Stability of D-brane Geometries And a Quantum Check of AdS/CFT

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Abstract

The first part of this thesis explores the stability of non-supersymmetric constructions using D-branes and M-branes. Guided mainly by intuition developed using the correspondence between gauge theory and gravity, known as AdS/CFT, we propose a precise relationship between thermodynamic and dynamical stability of non-extremal branes. We verify the conjecture explicitly for non-extremal M2-branes with angular momentum in planes perpendicular to the world-volume, in the limit of many M2-branes where the supergravity approximation is reliable. Next, we explore the stability of near-horizon geometries of extremal branes which are product geometries of anti-de Sitter space and positively curved Einstein spaces. Our main motivation is to answer the question: Do non-supersymmetric stable vacua exist? We find that the answer is *yes*. But for Type IIA strings in the presence of D8-branes and for a non-supersymmetric open string theory with gauge group USp(32) we find that spherical compactifications are unstable.

The second part of this thesis explores AdS/CFT predictions beyond the classical level. Such checks are usually hard to carry out, at least in the absence of supersymmetry. We find an interesting test which yields a manifestly finite answer without using supersymmetry. It involves calculating the one-loop vacuum energy of a tachyon field in anti-de Sitter space with boundary conditions corresponding to the presence of a double-trace operator in the dual field theory. Such an operator can lead to a renormalization group flow between two different conformal field theories related to each other by a Legendre transformation in the large N limit. The calculation of the one-loop vacuum energy enables us to verify the holographic c-theorem one step beyond the classical supergravity approximation.

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Chapter 1

Introduction

1.1 Why strings?

String theory proposes that the ultimate constituents of matter is not made up of point-like particles, but very tiny strings. In this theory, all interactions are supposed to occur by splitting and joining of strings. These strings are of two kinds – open or closed and are roughly 10^{-32} cm long. They live in ten spacetime dimensions; to make contact with our more familiar four dimensional world, it is assumed that six of the dimensions are compact and exceedingly tiny. The ordinary constituents of matter such as electrons, photons and so on, are just different modes of oscillation of a string. Why was such a seemingly radical theory needed?

In the 1960s, high energy experiments revealed the existence of a large number of hadrons with large angular momenta. Traditionally particles are viewed as excitations of a quantum field and are point-like. In four dimensions, if one demands that these quantum field theories be consistent at all energies, one is forced to accept the fact that there cannot be any particles with spin larger than one. In the jargon of particle physics, theories with large spin are either "free" (there are no interactions) or "non-renormalizable" (which means that such theories do not have any predictive power since the number of inputs needed are infinite). Moreover, it was also observed, that the square of the mass of these particles was proportional to the angular momentum – a property characteristic of rotating strings rather than particles. So people asked, "Could it be, that these particles are actually tiny strings?" And so string theory was born.

An attractive feature of string theory was that it had very nice ultra-violet properties and in fact could explain reasonably well some features of experimental data for very high energy scattering at small angles. Standard quantum field theories of particles normally predict a very hard behavior at short distances. A hand-waving way of explaining this is to say that because point particles have strictly zero size, there is a very definite observer independent event associated with two such particles colliding; for strings, which are objects having non-zero spatial extent, such an observer independent event simply does not exist. And so, strings can have arbitrarily high spins in their spectrum and still collide with finite cross-sections.

But this success was short-lived. It turned out that this new theory predicted a much softer behavior for fixed angle scattering at very high energies than was experimentally observed – if strings were getting scattered, the fall-off of scattering cross-section is predicted to be exponential in energy, in reality it was observed to have a power-law fall-off.

At this point, a new candidate theory known as Quantum Chromodynamics (QCD) was proposed which explained all that strings couldn't. The classical Langragian of QCD looks like a slightly more complicated version of electromagnetism – the gauge group here is the non-abelian group $SU(3)$ and the charges are carried by quarks which interact by exchanging gluons instead of photons. But, as we shall discuss shortly, its dynamics is very different from electromagnetism – at low energies QCD is a strongly coupled theory whereas the opposite is true for electromagnetism. This theory of quarks has been successful in explaining all known experimental observations to date.

So one might wonder, why strings? Well, the biggest stumbling block for any quantum field theory (of point particles) is in explaining nature at arbitrarily small length scales (or equivalently, arbitrarily high energies) is gravity. It is a well-known fact that a quantum field theory of gravity is highly non-renormalizable in four spacetime dimensions. At low energies this is not a problem because gravity is weak and so can be neglected, but at energies close to the Planck energy of 10^{19} GeV, gravitational interactions become comparable to the strong interactions and can no longer be ignored. Clearly then, a quantum field theory, such as QCD for instance, cannot explain physics all the way upto Planck energies. String theory is the only known quantum theory of gravity in four and higher spacetime dimensions. As we stated at the very beginning, the spectrum of a string corresponds to its different oscillation modes. It is a fact that all string theories have a massless, spin two particle in its spectrum. This can only be the graviton (the mediator of gravitational interactions). Although this belief that strings might be able to explain what gravity looks like at extremely high energies was a motivation which kept string theory alive, we'll shortly see that we've come full circle and in modern times string theory is believed to capture all the physics of certain gauge theories.

There is another problem. Even if we expect QCD to explain *only* strong interactions (and forget about gravity), we run into practical difficulties as we try to use the theory to explain phenomenon (like quark confinement) seen at the low energies of our present day accelerators. The reason for this is the fact that QCD is an "asymptotically free" theory which means that as one cranks up the energy, the coupling constant of QCD decreasing and asymptotes to zero at infinite energies. Conversely, at low energies this theory is strongly coupled and this makes traditional perturbative computations impossible. The only way out is by putting the theory on a lattice and then taking the continuum limit – a subject known as Lattice QCD. The drawback of such a route of investigation is that the physics is masked behind humongous numerical computations and even if we can explain all the observed properties of the strong force at low energies, we cannot claim to have *understood* the physics. Very recently, it was discovered that string theory just might be able to make analytic predictions of strongly coupled QCD-like theories amenable. Actually, it can achieve this for gauge theories with an infinite number of colors ($SU(N)$ gauge theories with $N \to \infty$). The hope then is to make pertubative corrections of order $1/N$ to this theory thereby making predictions for theories with $N = 3$. The idea that string theory, in some limit, might be dual to large N gauge theories was originally proposed by 't Hooft [1]. He theorized that the flux tubes of the gauge theory which stretch between quarks might actually be the dynamical strings of string theory. As we shall describe in the next section, the gravity/gauge theory duality or the AdS/CFT correspondence is the closest realization of that hope. In the next section we try to motivate this conjecture and explain what it means.

1.2 The AdS/CFT correspondence

't Hooft noticed [1] that the perturbative expansion of the Feynman diagrams of an $SU(N)$ gauge theory could be organized in terms of the dimensionless number $1/N$, so that in the large N limit computations simplify considerably. This is a slightly subtle business, because we need to know how to scale the coupling constant g_{YM} as we send $N \to \infty$. To figure that out, we make the assumption that the cutoff scale of the gauge theory is kept constant as we take the limit of large N. The beta-function equation for $SU(N)$ gauge theory is

$$\mu \frac{dg_{YM}}{d\mu} = -\frac{11}{3} N \frac{g_{YM}^3}{16\pi^2} + \mathcal{O}(g_{YM}^5)\,. \tag{1.1}$$

Clearly, to keep the leading terms of the same order when we send $N \to \infty$, we have to send $g_{YM} \to 0$ such that the combination $\lambda = g_{YM}^2 N$ is kept constant. The parameter λ is called the 't Hooft parameter.

The nice feature of a perturbative expansion in $1/N$ is that if the theory is written in double-line notation (this is just a trick where an adjoint field is replaced by a fundamental and an anti-fundamental field), the Feynman diagrams organize topologically with the dominant contribution being from the planar diagram, the next order contribution being from a diagram of genus one and so on. Each diagram can be shown to have a coefficient proportional to N^{χ} where χ is the Euler character of the surface and for closed oriented surfaces is equal to $2-2g$ where g is the number of handles of the surface. (For a particularly lucid derivation, see [2].) Clearly the planar diagram will contribute at an order N^2 more than a diagram with one hole, and in the large N limit we see that the planar diagram will dominate.

The link with string theory is that the perturbative expansion described above is exactly the same as one gets for closed oriented string theory with the string coupling constant taking the place of $1/N$ and the expansion is organized in terms of the topology of the string world-sheet. At this point one might object that the gauge theory expansion has surfaces which look more like fish-nets, while the string world-sheets are smooth and continuous. t'Hooft conjectured that in the large N limit, non-perturbative effects fill in the holes of the fish-net diagrams *exactly*, so the resemblance with string world-sheets became perfect. The requirement for demanding large N stems from the fact that the above string picture would be more accurate for weakly coupled (i.e. large N) string theory. So we have a heuristic argument of why one could expect any gauge theory to be dual to some weakly coupled string theory. [1] Exactly which string theory is a much harder question to answer and requires making educated guesses as we shall see below.

Depending on how bold one is, there are different forms of this gauge/string theory duality. In its strongest form it states that both these theories are equivalent for *all* values of g_{YM} and N. A weaker version of this conjecture says that the duality of the two theories holds only at fixed λ and infinite N. Since this is the same as sending the string coupling to zero, it means that we are dealing with classical string theory (no string loops). Finally the weakest version is to say that the theories become equivalent only in the limit $\lambda \rightarrow \infty$, which we'll see shortly corresponds on the string theory side to taking the low energy supergravity limit of string theory. It is this final limit that has been most tested, since it is the one where one faces the most manageable computations.

Let us next turn to the problem: Given a certain gauge theory, how does one go about guessing which string theory would be dual to it? For concreteness, we shall take the most famous example considered by Maldacena in [3]. Let us suppose we are asked to find the string theory dual to pure super Yang-Mills $SU(N)$ theory in four dimensions. The most suggestive route (and one which was taken historically) is a path that leads through

[1] Furthermore, since the string spectrum always contains the graviton, the string theory side of the correspondence must be a gravitational theory.

D-branes. D-branes are described as surfaces (for our purposes these surfaces are flat) on which open strings can end [4]. They have masses proportional to $1/g_s$, so at weak string coupling they become very heavy objects. In addition, they carry Ramond-Ramond charge. Just as a particle couples to a one form potential with a two form field strength, a p-dimensional brane called a Dp-brane, will couple to a $(p+2)$ form field strength. The low-energy limit of the theory describing a single D-brane is ordinary electromagnetism, i.e. a $U(1)$ gauge theory. If we have N separated D-branes however, the open strings can end on any one of them and we have a $(U(1))^N$ gauge theory. If we place all of these D-branes on top of each other, the gauge group gets enhanced and the theory is described by $SU(N)$ gauge theory. The supersymmetrized version of this is commonly called super Yang-Mills (SYM) theory. There is another picture of D-branes. Since they are heavy objects which carry charge, they curve space-time around them. For small curvatures (compared to the length of a typical string) we can ignore stringy effects and the theory is well approximated by classical supergravity (SUGRA). The equations one gets are just the classical Einstein-Maxwell equations and so the geometries look like charged, extended black holes. If these two pictures of D-branes are really equivalent, one is tempted to conjecture that $SU(N)$ SYM theory is dual to string theory in the background of these D-branes. The gauge theory lives is one less spacetime dimension than the dual string theory. This can be understood as follows: For a stack of N Dp-branes living in D dimensional spacetime (D is actually equal to 10, but we shall keep it arbitrary here), the gauge theory lives on the $p+1$ dimensional worldvolume of the branes. On the supergravity side, we have already mentioned that Dp-branes source a $p+2$ form field strength F_{p+2}. The flux of this Ramond-Ramond field spreads in the $D-p-1$ dimensional space around the Dp-branes. This space which is conformal to flat space, can be thought of as a cone over an S^{D-p-2} so that the flux lines pierce this sphere with Gauss' law telling us how many branes there are:

$$\int_{S^{D-p-2}} *_D F_{p+2} = N\,. \tag{1.2}$$

The bottomline is that our D dimensional space has been split into a $p+1$ dimensional part (on which the D-branes live), a radial coordinate (which measures how far we are from the D-branes) and a $D-p-2$ part which is a compact space (a sphere). One can dimensionally reduce this D dimensional theory in the Kaluza-Klein sense on the compact space to end up with an effective theory in $p+2$ dimensions. So in the end, as advertised, we have a $p+1$ dimensional gauge theory dual to a $p+2$ dimensional gravity theory. The fact that all of the dynamics of a $p+2$ dimensional quantum theory including gravity is encoded in a $p+1$ dimensional theory is reminiscent of a two dimensional hologram encoding information about three dimensions. It is for this reason that the equivalence of these two theories is often termed as "Holography", and is a concrete realization of a fact which had been proposed earlier by 't Hooft and Susskind for all quantum gravity theories. Having determined the dimension in which the dual quantum gravity theory lives, the next clue one uses in pinning down exactly which string theory is dual to a given gauge theory is global symmetries. Surely if two theories are dual to one another, both of them should have the same global symmetries.

For concreteness, we consider the most famous example first considered by Maldacena in [3] and made precise by Gubser, Klebanov, Polyakov in [5] and Witten in [6]. Suppose we are asked to find the string theory dual to super Yang-Mills $SU(N)$ theory in four dimensions. Here, the maximum number of supercharges that can be present is $2^{\frac{4}{2}} = 4$. The maximally supersymmetric theory is therefore known as $\mathcal{N}=4$ SYM theory. A quick guess for its string dual would be strings propagating in $AdS_5 \times S^5$, since $\mathcal{N}=4$ SYM is the low-energy effective field theory description of coincident D3-branes, and $AdS_5 \times S^5$ is the near-horizon limit of the supergravity background of the stack of branes. But it is instructive to do a more systematic analysis which we sketch now.

The fields present in the $\mathcal{N}=4$ theory are the gauge fields, 6 scalars and 4 fermions. There is a global $SU(4)$ R-symmetry which rotates the scalars and the fermions among themselves. Finally, this is an example of a theory whose coupling constant does not change with energy (even quantum mechanically). Such theories are known as conformally invariant theories. The conformal symmetry group in four dimensions is $SO(4,2)$. As we have just discussed above, the dual gravity theory would be a string theory effectively living as gravity in 5 dimensions. We have also argued that any purported dual must have the same global symmetries as the field theory which in this case is $SO(4,2) \times SU(4)$. For us this means that the background geometry in which strings propagate must have $SO(4,2)$ and $SU(4)$ as isometries. Anti-de Sitter space in d-dimensions (commonly known as AdS_d possesses $SO(d-1,2)$ symmetry. This is a negatively curved space and can be embedded in $d+1$ dimensional flat space as the hyperboloid

$$X_d^2 + X_{d+1}^2 - \sum_{i=1}^{d-1} X_i^2 = R^2 . \tag{1.3}$$

Since $SU(4) \sim SO(6)$, one is led to believe that the dual string theory lives in $AdS_5 \times S^5$. This led Maldacena to conjecture [3] that maximally supersymmetric $SU(N)$ SYM theory in four dimensions is dual to string theory (more precisely Type IIB string theory) on $AdS_5 \times S^5$. The strongest form of the conjecture proposes equivalence of the two theories for arbitrary values of the number of colors of the gauge theory and 't Hooft coupling $g_{YM}^2 N$. Testing this duality in this strong form is an extremely tall order: for large g_{YM} the gauge theory is strongly coupled and for small N the contribution of non-planar diagrams cannot be neglected. On the dual string theory side computations are equally hard to perform – the spectrum of strings propagating on $AdS_5 \times S^5$ is still an unsolved problem mainly because of the presence of Ramond-Ramond fields (although more recently, this problem has been solved in the plane-wave limit of this geometry). Thus, with the present machinery, we are forced to settle for less, and perform checks in certain limits.

To understand what such limits mean on both sides of the duality, the conjecture needs to be made more precise. Immediately after the appearance of [3], the authors in [5, 6] developed a dictionary to translate questions/results in one theory to meaningful questions/predictions in its dual theory. The statement that two theories are "equivalent" means that there exists a precise map between the states and fields on the string theory side and local, gauge invariant operators (which are the building blocks of any gauge theory) on the field theory side. Furthermore, there is also a well-defined correspondence between correlation functions of the two theories.

In order to present the dictionary, we need to set notation and discuss a little bit about the classical supergravity solution. The metric and the other non-zero field of Type IIB supergravity – the self-dual five form field strength (the dilaton is constant in this particular example) satisfy the classical Maxwell-Einstein equations. This approximation is valid in regions where the curvature of spacetime is much smaller than the string scale $\sqrt{\alpha'}$. For a stack of N D3-branes, these equations tell us that the metric is

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2)\,, \qquad (1.4)$$

and the flux of the 5-form field strength is proportional to the volume form on the S^5.

As one gets closer and closer to the horizon $r = 0$, energies get more and more red-shifted and since we have been talking about low energy limits, we should zoom on the metric near $r = 0$. It is comforting that this metric is precisely $AdS_5 \times S^5$ (just as we had expected from symmetry considerations) and changing the radial coordinates to $z = \frac{L^2}{r}$ we recognize the familiar form in Poincaré coordinates:

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dx^2 + dz^2\right) + L^2 d\Omega_5^2\,. \qquad (1.5)$$

Note that the five sphere and the AdS space have the same radius of curvature L, which is related to the number of D3-branes N, by the relation:

$$L^4 = 4\pi g_s N(\alpha')^2\,. \qquad (1.6)$$

This can be derived by demanding that the ADM tension of N extremal D3-branes is N times the tension of a single brane. Finally, since $g_s = g_{YM}^2$, we have the most celebrated piece of the dictionary relating the radius of curvature of AdS_5 to the 't Hooft coupling of the gauge theory:

$$\frac{L^4}{\alpha'^2} = 2g_{YM}^2 N\,. \qquad (1.7)$$

Let us find out the regions of validity of the two theories. The gravitational description is reliable when $L >> \sqrt{\alpha'}$, so using (1.7) we find that this description is valid for large 't Hooft coupling. The open string description, which is $SU(N)$ gauge theory, can be perturbatively handled when $g_{YM}^2 N << 1$, which is exactly the opposite limit (for a single

D-brane the gauge coupling is g_{YM}^2, but for N of them it is $g_{YM}^2 N$). This fact that the dual descriptions cover non-overlapping regions of parameter space is the cause of most of the excitement. It allows us to answer questions about strongly coupled gauge theories (recall that for asymptotically free theories like QCD, the low energy regime of the theory is usually inaccessible to perturbative calculations because it is strongly coupled) by mapping the problem to a weakly coupled gravity problem which can be solved by the usual methods.

While the fact that the regions of validity of the dual theories are non-overlapping is very powerful in making predictions, it makes tests of this conjecture quite hard. For example, let us suppose we know the dimension of a gauge theory operator at small 't Hooft coupling. Next, we do a gravity calculation and compute the mass of the mode which we've identified to be dual to this operator, and use the dictionary to translate this mass into the dimension of the operator. In general, this dimension will not agree with the value at weak coupling because the dimension of operators are generically expected to change under a renormalization group flow. In exceptional cases certain quantities might be protected by non-renormalization theorems and it is only in such instances that exact checks can be made. An example are field theory operators whose dimensions are protected either because of some conservation law (like current conservation) or because supersymmetry protects them from being renormalized. The dimensions of these operators can be converted using the dictionary to masses of modes arising from a Kaluza-Klein compactification of supergravity on compact spaces (for our case at hand this is the five sphere) and the two should agree. Ever since the conjecture was proposed, there has amassed an impressive body of evidence in its support (at least the weakest version). For a comprehensive review see [2] or for a shorter, more recent one [7].

The predictions made by classical supergravity calculations about the strong coupling behavior of gauge theories is valid only for both $N \to \infty$ and infinite 't Hooft coupling. To probe the region where N is still kept infinite, but the 't Hooft coupling is slowly dialed to finite values would mean from (1.7) that $\frac{\alpha'}{L^2}$ can no longer be ignored which means that we now have to consider (classical) strings moving in $AdS_5 \times S^5$. Finally, if we want to decrease N to finite values, we have to consider string loops which we've seen earlier to generate corrections of order $\frac{1}{N^2}$. This is why the strongest form of the conjecture, which is supposed to hold for *all* values of the 't Hooft coupling and N, is so difficult to verify.

The remainder of this thesis consists of the next three chapters each of which deals with aspects of AdS/CFT. Chapter 2 is based on the papers [8, 9] and deals with the connection between dynamical and thermodynamical stability of black holes and was inspired by the AdS/CFT duality. Chapter 3 is based on the papers [10, 11] and deals with the stability of different supergravity compactifications on positively curved Einstein manifolds with the non-compact space being anti-de Sitter. Chapter 4 is a non-trivial check of the AdS/CFT duality at an order beyond the classical level and is based on the paper [12].

Let us now briefly summarize the motivations and results of each of these chapters.

Thermodynamic stability vs. dynamical stability

The gravity/gauge theory correspondence is supposed to hold not only between the vacuum state of the gauge theory (the $\mathcal{N} = 4$ theory for example) and the undisturbed gravity background ($AdS_5 \times S^5$ in this case), but for arbitrary disturbances which leave the asymptotics of the AdS background intact. Indeed, since we are dealing with a quantum theory of gravity, in the path-integral we are integrating over all metrics which asymptote to AdS. In particular, this would include black hole excitations and from the point of the AdS/CFT correspondence indicates that the duality holds even when the two theories are heated up. For example, the Hawking-Page transition in gravity has been shown to correspond to the confinement-deconfinement transition in the field theory. In Chapter 2 we shall be guided by this duality to propose a precise relation between thermodynamic and dynamical stability of black branes. It is a familiar fact from general relativity in flat space that for black holes, these two types of stability do not agree. A well known example is the four dimensional Schwarzchild black hole in asymptotically flat space. It is stable against small metric perturbations and so in our terminology, is dynamically stable. On the other hand, it has negative specific heat for all values of mass and so is thermodynamically unstable. That the two types of stability do not agree may not seem very surprising – dynamical instability, when it exists, is a purely classical effect, while the thermodynamics of black holes is a quantum phenomenon (sending $\hbar \to 0$ causes the black hole to have zero temperature for instance).

The fact that the gravity/gauge duality should hold at non-zero temperatures would, however, seem to indicate that there should be a relation between the two types of instability we discussed above. The argument goes as follows: Consider a stack of black branes and suppose that the specific heat of the system becomes negative for some range of values of a conserved quantity, say charge, of the branes. This thermodynamic instability in the gravity theory should show up in the dual gauge theory possibly in the form of a phase transition. That should imply the existence of an exponentially growing mode which nucleates the new phase. Using the duality again, this should be mirrored in gravity by the existence of an exponentially growing mode in real time – which is the signature of a dynamical instability. Based on this line of reasoning, we conjecture that a black brane would be dynamically unstable precisely when it is thermodynamically unstable. We shall treat a particular example in detail; we find that the AdS/CFT intuition is indeed borne out (within limits of numerical accuracy). Later, a semi-classical proof was forwarded by Reall [13] and his method enabled an extension of the above conjecture to black holes with finite horizon size [14]. We shall outline both of these developments in the chapter. Why then are the thermodynamically unstable point black holes that we encounter in asymptotically flat space dynamically stable? We would like to argue that these have compact horizons (unlike black branes whose horizon areas are infinite) and so thermodynamic arguments are invalid.

Stability of AdS compactifications

We move on in the next chapter to discuss the stability of near-horizon geometries of extremal black p-branes. These are geometries of the form $AdS_{p+2} \times M^q$ with M^q a positively curved q dimensional Einstein manifold (Einstein spaces have the property that the Ricci tensor is proportional to the metric tensor – anti-de Sitter space and the sphere are common examples). We do our analysis for arbitrary values of $q = D - (p+2)$: the full theory lives in D spacetime dimensions so for string theory $D = 10$, for M-theory $D = 11$, for the more esoteric bosonic M-theory [15] $D = 27$. In the first part of chapter 3 we shall consider gravity coupled to only a form field. In the second part, we shall add a scalar potential to our Langrangian so that we have a cosmological term at the classical level. We shall view these theories as D dimensional theories compactified down to $p+2$ dimensions on the compact space M^q in the sense of Kaluza and Klein. The resulting $p+2$ dimensional effective theory lives in anti-de Sitter space. We compute the mass spectrum of the various scalar modes of the Kaluza-Klein tower of states. As a bonus, using the AdS/CFT duality we compute the dimensions of gauge theory operators which are dual to these modes in the gravity theory.

From the gravity/gauge theory duality perspective, the exercise of determining which compactifications are stable is important. Unstable ones will have pathologies associated with their field theory duals (they might not have any duals). We have mentioned how a theory of gravity in AdS is dual to a gauge theory living on the "boundary" of AdS. Which particular gauge theory depends on the compact space M^q. In the example that we considered in detail above, this space was S^5. That allowed 4 supercharges to be preserved. Field theories with less supersymmetry are interesting from a phenomenological perspective. This can be achieved by choosing less symmetric compact spaces. For instance, replacing S^5 by the coset space $(SU(2) \times SU(2))/U(1)$ (which is commonly known as T^{11}) leads to a dual gauge theory that has only $\mathcal{N} = 1$ susy.

The common examples of maximally supersymmetric solutions like $AdS_5 \times S^5$ or $AdS_7 \times S^4$ are known to be stable. These solutions have coupled scalar modes which are very close to the stability bound (we'll have an opportunity to discuss the precise criterion for stability in Chapter 3) and so one might wonder if choosing nonsupersymmetric vacua always makes them unstable. However, we find something quite different. We find that all but one of these modes are stable for *any* space M_q and this result is in fact independent of which Einstein space one chooses. The only potentially dangerous mode is one which locally conserves volume but changes the shape of the space. For product spaces, this is a balloon mode in which the volume of one part increases while the volume of the other decreases in a manner such that the total volume is locally conserved. For which spaces M^q this mode is unstable depends on the particular space. For a sphere of any dimension, this mode is stable. For $M_q = S^n \times S^{q-n}$ this mode is always unstable for $q < 9$, while if $M_q = T^{pq}$ then this mode is stable only for the supersymmetric case T^{11} and its smooth susy breaking $\mathbf{Z}_k$ quotient T^{kk}. We briefly comment on what this means for dual field theories in terms of the existence of infra-red fixed points close to the $\mathcal{N} = 1$ theory.

The results for the more general theories with a scalar potential added on is less rosy. We study in detail two such examples relevant to string theory: the massive IIA theory and Sugimoto's $USp(32)$ open string theory. In both of these cases, we find that compactifications on a sphere of the appropriate dimension are unstable. It remains to find stable non supersymmetric vacua of these theories.

Testing AdS/CFT beyond the classical level

In Chapter 4 we change gears and with the help of an example, set out to perform a check of the AdS/CFT duality beyond the leading order. Such an exercise is valuable because most checks of the correspondence has been at the level of classical supergravity. On the gauge theory side, computations beyond leading order, i.e. at order $1/N^2$, means that we are doing a loop computation in AdS. In general, such a loop computation is beset with problems. Since the gravity theory is highly non-renormalizable, loop computations would in general be very divergent. Even with the assumption that the underlying theory (here string theory) is a consistent theory of quantum gravity, one usually has to know how the full theory cancels divergences to propose a renormalization scheme to extract finite answers out of apparent infinities. There is also the difficulty of performing loop computations in string theory in the presence of Ramond-Ramond fields. However, we shall present an example where an answer that is manifestly finite can be extracted out of a loop computation in which a single scalar field closes in on itself. [2] With the assumption of an underlying consistent theory of quantum gravity, we can be assured that our answer is meaningful.

We start with the $\mathcal{N} = 4$ theory and deform it by adding a relevant double-trace operator. This leads to a renormalization group flow and we shall argue that the theory flows to a non-trivial infra-red fixed point which, in the large N limit, is related to the theory in the ultra-violet by a Legendre transformation. In four dimensions, there is a conjectured c-theorem due to Cardy [16] which states that the central charge of a theory decreases as one looks at lower and lower energy scales. This result has been proven for two dimensional theories by Zamolodchikov [17], but a direct proof in higher dimensions has resisted proof so far. Intuitively, such a result seems obvious because the central charge is a measure of the number of degrees of freedom of a field theory and therefore it should decrease in the infra-red as one integrates out massive degrees of freedom. The gravity/gauge duality allows one to map this statement into a well-defined problem in AdS. The central charge is closely related to the Weyl anomaly and the resulting c-theorem is often called the Holographic c-theorem. This theorem can be proved at tree-level on the gravity side if one uses the null-energy condition [18]. But at the quantum level it is not clear that one can talk meaningfully about the null energy condition, and indeed we'll use a method which does not rely on this condition at all. To our satisfaction, we shall find that at the level of the one-loop computation that we do – the c-theorem is obeyed in all dimensions. As a bonus, we shall make a prediction of the central charge of the CFT in the infra-red. Recently this prediction has been verified by an explicit field theory computation [19].

In the final chapter we conclude with some brief comments and outlook.

[2] This calculation is therefore done in quantum field theory and not string theory.

Chapter 2

Two Types of Black Hole Stability

2.1 Introduction

In the study of black hole physics, one is taught at the very outset that there are two distinct types of stability – thermodynamic and dynamical. Black holes which are not stable against small perturbations of the metric are said to be dynamically unstable. Such an instability causes the horizon to start to clump in real time. The most familiar black holes known from astrophysical contexts such as the four dimensional Schwarzschild and Kerr black holes in asymptotic flat space are known to be stable against such perturbations [20, 21]. However, if one adds an extra compact dimension, say a circle, and smears the black hole uniformly on this circle (so that its horizon has the topology $S^2 \times S^1$), then dynamical instability in the sense described above does set in, provided the radius of the circle is made sufficiently large compared to the horizon radius. This was found by Gregory and Laflamme [22] and we shall use the terms "dynamical instability" and "Gregory-Laflamme instability" interchangeably. Thermodynamic instability, on the other hand, would mean that a black hole has negative specific heat. Typically most black holes are unstable in this sense (at least for some range of values of conserved quantities like charge or mass). The ordinary Schwarzschild black hole is thermodynamically unstable for all values of mass. This feature that dynamical and thermodynamical stabilities do not agree is in fact typical for point black holes in asymptotic flat space.

One might think that indeed it is not very surprising that there is no relation between the two, since thermodynamical properties of black holes are quantum mechanical in nature (in the limit $\hbar \to 0$ black holes have infinite entropy, zero temperatures and so on) and it is not clear that any thermodynamic feature should be reflected in the classical Lorentzian time evolution.

The aim of the present chapter is to provide evidence that, contrary to expectations, there is in fact a precise relationship between the two. It is based on the papers [8, 9]. We shall argue that the thermodynamic prediction would agree with the dynamical one when the horizon is infinite in size. These are translationally invariant generalizations of point black holes called black branes. The rationale for considering infinite horizon size is that, thermodynamic quantities like entropy and free energy contain information about long-wavelength physics. We shall claim that in this limit both types of instabilities shall always set in precisely for the same critical values of charge. We shall support our claim with an explicit example - that of spinning M2-branes which are described by $\mathcal{N} = 8$ gauged supergravity. Within limits of numerical accuracy, the correspondence is impressive in that not only does thermodynamics correctly predict the value of the angular momentum (or equivalently the charge of the $\mathcal{N} = 8$ theory) at which dynamical instability sets in, but it also accurately predicts which mode would be unstable in a classical linearized analysis. For this example, the geometry of the spacetime is $AdS_4 \times S^7$, but the S^7 part of the metric does not participate. So for our purposes, it is just a point black hole in four dimensional anti-de Sitter space. To take the infinite volume limit, we shall consider the case when the radius of the black hole (or equivalently the mass) is much larger than the AdS radius. Our results cannot be applied to point black holes in asymptotically flat space, because there is no other length scale with which one can compare the horizon radius. Our black hole will also be charged (in the 11-d language of M-theory, these charges would represent the values of angular momentum of the stack of M2-branes) and our solution would be just the Reissner-Nordstrom black hole in anti-de Sitter space.

In Section 2.2 we shall first try to argue why it is plausible that there should be a relation between thermodynamic and dynamical instabilities. Section 2.3 contains a summary of the AdS_4-RN solution and some generalizations of it in $\mathcal{N} = 8$ gauged supergravity and in higher dimensions. Section 2.4 discusses the thermodynamic instability which occurs for large charge and explores via thermodynamic arguments the likely paths for time-evolution of the unstable solutions. In section 2.5, a linear perturbation analysis is carried out around the AdS_4-RN solution. Finally, we conclude with a short discussion which includes the present state of knowledge about the ultimate fate of unstable black strings.

2.2 Dynamical versus Thermodynamical Instability

The Gregory-Laflamme instability [22] is a classical instability of black brane solutions in which the mass tends to clump together non-uniformly. The intuitive explanation for this instability is that the entropy of an array of black holes is higher for a given mass than the entropy of the uniform black brane. The intuitive explanation leaves something to be desired, since it applies equally to near-extremal Dp-branes: scaling arguments establish that a sparse array of large black holes threaded by an extremal Dp-brane will be entropically favored over a uniform non-extremal Dp-brane; however it is not expected that near-extremal Dp-branes exhibit the type of instability found in [22]. It was checked in [23] that a Dp-brane which is far from extremality (that is, one whose tension is many times the extremal tension) does have an instability. It was also shown that the instability persists for charged black strings in five dimensions fairly close to extremality.[1] Less is known about the case of near-extremal D3-branes, M2-branes, and M5-branes, but one may take the absence of tachyons in the extensive AdS-glueball calculations ([24, 25] and subsequent works—see [2] for a review) as provisional evidence that these near-extremal branes are (locally) stable.[2]

[1]The charged black string studied in [23] happens to be thermodynamically unstable all the way down to extremality: the specific heat is negative. Thus (2.1) would lead us to believe that this non-extremal black string is always unstable. The extremal solution should be stable since it can be embedded in a supersymmetric theory as a BPS object.

[2]More properly, we should say that the near-extremal black brane solutions with many units of D3-brane, M2-brane, or M5-brane charge appear to be stable. A single brane has Planck scale curvatures near the horizon, so classical two-derivative gravity does not provide a reliable description. We will concern ourselves

In the AdS/CFT correspondence [3, 5, 6], one might at first think that the existence of a unitary field theory dual forbids an instability. But suppose we are at finite temperature, and that there is a thermodynamic instability in the field theory—like the onset of a phase transition. Then it is quite natural for some fluctuation mode (or modes) to grow exponentially in time, at least in a linearized analysis, as one nucleates the new phase. Exciting an unstable mode is a change in the state of the field theory, not its lagrangian; thus according to AdS/CFT there should be a *normalizable* mode in AdS which likewise grows exponentially with time [26]. This might be referred to as a "boundary tachyon," or a "tachyonic glueball," since in the gauge theory it corresponds to some bound state with negative mass-squared. We will prefer the term "dynamical instability," which is meant to convey that there is an instability in the Lorentzian time evolution of the black brane, in both its supergravity and dual field theory descriptions.

To sum up, the existence of a field theory dual makes plausible the following adaptation of the entropic justification for the Gregory-Laflamme instability:

For a black brane solution to be free of dynamical instabilities, it is necessary and sufficient for it to be locally thermodynamically stable. (2.1)

Here, local thermodynamic stability is defined as having an entropy which is concave down as a function of the mass and the conserved charges. This criterion was first used in a black brane context in [27], where it was found that spinning D3-branes could be made locally thermodynamically unstable if the ratio of the spin to the entropy was high enough. Further work in this direction, relevant to the current chapter, has appeared in [28, 29, 30, 31]. For a somewhat complementary point of view on the nature of the unstable solutions, see [32, 33].

The conjecture (2.1) is meant to be a local version of the argument about whether the array of black holes or the black brane has higher entropy; however it seems on more precarious ground since one may not be able to write down a non-uniform stationary solution that competes with the black brane entropically. Nonetheless, it was shown in [8] that (2.1)

exclusively with solutions which have a discrete parameter (M2-brane charge, for the most part) which can be dialed to infinity to suppress all corrections to classical gravity.

predicts with good accuracy the value of the charge where the four-dimensional anti-de Sitter Reissner-Nordstrom solution (AdS_4-RN) develops an instability.

In the large black hole limit, a dynamical instability appears when local thermodynamic stability is lost. The existence of a dynamical instability was the main result of [8]. It disproves the claim of [34, 35] that charged black holes in AdS are classically stable. As we explain in section 2.5, the instability persists some ways away from the large black hole limit, providing the first proven example of a black hole with a compact horizon and a pointlike singularity which exhibits a dynamical Gregory-Laflamme instability.[3] Such solutions are interesting from the point of view of Cosmic Censorship, and we discuss the possibility of forming a naked singularity, or at least regions of arbitrarily large curvatures. Our main result here is that adiabatic evolution toward maximum entropy does not lead to solutions which arise from making the mass smaller than some appropriate combination of the charges. Because entropic arguments appear to give good information not only on the existence of dynamical instabilities but also on the direction they point, it is reasonable to predict from our results that no perturbative analysis of a smooth black hole in AdS will demonstrate a violation of Cosmic Censorship.

The unstable mode of the AdS_4-RN solution does not involve fluctuations of metric at linear order. Rather, it involves the gauge fields and scalars of $\mathcal{N} = 8$ gauged supergravity. Because the metric is not fluctuating, it may seem odd to describe the process as a Gregory-Laflamme instability. But we claim that the instability we see is in the same "universality class" as instabilities where the horizon does fluctuate: to be more precise, if the charges of the black hole are made slightly unequal, then generically the instability will involve the metric. In fact, the metric does fluctuate in the equal charge case as well—only at a subleading order that is beyond the scope of our linearized perturbation analysis. We would in fact make the case that any dynamical instability of a black hole which leads to non-uniformities in charge or mass densities should be considered in the same category as

[3]Here we are referring to the existence of a local instability visible in a classical analysis. It has been observed [36] that the AdS-Schwarzschild solution times a sphere can have a lower entropy than a Schwarzschild black hole of the same mass which is localized on the sphere. This demonstrates global but not local instability, and suggests the possibility of tunneling from one configuration to the other.

the Gregory-Laflamme instability of uncharged black branes.

We emphasize that this chapter is concerned with the relation between local thermodynamic stability of stationary solutions and the stability of their classical evolution in Lorentzian time. It is known [37, 38, 39, 40, 41, 42] that black holes which are thermodynamically unstable have an unstable mode in the Euclidean time formalism. For spherically symmetric black holes this mode is an *s*-wave. The interpretation is that, for instance, an AdS-Schwarzschild black hole in contact with a thermal bath of radiation will not equilibrate with the bath if the specific heat of the black hole is negative. This beautiful story does not fall under the rubric of problems we are considering, because the processes by which equilibration takes place in Lorentzian time include Hawking radiation, which is non-classical. Rather, we are contemplating black holes or branes in isolation from other matter, in a classical limit where Hawking radiation is suppressed, and inquiring whether a stationary, uniform black object wants to stay uniform or get lumpy as Lorentzian time passes. It is less clear that there should be any relation between this dynamical question and local thermodynamic stability: for instance, a Schwarzschild black hole in asymptotically flat space is stable.[4] Yet we conjecture that (2.1) gives a precise relation when the black object has a non-compact translational symmetry.

After the conjecture was made by Gubser and this author, Reall [13] constructed a semi-classical proof of why thermodynamic instability for a black brane would necessarily imply that it would have a Gregory-Laflamme instability. Let us summarize his main arguments.

The first thing to notice is that if a black hole has negative specific heat, it must have a negative mode since it cannot be a local minimum of the Euclidean action. In the Euclidean path integral formalism, fluctuations of the metric contribute to the path integral for gravity (whose existence in the semi-classical limit can be justified). On explicitly performing this gaussian integral one pulls out a factor of $\frac{1}{\sqrt{\Delta_L}}$ where Δ_L is the Euclidean Lichnerowitz

[4]This stability is implied by classical no-hair theorems, see for example [43]. A more extensive list of references on no-hair theorems can be found in [44]. A consequence of the present work is that these theorems cannot be extended to charged black holes in AdS.

operator. Therefore, if one has a negative eigenvalue for Δ_L, i.e. for the equation

$$\Delta_L H_{\mu\nu} = \lambda H_{\mu\nu} \, , \tag{2.2}$$

if there is a solution with $\lambda < 0$, then the path-integral becomes ill-defined and signals the presence of a thermodynamic instability. So we see that if a system has negative specific heat, it means that the Euclidean Lichnerowitz operator has a negative eigenvalue.

To tie this to the issue of dynamical stability, one considers a p-brane in $p+d$ spacetime dimensions with the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \delta_{ij}dz^i dz^j \, . \tag{2.3}$$

where $g_{mu\nu}$ is the metric of the spacetime transverse to the brane (in Lorentzian signature) and z^i are the flat spatial worldvolume directions of the brane. For an uncharged brane, the metric perturbations in the transverse traceless gauge reduce to

$$\Delta_L h_{\mu\nu} = 0 \, . \tag{2.4}$$

Using the ansatz $h_{\mu\nu} = e^{ik_i z^i} H_{\mu\nu}(x)$ this reduces to

$$\Delta_L H_{mu\nu} = -k^2 H_{\mu\nu} \, . \tag{2.5}$$

where Δ_L is the d dimensional Lorentzian Lichnerowitz operator. The crucial point now is that, at the point where instability just sets in, the energy of that mode is zero. Recall that an unstable mode grows exponentially with time, so the square of its energy is negative; a stable mode is oscillatory in nature, so the square of its energy is greater than zero. Therefore the onset of instability occurs at zero energy. But this means that this mode is independent of time! So, for such a mode, Wick-rotation simply gives back the same mode (with a negative sign) and (2.5) becomes (2.2) with $\lambda = -k^2$. In this way, Reall showed that the presence of a negative specific heat would imply a negative value of λ which through $\lambda = -k^2$ could be used to get the threshold value of the wavelength of the dynamically unstable mode. An important point to note is that if the brane is compact, then for there to be an instability, its size must be larger than the wavelength of the threshold mode. So, for the compact case, the range of values of conserved quantities like charge, etc. for which the brane is thermodynamically unstable will always be larger than the range of values for which it is dynamically unstable. In the non-compact case however, it is obvious from the above discussion that the two regions of parameter space will always agree. Using this idea the link between the two types of stability for the compact case has been explored in [14].

Although we shall mainly focus on black holes in AdS and their black brane limits; the conjecture (2.1) is intended to apply equally to any black brane. The conjecture might even apply beyond the regime of validity of classical gravity. Any "sensible" gravitational dynamics should satisfy the Second Law of Thermodynamics, and (2.1) is motivated solely by intuition that Lorentzian time evolution should proceed so as to increase the entropy. (The stipulation of translational invariance prevents finite volume effects from vitiating simple thermodynamic arguments). For instance, it has recently been shown [45] that the near-extremal NS5-brane has a negative specific heat arising from genus one contributions on the string worldsheet (see also [46, 47], and [48] for related phenomena in 1+1-dimensional string theory).[5] This is not classical gravity, but (2.1) leads us to expect an instability in the Lorentzian time evolution of near-extremal NS5-branes.[6] The instability would drive the NS5-brane to a state in which the energy density is non-uniformly distributed over the world-volume.

2.3 The AdS_4-RN solution and its cousins

The bosonic part of the lagrangian for $\mathcal{N} = 8$ gauged supergravity [49, 50] in four dimensions involves the graviton, 28 gauge bosons in the adjoint of $SO(8)$, and 70 real scalars. Because of the scalar potential introduced by the gauging procedure, flat Minkowski space is not a vacuum solution of the theory; rather, AdS_4 is. It is known [51] that the maximally supersymmetric AdS_4 vacuum of $\mathcal{N} = 8$ gauged supergravity represents a consistent truncation of 11-dimensional supergravity compactified on S^7. The $AdS_4 \times S^7$ solution can be

[5] We thank D. Kutasov for bringing [45, 48] to our attention.

[6] We thank M. Rangamani for a number of discussions on this point.

obtained as the analytic completion of the near-horizon limit of a large number of coincident M2-branes.[7] Making the M2-branes near-extremal corresponds to changing AdS_4 to the AdS_4-Schwarzschild solution. Near-extremal M2-branes can also be given angular momentum in the eight transverse dimensions. There are four independent angular momenta, corresponding to the $U(1)^4$ Cartan subgroup of $SO(8)$: these reduce to electric charges in the AdS_4 description. The electrically charged black hole solutions can be obtained most efficiently by first making a consistent truncation of the full $\mathcal{N} = 8$ gauged supergravity theory to the $U(1)^4$ gauge fields plus three real scalars. Consistent truncation means that any solution of the reduced theory can be embedded in the full theory, with no approximations. For our purposes, it can be viewed as a sophisticated technique for generating solutions. The truncated bosonic lagrangian is

$$\begin{aligned}
\mathcal{L} &= \frac{\sqrt{g}}{2\kappa^2}\left[R - \sum_{i=1}^{3}\left(\frac{1}{2}(\partial\varphi_i)^2 + \frac{2}{L^2}\cosh\varphi_i\right) - 2\sum_{A=1}^{4} e^{\alpha^i_A\varphi_i}(F^{(A)}_{\mu\nu})^2\right] \\
\text{where} \quad \alpha^i_A &= \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.
\end{aligned} \tag{2.6}$$

We use the conventions of [52], in particular, the metric signature is $-+++$ and $G_4 = 1$. In [52] the electrically charged solutions were found to be

$$\begin{aligned}
ds^2 &= -\frac{F}{\sqrt{H}}dt^2 + \frac{\sqrt{H}}{F}dz^2 + \sqrt{H}z^2 d\Omega^2 \\
e^{2\varphi_1} &= \frac{h_1 h_2}{h_3 h_4} \quad e^{2\varphi_2} = \frac{h_1 h_3}{h_2 h_4} \quad e^{2\varphi_3} = \frac{h_1 h_4}{h_2 h_3} \\
F^{(A)}_{0z} &= \pm\frac{1}{\sqrt{8}h_A^2}\frac{Q_A}{z^2} \\
H &= \prod_{A=1}^{4} h_A \quad F = 1 - \frac{\mu}{z} + \frac{z^2}{L^2}H \quad h_A = 1 + \frac{q_A}{z} \\
Q_A &= \mu\cosh\beta_A\sinh\beta_A \quad q_A = \mu\sinh^2\beta_A
\end{aligned} \tag{2.7}$$

where the signs on the gauge fields can be chosen independently. We will lose nothing by choosing them all to be +. The quantities Q_A are the physical conserved charges, and they

[7] As stated in the introduction, taking the number of M2-branes large makes the geometry smooth on the Planck/string scale and thus suppresses corrections to classical two-derivative gravity.

correspond to the four independent angular momenta of M2-branes in eleven dimensions. The mass is [28]

$$M = \frac{\mu}{2} + \frac{1}{4}\sum_{A=1}^{4} q_A , \tag{2.8}$$

and the entropy is

$$S = \pi z_H^2 \sqrt{H(z_H)} \tag{2.9}$$

where z_H is the largest root of $F(z_H) = 0$. Only for sufficiently large μ do roots to this equation exist at all. When they don't, the solution is nakedly singular.

We will be most interested in the case where all four charges are equal, $q_A = q$. Then the solution can be written more conveniently in terms of a new radial variable, $r = z + q$, and it takes the form

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \\ F_{0r} &= \frac{Q}{\sqrt{8}r^2} \\ f &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2}, \end{aligned} \tag{2.10}$$

with the scalars set to 0. In (2.10), F_{0r} is the common value of all four gauge field strengths $F_{0r}^{(A)}$. The geometry (2.10) is a solution of pure Einstein-Maxwell theory with a cosmological constant: it is the AdS_4-RN solution.

There are related solutions to maximally supersymmetric gauged supergravity in five and seven dimensions, corresponding respectively to spinning D3-branes and spinning M5-branes. In the case of D3-branes, there are six transverse dimensions, the rotation group is $SO(6)$, the Cartan subalgebra is $U(1)^3$, and as a result there are three independent angular momenta (or charges in the Kaluza-Klein reduced description). In the case of M5-branes, there are five transverse dimensions, the rotation group is $SO(5)$, the Cartan subalgebra is $U(1)^2$, and there are two independent angular momenta/charges. We will record here only the Einstein frame metric in the Kaluza-Klein reduced description, in conventions where $G_N = 1$ and L is the radius of the asymptotic AdS space. For further information on these solutions, the reader is referred to [28, 32]. The metrics are

$$\begin{aligned}
AdS_5: \quad & ds^2 = -H^{-\frac{2}{3}} F dt^2 + H^{\frac{1}{3}} \left(\frac{dr^2}{F} + r^2 d\Omega_3 \right) \\
& H = \prod_{A=1}^{3} h_A \quad F = 1 - \frac{\mu}{r^2} + \frac{r^2}{L^2} H \quad h_A = 1 + \frac{q_A}{r^2} \\
AdS_7: \quad & ds^2 = -H^{-\frac{4}{5}} F dt^2 + H^{\frac{1}{5}} \left(\frac{dr^2}{F} + r^2 d\Omega_3 \right) \\
& H = \prod_{A=1}^{2} h_A \quad F = 1 - \frac{\mu}{r^4} + \frac{r^2}{L^2} H \quad h_A = 1 + \frac{q_A}{r^4} .
\end{aligned} \tag{2.11}$$

2.4 Thermodynamics

2.4.1 Generalities

Given the solutions (2.7) and (2.11), we may read off the entropy, the mass, and the conserved electric charges. Typically it is most straightforward to express these quantities in terms of the non-extremality parameter μ and the boost parameters β_A. However it is possible to eliminate μ and β_A and find a polynomial equation relating M, S, and the Q_A. This equation can be solved straightforwardly for M, but not in general for S. We will quote explicit results for $M = M(S, Q_1, \ldots, Q_n)$ in the next section. In this section we will discuss thermodynamic stability assuming that $M(S, Q_1, \ldots, Q_n)$ is known.

The microcanonical ensemble is usually specified by a function $S = S(M, Q_1, \ldots, Q_n)$. Assuming positive temperature (which is safe for regular black holes since the Hawking temperature can never be negative), one may always invert $M = M(S, Q_A)$ to $S = S(M, Q_A)$, where now we abbreviate $Q_1, \ldots, Q_n$ to Q_A. A standard claim in classical thermodynamics is that the entropy for "sensible" matter must be concave down as a function of the other extensive variables as in figure 2.1. Locally this means that the Hessian matrix,

$$\mathbf{H}^S_{M,Q_A} \equiv \begin{pmatrix} \frac{\partial^2 S}{\partial M^2} & \frac{\partial^2 S}{\partial M \partial Q_B} \\ \frac{\partial^2 S}{\partial Q_A \partial M} & \frac{\partial^2 S}{\partial Q_A \partial Q_B} \end{pmatrix} , \tag{2.12}$$

satisfies $\mathbf{H}^S_{M,Q_A} \leq 0$, *i.e.* it has no positive eigenvalues. To understand what this requirement means, consider the simplest case where $n = 0$ and $\partial^2 S / \partial M^2 > 0$. This is

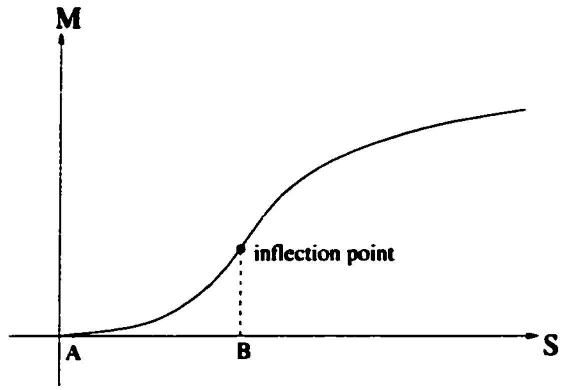


Figure 2.1: An example of a mass function whose convex hull is flat. The region we interpret as stable is from A to B.

the statement that the specific heat is negative. A substance with this property (in a non-gravitational setting, but equating mass with energy) is unstable: if we start at temperature T, then it is possible to raise the entropy without changing the total energy by having some regions at temperature $T + \delta T$ and others at $T - \delta T$. Since we are implicitly assuming a thermodynamic limit, it is irrelevant how big the domains of high and low temperature are. In a more refined description (*e.g.* Landau-Ginzburg theory), these domains might have a preferred size, or at least a minimal size.

In the more general setting of many independent thermodynamic variables, let us define intensive quantities

$$(y_0, y_1, \ldots, y_n) = (M/V, Q_1/V, \ldots, Q_n/V)\,, \tag{2.13}$$

where V is the volume. Suppose that $\mathbf{H}^S_{M,Q_A}$ has a positive eigenvector: $\mathbf{H}^S_{M,Q_A}\vec{v} = \lambda\vec{v}$ with $\lambda > 0$. Through a variation

$$y_j \to y_j + \epsilon v_j\,, \tag{2.14}$$

where ϵ is a function of position which integrates to 0, we can raise the entropy without changing the total energy or the conserved charges. Thus positive eigenvectors of $\mathbf{H}^S_{M,Q_A}$ indicate the way in which mass density and charge density tend to clump. Presumably the eigenvector with the most positive eigenvalue gives the dominant effect.

The stability requirement $\mathbf{H}^S_{M,Q_A} \leq 0$ may be rephrased as $\mathbf{H}^M_{S,Q_A} \geq 0$, where $\mathbf{H}^M_{S,Q_A}$ is the Hessian of M with respect to S and Q_A. This is easy to understand from a geometrical point of view. $\mathbf{H}^S_{M,Q_A} \leq 0$ says that all the principle curvatures of $S(M, Q_A)$ point toward negative S, or, equivalently, away from the point $(S, M, Q_A) = (\infty, 0, 0)$. Now, the point $(S, M, Q_A) = (0, \infty, 0)$ is on the opposite side of the co-dimension hypersurface defined by $S = S(M, Q_A)$ from $(S, M, Q_A) = (0, \infty, 0)$. Thus all principle curvatures should point toward $(0, \infty, 0)$, which means that $\mathbf{H}^M_{S,Q_A} \geq 0$. To determine the region of thermodynamic stability we may thus require $\det \mathbf{H}^M_{S,Q_A} > 0$, and then take the smallest connected components around points which are known to be stable.

While regions of stability are conveniently calculated from $\mathbf{H}^M_{S,Q_A}$, it is not clear that the eigenvector of $\mathbf{H}^S_{M,Q_A}$ with the largest positive eigenvalue can be read off easily from $\mathbf{H}^M_{S,Q_A}$. So it is useful to express $\mathbf{H}^S_{M,Q_A}$ directly in terms of derivatives of $M(S, Q_A)$:

$$\begin{aligned}
\frac{\partial^2 S}{\partial M^2} &= -\frac{1}{(\partial M/\partial S)^3}\frac{\partial^2 M}{\partial S^2} \\
\frac{\partial^2 S}{\partial Q_A \partial M} &= \frac{1}{(\partial M/\partial S)^3}\left[-\frac{\partial M}{\partial S}\frac{\partial^2 M}{\partial Q_A \partial S} + \frac{\partial M}{\partial Q_A}\frac{\partial^2 M}{\partial S^2}\right] \\
\frac{\partial^2 S}{\partial Q_A \partial Q_B} &= \frac{1}{(\partial M/\partial S)^3}\left[-\left(\frac{\partial M}{\partial S}\right)^2 \frac{\partial^2 M}{\partial Q_A \partial Q_B} - \frac{\partial^2 M}{\partial S^2}\frac{\partial M}{\partial Q_A}\frac{\partial M}{\partial Q_B}\right. \\
&\qquad \left. + \frac{\partial M}{\partial S}\left(\frac{\partial M}{\partial Q_A}\frac{\partial^2 M}{\partial Q_B \partial S} + \frac{\partial M}{\partial Q_B}\frac{\partial^2 M}{\partial Q_A \partial S}\right)\right].
\end{aligned} \tag{2.15}$$

A prescription for dealing with energy functions which violate the convexity condition $\mathbf{H}^M_{S,Q_A} \leq 0$ is the Maxwell construction, where one replaces $M(S, Q_A)$ with its convex hull (or $S(M, Q_A)$ by its convex hull—it's the same thing). This formal procedure is equivalent to allowing mixed phases where some domains have higher mass density or charge density than others. The energy functions resulting from charged black holes in AdS have the curious property that the convex hull is completely flat in some directions, so that chemical potentials (after taking the convex hull) are everywhere zero. This arises because, in certain directions, M rises slower than any nontrivial linear function of the other extensive variables. In this situation the Maxwell construction does not make much sense, because the mixed phases that it calls for have charges and mass concentrated arbitrarily highly in a small region, while the rest of the "sample" is at very low charge and mass density. A similar example in the simpler context of no conserved charges would be a mass function $M(S)$ like the one in figure 2.1. Here the natural physical interpretation is that the region between A and B represents a stable phase, while the region to the right of B is unstable toward clumping most of its energy into small regions. This tendency would presumably be cut off by some minimal length scale of domains. The mass functions obtained from charged black holes in AdS look roughly like figure 2.1 along some slices of the space of possible (S, Q_A). The interpretation we will offer is that the black holes are stable in the regime of parameters where convexity holds, and that they become dynamically unstable toward clumping their charge and energy outside this region.[8]

The line of thought summarized in the previous paragraph was already advanced in [29], but with only thermodynamic arguments to support it. A competing point of view was suggested in [34]: the black holes in question have no ergosphere (more precisely, there is a Killing vector field which is timelike everywhere outside the horizon), and this was argued to imply that there could be no superradiant modes, and hence no classical instability in the Lorentzian-time dynamics. The argument used the dominant energy condition, which need not always be satisfied by matter in AdS: in fact, the scalars φ_i in (2.6) violate the dominant energy condition because of their tachyonic potential (which however does satisfy the Breitenlohner-Freedman bound).

In [8], an explicit numerical calculation demonstrated the existence of a dynamical instability for certain AdS_4-RN black holes (related to spinning M2-branes with all four spins equal, as explained in the previous section). We will discuss this calculation at greater length in section 2.5. For now let us only remark that in the limit of large black holes, where the horizon area is infinite, the instability appears when thermodynamic stability is lost, up to a discrepancy of 0.7% which we suspect is numerical error. Furthermore, the combination

[8]There is a subtlety, discussed in [29], about the precise location of the boundary between stable and unstable regions. As the system approaches the inflection point at B, finite fluctuations might allow it to make small excursions into the unstable region. Working in a large N limit where classical supergravity applies on the AdS side of the duality seems to suppress such fluctuations.

of supergravity fields which became unstable indicated a change in local charge densities precisely in agreement with the analysis leading up to (2.14). Thus the conjecture (2.1) was tested to reasonably good accuracy along a two-parameter subspace (entropy and the common value of the four charges) of the five-parameter phase space. Further tests in AdS_4 are significantly more difficult because the metric usually enters in to the perturbation equations in a non-trivial way. However we will indicate in section 2.5 another case where the metric decouples. Tests in AdS_5 and AdS_7 can also be performed most easily in the equal charge case, but the analysis is somewhat more tedious because the spinor formalism is not as well worked out in higher dimensions (and probably is more cumbersome in any case).

Despite the absence of comprehensive tests, we will use (2.1) and the idea that black hole perturbations should follow the most unstable eigenvector of $\mathbf{H}^S_{M,Q_A}$ to propose in section 2.4.3 a qualitative picture of the evolution of unstable black holes in AdS. In brief, once the boundary of stability is passed, the independent charges tend to clump separately, as if they repelled one another but attracted themselves. But this is only an approximate tendency, with significant exceptions to be noted in section 2.4.3. The problem of finding out what actually happens to such unstable black holes/branes by an honest dynamical calculation is notoriously difficult. We shall have a little bit to say about this and what is known till date towards the end of this chapter.

2.4.2 Explicit formulas

It is possible to eliminate all the auxiliary quantities from (2.7), (2.8), and (2.9), and express M directly in terms of the entropy and the physical charges as

$$M = \frac{1}{2\pi^{\frac{3}{2}} L^2 \sqrt{S}} \left[\prod_{A=1}^{4} (S^2 + \pi L^2 S + \pi^2 L^2 Q_A{}^2) \right]^{\frac{1}{4}} . \tag{2.16}$$

We will often be interested in the limit of large black holes, $M/L \gg 1$. In this limit we have

$$M = \frac{1}{2\pi^{\frac{3}{2}} L^2 \sqrt{S}} \left[\prod_{A=1}^{4} (S^2 + \pi^2 L^2 Q_A{}^2) \right]^{\frac{1}{4}} . \tag{2.17}$$

with corrections suppressed by powers of M/L. As M/L approaches infinity, one obtains a black brane solution in the Poincare patch of AdS_4. Formally this limit can be taken by expanding (2.7) to leading order in small β_i, dropping the 1 from F, and replacing S^2 by R^2 in the metric.

As remarked in the previous section, local thermodynamic instability can be expressed as convexity of the function $M(S, Q_1, Q_2, Q_3, Q_4)$. By setting the Hessian of (2.17) equal to zero, we obtain the boundary separating the stable from the unstable region:

$$3S^8 - 2\pi^2 L^2 S^6 \sum_{A=1}^{4} Q_A^2 + \pi^4 L^4 S^4 \sum_{A<B} (Q_A Q_B)^2 - \pi^8 L^8 \prod_{A=1}^{4} Q_A^2 = 0 . \qquad (2.18)$$

First let us consider the case where the charges are pairwise set equal: $Q_1 = Q_3$ and $Q_2 = Q_4$. The above equation then factorises, giving us three relevant factors:

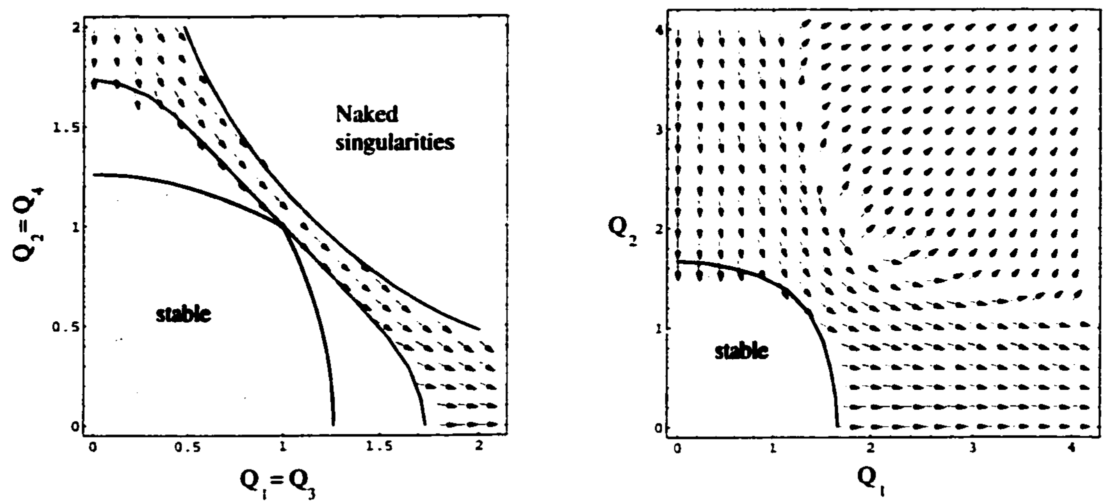
$$\left(S^2 - \pi^2 L^2 Q_1^2\right)\left(S^2 - \pi^2 L^2 Q_2^2\right)\left(S^2 - \frac{\pi^2 L^2}{6}(Q_1^2 + Q_2^2 + \sqrt{Q_1^4 + Q_2^4 + 14 Q_1^2 Q_2^2})\right) = 0 . \qquad (2.19)$$

When at least one of these factors become negative, $\mathbf{H}^M_{S,Q_A}$ develops a negative eigenvector and the black hole becomes thermodynamically unstable. A more convenient form may be obtained by eliminating S in favor of the mass M and introducing the dimensionless variable $\chi_i = \frac{Q_i}{M^{\frac{1}{3}} L^{\frac{1}{3}}}$. The above three equations in the new variables become

$$\begin{aligned}
&\left[\chi_1^4 + \chi_2^4 + 8\chi_1^2\chi_2^2 + (\chi_1^2 + \chi_2^2)\sqrt{\chi_1^4 + \chi_2^4 + 14\chi_1^2\chi_2^2}\right]^2 \\
&\qquad - 54\left(\chi_1^2 + \chi_2^2 + \sqrt{\chi_1^4 + \chi_2^4 + 14\chi_1^2\chi_2^2}\right) = 0 \\
&\chi_1^6 + 2\chi_1^2\chi_2^2(\chi_1^2 + \chi_2^2) - 4 = 0 \\
&\chi_2^6 + 2\chi_1^2\chi_2^2(\chi_1^2 + \chi_2^2) - 4 = 0 .
\end{aligned} \qquad (2.20)$$

The region depicting thermodynamically stable black holes is the intersection of the areas under the 3 curves as shown in figure 2.2(a).

The other relevant curve is the one separating nakedly singular solutions from regular black holes. The mathematical criterion for having a regular black hole solution is that the polynomial F in (2.7) should have a zero. In the large black hole limit, and in terms of χ_1



(a) (b)

Figure 2.2: Plots of the most unstable eigenvector of the Hessian matrix of $S(M, Q_1, Q_2, Q_3, Q_4)$. The inner curves are boundaries of stability. The outer curves (when they are present) denote the boundary between regular black branes and naked singularities.

and χ_2, this criterion reduces to

$$\chi_1^2\chi_2^2(\chi_1^8 + \chi_2^8) - 4\chi_1^4\chi_2^4(\chi_1^4 + \chi_2^4) + 132\chi_1^2\chi_2^2(\chi_1^2 + \chi_2^2) - 4(\chi_1^6 + \chi_2^6) + 6\chi_1^6\chi_2^6 - 432 = 0\,. \tag{2.21}$$

To determine if a black hole with given values of mass and charges is unstable, one first computes the values of χ_1 and χ_2 and locates this point in figure 2.2(a). The black hole is unstable if the point lies outside the shaded region depicting stable black holes but is within the boundary which separates black holes with naked singularities from those with a horizon. If the point lies in the unshaded (unstable) region of the plot without the vector field shown, it means that *within* each pair one charge wants to increase while the other decreases. The unstable eigenvector has no components along the hyperplane $Q_1 = Q_3$ and $Q_2 = Q_4$ and is not shown.

Finally, let us collect the thermodynamic results for the special case of all charges equal. We see that thermodynamic instability is present in the narrow region $1 < \chi < \sqrt{3}/2^{2/3}$. The associated eigenvector has the form $(0, 1, -1, 1, -1)$ where the components are along the axes M, Q_1, Q_2, Q_3, and Q_4 respectively: it looks like one pair of charges wants to increase while the other decreases. This can happen only locally, with each of the four charges conserved globally.

We'll also consider the case in which only two of the charges, Q_1 and Q_2, are non-zero. To get the region of thermodynamically stable black holes, we set $Q_3 = Q_4 = 0$ in (2.18):

$$3S^4 - 2\pi^2 L^2 S^2 (Q_1^2 + Q_2^2) + \pi^4 L^4 Q_1^2 Q_2^2 = 0 . \tag{2.22}$$

Just as we did in the previous case, we first eliminate S in favor of the mass M and then introduce the dimensionless variables $\chi_i = \frac{Q_i}{M^{\frac{1}{3}} L^{\frac{1}{3}}}$ to get:

$$\begin{aligned} &10(\chi_1^6 + \chi_2^6) + 21\chi_1^2\chi_2^2(\chi_1^2 + \chi_2^2) \\ &\qquad + \left(10\chi_1^4 + 10\chi_2^4 + 26\chi_1^2\chi_2^2\right)\sqrt{\chi_1^4 + \chi_2^4 - \chi_1^2\chi_2^2} - 432 = 0 . \end{aligned} \tag{2.23}$$

This is the boundary of the stable region, and is plotted in figure 2.2(b). Unlike the case of charges set equal pair-wise, black holes with two charges set to zero always have a horizon. This may be connected with the fact that there is a limit of rotating M2-branes with only two independent angular momenta nonzero which is a well-defined multi-center M2-brane solution, while with all angular momenta nonzero the corresponding limit is a singular configuration in eleven dimensions [53, 54, 55].

For black holes in AdS_5 and AdS_7, we will simply record here the mass in terms of the entropy and charges:

$$\begin{aligned} AdS_5: \quad M &= \frac{3}{2L^2(2\pi^4 S)^{2/3}} \left[\prod_{A=1}^{3} (4S^2 + \pi^4 L^2 Q_A^2) \right]^{1/3} \\ AdS_7: \quad M &= \frac{5}{4L^2(4\pi^9 S)^{2/5}} \left[\prod_{A=1}^{2} (16S^2 + \pi^6 L^2 Q_A^2) \right]^{2/5} . \end{aligned} \tag{2.24}$$

Stability analyses similar to the AdS_4 case can be carried out for AdS_5 and AdS_7. Some work along these lines was presented in [29], but the explicit expressions in (2.24) make the calculations much easier.

2.4.3 Adiabatic evolution

Tracking the evolution of unstable black holes in Lorentzian time is difficult. We have succeeded in establishing perturbatively the existence of a dynamical instability for the very special case of all charges equal: this is explained in section 2.5. This simplest case required the numerical solution of a fourth order ordinary differential equation with constraints at the horizon of the black hole and the boundary of AdS_4. Most other cases for black holes in AdS_4 involve fluctuations of the metric, which makes the analysis significantly harder. To investigate the instabilities beyond perturbation theory would require extensive numerical investigation of the second order PDE's that comprise the equations of motion of $\mathcal{N} = 8$ gauged supergravity.

The aim of this section is to use thermodynamic arguments to guess the qualitative features of the evolution of unstable black holes. Here we focus exclusively on the large black hole limit; however the conclusions may remain valid to an extent for finite size black holes with dynamical instabilities. The intuition is that knowing the entropy as a function of the other extensive parameters amounts to knowing the zero-derivative terms in an effective Landau-Ginzburg theory of the black hole (or of its dual field theory representation).

As explained in the paragraph around (2.13) and (2.14), an unstable eigenvector of $\mathbf{H}^S_{M,Q_A}$ (by which we mean one with positive eigenvalue) suggests a direction in which a black hole solution can be perturbed in order to raise entropy while keeping its total mass and conserved charges fixed; moreover it was shown in [8] (as we will explain in section 2.5) that the black hole's dynamical instability causes it to evolve in precisely the direction that the eigenvector indicates. The physics has no infrared cutoff, as is typical in Gregory-Laflamme setups, so we may hope that the charge and mass densities vary over long enough distance scales that we may continue to use the most unstable eigenvector of $\mathbf{H}^S_{M,Q_A}$ locally to determine the direction of the subsequent evolution. Following this line of thought to its logical conclusion leads us to the claim that the mass density and charge densities will locally evolve, subject to the constraints of conserving total energy and charge, from their initial values to values along a characteristic curve of the unstable vector field of $\mathbf{H}^S_{M,Q_A}$. This can only be approximately correct: finite wavelength distortions will occur, and it is not precisely right anyway to say that the time-evolution of Einstein's equations proceeds so as to maximize black hole entropy. Nevertheless it seems to us likely that a correct qualitative picture will emerge from tracking the flows generated by the most unstable eigenvector of $\mathbf{H}^S_{M,Q_A}$. At late times, or when charge and mass density are highly concentrated in small regions, another description is needed.

The characteristic curves of the most unstable eigenvector of $\mathbf{H}^S_{Q,M_A}$ may terminate in a region of stability, or in a region of naked singularities. Cosmic Censorship plus the conjectures of the previous paragraph suggest that the latter should never happen. This can be checked explicitly for the examples that we have. To this end, one can choose a generic value of charges and mass so that the black hole is almost naked, then determine the most unstable eigenvector of $\mathbf{H}^S_{M,Q_A}$, and then check that it is tangent to the surface separating naked singularities from regular black holes. We carried this out numerically for several cases and verified tangency; however we do not have a general argument. It appears, in fact, that the normal vector to the surface separating naked singularities from black holes is a stable eigenvector of $\mathbf{H}^S_{M,Q_A}$ (*i.e.* its eigenvalue is negative)—at least in the three-dimensional subspace with $Q_1 = Q_3$ and $Q_2 = Q_4$—so the obvious approach to an analytic demonstration that Cosmic Censorship is not violated by adiabatic evolution of black holes is to show that this normal vector is always a stable eigenvector of $\mathbf{H}^S_{M,Q_A}$. For now we content ourselves with the observation that in all the cases we have checked numerically, adiabatic evolution does stay in the region of regular black holes.

It is also possible that a characteristic curve becomes unstable at some point, in the sense that nearby characteristic curves diverge from it. To refine our previous claim, we may suppose that the black hole evolves along a bundle of nearby characteristic curves emanating from the original mass and charge density. This bundle may remain nearly one-dimensional, or it may split or become higher dimensional. We will not investigate the stability properties of the characteristic curves in any detail. Note that we are not attempting to specify any spatial or temporal properties of the evolution, only the range of mass and charge densities which form.

We present in figure 2.2 plots of unstable eigenvectors of the Hessian matrix $\mathbf{H}^S_{M,Q_A}$, projected onto a plane parametrized by two of the charges. From these vector fields, we may conclude that the different charges exhibit some tendency to separate from one another, but that this does not always happen, as in the upper right part of figure 2.2(b). The crucial point is that the unstable eigenvectors don't have a component normal to the boundary between naked singularities and regular black holes. Although this appears obvious from figure 2.2(a), the plot is slightly misleading in that the eigenvectors have been projected onto the plane of $Q_1 = Q_3$ and $Q_2 = Q_4$. One must preserve the components of vectors in the M direction to verify tangency.

When some angular momenta become large compared to the entropy for a spinning M2-brane solution, the geometry in eleven dimensions is approximately given by a rotating multi-center brane solution [53]. If one angular momentum is large, this multi-center solution is in the shape of a disk; if two are large and equal, it has the shape of a filled three-sphere. It seems clear that solutions of this form in an asymptotically flat eleven-dimensional spacetime are unstable toward fragmentation in the directions transverse to the M2-brane. This would mean that anti-de Sitter space fragments. In terms of the $SU(N)$ gauge theory, the disk corresponds to a $U(1)^{N-1}$ Higgsing, and in the fragmentation process some groups of $U(1)$'s try to come together to partially restore gauge invariance. It is not certain that such fragmentation occurs, particularly if the angular momentum density is large only locally. We merely indicate it as a possibility in the complicated late-time evolution of unstable black holes.

2.5 Existence of a dynamical instability

The existence of dynamical instabilities for thermodynamically unstable black branes should be completely generic. However, as mentioned already, the stability analysis is technically complicated for the general case of unequal charges: perturbations of the metric, four gauge fields, and three scalars lead to difficult coupled partial differential equations. Here we focus on the AdS_4-RN example, where the metric decouples and the problem can be reduced to a single gauge field and a single scalar. A formal argument relating thermodynamic and dynamical instability was suggested in [8], using the identification of the free energy with the

Euclidean supergravity action; however we have not yet succeeded in making this argument rigorous.

Because the unstable eigenvectors of $\mathbf{H}^{S}_{M,Q_A}$ (for all charges equal and sufficiently large) do not involve any change in the mass density, it is natural to expect that the perturbations that give rise to an unstable mode do not involve the metric.[9] More precisely, because of the form of the unstable eigenvectors, we expect that a relevant perturbation is

$$\delta F_A = \alpha^i_A \delta F \tag{2.25}$$

for some δF and fixed i, where the α^i_A were defined in (2.6). In section 2.4 we saw explicitly that $\delta Q_1 = \delta Q_3 = -\delta Q_2 = -\delta Q_4$ gave an unstable eigenvector; now we make a trivial alteration and focus on $\delta Q_1 = \delta Q_2 = -\delta Q_3 = -\delta Q_4$. Correspondingly we set $i = 1$ in (2.25).

The spectrum of linear perturbations to charged black holes in AdS has been considered before [56], but for the most part the perturbations under study were minimally coupled scalars. It is impractical to sift through the entire spectrum of supergravity looking for unstable modes (or tachyonic glueballs, in the language of [56]). The point of the previous paragraphs is that thermodynamics provides guidance not only on when to expect an instability, but also in which mode.

It is straightforward to start with the lagrangian in (2.6) and show that linearized perturbations to the equations of motion result in the following coupled equations:

$$\begin{gathered} d\delta F = 0 \qquad d * \delta F + d\delta\varphi_1 \wedge *F = 0 \\ \left[\Box + \frac{2}{L^2} - 8F^2_{\mu\nu}\right]\delta\varphi_1 - 16F^{\mu\nu}\delta F_{\mu\nu} = 0\,. \end{gathered} \tag{2.26}$$

Here $\Box = g^{\mu\nu}\nabla_\mu\partial_\nu$ is the usual scalar laplacian. F in (2.26) is the background field strength in (2.10): it is the common value of the four F_A. δF is *not* the variation in F itself; rather, the variation of the F_A is expressed in terms of δF in (2.25), with $i = 1$. The variation of the field strength is in a direction *orthogonal* to the background field strength of the AdS_4-RN

[9] Indeed, we suspect that the decoupling of the metric is possible precisely when there is an eigenvalue of $\mathbf{H}^{S}_{M,Q_A}$ which does not have a component in the M direction.

solution. The graviton decouples from the linearized perturbation equations: $\delta T_{\mu\nu}$ vanishes at linear order in δF because $\delta F_A \cdot F_A = 0$.[10]

For comparison, we write down the linearized equations for fluctuations of the other scalars:

$$\left[\Box + \frac{2}{L^2} - 8F_{\mu\nu}^2\right] \delta\varphi_i = 0 \tag{2.27}$$

for $i = 2, 3$. It was shown in [34] that any perturbation involving only matter fields satisfying the dominant energy condition could not result in a normalizable unstable mode (that is, a normalizable mode which grows exponentially in Lorentzian time). It was conjectured [34, 35] that in fact there was no classical instability at all. The scalars φ_i do not satisfy the dominant energy condition because of the potential term in (2.6). Thus the outcome of our calculations is not fore-ordained by general arguments, and we have a truly non-trivial check on the classical stability of highly charged black holes in $\mathcal{N} = 8$ gauged supergravity. In fact, our results turn out to be in conflict with the claim of classical stability in [34, 35].

Decoupling the equations in (2.26) is a chore greatly facilitated by the use of the dyadic index formalism introduced in [57]. For the reader interested in the details, we present an outline of the derivation in section 2.5.1. The final result is the fourth order ordinary differential equation (ODE)

$$\begin{aligned}\left(\frac{\omega^2}{f} + \partial_r f \partial_r - \frac{\ell(\ell+1)}{r^2}\right) r^3 \left(\frac{\omega^2}{f} + \partial_r f \partial_r - \frac{\ell(\ell+1)}{r^2} - \frac{2M}{r^3} + \frac{4Q^2}{r^4}\right) r\delta\tilde{\varphi}_1(r) = \\ 4Q^2 \left(\frac{\omega^2}{f} + \partial_r f \partial_r\right) \delta\tilde{\varphi}_1(r),\end{aligned} \tag{2.28}$$

where we have assumed the separated form $\delta\varphi_1 = \mathrm{Re}\, e^{-i\omega t} Y_{\ell m} \delta\tilde{\varphi}_1(r)$, where $Y_{\ell m}$ is the usual spherical harmonic on S^2. This is to be compared with the separated equation for the other scalars:

$$\left(\frac{\omega^2}{f} + \partial_r f \partial_r - \frac{\ell(\ell+1)}{r^2} - \frac{2M}{r^3} + \frac{4Q^2}{r^4}\right) r\delta\tilde{\varphi}_i(r) = 0 \tag{2.29}$$

for $i = 2, 3$.

[10] Besides the all-charges-equal case, we know of one other case where the metric decouples at linear order: $Q_1 = Q_3$ with $Q_2 = Q_4 = 0$. There may be other cases as well—presumably whenever $Q_A \cdot \delta Q_A = 0$ and $\delta S = 0$ for an unstable eigenvector $(\delta S, \delta Q_A)$ of $\mathbf{H}^M_{S,Q_A}$.

2.5.1 Dyadic index derivation of (2.28)

To derive (2.28) using the dyadic index formalism, it is convenient first to switch to $+---$ signature to avoid sign incompatibilities between the raising and lowering of dyadic and vector indices. One introduces a null tetrad of vectors, $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, defined so that $l^\mu n_\mu = -m^\mu m_\mu = 1$ and all other inner products vanish. Next define

$$\sigma^\mu_{\Delta\dot{\Delta}} = \begin{pmatrix} l^\mu & m^\mu \\ \bar{m}^\mu & n^\mu \end{pmatrix} \tag{2.30}$$

and set $D = l^\mu \partial_\mu$, $\Delta = n^\mu \partial_\mu$, $\delta = m^\mu \partial_\mu$, $\bar{\delta} = \bar{m}^\mu \partial_\mu$. Vector indices are converted into dyadic indices by setting $v_{\Delta\dot{\Delta}} = \sigma^\mu_{\Delta\dot{\Delta}} v_\mu$. Dyadic indices are raised and lowered using northwest contraction rules with $\epsilon_{01} = \epsilon^{01} = \epsilon_{\dot{0}\dot{1}} = \epsilon^{\dot{0}\dot{1}} = 1$. By demanding that $\sigma^\mu_{\Delta\dot{\Delta}}$ is covariantly constant, one can obtain a unique covariant derivative D_μ, whose action on a spinor is

$$D_\mu \psi_\Gamma = \partial_\mu \psi_\Gamma - \psi_\Sigma \gamma_\mu{}^\Sigma{}_\Gamma \,. \tag{2.31}$$

The so-called spin coefficients, $\gamma_{\Delta\dot{\Delta}\Sigma\Gamma} = \sigma^\mu_{\Delta\dot{\Delta}} \gamma_{\mu\Sigma\Gamma}$, are conventionally written as

$$\begin{aligned} \gamma_{0\dot{0}\Sigma\Gamma} &= \begin{pmatrix} \kappa & \epsilon \\ \epsilon & \pi \end{pmatrix} & \gamma_{0\dot{1}\Sigma\Gamma} &= \begin{pmatrix} \sigma & \beta \\ \beta & \mu \end{pmatrix} \\ \gamma_{1\dot{0}\Sigma\Gamma} &= \begin{pmatrix} \rho & \alpha \\ \alpha & \lambda \end{pmatrix} & \gamma_{1\dot{1}\Sigma\Gamma} &= \begin{pmatrix} \tau & \gamma \\ \gamma & \nu \end{pmatrix} . \end{aligned} \tag{2.32}$$

A less compressed presentation of dyadic index formalism can be found in [57, 20], and the appendix to [58].

For AdS_4-RN, a convenient choice of the null tetrad and the corresponding nonzero spin coefficients are as follows:

$$\begin{aligned} l^\mu &= (1/f, 1, 0, 0) & n^\mu &= \frac{1}{2}(1, -f, 0, 0) \\ m^\mu &= \frac{1}{r\sqrt{2}}(0, 0, 1, i\csc\theta) & \bar{m}^\mu &= \frac{1}{r\sqrt{2}}(0, 0, 1, -i\csc\theta) \end{aligned} \tag{2.33}$$

$$\rho = -\frac{1}{r} \quad \mu = -\frac{f}{2r} \quad \gamma = \frac{f'}{4} \quad \alpha = -\beta = -\frac{\cot\theta}{\sqrt{8}r} . \tag{2.34}$$

in (2.33) and (2.34) we have not yet taken the black brane limit. Taking this limit replaces $\csc\theta$ by 1 in (2.33) and sets $\alpha = \beta = 0$ in (2.34). Proceeding without the black brane limit, we trade the real antisymmetric tensor $F_{\mu\nu}$ for a complex symmetric tensor,

$$\Phi^{(0)}_{\Delta\Gamma} = \begin{pmatrix} \phi_0^{(0)} & \phi_1^{(0)} \\ \phi_1^{(0)} & \phi_2^{(0)} \end{pmatrix} \tag{2.35}$$

through the formula

$$4\sqrt{2}F_{\mu\nu}\sigma^{\mu}_{\Delta\dot{\Delta}}\sigma^{\nu}_{\Gamma\dot{\Gamma}} = \Phi^{(0)}_{\Delta\Gamma}\epsilon_{\dot{\Delta}\dot{\Gamma}} + \bar{\Phi}^{(0)}_{\dot{\Delta}\dot{\Gamma}}\epsilon_{\Delta\Gamma}\,. \tag{2.36}$$

The factor of $4\sqrt{2}$ in (2.35) is for convenience: the AdS_4-RN background has $\phi_1^{(0)} = Q/r^2$ and all other components zero. In the same way we trade in $\delta F_{\mu\nu}$ for $\Phi_{\Delta\Gamma}$, whose components are ϕ_0, ϕ_1, and ϕ_2, with a similar factor of $4\sqrt{2}$. Finally, we write φ in place of $\delta\varphi_1$ to avoid the ambiguity in the meaning of δ.

The first order equations for the gauge field in (2.26) can now be cast in dyadic form as follows:

$$D^{\Delta}{}_{\dot{\Gamma}}\Phi_{\Delta\Gamma} + \frac{1}{2}\partial^{\Delta\dot{\Delta}}\varphi(\Phi^{(0)}_{\Delta\Gamma}\epsilon_{\dot{\Delta}\dot{\Gamma}} + \bar{\Phi}^{(0)}_{\dot{\Delta}\dot{\Gamma}}\epsilon_{\Delta\Gamma}) = 0\,. \tag{2.37}$$

In components, these equations read

$$\begin{aligned}
(D-2\rho)\phi_1 - (\bar{\delta}-2\alpha)\phi_0 &= -\phi_1^{(0)}D\varphi \\
(\Delta+\mu-2\gamma)\phi_0 - \delta\phi_1 &= 0 \\
(D-\rho)\phi_2 - \bar{\delta}\phi_1 &= 0 \\
(\delta+2\beta)\phi_2 - (\Delta+2\mu)\phi_1 &= \phi_1^{(0)}\Delta\varphi\,.
\end{aligned} \tag{2.38}$$

It is possible to combine these equations into three second order equations in which only a single ϕ_i appears. Together with the scalar equation, these equations are equivalent to (2.26):

$$\begin{aligned}
\left[(D-3\rho)(\Delta+\mu-2\gamma) - \delta(\bar{\delta}-2\alpha)\right]\phi_0 &= -\phi_1^{(0)}\delta D\varphi \\
\left[(\Delta+3\mu)(D-\rho) - \bar{\delta}(\delta+2\beta)\right]\phi_2 &= -\phi_1^{(0)}\bar{\delta}\Delta\varphi \\
\left[(D-2\rho)(\Delta+2\mu) - (\delta+\beta-\alpha)\bar{\delta}\right]\phi_1 &= -\phi_1^{(0)}D\Delta\varphi \\
\left[\Box + \frac{2}{L^2} + 2(\phi_1^{(0)})^2\right]\varphi &= -4\phi_1^{(0)}\,\mathrm{Re}\,\phi_1
\end{aligned} \tag{2.39}$$

where we have made use of the fact that the spin coefficients are all real for AdS_4-RN. The equations (2.38) are a special case of (3.1)-(3.4) of [21]. The first and second equations of (2.39) are (3.5) and (3.7) of [21], and the third is derived in a similar manner. The fourth is the scalar equation in (2.26), but to preserve the definition of $\Box$ we write $\Box = -g^{\mu\nu}\nabla_\mu\partial_\nu$ in $+---$ conventions. The differential operators in the third equation of (2.39) are purely real (this takes a bit of checking for $(\delta+\beta-\alpha)\bar{\delta}$), so we can take the real and imaginary parts of this equation. The equation for $\operatorname{Im}\phi_1$ decouples from all the others. The equations for ϕ_0 and ϕ_2 are sourced by φ, but ϕ_0 and ϕ_2 do not otherwise enter; thus one can solve first for $\operatorname{Re}\phi_1$ and φ, and afterwards use the first and second equations in (2.39) to obtain ϕ_0 and ϕ_2. Since $\phi_1^{(0)}$ is nowhere vanishing, the last equation in (2.39) can be used to eliminate $\operatorname{Re}\phi_1$ algebraically. The final result is

$$[(D-2\rho)(\Delta+2\mu)-(\delta+\beta-\alpha)\bar{\delta}]\,\frac{1}{4\phi_1^{(0)}}\left[\Box+\frac{2}{L^2}+2(\phi_1^{(0)})^2\right]\varphi=\phi_1^{(0)}D\Delta\varphi\,. \qquad (2.40)$$

Plugging in the separated ansatz $\varphi = \operatorname{Re}\{e^{-i\omega t}Y_{\ell m}\delta\tilde{\varphi}_1(r)\}$, one easily obtains (2.28).

2.5.2 Numerical results from the fourth order equation

A dynamical instability exists if there is a normalizable, unstable solution to (2.28) or to (2.29). Neither of these equations admits a solution in closed form, so we have resorted to numerics. Briefly, the conclusion is that, in the black brane limit and within the limits of numerical accuracy, we find a single unstable mode for (2.28) precisely when $\chi > 1$, and no instabilities for (2.29). This is completely in accord with the intuition from thermodynamics: (2.29) represents a fluctuation that has nothing to do with the variation of charges that gave the unstable eigenvector of the Hessian matrix of $M(S,Q_1,Q_2,Q_3,Q_4)$. The unstable mode in (2.28) persists to finite size AdS_4-RN black holes, but eventually disappears for small enough black holes.

To carry out a numerical study of (2.28), the first step is to cast the equation in terms of a dimensionless radial variable u, a dimensionless charge parameter χ, a dimensionless mass parameter σ, and a dimensionless frequency $\tilde{\omega}$:

$$u=\frac{r}{M^{1/3}L^{2/3}} \qquad \chi=\frac{Q}{M^{2/3}L^{1/3}} \qquad \sigma=\left(\frac{L}{M}\right)^{2/3} \qquad \tilde{\omega}=\frac{\omega L^{4/3}}{M^{1/3}}\,. \qquad (2.41)$$

Then we have

$$\left(\frac{\tilde{\omega}^2}{\bar{f}} + \partial_u \bar{f} \partial_u - \sigma \frac{\ell(\ell+1)}{u^2}\right) u^3 \left(\frac{\tilde{\omega}^2}{\bar{f}} + \partial_u \bar{f} \partial_u - \sigma \frac{\ell(\ell+1)}{u^2} - \frac{2}{u^3} + \frac{4\chi^2}{u^4}\right) u \delta\tilde{\varphi}_1 =$$
$$4\chi^2 \left(\frac{\tilde{\omega}^2}{\bar{f}} + \partial_u \bar{f} \partial_u\right) \delta\tilde{\varphi}_1 \tag{2.42}$$
$$f = \sigma - \frac{2}{u} + \frac{\chi^2}{u^2} + u^2 .$$

Evidently, the dimensionless control parameters are ℓ (the partial wave number), σ, and χ. Using Mathematica, we solved (2.42) numerically via a shooting method, and obtained wavefunctions $\delta\tilde{\varphi}_1(r)$ which fall off like $1/r^2$ near the boundary of AdS_4 and at least as fast as $(r - r_H)^{|\omega|/f'(r_H)}$ near the horizon.

To check that the wavefunction is well behaved near the horizon[11] let us transform to Kruskal coordinates. The metric near the horizon is

$$ds^2 \approx -f'(r_H)(r - r_H)dt^2 + \frac{dr^2}{f'(r_H)(r - r_H)} + r_H^2 d\Omega_2^2 , \tag{2.43}$$

where r_H is the radius of the horizon. Dropping the S^2 piece and introducing a tortoise coordinate r_*, null coordinates $P_\pm$, and Kruskal coordinates (T, R) according to

$$\frac{dr_*}{dr} = \frac{1}{f'(r_H)(r - r_H)}$$
$$P_\pm = e^{\frac{1}{2} f'(r_H)(\pm t + r_*)} = \pm T + R , \tag{2.44}$$

one finds that the near-horizon metric is indeed regular:

$$ds_2^2 = -f'(r_H)(r - r_H)dt^2 + \frac{dr^2}{f'(r_H)(r - r_H)} = \frac{4}{f'(r_H)}(-dT^2 + dR^2) . \tag{2.45}$$

Having a radial wavefunction $\delta\tilde{\varphi}_1(r) = (r - r_H)^{|\omega|/f'(r_H)} \rho(r - r_H)$, where $\rho(r - r_H)$ remains bounded at the horizon, means that the time-dependent perturbation (with angular dependence suppressed) is

$$\delta\varphi_1(t, r) \sim (r - r_H)^{|\omega|/f'(r_H)} e^{|\omega| t} \rho(r - r_H) \sim P_+^{2|\omega|/f'(r_H)} \rho(P_+ P_-) , \tag{2.46}$$

which remains bounded as $P_- \to 0$. The black hole horizon is at $P_- = 0$, $P_+ > 0$ (see figure 2.3). Thus we see that the perturbation is small at the horizon in good coordinates,

[11] We thank G. Horowitz for suggesting that this check should be made.

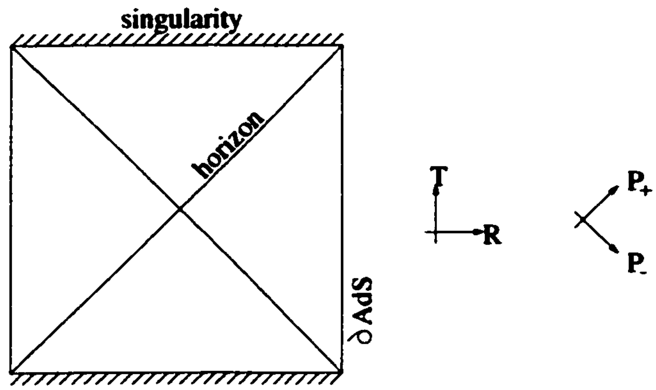


Figure 2.3: The Penrose diagram of a regular AdS black hole. We can take $T = R = P_+ = P_- = 0$ at the center of the diagram. The black hole horizon is the diagonal line going up and right from the origin.

at least for small P_+. (As the perturbation grows, the horizon eventually starts to fluctuate, but this is not an issue in the question of whether the instability exists).

A qualitative summary of our numerical results is displayed in figure 2.4(a). An example of a normalizable wave-function with negative ω^2 is shown in figure 2.4(b). Some points to note are:

- The boundary of the region of dynamical stability comes from instability in the $\ell = 1$ mode. The $\ell = 0$ mode is projected out by charge conservation. Higher ℓ modes become unstable in the upper left part of the shaded triangle in figure 2.4(b). The boundaries of dynamical instability for different ℓ all come together at $\sigma = 0$.

- At $\sigma = 0$, thermodynamic stability is lost at $\chi = 1$, whereas dynamical instability sets in at $\chi = 1.007$. We believe that the 0.7% discrepancy is due to numerical error.

- We have drawn the regions of dynamical instability and thermodynamic stability as disjoint in figure 2.4(a). In fact, our current numerics shows them overlapping by about 0.1% around $\sigma = 0.1$. We do not view this as significant because the numerical errors seem to be around 1%.

Finally, it is worth pointing out that the string theory program of computing black hole entropy via a microscopic state count in a field theory dual (see for example [59], or [60] for

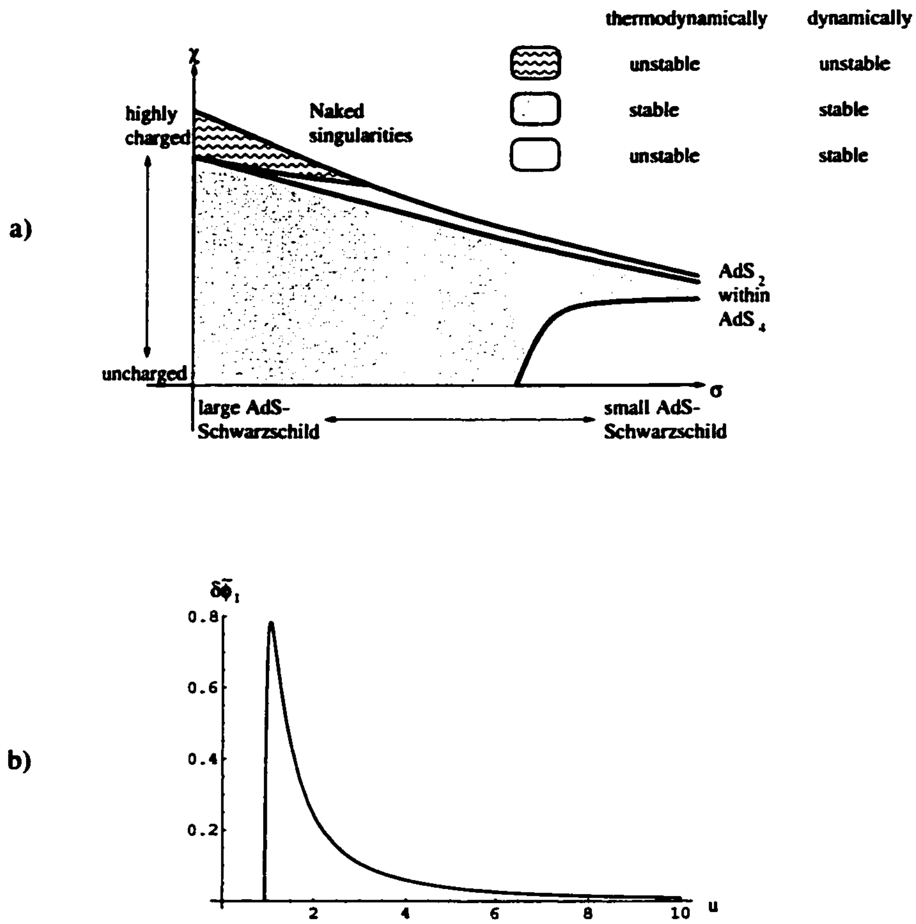


Figure 2.4: (a) A topologically correct representation of dynamical and thermodynamic stability in the whole χ-σ plane (but see the text regarding possible overlap of the two shaded regions). (b) A sample normalizable wave-function with negative ω^2: here $\sigma = 0.3$, $\chi = 0.96$, and $\bar{\omega}^2 = -0.281$.

a review) has proved hard to extend past the boundaries of thermodynamic stability. For instance, we have a good understanding of the entropy of near-extremal D3-branes [61, 62], but not of small Schwarzschild black holes in AdS. It seems to us that this is no accident: most sensible field theories have log-convex partition functions, and this translates into Hessian matrices $\mathbf{H}^S_{M,Q_A}$ which have no negative eigenvalues. Pushing past the boundary of thermodynamic stability in a field theory may be possible (particularly as one crosses a phase boundary and begins to nucleate the new phase), but doing so seems likely to produce dynamical instabilities in the Lorentzian time-evolution. This point of view has indeed informed our entire investigation.

A dual field theory description of a small Schwarzschild black hole in AdS must involve thermodynamic instability but no dynamical instabilities. We believe that finite volume effects in the field theory are essential in this regard: if one imagines a Landau-Ginzburg effective description of the field theory, then derivative terms must restore stability to a system whose infrared tendencies are controlled by the thermodynamic instability. Various properties of small AdS-Schwarzschild black holes have been explored (see for example [63, 64]), but the basic problem of reconciling thermodynamic instability with dynamical stability in the presence of a field theory dual remains to be addressed.

2.6 Conclusions

A common conception of the Gregory-Laflamme instability is that a uniform solution to Einstein's equations (plus matter) competes with a non-uniform solution, and the non-uniform solution sometimes wins out entropically. In such a situation, the generic expectation is that there is a first order tunneling transition from the uniform to the non-uniform state, which may take place very slowly due to a large energetic barrier. In fact, the original papers [22, 23] focused mainly on demonstrating the existence of unstable modes in a linearized perturbation analysis of the uniform solution. The distinction is between global and local stability. At the level of classical gravity/field theory, the latter concept is more meaningful, because with quantum effects suppressed it is impossible to tunnel away from a locally stable solution. The aim of this chapter has been to study local dynamical stability of black holes in anti-de Sitter space in relation to a particular notion of local thermodynamic stability, namely downward concavity of the entropy as a function of the other extensive variables. We reach two main conclusions:

1. In the limit of large black holes in AdS, dynamical and thermodynamic stability coincide. This conclusion is supported by numerical evidence. The small discrepancy between the observed onset of dynamical and thermodynamic instabilities is probably numerical error.

2. Dynamical instabilities persist for finite size black holes in AdS, down to horizon radii on the order of the AdS radius. The evidence is again only numerical, but we believe the final answer is correct and robust.

We regard point 1 as a partial verification of a rather more general conjecture, namely that black branes should have Gregory-Laflamme instabilities (in the local, dynamical sense of the papers [22, 23]) precisely when thermodynamic stability is lost. There is by now a lot of evidence that this conjecture is probably true. Also, as we sketched out briefly in Section 2.2, there is something close to a proof of the conjecture by Reall.

Point 2 is surprising because it is the first known example of a stationary black hole solution with a point-like singularity which exhibits a dynamical Gregory-Laflamme instability. Furthermore, it shows that no-hair theorems cannot always hold in anti-de Sitter space.

Is Cosmic Censorship really threatened by our analysis?[12] It is too early to say. Using the heuristic method of calculating the most unstable eigenvector of the Hessian of the entropy function, we have argued that adiabatic evolution of unstable black holes does not lead to nakedly singular solutions.

[12] If asymptotically flat spacetimes are part of the hypothesis of Cosmic Censorship, as is often the case, then of course no demonstration in global anti-de Sitter space is relevant. We prefer a broader interpretation of Cosmic Censorship—loosely speaking, that no observer who follows a timelike trajectory which never runs into singularities can receive signals from a singularity.

At the time of writing this chapter, the ultimate fate of these unstable black branes is still an open question. What we do know is that the horizon starts off by becoming lumpy and loses translational invariance. One might then think that since there is no natural scale at which this process can stop, the non-uniform brane would continue to become more and more non-uniform and finally the horizon would pinch off in some region. In a beautiful paper, Horowitz and Maeda have shown in [65] that there is an immediate difficulty to this line of reasoning. Their result is that the horizon cannot classically pinch off in finite affine time if the weak energy condition is satisfied. In fact, given any spacelike curve on the event horizon, if one evolves the curve along the null geodesic generators, its length cannot go to zero in finite affine parameter. The basic idea is to use the fact that the divergence θ of the null geodesic generators ℓ of the event horizon cannot become negative. This means that if part of the horizon is contracting, the orthogonal directions must be expanding. But this introduces a lot of shear σ_{MN} in the null geodesic congruence. One now uses the Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{D-2} - \sigma_{MN}\sigma^{MN} - R_{MN}\ell^M\ell^N \tag{2.47}$$

where λ is an affine parameter along the null geodesics and D is the total spacetime dimension. If the weak energy condition is satisfied, the right hand side is negative definite, so when the shear becomes large, θ decreases rapidly. One can show that if part of the horizon shrinks to zero size in finite affine parameter, θ must become negative which is a contradiction. If the above argument is correct, one is led to believe that the solution must settle down to a new static configuration without translational symmetry along the brane.

There are ways one could imagine bypassing the above argument; for instance if naked singularities form outside the horizon. At the very least, current numerical evidence indicates there might be problems associated with assuming that the end-point is a non-uniform brane. Gubser in [66], and more recently Wiseman [67] have provided numerical evidence that a more lumpy string is heavier than a less lumpy one, so that these non-uniform strings cannot be formed as a result of the decay of an unstable uniform string. The fate of the unstable black string is still not known and Kol has conjectured that the black string might undergo a topology changing transition and form a black hole (there has recently been some numerical evidence for this in [68]). In short, the fate is not known with full confidence.

Chapter 3

Stability of AdS Compactifications Without Supersymmetry

3.1 Introduction

The discovery of the *AdS*/CFT correspondence [3, 5, 6] (for a review see [2]) has led to renewed interest in the stability of geometries of the form $AdS_p \times M_q$ where AdS_p is anti-de Sitter spacetime and M_q is an Einstein space with positive Ricci tensor. Solutions of this type with a q-form field strength on M_q were first considered in higher dimensional supergravity theories by Freund and Rubin [69]. Due to the negative curvature of *AdS*, perturbative stability does not require the absence of all tachyonic modes. Instead, as Breitenlohner and Freedman (BF) first showed, scalars with $m^2 < 0$ may appear as long as their masses do not fall below a bound set by the curvature scale of *AdS* [70]. The issue of stability is important for understanding a possible dual conformal field theory (CFT) description. For stable solutions, the spectrum of masses directly yields the dimensions of certain operators in such a CFT. On the other hand, AdS vacua with some field(s) violating the BF bound need not have a well-defined field theory dual. Indeed, if one attempts to compute the two-point function of such a field, the result is highly cutoff-dependent. This is like having a lattice theory without a well-defined continuum limit. By extension, solutions to string theory or supergravity which are asymptotic to AdS vacua violating the bound may also be expected to have some pathology on the field theory side.

It is well known that for the standard ten and eleven dimensional maximally supersymmetric supergravity theories, 11D SUGRA on $AdS_4 \times S^7$ or $AdS_7 \times S^4$ and Type IIB SUGRA on $AdS_5 \times S^5$ are all stable. However, these solutions are all supersymmetric (SUSY), and simple nonsupersymmetric vacua like $AdS_4 \times M_n \times M_{7-n}$ [71] and $AdS_7 \times S^2 \times S^2$ [72] are known to be unstable. Furthermore, the SUSY examples have modes which either saturate the BF bound, or are very close to saturating it. This raises the question of the role that SUSY plays in ensuring stability of vacua of this type. (For earlier discussions of this question see *e.g.* [71], [72], [73].) This issue is of particular interest in light of the recent proposal of bosonic M-theory [15], a 27-dimensional theory which was hypothesized to appear as the strong-coupling limit of the bosonic string. Its low energy limit is assumed to be gravity coupled to a four-form field strength, which admits solutions of the form $AdS_4 \times S^{23}$ and $AdS_{23} \times S^4$. It was suggested that with these boundary conditions, bosonic M-theory might be holographically described by a (2+1)- or (21+1)-dimensional CFT. Thus, it is important to determine whether these solutions are stable.

One argument for the stability of $AdS_4 \times S^{23}$ [15] and more generally $AdS_p \times S^q$ is that these backgrounds are the near-horizon geometries of extremal black branes. However this is not completely satisfying for two reasons. First, although we expect extremal black branes to be stable, the appropriate positive mass theorem (stating roughly $M \geq Q$) has never been proven.[1] Second, as we will discuss later, one can construct extremal black brane solutions with unstable near horizon geometry by placing branes at the apex of appropriate cones. So, one needs to examine stability directly.

This chapter is based on the papers [10, 11]. In it we shall study the stability of general solutions of the form $AdS_p \times M_q$ in a theory of gravity coupled to a q-form field strength. When one expands the field equations to linear order, there are several types of modes. Some immediately decouple from the others, while the rest mix and must be

[1] Interestingly enough, if one tries to adapt Witten's spinorial approach, one succeeds only in the SUSY cases [74].

diagonalized. *A priori*, since the fundamental fields in $p+q$ dimensions are massless, and adding dependence on M_q should increase the mass, one might expect that the modes that don't mix should always be stable. Masses violating the BF bound might be expected, however, to arise in diagonalizing the coupled fluctuations — indeed, this is the origin of the modes that saturate or come very near to saturating the BF bound in the SUSY examples, so one might think that the absence of supersymmetry could push them over the edge.

Surprisingly, this is not what we find. It turns out that for any p and q and any Einstein space M_q, the coupled modes are always stable. Moreover, for S^q the lowest mass either saturates (q odd) or almost saturates (q even) the BF bound. This is not to say, however, that an arbitrary $AdS_p \times M_q$ background is stable. The dangerous mode turns out to be an unmixed scalar coming from the transverse, traceless metric perturbation on M_q. This is the only mode which is sensitive to the choice of Einstein manifold M_q. We shall study a few types of compact Einstein manifolds, starting with the simplest one – a sphere, then moving on to product spaces (so that M_q is a product of two compact spaces which are themselves Einstein manifolds) and finally we shall treat the important example in which M_q is the five dimensional coset space $(SU(2) \times SU(2))/(U(1))$ commonly known as T^{pq}.

If M_q is the round sphere S^q, it is easy to show that this mode is stable. In particular, the spacetimes of interest for bosonic M-theory, $AdS_4 \times S^{23}$ and $AdS_{23} \times S^4$, are stable. However, if $M_q = M_n \times M_{q-n}$ and $q < 9$, there is a mass violating the BF bound, corresponding to a mode which makes one factor grow while the other shrinks. This generalizes the instabilities of $AdS_4 \times M_n \times M_{7-n}$ and $AdS_7 \times S^2 \times S^2$, but also shows that this instability is limited to low dimensions. For $q \geq 9$, $AdS_p \times S^n \times S^{q-n}$ can be shown to be stable. The significance of the critical dimension $q = 9$ is not clear; it is sufficiently large that stable products cannot be realized in superstring/M-theory.

While the results for a sphere and product spaces are curious, there is a much stronger motivation for seriously investigating the question of stability of $AdS_5 \times T^{pq}$ compactifications. It is well-understood that the $AdS_5 \times T^{11}$ case is stable, since it is the supersymmetric near-horizon limit of D3-branes on a conifold [75]. Also, $AdS_5 \times T^{kk}$ must be stable (at least classically) since it is a smooth, supersymmetry-breaking $\mathbf{Z}_k$ quotient of T^{11}, as we shall review further below. On the other hand, T^{01} is a direct product space, $S^2 \times S^3$, and as we have already alluded, such product geometries are always unstable toward inflating one factor while deflating the other, provided the total number of compact dimensions is less than nine [71, 10]. The question then is whether T^{pq} is stable for some range of p/q close to 1, as occurs in the M^{pqr} case. For instance, there is an infinite family of compactifications of M-theory, $AdS_4 \times M^{pqr}$, which are stable but non-supersymmetric [76]. (M^{pqr} is a homogeneous Einstein 7-manifold describable as a coset space $(SU(3) \times SU(2) \times U(1))/(SU(2) \times U(1) \times U(1))$. The integers p, q, and r describe the embedding of $SU(2) \times U(1) \times U(1)$ in $SU(3) \times SU(2) \times U(1)$. In most cases the symmetry group of these spaces is $SU(3) \times SU(2) \times U(1)$.[2] For a certain range of p, q, and r, there is a BF-violating tachyon, and for the complementary range there is not.) In fact it is not: we shall demonstrate that all T^{pq} for $p \neq q$ are unstable by constructing the unstable mode explicitly. To a large extent this dashes the hope that renormalization group flows from the simplest $\mathbf{Z}_2$ orbifold of four-dimensional $\mathcal{N} = 4$ gauge theory might include infinitely many infrared fixed points.[3] (There is still the possibility that stable solutions exist with topology $S^2 \times S^3$ and with three-form field strengths and non-trivial dilaton: however these seem much more difficult to find).

The theories that we just discussed involve only gravity and a form field. There are physically interesting theories with an additional level of complexity – the presence of a scalar potential present at the classical level in addition to the above fields. Because of this, the results that we just stated above do not hold and a separate check has to be made. In the string theory context, the scalar could be a dilaton and the presence of the potential can be thought of as a cosmological term at the classical level. Two examples immediately come to mind – massive IIA supergravity and Sugimoto's $USp(32)$ open string theory [77].

The massive type IIA supergravity has a nonsupersymmetric vacuum of the form $AdS_4 \times$

[2]The two exceptions are $M^{101} = S^5 \times S^2$ and $M^{011} = CP^2 \times S^3$.

[3]The M^{pqr} manifolds are topologically distinct from one another for different values of $\frac{p}{q}$, so there is no question of whether one could flow from one to another.

S^6 [78], whose stability, to our knowledge, had never been investigated before. We show that the solution is unstable, with two modes violating the BF bound. To our knowledge, this is the first example of a theory where the product of AdS and a round sphere is unstable. The analysis is more involved here since there is a dilaton which mixes with some of the other modes, further complicating the coupled sector. Instabilities for more general $AdS_4 \times M_6$ can arise in several ways, but we show in particular that they do occur for $AdS_4 \times S^n \times S^{6-n}$.

Sugimoto's $USp(32)$ open string theory [77] also suffers the same fate. The obvious Freund-Rubin compactification for this theory $AdS_3 \times S^7$ is unstable. The reason to be interested in this vacuum is that it is the near-horizon limit of many coincident D1-branes in this theory. For the usual $SO(32)$ open string, the D1-brane turns out to be a non-perturbative construction of the $SO(32)$ heterotic string [79, 80]. There is no perturbative $USp(32)$ heterotic string in ten dimensions: the $USp(32)$ current algebra is too big to admit unitary representations with $c \leq 16$. Correspondingly, it is perhaps satisfying that we find fields which violate the BF bound in the $AdS_3 \times S^7$ vacuum of Sugimoto's theory. One may perhaps draw the general conclusion that string theories which are non-supersymmetric in their perturbative construction can suffer non-perturbative instabilities which prevent them from participating in weak-strong coupling dualities.

There is a vast literature on Kaluza-Klein theories, much of it in the context of higher dimensional supergravity, including a comprehensive review [81]. Our treatment of the harmonic analysis of fluctuations about $AdS_p \times M_q$ is most closely modeled on [82, 83, 84], and we have also consulted [71] and [85]. In Section 3.2 we present the general $AdS_p \times M_q$ background solution. The harmonic expansions for fluctuations and their linear equations of motion are discussed in Section 3.3. The mass spectra of the various fluctuations are analyzed in Sections 3.4-3.8. The more complicated case of massive type IIA supergravity and Sugimoto's theory is discussed in Section 3.9. The AdS_p mass spectra determine the dimensions of operators in hypothetical CFT_{p-1} dual field theories, and this is discussed in Section 3.10. In Section 3.11, we show that for some of the the unstable cases, the total energy (in the full nonlinear theory) is unbounded from below. We also speculate on the implications of our results for the stability of certain extremal black brane solutions. Conventions and properties of various differential operators are collected in an appendix.

3.2 Freund-Rubin Backgrounds

We start by considering classical $D = p + q$ dimensional gravity theory coupled to a q-form field strength. The action is given by:

$$S = \int d^p x d^q y \sqrt{-g} \left(R - \frac{1}{2q!} F_q^2 \right), \tag{3.1}$$

which leads to the equations of motion

$$R_{MN} = \frac{1}{2(q-1)!} F_{MP_2 \cdots P_q} F_N{}^{P_2 \cdots P_q} - \frac{(q-1)}{2(D-2)q!} g_{MN} F_q^2 . \tag{3.2}$$

$$d * F_q = 0 , \tag{3.3}$$

This theory supports a Freund-Rubin solution with the product metric

$$ds^2 = ds^2_{AdS_p} + ds^2_{M_q} , \tag{3.4}$$

describing a product of p-dimensional anti-de Sitter space with an Einstein manifold:

$$R_{\mu\nu} = -\frac{(p-1)}{L^2} g_{\mu\nu} , \tag{3.5}$$

$$R_{\alpha\beta} = \frac{(q-1)}{R^2} g_{\alpha\beta} , \tag{3.6}$$

and a background field strength on the compact space:

$$F_q = c \, \text{vol}_{M_q} . \tag{3.7}$$

We use $M, N, \ldots$ for indices on the full D-dimensional spacetime, while $\mu, \nu, \ldots$ are indices on AdS and $\alpha, \beta, \ldots$ are indices on M_q. The equations of motion (3.2), (3.3) relate the length scales L and R and the constant c:

$$c^2 = \frac{2(D-2)(q-1)}{(p-1)R^2} , \tag{3.8}$$

$$L = \frac{p-1}{q-1} R . \tag{3.9}$$

In the following six sections we shall study fluctuations of g_{MN} and F_q around this background. Among other things, we will conclude that the background is stable against these fluctuations when $M_q = S^q$, for arbitrary $p > 2$ and $q > 1$. If one wishes to embed the action (3.1) in a larger theory with additional fields, stability must be verified separately for the new modes. However let us note that the most tachyonic modes in the well-studied vacua of ten- and eleven-dimensional supergravities generally come from precisely the fields which support the solution. Thus, when these most "dangerous" modes come out stable, it suggests that the background is probably stable against all fluctuations.

3.3 Linearized equations of motion

3.3.1 Fluctuations

We are interested in studying the stability of linearized fluctuations around the background (3.4), (3.7). As we have discussed, anti-de Sitter space is stable even in the presence of tachyonic scalar fields, as long as their masses do not violate the Breitenlohner-Freedman bound:

$$m^2L^2 \geq -\frac{(p-1)^2}{4}\,. \tag{3.1}$$

The possibility that some tachyons could be acceptable in AdS_4 was first pointed out by Breitenlohner and Freedman [70], and extended to AdS_p by [86]. See also [87, 88] for early developments of this idea.

We consider the linearized fluctuations

$$\delta g_{\mu\nu} = h_{\mu\nu} = H_{\mu\nu} - \frac{1}{p-2} g_{\mu\nu} h^\alpha_\alpha\,, \tag{3.2}$$

$$\delta g_{\mu\alpha} = h_{\mu\alpha}\,, \qquad \delta g_{\alpha\beta} = h_{\alpha\beta}\,, \qquad \delta A_{q-1} = a_{q-1}\,, \quad \delta F_q \equiv f_q = da_{q-1}\,, \tag{3.3}$$

where we have defined a standard linearized Weyl shift on $h_{\mu\nu}$ in (3.2), and $F_q = dA_{q-1}$. It will be useful to decompose $H_{\mu\nu}$ and $h_{\alpha\beta}$ into trace and traceless parts:

$$H_{\mu\nu} = H_{(\mu\nu)} + \frac{1}{p} g_{\mu\nu} H^\rho_\rho\,, \qquad h_{\alpha\beta} = h_{(\alpha\beta)} + \frac{1}{q} g_{\alpha\beta} h^\gamma_\gamma\,, \tag{3.4}$$

where $g^{\mu\nu}H_{(\mu\nu)} = g^{\alpha\beta}h_{(\alpha\beta)} = 0$. To (mostly[4]) fix the internal diffeomorphisms and gauge freedom, we impose the de Donder-type gauge conditions

$$\nabla^\alpha h_{(\alpha\beta)} = \nabla^\alpha h_{\alpha\mu} = 0, \tag{3.5}$$

as well as the Lorentz-type conditions

$$\nabla^\alpha a_{\alpha\beta_2\ldots\beta_{q-1}} = \nabla^\alpha a_{\alpha\beta_2\ldots\beta_{q-2}\mu} = \cdots = \nabla^\alpha a_{\alpha\mu_2\ldots\mu_{q-1}} = 0. \tag{3.6}$$

A generic gauge potential $a_{\alpha_1\ldots\alpha_n\mu_{n+1}\ldots\mu_{q-1}}$, viewed as an n-form on M_q with additional AdS_p indices, can be expanded as the sum of an exact, a co-exact and a harmonic form on M_q by the Hodge decomposition theorem. The Lorentz conditions (3.6), which state that the form is co-exact, require the exact form in the decomposition to vanish, and hence the potentials can be expanded as co-exact forms (curls) and harmonic forms:

$$a_{\beta_1\ldots\beta_n\mu_{n+1}\mu_{q-1}} = \epsilon^{\alpha_1\alpha_2\ldots\alpha_{q-n}}{}_{\beta_1\ldots\beta_n}\nabla_{\alpha_1}b_{\alpha_2\ldots\alpha_{q-n}\mu_{n+1}\ldots\mu_{q-1}} + \beta^{harm}_{\beta_1\ldots\beta_n\mu_{n+1}\mu_{q-1}}. \tag{3.7}$$

When the compact space is an S^q there are no nontrivial harmonic forms, but they can appear for other M_q. In a compact notation, we may write (3.6) and (3.7) as

$$d_q *_q a = 0 \rightarrow a = *_q d_q b + \beta^{harm}, \tag{3.8}$$

where d_q and $*_q$ are the exterior derivative and Hodge dual with respect to the M_q space only.

With these gauge choices, we may expand the fluctuations in spherical harmonics as

$$\begin{aligned} H_{(\mu\nu)}(x,y) &= \sum_I H^I_{(\mu\nu)}(x)Y^I(y), \qquad H^\mu_\mu(x,y) = \sum_I H^I(x)Y^I(y), &(3.9)\\ h_{(\alpha\beta)}(x,y) &= \sum_I \phi^I(x)Y^I_{(\alpha\beta)}(y), \qquad h^\alpha_\alpha(x,y) = \sum_I \pi^I(x)Y^I(y), &(3.10)\\ h_{\mu\alpha}(x,y) &= \sum_I B^I_\mu(x)Y^I_\alpha(y), &(3.11)\\ a_{\beta_1\ldots\beta_{q-1}} &= \sum_I b^I(x)\,\epsilon^\alpha{}_{\beta_1\ldots\beta_{q-1}}\nabla_\alpha Y^I(y), &(3.12) \end{aligned}$$

[4] Besides unfixed p-dimensional diffeomorphisms and gauge transformations, extra conformal diffeomorphisms remain on S^q. These are related to the elimination of a $k = 1$ mode in the coupled scalar sector, as in section 3.4; for a discussion, see [83].

$$a_{\mu\beta_2\ldots\beta_{q-1}} = \sum_I b^I_\mu(x)\,\epsilon^{\alpha\beta}{}_{\beta_2\ldots\beta_{q-1}}\nabla_{[\alpha}Y^I_{\beta]}(y) + \sum_h \beta^h_\mu(x)\,\epsilon^{\alpha\beta}{}_{\beta_2\ldots\beta_{q-1}}Y^h_{[\alpha\beta]}\,, \quad (3.13)$$

$$\vdots$$

$$a_{\mu_1\ldots\mu_{q-1}} = \sum_I b^I_{\mu_1\ldots\mu_{q-1}}(x)Y^I(y)\,, \quad (3.14)$$

where I in each case is a generic label running over the possible spherical harmonics of the appropriate tensor type, and $h = 1\ldots b^n(M_q)$ runs over the harmonic n-forms on M_q for the gauge field with $(n-1)$ AdS_p indices. We have not included a term $\beta(x)$ in (3.12) since compact Riemannian Einstein spaces with positive curvature cannot possess harmonic one-forms; this is proved in the appendix. We will also find it convenient to define

$$b(x,y) \equiv \sum_I b^I(x)Y^I(y)\,, \qquad b_{\mu\alpha}(x,y) \equiv \sum_I b^I_\mu(x)Y^I_\alpha(y)\,. \quad (3.15)$$

3.3.2 Einstein equations and coupled form equations

We now consider the Einstein equations to linear order in fluctuations, as well as the form equations that mix with the graviton; the uncoupled form equations will be treated in section 3.7. We use the following notation: $\Box_x \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$, $\Box_y \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, and $\mathrm{Max}\, B_\mu \equiv \Box_x B_\mu - \nabla^\nu\nabla_\mu B_\nu$ is the Maxwell operator acting on vectors on AdS_q. Additionally, $\Delta_y \equiv -(d_q^\dagger d_q + d_q d_q^\dagger)$ is the Laplacian[5] acting on differential forms on M_q; for vectors, the explicit form is $\Delta_y Y_\alpha \equiv \Box_y Y_\alpha - R_\alpha{}^\beta Y_\beta$. Further, $f\cdot\epsilon \equiv f_{\alpha_1\cdots\alpha_q}\epsilon^{\alpha_1\cdots\alpha_q}/q!$.

For convenience, we present the linearized Ricci tensor in our conventions:

$$\begin{aligned} R^{(1)}_{MN} = -\frac{1}{2}[(\Box_x+\Box_y)h_{MN} + \nabla_M\nabla_N h^P_P - \nabla_M\nabla^P h_{PN} - \nabla_N\nabla^P h_{PM} \\ - 2R_{MPQN}h^{PQ} - R_M^P h_{NP} - R_N^P h_{MP}]\,. \end{aligned} \quad (3.16)$$

We employ Einstein's equations in their Ricci form, $R_{MN} = \bar{T}_{MN}$ with $\bar{T}_{MN} \equiv T_{MN} + \frac{1}{2-D}g_{MN}T^P_P$. For $R_{\mu\nu}$ we find

$$\begin{aligned} R^{(1)}_{\mu\nu} = -\frac{1}{2}[(\Box_x+\Box_y)(H_{\mu\nu} - \frac{1}{p-2}g_{\mu\nu}h^\gamma_\gamma) + \nabla_\mu\nabla_\nu(H^\rho_\rho - \frac{2}{p-2}h^\gamma_\gamma) \\ -\nabla_\mu\nabla^\rho(H_{\rho\nu} - \frac{1}{p-2}g_{\rho\nu}h^\gamma_\gamma) - \nabla_\nu\nabla^\rho(H_{\rho\mu} - \frac{1}{p-2}g_{\rho\mu}h^\gamma_\gamma) - 2R_{\mu\rho\sigma\nu}(H^{\rho\sigma} - \frac{1}{p-2}g^{\rho\sigma}h^\gamma_\gamma) \\ -R_\mu{}^\rho(H_{\rho\nu} - \frac{1}{p-2}g_{\rho\nu}h^\gamma_\gamma) - R_\nu{}^\rho(H_{\rho\mu} - \frac{1}{p-2}g_{\rho\mu}h^\gamma_\gamma)]\,, \end{aligned} \quad (3.17)$$

[5]The negative sign is standard in the Kaluza-Klein literature.

which must be equal to

$$\bar{T}_{\mu\nu}^{(1)} = -\frac{c^2(q-1)}{2(D-2)}h_{\mu\nu} - \frac{q(q-1)c^2}{2(D-2)q!}g_{\mu\nu}(-h^{\alpha\beta})\epsilon_{\alpha\gamma_2\cdots\gamma_q}\epsilon_\beta{}^{\gamma_2\cdots\gamma_q} - \frac{c(q-1)}{(D-2)}g_{\mu\nu}\,(f\cdot\epsilon)\,, \tag{3.18}$$

resulting in the equation

$$\begin{aligned}&-\frac{1}{2}[(\Box_x+\Box_y)H_{\mu\nu} + \nabla_\mu\nabla_\nu H^\rho_\rho - \nabla_\mu\nabla^\rho H_{\rho\nu} - \nabla_\nu\nabla^\rho H_{\rho\mu} - 2R_{\mu\rho\sigma\nu}H^{\rho\sigma} - R_\mu{}^\rho H_{\rho\nu} - R_\nu{}^\rho H_{\rho\mu}]\\ &\quad+\frac{1}{2(p-2)}g_{\mu\nu}(\Box_x+\Box_y)h^\gamma_\gamma - \frac{(q-1)^2}{(p-2)R^2}g_{\mu\nu}h^\gamma_\gamma + \frac{(q-1)^2}{(p-1)R^2}H_{\mu\nu} + \frac{q-1}{D-2}g_{\mu\nu}\Box_y\, cb = 0\,.\end{aligned} \tag{3.19}$$

For linearized $R_{\mu\alpha}$, we find

$$R_{\mu\alpha}^{(1)} = -\frac{1}{2}[\Box_x h_{\mu\alpha} - \nabla_\mu\nabla^\nu h_{\nu\alpha} - R_\mu{}^\nu h_{\nu\alpha} + \Box_y h_{\mu\alpha} - R_\alpha{}^\beta h_{\beta\mu} \tag{3.20}$$

$$-\nabla^\nu_\alpha h_{\nu\mu} + \nabla_\mu\nabla_\alpha(H^\rho_\rho - \frac{2}{p-2}h^\gamma_\gamma) - \nabla_\mu\nabla^\beta h_{\beta\alpha}]\,. \tag{3.21}$$

which is sourced by

$$\bar{T}_{\mu\alpha}^{(1)} = \frac{c}{2(q-1)!}f_{\mu\beta_2\cdots\beta_q}\epsilon_\alpha{}^{\beta_2\cdots\beta_q} - \frac{c^2(q-1)}{2(D-2)}h_{\mu\alpha} \tag{3.22}$$

$$= \frac{c}{2}\nabla_\mu\nabla_\alpha b + \frac{c}{2}(\Box_y b_{\mu\alpha} - R_\alpha{}^\beta b_{\mu\beta}) - \frac{c^2(q-1)}{2(D-2)}h_{\mu\alpha}\,. \tag{3.23}$$

For $R_{\alpha\beta}$ we have

$$\begin{aligned}R_{\alpha\beta}^{(1)} &= -\frac{1}{2}[(\Box_x+\Box_y)h_{(\alpha\beta)} - 2R_{\alpha\gamma\delta\beta}h^{(\gamma\delta)} - R_\alpha{}^\gamma h_{(\gamma\beta)} - R_\beta{}^\gamma h_{(\gamma\alpha)}\\ &+\frac{1}{q}g_{\alpha\beta}(\Box_x+\Box_y)h^\gamma_\gamma - (\frac{2}{q}+\frac{2}{p-2})\nabla_\alpha\nabla_\beta h^\gamma_\gamma + \nabla_\alpha\nabla_\beta H^\mu_\mu - \nabla_\alpha\nabla^\mu h_{\mu\beta} - \nabla_\beta\nabla^\mu h_{\mu\alpha}]\,.\end{aligned} \tag{3.24}$$

while on the right-hand side, we find

$$\begin{aligned}\bar{T}_{\alpha\beta}^{(1)} &= \frac{c}{2(q-1)!}(f_{\alpha\gamma_2\cdots\gamma_q}\epsilon_\beta{}^{\gamma_2\cdots\gamma_q} + f_{\beta\gamma_2\cdots\gamma_q}\epsilon_\alpha{}^{\gamma_2\cdots\gamma_q}) + \frac{c^2(q-1)}{2(q-1)!}(-h^{\gamma\delta})\epsilon_{\alpha\gamma\theta_3\cdots\theta_q}\epsilon_{\beta\delta}{}^{\theta_3\cdots\theta_q}\\ &-\frac{c^2(q-1)}{2(D-2)}(h_{(\alpha\beta)} + \frac{1}{q}g_{\alpha\beta}h^\gamma_\gamma) - \frac{c(q-1)}{(D-2)}g_{\alpha\beta}(f\cdot\epsilon) - \frac{q(q-1)c^2}{2(D-2)q!}g_{\alpha\beta}(-h^{\gamma\delta})\epsilon_{\gamma\theta_2\cdots\theta_q}\epsilon_\delta{}^{\theta_2\cdots\theta_q}\\ &= \frac{p-1}{D-2}g_{\alpha\beta}\Box_y\, cb + \frac{q-1}{R^2}h_{(\alpha\beta)} - \frac{(q-1)^2}{qR^2}g_{\alpha\beta}h^\gamma_\gamma\,,\end{aligned} \tag{3.25}$$

where we have used $(f\cdot\epsilon) = \Box_y b$ and $f_{\alpha_1\cdots\alpha_q} = (f\cdot\epsilon)\epsilon_{\alpha_1\cdots\alpha_q} = \epsilon_{\alpha_1\cdots\alpha_q}\Box_y b$.

We see that the modes of the graviton mix with the form modes b and b_μ. To solve the coupled systems, we must consider certain form equations as well. From the $\nabla^M F_{M\beta_2\ldots\beta_q}$

equation[6] , we find the expression

$$\nabla^M f_{M\beta_2\cdots\beta_q} - cg^{\mu\nu}\Gamma^{\gamma(1)}_{\mu\nu}\epsilon_{\gamma\beta_2\cdots\beta_q} - cg^{\gamma\delta}\Gamma^{\theta(1)}_{\gamma\delta}\epsilon_{\theta\beta_2\cdots\beta_q} - c(q-1)g^{\gamma\delta}\Gamma^{\theta(1)}_{\gamma\beta_2}\epsilon_{\delta\theta\beta_3\cdots\beta_q} = 0\,. \tag{3.26}$$

where we use the linearized Christoffel symbol,

$$\Gamma^{P(1)}_{MN} = \frac{1}{2}\left(\nabla_M h^P_N + \nabla_N h^P_M - \nabla^P h_{MN}\right)\,. \tag{3.27}$$

Contracting with the epsilon tensor on M_q, (3.26) becomes

$$(q-1)!\left(\nabla_\alpha[(\Box_x+\Box_y)b + \frac{c}{2}H^\mu_\mu - \frac{c(p-1)}{p-2}h^\gamma_\gamma] + \nabla^\mu[\Box_y b_{\mu\alpha} - R^{\ \beta}_\alpha b_{\mu\beta} - ch_{\mu\alpha}]\right) = 0\,. \tag{3.28}$$

Finally, from the $\nabla^M F_{M\mu\beta_3\ldots\beta_q}$ equation,

$$\nabla^M f_{M\mu\beta_3\cdots\beta_q} - cg^{\gamma\alpha}\Gamma^{\delta(1)}_{\gamma\mu}\epsilon_{\alpha\delta\beta_3\cdots\beta_q} = 0\,, \tag{3.29}$$

which reduces to

$$\begin{aligned}(q-2)!\Big[\Big(\Box_x+\Box_y-\frac{2(q-1)}{R^2}\Big)\nabla_{[\alpha}b_{\beta]\mu} - \nabla^\nu\nabla_\mu\nabla_{[\alpha}b_{\beta]\nu} - c\nabla_{[\alpha}B_{\beta]\mu} + 2R_{\alpha\ \beta}^{\ \gamma\ \delta}\nabla_{[\gamma}b_{\delta]\mu}\Big] \\ -(q-2)!D_{\beta_3}D^\nu a_{\mu\nu\beta_4\cdots\beta_q}\epsilon_{\alpha\beta}^{\ \ \beta_3\cdots\beta_q} + (q-2)!(\Box_x\beta_\mu - \nabla^\nu\nabla_\mu\beta_\nu) = 0\,.\end{aligned} \tag{3.30}$$

We now expand these fields in spherical harmonics and collect like terms. Below we present the results, collecting related equations and indicating the origin of each expression as follows: (E1), (E2) and (E3) for the AdS, mixed and M_q Einstein equations, and (F1) and (F2) for the form equations (3.28) and (3.30), respectively.

Equations for the coupled scalars π^I, b^I and H^I:

$$\text{(E3)}\quad \left[\left(\Box_x+\Box_y-\tfrac{2(q-1)^2}{R^2}\right)\pi^I + \Box_y\left(H^I - \tfrac{2(D-2)}{q(p-2)}\pi^I\right) + \tfrac{2q(p-1)}{(D-2)}\Box_y cb^I\right]Y^I = 0\,, \tag{3.31}$$

$$\text{(E3)}\quad \left(H^I - \tfrac{2(D-2)}{q(p-2)}\pi^I\right)\nabla_{(\alpha}\nabla_{\beta)}Y^I = 0\,, \tag{3.32}$$

$$\text{(F1)}\quad \nabla_\alpha\left(\Box_x b^I + \Box_y b^I + \tfrac{c}{2}H^I - \tfrac{c(p-1)}{(p-2)}\pi^I\right)Y^I = 0\,, \tag{3.33}$$

Equations for coupled vectors b^I_μ, B^I_μ:

$$\text{(E2)}\quad \left(\text{Max } B^I_\mu + \Delta_y B^I_\mu + \Delta_y cb^I_\mu - \tfrac{2(q-1)^2}{(p-1)R^2}b^I_\mu\right)Y^I_\alpha = 0\,, \tag{3.34}$$

[6] One may avoid explicit manipulation of Christoffel symbols by linearizing the equivalent equation $\partial_M\sqrt{-g}F^{MN_2\cdots N_q} = 0$.

$$\text{(F2)} \qquad \nabla_{[\alpha}\left(\text{Max } b^I_\mu + \Delta_y b^I_\mu - cB^I_\mu\right) Y^I_{\beta]} = 0\,. \tag{3.35}$$

$$\text{(F1)} \qquad \left(\nabla^\mu b^I_\mu \Delta_y - c\nabla^\mu B^I_\mu\right) Y^I_\alpha = 0\,, \tag{3.36}$$

$$\text{(E3)} \qquad (\nabla^\mu B^I_\mu)\nabla_{(\alpha} Y^I_{\beta)} = 0\,, \tag{3.37}$$

Equations for symmetric tensors $H^I_{\mu\nu}$:

$$\text{(E1)} \qquad (R^{(1)}_{\mu\nu}(H^I_{\rho\sigma}) - \tfrac{1}{2}\Box_y H^I_{\mu\nu} + \tfrac{(q-1)^2}{(p-1)R^2} H^I_{\mu\nu} + \tag{3.38}$$
$$\tfrac{1}{2(p-2)} g_{\mu\nu}(\Box_x + \Box_y)\pi^I - \tfrac{(q-1)^2}{(p-2)R^2} g_{\mu\nu}\pi^I + \tfrac{(q-1)}{(D-2)} g_{\mu\nu}\Box_y cb^I)Y^I = 0\,,$$

$$\text{(E2)} \qquad \left(-\nabla^\nu H^I_{\nu\mu} + \nabla_\mu H^I - \tfrac{(p+q-2)}{q(p-2)}\nabla_\mu \pi^I + \nabla_\mu cb^I\right)\nabla_\alpha Y^I = 0\,, \tag{3.39}$$

Note that in (3.38), $R^{(1)}_{\mu\nu}$ is the linearized Ricci tensor for AdS_p only, evaluated on the field $H_{\rho\sigma}$. Finally, there remain a few decoupled equations:

$$\text{(E3)} \quad \left[(\Box_x + \Box_y)\delta^\gamma_\alpha \delta^\delta_\beta - 2R_\alpha{}^{\gamma\delta}{}_\beta\right]\phi^I Y^I_{(\gamma\delta)} = 0\,. \tag{3.40}$$

$$\text{(F2)} \qquad (\text{Max } \beta^h_\mu) Y^h_{[\alpha\beta]} = 0\,, \tag{3.41}$$

$$\text{(F2)} \qquad (\nabla^\nu b^I_{\nu\mu})\nabla_{[\alpha}\nabla_{\beta]} Y^I = 0\,. \tag{3.42}$$

Notice that in passing from (3.30) to (3.35), we commuted the $\Box_y$ through the covariant derivative ∇_α, which not only produced precisely the Laplacian Δ_y acting on vectors, but also canceled all terms in (3.30) involving the Riemann tensor.

It is worth remarking that as a result, the properties of M_q enter into almost all these formulas only through the dimension q and the radius R. Consequently we will be able to treat these equations in a completely unified way, and prove that for generic $AdS_p \times M_q$ backgrounds, all the corresponding modes satisfy the Breitenlohner-Freedman bound and cannot destabilize the background. The sole exception is the equation (3.40) for the scalars coming from graviton modes on the compact space, which explicitly involves the Riemann tensor on M_q. There is thus no guarantee that the modes ϕ^I will possess the uniform stability properties for different choices of M_q. Indeed, we will find that for $M_q = S^q$ these modes are harmlessly positive mass for all q, while for any product $M_q = M_n \times M_{q-n}$ with $q < 9$ they contain an instability.

3.4 Coupled scalars

In this section, we consider the system of modes associated with the coupled scalars π^I, b^I and H^I, equations (3.31), (3.32) and (3.33).

For certain low-lying scalar spherical harmonics Y^I, some or all of their derivatives appearing in the equations of section 3.3.2 may vanish. Let us first treat the generic case where all derivatives of Y^I in (3.31), (3.32) and (3.33) are nonzero and hence the coefficients must vanish. Equation (3.32) then gives us a constraint which may be used to eliminate H^I in favor of π^I. Substituting into equation (3.33), we find

$$\left((\Box_x + \Box_y) b^I - c\frac{(q-1)}{q}\pi^I\right) Y^I = 0, \tag{3.1}$$

while the second term in parentheses vanishes in (3.31). We obtain from (3.31) and (3.1) the coupled system

$$L^2 \Box_x \begin{pmatrix} b^I/c \\ \pi^I \end{pmatrix} = (p-1)^2 \begin{pmatrix} \frac{\lambda^I}{(q-1)^2} & \frac{R^2}{q(q-1)} \\ \frac{4q\lambda^I}{(q-1)R^2} & \frac{\lambda^I}{(q-1)^2} + 2 \end{pmatrix} \begin{pmatrix} b^I/c \\ \pi^I \end{pmatrix}, \tag{3.2}$$

where $\Box_y Y^I = -\lambda^I Y^I / R^2$; that $\lambda^I \geq 0$ is straightforward and is shown in the appendix. On diagonalizing this matrix we obtain the mass spectrum

$$m^2 L^2 = \frac{(p-1)^2}{(q-1)^2}[\lambda + (q-1)(q-1 \pm \sqrt{4\lambda + (q-1)^2})]. \tag{3.3}$$

We now wish to analyze the spectrum (3.3) to check stability. Extrema of (3.3) occur for

$$1 \pm 2(q-1)(4\lambda + (q-1)^2)^{-1/2} = 0, \tag{3.4}$$

To satisfy (3.4) we must choose the negative sign, and we find a minimum at

$$\lambda = \frac{3}{4}(q-1)^2. \tag{3.5}$$

Substituting into (3.3), we find the elegant result that the minimum mass of the negative branch exactly saturates the Breitenlohner-Freedman bound independent of p and q,

$$m_{min}^2 L^2 = -\frac{1}{4}(p-1)^2 = m_{BF}^2 L^2. \tag{3.6}$$

Since the positive branch leads to manifestly positive masses, we have proven there can be no unstable modes in this sector, at least for modes associated to generic spherical harmonics. We shall complete the proof by treating the special cases momentarily.

Although the spectrum (3.3) always saturates the BF bound as a smooth function of λ, there need not be physical states at the minimum, since only discrete values of λ appear for given M_q. If $M_q = S^q$, then the eigenvalues of the spherical harmonics are $\lambda = k(k+q-1)$, for integer $k \geq 0$, and the mass formulas for the two branches take on the form

$$m_-^2 L^2 = \frac{(p-1)^2}{(q-1)^2} k(k-q+1)\,, \qquad m_+^2 L^2 = \frac{(p-1)^2}{(q-1)^2}(k+2(q-1))(k+q-1)\,. \tag{3.7}$$

The minimum (3.5) occurs for S^q at $k = (q-1)/2$ in the minus branch. We notice that whenever q is odd, there will be a mode with precisely the Breitenlohner-Freedman mass, while for q even the lightest-mass states from this sector will appear just above the bound. This is consistent with what is already known about $AdS_4 \times S^7$ and $AdS_7 \times S^4$ [85, 82, 84].

Let us now examine the special cases. For $k = 1$ on S^q, $\nabla_{(\alpha}\nabla_{\beta)} Y^I = 0$ and we cannot use (3.32); this only occurs for maximally symmetric spaces, and hence is not a concern for other M_q, where nonconstant Y^I can be treated as above. Following [84] we may deal with this in one of two ways: either using a residual gauge invariance to impose the constraint anyway, or explicitly evaluating the remaining equations and showing that one linear combination drops out. We shall do the latter; for a discussion of the former, see [83].

We now consider equation (3.33) as a constraint to eliminate H^I in favor of π^I and b^I. The remaining equation (3.31) becomes

$$\left(\Box_x + \frac{3q-2}{q}\Box_y - \frac{2(q-1)^2}{R^2}\right)\pi^I - \frac{R^2(p-1)}{(q-1)(D-2)}\left(\Box_x + \Box_y - \frac{2q(q-1)}{R^2}\right)\Box_y c b^I = 0\,. \tag{3.8}$$

In the case of the sphere, $\Box_y = -q/R^2$ and we find an equation for a single mode,

$$\left(\Box_x - \frac{q(2q-1)}{R^2}\right)\left(\pi^I + \frac{q(p-1)}{(q-1)(D-2)} c b^I\right) = 0\,, \tag{3.9}$$

which has the same mass as one would obtain from naively substituting $k = 1$ into the positive branch of (3.7).

For constant Y^I on any M_q, all derivatives of Y^I vanish and the only nontrivial equation is (3.31), which reduces to

$$\left(\Box_x - \frac{2(q-1)^2}{R^2}\right)\pi^I = 0, \tag{3.10}$$

where again the mass matches what one obtains by substituting $k = \lambda = 0$ into the positive branch of (3.3). Thus we learn that a proper treatment extends the positive branch of (3.3) down to $k = 0$, while the negative branch truncates at $k = 2$ for S^q and $k = 1$ for other M_q.

The only remaining scalar fields associated to modes of the graviton are the ϕ^I, which obey the uncoupled equation (3.40). These shall turn out to be the modes that can threaten stability. We shall return to these in section 3.8.

3.5 Coupled vectors

We now consider the graviphoton B_μ and the form mode b_μ with which it mixes. We expect to find a massless vector for each Killing vector on M_q as well as a tower of massive fields, and indeed this is what we obtain. An additional $b^2(M_q)$ massless vectors arise from the gauge potential, where $b^2(M_q)$ is the second Betti number.

The relevant equations are (3.34), (3.35), (3.36), and (3.37). One readily sees that (3.36) can be obtained from the divergence of (3.35). We obtain the following coupled system from (3.34) and (3.35):

$$L^2 \text{ Max} \begin{pmatrix} cb^I_\mu \\ B^I_\mu \end{pmatrix} = (p-1)^2 \begin{pmatrix} \frac{\kappa^I}{(q-1)^2} & \frac{2(D-2)}{(p-1)(q-1)} \\ \frac{\kappa^I}{(q-1)^2} & \frac{\kappa^I}{(q-1)^2} + \frac{2}{p-1} \end{pmatrix} \begin{pmatrix} cb^I_\mu \\ B^I_\mu \end{pmatrix}, \tag{3.1}$$

where $\Delta_y Y^I_\alpha = -\kappa^I Y^I_\alpha / R^2$. The masses that result are

$$m^2 L^2 = \frac{(p-1)^2}{(q-1)^2}\kappa + (p-1)\left(1 \pm \sqrt{1 + 2\frac{p-1}{(q-1)^3}(D-2)\kappa}\right). \tag{3.2}$$

On a general Einstein space, we may derive the bound $\kappa^I \geq 2(q-1)$, with equality when Y^I_α is a Killing vector, by considering $\int d^q y\, S_{\alpha\beta} S^{\alpha\beta} \geq 0$ with $S_{\alpha\beta} \equiv \nabla_\alpha Y_\beta + \nabla_\beta Y_\alpha$ (see appendix). For these Killing modes, the masses on the negative branch of (3.2) vanish. Hence we do indeed find a massless vector for each isometry of the compact space M_q. For Killing modes (3.37) is trivially satisfied and does not constrain the vector fields.

The positive branch for $\kappa = 2(q-1)$ yields a positive mass, and one can show that for both branches (3.2) monotonically increases with κ for $\kappa \geq 2(q-1)$. Thus all vector modes are either massless or have positive mass. For the non-Killing modes (3.36) and (3.37) provide the usual divergence-free condition for massive vectors, while for the massive modes associated to the Killing vectors (3.36) accomplishes this by itself.

When the cohomology $H^2(M_q)$ is nontrivial, harmonic 2-forms $Y^h_{[\alpha\beta]}$ give rise to $b^2(M_q)$ additional massless vectors β^h_μ, as we see from equation (3.41).

3.6 Graviton and tensor fields

We now establish the existence of the p-dimensional graviton and demonstrate the stability of the tower of massive symmetric two-index tensors. The graviton comes from the constant Y^I mode of equation (3.38). Using (3.10), this reduces to

$$R^{(1)}_{\mu\nu}(H^I_{\rho\sigma}) + \frac{p-1}{L^2} H^I_{\mu\nu} = 0\,, \tag{3.1}$$

which is the correct fluctuation equation for a linearized graviton in AdS_p.

For generic Y^I, the trace and longitudinal parts of (3.38) are redundant given (3.31), (3.32) (3.33), and (3.39), which express the trace and divergence of $H_{\mu\nu}$ in terms of π and b. One can use these equations to reduce (3.38) to

$$\left[(\Box_x + \Box_y + \frac{2}{L^2})H^I_{(\mu\nu)} - 2\nabla_{(\mu}\nabla_{\nu)}cb^I\right] Y^I = 0\,. \tag{3.2}$$

A massive tensor field of mass m^2 is described by a field $\phi_{(\mu\nu)}$ which satisfies the wave equation and transversality constraints

$$(\Box_x - m^2)\varphi_{(\mu\nu)} = 0\,, \tag{3.3}$$

$$\nabla^\mu \varphi_{(\mu\nu)} = 0\,. \tag{3.4}$$

To bring (3.2) to this form, we follow [84]. Define $\phi_{(\mu\nu)}$ in terms of $H_{(\mu\nu)}$ by

$$H_{(\mu\nu)} = \phi_{(\mu\nu)} + \nabla_{(\mu}\nabla_{\nu)}(ub + v\pi)\,, \tag{3.5}$$

where u and v are constants which can be determined by the following procedure, which we outline without full detail. The first step is to substitute (3.5) into (3.2) and require that $\phi_{(\mu\nu)}$ satisfy (3.3) with mass $m_I^2 = \lambda^I/R^2 - 2/L^2$, where $-\lambda^I/R^2$ is as usual the eigenvalue of $\Box_y$ on Y^I. The remaining terms are required to cancel which gives one condition to determine u and v. The second condition is obtained by applying ∇^μ to (3.5). The left side is expressed in terms of b and π using (3.32) and (3.39), and one imposes (3.4). After commuting covariant derivatives, one finds two scalar conditions. Both contain the term $\Box_x(ub + v\pi)$ which may be eliminated between them. The constants u and v may then be obtained by requiring that coefficients of the independent fields $b(x)$ and $\pi(x)$ vanish. The results are

$$u = \frac{2c(D-2)(p-2)}{(q-1)L^2(\frac{\lambda^I}{R^2} - \frac{p-2}{L^2})} \tag{3.6}$$

$$v = -\frac{D-2}{q(p-1)(\frac{\lambda^I}{R^2} - \frac{p-2}{L^2})}. \tag{3.7}$$

Strictly speaking the argument above does not apply to the $k = 1$ graviton mode on S^q since it uses the constraint (3.32) which no longer follows from the Einstein equations. The simplest way to extend the argument is to use the unfixed conformal diffeomorphisms discussed in [84] to impose the constraint for $k = 1$. The argument then goes through unchanged.

The apparent tensor mass m_I^2 is not positive for all geometries $AdS_p \times M_q$. However [84] one can examine $R_{\mu\nu}^{(1)}$ in (3.1) to see that the graviton itself has an apparent mass $-2/L^2$. When this is subtracted one sees that higher tensor modes have positive mass λ^I/R^2. These modes transform in unitary representations of the AdS_p isometry group, and we have stability.

3.7 Uncoupled form fluctuations

As we saw, the gauge potentials with zero and one indices on AdS_p mix with the graviton scalars and vectors. The remaining form fields are decoupled. It is easiest to treat them using a differential form notation. Thanks to the gauge condition (3.8), these may be written

$$a(x,y) = \sum_I b^I(x) *_q d_q Y^I(y) + \sum_h \beta^h(x) Y^h(y)\,. \tag{3.1}$$

The linearized equation of motion is simply

$$d * da = 0\,. \tag{3.2}$$

Consider first the form $Y^I(y)$ with $n \geq 2$ indices on M_q; the field b^I then has n indices on AdS_p. Evaluating (3.2) and using the identities $*(A_m(x)B_n(y)) = (-1)^{n(p-m)} *_p (A_m) *_q (B_n)$ and $d_q *_q Y^I = 0$, we arrive at the equations

$$(d_p *_p d_p b^I) d_q Y^I + (-1)^{n^2} (*_p b^I) d_q \Delta_y Y^I = 0\,, \tag{3.3}$$

$$(d_p *_p b^I) \Delta_y Y^I = 0\,. \tag{3.4}$$

Equation (3.4) already appeared for the form with 2 indices on AdS_p as (3.42). It follows from (3.4) that (3.3) reduces to

$$\left(\Delta_x - \frac{\kappa^I}{R^2}\right) b^I = 0\,, \tag{3.5}$$

where Δ_x is the Laplacian on AdS_p and $\Delta_y Y^I = -\kappa^I Y^I / R^2$ is the eigenvalue of the Laplacian on M_q, as before. Thus these are standard positive-mass modes resulting from the dimensional reduction.

The harmonic modes are even simpler; we find

$$(d_p *_p \beta^h) Y^h = 0\,. \tag{3.6}$$

Thus we have a massless form of appropriate rank for each cohomology class, as expected.

One potential modification of the action (3.1) is the addition of a Chern-Simons term

$$\Delta S \sim \int A_{q-1} \wedge (F_q)^n\,, \tag{3.7}$$

where the wedge product is understood. Naturally, this is only possible when q is even, and when an integer n satisfying $nq = p + 1$ can be found. (For $p = 23$, $q = 4$, one may add a CS term with $n = 6$.) Notice that such a term breaks the duality symmetry between a theory with F_q, which we have used, and a dual F_p; results for the rest of this chapter would be identical had we used F_p, but not in this instance. The modified action (3.7) leaves Einstein's equations unchanged, and modifies the form equation to

$$d * F_q = \gamma (F_q)^n \,, \tag{3.8}$$

for some constant γ. In supersymmetric theories like 11-dimensional supergravity, the constant γ is fixed by supersymmetry. Absent supersymmetry or some other guiding principle, there is no preferred choice of γ. For $n \geq 2$ our solution (3.4), (3.7) is still valid since $F_q \wedge F_q$ vanishes. (For $n = 1$, on the other hand, the Freund-Rubin background is not a solution.) Because $F_q \wedge F_q$ vanishes, (3.8) will begin to differ from (3.3) only at the $n-1$ order in perturbations. Hence, our linearized analysis will only be affected if $n = 2$. Furthermore, for $f_q \wedge F_q$ to be nonzero, the fluctuation f_q must be polarized entirely along AdS_p. Hence, the addition of the term (3.7) can affect our analysis for only the single mode (3.14). We find the equation

$$(\Delta_x + \Delta_y - 2c\gamma *_p d_p)\, b^I Y^I = 0 \,. \tag{3.9}$$

We notice that $(*_p d_p)^2 = \Delta_x$ (for dimensions where (3.7) is possible). We can thus factorize (3.9) into

$$\begin{gathered}(*_p d_p + m_1)(*_p d_p + m_2)\, b^I Y^I = 0 \,, \\ m_1 + m_2 = -2c\gamma \,, \qquad m_1 m_2 = -\kappa/R^2 \,,\end{gathered} \tag{3.10}$$

with the solution

$$m_1 = -c\gamma + \sqrt{c^2\gamma^2 + \frac{\kappa}{R^2}} \,, \qquad m_2 = -c\gamma - \sqrt{c^2\gamma^2 + \frac{\kappa}{R^2}} \,. \tag{3.11}$$

There will be two towers, one annihilated by each of the factors in (3.10). The second-order equations are

$$(\Delta_x - m_i^2)\, b^I Y^I = 0 \,, \tag{3.12}$$

for $i = 1, 2$, and we see that m_i^2 are non-tachyonic masses regardless of γ.

3.8 Metric perturbations on M_q and stability

All the modes we have considered thus far have masses within the bounds for stability; moreover, we were able to show this for $AdS_p \times M_q$ where M_q is an arbitrary q-dimensional Einstein manifold. The only fields we have not considered as yet come from the traceless modes of the graviton on M_q, and satisfy (3.40), which we repeat here:

$$\left[(\Box_x + \Box_y)\delta_\alpha^\gamma \delta_\beta^\delta - 2R_\alpha{}^{\gamma\delta}{}_\beta\right] \phi^I \, Y^I_{(\gamma\delta)} \quad = \quad 0\,. \tag{3.1}$$

It is possible to rewrite equation (3.1) in terms of the Lichnerowicz operator Δ_L and the Ricci tensor:

$$[\Box_x + \Delta_L + \frac{2(q-1)}{R^2}]\phi^I \, Y^I_{(\alpha\beta)} \quad = \quad 0\,. \tag{3.2}$$

but since Δ_L does not obey a universal inequality as $\Box_y$ and Δ_y do, this form is not as useful. The presence of the Riemann tensor indicates that (3.1) can have different properties depending on the particular choice of M_q.

One may wonder about other fluctuations obeying (3.1), and whether they may place even more stringent constraints on the requirements for stability. The field (3.7) is the lowest in a tower of modes that are traces on each individual space in the product, but traceless overall. Higher excitations will have more positive masses from the $\Box_y$ term. The remaining modes are traceless on each M_n and M_{q-n}, namely $h_{(ab)}$, $h_{(ij)}$ and h_{ai}. For h_{ai} we find the universal result

$$(\Box_x + \Box_y)h_{ai} = 0\,, \tag{3.3}$$

which is obviously stable. while for either of the other two we have effectively a copy of equation (3.1) but involving the Riemann tensor of just one of the spaces in the product:

$$\left[(\Box_x + \Box_y)\delta_a^c \delta_b^d - 2R_a{}^{cd}{}_b\right] h_{(cd)} \quad = \quad 0\,. \tag{3.4}$$

and similar for $h_{(ij)}$. This obviously depends on the details of M_n. One observation we can make is that if M_n itself is a product (and so the original compact space M_q is a product of three or more manifolds), a mode analogous to (3.7) will have a mass $m^2 = 2(n-1)/R_1^2 = 2(q-1)/R^2$, where the last equality comes from (3.6), and thus will be unstable precisely when (3.7) is; and hence no new instability automatically arises for products of three or more spaces beyond that already generically present for a product of two.

After all these generalities, let us see what we find for the three examples that we promised to treat – the round sphere, product spaces of Einstein manifolds (the resulting space is also Einstein), and T^{pq}.

For $M_q = S^q$, the Riemann tensor has the maximally symmetric form $R_{\alpha\beta\gamma\delta} = (g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})/R^2$. Equation (3.1) reduces to

$$\left[(\Box_x + \Box_y) - \frac{2}{R^2}\right]\phi^I\, Y^I_{(\alpha\beta)} \;=\; 0\,. \tag{3.5}$$

All these modes are manifestly positive-mass. We thus complete our demonstration of the stability of the $AdS_p \times S^q$ background for all p and q.

In [71] and [72] it was pointed out that $AdS_4 \times M_n \times M_{7-n}$ and $AdS_7 \times S^2 \times S^2$, respectively, were unstable to a perturbation in which one compact space becomes uniformly larger and the other smaller keeping the total volume fixed. We now generalize this to an arbitrary product of Einstein spaces $M_q = M_n \times M_{q-n}$ with $n \geq 2$. Let a, b denote indices on M_n and i, j denote indices on M_{q-n}. If the radii of the spaces are R_1 and R_2, requiring that the total compact space is also Einstein imposes the relation

$$\frac{n-1}{R_1^2} = \frac{q-n-1}{R_2^2} = \frac{q-1}{R^2}\,. \tag{3.6}$$

Consider now the mode

$$h_{ab} = \frac{1}{n} g_{ab}\phi(x)\,, \qquad h_{ij} = -\frac{1}{q-n} g_{ij}\phi(x)\,, \tag{3.7}$$

which satisfies $h^\alpha_\alpha = 0$ as well as the gauge condition (3.5) and therefore obeys (3.1). This perturbation increases the radius of one of the Einstein spaces and decreases the radius of the other keeping the total volume constant (to first order). Evaluating (3.1), we find

$$\left[\Box_x + \frac{2(q-1)}{R^2}\right]\phi^I = 0\,. \tag{3.8}$$

Thus this mode has the mass

$$m^2L^2 = -\frac{2(p-1)^2}{(q-1)} = \frac{8}{q-1}m_{BF}^2L^2\,. \qquad (3.9)$$

Consequently the Breitenlohner-Freedman bound (3.1) is violated for $q < 9$. This result is independent of p, and depends on the internal space only in that it is a product of Einstein spaces that is itself Einstein with total dimension q; in particular the relative dimension of the spaces is irrelevant.

The case of $AdS_5 \times T^{pq}$ is sufficiently rich to merit a separate subsection and we turn to this next. We shall find that only the supersymmetric case and its smooth supersymmetry breaking $\mathbf{Z_p}$ quotient is stable. Thus it still remains to find a stable, non-supersymmetric anti-de Sitter compactification of type IIB supergravity which is not locally isometric to a supersymmetric one. Infinitely many such compactifications of eleven-dimensional supergravity to $AdS_4 \times M_7$ have long been known, as we remarked in the introduction. This problem of non-supersymmetric AdS_5 vacua takes on a new interest in light of AdS/CFT, because it corresponds to discovering four-dimensional, non-supersymmetric, strong-coupling fixed points of the renormalization group.

3.8.1 Stability Analysis for Compactifications on T^{pq}

Let us now examine the issue of stability of Type IIB supergravity compactified on the manifold commonly known as T^{pq}. This is a five-dimensional Einstein manifold which is a coset of $SU(2) \times SU(2)$ by a $U(1)$ whose generator can be written as $p\Sigma_3 + q\tilde{\Sigma}_3$, where Σ_3 and $\tilde{\Sigma}_3$ are generators of the two $SU(2)$'s. The integers p and q describe the winding numbers of the $U(1)$ fiber over the two spheres. The most general metric on T^{pq} consistent with $SU(2) \times SU(2) \times U(1)$ isometry is:

$$ds^2 = a^2(dy_1^2+\sin^2 y_1 dy_2^2)+b^2(dy_3^2+\sin^2 y_3 dy_4^2)+c^2(dy_5-p\cos y_1 dy_2-q\cos y_3 dy_4)^2\,. \qquad (3.10)$$

where a, b, and c are constants, y_1 and y_3 range from 0 to π, y_2 and y_4 range from 0 to 2π, and y_5 ranges from 0 to 4π. Conditions on a, b, and c for the metric (3.10) to be Einstein were discussed in [89], and we will recap some of the relevant points as they will apply to our subsequent analysis. We will assume throughout that p and q are relatively prime, and then in the last paragraph of this section address the case where they are not.

Let us choose the following basis of 1-forms:

$$E^1 = a dy_1 , \quad E^2 = a \sin y_1 dy_2 , \quad E^3 = b dy_3 , \quad E^4 = b \sin y_3 dy_4 , \\ E^5 = c(dy_5 - p\cos y_1 dy_2 - q\cos y_3 dy_4) . \tag{3.11}$$

The spin-coefficients in this basis are:

$$\omega_{12} = -\frac{1}{a}\cot y_1 E^2 - \frac{pc}{2a^2}E^5 , \quad \omega_{13} = \omega_{14} = 0 , \quad \omega_{15} = -\frac{pc}{2a^2}E^2 , \quad \omega_{25} = \frac{pc}{2a^2}E^1 , \\ \omega_{34} = -\frac{1}{b}\cot y_3 E^4 - \frac{qc}{2b^2}E^5 , \quad \omega_{23} = \omega_{24} = 0 , \quad \omega_{35} = -\frac{qc}{2b^2}E^4 , \quad \omega_{45} = \frac{qc}{2b^2}E^3 . \tag{3.12}$$

The curvature components are calculated using the relation:

$$R^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu . \tag{3.13}$$

Only a few of them need to be actually computed. The remaining ones can be found using the symmetry of the metric and the symmetric and anti-symmetric properties of the Riemann tensor. However, for completeness we list all of the components of the curvature 2-form:

$$R^1{}_2 = \left(\frac{1}{a^2} - \frac{3p^2c^2}{4a^4}\right)E^1E^2 - \frac{pqc^2}{2a^2b^2}E^3E^4 , \quad R^1{}_3 = -\frac{pqc^2}{4a^2b^2}E^2E^4 , \quad R^1{}_4 = \frac{pqc^2}{4a^2b^2}E^2E^3 , \\ R^3{}_4 = \left(\frac{1}{b^2} - \frac{3q^2c^2}{4b^4}\right)E^3E^4 - \frac{pqc^2}{2a^2b^2}E^1E^2 , \quad R^2{}_4 = -\frac{pqc^2}{4a^2b^2}E^1E^3 , \quad R^2{}_3 = \frac{pqc^2}{4a^2b^2}E^1E^4 , \\ R^1{}_5 = \frac{p^2c^2}{4a^4}E^1E^5 , \quad R^2{}_5 = \frac{p^2c^2}{4a^4}E^2E^5 , \quad R^3{}_5 = \frac{q^2c^2}{4b^4}E^3E^5 , \quad R^4{}_5 = \frac{q^2c^2}{4b^4}E^4E^5 . \tag{3.14}$$

Finally, we have to demand that the metric above is Einstein. In the orthonormal basis that we chose above, this condition is simply that $R^i{}_j = \Lambda\delta^i{}_j$, where Λ is the constant of proportionality between the Ricci tensor and the metric. This yields three equations relating the constants Λ, a, b and c:

$$\Lambda = \frac{2a^2 - p^2c^2}{2a^4} = \frac{2b^2 - q^2c^2}{2b^4} = \frac{(a^4q^2 + b^4p^2)c^2}{2a^4b^4} . \tag{3.15}$$

For convenience, let us work in units where the radius of one of the spheres is set equal to unity, i.e. $a \equiv 1$. In these units the other constants b and c are:

$$b^2 = \frac{1}{3\Lambda - 1} \qquad c^2 = \frac{2(1-\Lambda)}{p^2}. \tag{3.16}$$

It is also helpful to express the ratio $\frac{q}{p}$ in terms of Λ:

$$\left(\frac{q}{p}\right)^2 = \frac{2\Lambda - 1}{(1-\Lambda)(3\Lambda-1)^2}. \tag{3.17}$$

Looking at the last expression we see that Λ varies between $\frac{1}{2}$ and 1. So, given any manifold T^{pq} we first evaluate the ratio $\frac{q}{p}$ and then using (3.17) compute Λ. For instance, the space T^{11} has $\Lambda = \frac{2}{3}$. All questions about stability can be answered in terms of values of Λ.

In [10] it was shown that for an arbitrary Einstein manifold, the masses of the scalar modes resulting from a mixing of 3 scalars: the trace of the metric on AdS_5, the trace on M_5 and another scalar which arises from the fluctuations of the five form field strength never violate the stability bound - they can at most saturate it. The masses of the coupled scalar modes are:

$$m^2L^2 = \lambda + 16 \pm 8\sqrt{\lambda + 4}, \tag{3.18}$$

where $\Box_y Y \equiv -\lambda Y/R^2$ and Y is a scalar harmonic on T^{pq}. By $\Box_y$ we will always mean $\nabla^\alpha\nabla_\alpha$. Minimizing with respect to λ we find that the least massive mode corresponds to $\lambda = 12$. Moreover, this mode just saturates the stability bound $m^2L^2 \geq -4$. The isometry group of T^{pq} is $SU(2) \times SU(2) \times U_R(1)$ so the eigenvalues of the scalar Laplacian on T^{pq} are expressed in terms of the eigenvalues j_1, j_2, r corresponding to the two $SU(2)$'s and the $U(1)_R$ [90, 91, 92]:

$$\frac{\lambda}{R^2} = \frac{r^2}{c^2} + \frac{1}{a^2}\left[j_1(j_1+1) - (pr)^2\right] + \frac{1}{b^2}\left[j_2(j_2+1) - (qr)^2\right]. \tag{3.19}$$

with $j_1 \geq pr$ and $j_2 \geq qr$. Let us examine this for the special case of T^{11}. Here we have from (3.17) and (3.16) $\Lambda = \frac{2}{3}$, $a = b = 1$, and $c^2 = \frac{2}{3}$. Since $\Lambda = \frac{4}{R^2}$ from (3.6) and from (3.9) $R = L$, the expression for the eigenvalue of the scalar harmonic on T^{11} simplifies to:

$$\lambda = 6\left[j_1(j_1+1) + j_2(j_2+1) - \frac{r^2}{2}\right]. \tag{3.20}$$

The value of $\lambda = 12$ is thus satisfied for $(j_1, j_2, r) = (1, 0, 0)$ and $(0, 1, 0)$.

None of these coupled scalar modes can violate the stability bound for any T^{pq}. Indeed, we have already shown in the first part of this chapter that the only mode which could potentially violate the stability bound is the traceless graviton mode:

$$[(\Box_x + \Box_y)\delta_c^a \delta_d^b - 2R_c{}^{ab}{}_d]\,(\phi(x)Y_{ab}(y)) = 0\,, \tag{3.21}$$

where as usual, $\Box_y = \nabla^a \nabla_a$ and $\Box_x = \nabla^\mu \nabla_\mu$. This equation may be rewritten in terms of the Ricci tensor and the Lichnerowicz operator Δ_L. The action of the latter on symmetric tensors Y_{ab} is defined as:

$$\Delta_L Y_{ab} \equiv \Box_y Y_{ab} - 2R_a{}^{cd}{}_b Y_{cd} - 2R_{(a}{}^c Y_{b)c}\,. \tag{3.22}$$

Here and below, $(\ldots)$ indicates symmetrization: $(ab) = (ab + ba)/2$. Using the definition (3.22), the fluctuation equation for the symmetric traceless graviton modes (3.21) is simply:

$$[\Box_x + \Delta_L + \frac{2(q-1)}{R^2}]\,(\phi Y_{ab}) \;=\; 0\,. \tag{3.23}$$

Since we're dealing with $AdS_5 \times M_5$, $p = q = 5$ and from (3.9) we have $R = L$. The Breitenlohner-Freedman bound is $m^2L^2 \geq -4$. Assembling all these facts together we can translate the BF bound from a bound on the mass to one on the eigenvalue of the Lichnerowicz operator acting on symmetric tensors Y_{ab}:

$$\Delta_L Y_{ab} = \lambda Y_{ab}\,. \tag{3.24}$$

For stability we must have:

$$\lambda L^2 \leq -4\,. \tag{3.25}$$

It is somewhat painful to diagonalize the Lichnerowicz operator directly. Fortunately, a trick employed in [76] works here as well.[7] The key is to use the identity:

$$\int dV\, Y^{ab}\Delta_L Y_{ab} = \int dV\,[-4Y^{ab}R_{acdb}Y^{cd} - 4\Lambda Y^{ab}Y_{ab} - 3\nabla^{(a}Y^{bc)}\nabla_{(a}Y_{bc)}]\,. \tag{3.26}$$

[7] We thank C. Pope for a communication which brought this paper to our attention.

which can easily be demonstrated by writing out explicitly $Y^{ab}\Delta_L Y_{ab}$ and $\nabla^{(a}Y^{bc)}\nabla_{(a}Y_{bc)}$, simplifying, and integrating by parts. Let us diagonalize the Riemann tensor by solving the eigenvalue equation:

$$R_a{}^{bc}{}_d Y_{bc} = \kappa Y_{ad} \,. \tag{3.27}$$

This would involve diagonalizing a 15×15 matrix. One of the eigenvectors would be pure trace, so that we'll be left with 14 traceless eigenvectors. Using (3.25) and (3.26) our stability bound now reads: The geometry would be stable if every κ satisfies:

$$\kappa \geq \frac{1}{L^2} - \Lambda \,. \tag{3.28}$$

From (3.6) we have $\Lambda = \frac{4}{R^2}$ and since $R = L$, the above bound can be expressed solely in terms of Λ as:

$$\kappa_{min} \geq -\frac{3}{4}\Lambda \,. \tag{3.29}$$

where κ_{min} is the least of the 14 eigenvalues. Thus we have reduced the problem of solving the complicated equation (3.21) into a simple one of diagonalizing the Riemann tensor. On account of the simple metric and the symmetries involved, there is very little mixing of the modes and the problem is sufficiently simple to be solved by hand. The eigenvectors and eigenvalues are shown in table 3.1. In this table, $\alpha_\pm$, $\beta_\pm$ and $\gamma_\pm$ are constants given by

$$\begin{aligned}
\alpha_\pm &= 4(1-2\Lambda)(7\Lambda - 4 \mp \sqrt{49\Lambda^2 - 60\Lambda + 20}) \,, \\
\beta_\pm &= 2(1-2\Lambda)(-7\Lambda + 2 \pm \sqrt{49\Lambda^2 - 60\Lambda + 20}) \,, \\
\gamma_\pm &= (2 - \Lambda \pm \sqrt{49\Lambda^2 - 60\Lambda + 20})(-7\Lambda + 2 \pm \sqrt{49\Lambda^2 - 60\Lambda + 20}) \,,
\end{aligned} \tag{3.30}$$

and we have also defined

$$X_{ab}^{cd} = \delta^c_{(a}\delta^d_{b)} \,. \tag{3.31}$$

We have not written down the eigenvalues for the last two modes in the table in terms of a, b, c, p and q because the expressions are quite lengthy. Remembering that $\frac{1}{2} \leq \Lambda \leq 1$ we find that there is only one mode (the last one in the table) which can violate the bound (3.29). For this potentially dangerous mode we find that only $\Lambda = \frac{2}{3}$ (which corresponds to the manifold T^{11}) saturates the bound, while all other values of Λ lead to masses which

Eigenvectors	Eigenvalues	Eigenvalues in units $a=1$
$X^{12}_{ab},\ X^{11}_{ab}-X^{22}_{ab}$	$\frac{1}{a^2}-\frac{3p^2c^2}{4a^4}$	$\frac{1}{2}(3\Lambda-1)$
$X^{34}_{ab},\ X^{33}_{ab}-X^{44}_{ab}$	$\frac{1}{b^2}-\frac{3q^2c^2}{4b^4}$	$\frac{1}{2}$
$X^{13}_{ab}+X^{24}_{ab},\ X^{14}_{ab}+X^{23}_{ab}$	$\frac{3pqc^2}{4a^2b^2}$	$\frac{3}{2}\sqrt{(1-\Lambda)(2\Lambda-1)}$
$X^{13}_{ab}-X^{24}_{ab},\ X^{14}_{ab}-X^{23}_{ab}$	$-\frac{3pqc^2}{4a^2b^2}$	$-\frac{3}{2}\sqrt{(1-\Lambda)(2\Lambda-1)}$
$X^{15}_{ab},\ X^{25}_{ab}$	$\frac{p^2c^2}{4a^4}$	$\frac{1}{2}(1-\Lambda)$
$X^{35}_{ab},\ X^{45}_{ab}$	$\frac{q^2c^2}{4b^4}$	$(\Lambda-\frac{1}{2})$
$\alpha_+(X^{11}_{ab}+X^{22}_{ab})+\beta_+(X^{33}_{ab}+X^{44}_{ab})+\gamma_+X^{55}_{ab}$		$\frac{1}{4}(-\Lambda+\sqrt{49\Lambda^2-60\Lambda+20})$
$\alpha_-(X^{11}_{ab}+X^{22}_{ab})+\beta_-(X^{33}_{ab}+X^{44}_{ab})+\gamma_-X^{55}_{ab}$		$\frac{1}{4}(-\Lambda-\sqrt{49\Lambda^2-60\Lambda+20})$

Table 3.1: Eigenvectors and eigenvalues of the Riemann tensor, as defined in (3.27).

violate the bound. This tells us that the only stable compactification on T^{pq} manifolds with p and q relatively prime turns out also to be the only one which preserves supersymmetry. (Note that the modes $X^{13}_{ab}-X^{24}_{ab}$ and $X^{14}_{ab}-X^{23}_{ab}$ saturate the bound for T^{11} while for all other T^{pq} have eigenvalues which are above the bound. This might lead us to suspect that for T^{11} these modes have masses $m^2L^2=-4$. But on careful examination we find that they do not satisfy the Killing tensor equation $\nabla_{(a}Y_{bc)}=0$. Therefore, according to (3.26) they have masses $m^2L^2>-4$).

A word now about what the unstable mode (or in the case of T^{11} the marginally stable mode) looks like. For T^{11}, we can use (3.30) to evaluate the constants $\alpha_-=-\frac{8}{3}$, $\beta_-=\frac{8}{3}$, and $\gamma_-=0$ so that the eigenvector which just saturates the bound is simply diag$(-1,-1,1,1,0)$. Geometrically this is a fluctuation in which one S^2 expands while the other shrinks with the length of the $U(1)$ fiber unchanged. For generic T^{pq} however, such a simple picture is not obtained. As an example, let us consider the unstable mode of T^{12}. For this manifold, using (3.17) we find $\Lambda\approx 0.9331$, and using (3.30) we have $\alpha_-\approx -17.73$, $\beta_-\approx 12.33$, and $\gamma_-\approx 10.80$ so this fluctuation makes one S^2 shrink and the other expand accompanied by an elongation of the fiber.

To summarize, for T^{11} we have found a total of seven modes which saturate the stability bound. Six of them come from the coupled scalar modes, and the remaining one is a traceless graviton mode. All of these modes have masses $m^2L^2 = -4$. Using the relation between the mass and the dimension of the corresponding operators in the dual field theory, $\Delta = \frac{1}{2}[(p-1) \pm \sqrt{(p-1)^2 + 4m^2L^2}]$ (here p is the dimension of AdS), we find that the operators have scaling dimension 2. In the next section we shall examine in some detail the issue of identifying these operators in the dual field theory according to the AdS/CFT correspondence.

In the above analysis we have only shown that if the inequality (3.29) is satisfied, then stability is guaranteed. Let us now prove that if the inequality is violated, then we necessarily have instability in the traceless graviton sector. Looking at (3.26) we find that we have to demonstrate that the putative unstable mode is also a Killing tensor obeying $\nabla_{(a}Y_{bc)} = 0$. To prove this, we make the following observations. First, if we restrict all three indices a, b, and c to lie in the four manifold which are the two S^2's of T^{pq} (i.e. these indices are allowed to run from 1 to 4), then a constant, diagonal tensor of the form:

$$Y_{ab} = \mathrm{diag}(\alpha, \alpha, \beta, \beta, \gamma)\,, \tag{3.32}$$

with $2\alpha + 2\beta + \gamma = 0$ is covariantly constant. Second, we notice that the spin-coefficients of the metric written down in (3.12) can be split up in the following way:

$$\omega^5_{ab} = \omega^4_{ab} + c_{ab}E^5 \qquad \omega^5_{a5} = c_{ab}E^b\,, \tag{3.33}$$

where ω^5 refers to the full spin connection, ω^4 refers to the part on the two S^2's, and c_{ab} is antisymmetric in a and b. Using this fact and the form of the constant diagonal tensor, one can show that indeed this eigentensor satisfies the Killing condition. So we have demonstrated that the only stable $AdS_5 \times T^{pq}$ compactification with p and q relatively prime is on T^{11}.

To extend the discussion to the case of p and q not relatively prime, a topological point should be made first: if $\gcd(p,q) = k \neq 1$, then T^{pq} is topologically $S^2 \times S^3/\mathbf{Z_k}$, where the

$\mathbf{Z}_k$ acts freely on the Hopf fiber of S^3. The $k = 1$ case of this statement follows based on arguments given in [93]; the $k > 1$ statement follows as a corollary when one notes that modding out by the $U(1)$ generated by $p\Sigma_3 + q\tilde{\Sigma}_3$ can be accomplished by first modding out by the $U(1)$ generated by $(p\Sigma_3 + q\tilde{\Sigma}_3)/k$ and then dividing by $\mathbf{Z}_k$. Moreover, T^{pq} is metrically a quotient of $T^{p/k,q/k}$ by $\mathbf{Z}_k$ acting on the $U(1)$ fiber. Thus one can flow without encountering a topological obstruction from any T^{pq} to any other precisely when $\gcd(p, q)$ remains unchanged. In each class of manifolds T^{pq} with $\gcd(p, q) = k$ fixed, there is precisely one which is classically stable, namely T^{kk}. Only for $k = 1$ is any supersymmetry preserved. The perturbation analysis on T^{pq} could be carried out by considering $\mathbf{Z}_k$ invariant functions on $T^{p/k,q/k}$. The orbifolding by k also has a well-defined meaning on the gauge theory side, resulting for T^{kk} in a theory with gauge group $SU(N)^{2k}$ and some complicated matter. Details of counting and the operator map could be pursued for the case of general k, but in Section 3.8.2 we will do so only for $k = 1$.

3.8.2 An Aside: The Operator Map for T^{11}

In the previous section we saw that the Freund-Rubin compactification of Type IIB on $AdS_5 \times T^{11}$ have three modes which have masses which just saturate the stability bound. Recall that two of these modes came from the coupled scalars and the remaining one is a traceless graviton mode. According to the AdS/CFT correspondence, these modes should correspond to operators whose dimension is protected and would therefore be either chiral primaries or conserved currents. So let us try to find the dual operators.

The compactification of Type IIB SUGRA on T^{11} has $SU(2,2|1)$ symmetry. Let us try to put these fluctuations into $SU(2,2|1)$ supermultiplets. A unitary highest weight representation of $SU(2,2|1)$ can be decomposed into a direct sum of unitary highest weight representations of the bosonic subalgebra $SU(2,2) \times U_R(1)$ whose maximal compact subalgebra is $U(1) \times SU(2) \times SU(2) \times U_R(1)$ [94, 95]. The first $U(1)$ is the energy and the last one is the R-charge. So the highest weight representations of $SU(2,2|1)$ are labelled by four quantum numbers $D(E_0, s_1, s_2; r)$. In addition to these, there are of course, the quantum numbers associated with the symmetry of the isometry group of T^{11} which is $SU(2) \times SU(2) \times U_R(1)$. Note that this $U_R(1)$ is the same $U_R(1)$ as the one associated to the R-charge. We'll call the additional quantum numbers due to these last two $SU(2)$'s as (j_1, j_2). So our first task is to find out all the six quantum numbers of the fluctuation mode in question $(E_0, s_1, s_2; r, j_1, j_2)$.

From the $5d$ AdS point of view all of the modes in question, including Y_{ab}, are scalars. So, the spin quantum numbers $s_1 = s_2 = 0$. To compute the AdS energy, we use the relation between mass and energy for a scalar in $5d$ AdS space:

$$E_0 = 2 \pm \sqrt{4 + m^2L^2}\,. \tag{3.34}$$

We found above that these modes have masses $m^2L^2 = -4$, so $E_0 = 2$. Finally, to obtain the value of r, we note that for a representation of $SU(2,2|1)$ to be unitary, there are inequalities among the four quantum numbers E_0, s_1, s_2, and r. The relevant one for our purposes here is $E_0 \geq 2s_2 + \frac{3}{2}r + 2$ which fixes $r = 0$. We observe now that this set of quantum numbers $(2, 0, 0, 0)$ satisfies 3 multiplet shortening conditions [94] (which is what is expected for a field saturating the unitarity bound):

$$E_0 - 2s_1 + \frac{3}{2}r - 2 = 0 \qquad E_0 - 2s_2 + \frac{3}{2}r - 2 = 0 \qquad s_2 = 0\,. \tag{3.35}$$

So we get the following multiplet with only 4 fields present:

E_0/R	$r=-1$	$r=0$	$r=1$
2		$(0,0)$	
$\frac{3}{2}$	$(\frac{1}{2},0)$		$(0,\frac{1}{2})$
3		$(\frac{1}{2},\frac{1}{2})$	

where the quantities in the table refer to the quantum numbers (s_1, s_2). For completeness, we note that the masses of these fields can be calculated using the relations:

$$\begin{aligned} \left(\frac{1}{2},0\right)\left(0,\frac{1}{2}\right) \qquad & m = |E_0 - 2| = \frac{1}{2}\,, \\ \left(\frac{1}{2},\frac{1}{2}\right) \qquad & m^2 = (E_0 - 1)(E_0 - 3) = 0\,. \end{aligned} \tag{3.36}$$

Let us now turn to the field theory realization of this. The conformal field theory dual to the supergravity theory has two doublets of chiral superfields A_i, B_j $(i,j = 1,2)$ transforming in the $(N,\bar{N})$ and $(\bar{N},N)$ representations of $SU(N) \times SU(N)$. These fields both have R-charge 1. The global symmetry group $SU(2) \times SU(2)$ quantum numbers for these fields are $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively. According to the AdS/CFT correspondence, each supergravity field with quantum numbers $(E_0, s_1, s_2; r)$ is mapped to a conformal field with scaling dimension $\Delta = E_0$, Lorentz quantum numbers of an $SL(2,C)$ representation (s_1, s_2), and an R-symmetry charge r. Since we had multiplet shortening, we know that the dimension of the corresponding superfield would be protected, and furthermore we also determined its dimension to be 2. There are natural field theory candidates with the desired properties to be dual to the scalars we have found.[8] Namely, consider the real superfields:[9]

$$
\begin{aligned}
J_A &= \mathrm{Tr}\, A_{(i} e^{V_2} A^*_{j)} e^{V_1} \qquad J_B = \mathrm{Tr}\, B_{(i} e^{V_1} B^*_{j)} e^{V_2} \\
J_{\text{baryon}} &= \mathrm{Tr}\, A_{[1} e^{V_2} A^*_{2]} e^{V_1} - \mathrm{Tr}\, B_{[1} e^{V_1} B^*_{2]} e^{V_2} \,.
\end{aligned}
\tag{3.37}
$$

The vector component of each of these is a conserved current, by Noether's theorem: J_A is associated with the global $SU(2)$ rotating A_1 and A_2; J_B is associated with the other global $SU(2)$; and J_{baryon} is associated with the unbroken $U(1)_{\text{baryon}}$. The scalar component of each of these superfields, call them $\mathcal{O}_A$, $\mathcal{O}_B$, and $\mathcal{O}_{\text{baryon}}$, have protected dimension 2 and R-charge 0. The operator $\mathcal{O}_{\text{baryon}}$ was discussed in [96] in the context of resolving the conifold. Of the supergravity modes with masses $m^2L^2 = -4$, we conjecture that the baryon current is the one which is dual to the traceless graviton mode, while J_A and J_B are dual to the coupled scalar modes. The global $SU(2) \times SU(2)$ charges support this expectation. Moreover, both the traceless graviton fluctuation that we examined in the previous section and its proposed operator dual J_{baryon} flip sign on interchanging the S^2's, i.e. they are both $\mathbf{Z_2}$ odd.

[8] We thank M. Strassler for useful communications which helped us identify the field theory operators.

[9] In (3.37), V_1 and V_2 are the real superfields that include the $SU(N) \times SU(N)$ gauge fields. There is a notational subtlety: A^*_i transforms as a doublet of $SU(2)$, and we have omitted the ϵ_{ij} which would usually be inserted to make the group action come out right.

3.9 *AdS* vacua of theories with a cosmological term

In this section we shall consider theories with a cosmological term in the action. The two examples that we shall consider in detail are the AdS_4 vacua of massive IIA theory and the Sugimoto theory.

3.9.1 AdS_4 vacua of massive IIA

Massive type IIA supergravity has $AdS_4 \times M_6$ vacua [78] which are non-supersymmetric and whose stability, to our knowledge, had not been investigated before.[10] Even the existence of these solutions is non-trivial, since there is a potential term for the dilaton which pushes it toward weak coupling. What makes $AdS_4 \times M_6$ vacua possible is that a uniform RR field strength, F_4 or F_6 according to taste, pushes the dilaton toward strong coupling, and there is an extremum of this total potential where the dilaton can be constant.

The extremum is in fact a maximum, but it doesn't make sense to ask whether second derivative of the total dilaton potential alone satisfies the BF bound, because the dilaton couples non-trivially to the form and to the graviton. This mixing means that the coupled scalars sector requires a more intricate analysis than before. The result will be that the apparent s-wave tachyon coming from a naive analysis of the dilaton potential is completely erased (effectively, it is a gauge artifact), but for S^6 there is a d-wave and an f-wave mode which violates the BF bound, rendering this vacuum unstable! To our knowledge, this is the first time that a product of AdS and a round sphere is unstable. We also show that for $M_6 = S^n \times S^{6-n}$ the BF bound is violated within the coupled scalar sector, as well as having the same purely gravitational instability found earlier, where one factor shrinks while the other grows.

The remaining modes, outside the coupled scalar sector, satisfy the same equations as in the generic $AdS_p \times M_q$ systems we already considered. Thus the traceless graviton on

[10] There is also a *supersymmetric* (and necessarily stable) vacuum which is a fibration of AdS_6 over S^4 with a non-trivial dilaton. It is the near-horizon geometry of the D4-D8 system [97]. It would be interesting to explore the properties of this background as well as generalizations of it where S^4 is replaced by other manifolds, but we will not do so here.

M_6 joins the coupled scalars as a possible source of instability. We do not analyze other Einstein manifolds M_6 explicitly, but we provide the tools needed for such an analysis. It is still possible that there exist stable $AdS_4 \times M_6$ vacua.

To make the discussion similar to our previous analysis, let us express the action for massive IIA in terms of a six-form field strength, which is essentially the Hodge dual of the usual four-form:

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\xi^2 F_6^2 - \frac{m^2}{8}\xi^{-10}\right] \qquad \text{where } \xi = e^{-\phi/4}, \tag{3.1}$$

and we include a $1/6!$ in the definition of F_6^2, as in [98]. We also include a factor of $1/q!$ in the inner product of forms, $\omega_q \cdot \tilde{\omega}_q$. The equations of motion are

$$\begin{aligned}
&R_{MN} = \frac{m^2}{64}\xi^{-10}g_{MN} + \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{\xi^2}{2\cdot 5!}F_{MP_1P_2P_3P_4P_5}F_N{}^{P_1P_2P_3P_4P_5} - \frac{5}{16}\xi^2 g_{MN}F_6^2\,,\\
&\Box\,\phi - \frac{5}{16}m^2\xi^{-10} + \frac{\xi^2}{4}F_6^2 = 0\,,\\
&d * \xi^2 F_6 = 0\,,
\end{aligned} \tag{3.2}$$

and there is an $AdS_4 \times M_6$ background with $\phi = 0$, $F_6 = c\,\mathrm{vol}_{M_6}$. We readily derive the relations

$$c^2 = F_6^2 = \frac{5}{4}m^2 = \frac{10}{L^2} = \frac{25}{R^2}, \tag{3.3}$$

where L is the radius of curvature of AdS_4, such that $R_{\mu\nu} = -\frac{3}{L^2}g_{\mu\nu}$, and R is the radius of curvature of M_6, such that $R_{\alpha\beta} = \frac{5}{R^2}g_{\alpha\beta}$.

Just as for $AdS_p \times M_q$, we wish to linearize around the background to obtain the mass spectrum. For the coupled scalar sector, we wish to focus on perturbations of the form $g_{MN} \to g_{MN} + h_{MN}$ with $h_{\mu\nu} = \frac{1}{4}g_{\mu\nu}h^\lambda_\lambda$ and $h_{\alpha\beta} = \frac{1}{6}g_{\alpha\beta}h^\gamma_\gamma$. Also let $\delta\phi$ be the perturbation in ϕ and let f_6 be the perturbation in F_6, where, as before, we write

$$h^\alpha_\alpha = \pi\,, \quad f_6 = da_5\,, \quad \text{where} \quad a_5 = *_6 db\,. \tag{3.4}$$

The algebraic relation $h^\mu_\mu + h^\alpha_\alpha = \frac{1}{3}h^\alpha_\alpha$ follows from the symmetric traceless part of the Einstein equations, as before. It is now possible to derive coupled second order equations relating $\delta\phi$, b, and π from the variations of the R^α_α Einstein equation, the scalar equation, and the form equation, using the algebraic relation when needed to eliminate h^μ_μ in favor of h^α_α. We use a form notation in this section for convenience.

The R^α_α equation is

$$R^\alpha_\alpha = \frac{3}{4L^2}\xi^{-10} + \frac{9}{8}\xi^2 F_6^2 + \frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi\,. \tag{3.5}$$

Using (3.16), (3.3), and (3.4), we find

$$\begin{aligned}\delta R^\alpha_\alpha &= -\frac{2}{L^2}h^\alpha_\alpha - \frac{1}{2}(\Box_x + \Box_y)h^\alpha_\alpha - \frac{1}{2}\Box_y(h^\mu_\mu + h^\alpha_\alpha) + \frac{1}{6}\Box_y h^\alpha_\alpha \\ &= -\frac{15}{4L^2}\delta\phi + \frac{9}{4}c\Box_y b - \frac{45}{4L^2}h^\alpha_\alpha\,,\end{aligned} \tag{3.6}$$

where we have used the fact that $\Box_y = *_6 d *_6 d$ acting on b. The algebraic relation allows us to simplify this to

$$(\Box_x + \Box_y)\pi - \frac{37}{2L^2}\pi - \frac{15}{2L^2}\delta\phi + \frac{9}{2}c\Box_y b = 0\,. \tag{3.7}$$

For the scalar, the equation of motion is

$$\Box\phi - \frac{5}{2L^2}\xi^{-10} + \frac{\xi^2}{4}F_6^2 = 0\,. \tag{3.8}$$

Linear variation around the background gives

$$(\Box_x + \Box_y)\delta\phi - \frac{25}{4L^2}\delta\phi - \frac{1}{8}\delta\phi F_6^2 + \frac{1}{2}F_6\cdot f_6 - \frac{1}{4}h^{\alpha\beta}F_{\alpha\gamma_1\ldots\gamma_5}F_\beta{}^{\gamma_1\ldots\gamma_5}\frac{1}{5!} = 0\,, \tag{3.9}$$

which upon simplification and use of $F_6 \cdot f_6 = c\Box_y b$ becomes

$$(\Box_x + \Box_y)\delta\phi - \frac{15}{2L^2}\delta\phi + \frac{1}{2}c\Box_y b - \frac{5}{2L^2}\pi = 0\,. \tag{3.10}$$

The variation of the form equation is

$$d(\delta *)F_6 - d * \frac{1}{2}\delta\phi F_6 + d * f_6 = 0\,, \tag{3.11}$$

where $\delta *$ indicates the variation in the Hodge dual. After some algebra this becomes

$$\frac{c}{2}d(h^\mu_\mu - h^\alpha_\alpha - \delta\phi)\wedge \mathrm{vol}_4 + d(\Box_x + \Box_y)b \wedge \mathrm{vol}_4 = 0\,, \tag{3.12}$$

and so, using the algebraic relation, we obtain

$$(\Box_x + \Box_y)b - \frac{5c}{6}\pi - \frac{c}{2}\delta\phi = 0\,. \tag{3.13}$$

Gathering everything together, setting $b = cL^2B$ for convenience, and recalling that $c^2 = 10/L^2$, one obtains the following system of equations:

$$\begin{aligned}
(\Box_x + \Box_y)B - \frac{5}{6L^2}\pi - \frac{1}{2L^2}\delta\phi = 0 \\
(\Box_x + \Box_y)\pi - \frac{37}{2L^2}\pi - \frac{15}{2L^2}\delta\phi + 45\Box_y B = 0 \\
(\Box_x + \Box_y)\delta\phi - \frac{5}{2L^2}\pi - \frac{15}{2L^2}\delta\phi + 5\Box_y B = 0\,.
\end{aligned} \tag{3.14}$$

This results in

$$L^2\Box_x \begin{pmatrix} B \\ \pi \\ \delta\phi \end{pmatrix} = \begin{pmatrix} \frac{2}{5}\lambda & \frac{5}{6} & \frac{1}{2} \\ 18\lambda & \frac{2}{5}\lambda + \frac{37}{2} & \frac{15}{2} \\ 2\lambda & \frac{5}{2} & \frac{2}{5}\lambda + \frac{15}{2} \end{pmatrix} \begin{pmatrix} B \\ \pi \\ \delta\phi \end{pmatrix}, \tag{3.15}$$

where as before $-R^2\Box_y Y^I = \lambda Y^I$. We find the mass eigenvalues

$$m^2L^2 = \frac{2}{5}\lambda + 6\,, \quad \frac{2}{5}\lambda + 10 + 2\sqrt{25+4\lambda}\,, \quad \text{and } \frac{2}{5}\lambda + 10 - 2\sqrt{25+4\lambda}\,. \tag{3.16}$$

The Breitenlohner-Freedman bound for $p = 4$ is $m^2L^2 \geq -9/4$. We see that the first two towers in (3.16) are harmless (in fact they're not even tachyonic), but the third tower will violate the BF bound if some value of λ falls in the interval

$$\lambda_{\text{unstable}} \in \left(\frac{155}{8} - 5\sqrt{\frac{5}{2}}, \frac{155}{8} + 5\sqrt{\frac{5}{2}}\right) \approx (11.47, 27.28)\,. \tag{3.17}$$

For S^6, we have $\lambda = k(k+5)$, for which $k = 2, 3$ gives values in the interval (3.17). Thus for both d- and f-waves, the eigen-combinations of B, π, and $\delta\phi$ corresponding to the third eigenvalue in (3.16) are unstable modes of the $AdS_4 \times S^6$ solution. They have the common mass $m^2L^2 = -12/5$.

It is interesting that in fact all values of m^2L^2 that occur for $AdS_4 \times S^6$ in the coupled scalar sector are rational: upon substituting $\lambda = k(k+5)$ into (3.16), we obtain

$$m^2L^2 = \frac{2k^2}{5} + 2k + 6\,, \quad \frac{2k^2}{5} + 6k + 20\,, \quad \text{and } \frac{2k^2}{5} - 2k\,. \tag{3.18}$$

However the corresponding dimensions of operators in a hypothetical three-dimensional CFT are not rational.

Instabilities can occur in the coupled scalar sector of other M_q as well. As an example, consider $M_6 = S^n \times S^{6-n}$. For product spherical harmonics on the two spheres labeled by (k_1, k_2), we find several unstable modes in the interval (3.17): $(1,1)$, $(0,2)$ and $(1,2)$ for $n=2$, and $(1,1)$, $(2,0)$ and $(0,2)$ for $n=3$.

As in section 3.4, the constraint relating h^μ_μ and h^α_α no longer obtains for the $k=1$ case on S^6, so a more careful analysis must be performed. Without imposing the algebraic constraint, the dilaton equation (3.10) is unmodified, while equations (3.6) and (3.12) become

$$(\Box_x + \Box_y)\pi + \Box_y(H+\pi) - \frac{1}{3}\Box_y\pi - \frac{37}{2L^2}\pi - \frac{15}{2L^2}\delta\phi + \frac{9}{2}c\Box_y b = 0, \tag{3.19}$$

$$(\Box_x + \Box_y)b - \frac{c}{2}\pi + \frac{c}{2}H - \frac{c}{2}\delta\phi = 0. \tag{3.20}$$

For $k=1$, we have $\Box_y = -12/5L^2$. The dilaton equation (3.10) then becomes

$$\begin{aligned}(\Box_x - \frac{99}{10L^2})\delta\phi - \frac{5}{2L^2}\pi - \frac{6}{5L^2}cb &= \\ (\Box_x - \frac{99}{10L^2})\delta\phi - \frac{5}{2L^2}\sigma &= 0,\end{aligned} \tag{3.21}$$

which defines $\sigma \equiv \pi + \frac{12}{25}cb$. Next, using (3.20) we can show that

$$\Box_x\pi + \Box_y H = \Box_x\sigma - \frac{12}{5L^2}\sigma - \frac{12}{5L^2}\delta\phi, \tag{3.22}$$

which allows us to write equation (3.19) as

$$\Box_x\sigma - \frac{249}{10L^2}\sigma - \frac{99}{10L^2}\delta\phi = 0. \tag{3.23}$$

As in the examples without a coupled dilaton, one linear combination of fields has dropped out of the $k=1$ system. We can now diagonalize the equations (3.21) and (3.23). We discover the mass eigenvalues

$$m^2L^2 = \frac{42}{5}, \qquad m^2L^2 = \frac{132}{5}, \tag{3.24}$$

which coincide with the $k=1$ masses in the first two towers of (3.18).

The constant Y^I sector is straightforward for all M_6. The form equation no longer obtains, and the b mode does not exist, leaving only the equations

$$\Box_x \pi = \frac{37}{2L^2}\pi + \frac{15}{2L^2}\delta\phi, \tag{3.25}$$

$$\Box_x \delta\phi = \frac{15}{2L^2}\delta\phi + \frac{5}{2L^2}\pi, \tag{3.26}$$

with corresponding positive-mass eigenvalues

$$m^2L^2 = 6, \qquad m^2L^2 = 20. \tag{3.27}$$

These are exactly the masses obtained from the first two towers in (3.16) with $k = \lambda = 0$. Thus the general "rule of thumb" (valid in all cases we have considered, as well as in the familiar supersymmetric examples) is that one simply drops the most tachyonic mode from the first two partial waves in the coupled scalar sector.

It is not hard to see that the remaining equations of motion are basically unmodified from the analysis of previous sections. The dilaton fluctuation $\delta\phi$ cannot appear in the other polarizations of the form equation, where the background field strength vanishes. Hence these are unchanged from before. In the Einstein equations, it is straightforward that $\delta\phi$ does not appear in the $R_{\mu\alpha}$ equation or in parts of the $R_{\alpha\beta}$ equation other than those treated already by considering the trace. Owing to the relations (3.3) arising from the requirement that the compact space is Einstein, these equations are identical to those we already studied once written in terms of L. The dilaton fluctuation and the other scalars do appear in the $R_{\mu\nu}$ equation, analogous to the appearance of π, b and H in (3.38), but this leads only to a scalar expression linearly dependent on the ones we have considered earlier.

Consequently, we can employ the work we have already done wholesale. In particular, we again have the potential source of instability from the set of scalars ϕ^I, obeying equation (3.1). Hence we learn that general product spaces are again unstable against having one factor shrink while the other grows.

3.9.2 Sugimoto's $USp(32)$ open string theory

The next example of an unstable Freund-Rubin compactification that we shall consider arises in the $USp(32)$ open string theory considered in [77]. As we shall discover, the modes which are unstable come from a mixing of the trace of the metric on AdS_3, the trace on S^7, and another scalar arising from the fluctuations of the form field. Thus, this instability is of the same type as the one for the Freund-Rubin compactification of massive Type IIA supergravity on $AdS_4 \times S^6$ [10]. Since the computation is exactly the same as the one for Type IIA that we discussed in the previous section, we shall be somewhat brief here.

The low-energy effective action of the Sugimoto theory in the string frame is [99]:

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{G}\left[e^{-2\phi}\left(R + 4(\partial\phi)^2\right) - \frac{1}{12}F_3^2 - \alpha e^{-\phi}\right]. \tag{3.28}$$

where α for our purposes is just a constant. In our conventions, $F_3^2 = F_{MNP}F^{MNP}$. To bring the action (3.28) into Einstein frame, we rescale the metric as $g_{MN} = e^{\frac{-\phi}{2}}G_{MN}$. The action then becomes:

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{g}\left[R - \frac{1}{2}(\partial\phi)^2 - \alpha e^{\frac{3}{2}\phi} - \frac{1}{12}e^{\phi}F_3^2\right]. \tag{3.29}$$

The scalar equation of motion which follows from this action is:

$$\Box\,\phi - \frac{3}{2}\alpha e^{\frac{3}{2}\phi} - \frac{1}{12}e^{\phi}F_3^2 = 0\,. \tag{3.30}$$

For a constant ϕ background, this equation can have a solution if we use the Freund-Rubin ansatz $F_{\mu\nu\rho} = f\epsilon_{\mu\nu\rho}$, i.e. the three-form is along the AdS part. So we find that $AdS_3 \times S^7$ is indeed a solution with $\phi = 0$.

For convenience, let us dualize the three-form and use a seven-form instead. The action (which is what we shall be using from now on) is:

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2\cdot 7!}e^{-\phi}F_7^2 - \alpha e^{\frac{3}{2}\phi}\right]. \tag{3.31}$$

The background geometry has $\phi = 0$. The equations of motion are:

$$R_{MN} = \frac{1}{2\cdot 6!}e^{-\phi}F_{MP_1\cdots P_5}F_N{}^{P_1\cdots P_5} - \frac{3}{8\cdot 7!}e^{-\phi}F_7^2 g_{MN} + \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{1}{8}\alpha e^{\frac{3}{2}\phi}g_{MN}\,, \tag{3.32}$$

$$d * (e^{-\phi} F_7) = 0\,, \tag{3.33}$$

$$\Box\,\phi + \frac{1}{2\cdot 7!} e^{-\phi} F_7^2 - \frac{3}{2}\alpha e^{\frac{3}{2}\phi} = 0\,. \tag{3.34}$$

We want to express all the parameters in terms of the AdS radius L. For the background $F_7 = c\,\mathrm{vol}_{S^7}$ and $\phi = 0$ so (3.34) gives $c^2 = \frac{1}{7!}F_7^2 = 3\alpha$. The Einstein equation yields $\alpha = \frac{2}{L^2} = \frac{12}{R^2}$. So, the ratio of the radii is $R^2 = 6L^2$. Let us now proceed to get the mass spectrum of the scalars. Tracing over the indices on the sphere in (3.32) gives:

$$R_\alpha^\alpha = \frac{7}{4L^2} e^{\frac{3}{2}\phi} + \frac{7}{8\cdot 7!} e^{-\phi} F_7^2 + \frac{1}{2}\partial^\alpha \phi \partial_\alpha \phi\,. \tag{3.35}$$

When expanded to linear order, the two sides of the above equation yield:

$$-\frac{1}{L^2}\pi - \frac{1}{2}(\Box_x + \Box_y)\pi = -\frac{21}{8L^2}\delta\phi + \frac{7}{4}c\Box_y b - \frac{21}{4L^2}\pi\,. \tag{3.36}$$

where π denotes the trace of the metric fluctuation on S^7, and the fluctuation of the 7-form field strength is expressed as $\delta F_7 = da_6$ with $a_6 = *_7 db$. On simplification this finally gives:

$$(\Box_x + \Box_y)\pi - \frac{17}{2L^2}\pi + 21\Box_y B - \frac{21}{4L^2}\delta\phi = 0\,, \tag{3.37}$$

where we've introduced the notation $b \equiv cL^2 B$. For the scalar fluctuations we expand (3.34) to linear order:

$$(\Box_x + \Box_y)\delta\phi - \frac{15}{2L^2}\delta\phi - \frac{3}{L^2}\pi + 6\Box_y B = 0\,. \tag{3.38}$$

The form equation (3.33) expanded to linear order yields after a little algebra:

$$(\Box_x + \Box_y)B - \frac{6}{7L^2}\pi - \frac{1}{L^2}\delta\phi = 0\,. \tag{3.39}$$

Assembling all the three equations, and assuming that B, π, and $\delta\phi$ are eigenvectors of $\Box_y$ with eigenvalue $-\lambda/R^2$, we obtain the following mass matrix equation:

$$\Box_x \begin{pmatrix} B \\ \pi \\ \delta\phi \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{R^2} & \frac{6}{7L^2} & \frac{1}{L^2} \\ \frac{21\lambda}{R^2} & \frac{\lambda}{R^2} + \frac{17}{2L^2} & \frac{21}{4L^2} \\ \frac{6\lambda}{R^2} & \frac{3}{L^2} & \frac{\lambda}{R^2} + \frac{15}{2L^2} \end{pmatrix} \begin{pmatrix} B \\ \pi \\ \delta\phi \end{pmatrix}\,. \tag{3.40}$$

On diagonalizing the matrix, and using the relation $R^2 = 6L^2$ to eliminate R^2, we obtain the eigenvalues (mass squared) $m^2L^2 = \frac{\lambda+24}{6}$, $\frac{\lambda+36+12\sqrt{\lambda+9}}{6}$, and $\frac{\lambda+36-12\sqrt{\lambda+9}}{6}$. Only states in the last tower can be tachyonic. On S^7, the spherical harmonics have eigenvalues $\lambda = k(k+6)$. The dangerous tower of states when expressed in terms of k become, $m^2L^2 = \frac{k(k-6)}{6}$. Remembering that the BF bound for this system is $m^2L^2 \geq -1$, we see that the modes $k = 2$, 3, and 4 violate the bound. The presence of three unstable modes makes it considerably more difficult to find a stable compactification where S^7 is replaced by some other seven-manifold M_7: there is a fairly wide range of eigenvalues for the laplacian on M_7 which would lead through (3.40) to an unstable mode.

3.10 Possible CFT duals

As discussed in the introduction, this investigation was motivated by the proposal [15] that the case $D = p+q = 27$ with a 4-form field is the low-energy limit of a "bosonic M-theory," and that its $AdS_4 \times S^{23}$ compactification has a CFT$_3$ dual in the framework of the AdS/CFT correspondence. Since an $AdS_p \times S^q$ compactification has been shown to be stable, it is interesting to speculate in general about possible CFT$_d$ duals (with $d = p-1$). We give a very heuristic discussion which emphasizes the pattern of operator dimensions.

For scalar operators the basic AdS/CFT relation $\Delta(\Delta - d) = m^2L^2$ admits the two roots

$$\Delta_\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m^2L^2}. \tag{3.1}$$

If the mass satisfies the inequality $m^2L^2 \geq -\frac{d^2}{4} + 1$, then only the assignment Δ_+ obeys the unitarity bound $\Delta \geq \frac{d}{2} - 1$. (This bound is saturated for a free massless scalar field in d dimensions). But for $-\frac{d^2}{4} \leq m^2L^2 \leq -\frac{d^2}{4} + 1$, both Δ_+ and Δ_- are, *a priori*, consistent choices for the scale dimension of the dual operator. On general grounds it seems most natural to choose the larger of the two dimensions, Δ_+, as the dimension of the operator, because only then can one compute correlators by straightforwardly imposing a boundary condition on the larger of the two linearly independent solutions of the scalar. If Δ_- is chosen as the dimension, then to obtain field theory correlators one must make a Legendre transform of the Δ_+ results. These points were discussed in [96], where also a particular example was exhibited where the Δ_- dimension was needed. In this example, the field theory was supersymmetric, and the operator was a chiral primary, so its anomalous dimension could be worked out purely on field theory grounds as the sum of the the anomalous dimensions of its factors. The computation is rigorous because all the dimensions are dictated by a $U(1)_R$ current which is obviously additive.

The mass eigenvalues of coupled scalars of general $AdS_p \times S^q$ compactifications are given in (3.7). Since $m_+^2 > 0$, the operator duals of positive branch scalars have the unique dimension assignments

$$\Delta = \frac{p-1}{2}\left[1 + \frac{2}{q-1}\left(k + \frac{3(q-1)}{2}\right)\right] . \tag{3.2}$$

For the negative branch of the scalar mass spectrum, there are the two possibilities

$$\Delta_\pm = \frac{p-1}{2}\left[1 \pm \frac{2}{q-1}\left|k - \frac{q-1}{2}\right|\right] . \tag{3.3}$$

In accord with the discussion in the previous paragraph the negative root is a possible choice in the range

$$\left|k - \frac{q-1}{2}\right| \leq \frac{q-1}{d} . \tag{3.4}$$

Recall that k indicates the $SO(q+1)$ representation formed from k factors of the vector, then symmetrized with the trace removed.

For the purposes of orientation, let us recall a familiar result for $AdS_5 \times S^5$. Here the chiral primary operators are $\operatorname{tr} X^{(I_1} \cdots X^{I_k)}$ in $\mathcal{N} = 4$ super-Yang-Mills theory, where $(I_1 \ldots I_k)$ indicates the symmetric traceless combination. Their AdS duals are the coupled fluctuations of the metric and the five-form on the negative branch that leads to (3.3). The dimensions are $\Delta(k) = k = 2, 3, 4, 5, \ldots$, and one always chooses Δ_+. The anomalous dimensions vanish: $\Delta(k) = k$ is the free-field result. A similar story holds for $AdS_4 \times S^7$, with $\Delta(k) = k/2$, except that one must choose Δ_- for $k = 2$. Some of these operators are thought of as coming from $\operatorname{tr} X^{(I_1} \cdots X^{I_k)}$ on coincident D2-branes, and for the others one must dualize the vector boson into an eighth scalar. Free field counting still applies, and it can be backed up by a supersymmetry argument as for the $AdS_5 \times S^5$ case. Lastly, for $AdS_7 \times S^4$, the dimensions are $\Delta(k) = 2k$, and one always chooses Δ_+. A free field understanding is lacking in this mysterious (2,0) theory, but as before a link can be established between the R-symmetry and the dimension which guarantees that $\Delta(k)$ is linear in k.

Let us begin the discussion of the spectra for general p and q by observing that it is doubly remarkable that both the quadratic equation for scalar masses and the equation $\Delta(\Delta - d) = m^2 L^2$ have rational roots in the general case. This is an aesthetically pleasing point for a putative CFT dual, but unfortunately it is the end of the good news.

Focusing on the negative branch (3.3) makes sense, since these were the simplest operators in cases which we understand. Starting with our free field prejudices, we might suspect that the k'th operator would be expressible as $\operatorname{tr} X^{(I_1} \cdots X^{I_k)}$, and that its dimension $\Delta(k)$ is linear in k. Then we arrive at $\Delta(k) = \frac{p-1}{q-1}k$. For example, $\Delta(k) = \frac{3}{22}k$ for $AdS_4 \times S^{23}$. This does not make sense because $k = 2$ gives $\Delta = \frac{3}{11} < \frac{1}{2}$, the free scalar dimension. That is, we tried to choose Δ_- in a range where only Δ_+ was possible. The general result is that a linear spectrum of dimensions $\Delta(k)$ is permitted provided

$$q - 1 \leq \frac{4(p-1)}{p-3}. \tag{3.5}$$

If this inequality fails, as in the case $AdS_4 \times S^{23}$, then some operators of low $SO(q+1)$ charge will have a larger dimension than operators of higher $SO(q+1)$ charge, which we may view as a failure of the free-field intuition that singlet operators are built from fundamental fields whose dimensions add. It does not mean, however, that there can't be a CFT dual: for instance, it is consistent with the unitarity bound to choose Δ_+ uniformly, which produces a spectrum $\Delta(k)$ with a kink about $k = \frac{q-1}{2}$. More arcane choices may also be imagined. In the absence of supersymmetry or some input from field theory, we have no way of deciding between the alternatives.

Let us now discuss the spectra of coupled vectors for general $AdS_p \times S^q$ compactifications. Inserting the eigenvalue formula $\kappa = (k+1)(k+q-2)$ for vector spherical harmonics in

(3.2), we find the masses

$$m^2L^2 = \frac{(p-1)^2}{(q-1)^2}(k+1)(k+q-2)+(p-1)\left(1 \pm \sqrt{1+2\frac{p-1}{(q-1)^3}(p+q-2)(k+1)(k+q-2)}\right). \tag{3.6}$$

These mass eigenvalues are generically irrational (although they are rational for the supersymmetric compactifications $AdS_4 \times S^7$, $AdS_7 \times S^4$ and $AdS_5 \times S^5$.) Irrationality persists for vector scale dimensions (except for Killing vectors, where $m^2 = 0$)

$$\Delta = \frac{1}{2}[d + \sqrt{(d-2)^2 + 4m^2}]. \tag{3.7}$$

In particular, $AdS_4 \times S^{23}$ has irrational masses and dimensions for massive vectors.

It is certainly remarkable that the scalars dual to chiral primary operators in the well-understood $AdS_5 \times S^5$, $AdS_4 \times S^7$, and $AdS_7 \times S^4$ vacua still lead to rational dimensions for general p and q. If (3.5) is violated and a linear spectrum of dimensions is impossible for scalars, then it seems difficult to imagine a concise understanding based on a Lagrangian. The fact that massive vector modes generically have irrational dimensions also makes it seem less likely that a purely field theoretic formulation of the putative dual CFT will be accessible in the near future.

The $AdS_4 \times S^6$ compactification presents an even less rosy picture, in that the BF bound is violated. Obvious candidates for a brane realization of this vacua (involving D2-branes and D8-branes) seem also to be unstable, only the instability is usually in the form of a tadpole instead of a tachyon. It would be very interesting if a stable $AdS_4 \times M_6$ vacuum could be found for appropriate M_6, corresponding to some analyzable type I′ brane configuration. It would also be satisfying if one could start with some unstable D2-D8 construction and show that in an appropriate near-horizon limit the brane instability reduces to the violations of the BF bound that we have observed.[11]

Finally, let us extend some remarks on thermodynamics made in [15] for the $AdS_4 \times S^{23}$ and $AdS_{23} \times S^4$ cases. An obvious measure of the number of degrees of freedom in a CFT in $p-1$ dimensions is the ratio $c_{\text{thermo}} = S/(VT^{p-1})$. In the $p+q$-dimensional theory, there

[11] We thank O. Bergman and A. Brandhuber for discussions on these and related points.

are solutions with both magnetic and electric charge under the field strength F_q, so there is flux quantization, and we can ask how c_{thermo} scales with N, the number of flux quanta through the compact space. For $AdS_p \times S^q$, we can reason out this scaling by recalling that in an asymptotically flat solution, the number of branes enters the harmonic function in the metric as $H = 1 + c_1 N(\ell_{\text{Pl}}/r)^{q-1}$, where c_1 is some dimensionless constant. Thus L and R scale as $N^{1/(q-1)}\ell_{\text{Pl}}$. In a near-extremal solution, the Bekenstein-Hawking entropy scales as $(L/\ell_{\text{Pl}})^{p+q-2}$, whereas the Hawking temperature does not scale with ℓ_{Pl} at all. Putting everything together, one finds

$$c_{\text{thermo}} \sim N^{(p+q-2)/(q-1)} . \tag{3.8}$$

This specializes to the odd results $c_{\text{thermo}} \sim N^{25/22}$ for $AdS_4 \times S^{23}$ and $c_{\text{thermo}} \sim N^{25/3}$ for $AdS_{23} \times S^4$. These peculiar fractions do not bring any known CFT's to mind, but at least they represent something to shoot for in constructing putative duals of $AdS_p \times M_q$.

3.11 Implications for Extremal Black Branes and Negative Energy

As mentioned above, the (nondilatonic) theories of gravity (3.1) all contain charged black brane solutions, where the charge is obtained by integrating F_q over an S^q surrounding the brane. (For the general solution, see [74].) In particular, there are extremal black branes, with metric

$$ds^2 = H^{-\frac{2}{p+1}}(-dt^2 + d\mathbf{y}\cdot d\mathbf{y}) + H^{\frac{2}{q-1}}(dr^2 + r^2 d\Omega_q) \tag{3.1}$$

where H is the harmonic function $H(r) = 1 + c_1 N(\ell_{\text{Pl}}/r)^{q-1}$. The near horizon limit is just $AdS_p \times S^q$. So the stability we have found for $AdS_p \times S^q$ for all p and q is consistent with the expected stability of extremal solutions. However, we have also seen that $AdS_p \times M_n \times M_{q-n}$ is unstable, when $q < 9$ and M_n, M_{q-n} are Einstein spaces. These can also arise as the near horizon limit of a type of extremal black brane as follows. Consider the cone over $M_n \times M_{q-n}$

$$ds^2 = dr^2 + r^2(d\sigma^2_{M_n} + d\sigma^2_{M_{q-n}}) \tag{3.2}$$

This space is Ricci flat, and has a curvature singularity at the apex $r = 0$. (Even though the curvature goes to zero for large r, this space is not asymptotically flat in the usual sense since the curvature only falls off like r^{-2}.) Suppose one places a stack of branes at the apex of the cone, extended in the orthogonal directions. The resulting exact solution is obtained by simply replacing the flat transverse metric in (3.1) with the cone metric (3.2).

One might have expected this new solution to be stable, since it is the extremal limit of a family of black brane solutions. However it is easy to see that it is not (at least for $q < 9$). The near horizon limit is now $AdS_p \times M_n \times M_{q-n}$ which is unstable to a perturbation (equation (3.7)) that goes to zero asymptotically in AdS_p. So a similar perturbation with support very close to the horizon of the extremal black brane will also grow exponentially. This is independent of the change in boundary conditions at infinity since, in the Poincare coordinates appropriate to the near horizon geometry of AdS_p, a scalar field near the horizon has a unique evolution inside a spacetime region that includes infinite Poincare time. One might object that extremal black branes are always unstable in the sense that adding a small amount of energy causes them to become nonextremal[12] , and the horizon moves from an infinite distance to a finite distance (in spacelike directions). However, as we will see, our perturbation is very different in that it can actually decrease the mass.

A natural question to ask is what does this instability lead to? As we have seen, the unstable mode causes one factor, say M_n, to shrink in size and the other to grow. So one might expect that in the full nonlinear evolution, M_n simply shrinks to zero size. A serious difficulty with this picture was discussed by Horowitz and Maeda [65] – if the weak energy condition is satisfied, event horizons cannot have collapsing cycles. We sketched their main argument in Section 2.6 of chapter 2. The instability we are discussing can be viewed as an extremal analog of the Gregory-Laflamme instability. Since our theory satisfies the weak energy condition, and the result in [65] does not require that the horizon is nonextremal, it can also be applied to our case. Thus, M_n cannot shrink to zero size, and there must be

[12]This is true for branes of finite extent. For infinite branes, one needs nonzero energy density to become nonextremal.

another static solution whose near horizon geometry is not $AdS_p \times M_n \times M_{q-n}$. [13]

Strictly speaking, the near horizon limit of the black brane solution includes only part of AdS_p (the region covered by the Poincare coordinates). Suppose we now consider the global solution $AdS_p \times M_n \times M_{q-n}$ and ask what happens if we perturb it in the unstable direction. As a first step toward answering this question, we show that there are solutions in the full nonlinear theory which are asymptotically $AdS_p \times M_n \times M_{q-n}$ and have arbitrarily negative energy (where, as usual, we measure energy relative to AdS_p). Since the perturbation violates the BF bound, it is clear we can lower the energy slightly by turning on this mode. To show the energy can be arbitrarily negative, it suffices to construct suitable initial data. Consider the spatial metric

$$ds^2 = \left[\frac{r^2}{L^2} + 1 - \frac{m(r)}{r^{p-3}}\right]^{-1} dr^2 + r^2 d\Omega_{p-2} + e^{(q-n)\phi(r)} d\sigma_{M_n} + e^{-n\phi(r)} d\sigma_{M_{q-n}} \tag{3.3}$$

so $m = 0, \phi = 0$ corresponds to the metric on a static surface (in global coordinates) for $AdS_p \times M_n \times M_{q-n}$. The total mass is proportional to $m(\infty)$. Notice that the volume of the q-dimensional internal space is independent of ϕ. This is a nonlinear generalization of the perturbation we considered in section 3.8. We again set $F_q = c \, \mathrm{vol}_{M_q}$. If we set all time derivatives to zero, the only constraint on this initial data is the Hamiltonian constraint of general relativity which implies that the scalar curvature of (3.3) must be $c^2/2$ where c^2 is given by (3.8). This yields a first order differential equation which can be used to solve for $m(r)$ in terms of $\phi(r)$. If we assume ϕ is everywhere small, this equation becomes

$$\frac{m'}{r^{p-2}} \propto \left[\frac{r^2}{L^2} + 1 - \frac{m(r)}{r^{p-3}}\right](\phi')^2 - \frac{2(p-1)^2}{(q-1)L^2}\phi^2 \tag{3.4}$$

The right hand side resembles the energy density of the linearized unstable mode (3.7) except that the ϕ' term involves the corrected spatial metric. Since the term involving $m(r)$ on the right hand side only decreases the energy density we can get an upper limit on the mass by dropping it. One can now explicitly find $\phi(r)$ so that $m(\infty)$ is arbitrarily negative.

[13] One might worry that there will be a problem applying the result in [65] since the unstable extremal black brane is not asymptotically flat in the usual sense. However, even though null infinity is not well defined, one can still define the event horizon as the boundary of the past of a surface at large r, and the result will still apply.

For example, if $q < 9 - (8/p)$, one can take $\phi = \phi_0 e^{-r/a}$. The total mass is negative for large a, and goes to minus infinity as $a \to \infty$.

If we start with $AdS_p \times M_n \times M_{q-n}$ and perturb it slightly, the energy will be only slightly negative. As we have just seen, this is very far from the minimum energy solution. A priori, one might expect M_n to collapse down to zero size in finite time. This will produce a curvature singularity. It is unlikely that this singularity is naked, since we don't expect cosmic censorship to be violated so easily in the higher dimensional theory of gravity we are considering. It may form a black hole, or in light of the horizon results, M_n may not collapse down at all. In the latter case, since we are using reflecting boundary conditions at infinity (appropriate for the AdS/CFT correspondence), the solution may not settle down to any static configuration. It would be interesting to investigate this further.

We have not considered the massive IIA theory in this section. It would also be interesting to investigate the implications of the instability of $AdS_4 \times S^6$ for negative energies and extremal black branes in this theory.

Chapter 4

A Test of AdS/CFT beyond the classical level

4.1 Double-trace operators and One Loop Vacuum Energy in AdS/CFT

We mentioned in the introduction that the AdS/CFT correspondence [3, 5, 6] relates a d-dimensional quantum field theory to a $(d+1)$-dimensional gravitational theory, the most notable example being $\mathcal{N}=4$, $d=4$ super-Yang-Mills theory and type IIB string theory on $AdS_5 \times S^5$. Most of the checks and predictions of this duality have been at the level of classical supergravity. It is particularly difficult to carry out meaningful loop computations in AdS, corresponding to $1/N$ corrections in the gauge theory, simply because the supergravity theory is highly non-renormalizable, and the Ramond-Ramond fields make computations in the string genus expansion unwieldy at best. The aim of this chapter, which is based on the paper [12], is to obtain a simple one-loop result in AdS that is finite in any dimension. The result is an expression for the difference of the vacuum energies that arises from changing boundary conditions on a tachyonic scalar field with mass in a particular range.

The inspiration for this computation came from Witten's treatment [100] of multi-trace deformations of the gauge theory lagrangian and their dual descriptions in asymptotically anti-de Sitter space. Such a dual description was also discussed in [101]; however, our treatment will follow [100] more closely. Earlier work describing the same gauge theory deformations in terms of non-local terms in the string worldsheet action appeared in [102, 103]. To be definite, suppose one were to add to the gauge theory lagrangian a term $\frac{f}{2}\mathcal{O}^2$ where $\mathcal{O}$ is a single trace operator with dimension 3/2, dual to a scalar field ϕ whose mass satisfies $m^2L^2 = -15/4$.[1] The coefficient f has dimensions of mass, so $\frac{f}{2}\mathcal{O}^2$ is a relevant deformation, and there is a renormalization group (RG) flow starting from a UV fixed point where $f = 0$. The endpoint of this flow is, plausibly, an IR fixed point whose correlators are related to those of the original $f = 0$ theory, in the large N limit, by a Legendre transformation in a manner explained in [96].[2] In particular, the scalar that was for $f = 0$ related to the operator $\mathcal{O}$ of dimension 3/2, is at the IR fixed point related to an operator $\tilde{\mathcal{O}}$ of dimension 5/2.

How is all this reflected in AdS? According to [100], the addition of $\frac{f}{2}\mathcal{O}^2$ amounts to specifying particular linear boundary conditions on the scalar ϕ at the boundary of AdS. At the classical level, these boundary conditions are consistent with the original AdS_5 solution with $\phi = 0$. Superficially, this looks like a puzzle, since we were expecting an RG flow. In fact, conformal invariance is violated by the $\mathcal{O}^2$ deformation, but at leading order in N its effects are restricted to certain correlators that we will describe in section 4.2. The crux of the matter is that it is impossible to satisfy the boundary conditions on ϕ with a $SO(4,2)$-invariant bulk-to-bulk propagator, except when $f = 0$ or ∞. This gives rise to one loop effects that cause deviations from AdS_5.

Although we will not obtain the full one-loop corrected solution corresponding to an RG flow due to the $\frac{f}{2}\mathcal{O}^2$ deformation, we will consider its endpoints and perform a one-loop supergravity check of the c-theorem. This "theorem," conjectured in four dimensions by Cardy [16] as a generalization of Zamolodchikov's celebrated two-dimensional c-theorem [17], has been shown to follow from AdS/CFT at the level of classical supergravity provided

[1]Such a situation could arise in the theory dual to D3-branes at the tip of a conifold [75], where there are indeed dimension 3/2 color singlet operators.

[2]We will discuss further in section 4.2 the reasoning behind the claim that the flow ends at an IR fixed point, as well as some caveats.

the null energy condition holds [104, 18] (see also [105] for earlier work in this direction). The magnitude of the vacuum energy of AdS_5, measured in five-dimensional Planck units, is proportional to an appropriate central charge raised to the $-2/3$ power. So the vacuum energy should be more negative in the infrared than in the ultraviolet, and at the classical level, that is what is shown in [104, 18] (actually, the arguments on the AdS are dimension-independent, though it is not entirely clear how to translate the "holographic" central charge into field theory language in the case of odd-dimensional CFT's). At the quantum level, the arguments of [104, 18] have no force because it's not clear that the null energy condition is valid or even relevant. So an explicit loop calculation is appropriate. All that is needed is the one-loop contribution of the scalar ϕ to the vacuum energy. This quantity is divergent, but the difference between imposing the two simple boundary conditions (described above as $f \to 0$ and $f \to \infty$) gives a finite result. The contributions of all other fields can be ignored because they do not change at the one loop level as one changes the boundary conditions on ϕ. Also, because we only desire a one-loop vacuum amplitude, we may entirely ignore interactions of the scalar with other fields, and work simply with the free action

$$S = \int d^5 z \sqrt{g} \left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \right) . \tag{4.1}$$

where we work in mostly plus signature, so that the metric of AdS_5 on the Poincaré patch is

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2 \right) . \tag{4.2}$$

For definiteness, our discussion has focused on AdS_5 and a scalar with a particular mass; however, the results we will obtain can be presented with considerable generality for AdS_{d+1}, as we will describe. For odd d, the formulas for the vacuum energy are much more complicated, and for the sake of efficiency we check the sign via numerics.

The organization of the chapter is as follows. In section 4.2 we briefly review the prescription of [100] for treating multi-trace operators, and we demonstrate that general boundary conditions are incompatible with $SO(4,2)$-invariance of the scalar propagator. In section 4.3 we compute the finite change in the one-loop vacuum energy discussed above, and make some remarks on the interpolating geometry connecting the two anti-de Sitter endpoints. We conclude in section 4.4 by extracting the prediction for the central charge, and observing that the c-theorem is obeyed.

4.2 Multi-trace operators and scalar propagators

The proposal of [100] is a natural generalization of the original prescription for computing correlators [5, 6], and it should in principle be derivable from it: see [106] for a more precise discussion. Suppose one starts with the complete set $\mathcal{O}_a$ of independent, local, color-singlet, normalized, single-trace operators: for $\mathcal{N}=4$ super-Yang-Mills theory these would include, for example, $\frac{1}{N}\operatorname{tr} X_1 X_2$ and $\frac{1}{N}\operatorname{tr} F_{\mu\nu}\nabla_\rho\lambda_1$. The action can be written as $I = N^2 W(\mathcal{O}_a)$ for some functional W, which for $\mathcal{N}=4$ super-Yang-Mills would be the integral of a linear function of those $\mathcal{O}_a$ which are Lorentz scalars. The general belief is that the $\mathcal{O}_a$ can be put into one-to-one correspondence with the quantum states of type IIB string theory in AdS_5.[3]

Restricting ourselves to scalars in AdS_5, we have the standard relation $\Delta_a(\Delta_a - d) = m_a^2 L^2$ relating the dimension of $\mathcal{O}_a$ to the mass of the field ϕ_a. Writing the metric for the Poincaré patch of AdS_5 as

$$ds^2 = \frac{L^2}{r^2}\left(-dt^2 + \sum_{i=0}^{d-2} dx_i^2 + dr^2\right), \tag{4.3}$$

we have boundary asymptotics for ϕ_a as follows:

$$\phi_a \sim \alpha_a(x) r^{d-\Delta_a} + \beta_a(x) r^{\Delta_a} \qquad \text{for } r \to 0. \tag{4.4}$$

The prescription of [100] is to replace $W(\mathcal{O}_a)$ by $W(\beta_a)$ and impose the following boundary conditions:

$$\alpha_a(x) = \frac{\delta W}{\delta\beta_a(x)}. \tag{4.5}$$

[3]There is considerable subtlety in this claim. It has been demonstrated that the Kaluza-Klein tower of supergravity modes in $AdS_5 \times S^5$ is in correspondence with the chiral primaries of $\mathcal{N}=4$ super-Yang-Mills and their descendants; and the duals of certain non-perturbative states have been found, such as dibaryons (see for example [107] and giant gravitons [108]. Evidence is growing that the operator-state map extends faithfully to excited string states (see for example [109, 110]). Since the states in question can sometimes be extended across most of AdS_5 (as in [110]), it is not entirely clear that a second quantized treatment in terms of local fields is appropriate; but this is scarcely relevant to the situation at hand, since extended states are very massive, and we're interested only in tachyons.

The partition function of the gravitational theory in AdS, subject to the boundary conditions (4.5), is then supposed to equal the partition function of the gauge theory.

The simplest non-trivial example is double trace operators: most simply, $\mathcal{O}^2$ where the scalar operator $\mathcal{O}$ has dimension Δ between $\frac{d}{2} - 1$ and $d/2$. The lower bound $\frac{d}{2} - 1$ is the dimension of a free scalar field and is the minimum dimension required to satisfy unitarity. The upper bound of $\frac{d}{2}$ is chosen so that the double trace operator is relevant. Then W includes a term $\frac{f}{2} \int d^d x \, \mathcal{O}^2$. This brings us back to the discussion initiated in the introduction: nonzero f plausibly drives the field theory from a UV fixed point where the boundary conditions are $\alpha = 0$ to an IR fixed point where the boundary conditions are $\beta = 0$. Since these two fixed points will be the focus of section 4.3, let us introduce an additional convenient notation: Δ_+ and Δ_- are the two solutions to $\Delta(\Delta - d) = m^2 L^2$, with Δ_- being the lesser of the two (and thus in the aforementioned range, from $\frac{d}{2} - 1$ to $d/2$). Clearly $\Delta_+ = d - \Delta_-$.

When $\Delta_- < d/2$, the addition of a trace-squared operator $\mathcal{O}^2$, where $\mathcal{O}$ has dimension $\Delta = \Delta_-$, is a relevant deformation, so conformal invariance must be broken in the gauge theory. The results of [100] for $d = 4$ and $\Delta_- = 2$ suggests that even when $\Delta_- = d/2$ there is a logarithmic RG flow. The simplest indication of the breaking of conformal invariance in supergravity is that the bulk-to-bulk propagator for the scalar ϕ dual to $\mathcal{O}$ cannot be $SO(4,2)$-invariant. We will now demonstrate this claim.

The propagator in question can be defined as

$$iG(z, z') = \langle 0 | T\{\phi(z)\phi(z')\} | 0 \rangle \,, \tag{4.6}$$

and it satisfies the equation of motion

$$(\Box - m^2) G(z, z') = \delta^{d+1}(z - z') \,, \tag{4.7}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, and the delta function includes a $1/\sqrt{g}$ in its definition, so that

$$\int d^{d+1}z \sqrt{g}\, f(z) \delta(z - z') = f(z') \tag{4.8}$$

for any continuous function $f(z)$. If the propagator is to respect $SO(4,2)$ invariance, it must be a function only of the geodesic distance $\sigma(z, z')$, which is known to be

$$\sigma(z, z') = L \log\left(\frac{1+\sqrt{1-\zeta^2}}{\zeta}\right) \qquad \text{where} \qquad \zeta = \frac{2rr'}{r^2 + r'^2 - (t-t')^2 + (\vec{x}-\vec{x}')^2}, \quad (4.9)$$

where L is the radius of AdS. The only solutions to (4.8) which are functions only of ζ are $G(z, z') = pG_{\Delta_-} + (1-p)G_{\Delta_+}$ where for any Δ (cf. [111, 112]),[4]

$$iG_\Delta = \frac{\Gamma(\Delta)}{2^\Delta \pi^{d/2} L^{d-1}(2\Delta - d)\Gamma(\Delta - \frac{d}{2})} \zeta^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - \frac{d}{2} + 1; \zeta^2\right). \qquad (4.10)$$

where F is the hypergeometric function. By keeping z' fixed while z approaches the boundary of AdS, it is straightforward to verify that for no choice of $p \in (0,1)$ and $f \in (0, \infty)$ does the propagator $G(z, z') = pG_{\Delta_-} + (1-p)G_{\Delta_+}$ satisfy the boundary conditions (4.5), which in our case amount to $\alpha = f\beta$. For $p = 0$ and $f = 0$ the boundary conditions are satisfied with $SO(4,2)$ invariance preserved, corresponding to a fixed point of RG where ϕ is dual to an operator $\mathcal{O}$ with dimension Δ_-. Let us call this the Δ_- theory. And for $p = 1$ and $f = \infty$ (formally speaking), again the boundary conditions are satisfied with $SO(4,2)$ invariance, and now ϕ corresponds to an operator $\tilde{\mathcal{O}}$ with dimension Δ_+: this we will call the Δ_+ theory.

It was already remarked in [100] that a renormalization group flow should interpolate between the Δ_- theory in the UV and the Δ_+ theory in the IR. This is in fact a somewhat subtle claim: why should we think that the RG flow initiated by adding $\frac{f}{2}\mathcal{O}^2$ ends up at a non-trivial IR fixed point? We can argue as follows: the Legendre transformation prescription of [96] guarantees that the IR fixed point exists, at least in the large N limit. The existence of a fixed point of RG is a generic phenomenon, so $1/N$ corrections should not spoil the claim, nor should they greatly alter the location of the fixed point in the space of possible couplings. Since a naive scaling argument (just looking at the dimension of f) tells us that the RG flow should end up at the desired IR fixed point if we ignore all $1/N$ corrections, it should be that *some* RG flow exists close to the approximate one we naively identified, ending at the non-trivial IR fixed point. A significant caveat to this reasoning is

[4]The expression for $G(z, z')$ above differs by a sign from that in [112, 111] because the latter define the Green's function as $-iG(z, z') = \langle 0|\mathrm{T}\phi(z)\phi(z')|0\rangle$.

that AdS/CFT examples often (in fact, nearly always in the literature so far) have exactly marginal deformations. A *line* of fixed points of RG is *not* a generic phenomenon, and $1/N$ effects in the absence of supersymmetry generically could destroy such a line. Only one point could be left after $1/N$ effects are included; or, worse yet, only a point infinitely far out in coupling space could be left. Translated into supergravity terms, these remarks mean that the one-loop contribution to the potential could source the dilaton or other moduli, possibly leaving no extrema at finite values of the fields. If there are no such moduli in the first place (as perhaps one would expect for a truly *generic* non-supersymmetric quantum field theory with an AdS dual), then this caveat is not a problem. In practice, however, it is likely to interfere with constructing explicit string theory examples of the RG flow discussed in this chapter. For the remainder of our discussion, we will ignore the caveat.

Since the renormalization group flow is non-trivial, it is natural to expect that the supergravity geometry deviates from AdS. The surprise is that this does *not* happen classically. Roughly, this can be understood in field theory terms as a reflection of the fact that n-point functions involving only the stress energy tensor do not receive corrections at leading order in N.[5] At subleading order in N, or at one-loop in supergravity, deviations from AdS must occur, simply because a one-loop diagram where the $SO(4,2)$-non-invariant scalar propagator closes upon itself must give rise to an effective potential that varies over spacetime. Entertainingly, there is no classical scalar field which is varying; rather, the variation in the potential arises on account of proximity to the boundary. This is in contrast to previously studied examples of RG flow in AdS_5 (for instance [104, 18]) where the flow is described in terms of scalars in the five-dimensional supermultiplet of the graviton with non-trivial dependence on radius.

There should be a solution to the one-loop-corrected supergravity lagrangian interpolating between one asymptotically AdS region near the boundary, corresponding to the Δ_- UV fixed point, and a different one in the interior, corresponding to the Δ_+ IR fixed point.

[5]Correlation functions which *do* receive corrections at leading order in N when $\frac{f}{2}\mathcal{O}^2$ is added to the lagrangian are precisely those which pick up contributions from factorized forms $\langle\mathcal{O}\ldots\rangle\langle\mathcal{O}\ldots\rangle$, where the dots indicate any arrangement of the operators involved in the original correlator.

For instance, one could require that the symmetries of $\mathbf{R}^{3,1}$ be preserved in the solution, which must then have the form

$$ds^2 = e^{2A(r)}(-dt^2 + d\vec{x}^2) + dr^2 , \tag{4.11}$$

where $A(r) \rightarrow r/L_{\mp}$ as $r \rightarrow \pm\infty$. (Another choice would be to require the symmetries of $\mathbf{S}^3 \times \mathbf{R}$, which should lead to a solution with the conformal structure of global AdS). We will not find the full interpolating solution, but we will explore some properties of its AdS endpoints. We will be particularly interested in the central charge of the CFT's dual to the two endpoints. To the leading non-trivial order, these may be computed as a one-loop saddle-point approximation to the supergravity "path integral" (supposing that such an object exists), but without deforming the AdS background itself.

4.3 One loop vacuum energy for the tachyon field

The full classical action that we wish to consider is

$$S = \frac{1}{2\kappa^2}\int d^{d+1}z\sqrt{g}\,(R - \Lambda_0) + \int d^{d+1}z\sqrt{g}\left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right) . \tag{4.12}$$

Here Λ_0 is a negative constant. The scalar is subject to the boundary conditions

$$\phi \sim \alpha r^{d-\Delta} + \beta r^{\Delta} \qquad \text{where} \qquad \alpha = f\beta . \tag{4.13}$$

As remarked previously, AdS_{d+1} with $\phi = 0$ and $1/L^2 = -\frac{\Lambda_0}{d(d-1)}$ is a classical solution to the equations of motion from (4.13), but we expect that once one-loop effects are accounted for, this solution is corrected to an interpolation between AdS_{d+1} spaces in the UV and IR with slightly different radii. The one-loop scalar bubble diagram corrects the gravitational lagrangian by an amount $\delta\mathcal{L}$, where

$$-\sqrt{g}^{-1}\delta\mathcal{L} = V = -\frac{i}{2}\,\mathrm{tr}\log(-\Box + m^2) . \tag{4.14}$$

Our main computation will be to evaluate this correction in the unperturbed background. In principle, one could go on to find the interpolating geometry perturbatively in the small parameter $\kappa\Lambda_0^{(d-1)/2}$. This would require separating $\delta\mathcal{L}$ into contributions to the cosmological term and two- and four-derivative expression in the metric—a much more involved computation than simply evaluating (4.14) in the unperturbed background. For brevity, we will use the notation V in preference to $\delta\mathcal{L}$ for the scalar self-energy (4.14), despite the fact that in the full background-independent form involves derivative terms as well as finite non-local terms. V is divergent, but we assume that the action (4.12) is part of well-defined theory of quantum gravity (presumably, a compactification of string theory or M-theory), so that all loop divergences are canceled in some physical way, leaving only finite renormalization effects. It may be that in the full theory, Λ_0 is just the extremal value of a classical potential function of several scalars; if so, then we are operating on the understanding that the second derivative of this potential function with respect to ϕ vanishes at $\phi = 0$ (that is, we've soaked up any such second derivative into what we call m^2 in (4.12)).

In general, it is difficult to compute one-loop corrections in an effective theory without knowing precisely how the full theory cancels divergences. Results obtained for a chiral anomaly in supergravity [113] for $AdS_5 \times S^5$ can be used to show that the central charge is corrected at one loop in supergravity, leading to $c \propto N^2 - 1$, as appropriate for $SU(N)$ super-Yang-Mills, rather than $c \propto N^2$ (the leading order result). Thus in this case, the difficulties were overcome. Our situation is more generic, in that we do not depend on supersymmetry or a special spectrum of operators. What we are nevertheless able to do is to determine the finite difference between V in the case where $f = 0$ in (4.13) and the case where $f = \infty$. This we will then translate into a change in the central charge as one flows from the UV (the Δ_- theory) to the IR (the Δ_+ theory). What makes the computation clean is that at one loop, we do not have to worry about interactions of the scalar with other fields, and the only relevant diagram is the one where a single scalar propagator closes on itself, with no vertices.

4.3.1 Vacuum energy in limiting regions of AdS

The computation of the one-loop contribution to the vacuum energy by a scalar in curved space, like in flat space, amounts to summing the logarithm of the eigenvalues of the Klein-Gordon operator. A more easily computable expression is obtained by expressing the result in terms of an integral of the Green's function with respect to some parameter such as proper time or mass.[6] All of this is quite standard, so we just write down the result, referring the reader to [115] pp. 156-158 for a derivation: if the propagator $G(z, z'; m^2, f)$ is defined by

$$(\Box_z - m^2)G(z, z'; m^2, f) = \delta^{d+1}(z - z'), \tag{4.15}$$

(with the delta-function including a $\sqrt{g}$ factor as in (4.8)) together with boundary conditions (4.13), as discussed in section 4.2, then formally,

$$V(z; m^2, f) = -\frac{i}{2} \lim_{z \to z'} \int_{m^2}^{\infty} d\tilde{m}^2 \, G(z, z'; \tilde{m}^2, f), \tag{4.16}$$

and for the cases $f = 0, \infty$, the fact that we can make the scalar propagator $SO(4, 2)$ invariant means that V will be independent of the position z.[7] The formula (4.16) is problematic because for large masses, $G(z, z', \tilde{m}^2, 0)$ diverges at the boundary of AdS. This is unusual: the typical situation for quantum field theory in curved spacetime is that quantities become well-defined in the limit where masses are much larger that the inverse radius of curvature. Thus, instead of using (4.16), a well-defined procedure is to integrate down to the Breitenlohner-Freedman bound which is the smallest mass possible with normalizable modes in AdS. Thus we obtain

$$V(z; m^2, f) = V(z; m_{BF}^2, f) + \frac{i}{2} \lim_{z \to z'} \int_{m_{BF}^2}^{m^2} d\tilde{m}^2 \, G(z, z'; \tilde{m}^2, f), \tag{4.17}$$

where $m_{BF}^2 L^2 = -d^2/4$ is the Breitenlohner-Freedman bound. (For a derivation see the Appendix). It is possible to argue that $V(z; m_{BF}^2, f)$ is the same for $f = 0$ and $f = \infty$.

[6] For a different method of computing the effective potential based on the technique of Zeta-function regularization see [114].

[7] Actually, we have tucked an additional complication into our notation: V is, more properly, minus the one-loop correction to the full gravitational lagrangian, and as such includes not just a scalar piece, but also terms depending on curvatures. For the central charge computation, as we shall explain, the relevant quantity is the sum of all these terms evaluated on AdS.

Indeed, the eigenmodes for a tachyon of mass m^2 with boundary conditions specified by $f = 0$ is given by $\omega = \Delta_- + \ell + 2n$ and that specified by $f = \infty$ is given by $\omega = \Delta_+ + \ell + 2n$ [70], where ℓ is the orbital angular momentum quantum number and n is the radial quantum number. But for a scalar with mass saturating the BF bound, $\Delta_+ = \Delta_- = \frac{d}{2}$. So from a viewpoint of canonical quantization it seems inevitable that $V(z; m_{BF}^2, 0) - V(z; m_{BF}^2, \infty) = 0$. We can argue further that for general f the eigenfunctions would be a linear combination of those with $f = 0$ and $f = \infty$. That would again imply that for $\Delta = \frac{d}{2}$, the eigenvalues are unchanged. So we conclude that the $V(z; m_{BF}^2, f) - V(z; m_{BF}^2, 0) = 0$ for all values of f.

Thus we are led to the formula that we will really use for computation:

$$V_+ - V_- = \frac{i}{2} \int_{m_{BF}^2}^{m^2} d\tilde{m}^2 \left[G_{\tilde{\Delta}_+}(z,z) - G_{\tilde{\Delta}_-}(z,z) \right] + V(z; m_{BF}^2, \infty) - V(z; m_{BF}^2, 0) \,. \quad (4.18)$$

where $V_+ = V(z, m^2, \infty)$ and $V_- = V(z, m^2, 0)$. We have used the fact that $G_{\tilde{\Delta}_+}(z, z')$, as defined in (4.10), is precisely $G(z, z'; \tilde{m}^2, \infty)$, while $G_{\tilde{\Delta}_-}(z, z') = G(z, z'; \tilde{m}^2, 0)$. In light of the argument of the previous paragraph, the terms outside the integral cancel. The advantage of (4.18) is that $G_{\tilde{\Delta}_+}(z,z) - G_{\tilde{\Delta}_-}(z,z)$ is finite, so that the final answer is also manifestly finite. We have confidence that no other finite renormalization effects can slip in to the calculation, because the only thing that changes between the Δ_- and Δ_+ vacua is the boundary condition on ϕ.

As a warm-up let us first carry out the computation for AdS_5. To get the value of $G_{\tilde{\Delta}_+}(z,z) - G_{\tilde{\Delta}_-}(z,z)$ for coincident points one has to first express the Green's functions in terms of the geodesic distance σ. From (4.9) we see that in terms of the variable ζ the geodesic separation is given by $\cosh(\frac{\sigma}{L}) = \frac{1}{\zeta}$ so we rewrite the propagator (4.10) in terms of σ and then expand $i \left[G_{\tilde{\Delta}_+}(z,z) - G_{\tilde{\Delta}_-}(z,z) \right]$ in a power series in powers of $\frac{\sigma}{L}$. The answer is finite and in the limit $\frac{\sigma}{L} \to 0$, for AdS_5 we obtain the simple expression:

$$i \left[G_{\tilde{\Delta}_+}(z,z) - G_{\tilde{\Delta}_-}(z,z) \right] = -i \left[G_{\tilde{\Delta}}(z,z) - G_{4-\tilde{\Delta}}(z,z) \right] = -\frac{(\tilde{\Delta}-1)(\tilde{\Delta}-2)(\tilde{\Delta}-3)}{12\pi^2 L^3} \,. \quad (4.19)$$

The difference in the vacuum energies using (4.18) is therefore

$$\begin{aligned} V_+ - V_- &= \frac{i}{2}\int_{m_{BF}{}^2}^{m_0^2} d\tilde{m}^2 \left[G_{\tilde{\Delta}_+}(z,z) - G_{\tilde{\Delta}_-}(z,z)\right] \\ &= -\frac{1}{2}\int_2^{\Delta_-} \frac{d\tilde{\Delta}}{L^2}\left[2(\tilde{\Delta}-2)\frac{(\tilde{\Delta}-1)(\tilde{\Delta}-2)(\tilde{\Delta}-3)}{12\pi^2 L^3}\right] \\ &= -\frac{1}{12\pi^2 L^5}\int_0^{\Delta_- -2} d\tilde{\nu}\left[\tilde{\nu}^2(\tilde{\nu}^2-1)\right] = \frac{1}{12\pi^2 L^5}\left[\frac{(\Delta_- -2)^3}{3} - \frac{(\Delta_- -2)^5}{5}\right], \end{aligned} \tag{4.20}$$

where in the second line we have used $\tilde{m}^2 L^2 = \tilde{\Delta}(\tilde{\Delta}-4)$ and the fact that $\Delta_{BF} = \frac{d}{2} = 2$. Since $\Delta_- < 2$ we find that $V_+ - V_- < 0$, and therefore $c_- > c_+$ in agreement with the field theory prediction.

It is straightforward to generalize this for any odd-dimensional anti de-Sitter spacetime because for d even, the difference $i[G_{\tilde{\Delta}_+}(z,z) - G_{\tilde{\Delta}_-}(z,z)]$ is quite simple in form. Before writing this down, for convenience, let us define $d \equiv 2k$ so that the spacetime is AdS_{2k+1}. In terms of k, $i\left[G_{\tilde{\Delta}_+}(z,z) - G_{\tilde{\Delta}_-}(z,z)\right]$ is:

$$i\left[G_{\tilde{\Delta}_+}(z,z) - G_{\tilde{\Delta}_-}(z,z)\right] = -i\left[G_{\tilde{\Delta}}(z,z) - G_{d-\tilde{\Delta}}(z,z)\right] = -\frac{(-1)^k}{n_k \pi^k L^{d-1}}\prod_{i=1}^{2k-1}(\tilde{\Delta}-i). \tag{4.21}$$

where $n_k = 2^k(2k-1)!!$.

The difference in the vacuum energies is therefore

$$\begin{aligned} V_+ - V_- &= \frac{i}{2}\int_{m_{BF}{}^2}^{m_0^2} d\tilde{m}^2 \left[G_{\tilde{\Delta}_+}(z,z) - G_{\tilde{\Delta}_-}(z,z)\right] \\ &= \frac{1}{2}\int_{\Delta_-}^{k} \frac{d\tilde{\Delta}}{L^2}\left[2(\tilde{\Delta}-k)\frac{(-1)^k}{n_k\pi^k L^{d-1}}\prod_{i=1}^{2k-1}(\tilde{\Delta}-i)\right], \end{aligned} \tag{4.22}$$

where in the second line we have used $\tilde{m}^2 L^2 = \tilde{\Delta}(\tilde{\Delta}-d)$ and the fact that $\Delta_{BF} = \frac{d}{2} = k$. Shifting the variable of integration by introducing a new variable $\tilde{\nu} \equiv \tilde{\Delta}_- - k$, the integrand can be written down in a terms of the Pochhammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)}$:

$$2(\tilde{\Delta}-k)\frac{(-1)^k}{n_k\pi^k L^{d+1}}\prod_{i=1}^{2k-1}(\tilde{\Delta}-i) = \frac{(-1)^k}{n_k\pi^k L^{d+1}}\prod_{i=0}^{k-1}\left(\tilde{\nu}^2 - i^2\right) = \frac{1}{n_k\pi^k L^{d+1}}(\tilde{\nu})_k(-\tilde{\nu})_k. \tag{4.23}$$

The factor $(-1)^k$ was nullified by an extra factor of $(-1)^k$ from the product. Assembling all of this, we finally have

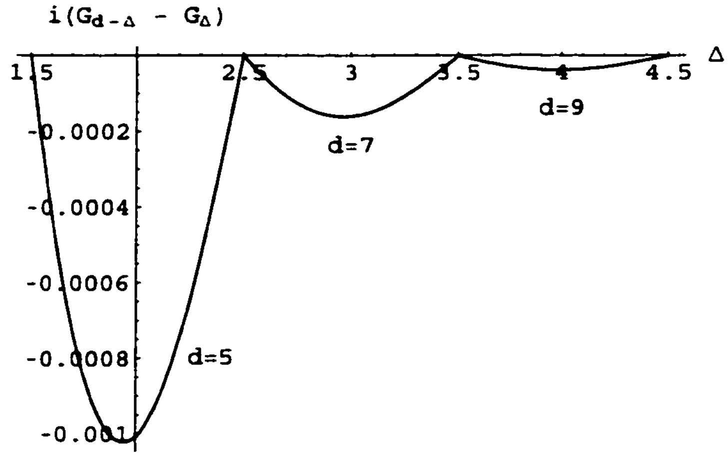


Figure 4.1: $i(G_{d-\Delta} - G_{\Delta})$ as a function of Δ, for AdS_6, AdS_8, and AdS_{10} (corresponding to $d = 5, 7, 9$), in units where $L = 1$.

$$V_+ - V_- = \frac{1}{2n_k \pi^k L^{d+1}} \int_{\nu}^{0} d\bar{\nu} \left[(\bar{\nu})_k (-\bar{\nu})_k \right] , \tag{4.24}$$

where we recall that $n_k = 2^k(2k-1)!!$. The lower limit of integration ν depends on the value of Δ_-. Since $k \le \Delta_- \le k-1$, the range of ν is $-1 \le \nu \le 0$. The function $(\nu)_k(-\nu)_k < 0$ for all k and $-1 \le \nu \le 0$. So for any odd-dimension anti de-Sitter spacetimes we have shown that $V_+ - V_- < 0$.

For even dimensional spacetimes, an analytic proof seems cumbersome, so we resorted to numerics. As an explicit example, figure 4.1 shows a plot of $i(G_{d-\Delta} - G_{\Delta})$ as a function of Δ for several even-dimensional anti-de Sitter spacetimes. In each dimension, we've plotted the integrand of (4.18) for $d/2 - 1 < \Delta < d/2$. Since the integrand is always negative on this range, we conclude that $V_+ < V_-$ in accordance with the c-theorem intuition. This is also true for $d = 3$, and we believe it is true generally.

4.3.2 Vacuum energy throughout AdS

The results of the previous section were stated in terms of $V_+ - V_- = V(z; m^2, \infty) - V(z; m^2, 0)$ (both terms were in fact independent of the position z in AdS). Here we would like to investigate $V(z; m^2, f)$ for finite f. This quantity diverges, but $V(z; m^2, f) - V_-$ is finite. We will be able to verify the formulas

$$\lim_{z_0\to 0}\left[V(z;m^2,f)-V_-\right]=0\,,\qquad \lim_{z_0\to\infty}\left[V(z;m^2,f)-V_-\right]=V_+-V_-\,,\tag{4.25}$$

which we consider intuitively obvious since $\frac{f}{2}\mathcal{O}^2$ is a relevant operator in the CFT, and therefore unimportant in the UV but important in the IR.

As a first step, one needs the Green's function for the scalar obeying mixed boundary conditions for all values of f (not just the ones for $f=0$ and $f=\infty$ that we wrote down earlier). This would be needed to compute the vacuum energy contribution due to the bubble diagram. The one-loop corrected action would then induce corrections in the geometry which can be computed from the Einstein equations. Let us work in Euclidean AdS to get the Green's function $G_E(x,y;f)$ which we shall Wick rotate to obtain $G(x,y;f)$ in Minkowski signature. We shall follow the canonical method of obtaining Green's functions. In Poincaré coordinates the scalar wave equation is

$$\left[x_0^2(\vec{\partial}^2+\partial_0^2)-x_0(d-1)\partial_0-m^2\right]\phi(x_0,\vec{x})=0\,,\tag{4.26}$$

where from now on we shall denote the radial direction by x_0 or y_0 and $\vec{x}$ is a vector with components along the d remaining directions. The two linearly independent solutions to this equation are: $\phi_1=x_0^{\frac{d}{2}}e^{-i\vec{k}\cdot\vec{x}}I_\nu(kx_0)$ and $\phi_2=x_0^{\frac{d}{2}}e^{-i\vec{k}\cdot\vec{x}}I_{-\nu}(kx_0)$ where $\nu=\sqrt{m^2L^2+\frac{d^2}{4}}$. In the notation of our previous sections, the Green's function obeys the equation

$$(\Box-m^2)G_E(x,y;f)=\delta^{d+1}(x-y)\,,\tag{4.27}$$

where we remind ourselves that the delta function includes a $\frac{1}{\sqrt{g}}$ in its definition. The right hand side is zero for $x_0\neq y_0$, so we have

$$\begin{aligned}G_E(x,y)&=A_1\phi_1(x)+A_2\phi_2(x)\qquad\text{for}\qquad x_0<y_0\\&=B_1\phi_1(x)+B_2\phi_2(x)\qquad\text{for}\qquad x_0>y_0\,.\end{aligned}\tag{4.28}$$

The boundary behavior of the scalar we're interested in is: $\phi(x_0,\vec{x})=f\beta(\vec{x})x_0^{\frac{d}{2}+\nu}+\beta(\vec{x})x_0^{\frac{d}{2}-\nu}$. We choose our ϕ_1 and ϕ_2 so that they have the right boundary behavior and also require that the Green's function not diverge in the bulk (large values of the radial coordinate x_0) for two non-coincident points. One convenient choice of ϕ_1 and ϕ_2 is:

$$\phi_1 = x_0^{\frac{d}{2}} e^{-i\vec{k}.\vec{x}} \left(I_\nu(kx_0) + f(\frac{2}{k})^{2n} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} I_\nu(kx_0) \right) \qquad \text{and} \qquad \phi_2 = x_0^{\frac{d}{2}} e^{-i\vec{k}.\vec{x}} K_\nu(kx_0)\,, \tag{4.29}$$

so that ϕ_1 satisfies the boundary condition for small x_0 and ϕ_2 is finite in the bulk. From the asymptotics of Bessel functions, we see that ϕ_1 diverges as $x_0 \to \infty$ and ϕ_2 diverges as $x_0 \to 0$. This forces us to set $A_2 = B_1 = 0$ in (4.28). The remaining two constants are determined by integrating (4.27) twice which gives us two conditions: (i) the Green's function is continuous at $x_0 = y_0$, and (ii) its radial derivative has a jump discontinuity of $\frac{1}{x_0^{d-1}}$ at $x_0 = y_0$. This yields

$$A_1 = \frac{\phi_2(y_0)}{W[\phi_1(y_0), \phi_2(y_0)]} \qquad B_2 = \frac{\phi_1(y_0)}{W[\phi_1(y_0), \phi_2(y_0)]}\,, \tag{4.30}$$

where $W[\phi_1(y_0), \phi_2(y_0)]$ is the Wronskian. For our choice of ϕ_1 and ϕ_2 the Wronskian is:

$$W[\phi_1(y_0), \phi_2(y_0)] = -\frac{\Gamma(1-\nu) + f(\frac{2}{k})^{2\nu}\Gamma(1+\nu)}{\Gamma(1-\nu)} \left(\frac{L}{y_0}\right)^{d-1}\,, \tag{4.31}$$

so combining (4.28), (4.30), and (4.31) we obtain the Green's function:

$$G_E(x,y;f) = -\int \frac{d\kappa_E d^{d-1}k}{(2\pi)^d} \frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}(x_0 y_0)^{\frac{d}{2}} K_\nu(ky_0)}{\left(1 + (\frac{2}{k})^{2\nu} f \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\right) L^{d-1}} \left[I_{-\nu}(kx_0) + f\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}(\frac{2}{k})^{2\nu} I_\nu(kx_0) \right] \tag{4.32}$$

for $x_0 < y_0$ and a similar expression for $x_0 > y_0$. In the above equation, κ_E is the temporal component of momentum. Finally, we Wick rotate this component $\kappa_E = ik$ to get the Green's function in Minkowski signature:

$$iG(x,y;f) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}(x_0 y_0)^{\frac{d}{2}} K_\nu(ky_0)}{\left(1 + (\frac{2}{k})^{2\nu} f \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\right) L^{d-1}} \left[I_{-\nu}(kx_0) + f\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}(\frac{2}{k})^{2\nu} I_\nu(kx_0) \right] \tag{4.33}$$

The integral for general values of f, d and ν is hard. For $f = 0$ and $f = \infty$ it can be evaluated and the result is an expression which is related to (4.10) by a quadratic hypergeometric transformation [116]. A little bit more can be said about the radial dependence of the one-loop vacuum energy. This latter quantity depends on the Green's function for coincident points $G(x,x;f)$. We saw before that this divergent quantity was best handled by subtracting out $G(x,x;0)$. The result is then finite:

$$i\left[G(x,x;f)-G(x,x;0)\right]=-\frac{1}{2^{d-2}\pi^{\frac{d}{2}}L^{d-1}\Gamma(\nu)\Gamma(1-\nu)\Gamma(\frac{d}{2})}\int_0^\infty d\tilde{k}\tilde{k}^{d-1}\frac{\tilde{f}}{\tilde{k}^{2\nu}+\tilde{f}}[K_\nu(\tilde{k})]^2. \tag{4.34}$$

where $\tilde{f}=2^{2\nu}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}fx_0^{2\nu}$ and $\tilde{k}=kx_0$. Note that the excess vacuum energy depends on the radial coordinate x_0 in the particular combination $fx_0^{2\nu}$.

In order to make any further progress, one would need to first compute the momentum integral and then integrate over ν to obtain the vacuum energy. We argued earlier that $V(x;m^2_{BF},f)-V(x;m^2_{BF},0)=0$ for all values of f, so using (4.18) and (4.34) we have:

$$\begin{aligned}
V(x;m^2,f)-V(x;m^2,0)&=\frac{i}{2}\int_{m^2_{BF}}^{m_0^2}d\tilde{m}^2\left[G(x,x;f)-G(x,x;0)\right]\\
&=\frac{i}{2}\int_0^\nu\frac{d\tilde{\nu}^2}{L^2}\left[G(x,x;f)-G(x,x;0)\right]\\
&=-\frac{1}{2^{d-2}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})L^{d+1}}\int_0^\nu d\tilde{\nu}\frac{\tilde{\nu}}{\Gamma(\tilde{\nu})\Gamma(1-\tilde{\nu})}\int_0^\infty d\tilde{k}\frac{\tilde{k}^{d-1}\tilde{f}}{\tilde{k}^{2\nu}+\tilde{f}}\left[K_\nu(\tilde{k})\right]^2.
\end{aligned} \tag{4.35}$$

where we remind ourselves that $\tilde{f}=2^{2\nu}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}fx_0^{2\nu}$. The double integral is difficult to perform explicitly. However, it is not hard to show from (4.35) that $V(x;m^2,f)$ decreases monotonically as f increases from 0 to ∞. To see this we note that the integrand depends on x_0 only through $\tilde{f}$ and since the integrand is a monotonic function of $\tilde{f}$, clearly $V(x;m^2,f)$ decreases monotonically with increasing f.

4.4 Conclusions

The upshot of section 4.3.1 was an evaluation of the change in the one-loop self-energy, V_+-V_-, between the IR and UV endpoints of a holographic RG flow. We would now like to convert this into a change in the central charge of the dual field theory.

In [117], the central charge was obtained by holographically computing the Weyl anomaly: on the field theory side,

$$\delta W[g_{\mu\nu}]=\frac{1}{2}\int d^4x\sqrt{g}\,\omega\langle T^\mu_\mu\rangle \tag{4.36}$$

upon a conformal variation $g_{\mu\nu} \to e^{2\omega} g_{\mu\nu}$, where W is the generating functional for connected Green's functions. At the one-loop level, the prescription of [5, 6] asserts that W is the classical supergravity action. The exact statement is that the partition functions of string theory and gauge theory coincide (subjected to boundary conditions and source terms in the usual way). In the calculation of [117], the supergravity action integral is evaluated with a radial cutoff, where the choice of radius amounts to a choice of metric within a conformal class. The supergravity lagrangian evaluates to a constant in AdS, and the central charge is proportional to this constant.[8] All that we need to do in order to correct the central charge computation at one loop is to ask by how much the one-loop-corrected lagrangian differs from the tree-level lagrangian, when evaluated in AdS. The tree level and one-loop lagrangians will stand in the same ratio as the leading large N central charge and its $1/N$-corrected counterpart.

The tree level lagrangian is

$$\sqrt{g}^{-1}\mathcal{L}_{\text{tree}} = \frac{1}{\kappa_{d+1}^2}(R - \Lambda_0) = -\frac{2d}{\kappa_{d+1}^2 L^2}\,. \tag{4.37}$$

The calculation indicated by the discussion in the previous paragraph is

$$\frac{c_{\text{corrected}}}{c_{\text{tree}}} = \frac{\mathcal{L}_{\text{tree}} + \delta\mathcal{L}}{\mathcal{L}_{\text{tree}}}\,, \tag{4.38}$$

where $\delta\mathcal{L} = -\sqrt{g}V$ is the one-loop correction to the lagrangian that we computed in section 4.3. Because we are only able to compute V up to an additive constant that is independent of boundary conditions, the only meaningful ratio that we can form is

$$\frac{c_+}{c_-} = \frac{\mathcal{L}_{\text{tree}} - \sqrt{g}V_+}{\mathcal{L}_{\text{tree}} - \sqrt{g}V_-} = 1 + \frac{V_- - V_+}{\sqrt{g}^{-1}\mathcal{L}_{\text{tree}}}\,. \tag{4.39}$$

so that

$$\frac{c_+ - c_-}{c_-} = (V_+ - V_-)\left(\frac{\kappa_{d+1}^2 L^2}{2d}\right)\,. \tag{4.40}$$

To check if c_- is indeed greater than c_+, all that we have to show is that $V_+ < V_-$. But that is exactly what we saw above.

[8] *A priori*, one might worry that boundary terms in the supergravity action also contribute to the central charge. That this does not happen depends on the circumstance, noted in [117], that the only log-divergent terms in the supergravity calculation arise from the integral of the bulk action.

As an example, in AdS_5, we obtain from (4.22) and (4.40) the result

$$\frac{c_+ - c_-}{c_-} = \frac{\kappa_5^2}{192\pi^2 L^3} \left[\frac{(\Delta_- - 2)^3}{3} - \frac{(\Delta_- - 2)^5}{5} \right] . \tag{4.41}$$

One can go further and translate the function $V(z; m^2, f) - V_-$ into a correction to the central charge whose scale dependence is monotonic. It is not clear how well-defined such a function can be on the supergravity side: because the bulk theory includes gravity, it has no local observables. Poetically, we would like to relate this to the fact that renormalization group effects in field theory are scheme-dependent—but it is difficult to make this precise.

It would be interesting to see how the construction discussed in this chapter might be realized as part of a compactification of string theory to four dimensions, along the lines of [118, 119]. One of the most interesting questions in that context is one that we glossed over here: before considering the loop effects in supergravity, one generally expects a moduli space of vacua, and this statement probably translates into field theory terms as the existence of a line of fixed points. Mapping the lifting of moduli into field theory terms might at least gain us a restatement of the moduli problem in terms of the existence of isolated fixed points of the renormalization group.

Chapter 5

Conclusions and Outlook

In this short chapter it might be useful to speculate on some future avenues of exploration based on research detailed in the previous chapters.

The first part of the thesis contains the author's contribution to the development of the understanding of the instabilities of non-extremal black branes. At the time of writing this thesis, a lot of key questions in this field remain unanswered. The most important of these is the ultimate fate of unstable black branes.

The linearized equations indicate that the horizon initially starts out by becoming non-uniform. There is a lot of debate in the community about whether the end-point of evolution is a configuration which is not translationally invariant. The best numerical evidence at the present time seems to indicate that non-uniform branes are probably not the end-products of decay – this then would mean that some violent phenomenon takes place towards the end. It would be curious to find out what exactly happens in the end. Does a naked singularity form, which renders the proof of [65] inapplicable? Dynamical violations of cosmic censorship would be a truly novel phenomenon from the point of classical general relativity. It is to be warned though, that once curvatures in some region of spacetime get very large and comparable to $1/\sqrt{\alpha'}$, stringy corrections have to be taken into account.

From a particle physics point of view, there are some extensions to Chapter 4 which are well worth undertaking. Let us discuss schematically some of these.

It is rewarding to look for and study RG flows where supersymmetry is broken along the flow. The attractive incentive is the ability to make quantitative predictions (like the central charge in Chapter 4) about a non-supersymmetric theory if one can identify its supersymmetric high-energy origin. Quantities in the non-supersymmetric theory would then be obtained as corrections (hopefully finite) to their values in the susy theory.

An embarrassing problem in string theory is the moduli problem. Simply stated, one finds that when one looks for supersymmetric vacua of string theory, one ends up with an entire family of them, which are connected continuously. This means that there is no preferred choice of a single vacua of the theory. It is expected that breaking supersymmetry would result in "lifting the flat directions" and leaving behind a single point in moduli space. This would be the true vacuum. It is also possible that multiple disconnected vacua are left behind, but even then, on general grounds, one expects a single one to be the global minimum in energy.

The RG flow that we considered in Chapter 4 might be viewed as a restatement of the moduli problem. The known examples of flows which preserve supersymmetry result in a line of fixed points. At leading non-trivial order in N, there seems to be an RG flow from any fixed point in the UV to one in the IR; but subleading $1/N$ corrections should be expected to make such non-supersymmetric IR fixed points unique, or to get rid of them altogether. It is useful to investigate examples like the one we discussed and find out if non-supersymmetric fixed points do indeed always occur in isolation. Learning how the absence of supersymmetry causes the line to shrink to a single point (or isolated points) might then shed some light on how string theory chooses its unique vacuum.

In any case, if one hopes to use the AdS/CFT correspondence to learn about QCD, one has to learn how to do loop computations which would result in corrections of order $1/N^2$ to the answer for large N. This requires doing loop computations in AdS. Searching for examples which can be calculated in a controlled fashion is therefore a fruitful exercise. It is possible that in most of the cases some amount of supersymmetry would remain unbroken, but one might take an optimistic viewpoint and hope that generic predictions would still hold good for their non-supersymmetric cousins.

Appendix A

Conventions and Notation

Here we collect conventions and a few properties of the differential operators we employ. We work in a metric of signature $(- + + \cdots +)$ and define the Ricci tensor in terms of the Riemann tensor by $R_{MN} \equiv R^P{}_{MPN}$.

The Hodge-de Rham Laplacian $\Delta_y = -(d^\dagger d + dd^\dagger)$ is negative-definite, but in the case of a compact Riemannian Einstein space of positive curvature a more stringent bound can be derived for the case of one-forms. We use $-R^2\Delta_y Y^I \equiv \kappa^I Y^I$, and for the ordinary Laplacian $\Box_y \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, $-R^2\Box_y Y^I \equiv \lambda^I Y^I$. For scalar spherical harmonics Y^I, $\Box_y = \Delta_y$, and a vanishing eigenvalue always exists corresponding to $Y^I = \text{const.}$[1] For one-forms, we may consider

$$\begin{aligned} 0 &\leq \int (\nabla^\alpha Y^{I\beta} + \nabla^\beta Y^{I\alpha})(\nabla_\alpha Y^I_\beta + \nabla_\beta Y^I_\alpha) = 2\int \nabla^\alpha Y^{I\beta}(\nabla_\alpha Y^I_\beta + \nabla_\beta Y^I_\alpha) \quad \text{(A.1)} \\ &= -2\int Y^{I\beta}(\Box_y + \frac{q-1}{R^2})Y^I_\beta = -2\int Y^{I\beta}(\Delta_y + \frac{2(q-1)}{R^2})Y^I_\beta , \end{aligned}$$

proving $\kappa^I \geq 2(q-1)$; furthermore, equality occurs for $(\nabla_\alpha Y^I_\beta + \nabla_\beta Y^I_\alpha) = 0$, which is the condition for Y^I_β to be a Killing vector. Additionally, the absence of harmonic one-forms Y^h_α on a compact Einstein space of positive curvature may be proved as follows. Any harmonic one-form must satisfy $\nabla^\alpha Y^h_\alpha = 0 = \nabla_\alpha Y^h_\beta - \nabla_\beta Y^h_\alpha$, so

$$0 = \int \nabla^\alpha Y^{h\beta}(\nabla_\alpha Y^h_\beta - \nabla_\beta Y^h_\alpha) = \int \left(\nabla^\alpha Y^{h\beta}\nabla_\alpha Y^h_\beta + \frac{q-1}{R^2} Y^{h\beta} Y^h_\beta\right) , \quad \text{(A.2)}$$

[1] One can derive the bound $\lambda^I \geq q$ for nonconstant Y^I [81].

which is impossible as the right-hand side is a sum of a nonnegative and a positive quantity.

For the case of S^q, the eigenvalues λ^I of the ordinary Laplacian $\Box_y$ for the various tensor harmonics are

Tensor harmonic	λ^I	Range of k
Y^I	$k(k+q-1)$	$k \geq 0$
Y^I_α	$k(k+q-1)-1$	$k \geq 1$
$Y^I_{[\alpha_1 \cdots \alpha_n]}$	$k(k+q-1)-n$	$k \geq 1$
$Y^I_{(\alpha\beta)}$	$k(k+q-1)-2$	$k \geq 2$

while for the Hodge-de Rham Laplacian acting on vectors, we obtain

$$\kappa^I = (k+1)(k+q-2)\,, \qquad k \geq 1\,. \tag{A.3}$$

Appendix B

Derivation of Modified Expression for One Loop Vacuum Energy

In this appendix we shall sketch the derivation of (4.17). Our starting point is the familiar field theory result that the one-loop effective potential is

$$V(z; m^2, f) = -\frac{i}{2} \mathrm{tr} \log(-\Box + m^2) \,. \tag{B.1}$$

We shall denote the Klein-Gordon operator $(-\Box + m^2)$ by $\hat{K}(m^2, f)$ and as an operator, it is related to our definition of the Green's function (4.15) by $\hat{G}(m^2, f) = -[\hat{K}(m^2, f)]^{-1}$. The representations of operators such as $\hat{G}(m^2, f)$ in an orthonormal basis shall be denoted by the obvious notation: $\langle z|\hat{G}(m^2, f)|z\rangle = G(z, z'; m^2, f)$. In terms of the Green's function, the effective potential is then

$$V(z; m^2, f) = \frac{i}{2} \lim_{z' \to z} \log[-G(z, z'; m^2, f)] \,. \tag{B.2}$$

We shall use the Schwinger proper-time formalism to evaluate this. One needs two simple operator relations both of which follow from the relation between $\hat{G}(m^2, f)$ and $\hat{K}(m^2, f)$

$$\begin{aligned} \hat{G}(m^2, f) &= -i \int_0^\infty e^{-is\hat{K}(m^2, f)} ds \\ \log[-\hat{G}(m^2, f)] &= \int_0^\infty \frac{e^{-is\hat{K}(m^2, f)}}{is} i ds + \gamma \,, \end{aligned} \tag{B.3}$$

where γ is the Euler's constant. For the effective potential, we see from (B.2) that we need $\log[-\hat{G}(m^2,f)]$ which differs by a factor of is from the integral representation of $\hat{G}(m^2,f)$ above.

To proceed any further we need the DeWitt-Schwinger representation of the Green's function (the reader is referred to [115] pg. 75 for a derivation)

$$G(z,z';m^2,f) = -i\frac{\sqrt{M(z,z')}}{(4\pi is)^{\frac{d+1}{2}}}\int_0^\infty ids e^{-im^2s+\frac{\eta(z,z')}{2is}}F(z,z';is)\,, \tag{B.4}$$

where $\eta(z,z')$ is one-half the proper distance between the points z and z', and $M(z,z') = -\det[\partial_\mu\partial_\nu\eta(z,z')]$. For our purposes we shall just need to use the fact that the only place where the mass appears is in the exponent and integrating with respect to m^2 will bring down an extra factor of is that we need. So integrating both sides of (B.4) between two arbitrary masses m_1^2 and m_2^2 and using (B.3) we obtain

$$\int_{m_1{}^2}^{m_2{}^2} d\tilde{m}^2[-G(z,z';\tilde{m}^2,f)] = -\log[-G(z,z';m_2^2,f)] + \log[-G(z,z';m_1^2,f)]\,. \tag{B.5}$$

In the usual treatment one chooses one of the masses to be infinite, but as we explained in the main text, this cannot be done here. Instead of integrating toward heavier masses, we integrate in the opposite direction down to the Breitenlohner-Freedman bound. Therefore, we set $m_1^2 = m_{\rm BF}^2$ and $m_2^2 = m^2$ in (B.5) and use (B.2) to get

$$V(z;m^2,f) = \frac{i}{2}\lim_{z\to z'}\int_{m_{\rm BF}^2}^{m^2} d\tilde{m}^2 G(z,z';m^2,f) + V(z;m_{\rm BF}^2,f)\,. \tag{B.6}$$

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