}+2s_{23}+2s_{24}\right)-10m^4+s_{13}s_{14}\\ m^2\left(-2s_{12}-5s_{14}-2s_{23}-2s_{24}\right)+10m^4+s_{14}\left(s_{12}+s_{13}+s_{23}+s_{24}\right)\\ m^2\left(2s_{12}-4s_{14}+2s_{23}+2s_{24}\right)-10m^4+s_{14}\left(s_{13}+s_{14}\right)\\ 0\\ m^2\left(2s_{12}+5s_{14}+2s_{23}+2s_{24}\right)-10m^4+s_{14}\left(-s_{12}-s_{13}-s_{23}-s_{24}\right)\\ m^2\left(4s_{12}+5s_{14}+4s_{23}+4s_{24}\right)-20m^4+s_{14}\left(-s_{12}-s_{23}-s_{24}\right)\\ s_{14}\left(4m^2-s_{14}\right)\\ -m^2\left(2s_{12}+9s_{14}+2\left(s_{23}+s_{24}\right)\right)+10m^4+s_{14}\left(s_{12}+s_{13}+s_{14}+s_{23}+s_{24}\right)\\ 2m^2\left(s_{12}-2s_{14}+s_{23}+s_{24}\right)-10m^4+s_{14}\left(s_{13}+s_{14}\right) \end{bmatrix},$$

$$e_3=\frac{1}{2}\begin{bmatrix} -m^2\left(7s_{12}+5s_{13}+2\left(2s_{14}+s_{23}+s_{24}\right)\right)+10m^4+s_{12}^2+s_{14}s_{23}+s_{13}\left(2s_{14}+s_{24}\right)+s_{12}\left(2s_{13}+s_{14}+s_{23}+s_{24}\right)\\ -m^2\left(s_{12}+3s_{13}+4s_{14}+s_{23}+s_{24}\right)+5m^4+s_{12}s_{14}+s_{14}s_{23}+s_{13}\left(2s_{14}+s_{24}\right)\\ \left(s_{13}-s_{14}\right)\left(-\left(m^2-s_{12}\right)\right)\\ 0\\ 4m^2s_{14}+m^2s_{23}+m^2s_{24}+s_{12}\left(m^2-s_{14}\right)+s_{13}\left(3m^2-2s_{14}-s_{24}\right)-5m^4-s_{14}s_{23}\\ \left(m^2-s_{12}\right)\left(5m^2-s_{12}-2s_{13}-s_{23}-s_{24}\right)\\ -m^2\left(7s_{12}+4s_{13}+5s_{14}+2s_{23}+2s_{24}\right)+10m^4+s_{12}^2+s_{14}s_{23}+s_{13}\left(2s_{14}+s_{24}\right)+s_{12}\left(s_{13}+2s_{14}+s_{23}+s_{24}\right)\\ -\left(m^2-s_{12}\right)\left(5m^2-s_{12}-s_{13}-s_{14}-s_{23}-s_{24}\right)\\ \left(s_{13}-s_{14}\right)\left(-\left(m^2-s_{12}\right)\right) \end{bmatrix},$$

$$e_4=\begin{bmatrix} -2m^2\left(s_{12}+s_{13}+s_{23}\right)+8m^4+s_{13}s_{23}\\ m^2\left(11s_{12}+6s_{13}+4s_{14}+11s_{23}+4s_{24}\right)-28m^4-s_{13}\left(s_{12}+s_{23}+s_{24}\right)-\left(s_{12}+s_{23}\right)\left(s_{12}+s_{14}+s_{23}+s_{24}\right)\\ -m^2\left(11s_{12}+6s_{13}+2s_{14}+4s_{23}+4s_{24}\right)+26m^4+s_{13}\left(s_{12}+s_{23}+s_{24}\right)+s_{12}\left(s_{12}+s_{14}+s_{23}+s_{24}\right)\\ 0\\ -m^2\left(11s_{12}+6s_{13}+4s_{14}+11s_{23}+4s_{24}\right)+28m^4+s_{13}\left(s_{12}+s_{23}+s_{24}\right)+\left(s_{12}+s_{23}\right)\left(s_{12}+s_{14}+s_{23}+s_{24}\right)\\ -m^2\left(13s_{12}+8s_{13}+4s_{14}+13s_{23}+4s_{24}\right)+36m^4+\left(s_{12}+s_{23}\right)\left(s_{12}+s_{14}+s_{23}+s_{24}\right)+s_{13}\left(s_{12}+2s_{23}+s_{24}\right)\\ m^2\left(9s_{12}+2\left(2s_{13}+s_{14}+s_{23}+2s_{24}\right)\right)-18m^4-s_{13}\left(s_{12}+s_{24}\right)-s_{12}\left(s_{12}+s_{14}+s_{23}+s_{24}\right)\\ m^2\left(2s_{12}+2s_{13}+2s_{14}+9s_{23}\right)-10m^4-s_{13}s_{23}-s_{23}\left(s_{12}+s_{14}+s_{23}+s_{24}\right)\\ -m^2\left(11s_{12}+6s_{13}+2s_{14}+4s_{23}+4s_{24}\right)+26m^4+s_{13}\left(s_{12}+s_{23}+s_{24}\right)+s_{12}\left(s_{12}+s_{14}+s_{23}+s_{24}\right) \end{bmatrix},$$

$$e_5=\begin{bmatrix} -2m^2\left(s_{12}+s_{13}+s_{23}\right)+8m^4+s_{12}s_{13}\\ s_{12}\left(2m^2-s_{14}\right)-\left(4m^2-s_{23}\right)\left(2m^2-s_{14}\right)+s_{13}\left(s_{24}-2m^2\right)\\ -2m^2\left(2s_{12}+s_{13}+s_{14}+s_{23}+s_{24}\right)+16m^4+s_{12}s_{13}+s_{12}s_{14}\\ 0\\ s_{12}\left(s_{14}-2m^2\right)+\left(2m^2-s_{14}\right)\left(4m^2-s_{23}\right)+s_{13}\left(2m^2-s_{24}\right)\\ -4m^2s_{23}+s_{12}\left(-4m^2+s_{13}+s_{14}\right)+s_{14}\left(s_{23}-4m^2\right)+16m^4-s_{13}s_{24}\\ 2m^2\left(-4m^2+s_{14}+s_{24}\right)+s_{12}\left(2m^2-s_{14}\right)\\ -2m^2\left(s_{13}-s_{14}-s_{23}+s_{24}\right)-s_{14}s_{23}+s_{13}s_{24}\\ -2m^2\left(2s_{12}+s_{13}+s_{14}+s_{23}+s_{24}\right)+16m^4+s_{12}s_{13}+s_{12}s_{14} \end{bmatrix},$$

$$e_6=\begin{bmatrix} -m^2\left(7s_{12}+2\left(2s_{14}+s_{23}+s_{24}\right)\right)+10m^4+s_{12}^2+s_{14}s_{23}+s_{12}\left(s_{14}+s_{23}+s_{24}\right)\\ s_{14}\left(s_{23}-4m^2\right)+\left(2m^2-s_{24}\right)\left(5m^2-s_{23}-s_{24}\right)+s_{12}\left(-2m^2+s_{14}+s_{24}\right)\\ 2m^2\left(5m^2-s_{23}-s_{24}\right)+s_{12}\left(-7m^2+s_{23}+s_{24}\right)+s_{14}\left(-4m^2+s_{23}+s_{24}\right)+s_{12}^2\\ 0\\ s_{14}\left(4m^2-s_{23}\right)+s_{12}\left(2m^2-s_{14}-s_{24}\right)-\left(2m^2-s_{24}\right)\left(5m^2-s_{23}-s_{24}\right)\\ \left(s_{12}-s_{24}\right)\left(-5m^2+s_{12}+s_{23}+s_{24}\right)\\ s_{14}\left(s_{12}-s_{24}\right)\\ \left(2m^2-s_{24}\right)\left(5m^2-s_{23}-s_{24}\right)+s_{12}\left(s_{24}-2m^2\right)+s_{14}\left(-4m^2+s_{23}+s_{24}\right)\\ 2m^2\left(5m^2-s_{23}-s_{24}\right)+s_{12}\left(-7m^2+s_{23}+s_{24}\right)+s_{14}\left(-4m^2+s_{23}+s_{24}\right)+s_{12}^2 \end{bmatrix}.$$

We have verified numerically that all of $e_1 ... e_6$ are parallel to the null vector, e_0, when $\epsilon(1,3,4) = 0$.

Another choice of q_i and e_i is the following:

$$q_1 = u_1 - u_2 + u_3 + u_5 + 2u_6 - u_8 + u_9 \ ,$$

$$q_2 = u_1 - u_2 + u_5 + 2u_6 + u_7 - 2u_8 \ ,$$

$$q_3 = 2u_1 - 3u_2 + 2u_3 + 3u_5 + 5u_6 - 3u_8 + 2u_9 \ ,$$

$$q_4 = u_1 - 2u_2 + 2u_3 + 2u_5 + 3u_6 - u_7 - u_8 + 2u_9 \ ,$$

$$q_5 = -u_1 + u_2 - u_3 - u_5 - 2u_6 + u_8 - u_9 \ ,$$

$$q_6 = u_1 - 2u_2 + 2u_5 + 3u_6 + u_7 - 3u_8 \ ,$$

$$e_1 = \begin{bmatrix} -s_{23}\left(-13m^2 + 3s_{12} - s_{13} + s_{14} + 2s_{23} + 2s_{24}\right) \\ -2m^2 s_{14} - 7m^2 s_{23} + s_{13}\left(8m^2 + s_{23}\right) + s_{12}\left(8m^2 - 2s_{13} + s_{23}\right) - 15m^4 - s_{12}^2 - s_{13}^2 + s_{14}^2 + 2s_{23}^2 - s_{14}s_{23} + 2s_{23}s_{24} \\ -8m^2 s_{14} + 5m^2 s_{23} - 5m^2 s_{24} + s_{12}\left(-18m^2 + 2s_{13} + 2s_{14} - s_{23} + s_{24}\right) + s_{13}\left(-8m^2 + s_{23} + s_{24}\right) + 40m^4 + 2s_{12}^2 - 2s_{23}^2 + s_{14}s_{23} + s_{14}s_{24} - 2s_{23}s_{24} \\ -5m^2\left(2s_{12} + 2s_{14} + 3s_{23} + s_{24}\right) + 25m^4 + s_{12}^2 - s_{13}^2 + s_{14}^2 + 2s_{23}^2 + s_{13}s_{23} + s_{14}s_{23} + s_{13}s_{24} + s_{14}s_{24} + 2s_{23}s_{24} + s_{12}\left(2s_{14} + 3s_{23} + s_{24}\right) \\ -m^2\left(18s_{12} + 8s_{13} + 8s_{14} + 8s_{23} + 5s_{24}\right) + 40m^4 + 2s_{12}^2 + 2s_{14}s_{23} + s_{13}s_{24} + s_{14}s_{24} + s_{12}\left(2s_{13} + 2s_{14} + 2s_{23} + s_{24}\right) \\ -8m^2 s_{14} + 5m^2 s_{23} - 5m^2 s_{24} + s_{12}\left(-18m^2 + 2s_{13} + 2s_{14} - s_{23} + s_{24}\right) + s_{13}\left(-8m^2 + s_{23} + s_{24}\right) + 40m^4 + 2s_{12}^2 - 2s_{23}^2 + s_{14}s_{23} + s_{14}s_{24} - 2s_{23}s_{24} \\ -2m^2 s_{14} - 7m^2 s_{23} + s_{13}\left(8m^2 + s_{23}\right) + s_{12}\left(8m^2 - 2s_{13} + s_{23}\right) - 15m^4 - s_{12}^2 - s_{13}^2 + s_{14}^2 + 2s_{23}^2 - s_{14}s_{23} + 2s_{23}s_{24} \\ 0 \\ m^2\left(-8s_{12} - 8s_{13} + 2s_{14} + 20s_{23}\right) + 15m^4 + s_{12}^2 + s_{13}^2 - s_{14}^2 - 4s_{23}^2 + 2s_{12}\left(s_{13} - 2s_{23}\right) - 4s_{23}s_{24} \end{bmatrix},$$

$$e_2 = \begin{bmatrix} s_{23}\left(-5m^2 + 2s_{12} + s_{14} + 2s_{24}\right) \\ m^2\left(s_{12} + s_{13} + 6s_{14} - 3s_{23}\right) - 5m^4 - s_{14}^2 - s_{12}s_{14} - s_{13}s_{14} + s_{14}s_{23} + 2s_{23}s_{24} \\ m^2\left(7s_{12} + 2s_{13} + 2s_{14} - s_{23} + 5s_{24}\right) - 10m^4 - s_{12}^2 - s_{13}s_{24} - s_{14}s_{24} - s_{12}\left(s_{13} + s_{14} - s_{23} + s_{24}\right) \\ m^2\left(8s_{12} + 3s_{13} + 8s_{14} + s_{23} + 5s_{24}\right) - 15m^4 - s_{12}^2 - s_{14}^2 - s_{13}s_{14} - s_{13}s_{24} - s_{14}s_{24} - s_{12}\left(s_{13} + 2s_{14} + s_{23} + s_{24}\right) \\ m^2\left(7s_{12} + 2s_{13} + 2s_{14} + 4s_{23} + 5s_{24}\right) - 10m^4 - s_{12}^2 - s_{14}s_{23} - s_{13}s_{24} - s_{14}s_{24} - 2s_{23}s_{24} - s_{12}\left(s_{13} + s_{14} + s_{23} + s_{24}\right) \\ m^2\left(7s_{12} + 2s_{13} + 2s_{14} - s_{23} + 5s_{24}\right) - 10m^4 - s_{12}^2 - s_{13}s_{24} - s_{14}s_{24} - s_{12}\left(s_{13} + s_{14} - s_{23} + s_{24}\right) \\ m^2\left(s_{12} + s_{13} + 6s_{14} - 3s_{23}\right) - 5m^4 - s_{14}^2 - s_{12}s_{14} - s_{13}s_{14} + s_{14}s_{23} + 2s_{23}s_{24} \\ 0 \\ -m^2\left(s_{12} + s_{13} + 6s_{14} + 2s_{23}\right) + 5m^4 + s_{14}^2 + s_{13}s_{14} + s_{12}\left(s_{14} + 2s_{23}\right) \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 2m^2 s_{14} + s_{23}\left(-6m^2 + s_{12} + s_{24}\right) - 2m^4 + s_{23}^2 \\ -m^2 s_{24} + s_{23}\left(7m^2 - s_{12} - s_{13} - s_{24}\right) + s_{14}\left(s_{24} - 2m^2\right) + 2m^4 - s_{23}^2 \\ 2m^2\left(s_{13} + s_{14} - 3\left(s_{23} + s_{24}\right)\right) - 4m^4 + s_{23}^2 + s_{24}^2 + s_{12}s_{23} + s_{12}s_{24} + 2s_{23}s_{24} \\ s_{13}\left(2m^2 - s_{23}\right) + \left(s_{23} - s_{24}\right)\left(7m^2 - s_{12} - s_{23} - s_{24}\right) + s_{14}\left(s_{24} - 2m^2\right) \\ 2m^2 s_{13} + s_{24}\left(-6m^2 + s_{12} + s_{23}\right) - 2m^4 + s_{24}^2 \\ 2m^2\left(s_{13} + s_{14} - 3\left(s_{23} + s_{24}\right)\right) - 4m^4 + s_{23}^2 + s_{24}^2 + s_{12}s_{23} + s_{12}s_{24} + 2s_{23}s_{24} \\ -m^2 s_{24} + s_{23}\left(7m^2 - s_{12} - s_{13} - s_{24}\right) + s_{14}\left(s_{24} - 2m^2\right) + 2m^4 - s_{23}^2 \\ 0 \\ m^2 s_{24} + s_{14}\left(4m^2 - s_{24}\right) + s_{23}\left(-13m^2 + 2s_{12} + s_{13} + 2s_{24}\right) - 4m^4 + 2s_{23}^2 \end{bmatrix},$$

$$e_4 = \begin{bmatrix} s_{12}\left(-9m^2+s_{13}+s_{14}+2s_{23}+s_{24}\right)+\left(4m^2-s_{23}\right)\left(5m^2-s_{14}-s_{24}\right)+s_{13}\left(s_{24}-4m^2\right)+s_{12}^2 \\ m^2\left(s_{12}+s_{13}+s_{14}-3s_{23}+s_{24}\right)-5m^4+s_{14}s_{23}-s_{14}s_{24}+s_{23}s_{24} \\ -m^2\left(2s_{12}+2s_{13}+2s_{14}+s_{23}-5s_{24}\right)+10m^4-s_{24}^2+s_{12}\left(s_{23}-s_{24}\right)-s_{23}s_{24} \\ m^2\left(8s_{12}+3s_{13}+3s_{14}+s_{23}+10s_{24}\right)-15m^4-s_{12}^2-s_{24}^2-s_{13}s_{24}-s_{14}s_{24}-s_{23}s_{24}-s_{12}\left(s_{13}+s_{14}+s_{23}+2s_{24}\right) \\ m^2\left(7s_{12}+2s_{13}+2s_{14}+4s_{23}+9s_{24}\right)-10m^4-s_{12}^2-s_{24}^2-s_{14}s_{23}-s_{13}s_{24}-2s_{23}s_{24}-s_{12}\left(s_{13}+s_{14}+s_{23}+2s_{24}\right) \\ -m^2\left(2s_{12}+2s_{13}+2s_{14}+s_{23}-5s_{24}\right)+10m^4-s_{24}^2+s_{12}\left(s_{23}-s_{24}\right)-s_{23}s_{24} \\ m^2\left(s_{12}+s_{13}+s_{14}-3s_{23}+s_{24}\right)-5m^4+s_{14}s_{23}-s_{14}s_{24}+s_{23}s_{24} \\ 0 \\ -m^2\left(10s_{12}+5s_{13}+5s_{14}+2s_{23}+5s_{24}\right)+25m^4+s_{12}^2+s_{13}s_{24}+s_{14}s_{24}+s_{12}\left(s_{13}+s_{14}+2s_{23}+s_{24}\right) \end{bmatrix},$$

$$e_5 = \begin{bmatrix} s_{12}\left(-7m^2+s_{13}+s_{14}+2s_{23}+s_{24}\right)+\left(2m^2-s_{23}\right)\left(5m^2-s_{14}-2s_{24}\right)+s_{13}\left(s_{24}-2m^2\right)+s_{12}^2 \\ -m^2\left(s_{12}+s_{13}+s_{14}+3s_{23}+4s_{24}\right)+5m^4+s_{14}s_{23}+s_{12}s_{24}+\left(s_{13}+2s_{23}\right)s_{24} \\ \left(s_{23}-s_{24}\right)\left(s_{12}-m^2\right) \\ -\left(m^2-s_{12}\right)\left(5m^2-s_{12}-s_{13}-s_{14}-s_{23}-s_{24}\right) \\ m^2\left(7s_{12}+2s_{13}+2s_{14}+4s_{23}+5s_{24}\right)-10m^4-s_{12}^2-s_{14}s_{23}-\left(s_{13}+2s_{23}\right)s_{24}-s_{12}\left(s_{13}+s_{14}+s_{23}+2s_{24}\right) \\ \left(s_{23}-s_{24}\right)\left(s_{12}-m^2\right) \\ -m^2\left(s_{12}+s_{13}+s_{14}+3s_{23}+4s_{24}\right)+5m^4+s_{14}s_{23}+s_{12}s_{24}+\left(s_{13}+2s_{23}\right)s_{24} \\ 0 \\ \left(m^2-s_{12}\right)\left(5m^2-s_{12}-s_{13}-s_{14}-2s_{23}\right) \end{bmatrix},$$

$$e_6 = \begin{bmatrix} -2m^2s_{14}+s_{23}\left(2m^2-s_{12}-s_{24}\right)+2m^4 \\ s_{14}\left(2m^2-s_{23}-s_{24}\right)-\left(m^2-s_{23}\right)\left(2m^2-s_{24}\right) \\ 2m^2\left(s_{12}+s_{23}+3s_{24}\right)-6m^4-s_{24}^2-s_{23}s_{24}-s_{12}\left(s_{23}+s_{24}\right) \\ s_{12}\left(2m^2-s_{24}\right)-\left(5m^2-s_{23}-s_{24}\right)\left(2m^2-s_{24}\right)+s_{14}\left(4m^2-s_{23}-s_{24}\right) \\ 2m^2\left(s_{12}+s_{14}+3s_{24}\right)-8m^4-s_{24}^2-s_{12}s_{24} \\ 2m^2\left(s_{12}+s_{23}+3s_{24}\right)-6m^4-s_{24}^2-s_{23}s_{24}-s_{12}\left(s_{23}+s_{24}\right) \\ s_{14}\left(2m^2-s_{23}-s_{24}\right)-\left(m^2-s_{23}\right)\left(2m^2-s_{24}\right) \\ 0 \\ -m^2s_{24}+s_{14}\left(-4m^2+s_{23}+s_{24}\right)+4m^4-s_{12}s_{23} \end{bmatrix}.$$

We found numerically that for this choice all e_i are parallel to e_0 when $\epsilon(2,3,4)=0$.

H.2 Special Kinematics of 3D Topologically Massive Theories

The on-shell polarisation vectors of topologically massive theories satisfy the equations of motion given in (2.123), while the external momenta should be on-shell, that is, $p_i^2 = -m_i^2$. Furthermore, special relations can arise in three spacetime dimensions as we will see in the following.

3-point Amplitudes From (2.123), we have the following relations for the polarisation vectors of the three external states 1, 2 and 3:

$$\begin{aligned}
&\varepsilon_{1\mu} + \frac{i}{m}\epsilon_{\mu\nu\rho}p_1^\nu \varepsilon_{1\rho} = 0 \ , \\
&\varepsilon_{2\mu} + \frac{i}{m}\epsilon_{\mu\nu\rho}p_2^\nu \varepsilon_2 \rho = 0 \ , \\
&\varepsilon_{3\mu} + \frac{i}{m}\epsilon_{\mu\nu\rho}p_3^\nu \varepsilon_3 \rho = 0 \ .
\end{aligned} \tag{H.1}$$

By contracting the first line with ε_2 and ε_3 and using the second and third term respectively, we get the following relations:

$$ee_{12} = -\frac{2}{m^2}ep_{21}ep_{12} \ , \quad ee_{13} = -\frac{2}{m^2}ep_{12}ep_{32} \ . \tag{H.2}$$

Similarly, contracting the second line with ε_3 and using the last line we get:

$$ee_{23} = -\frac{2}{m^2}ep_{21}ep_{23} \ . \tag{H.3}$$

This shows that

$$ee_{ij}m^2 = 2ep_{ij}ep_{ji} \ . \tag{H.4}$$

We use these relations to derive (2.145).

4-point Amplitudes At 4-points, the analytic manipulations become more involve and we proceed to use a numerical approach to find the on-shell relations between polarisation vectors and momenta. We have checked that the following relations

$$\begin{aligned}
\frac{ee_{12}ee_{34}}{ee_{14}ee_{23}} &= \frac{1}{(s+t)^4}\Bigg(-8m^2s(s-3t)(s+t) + s^2(s+t)^2 + 32m^3(-s+t)\sqrt{-stu} \\
&\qquad +8ms(s+t)\sqrt{-stu} + 16m^4(s^2-6st+t^2)\Bigg) \ , \\
\frac{ee_{13}ee_{24}}{ee_{14}ee_{23}} &= \frac{1}{(s+t)^4}\Bigg(8m^2t(3s-t)(s+t) + t^2(s+t)^2 + 16m^4(s^2-6st+t^2) \\
&\qquad -8m\sqrt{-stu}(4m^2(s-t)+t(s+t))\Bigg) \ ,
\end{aligned} \tag{H.5}$$

are satisfied when using random on-shell kinematics as described in Appendix H.4. These relations can also be checked in the Breit coordiante system (2.148), where the Mandelstam variables are related by (2.149). Once these two relations are imposed, we can show analytically that the 4-point TMG amplitude is the double copy of the 4-point TMYM one.

5-point Amplitudes At 5-points, we will use a special property of 3D, namely, the fact that any anti-symmetric tensor with 4 indices is identically zero. It is specially useful to look at the case

$$\epsilon_{[\mu\nu\rho}(p_i)_{\sigma]} = 0 \ . \tag{H.6}$$

By contracting this relation with $p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$ and choosing different i's, we obtain four relations between the s_{ij} Mandelstam variables and $\epsilon(i,j,k) = \epsilon_{\mu\nu\sigma} p_i^\mu p_j^\nu p_k^\sigma$. From them, we can write s_{12}, s_{13}, s_{14}, s_{23} in terms of $\epsilon(i,j,k)$ and s_{24}. For example:

$$s_{13} = \frac{\epsilon(1,2,3)\epsilon(1,3,4)\big(s_{24}-2m^2\big)+m^2\epsilon(1,2,3)^2-m^2\big(\epsilon(1,2,4)^2-2\epsilon(2,3,4)\epsilon(1,2,4)-\epsilon(1,3,4)^2+\epsilon(2,3,4)^2\big)}{\epsilon(1,2,4)\epsilon(2,3,4)} . \tag{H.7}$$

To find the remaining s_{24} in terms of $\epsilon(i,j,k)$ we can consider products of two $\epsilon(i,j,k)$ and expand the double ϵ in terms of the metric, that is,

$$\begin{aligned}\epsilon(1,2,3)\epsilon(1,2,3) &= \epsilon_{\mu\nu\rho}\epsilon_{\alpha\beta\gamma}(p_1^\mu p_2^\nu p_3^\rho)(p_1^\alpha p_2^\beta p_3^\gamma), \\ &= \frac{1}{4}\left(-16m^6 + 8m^4(s_{12}+s_{13}+s_{23}) - m^2(s_{12}+s_{13}+s_{23})^2 + s_{12}s_{13}s_{23}\right) \ .\end{aligned} \tag{H.8}$$

By using the previously derived relations for s_{12}, s_{13}, s_{14}, and s_{23}; we can derive an expression for s_{24} purely in terms of $\epsilon(i,j,k)$.

H.3 4-point TMG Amplitude

In this appendix we show the explicit expression for the 4-point TMG amplitude in a general coordinate system and in the Breit coordinate system. In terms of the Mandelstam variables, the four-point amplitude of TMG reads

$$
\begin{aligned}
&M_4 = \\
&\Bigg(- ee_{13}^2 ee_{24}^2 \left(m^2 - s\right) \Bigg(80 \left(3m^2 - s\right) \left(s - 4m^2\right)^2 m^8 + 4 \left(4m^2 - s\right) \left(1384m^6 - 897sm^4 + 179s^2m^2 - 8s^3\right) tm^4 \\
&+ \left(22128m^6 - 5568sm^4 + 729s^2m^2 - 119s^3\right) t^3m^2 - 4 \left(7552m^8 - 3258sm^6 + 568s^2m^4 - 85s^3m^2 + 8s^4\right) t^2m^2 \\
&- 6 \left(4m^2 + s\right) t^6 + \left(2m^2 - s\right) \left(76m^2 + 3s\right) t^5 - \left(5360m^6 - 916sm^4 + 154s^2m^2 + s^3\right) t^4 \Bigg) s^2 \\
&+ ee_{12}ee_{34} \left(m^2 - t\right) t \Bigg(ee_{14}ee_{23} \Bigg(- 1477632m^{16} + 256(5182s + 6493t)m^{14} - 64 \left(5444s^2 + 15491ts + 10889t^2\right) m^{12} \\
&+ 16 \left(984s^3 + 3973ts^2 + 12776t^2s + 7851t^3\right) m^{10} + 4 \left(5760s^4 + 10402ts^3 + 13539t^2s^2 + 1045t^3s - 1568t^4\right) m^8 \\
&- 4 \left(1600s^5 + 5084ts^4 + 5866t^2s^3 + 3305t^3s^2 + 971t^4s + 151t^5\right) m^6 \\
&+ \left(512s^6 + 1976ts^5 + 4732t^2s^4 + 2798t^3s^3 + 152t^4s^2 - 11t^5s + 24t^6\right) m^4 + \\
&t \left(128s^6 + 360ts^5 + 232t^2s^4 - 249t^3s^3 + 26t^4s^2 + 22t^5s + 4t^6\right) m^2 + st^3 \left(-2s^4 - 5ts^3 + 10t^2s^2 + 2t^3s + t^4\right) \Bigg) \\
&+ ee_{12}ee_{34} \Bigg(- 768(486s - 481t)m^{14} + 64 \left(3360s^2 - 49ts - 6493t^2\right) m^{12} \\
&+ 16 \left(-600s^3 - 613ts^2 + 10912t^2s + 10889t^3\right) m^{10} - 4 \Bigg(3072s^4 + 7778ts^3 + 8855t^2s^2 + 15613t^3s \\
&+ 7851t^4 \Bigg) m^8 + 4 \left(1088s^5 + 1948ts^4 + 2314t^2s^3 + 491t^3s^2 + 1282t^4s + 392t^5\right) m^6 \\
&+ \left(-512s^6 - 952ts^5 - 492t^2s^4 + 2178t^3s^3 + 1192t^4s^2 + 434t^5s + 151t^6\right) m^4
\end{aligned}
$$

$$
\begin{aligned}
&- t\left(128s^6 + 488ts^5 + 696t^2s^4 + 343t^3s^3 + 27t^4s^2 + 29t^5s + 6t^6\right) m^2\\
&- t^3(2s+t)\left(-s^4 - 4ts^3 + t^2s^2 + t^3s + t^4\right)\Bigg)\Bigg)\\
&+ ee_{13}ee_{24}\Bigg(ee_{14}ee_{23}\left(s-m^2\right)\Bigg(- 8\left(s-4m^2\right)^2\left(s-3m^2\right)\left(3s-22m^2\right)m^6\\
&- 4\left(s-4m^2\right)\left(-3064m^6 + 2221sm^4 - 517s^2m^2 + 36s^3\right) tm^4 + 2\Bigg(29184m^8 - 17444sm^6 + 3976s^2m^4 - 437s^3m^2\\
&+ 21s^4\Bigg)t^2m^2 + 6\left(4m^2+s\right)t^6 + \left(-248m^4 + 118sm^2 - 3s^2\right)t^5 + \left(5392m^6 - 1028sm^4 + 204s^2m^2 - 5s^3\right)t^4\\
&+ \left(-30256m^8 + 10336sm^6 - 1541s^2m^4 + 163s^3m^2 - 2s^4\right)t^3\Bigg)s^2 + ee_{12}ee_{34}\Bigg(768\left(5s^2 + 486ts - 481t^2\right)m^{16}\\
&- 64\left(92s^3 + 4229ts^2 + 8651t^2s - 10822t^3\right)m^{14} + 16\left(123s^4 + 1027ts^3 + 29747t^2s^2 + 14110t^3s - 26039t^4\right)m^{12}\\
&+ 4\left(66s^5 + 7447ts^4 - 11718t^2s^3 - 62062t^3s^2 - 15385t^4s + 25368t^5\right)m^{10}\\
&- 4\left(52s^6 + 2394ts^5 + 4948t^2s^4 + 2145t^3s^3 - 16136t^4s^2 - 4706t^5s + 1901t^6\right)m^8\\
&+ \left(24s^7 + 716ts^6 + 6422t^2s^5 + 13743t^3s^4 + 9891t^4s^3 - 7861t^5s^2 - 2569t^6s - 484t^7\right)m^6\\
&+ t\left(16s^7 - 972ts^6 - 2312t^2s^5 - 2829t^3s^4 - 25t^4s^3 - 317t^5s^2 - 73t^6s + 36t^7\right)m^4\\
&+ t^2\left(86s^7 + 263ts^6 + 259t^2s^5 - 50t^3s^4 - 100t^4s^3 + 23t^5s^2 + 13t^6s + 4t^7\right)m^2\\
&+ st^3\left(2s^6 + 9ts^5 + 15t^2s^4 + 20t^3s^3 + 6t^4s^2 + 3t^5s + t^6\right)\Bigg)\Bigg)\Bigg)\times\\
&\frac{-i}{128m^2(m^2-s)s^2(m^2-t)t(-4m^2+s+t)^2(-3m^2+s+t)}\,.
\end{aligned}
\tag{H.9}
$$

In order to see the double copy relation explicitly, we write this amplitude in the Breit coordinate system which was defined in (2.148). The amplitude largely simplifies and is now given by

$$
\begin{aligned}
M_4 =& \frac{-m}{2\lambda^2p^2\left(\lambda^2+m^2\right)^2\left(m^2+4p^2\right)\left(3m^4 + 4m^2\left(\lambda^2+p^2\right) + 4\lambda^2p^2\right)}\times\\
&\Bigg(30E\lambda m^{10}\left(\lambda^2p - 4p^3\right) + 3im^{11}\left(\lambda^4 + 16p^4 - 96\lambda^2p^2\right) + 8E\lambda^5p^5\left(\lambda^4 - 32p^4 - 4\lambda^2p^2\right)\\
&+ Em^8\left(-880\lambda p^5 + 366\lambda^3p^3 + 88\lambda^5p\right) + 8E\lambda^3m^2p^3\left(-8\lambda^6 + 320p^6 + 8\lambda^2p^4 + \lambda^4p^2\right)\\
&+ 2im^9\left(5\lambda^6 + 152p^6 - 502\lambda^2p^4 - 367\lambda^4p^2\right) + 2i\lambda^4mp^4\left(18\lambda^6 + 640p^6 + 416\lambda^2p^4 - 17\lambda^4p^2\right)\\
&+ 2E\lambda m^6p\left(41\lambda^6 - 960p^6 + 856\lambda^2p^4 + 148\lambda^4p^2\right) + im^7\left(11\lambda^8 + 512p^8 - 3396\lambda^2p^6 - 667\lambda^4p^4 - 652\lambda^6p^2\right)\\
&+ 2E\lambda m^4p\left(12\lambda^8 - 640p^8 + 1776\lambda^2p^6 + 552\lambda^4p^4 - 131\lambda^6p^2\right)\\
&- i\lambda^2m^3p^2\left(56\lambda^8 + 2560p^8 - 768\lambda^2p^6 - 1352\lambda^4p^4 - 77\lambda^6p^2\right)\\
&+ 2im^5\left(2\lambda^{10} + 128p^{10} - 2464\lambda^2p^8 - 413\lambda^4p^6 + 217\lambda^6p^4 - 131\lambda^8p^2\right)\Bigg)\,.
\end{aligned}
\tag{H.10}
$$

In this coordinate system, the shifted kinematic factors of TMYM, (2.134), read

$$
\begin{aligned}
\hat{n}_s =& \frac{-i\lambda}{mp\left(p^2-E^2\right)\left(-E^2+m^2+p^2\right)} \times \\
& \Bigg(-5\lambda m^5 p - \lambda m^3 p\left(\lambda^2+31p^2\right) - 2i\lambda^2 p^2 \sqrt{m^2+p^2}\left(\lambda^2+4p^2\right) + i m^4 \sqrt{m^2+p^2}\left(\lambda^2+2p^2\right) \\
& m\left(-16\lambda p^5 - 9\lambda^3 p^3 + 4\lambda^5 p\right) + +i m^2 \sqrt{m^2+p^2}\left(\lambda^4+8p^4+2\lambda^2 p^2\right) \\
& E\Bigg(2im^4\left(\lambda^2+5p^2\right) + \lambda m p \sqrt{m^2+p^2}\left(3\lambda^2-16p^2\right) + 5\lambda m^3 p \sqrt{m^2+p^2} - 2i\lambda^2 p^2\left(\lambda^2+4p^2\right) \\
& + 2im^2\left(\lambda^4+4p^4-7\lambda^2 p^2\right) \Bigg)\Bigg) , \\
\hat{n}_t =& -\frac{2\lambda\sqrt{m^2+p^2}}{mp\left(p^2-E^2\right)\left(-E^2+m^2+p^2\right)} \times \\
& \left(-5iE\lambda m^3 p + iE\lambda m\left(16p^3-3\lambda^2 p\right) + m^4\left(\lambda^2+2p^2\right) + m^2\left(\lambda^4+8p^4+2\lambda^2 p^2\right) - 2\lambda^2 p^2\left(\lambda^2+4p^2\right) \right) , \\
\hat{n}_u =& -\hat{n}_s - \hat{n}_t \ .
\end{aligned}
\tag{H.11}
$$

As mentioned before, by plugging in these kinematic factors in (2.147) and using (2.149), we get the amplitude of TMG (H.10).

H.4 Numerical Method for Random Kinematics

In this section we explain how we generate the numerical 3D on-shell kinematics and give some specific values that we used to check the double copy of TMYM at 5-point. We perform all computations over finite fields. This way we avoid numerical errors and make the calculations more efficient. We consider the field of integers modulo p, a prime number which is equal to $p = 2147483497$ in this work. Computations over finite fields are common and have been used to reconstruct polynomials in kinematics variables in loop QCD calculations [112, 163, 164, 165]. We refer the reader to [111, 112] for a detailed explanation.

We generate random kinematics such that the two following conditions are satisfied:

- The momenta are on-shell and conserved.
- The polarisation vectors satisfy the constraint (2.123).

The components of each external momenta are related by the on-shell condition,

$$p^{\mu} = (p_0, p_1, p_2) \ , \quad -p_0^2 + p_1^2 + p_2^2 = -m^2 \ , \tag{H.12}$$

and the second condition is satisfied for the polarisation vector built out of these components as,

$$\epsilon^{\mu} = \left(-\frac{(m - p_0 - p_1 + ip_2)(m + p_0 + p_1 + ip_2)}{2m(p_0 + p_1)}, \frac{(p_0 + p_1)^2 + (m + ip_2)^2}{2m(p_0 + p_1)}, -i + \frac{p_2}{m} \right) \ . \tag{H.13}$$

We have included ten of the random kinematics that we used to calculate the 5-point TMG in Table H.1 and the unshifted numerators of TMYM in Table 2. The unshifted numerators of TMYM double copy to the 5-point TMG amplitude using (2.12).

	m	s_{12}	s_{13}	s_{14}	s_{23}	M_5^{TMG}
1	384817470	1158823430	345329619	1397768610	264965055	$1010590219\kappa^3$
2	1250040736	652270246	1821369346	1622372086	739244825	$1355550730\kappa^3$
3	800857604	2035880423	1968515133	1224440350	664321872	$526697979\kappa^3$
4	1150467713	2060321774	82539557	702220445	431821399	$467871508\kappa^3$
5	158667339	2051154971	369848949	890093650	756917203	$475230586\kappa^3$
6	1916307032	541901353	2099692150	150737937	425995603	$1278139921\kappa^3$
7	1662283157	938971574	50758705	928659888	1820858158	$108642017\kappa^3$
8	1078072319	1151859367	1186765675	110159710	209051438	$1638775080\kappa^3$
9	231108131	1439516500	572657143	405624245	68286568	$266492730\kappa^3$
10	1849710816	247156271	155877255	2085263836	62583717	$966750502\kappa^3$

Table H.1: Examples of the kinematic values used to calculate the unshifted numerators of TMYM and the 5-point TMG. The values are in the field of integers modulo $p = 2147483497$. The remaining Mandelstam variable, s_{24}, can be obtained by requiring that (2.64) is zero in 3D.

	1	2	3	4	5	6	7	8	9	10
n_1	1679102633	348983868	399241281	842732794	300495714	332245457	2078791969	405757021	704097804	1147807466
n_2	317552067	1079962755	683351824	2113803420	650623635	1651377807	443295016	1307896649	546715501	33350122
n_3	1771495024	671539061	121827398	1212918710	877929848	1648524257	1751453994	715853795	437788195	1673049330
n_4	1358547387	1615179844	1720845160	1032631735	698981299	676408057	1421711641	586269609	1143032868	1144909226
n_5	211933474	81190884	921568849	1552408865	1716179733	1587978966	1832576367	503370947	1216904523	1157374746
n_6	1117522969	1640095150	1919642084	998205023	1533934317	1763892991	1547349116	1008373879	1419527987	1656418171
n_7	1479358736	259089853	155577739	1231349959	1543395860	88931139	114315550	998844655	1430903268	310456086
n_8	1654319285	823920651	859957704	2079622430	322878784	1433806805	651369019	638654072	1572317114	1484455868
n_9	925637889	1084982689	1510593127	1223632823	991858505	1585631481	1130701506	1854538988	845822321	107084633
n_{10}	241485408	181500425	2095141802	1445698561	946176777	1036495597	924128220	665606563	1436242466	167117366
n_{11}	1913208290	547913677	1461674584	601184017	391951191	112316195	407293624	777544715	908354210	719884700
n_{12}	1372613275	659324877	523294703	785456003	1120871918	2024095541	618177398	2047361914	1454336818	1233365105
n_{13}	1789815668	1037075753	209962576	493974229	1279666813	1941726328	1377397281	1546104552	2031446261	1379695040
n_{14}	1244173774	1475461191	1520770510	1520373298	1781914190	211939090	1680738019	1229996023	591423398	2134631466
n_{15}	1806462521	318300183	733389430	1399767053	630210267	1986264353	1729317645	1452460649	488881236	1508026444

Table H.2: Numerical values for the unshifted kinematic factors of the 5-point TMYM.

H.5 BCJ Relation in Terms of Partial Amplitudes

The BCJ relation in terms of colour ordered partial amplitudes takes the following form:

$$U.\begin{bmatrix} A_5[12345] \\ A_5[12435] \\ A_5[13245] \\ A_5[13425] \\ A_5[14235] \\ A_5[14325] \end{bmatrix} = 0 \ , \tag{H.14}$$

where $U = \{u_1, ..., u_6\}$ in this basis is given as:

$$\begin{aligned} u_1 = &\left(m^2 - s_{12}\right)\Big(\epsilon(1,2,4)\left(-5m^2 + s_{12} + s_{14} + s_{24}\right) + m^2\epsilon(1,2,3) - s_{13}\epsilon(1,3,4) + s_{24}\epsilon(1,3,4) \\ &- s_{14}\epsilon(2,3,4) + s_{23}\epsilon(2,3,4)\Big) \ , \end{aligned}$$

$$
\begin{aligned}
u_2 =\; & 7m^2 s_{24}\epsilon(1,2,3) + m^2 s_{14}\epsilon(1,2,4) + m^2 s_{24}\epsilon(1,2,4) + 3m^2 s_{14}\epsilon(1,3,4) + 2m^2 s_{24}\epsilon(1,3,4) \\
& - m^2 s_{24}\epsilon(2,3,4) + s_{23}\left(m^2(\epsilon(1,2,3) + 3\epsilon(1,3,4) - \epsilon(2,3,4)) - s_{14}\epsilon(1,3,4) - s_{24}(\epsilon(1,2,3) + \epsilon(1,3,4))\right) \\
& + s_{12}\Big(3m^2\epsilon(1,2,3) + 4m^2\epsilon(1,2,4) + 4m^2\epsilon(1,3,4) - 9m^2\epsilon(2,3,4) - s_{23}\epsilon(1,3,4) - s_{14}(\epsilon(1,2,4) + \epsilon(1,3,4) \\
& - \epsilon(2,3,4)) - s_{24}(2\epsilon(1,2,3) + \epsilon(1,2,4) + \epsilon(1,3,4) - \epsilon(2,3,4)) + s_{23}\epsilon(2,3,4)\Big) - 7m^4\epsilon(1,2,3) \\
& - 3m^4\epsilon(1,2,4) - 5m^4\epsilon(1,3,4) + 7m^4\epsilon(2,3,4) - s_{24}^2\epsilon(1,2,3) - s_{14}s_{24}\epsilon(1,3,4) - s_{12}^2(\epsilon(1,2,4) + \epsilon(1,3,4) \\
& - 2\epsilon(2,3,4)) - s_{14}s_{24}\epsilon(2,3,4) \ ,
\end{aligned}
$$

$$
\begin{aligned}
u_3 =\; & 7m^2 s_{24}\epsilon(1,2,3) + m^2 s_{14}\epsilon(1,2,4) - 3m^2 s_{13}\epsilon(1,3,4) - 4m^2 s_{24}\epsilon(1,3,4) - m^2 s_{13}\epsilon(2,3,4) - m^2 s_{14}\epsilon(2,3,4) \\
& - m^2 s_{24}\epsilon(2,3,4) + s_{12}\Big(2m^2\epsilon(1,2,3) + 5m^2\epsilon(1,2,4) - 3m^2\epsilon(1,3,4) - 5m^2\epsilon(2,3,4) - s_{24}(2\epsilon(1,2,3) \\
& + \epsilon(1,2,4)) + s_{13}\epsilon(1,3,4) + s_{14}(\epsilon(2,3,4) - \epsilon(1,2,4))\Big) + s_{23}\Big(m^2(2\epsilon(1,2,3) - 3\epsilon(1,3,4) - \epsilon(2,3,4)) \\
& + s_{24}(\epsilon(1,3,4) - \epsilon(1,2,3)) + s_{13}(\epsilon(1,3,4) + \epsilon(2,3,4))\Big) - 8m^4\epsilon(1,2,3) - 2m^4\epsilon(1,2,4) + 11m^4\epsilon(1,3,4) \\
& + 7m^4\epsilon(2,3,4) - s_{24}^2\epsilon(1,2,3) + s_{13}s_{24}\epsilon(1,3,4) + s_{12}^2(\epsilon(2,3,4) - \epsilon(1,2,4)) \ ,
\end{aligned}
$$

$$
u_4 = (\epsilon(1,2,4) + \epsilon(1,3,4) - \epsilon(2,3,4))\left(-\left(m^2 - s_{12}\right)\right)\left(4m^2 - s_{12} - s_{14} - s_{24}\right) \ ,
$$

$$
\begin{aligned}
u_5 =\; & m^2 s_{14}(\epsilon(1,2,4) + 2\epsilon(1,3,4)) + s_{12}\Big(m^2(\epsilon(1,2,3) + 5\epsilon(1,2,4) + 5\epsilon(1,3,4) - 4\epsilon(2,3,4)) \\
& - s_{24}(\epsilon(1,2,3) + \epsilon(1,2,4) + \epsilon(1,3,4)) - s_{14}(\epsilon(1,2,4) + \epsilon(1,3,4) - \epsilon(2,3,4))\Big) \\
& + s_{24}\Big(m^2(2\epsilon(1,2,3) + 2\epsilon(1,2,4) + 3\epsilon(1,3,4) + \epsilon(2,3,4)) - s_{14}(\epsilon(1,3,4) + \epsilon(2,3,4))\Big) + m^4(-(2\epsilon(1,2,3) \\
& + 5\epsilon(1,2,4) + 6\epsilon(1,3,4) - 2\epsilon(2,3,4))) - s_{12}^2(\epsilon(1,2,4) + \epsilon(1,3,4) - \epsilon(2,3,4)) \ ,
\end{aligned}
$$

$$
u_6 = m^2\Big(- 4m^2\epsilon(1,2,3) - 3m^2\epsilon(1,2,4) + 6m^2\epsilon(1,3,4) + 6m^2\epsilon(2,3,4) - 2s_{13}\epsilon(1,3,4) + s_{24}(2\epsilon(1,2,3)
$$

$$+ \epsilon(1,2,4) + \epsilon(1,3,4)) + s_{14}(\epsilon(1,2,4) - \epsilon(2,3,4)) - s_{13}\epsilon(2,3,4)\Big) + s_{12}\Big(m^2\epsilon(1,2,3) + 5m^2\epsilon(1,2,4)$$

$$- 2m^2\epsilon(1,3,4) - 5m^2\epsilon(2,3,4) + s_{13}\epsilon(1,3,4) - s_{24}(\epsilon(1,2,3) + \epsilon(1,2,4) + \epsilon(1,3,4))$$

$$+ s_{14}(\epsilon(2,3,4) - \epsilon(1,2,4))\Big) + s_{23}\left(m^2(\epsilon(1,2,3) - 2\epsilon(1,3,4) - \epsilon(2,3,4)) + s_{13}(\epsilon(1,3,4) + \epsilon(2,3,4))\right)$$

$$+ s_{12}^2(\epsilon(2,3,4) - \epsilon(1,2,4)) \ .$$

Appendix I

Derivation of Eikonal Resummation

Here we will show that (3.16) and (3.17) are valid in topological massive gravity by following the same steps as in [136]. Assuming that only ladder diagrams contribute, the $n-1$ loop integrands are obtained by multiplying n factors of two graviton-scalar-scalar vertices, contracted with a graviton propagator, together with scalar propagators, see figure I.1.

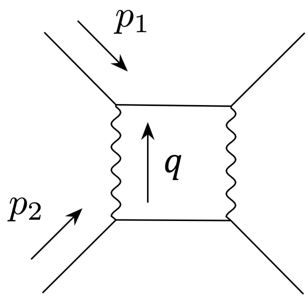

Figure I.1: Example of a box diagram that appears within the ladder diagrams contributing in the eikonal limit. Here, the gravitons correspond to the rungs of the ladder.

Two scalar-scalar-graviton vertices contracted with a graviton propagator of momentum $q^\mu = (q^u, q^v, q^y)$ in the eikonal limit give

$$-\frac{s^2}{4}\frac{F(q)}{q^4(q^2+m^2)}, \tag{I.1}$$

where m is graviton mass and

$$F(q) = \kappa^2 \left(-i(q^u q^v)^2 + 2q^u q^v q^y m - 2(q^y)^2(q^y - im)m\right) . \tag{I.2}$$

Assuming that the momentum in the scalar propagators can be approximated as $(p+k)^2 \to 2p{\cdot}k$, where p is p_1 or p_2 and k is any loop momentum, the sum of $n-1$ loop diagrams gives the following:

$$\begin{aligned} i\mathcal{M}_{n-1} = & \int \prod_{i=1}^{n} \left(\frac{d^3 q_i}{(2\pi)^3} \frac{-s^2 F(q_i)}{4q_i^4(q_i^2+m^2)} \right) (2\pi)^3 \delta^3 \left(p_3 + p_1 + \sum_{i=i}^{n} q_i \right) \\ & \times \frac{-i}{2p_1 \cdot q_1 - i\epsilon} \frac{-i}{2p_1 \cdot (q_1+q_2) - i\epsilon} \cdots \frac{-i}{2p_1 \cdot (q_1 + q_2 + \cdots + q_{n-1}) - i\epsilon} \\ & \times \sum_{\sigma \in S_n} \frac{-i}{-2p_2 \cdot q_{\sigma(1)} - i\epsilon} \frac{-i}{-2p_2 \cdot (q_{\sigma(1)} + q_{\sigma(2)}) - i\epsilon} \cdots \frac{-i}{-2p_2 \cdot (q_{\sigma(1)} + q_{\sigma(2)} + \cdots + q_{\sigma(n-1)}) - i\epsilon} . \end{aligned} \tag{I.3}$$

Using light-cone coordinates and taking the eikonal approximation we can write $p_1 \cdot q = -p^v q^u$ and $p_2 \cdot q = -p^u q^v$; hence

$$\begin{aligned} i\mathcal{M}_{n-1} = & \frac{1}{(4p^v p^u)^{n-1}} \int \prod_{i=1}^{n} \left(\frac{dq_i^v dq_i^u dq_i^y}{(2\pi)^3} \frac{-s^2 F(q_i)}{4q_i^4(q_i^2+m^2)} \right) (2\pi)^3 \delta \left(q + \sum_{i=i}^{n} q_i^y \right) \\ & \times \delta \left(\sum_{i=i}^{n} q_i^u \right) \frac{-i}{-q_1^u - i\epsilon} \frac{-i}{-(q_1^u + q_2^u) - i\epsilon} \cdots \frac{-i}{-(q_1^u + q_2^u + \cdots + q_{n-1}^u) - i\epsilon} \\ & \times \delta \left(\sum_{i=i}^{n} q_i^v \right) \sum_{\sigma \in S_n} \frac{-i}{q_{\sigma(1)}^v - i\epsilon} \frac{-i}{(q_{\sigma(1)}^v + q_{\sigma(2)}^v) - i\epsilon} \cdots \frac{-i}{(q_{\sigma(1)}^v + q_{\sigma(2)}^v + \cdots + q_{\sigma(n-1)}^v) - i\epsilon} . \end{aligned} \tag{I.4}$$

We now make use of the following identity,

$$\begin{aligned} & \lim_{\epsilon \to 0} \delta(x_1 + x_2 + \cdots + x_n) \sum_{\text{œ} \in \mathrm{S_n}} \frac{1}{x_{\sigma(1)} \pm i\epsilon} \frac{1}{x_{\sigma(1)} + x_{\sigma(2)} \pm i\epsilon} \cdots \frac{1}{x_{\sigma(1)} + x_{\sigma(2)} + \cdots x_{\sigma(n-1)} \pm i\epsilon} \\ & = (\mp 2\pi i)^{n-1} \delta(x_1)\delta(x_2)\cdots\delta(x_n) , \end{aligned} \tag{I.5}$$

on the last line of (I.4) to get

$$i\mathcal{M}_{n-1} = \left(\frac{2\pi i}{4p^v p^u}\right)^{n-1} \int \prod_{i=1}^{n} \left(\frac{dq_i^v dq_i^u dq_i^y}{(2\pi)^3} \frac{-s^2 F(q_i)}{4q_i^4(q_i^2+m^2)}\right) (2\pi)^3 \delta\left(q + \sum_{i=i}^{n} q_i^y\right)$$
$$\times \delta\left(\sum_{i=i}^{n} q_i^u\right) \frac{1}{q_1^u + i\epsilon} \frac{1}{(q_1^u+q_2^u)+i\epsilon} \cdots \frac{1}{(q_1^u+q_2^u+\cdots+q_{n-1}^u)+i\epsilon}$$
$$\times \prod_{i=1}^{n} \delta\left(q_i^v\right) . \tag{I.6}$$

Performing all q_i^v integrals sets all q_i^v to zero, so $q_i^2 = -2q_i^v q_i^u + (q_i^y)^2 \to (q_i^y)^2$, and we get

$$i\mathcal{M}_{n-1} = \left(\frac{2\pi i}{4p^v p^u}\right)^{n-1} \int \prod_{i=1}^{n} \left(\frac{dq_i^u dq_i^y}{(2\pi)^3} \frac{-s^2 F(\{q_i^u, 0, q_i^y\})}{4(q_i^y)^4((q_i^y)^2+m^2)}\right) (2\pi)^3 \delta\left(q + \sum_{i=i}^{n} q_i^y\right)$$
$$\times \delta\left(\sum_{i=i}^{n} q_i^u\right) \frac{1}{q_1^u + i\epsilon} \frac{1}{(q_1^u+q_2^u)+i\epsilon} \cdots \frac{1}{(q_1^u+q_2^u+\cdots+q_{n-1}^u)+i\epsilon} . \tag{I.7}$$

We can symmetrize the second line by summing over all permutations of labels and dividing by $n!$ to get

$$i\mathcal{M}_{n-1} = \frac{1}{n!}\left(\frac{2\pi i}{4p^v p^u}\right)^{n-1} \int \prod_{i=1}^{n} \left(\frac{dq_i^u dq_i^y}{(2\pi)^3} \frac{-s^2 F(\{q_i^u, 0, q_i^y\})}{4(q_i^y)^4((q_i^y)^2+m^2)}\right) (2\pi)^3 \delta\left(q + \sum_{i=i}^{n} q_i^y\right)$$
$$\times \delta\left(\sum_{i=i}^{n} q_i^u\right) \sum_{\sigma\in S_n} \frac{1}{q_1^u + i\epsilon} \frac{1}{(q_1^u+q_2^u)+i\epsilon} \cdots \frac{1}{(q_1^u+q_2^u+\cdots+q_{n-1}^u)+i\epsilon} . \tag{I.8}$$

Then applying the identity (I.5) on the last line we get

$$i\mathcal{M}_{n-1} = \frac{1}{n!}\left(\frac{2\pi i}{4p^v p^u}\right)^{n-1} \int \prod_{i=1}^{n} \left(\frac{dq_i^u dq_i^y}{(2\pi)^3} \frac{-s^2 F(\{q_i^u, 0, q_i^y\})}{4(q_i^y)^4((q_i^y)^2+m^2)}\right) (2\pi)^3 \delta\left(q + \sum_{i=i}^{n} q_i^y\right)$$
$$\times (-2\pi i)^{n-1} \prod_{i=1}^{n} \delta\left(q_i^u\right) . \tag{I.9}$$

Now performing q_i^u integrals gives

$$i\mathcal{M}_{n-1} = \frac{1}{n!}\left(\frac{(2\pi)^2}{4p^v p^u}\right)^{n-1} \int \prod_{i=1}^{n} \left(\frac{dq_i^y}{(2\pi)^3} \frac{-s^2 F(\{0, 0, q_i^y\})}{4(q_i^y)^4((q_i^y)^2+m^2)}\right) (2\pi)^3 \delta\left(q + \sum_{i=i}^{n} q_i^y\right) . \tag{I.10}$$

From (I.2) we get $F(\{0,0,q_i^y\}) = -2\kappa^2(q^y)^2(q^y - im)m$, and we can write

$$\delta\left(q + \sum_{i=i}^{n} q_i^y\right) = \int_{-\infty}^{\infty} \frac{db}{2\pi} e^{-ib\left(q+\sum_{i=i}^{n} q_i^y\right)} \tag{I.11}$$

so finally we get

$$i\mathcal{M}_{n-1} = \frac{1}{n!}\left(\frac{1}{2s}\right)^{n-1} \int_{-\infty}^{\infty} dbe^{-ibq}\left(\int_{-\infty}^{\infty} \frac{dq^y}{2\pi} \frac{\kappa^2 s^2 m}{2(q^y)^2(q^y+im)} e^{-ibq^y}\right)^n , \tag{I.12}$$

where we used the fact that $s = 2p^v p^u$ in the eikonal limit. The term in the second integral can be written as

$$\frac{\kappa^2 s^2 m}{2(q^y)^2(q^y+im)} = iM_{\text{tree}}(s, t = -(q^y)^2), \tag{I.13}$$

where M_{tree} is the eikonal limit of tree level 2-2 scalar scattering amplitude. Then summing all loop diagrams gives the full eikonal amplitude:

$$i\mathcal{M}_{\text{eik}} = 2s \int_{-\infty}^{\infty} dbe^{-ibq} \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{i}{2s}\int_{-\infty}^{\infty} \frac{dq^y}{2\pi} M_{\text{tree}}(s, t = -(q^y)^2) e^{-ibq^y}\right)^n = 2s \int_{-\infty}^{\infty} dbe^{-ibq}\left(e^{i\delta} - 1\right), \tag{I.14}$$

where

$$\delta = \frac{1}{2s}\int_{-\infty}^{\infty} \frac{dq^y}{2\pi} M_{\text{tree}}(s, t = -(q^y)^2) e^{-ibq^y} . \tag{I.15}$$

is commonly referred to as the phase shift. Therefore, we have proved that the TMG amplitudes exponentiate in the eikonal limit, which to the best of our knowledge has not been proven before.

The calculation of the eikonal amplitude in TME is almost identical to that of TMG but now two scalar-scalar-photon vertices contracted with photon propagator gives

$$\frac{sF(q)}{q^2(q^2+m^2)}, \tag{I.16}$$

where

$$F(q) = 2g^2Q^2(-iq^u q^v + q^y(iq^y + m)) \tag{I.17}$$

instead of (I.1) and (I.2). Now, repeating the same steps as before we get

$$i\mathcal{A}_{n-1} = \frac{1}{n!}\left(\frac{(2\pi)^2}{4p^v p^u}\right)^{n-1} \int \prod_{i=1}^{n} \left(\frac{dq_i^y}{(2\pi)^3}\frac{sF(\{0,0,q_i^y\})}{(q_i^y)^2((q_i^y)^2+m^2)}\right)(2\pi)^3\delta\left(q+\sum_{i=i}^{n} q_i^y\right) . \tag{I.18}$$

Using $F(\{0,0,q_i^y\}) = i2g^2Q^2q^y(q^y - im)$ and the expression for the tree-level scattering amplitude in the eikonal limit,

$$\frac{2isg^2Q^2}{q^y(q^y+im)} = iA_{\text{tree}}(s, t = -(q^y)^2) \ , \tag{I.19}$$

we get the same expression as in TMG case

$$i\mathcal{A}_{\text{eik}} = 2s\int_{-\infty}^{\infty} dbe^{-ibq}\left(e^{i\delta}-1\right), \tag{I.20}$$

where the phase shift reads

$$\delta = \frac{1}{2s}\int_{-\infty}^{\infty}\frac{dq^y}{2\pi}A_{\text{tree}}(s, t = -(q^y)^2)e^{-ibq^y}. \tag{I.21}$$

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