

SECOND-ORDER PERTURBATION THEORY FOR ACCELERATORS†

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The nonlinear effects on the dynamical systems with emphasis on nonlinear resonances are investigated using second-order perturbation theory in two dimensions. We have solved the equations of motion and derived expressions that yield information about nonlinear contributions to the dynamics of particles in an accelerator, including the perturbation of tune; emittance growth, Hamiltonian resonance strength, generating function resonance strength, fixed points, Chirikov criteria, island width, etc. Furthermore, we have derived symplectic expressions for calculating the emittance and phase which can be used as a faster alternative to tracking. This formalism was implemented in a code, NONLIN, that can be used to study nonlinear effects in accelerators.

1. INTRODUCTION

We investigate the nonlinear effects on dynamical systems with emphasis on nonlinear resonances. We begin with the equations of motion, from which we find the Hamiltonian for a particle in an accelerator as we introduce the concept of the invariant of the motion. Defining the Frenet-Serret coordinates (for a particle in an orbit), we develop the general expression for the vector potential. Using action-angle variables and canonical perturbation theory for accelerators (similar to but more general than Ref. 1), we simulate the nonlinear resonances by inclusion of sextupoles and octupoles in two dimensions. Thereby, we have solved the equations of motion and derived expressions that yield information about nonlinear contributions to the dynamics of particles in an accelerator, including the perturbation of tune, emittance growth, Hamiltonian resonance strength, generating function resonance strength, fixed points, Chirikov criteria, island width, etc.

We have implemented some of our results in an algorithm (a preliminary version of NONLIN), which can be used to study nonlinear effects in accelerators, e.g., resonances.

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2. EQUATIONS OF MOTION AND THE HAMILTONIAN FOR A PARTICLE IN AN ACCELERATOR

The trajectory for a particle in an accelerator can be found from the Lorentz force, i.e.,

$$\frac{d}{dt}\mathbf{p} = q\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right], \quad (1)$$

where \mathbf{p} is the momentum, \mathbf{E} and \mathbf{B} are the electric and magnetic field, \mathbf{v} is the velocity, q is the charge, c is the speed of light, and t is the time (all in the laboratory frame of reference).

It is convenient to express the electric and magnetic fields in terms of the vector (\mathbf{A}) and scalar (ϕ) potentials:

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}, \quad (2)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (3)$$

Since the equations of motion can be obtained from the Hamiltonian H using Hamilton's equations,

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad (4)$$

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad (5)$$

with

$$H = \sum p_i \dot{x}_i - L(x_i, p_i, t) \quad (6)$$

[where L is the Lagrangian given by Lagrange's equations, i.e.,

$$\frac{d}{dt}\nabla_{\mathbf{v}}L - \nabla L = 0, \quad (7)$$

$$v_i \equiv \dot{x}_i = \frac{dx_i}{dt},$$

and $\nabla_{\mathbf{v}}$ is the gradient in \mathbf{v} space, with $(\mathbf{v}, \mathbf{x}, t)$ as the independent variables (basis)], the Hamiltonian that produces Eq. (1) as the equations of motion can be found by first substituting Eqs. (3) into Eq. (1):

$$\frac{d}{dt}\mathbf{p} = q\left[-\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} + \frac{1}{c}\mathbf{v} \times (\nabla \times \mathbf{A})\right]. \quad (8)$$

Then, after some manipulation, we get

$$\frac{d}{dt}\left(\mathbf{p} + \frac{q}{c}\mathbf{A}\right) + q\nabla \cdot \left(\phi - \frac{1}{c}\mathbf{v} \cdot \mathbf{A}\right) = 0. \quad (9)$$

Comparing this [Eq. (9)] and Lagrange's equations [Eq. (7)], we obtain the following Lagrangian:

$$L = T - q\phi + \frac{q}{c} \mathbf{A} \cdot \mathbf{v}, \quad (10)$$

where $\nabla_{\mathbf{v}} T = \mathbf{p}$ [since $\mathbf{p} = m\mathbf{v}\gamma$, then $T = -mc^2/\gamma$, where $\gamma = 1/(1 - v^2/c^2)^{1/2}$].

The Hamiltonian can be found from the Lagrangian by changing the base variables from $(\mathbf{v}, \mathbf{x}, t)$ to $(\mathbf{P}, \mathbf{x}, t)$, where $\mathbf{P} = (\mathbf{p} + (q/c)\mathbf{A})$ is the canonical momentum; using Eq. (6) and the above Lagrangian [Eq. (10)]:

$$H = c \sqrt{\left(\mathbf{P} - \frac{q}{c} \mathbf{A}\right)^2} + m_0^2 c^2 + q\phi. \quad (11)$$

Furthermore, through this Hamiltonian formulation, the concept of an invariant of the motion can be introduced. We first express the total differential of the Lagrangian and the Hamiltonian:

$$dL = (\nabla_{\mathbf{v}} L) \cdot d\mathbf{v} + (\nabla L) \cdot d\mathbf{x} + \frac{\partial L}{\partial t} dt \quad (12)$$

and

$$dH = (\nabla_{\mathbf{P}} H) \cdot d\mathbf{P} + (\nabla H) \cdot d\mathbf{x} + \frac{\partial H}{\partial t} dt. \quad (13)$$

Defining the independent canonical variable \mathbf{P} to be

$$\mathbf{P} = \nabla_{\mathbf{v}} L \quad (14)$$

and, since

$$\mathbf{P} \cdot d\mathbf{v} = d(\mathbf{P} \cdot \mathbf{v}) - \mathbf{v} \cdot d\mathbf{P}, \quad (15)$$

$$\nabla(\mathbf{P} \cdot \mathbf{v}) = 0, \quad (16)$$

then Eqs. (12) and (13) become equivalent expressions (leading to the Hamiltonian formulation Eqs. (4)–(6), if

$$H = \mathbf{P} \cdot \mathbf{v} - L, \quad (17)$$

$$v = \nabla_{\mathbf{P}} H, \quad (18)$$

$$\frac{d}{dt} \mathbf{P} = -\nabla L = \nabla H. \quad (19)$$

Thus, if H is independent of time (i.e., $dH/dt = 0$), then H is called an invariant of the motion. Other invariants may also exist, e.g., emittance.¹

It is useful for accelerators to express the Hamiltonian [Eq. (11)] in terms of the Frenet-Serret coordinate system (x, s, z) , (for particles in an orbit) shown in Fig. 1, such that

$$\frac{d}{ds} \hat{x} = \frac{\hat{s}}{\rho(s)}, \quad (21)$$

$$\frac{d}{ds} \hat{s} = -\frac{\hat{x}}{\rho(s)}, \quad (22)$$

$$\frac{d}{ds} \hat{z} = 0, \quad (23)$$

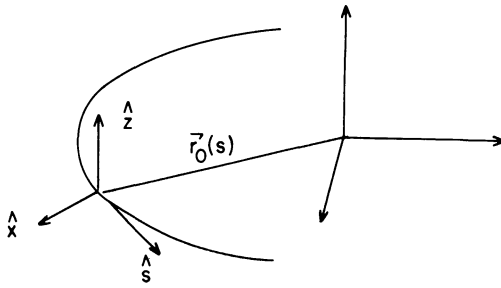


FIGURE 1 Coordinate system.

where $\hat{\cdot}$ defines a unit vector and $\rho(s)$ is the radius of curvature (e.g., in a bending magnet), which may vary along the curve. In this coordinate system, a particle that is x units along the \hat{x} direction will have a radius of curvature equal to $x + \rho(s)$. Thus, the Hamiltonian becomes

$$H = c \sqrt{\frac{\left(P_s - \frac{q}{c} A_s\right)^2}{\left(1 + \frac{x}{\rho}\right)^2} + \left(P_x - \frac{q}{c} A_x\right)^2 + \left(P_z - \frac{q}{c} A_z\right)^2 + M_0^2 c^2 + q\phi}, \quad (24)$$

where the length l (as well as the momentum) the particle travels along the orbit varies with the radius of curvature, i.e.,

$$l = \frac{x + \rho}{\rho} s = s \left(1 + \frac{x}{\rho}\right), \quad (25)$$

$$P_s = (P \cdot \hat{s}) \left(1 + \frac{x}{\rho}\right), \quad (26)$$

and

$$A_s = (A \cdot \hat{s}) \left(1 + \frac{x}{\rho}\right). \quad (27)$$

P_x and P_z are projections of momentum along the x and z directions.

In an accelerator, the electric and magnetic fields are periodic with s (the length the particle travels when on an equilibrium orbit); however, since the fields are not in general simple functions of time t , it is useful to change the independent variable from t to s (as the new “time” variable). Thus, the new Hamiltonian should have the following equations of motion:

$$\frac{dx}{ds} = \frac{\partial P_x}{\partial H} = - \frac{\partial P_s}{\partial P_x} \Big|_H = \frac{\partial \vec{H}_0}{\partial P_x}, \quad (28)$$

$$\frac{dP_x}{ds} = -\frac{\frac{\partial H}{\partial x}}{\frac{\partial P_s}{\partial P_s}} = \frac{\partial P_s}{\partial x} \Big|_H = -\frac{\partial \bar{H}_0}{\partial x}, \quad (29)$$

$$\frac{dz}{ds} = \frac{\frac{\partial H}{\partial P_z}}{\frac{\partial P_s}{\partial P_s}} = -\frac{\partial P_s}{\partial P_z} \Big|_H = \frac{\partial \bar{H}_0}{\partial P_z}, \quad (30)$$

$$\frac{dP_z}{ds} = -\frac{\frac{\partial H}{\partial z}}{\frac{\partial P_s}{\partial P_s}} = \frac{\partial P_s}{\partial z} \Big|_H = -\frac{\partial \bar{H}_0}{\partial z} \quad (31)$$

$$\frac{dt}{ds} = \frac{\partial P_s}{\partial H} = \frac{\partial H_0}{\partial(-H)}, \quad (32)$$

$$\frac{dH}{ds} = \frac{\frac{\partial H}{\partial t}}{\frac{\partial P_s}{\partial P_s}} = \frac{\partial P_s}{\partial t} \Big|_H = -\frac{\partial \bar{H}_0}{\partial t}. \quad (33)$$

This implies the new Hamiltonian (\bar{H}_0) to be

$$\bar{H}_0 = -P_s(x, P_x, z, P_z, t, -H, s), \quad (34)$$

with $(t, -H)$ forming a set of conjugate variables.

We further simplify the Hamiltonian by considering the case when there are no electric fields and only transverse magnetic fields, i.e., $\phi = A_x = A_z = 0$; then \bar{H}_0 reduces to

$$\bar{H}_0 = -\left(1 + \frac{x}{\rho}\right) \sqrt{P^2 - P_x^2 - P_z^2} - \frac{q}{c} A_s, \quad (35)$$

$$P = \sqrt{H^2 - m^2 c^4}/c \quad (36)$$

(note, as long as there is no time dependence, H is a constant of the motion). Since $P \gg P_x, P_z$ and $\rho \gg x$, then

$$\bar{H}_0 = -P \left(1 + \frac{x}{\rho}\right) + \frac{P_x^2}{2P} + \frac{P_z^2}{2P} - \frac{q}{c} A_s + \text{higher-order terms.} \quad (37)$$

The form of the vector potential $A_s(x, z, s)$ is restricted since it must obey Maxwell's equations, i.e.,

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = 0, \quad (38)$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0, \quad (39)$$

with the vertical field B_z (on the plane) given by the following Taylor expansion:

$$B_z(x, z = 0) = B\rho \left(\frac{1}{\rho} + Kx + S \frac{x^2}{2!} + O \frac{x^3}{3!} + \dots \right), \quad (40)$$

which in turn defines the quadrupole strength K , sextupole strength S , octupole strength O , etc.

Equations (39)–(40) implies (with $A_x = A_z = 0$) that

$$\frac{\partial A_s}{\partial s} = 0 \quad (41)$$

and

$$\frac{\partial}{\partial x} \left(1 + \frac{x}{\rho} \right)^{-1} \frac{\partial}{\partial x} \left(1 + \frac{x}{\rho} \right) A_s + \frac{\partial^2 A_s}{\partial z^2} = 0. \quad (42)$$

Hence, A_s becomes

$$A_s = -B\rho \left[\frac{1}{\rho} \left(x - \frac{x^2}{2\rho} + \frac{x^3}{2\rho^2} - \frac{x^4}{2\rho^3} + \dots \right) + K \left(\frac{x^2 - z^2}{2} - \frac{x^3}{6\rho} + \frac{4x^4 - z^4}{24\rho^2} + \dots \right) \right. \\ \left. + S \left(\frac{x^3 - 3xz^2}{6} - \frac{x^4 - z^4}{24\rho} + \dots \right) + O \left(\frac{x^4 - 6x^2z^2 + z^4}{24} + \dots \right) + \dots \right], \quad (43)$$

where (assuming a separated-function accelerator) the contribution due to K/ρ , K/ρ^2 , etc., (as well as the terms of the order of $1/\rho^3$) are negligible.

Thus, for illustration we will use the following potential:

$$A_s = -\frac{P_0 c}{q} \left[\frac{x}{\rho} + \left(\frac{1}{\rho^2} - K \right) \frac{x^2}{2} + K \frac{z^2}{2} \right. \\ \left. + \frac{S}{6} (x^3 - 3xz^2) + \frac{O}{24} (x^4 - 6x^2z^2 + z^4) + \dots \right], \quad (44)$$

where P_0 is the momentum of the beam on the equilibrium orbit. Then, Eq. (37) becomes

$$\tilde{H}_0 = -P - (P - P_0) \frac{x}{\rho} + \frac{P_x^2}{2P} + \frac{P_z^2}{2P} + P_0 \left(\frac{1}{\rho^2} - K \right) \frac{x^2}{2} + P_0 K \frac{z^2}{2} \\ + P_0 \frac{S}{6} (x^3 - 3xz^2) + P_0 \frac{O}{24} (x^4 - 6x^2z^2 + z^4). \quad (45)$$

The coefficients ρ , K , S , O , etc. (generally expressed as functions of s), in the vector potential A_s [Eq. (44)] must be constant in order to satisfy Maxwell's equations, Eqs. (38)–(39). If these constants are made piecewise constant in "time" s for each magnet, the Maxwell's equations will still hold, except at the ends of the magnets where the field changes, resulting in the addition of a longitudinal component of the field. These edges are often modeled by discontinuous functions, where a matching condition is used.

Finally, we normalize the Hamiltonian with the following transformation:

$$H_0 = \frac{\tilde{H}_0}{P}, \quad (46)$$

$$p_x = \frac{P_x}{P}, \quad (47)$$

$$p_z = \frac{P_z}{P}, \quad (48)$$

and assume that the momentum of the particle $P = P_0$ (i.e., the design momentum). Note that the constant term P in Eq. (44) has been dropped, since it does not affect the equations of motion, leaving

$$H_0 = \frac{p_x^2}{2} + \frac{p_z^2}{2} + \left(\frac{1}{\rho^2(s)} - K(s) \right) \frac{x^2}{2} + K(s) \frac{z^2}{2} + \frac{S(s)}{6} (x^3 - 3xz^2) + \frac{O(s)}{24} (x^4 - 6x^2z^2 + z^4) + \dots \quad (49)$$

The solutions to the equations of motion obtained from this Hamiltonian [Eq. (49)] is sought but are nontrivial (due to the presence of nonlinear terms) and are discussed in the following sections.

3. PERTURBATION THEORY

The Hamiltonian describing an accelerator with sextupoles and octupoles can be represented (from Section 2):

$$H_0 = \frac{1}{2}(p_x^2 + p_z^2) + \left(\frac{1}{\rho^2(s)} - K(s) \right) \frac{x^2}{2} + K(s) \frac{z^2}{2} + \frac{S(s)}{6} (x^3 - 3xz^2) + \frac{O(s)}{24} (x^4 - 6x^2z^2 + z^4), \quad (50)$$

where x and z represent the transverse particle position (shown in Fig. 1) with respect to the equilibrium orbit, and p_x and p_z are the conjugate momenta. $S(s)$ and $O(s)$ are given by

$$S(s) = \frac{1}{B\rho} \left. \frac{d^2 B_z}{dx^2} \right|_{x,z=0} \quad (51)$$

and

$$O(s) = \frac{1}{B\rho} \left. \frac{d^3 B_z}{dx^3} \right|_{x,z=0} \quad (52)$$

B_z is the vertical component of the B field. Note that the field due to the dipole is a predominantly vertical field and may have sextupole and octupole terms as defined in Eqs. (51) and (52).

In order to study the effects of the nonlinear elements we use the canonical transformations to action-angle variables² with the generating function F :

$$F(x, z, \phi_x, \phi_z, s) = -\frac{z^2}{2\beta_z(s)} \left[\tan \phi_z - \frac{\beta'_z(s)}{2} \right] - \frac{x^2}{2\beta_x(s)} \left[\tan \phi_x - \frac{\beta'_x(s)}{2} \right]. \quad (53)$$

Where ϕ_x and ϕ_z are the angle variables, primes represent d/ds , and $\beta_x(s)$ and $\beta_z(s)$ (the horizontal and vertical beta functions) are solutions to

$$\frac{\beta_x \beta_x''}{2} - \frac{(\beta_x')^2}{4} + \left(\frac{1}{\rho^2} - K(s) \right) \beta_x^2 = 1 \quad (54)$$

and

$$\frac{\beta_z \beta_z''}{2} - \frac{(\beta_z')^2}{4} + K(s) \beta_z^2 = 1. \quad (55)$$

The action variables J_x and J_z are

$$J_x = -\frac{\partial F}{\partial \phi_x} = \frac{x^2}{2\beta_x(s)} \sec^2 \phi_x, \quad (56)$$

$$J_z = -\frac{\partial F}{\partial \phi_z} = \frac{z^2}{2\beta_z(s)} \sec^2 \phi_z, \quad (57)$$

which implies that

$$x = \sqrt{2J_x \beta_x(s)} \cos \phi_x, \quad (58)$$

$$z = \sqrt{2J_z \beta_z(s)} \cos \phi_z. \quad (59)$$

The emittance $E_{x,z}$ is an invariant of the motion in an accelerator without sextupoles, octupoles, etc., and the conjugate momenta are proportional to the beam emittances ($2J_x = E_x/\pi$ and $2J_z = E_z/\pi$). Therefore, without the nonlinear elements, J_x and J_z are invariants of the motion.

The transformed Hamiltonian becomes

$$\begin{aligned} H_1 = H_0 + \frac{\partial F}{\partial s} = & \frac{J_x}{\beta_x(s)} + \frac{J_z}{\beta_z(s)} + \frac{S(s)}{3} \sqrt{2J_x \beta_x(s)} \cos \phi_x [J_x \beta_x(s) \cos^2 \phi_x \\ & - 3J_z \beta_z(s) \cos^2 \phi_z] + \frac{O(s)}{6} [J_x^2 \beta_x^2(s) \cos^4 \phi_x - 6J_x J_z \beta_x(s) \beta_z(s) \cos^2 \phi_x \cos^2 \phi_z \\ & + J_z^2 \beta_z^2(s) \cos^4 \phi_z]. \end{aligned} \quad (60)$$

Next, we search for a new generating function G that eliminates the ϕ_x and ϕ_z dependencies in a new Hamiltonian H_2 so that new action variables (K_x and K_z) including nonlinear effects are invariants of the motion. We used the following generating function:

$$\begin{aligned} G(K_x, K_z, \phi_x, \phi_z, s) = & \phi_x K_x + \phi_z K_z + K_x^{3/2} w_1(\phi_x, \phi_z, s) + K_x^{1/2} K_z w_2(\phi_x, \phi_z, s) \\ & + K_x^2 v_1(\phi_x, \phi_z, s) + K_x K_z v_2(\phi_x, \phi_z, s) + K_z^2 v_3(\phi_x, \phi_z, s). \end{aligned} \quad (61)$$

Hence, H_1 transforms into

$$H_2 = \frac{K_x}{\beta_x} + \frac{K_z}{\beta_z} + a(s)K_x^2 + b(s)K_xK_z + c(s)K_z^2 + O(K^{5/2}) \quad (62)$$

where $a(s)$, $b(s)$, and $c(s)$ are given in Appendix B [by Eqs. (B-15), (B-22), and (B-43)], and the functions $w_1(\phi_x, \phi_z, s)$, $w_2(\phi_x, \phi_z, s)$, $v_1(\phi_x, \phi_z, s)$, etc., in the generating function G [Eq. (57)] must satisfy the following equations:

$$0 = \frac{1}{\beta_x(s)} \frac{\partial w_1}{\partial \phi_x} + \frac{1}{\beta_z(s)} \frac{\partial w_1}{\partial \phi_z} + \frac{\partial w_1}{\partial s} + \frac{\sqrt{2}}{3} S(s) \beta_x^{3/2}(s) \cos^3 \phi_x, \quad (63)$$

$$0 = \frac{1}{\beta_x(s)} \frac{\partial w_2}{\partial \phi_x} + \frac{1}{\beta_z(s)} \frac{\partial w_2}{\partial \phi_z} + \frac{\partial w_2}{\partial s} - \sqrt{2\beta_x(s)} S(s) \beta_z(s) \cos \phi_x \cos^2 \phi_z, \quad (64)$$

$$a(s) = \frac{1}{\beta_x} \frac{\partial v_1}{\partial \phi_x} + \frac{1}{\beta_z(s)} \frac{\partial v_1}{\partial \phi_z} + \frac{\partial v_1}{\partial s} + \frac{\sqrt{2}}{2} S(s) \beta_x^{3/2}(s) \frac{\partial w_1}{\partial \phi_x} \cos^3 \phi_x - \sqrt{2\beta_x(s)} S(s) \beta_z \cos \phi_x \cos^2 \phi_z \frac{\partial w_1}{\partial \phi_z} + \frac{O(s)}{6} \beta_x^2(s) \cos^4 \phi_x, \quad (65)$$

$$b(s) = \frac{1}{\beta_x(s)} \frac{\partial v_2}{\partial \phi_x} + \frac{1}{\beta_z(s)} \frac{\partial v_2}{\partial \phi_z} + \frac{\partial v_2}{\partial s} + \frac{\sqrt{2}}{2} S(s) \beta_x^{3/2}(s) \frac{\partial w_2}{\partial \phi_x} \cos^3 \phi_x - \sqrt{\frac{\beta_x}{2}} S(s) \beta_z \left(\frac{\partial w_1}{\partial \phi_x} + 2 \frac{\partial w_2}{\partial \phi_z} \right) \cos \phi_x \cos^2 \phi_z - O(s) \beta_x(s) \beta_z(s) \cos^2 \phi_x \cos^2 \phi_z, \quad (66)$$

$$c(s) = \frac{1}{\beta_x(s)} \frac{\partial v_3}{\partial \phi_x} + \frac{1}{\beta_z(s)} \frac{\partial v_3}{\partial \phi_z} + \frac{\partial v_3}{\partial s} - \sqrt{\frac{\beta_x}{2}} S(s) \beta_z(s) \frac{\partial w_2}{\partial \phi_x} \cos \phi_x \cos^2 \phi_z + \frac{O(s)}{6} \beta_z^2(s) \cos^4 \phi_z. \quad (67)$$

To eliminate the ϕ_x and ϕ_z dependent terms, we find the functions w_1 , w_2 , v_1 , v_2 , and v_3 such that the coefficients of $K_x^{n/2}$, $K_z^{m/2}$ are either zero or functions of s (where m and n are integers and $(m+n) \leq 4$). This implies H_2 is no longer a function of ϕ_x or ϕ_z to second order in K ; K_x and K_z become (approximately) invariants of the motion. That is,

$$\frac{dK_x}{ds} = -\frac{dH_2}{\partial \phi_x} \cong 0 \quad (68)$$

$$\frac{dK_z}{ds} = -\frac{\partial H_2}{\partial \phi_z} \cong 0 \quad (69)$$

Note that, for stable motion, K_x and K_z are of the order of emittance, therefore they are small.

The method for solving Eqs. (63) through (67) for w_1 , w_2 , v_1 , v_2 , and v_3 are given in Appendices A and B. Note that these solutions can be expressed in a Fourier series from which we can obtain information on resonances. The generating function in Eq. (3.12) is linear in w_1 , w_2 , v_1 , v_2 , and v_3 and can be

expressed as a Fourier series:

$$G = K_x \phi_x + K_z \phi_z + \sum_k [g_k^c(K_x, K_z, s) \cos(n_{xk} \phi_x + n_{zk} \phi_z) + g_k^s(K_x, K_z, s) \sin(n_{xk} \phi_x + n_{zk} \phi_z)]. \quad (70)$$

From this G , we can define a generating function resonance strength R_k as follows:

$$R_k(K_x, K_z, s) = \sqrt{[g_k^c(K_x, K_z, s)]^2 + [g_k^s(K_x, K_z, s)]^2} \cdot |\sin \pi(n_{xk} \nu_x + n_{zk} \nu_z)| \quad (71)$$

such that, when on resonance, R_k reduces to the Hamiltonian resonance strength. These resonance strengths (R_k) can be seen to be directly related to emittance growth, since from the generating function G we have

$$E_x = 2\pi J_x = 2\pi \frac{\partial G}{\partial \phi_x}, \quad (72a)$$

$$E_z = 2\pi J_z = 2\pi \frac{\partial G}{\partial \phi_z}. \quad (72b)$$

From these we can estimate the maximum growth of the beam emittance as

$$E_x \leq 2\pi \left[K_x + \sum_k \left| n_{xk} \frac{R_k(K_x, K_z, s)}{\sin \pi(n_{xk} \nu_x + n_{zk} \nu_z)} \right| \right], \quad (73a)$$

$$E_z \leq 2\pi \left[K_z + \sum_k \left| n_{zk} \frac{R_k(K_x, K_z, s)}{\sin \pi(n_{xk} \nu_x + n_{zk} \nu_z)} \right| \right]. \quad (73b)$$

As long as the tunes are far from any resonances and K_x and K_z are small, Eqs. (73) give the upper limit of the emittance growth.

Furthermore, the betatron tune is perturbed due to these nonlinear terms. To find the tune, we return to the transformed Hamiltonian H_2 [Eq. (62)]. The equation of motion for the phase advance of H_2 is

$$\frac{d}{ds} \psi_x = \frac{\partial H}{\partial K_x} = \frac{1}{\beta_x} + 2a(s)K_x + b(s)K_z, \quad (74a)$$

$$\frac{d}{ds} \psi_z = \frac{\partial H}{\partial K_z} = \frac{1}{\beta_z} + b(s)K_x + 2c(s)K_z, \quad (74b)$$

where ψ_x and ψ_z are the transformed angle variables of Hamiltonian H_2 and the coefficients $a(s)$, $b(s)$, $c(s)$ are given in Appendix B. Since (as discussed before) K_x and K_z are invariants of the motion, the perturbed betatron tune is found to be

$$\begin{aligned} \nu_x &= \nu_x^0 + 2\alpha_{xx}K_x + 2\alpha_{xz}K_z, \\ \nu_z &= \nu_z^0 + 2\alpha_{zx}K_x + 2\alpha_{zz}K_z, \end{aligned} \quad (75)$$

where

$$\nu_x^0 = \frac{1}{2\pi} \int_0^c \frac{dt}{\beta_x(t)}, \quad (76)$$

$$v_z^0 = \frac{1}{2\pi} \int_0^C \frac{dt}{\beta_z(t)}, \tag{77}$$

and

$$\alpha_{xx} = \frac{1}{\pi} \int_0^C a(t) dt, \tag{78}$$

$$\alpha_{xz} = \frac{1}{2\pi} \int_0^C b(t) dt, \tag{79}$$

$$\alpha_{zz} = \frac{1}{\pi} \int_0^C c(t) dt, \tag{80}$$

with C being the circumference of the accelerator.

The perturbative treatment given above works fine as long as we are far from resonance.

4. NEAR RESONANCE

The perturbative approach given in Section 3 breaks down when we approach a resonance, i.e., $\delta \cong 0$, where,

$$\delta \equiv n_x v_x + n_z v_z - p \tag{81}$$

and the integers n_x , n_z , and p classify the given resonance. The order of this resonance is defined as $N \equiv |n_x| + |n_z|$ which depends on how far we can expand the Hamiltonian. Equation (50) illustrates a Hamiltonian with sextupoles and octupoles. The sextupoles produce third-order resonances whereas the octupoles produce fourth-order resonances [see Eq. (60)]. The third-order sextupole resonances in the Hamiltonian of Eq. (60) can be removed by using the generating function given by Eq. (61) with $v_1 = v_2 = v_3 = 0$, and w_1 and w_2 given in Appendix B. In the new Hamiltonian, we see the second-order sextupole terms contributing to the fourth-order resonances (as well as the octupole terms). We can continue finding higher-order resonances by using higher-order transformations similar to those described above. The general form of the new Hamiltonian is

$$H = \frac{I_x}{\beta_x(s)} + \frac{I_z}{\beta_z(s)} + \alpha(I_x, I_z, s) + h^c(I_x, I_z, s) \cos(n_x \phi_x + n_z \phi_z) + h^s(I_x, I_z, s) \sin(n_x \phi_x + n_z \phi_z), \tag{82}$$

where (I_x, ϕ_x) and (I_z, ϕ_z) are the action-angle variables and assume we are near a particular resonance and far enough away from other resonances so that their effects are negligible.

Next, we solve the equations of motion obtained from the Hamiltonian in Eq. (82) to find the behavior of the system near a resonance. We first remove some of the "time" dependence using the following generating function F [and later

another generating function G , given by Eq. (103), will remove the rest]:

$$F(L_x, L_z, \phi_x, \phi_z, s) = L_x \left[\phi_x + \frac{2\pi}{C} v_x^0 s - \int_0^s \frac{dt}{\beta_x(t)} \right] + L_z \left[\phi_z + \frac{2\pi}{C} v_z^0 s - \int_0^s \frac{dt}{\beta_z(t)} \right] \\ + \frac{s}{C} \int_0^C \alpha(L_x, L_z, t) dt - \int_0^s \alpha(L_x, L_z, t) dt, \quad (83)$$

where v_x^0 and v_z^0 are the unperturbed betatron tunes defined in Eq. (71) and (L_x, ψ_x) and (L_z, ψ_z) are the new action-angle variables. From this generating function, the new variables are related to the old variables as

$$I_x = L_x, \quad (84)$$

$$I_z = L_z, \quad (85)$$

$$\psi_x = \phi_x + \frac{2\pi}{C} v_x^0 s - \int_0^s \frac{dt}{\beta_x(t)} + \frac{\partial}{\partial L_x} \left[\frac{s}{C} \int_0^C \alpha(L_x, L_z, t) dt - \int_0^s \alpha(L_x, L_z, t) dt \right], \quad (86)$$

$$\psi_z = \phi_z + \frac{2\pi}{C} v_z^0 s - \int_0^s \frac{dt}{\beta_z(t)} + \frac{\partial}{\partial L_z} \left[\frac{s}{C} \int_0^C \alpha(L_x, L_z, t) dt - \int_0^s \alpha(L_x, L_z, t) dt \right]. \quad (87)$$

Thus, the new Hamiltonian becomes

$$H_1 = H + \frac{\partial F}{\partial s}, \quad (86)$$

$$H_1 = \frac{2\pi}{C} v_x^0 L_x + \frac{2\pi}{C} v_z^0 L_z + \frac{1}{C} \int_0^C \alpha(L_x, L_z, t) dt + h^c(L_x, L_z, s) \cos \left\{ n_x [\psi_x + \xi_x(s)] \right. \\ \left. + n_z [\psi_z + \xi_z(s)] - \frac{2\pi}{C} (n_x v_x + n_z v_z) s \right\} + h^s(L_x, L_z, s) \sin \left\{ n_x [\psi_x + \xi_x(s)] \right. \\ \left. + n_z [\psi_z + \xi_z(s)] - \frac{2\pi}{C} (n_x v_x + n_z v_z) s \right\}, \quad (89)$$

where

$$\xi_x(s) = \int_0^s \frac{dt}{\beta_x(t)} + \frac{\partial}{\partial L_x} \int_0^s \alpha(L_x, L_z, t) dt, \quad (90)$$

$$\xi_z(s) = \int_0^s \frac{dt}{\beta_z(t)} + \frac{\partial}{\partial L_z} \int_0^s \alpha(L_x, L_z, t) dt, \quad (91)$$

$$v_x = v_x^0 + \frac{1}{2\pi} \frac{\partial}{\partial L_x} \int_0^C \alpha(L_x, L_z, t) dt, \quad (92)$$

$$v_z = v_z^0 + \frac{1}{2\pi} \frac{\partial}{\partial L_z} \int_0^C \alpha(L_x, L_z, t) dt. \quad (93)$$

Equation (89) still contains many resonance terms. Since $\xi_x(s) - (2\pi/C)v_x s$

and $\xi_z(s) - (2\pi/C)v_z s$ are periodic in s , then the “time”-dependent term in Eq. (89) can be expanded in a Fourier series about s . Defining the coefficients of two Fourier series [one multiplying $\cos(n_x\psi_x + n_z\psi_z)$ and the other multiplying $\sin(n_x\Psi_x + n_z\Psi_z)$, where a bar is used to distinguish the coefficients of the latter series] as

$$h_k^c(L_x, L_z) = \frac{2}{C} \int_0^C \left\{ h^c(L_x, L_z, t) \cos \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} (n_x v_x + n_z v_z) t \right] \right. \\ \left. + h^s(L_x, L_z, t) \sin \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} (n_x v_x + n_z v_z) t \right] \right\} \cos \left(\frac{2\pi}{C} kt \right) dt, \quad (94)$$

and

$$h_k^s(L_x, L_z) = \frac{2}{C} \int_0^C \left\{ h^c(L_x, L_z, t) \cos \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} (n_x v_x + n_z v_z) t \right] \right. \\ \left. + h^s(L_x, L_z, t) \sin \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} (n_x v_x + n_z v_z) t \right] \right\} \sin \left(\frac{2\pi}{C} kt \right) dt, \quad (95)$$

$$\bar{h}_k^c(L_x, L_z) = \frac{2}{C} \int_0^C \left\{ -h^c(L_x, L_z, t) \sin \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} (n_x v_x + n_z v_z) t \right] \right. \\ \left. + h^s(L_x, L_z, t) \cos \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} (n_x v_x + n_z v_z) t \right] \right\} \cos \left(\frac{2\pi}{C} kt \right) dt, \quad (96)$$

$$\bar{h}_k^s(L_x, L_z) = \frac{2}{C} \int_0^C \left\{ -h^c(L_x, L_z, t) \sin \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} (n_x v_x + n_z v_z) t \right] \right. \\ \left. + h^s(L_x, L_z, t) \cos \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} (n_x v_x + n_z v_z) t \right] \right\} \sin \left(\frac{2\pi}{C} kt \right) dt, \quad (97)$$

the Hamiltonian of Eq. (99) can be written as

$$H_1 = \frac{2\pi}{C} v_x^0 L_x + \frac{2\pi}{C} v_z^0 L_z + \frac{1}{C} \int_0^C \alpha(L_x, L_z, t) dt \\ + \sum_k \left\{ \left[h_k^c(L_x, L_z) \cos \left(\frac{2\pi}{C} ks \right) + h_k^s(L_x, L_z) \sin \left(\frac{2\pi}{C} ks \right) \right] \cos(n_x \psi_x + n_z \psi_z) \right. \\ \left. + \left[\bar{h}_k^c(L_x, L_z) \cos \left(\frac{2\pi}{C} ks \right) + \bar{h}_k^s(L_x, L_z) \cdot \sin \left(\frac{2\pi}{C} ks \right) \right] \sin(n_x \psi_x + n_z \psi_z) \right\} \quad (98)$$

or

$$H_1 = \frac{2\pi}{C} v_x^0 L_x + \frac{2\pi}{C} v_z^0 L_z + \frac{1}{C} \int_0^C \alpha(L_x, L_z, t) dt \\ + \sum_k \left[\frac{h_k^c(L_x, L_z) + \bar{h}_k^s(L_x, L_z)}{2} \cdot \cos \left(n_x \psi_x + n_z \psi_z - \frac{2\pi}{C} ks \right) \right. \\ \left. + \frac{h_k^c(L_x, L_z) - \bar{h}_k^s(L_x, L_z)}{2} \cos \left(n_x \psi_x + n_z \psi_z + \frac{2\pi}{C} ks \right) \right]$$

$$\begin{aligned}
& + \frac{h_k^s(L_x, L_z) - \bar{h}_k^c(L_x, L_z)}{2} \sin \left(n_x \psi_x + n_z \psi_z - \frac{2\pi}{C} ks \right) \\
& + \frac{h_k^s(L_x, L_z) + \bar{h}_k^c(L_x, L_z)}{2} \cdot \sin \left(n_x \psi_x + n_z \psi_z + \frac{2\pi}{C} ks \right) \Big]. \quad (99)
\end{aligned}$$

The greatest perturbation to the motion is expected to come from the term that has the slowest-varying argument of sin or cos. This is the term $k = p$, where p is chosen such that $n_x v_x + n_z v_z - p \cong 0$. For simplicity, let

$$\begin{aligned}
A(L_x, L_z) & \equiv C \frac{h_p^c(L_x, L_z) + \bar{h}_p^s(L_x, L_z)}{2} \\
& = \int_0^C \left\{ h^c(L_x, L_z, t) \cos \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} \delta t \right] \right. \\
& \quad \left. + h^s(L_x, L_z, t) \sin \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} \delta t \right] \right\} dt, \quad (100)
\end{aligned}$$

$$\begin{aligned}
B & \equiv C \frac{h_p^s(L_x, L_z) - \bar{h}_p^c(L_x, L_z)}{2} \\
& = \int_0^C \left\{ -h^c(L_x, L_z, t) \sin \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} \delta t \right] \right. \\
& \quad \left. + h^s(L_x, L_z, t) \cos \left[n_x \xi_x(t) + n_z \xi_z(t) - \frac{2\pi}{C} \delta t \right] \right\} dt, \quad (101)
\end{aligned}$$

where $\delta \equiv n_x v_x + n_z v_z - p$. Then the Hamiltonian becomes (approximately)

$$\begin{aligned}
H_1 & = \frac{2\pi}{C} v_x^0 L_x + \frac{2\pi}{C} v_z^0 L_z + \frac{1}{C} \int_0^C \alpha(L_x, L_z, t) dt + \frac{1}{C} A(L_x, L_z) \\
& \cdot \cos \left(n_x \psi_x + n_z \psi_z - \frac{2\pi}{C} ps \right) + \frac{1}{C} B(L_x, L_z) \sin \left(n_x \psi_x + n_z \psi_z - \frac{2\pi}{C} ps \right). \quad (102)
\end{aligned}$$

The equation of motion for the above Hamiltonian can be solved exactly in order to find its stable and unstable motion. This is done using a second canonical transformation (which eliminates the "time" variable) given by the generating function

$$G(D_x, D_z, \psi_x, \psi_z) = \begin{cases} D_x \left(n_x \psi_x + n_z \psi_z - \frac{2\pi}{C} ps \right) + D_z \psi_z & \text{for } n_x \neq 0 \\ D_x \psi_x + D_z \left(n_z \psi_z - \frac{2\pi}{C} ps \right) & \text{for } n_x = 0 \end{cases}, \quad (103)$$

where (D_x, ζ_x) and (D_z, ζ_z) are the new action-angle variables, related to the old variables as follows:

$$L_x = \begin{cases} n_x D_x & \text{for } n_x \neq 0 \\ D_x & \text{for } n_x = 0 \end{cases}, \quad (104)$$

$$L_z = \begin{cases} n_z D_x + D_z & \text{for } n_x \neq 0 \\ n_z D_z & \text{for } n_x = 0 \end{cases} \quad (105)$$

$$\zeta_x = \begin{cases} n_x \psi_x + n_z \psi_z - \frac{2\pi}{C} ps & \text{for } n_x \neq 0 \\ \psi_x & \text{for } n_x = 0 \end{cases}, \quad (106)$$

$$\zeta_z = \begin{cases} \psi_z & \text{for } n_x \neq 0 \\ n_z \psi_z - \frac{2\pi}{C} ps & \text{for } n_x = 0 \end{cases}. \quad (107)$$

Then the transformed Hamiltonian becomes

$$H_2 = \begin{cases} \frac{2\pi}{C} (n_x v_x^0 + n_z v_z^0 - p) D_x + \frac{2\pi}{C} v_z^0 D_z + \frac{1}{C} \int_0^C \alpha(n_x D_x, n_z D_x + D_z, t) dt \\ + \frac{1}{C} A(n_x D_x, n_z D_x + D_z) \cos \zeta_x + \frac{1}{C} B(n_x D_x, n_z D_x + D_z) \sin \zeta_x & \text{(for } n_x \neq 0) \\ \frac{2\pi}{C} v_x^0 D_x + \frac{2\pi}{C} (n_z v_z^0 - p) D_z + \frac{1}{C} \int_0^C \alpha(D_x, n_z D_z, t) dt \\ + \frac{1}{C} A(D_x, n_z D_z) \cos \zeta_z + \frac{1}{C} B(D_x, n_z D_z) \sin \zeta_z & \text{(for } n_x = 0) \end{cases} \quad (108)$$

This new Hamiltonian is an invariant of the motion, and since it only depends on one angle (phase) variable, we have for

(i) $n_x \neq 0$:

(1) $D_z = \text{constant}$,

$$(2) \frac{2\pi}{C} (n_x v_x^0 + n_z v_z^0 - p) D_x + \frac{1}{C} \int_0^C \alpha(n_x D_x, n_z D_x + D_z, t) dt + \frac{1}{C} A(n_x D_x, n_z D_x + D_z) \cos \zeta_x + \frac{1}{C} B(n_x D_x, n_z D_x + D_z) \sin \zeta_x = \text{constant}. \quad (109a)$$

(ii) $n_x = 0$:

(1) $D_x = \text{constant}$,

$$(2) \frac{2\pi}{C} (n_z v_z^0 - p) D_z + \frac{1}{C} \int_0^C \alpha(D_x, n_z D_z, t) dt + \frac{1}{C} A(D_x, n_z D_z) \cos \zeta_z + \frac{1}{C} B(D_x, n_z D_z) \sin \zeta_z = \text{constant}. \quad (109b)$$

From the above equations, the motion can be found (analytically or via the phase plots). Furthermore, we can study the motion through the analysis of the fixed points given below.

Fixed Points: The points at which there is no motion are defined as fixed

points. For

(i) $n_x \neq 0$:

$$\begin{aligned} \frac{d}{ds} \zeta_x &= \frac{\partial H_2}{\partial D_x} = \frac{2\pi}{C} \delta + \frac{1}{C} \frac{\partial}{\partial D_x} A(n_x D_x, n_z D_x + D_z) \cos \zeta_x \\ &+ \frac{1}{C} \frac{\partial}{\partial D_x} B(n_x D_x, n_z D_x + D_z) \sin \zeta_x = 0, \end{aligned} \quad (110a)$$

where δ is the distance from the resonance to the fixed points and is a function of the action variables, and

$$\begin{aligned} \frac{d}{ds} D_x &= -\frac{\partial H_2}{\partial \zeta_x} = \frac{1}{C} A(n_x D_x, n_z D_x + D_z) \sin \zeta_x \\ &- \frac{1}{C} B(n_x D_x, n_z D_x + D_z) \cos \zeta_x = 0. \end{aligned} \quad (110b)$$

(ii) $n_x = 0$:

$$\begin{aligned} \frac{d}{ds} \zeta_z &= \frac{\partial H_2}{\partial D_z} = \frac{2\pi}{C} \delta + \frac{1}{C} \frac{\partial}{\partial D_z} A(D_x, n_z D_z) \cos \zeta_z \\ &+ \frac{1}{C} \frac{\partial}{\partial D_z} B(D_x, n_z D_z) \sin \zeta_z = 0 \end{aligned} \quad (110c)$$

and

$$\frac{d}{ds} D_z = -\frac{\partial H_2}{\partial \zeta_z} = \frac{1}{C} A(D_x, n_z D_z) \sin \zeta_z - \frac{1}{C} B(D_x, n_z D_z) \cos \zeta_z = 0. \quad (110d)$$

Both of the above cases ($n_x \neq 0, n_x = 0$) will lead to the following solution for fixed points, i.e., for all values of n_x , we have [by first solving Eq. (110b) or (110d) and substituting the solutions into Eq. (110a) or (110c)],

$\delta(L_x, L_z)$

$$= \pm \frac{n_x \frac{\partial}{\partial L_x} [A^2(L_x, L_z) + B^2(L_x, L_z)] + n_z \frac{\partial}{\partial L_z} [A^2(L_x, L_z) + B^2(L_x, L_z)]}{4\pi \sqrt{A^2(L_x, L_z) + B^2(L_x, L_z)}}. \quad (111)$$

Equations (110b) and (110d) each has $2N$ unique solutions in action-angle variables (L_x, ψ_x) [and in Eq. (110d), (L_z, ψ_z)], leading to $2N$ fixed points (which may be stable or unstable).

Each stable fixed point can have stable motion, described by a closed trajectory, about it, except when the motion becomes so large that it can go through an unstable fixed point. These stable regions are called islands. It is of interest to estimate the size of these islands when on resonance, since this may give clues as to the importance of the nearby resonances which were neglected. To estimate the island width, we use the invariant given by Eq. (109a). The action at the stable (D_s) and unstable (D_u) fixed points are defined such that when the action $D_x = D_s$ then we are on a stable fixed point; however, when $D_x = D_u$ we will be on an unstable fixed point.

From Eq. (109a) we have (at fixed points)

$$\begin{aligned} \frac{2\pi}{C}(n_x v_x^0 + n_z v_z^0 - p)D_u + I(D_u, D_z) \pm M(D_u, D_z) \\ = \frac{2\pi}{C}(n_x v_x^0 + n_z v_z^0 - p)D_s + I(D_s, D_z) \mp M(D_s, D_z), \end{aligned} \quad (112)$$

where

$$I(D_x, D_z) = \frac{1}{C} \int_0^C \alpha(n_x D_x, n_z D_x + D_z, t) dt \quad (113)$$

$$M(D_x, D_z) = \frac{1}{C} \sqrt{A^2(n_x D_x, n_z D_x + D_z) + B^2(n_x D_x, n_z D_x + D_z)} \quad (114)$$

and the sign in Eq. (112) is chosen depending on whether the case is stable or unstable; M is the ‘‘Hamiltonian resonance strength,’’ which is a positive quantity and does not change sign with change in action. In some cases, the upper signs in Eq. (112) determine the stability conditions, in which case D_s is greater than D_u . Expanding the $I(D_x, D_z)$ and $M(D_x D_z)$ in a Taylor series about D_u , Eq. (112) becomes

$$\begin{aligned} \frac{2\pi}{C}(n_x v_x^0 + n_z v_z^0 - p)D_u + I(D_u, D_z) \pm M(D_u, D_z) \\ = \frac{2\pi}{C}(n_x v_x^0 + n_z v_z^0 - p)D_s + I(D_u, D_z) + \frac{\partial I(D_u, D_z)}{\partial D_u}(D_s - D_u) \\ + \frac{1}{2} \frac{\partial^2 I(D_u, D_z)}{\partial D_u^2} (D_s - D_u)^2 \mp M(D_u, D_z) \mp \frac{\partial M(D_u - D_z)}{\partial D_u} (D_s - D_u) \\ \mp \frac{1}{2} \frac{\partial^2 M(D_u, D_z)}{\partial D_u^2} (D_s - D_u)^2 + 0[(D_s - D_u)^3]. \end{aligned} \quad (115)$$

Since the bandwidth at the unstable fixed point is

$$\delta_u = n_x v_x + n_z v_z - p = n_x v_x^0 + n_z v_z^0 - p + \frac{C}{2\pi} \frac{\partial I(D_u, D_z)}{\partial D_u}, \quad (116)$$

or from Eq. (111) δ_u can be written as

$$\delta_u = \mp \frac{C}{2\pi} \frac{\partial}{\partial D_x} M(D_x, D_z) \Big|_{D_x=D_u}, \quad (117)$$

then Eqs. (115) through (117) lead to the following equations with $(D_s - D_u)$ small:

$$0 = \frac{1}{2} \left(\frac{\partial^2 I}{\partial D_u^2} \mp \frac{\partial^2 M}{\partial D_u^2} \right) (D_s - D_u)^2 \mp 2 \frac{\partial M}{\partial D_u} (D_s - D_u) \mp 2M(D_u, D_s). \quad (118)$$

Solving for $D_s - D_u$ we obtain

$$D_s - D_u = \begin{cases} 2 \left[\frac{\pm \frac{\partial M}{\partial D_u} + \sqrt{\left(\frac{\partial M}{\partial D_u}\right)^2 - M \frac{\partial^2 M}{\partial D_u^2} \pm M \frac{\partial^2 I}{\partial D_u^2}}}{\frac{\partial^2 I}{\partial D_u^2} \mp \frac{\partial^2 M}{\partial D_u^2}} \right] & (119a) \\ 2 \left[\frac{\pm \frac{\partial M}{\partial D_u} - \sqrt{\left(\frac{\partial M}{\partial D_u}\right)^2 - M \frac{\partial^2 M}{\partial D_u^2} \pm M \frac{\partial^2 I}{\partial D_u^2}}}{\frac{\partial^2 I}{\partial D_u^2} \mp \frac{\partial^2 M}{\partial D_u^2}} \right] & (119b) \end{cases}$$

The signs in the discriminants of the above equations (which determine stability conditions and describe the stable and unstable fixed points) must be chosen such that the discriminant

$$\left(\frac{\partial M}{\partial D_u}\right)^2 - M \frac{\partial^2 M}{\partial D_u^2} \pm M \frac{\partial^2 I}{\partial D_u^2}$$

in Eqs. (119) is positive.

Given the island width [Eq. (119)] of two nearby resonances, a criterion determining whether the resonances overlap (or may be treated separately) can be obtained. If the island width ($D_s - D_u$) is large enough to cause the bandwidth [δ given by Eq. (81)] to cross another resonance, then the resonances overlap. The total change in bandwidth $\Delta\delta$ due to change in action across the entire island can be found from Taylor series expansion of $\delta(D_x, D_z)$ about $D_x = D_u$ as

$$\begin{aligned} \delta &= \delta_u + \frac{\partial \delta}{\partial D_u} (D_s - D_u) \\ &= \delta_u + \frac{C}{2\pi} \frac{\partial^2 I}{\partial D_u^2} (D_s - D_u) \end{aligned} \tag{120}$$

or

$$\Delta\delta = \frac{2C}{\pi} \left| \frac{\frac{\partial^2 I}{\partial D_u^2} \sqrt{\left(\frac{\partial M}{\partial D_u}\right)^2 - M \frac{\partial^2 M}{\partial D_u^2} \pm M \frac{\partial^2 I}{\partial D_u^2}}}{\frac{\partial^2 I}{\partial D_u^2} \mp \frac{\partial^2 M}{\partial D_u^2}} \right|. \tag{121}$$

If there exists a nearby resonance with bandwidth δ_r , which satisfies the following criterion:

$$\delta_r \gg \frac{2C}{\pi} \left| \frac{\frac{\partial^2 I}{\partial D_u^2} \sqrt{\left(\frac{\partial M}{\partial D_u}\right)^2 - M \frac{\partial^2 M}{\partial D_u^2} \pm M \frac{\partial^2 I}{\partial D_u^2}}}{\frac{\partial^2 I}{\partial D_u^2} \mp \frac{\partial^2 M}{\partial D_u^2}} \right|, \tag{122}$$

then the resonances do not overlap (are isolated) and can be treated separately.

In the case where we can neglect the contribution of $\partial M/\partial D_u$ and $\partial^2 M/\partial D_u^2$ in the above equation, Eq. (122) reduces to the Chirikov (overlap) criterion. There are other variations of this criterion, e.g.,

$$\delta_r \gg \frac{3C}{\pi} \left| \frac{\partial^2 I}{\partial D_u^2} \frac{\sqrt{\left(\frac{\partial M}{\partial D_u}\right)^2 - M \frac{\partial^2 M}{\partial D_u^2} \pm M \frac{\partial^2 I}{\partial D_u^2}}}{\frac{\partial^2 I}{\partial D_u^2} \mp \frac{\partial^2 M}{\partial D_u^2}} \right|, \quad (123)$$

which leads to Green's criterion when the contribution of $\partial M/\partial D_u$ and $\partial^2 M/\partial D_u^2$ are negligible.

An alternate method of determining the behavior near resonances is through stop-band widths given elsewhere.³

5. CONCLUSION

In closing, we have described an algorithm to obtain information about nonlinear contributions to the dynamics of particles in an accelerator. This information includes the perturbation to the linear betatron tune and the growth in emittance, and (when on resonance) resonance strength (both in the generating function and Hamiltonian), stop-band width, fixed points, island width, and the resonance overlap criteria (e.g., Chirikov criterion, Greene's criterion, etc).

This algorithm was illustrated using two-dimensional systems (circular particle accelerators) and was implemented in the code NONLIN.^{4,5} The results obtained from this program were compared with HARMON.^{5,7}

Further extensions of this method have been successfully used to compute the variation of action, smear (i.e., spread of phase points about nominally invariant phase trajectory), and linear aperture.⁸ The results agree quite well with the results obtained from tracking programs (PATRICIA and ORBIT),⁹ thus providing an alternate method to tracking.

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APPENDIX A

In canonical transformation of the Hamiltonian, we often come across equations of the following form:

$$a_x(s) \frac{\partial w}{\partial \phi_x} + a_z(s) \frac{\partial w}{\partial \phi_z} + \frac{\partial w}{\partial s} + F(\phi_x, \phi_z, s) = 0, \quad (\text{A-1})$$

with $F(\phi_x, \phi_z, s)$ expandable in a Fourier series in ϕ_x and ϕ_z ; $F(\phi_x, \phi_z, s)$, $a_x(s)$, and $a_z(s)$ periodic in s with period L (e.g., $L = \text{circumference}$). [Note that we can solve Eq. (A-1) with more than two dimensions in ϕ].

We begin by expanding $F(\phi_x, \phi_z, s)$ in a Fourier series as:

$$F(\phi_x, \phi_z, s) = \sum_{n_z=-\infty}^{\infty} \sum_{n_x=-\infty}^{\infty} F_{n_x n_z}(s) e^{i(n_x \phi_x + n_z \phi_z)}, \quad (\text{A-2})$$

where

$$F_{n_x n_z}(s) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(\phi_x, \phi_z, s) e^{-i(n_x \phi_x + n_z \phi_z)} d\phi_x d\phi_z, \quad (\text{A-3})$$

and assume that $w(\phi_x, \phi_z, s)$ can be written in a Fourier series as

$$w(\phi_x, \phi_z, s) = \sum_{n_z=-\infty}^{\infty} \sum_{n_x=-\infty}^{\infty} w_{n_x n_z}(s) e^{i(n_x \phi_x + n_z \phi_z)}. \quad (\text{A-4})$$

The dependence on ϕ_x and ϕ_z in Eq. (A-1) can be removed by substituting Eqs. (A-2) and (A-4) into (A-1) and collecting terms, leading to

$$0 = \frac{d}{ds} w_{n_x n_z}(s) + i(n_x a_x + n_z a_z) w_{n_x n_z}(s) + F_{n_x n_z}(s). \quad (\text{A-5})$$

This can be solved by transforming $w_{n_x n_z}(s)$ into $Z_{n_x n_z}(s)$ as

$$Z_{n_x n_z}(s) = e^{+i[n_x \psi_x(s) + n_z \psi_z(s)]} w_{n_x n_z}(s), \quad (\text{A-6})$$

where

$$\psi_x(s) = \int_0^s a_x(t) dt, \quad (\text{A-7})$$

$$\psi_z(s) = \int_0^s a_z(t) dt. \quad (\text{A-8})$$

This leads to

$$\frac{d}{ds} Z_{n_x n_z}(s) = -e^{i(n_x \psi_x + n_z \psi_z)} F_{n_x n_z}(s). \quad (\text{A-9})$$

Before solving Eq. (A-9), we first consider boundary conditions which are periodic. Due to the periodicity in $a_x(s)$ and $a_z(s)$ we have

$$\psi_x(s + L) = \psi_x(s) + 2\pi v_x, \tag{A-10a}$$

$$\psi_z(s + L) = \psi_z(s) + 2\pi v_z, \tag{A-10b}$$

where

$$v_x = \frac{1}{2\pi} \int_0^L a_x(s) ds, \tag{A-11a}$$

and

$$v_z = \frac{1}{2\pi} \int_0^L a_z(s) ds. \tag{A-11b}$$

Since $F_{n_x n_z}(s)$ is periodic in s , then at $(s + L)$ Eq. (A-9) becomes

$$\begin{aligned} \frac{d}{ds} Z_{n_x n_z}(s + L) \\ = - e^{2\pi i(n_x v_x + n_z v_z)} e^{i(n_x \psi_x + n_z \psi_z)} F_{n_x n_z}(s) = e^{2\pi i(n_x v_x + n_z v_z)} \frac{d}{ds} Z_{n_x n_z}(s). \end{aligned} \tag{A-12}$$

Integrating Eq. (A-12) and manipulating leaves

$$Z_{n_x n_z}(s + L) - Z_{n_x n_z}(s) = 2i \sin \pi(n_x v_x + n_z v_z) e^{\pi i(n_x v_x + n_z v_z)} Z_{n_x n_z}(s). \tag{A-13}$$

From Eq. (A-9)

$$Z_{n_x n_z}(s + L) - Z_{n_x n_z}(s) = - \int_s^{s+L} e^{i[n_x \psi_x(t) + n_z \psi_z(t)]} F_{n_x n_z}(t) dt. \tag{A-14}$$

Thus,

$$Z_{n_x n_z}(s) = \frac{i}{2 \sin \pi(n_x v_x + n_z v_z)} \int_s^{s+L} e^{i\{n_x[\psi_x(t) - \pi v_x] + n_z[\psi_z(t) - \pi v_z]\}} F_{n_x n_z}(t) dt, \tag{A-15}$$

or $w_{n_x n_z}(s)$ becomes

$$\begin{aligned} w_{n_x n_z}(s) = \frac{i}{2 \sin \pi(n_x v_x + n_z v_z)} \\ \times \int_s^{s+L} e^{i\{n_x[\psi_x(t) - \psi_x(s) - \pi v_x] + n_z[\psi_z(t) - \psi_z(s) - \pi v_z]\}} F_{n_x n_z}(t) dt. \end{aligned} \tag{A-16}$$

Translating Eqs. (A-2) and (A-3) from the exponential Fourier series to sin-cos Fourier series (where superscript c, s implies coefficients of cosine and sine terms respectively) and defining

$$F_{n_x n_z}^c(s) = F_{n_x n_z}(s) + F_{-n_x -n_z}(s) \tag{A-17}$$

and

$$F_{n_x n_z}^s(s) = i[F_{n_x n_z}(s) - F_{-n_x -n_z}(s)], \tag{A-18}$$

with similar definitions for $w_{n_x n_z}^c(s)$ and $w_{n_x n_z}^s(s)$, leads to

$$w_{n_x n_z}^c(s) = \frac{1}{2 \sin \pi(n_x v_x + n_z v_z)} \int_s^{s+L} [F_{n_x n_z}^s(t) \cos \{n_x[\psi_x(t) - \psi_x(s) - \pi v_x] + n_z[\psi_z(t) - \psi_z(s) - \pi v_z]\} - F_{n_x n_z}^c(t) \sin \{n_x[\psi_x(t) - \psi_x(s) - \pi v_x] + n_z[\psi_z(t) - \psi_z(s) - \pi v_z]\}] dt \quad (\text{A-19})$$

and

$$w_{n_x n_z}^s(s) = -\frac{1}{2 \sin \pi(v_x n_x + v_z n_z)} \int_s^{s+L} [F_{n_x n_z}^c(t) \cos \{n_x[\psi_x(t) - \psi_x(s) - \pi v_x] + n_z[\psi_z(t) - \psi_z(s) - \pi v_z]\} + F_{n_x n_z}^s(t) \sin \{n_x[\psi_x(t) - \psi_x(s) - \pi v_x] + n_z[\psi_z(t) - \psi_z(s) - \pi v_z]\}] dt \quad (\text{A-20})$$

We note that $w(\phi_x, \phi_z, s)$ is also periodic in s with period L .

APPENDIX B

In Section 3, from the second-order perturbation of the Hamiltonian, we arrived at the following five equations [Eqs. (63) through (67)]:

$$0 = \frac{1}{\beta_x(s)} \frac{\partial w_1}{\partial \phi_x} + \frac{1}{\beta_z(s)} \frac{\partial w_1}{\partial \phi_z} + \frac{\partial w_1}{\partial s} + \frac{\sqrt{2}}{3} S(s) \beta_x^{3/2}(s) \cos^3 \phi_x, \quad (\text{B-1})$$

$$0 = \frac{1}{\beta_x(s)} \frac{\partial w_2}{\partial \phi_x} + \frac{1}{\beta_z(s)} \frac{\partial w_2}{\partial \phi_z} + \frac{\partial w_2}{\partial s} - \sqrt{2} \beta_x(s) S(s) \beta_z(s) \cos \phi_x \cos^2 \phi_z, \quad (\text{B-2})$$

$$a(s) = \frac{1}{\beta_x(s)} \frac{\partial v_1}{\partial \phi_x} + \frac{1}{\beta_z(s)} \frac{\partial v_1}{\partial \phi_z} + \frac{\partial v_1}{\partial s} + \frac{\sqrt{2}}{2} S(s) \beta_x^{3/2}(s) \frac{\partial w_1}{\partial \phi_x} \cos^3 \phi_x - \sqrt{2} \beta_x(s) S(s) \beta_z(s) \cos \phi_x \cos^2 \phi_z \frac{\partial w_1}{\partial \phi_z} + \frac{O(s)}{6} \beta_x^3(s) \cos^4 \phi_x, \quad (\text{B-3})$$

$$b(s) = \frac{1}{\beta_x(s)} \frac{\partial v_2}{\partial \phi_x} + \frac{1}{\beta_z(s)} \frac{\partial v_2}{\partial \phi_z} + \frac{\partial v_2}{\partial s} + \frac{\sqrt{2}}{2} S(s) \beta_x^{3/2}(s) \frac{\partial w_2}{\partial \phi_x} \cos^3 \phi_x - \sqrt{\frac{\beta_x}{2}} S(s) \beta_z(s) \left(\frac{\partial w_1}{\partial \phi_x} + 2 \frac{\partial w_2}{\partial \phi_z} \right) \cos \phi_x \cos^2 \phi_z - O(s) \beta_x(s) \beta_z(s) \cos^2 \phi_x \cos^2 \phi_z, \quad (\text{B-4})$$

$$c(s) = \frac{1}{\beta_x(s)} \frac{\partial v_3}{\partial \phi_x} + \frac{1}{\beta_z(s)} \frac{\partial v_3}{\partial \phi_z} + \frac{\partial v_3}{\partial s} - \sqrt{\frac{\beta_x(s)}{2}} S(s) \beta_z \frac{\partial w_2}{\partial \phi_x} \cos \phi_x \cos^2 \phi_z + \frac{O(s)}{6} \beta_z^2 \cos^4 \phi_z, \quad (\text{B-5})$$

where $a(s)$, $b(s)$, and $c(s)$ are given in Eqs. (B-15), (B-22), and (B-43), respectively.

Appendix A shows how to solve these five equations once we expand $F(\phi_x, \phi_z, s)$ of each equation in a Fourier series, to obtain their corresponding Fourier coefficients. In this appendix, we list these coefficients.

First we solve Eqs. (B-1) and (B-2). Using these solutions we solve the other three equations.

For Eq. (B-1) the Fourier coefficients are

$$F_{10}^c(s) = \frac{\sqrt{2}}{4} S(s) \beta_x^{3/2}(s), \quad (\text{B-6})$$

$$F_{30}^c(s) = \frac{\sqrt{2}}{12} S(s) \beta_x^{3/2}(s), \quad (\text{B-7})$$

and all others are zero. From Appendix A, we see that the solution will be of the form

$$w_1(\phi_x, \phi_z, s) = A_1(s) \cos \phi_x + A_3(s) \cos 3\phi_x + B_1(s) \sin \phi_x + B_3(s) \sin 3\phi_x, \quad (\text{B-8})$$

where

$$A_1(s) = -\frac{1}{2 \sin \pi v_x} \int_s^{s+L} F_{10}^c(t) \sin(\psi_x(t) - \psi_x(s) - \pi v_x) dt, \quad (\text{B-9})$$

with

$$\psi_x(s) = \int_0^s \frac{dt}{\beta_x(t)} \quad (\text{B-10})$$

and similar expressions for the other coefficients.

From Eq. (B-2), the Fourier coefficients become

$$F_{1-2}^c(s) = -\frac{\sqrt{2}}{4} S(s) \sqrt{\beta_x(s)} \beta_z(s), \quad (\text{B-11})$$

$$F_{10}^c(s) = -\frac{\sqrt{2}}{2} S(s) \sqrt{\beta_x(s)} \beta_z(s), \quad (\text{B-12})$$

$$F_{12}^c(s) = -\frac{\sqrt{2}}{4} S(s) \sqrt{\beta_x(s)} \beta_z(s), \quad (\text{B-13})$$

with a solution in the form

$$w_2(\phi_x, \phi_z, s) = C_{-2}(s) \cos(\phi_x - 2\phi_z) + C_0(s) \cos \phi_x + C_2 \cos(\phi_x + 2\phi_z) + D_{-2}(s) \sin(\phi_x - 2\phi_z) + D_0(s) \sin \phi_x + D_2(s) \sin(\phi_x + 2\phi_z). \quad (\text{B-14})$$

From these two solutions, we could find the Fourier coefficients for the latter three equations.

To compute the v_1 function given by Eq. (B-3), the Fourier coefficients are

$$F_{00}^c(s) = \frac{3}{16} \sqrt{2} \beta_x^{3/2}(s) S(s) [B_1(s) + B_3(s)] + \frac{O(s)}{16} \beta_x^2(s) = a(s), \quad (\text{B-15})$$

$$F_{20}^c(s) = \frac{\sqrt{2}}{16} \beta_x^{3/2}(s) S(s) [4B_1(s) + 9B_3(s)] + \frac{O(s)}{12} \beta_x^2(s), \quad (\text{B-16})$$

$$F_{20}^s(s) = -\frac{\sqrt{2}}{16} \beta_x^{3/2}(s) S(s) [2A_1(s) + 9A_3(s)], \quad (\text{B-17})$$

$$F_{40}^c(s) = +\frac{\sqrt{2}}{16} \beta_x^{3/2}(s) S(s) [B_1(s) + 9B_3(s)] + \frac{O(s)}{48} \beta_x^2(s), \quad (\text{B-18})$$

$$F_{40}^s(s) = -\frac{\sqrt{2}}{16} \beta_x^{3/2}(s) S(s) [A_1(s) + 9A_3(s)], \quad (\text{B-19})$$

$$F_{60}^c(s) = +\frac{3}{16} \sqrt{2} \beta_x^{3/2}(s) S(s) B_3(s), \quad (\text{B-20})$$

$$F_{60}^s = -\frac{3}{16} \sqrt{2} \beta_x^{3/2}(s) S(s) A_3(s), \quad (\text{B-21})$$

where $A_1(s)$, $A_3(s)$, $B_1(s)$, and $B_3(s)$ are the coefficients from the solution of w_1 [Eq. (B-8)].

To compute the v_2 function given by Eq. (B-4), the Fourier coefficients are

$$\begin{aligned} F_{00}^c(s) &= \frac{\sqrt{2\beta_x(s)}}{16} S(s) \{ \beta_z(s) [4D_{-2}(s) - 4D_2(s) - 2B_1(s)] + 3\beta_x(s) D_0(s) \} \\ &\quad - \frac{O(s)}{4} \beta_x(s) \beta_z(s) = b(s), \end{aligned} \quad (\text{B-22})$$

$$\begin{aligned} F_{20}^c(s) &= \frac{\sqrt{2\beta_x(s)}}{16} S(s) \{ \beta_z(s) [4D_{-2}(s) - 4D_2(s) - 6B_3(s) - 2B_1(s)] \\ &\quad + 4\beta_x(s) D_0(s) \} - \frac{O(s)}{4} \beta_x(s) \beta_z(s), \end{aligned} \quad (\text{B-23})$$

$$F_{20}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s) \{ \beta_z(s) [6A_3(s) + 2A_1(s) + 4C_2(s) - 4C_{-2}(s)] - 2\beta_x(s) C_0(s) \}, \quad (\text{B-24})$$

$$F_{40}^c(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s) [\beta_x(s) D_0(s) - 6\beta_z(s) B_3(s)], \quad (\text{B-25})$$

$$F_{40}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s) [6\beta_z(s) A_3(s) - \beta_x(s) C_0(s)], \quad (\text{B-26})$$

$$\begin{aligned} F_{02}^c(s) &= \frac{\sqrt{2\beta_x(s)}}{16} S(s) \{ [3\beta_x(s) - 8\beta_z(s)] D_2(s) + [3\beta_x(s) + 8\beta_z(s)] D_{-2}(s) \\ &\quad - 2\beta_z(s) B_1(s) \} - \frac{O(s)}{4} \beta_x(s) \beta_z(s), \end{aligned} \quad (\text{B-27})$$

$$F_{02}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s) \{ [8\beta_z(s) - 3\beta_x(s)] C_2(s) + [8\beta_z(s) + 3\beta_x(s)] C_{-2}(s) \}, \quad (\text{B-28})$$

$$F_{04}^c(s) = \frac{\sqrt{2\beta_x(s)}}{4} S(s) \beta_z(s) [D_{-2}(s) - D_2(s)], \quad (\text{B-29})$$

$$F_{04}^s(s) = \frac{\sqrt{2\beta_x(s)}}{4} S(s)\beta_z(s)[C_{-2}(s) + C_2(s)], \quad (\text{B-30})$$

$$F_{4-2}^c(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)[\beta_x(s)D_{-2}(s) - 3\beta_z(s)B_3(s)], \quad (\text{B-31})$$

$$F_{4-2}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)[3\beta_z(s)A_3(s) - \beta_x(s)C_{-2}(s)], \quad (\text{B-32})$$

$$F_{2-2}^c(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)\{\beta_x(s)D_2(s) + [8\beta_z(s) + 3\beta_x(s)]D_{-2}(s) - \beta_z(s)[3B_3(s) + B_1(s)]\} - \frac{O(s)}{8} \beta_x(s)\beta_z(s), \quad (\text{B-33})$$

$$F_{2-2}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)\{\beta_x(s)C_2(s) - [8\beta_z(s) + 3\beta_x(s)]C_{-2}(s) + \beta_z(s)[3A_3(s) + A_1(s)]\}, \quad (\text{B-34})$$

$$F_{22}^c(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)\{[3\beta_x(s) - 8\beta_z(s)]D_2(s) + \beta_x(s)D_{-2}(s) - \beta_z(s)[3B_3(s) + B_1(s)]\} - \frac{O(s)}{8} \beta_x(s)\beta_z(s), \quad (\text{B-35})$$

$$F_{22}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)\{[8\beta_z(s) - 3\beta_x(s)]C_2(s) + \beta_x(s)C_{-2}(s) + \beta_z(s)[3A_3(s) + A_1(s)]\}, \quad (\text{B-36})$$

$$F_{42}^c(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)[\beta_x(s)D_2(s) - 3\beta_z(s)B_3(s)], \quad (\text{B-37})$$

$$F_{42}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)[3\beta_z(s)A_3(s) - \beta_x(s)C_2(s)], \quad (\text{B-38})$$

$$F_{2-4}^c(s) = \frac{\sqrt{2\beta_x(s)}}{4} S(s)\beta_z(s)D_{-2}(s), \quad (\text{B-39})$$

$$F_{2-4}^s(s) = -\frac{\sqrt{2\beta_x(s)}}{4} S(s)\beta_z(s)C_{-2}(s), \quad (\text{B-40})$$

$$F_{24}^c(s) = -\frac{\sqrt{2\beta_x(s)}}{4} S(s)\beta_z(s)D_2(s), \quad (\text{B-41})$$

$$F_{24}^s(s) = \frac{\sqrt{2\beta_x(s)}}{4} S(s)\beta_z(s)C_2(s), \quad (\text{B-42})$$

where the coefficient of the w_1 and w_2 functions were used.

Finally, we give the coefficients needed to compute the v_3 function given by

Eq. (B-5) as

$$F_{00}^c(s) = -\frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)[D_2(s) + 2D_0(s) + D_{-2}(s)] + \frac{O(s)}{16} \beta_z^2(s) = c(s), \quad (\text{B-43})$$

$$F_{20}^c(s) = -\frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)[D_2(s) + 2D_0(s) + D_{-2}(s)], \quad (\text{B-44})$$

$$F_{20}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)[C_2(s) + 2C_0(s) + C_{-2}(s)], \quad (\text{B-45})$$

$$F_{02}^c(s) = -\frac{\sqrt{2\beta_x(s)}}{8} S(s)\beta_z(s)[D_2(s) + D_0(s) + D_{-2}(s)] + \frac{O(s)}{12} \beta_z^2(s), \quad (\text{B-46})$$

$$F_{02}^s(s) = \frac{\sqrt{2\beta_x(s)}}{8} S(s)\beta_z(s)[C_2(s) - C_{-2}(s)], \quad (\text{B-47})$$

$$F_{04}^c(s) = -\frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)[D_2(s) + D_{-2}(s)] + \frac{O(s)}{48} \beta_z^2(s), \quad (\text{B-48})$$

$$F_{04}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)[C_2(s) - C_{-2}(s)], \quad (\text{B-49})$$

$$F_{2-2}^c(s) = -\frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)[D_0(s) + 2D_{-2}(s)], \quad (\text{B-50})$$

$$F_{2-2}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)[C_0(s) + 2C_{-2}(s)], \quad (\text{B-51})$$

$$F_{22}^c(s) = -\frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)[2D_2(s) + D_0(s)], \quad (\text{B-52})$$

$$F_{22}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)[2C_2(s) + C_0(s)], \quad (\text{B-53})$$

$$F_{2-4}^c(s) = -\frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)D_{-2}(s), \quad (\text{B-54})$$

$$F_{2-4}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)C_{-2}(s), \quad (\text{B-55})$$

$$F_{24}^c(s) = -\frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)D_2(s), \quad (\text{B-56})$$

$$F_{24}^s(s) = \frac{\sqrt{2\beta_x(s)}}{16} S(s)\beta_z(s)C_2(s), \quad (\text{B-57})$$

where, we used the coefficient from w_2 . Some of these coefficients were obtained using the program MACSYMA.¹⁰ Note the F_{00}^c term that appears in each v_i , $i = 1, 2, 3$ does not represent resonances but contributes to the perturbation of the tune.