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DIFFERENTIAL MAPS, DIFFERENCE MAPS, INTERPOLATED MAPS, AND LONG-TERM PREDICTION

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Mapping techniques appear to be attractive for the long-term prediction of particle motion in accelerators. Here we apply such methods to an exactly solveable example, the simple pendulum, and show that a numerical interpolation map predicts the evolution more accurately than an analytically derived differential map of the same order. Even so, in the presence of appreciable nonlinearity, the impracticality of achieving accurate prediction beyond some hundreds of cycles of oscillation is shown. These results may suggest that caution be used in claims of accuracy for predictions of long-term stability in the admittedly different situation in an accelerator.

1. INTRODUCTION

Linear maps have long been basic to accelerator theory. Accelerator lattice descriptions may be complicated and special, and a linear map (i.e., transfer matrix) represents an elegant distillation of this complexity.

In an attempt to study the reliability of nonlinear mapping techniques, we examined the case of a nonlinear mechanical system, the simple pendulum (for which the exact solution is known), using various mapping techniques. This system forms a rather faithful analog for longitudinal oscillation of particles in an accelerator but is admittedly not a good model for transverse motion.

Investigated were differential maps, difference maps, and interpolated maps. Differential maps are Taylor-series expansions of the exact map; the coefficients are assumed to be obtainable exactly by analytic formulas. Difference maps are the same, but the coefficients are obtained by numerical differentiation. Interpolated maps are obtained by interpolating exact maps.

Sophisticated methods have been applied to the problem of finding differential maps for complicated lattices, and elegant results have been obtained. Examples are MARYLIE,[1] developed by Dragt and others, and the more-powerful methods of Forest.[2] Also, the differential algebra methods of Berz[3] have been used to obtain high-order maps that have been shown to be correct by comparison with exact solutions evaluated to the same order.[4] With the

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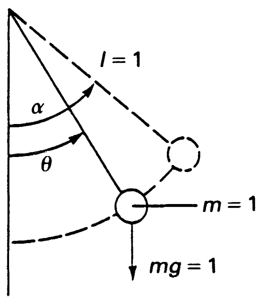
accelerator tracking program TEAPOT[5] (which uses exact analytical propagation formulas and does not regularly use transfer matrices or any other kind of map for particle tracking) difference maps can be obtained numerically for complicated lattices.

In what follows the performance of these three types of map for long-term prediction of pendulum motion was compared.

2. MATHEMATICAL FORMULATION AND EXACT SOLUTION

The system analysed is shown in the diagram. The maximum swing angle is α, and the instantaneous angle is θ. The gravity constant g, pendulum length l, and mass m are set equal to 1.



The kinetic energy is given by $T = \dot{\theta}^2/2$, and the potential energy is given by $V = 2\sin^2\theta/2$. Using standard notation,[6] the total energy E is expressed in terms of a parameter k so that

$$E \equiv 2k^2 = T + V = 2\sin^2\alpha/2. \tag{1}$$

After the change of variable $\sin\theta/2 = k\sin\phi$, the equation of motion is transformed to

$$\dot{\phi}^2 = 1 - k^2\sin^2\phi. \tag{2}$$

The solution of this equation is

$$u = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}, \tag{3}$$

where the symbol u is used for time to conform to standard notation of elliptic functions.[6] The particular elliptic functions used are $\mathrm{sn}\, u$, $\mathrm{cn}\, u$, and $\mathrm{dn}\, u$. The solution of Eq. (3) is

$$\mathrm{sn}\, u = \sin\phi, \tag{4}$$

or, in terms of the original variable,

$$\theta = 2\sin^{-1}(k\,\mathrm{sn}\, u). \tag{5}$$

In our units, the small amplitude motion is periodic in u with period 2π.

To make the mapping as simple as possible, we use the variables:

$$\begin{aligned} x &= \sqrt{V/E} = k^{-1} \sin \theta/2 \\ p &= \sqrt{T/E}. \end{aligned} \tag{6}$$

Combining the energy conservation equation

$$V + T = E \quad \text{or} \quad x^2 + p^2 = 1 \tag{7}$$

with the identity $\text{sn}^2\, u + \text{cn}^2\, u = 1$, it follows that the system point (x, p) in phase space is restricted to the unit circle. We take the pendulum to be swinging through the origin at $u = 0$; with this initial condition,

$$\begin{aligned} x &= \text{sn}\, u \\ p &= \text{cn}\, u. \end{aligned} \tag{8}$$

To obtain the relationship of these coordinates to canonical coordinates, the Lagrangean is written in terms of x and $\dot{x}$,

$$L = \frac{2k^2\dot{x}^2}{1 - k^2x^2} - 2k^2x^2. \tag{9}$$

The momentum conjugate to x is

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{4k^2\dot{x}}{1 - k^2x^2}, \tag{10}$$

and the Hamiltonian is

$$H = 2k^2x^2 + p_x^2(1 - k^2x^2)/(8k^2). \tag{11}$$

Finally then, the relationship of the variable p to the canonical variable p_x is

$$p = p_x\sqrt{1 - k^2x^2}/(4k^2). \tag{12}$$

3. THE EXACT TRANSFER MAP

When the time evolves from u to $u + v$, substitution into Eq. (8) shows that the phase space point moves as

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} \text{sn}\, u \\ \text{cn}\, u \end{pmatrix} \rightarrow \begin{pmatrix} x_v \\ p_v \end{pmatrix} = \begin{pmatrix} \text{sn}\,(u + v) \\ \text{cn}\,(u + v) \end{pmatrix}. \tag{13}$$

Using "addition formulas",[7] valid for elliptic functions, we can express evolution by the exact nonlinear mapping

$$\begin{pmatrix} x_v \\ p_v \end{pmatrix} = \begin{pmatrix} \text{cn}\, u\, \text{dn}\, v & \text{sn}\, v(1 - k^2x^2)^{1/2} \\ -\,\text{sn}\, u\, \text{dn}\, v(1 - k^2x^2)^{1/2} & \text{cn}\, v \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \frac{1}{1 - k^2x^2\, \text{sn}^2\, v} \tag{14}$$

To predict the long-term motion of the pendulum this map could be iterated, but of course it is simpler just to use Eq. (8). When we attempt to track accurately over, for example, 10^{10} cycles we seem to face the task of evaluating expressions such as $\mathrm{sn}\, 10^{10}$. In fact, since whole cycles can be subtracted, assuming that the period is known with sufficient accuracy, the exact motion can be predicted with an accuracy millions of times better than can be obtained with any of the approximate maps discussed.

4. THE DIFFERENTIAL MAP

The x-dependent factors appearing in Eq. (14) can be expanded easily. For example, setting $a = k^2x^2$ and $b = k^2x^2 \,\mathrm{sn}^2\, v$ we have

$$(1-a)^{1/2}(1-b)^{-1} = 1 + (b - a/2) + (b^2 - ba/2 - a^2/8) + \cdots \tag{15}$$

The last term exhibited is quartic in x, but the formula is easily extended. For lattice calculations it is impractical[7] to go much beyond terms of order x^8, and that is how far we carried calculations in this paper.

This map exhibits especially simple features. For one thing it is independent of p. This is not basic, since $x^2 = 1 - p^2$. Also, only even powers of x appear. To be analogous in this regard an accelerator lattice would contain no sextupoles or other magnets having an even number of poles. Clearly then, the pendulum is not analogous to general nonlinear betatron motion. The leading qualitative manifestation of this lack of analogy is probably that the pendulum frequency is perturbed in the lowest nonvanishing order of perturbation; the tune of a lattice containing only sextupoles is unperturbed in lowest order. The pendulum is, however, closely analogous to longitudinal particle motion.

In practice differential maps are necessarily truncated at some order; this is made manifest by nonsymplectic behavior in the following order. For the pendulum description this corresponds to the wandering of the phase point off the unit circle. In this case, the frequently employed practice of forcing symplecticity artifically would be accomplished by normalizing the phase space point to unity after each iteration. This was, however, not done. Rather, the deviation from the known analytic solution was used as a measure of error.

5. INTERPOLATED MAPS

Any one of the four matrix elements of Eq. (14) can be written as a function $f(x)$. In preparation for calculating what will be called an interpolated map, this function can be evaluated at points on a regular grid. Since the functions do not depend on p we need only use points on the x-axis, and for simplicity we use equally spaced values, $0, \pm x_{\mathrm{typ}}, \pm 2x_{\mathrm{typ}}, \ldots$. Here we use the somewhat clumsy notation of TEAPOT.[5] The value x_{typ} is a free parameter to be specified later. If there were p-dependence there would be another parameter, p_{typ}.

For writing difference formulas we introduce the notation $f_n = f(nx_{\mathrm{typ}})$. We express x in units of x_{typ} by introducing the variable $r = x/x_{\mathrm{typ}}$. The function $f(x(r))$ can be expressed approximately in terms of the tabulated values f_n by a Lagrangian interpolation formula. Since our functions are all even in r, only values f_n with positive n are required. Thus,

$$\begin{aligned}
f^{(0)} &= f_0 \\
f^{(2)} &= -(r^2-1)f_0 + r^2 f_1 \\
f^{(4)} &= \frac{(r^2-1)(r^2-4)}{4}f_0 - \frac{r^2(r^2-4)}{3}f_1 + \frac{r^2(r^2-1)}{12}f_2 \\
f^{(6)} &= -\frac{(r^2-1)(r^2-4)(r^2-9)}{36}f_0 + \frac{r^2(r^2-4)(r^2-9)}{24}f_1 \\
&\quad - \frac{r^2(r^2-1)(r^2-9)}{60}f_2 + \frac{r^2(r^2-1)(r^2-4)}{360}f_3 \cdots .
\end{aligned} \tag{16}$$

By calculating each of the matrix elements of Eq. (14) using these formulas, we obtain interpolated maps of successively higher order.

6. DIFFERENCE MAPS

For our problem, any particular differential map element can be expressed by a series,

$$f(x) = R + Ux^2 + Wx^4 + Yx^6 + \cdots \tag{17}$$

The vanishing of odd powers has been explained above. By comparison with Eq. (16); approximate difference formulas can be written for the Taylor expansion coefficients of the differential map. For example, from $f^{(4)}$ we obtain

$$\begin{aligned}
R &\approx R^{(4)}(x_{\text{typ}}) = f_0 \\
U &\approx U^{(4)}(x_{\text{typ}}) = \frac{-15f_0 + 16f_1 - f_2}{12x_{\text{typ}}^2} \\
W &\approx W^{(4)}(x_{\text{typ}}) = \frac{3f_0 - 4f_1 + f_2}{12x_{\text{typ}}^4} .
\end{aligned} \tag{18}$$

To obtain a rigorous connection between these newly introduced, x_{typ}-dependent expressions with the corresponding coefficients of the differential map, it is clearly necessary to take the limit $x_{\text{typ}} \to 0$. That cannot be done numerically, but, depending on the computer word length of the computer being used, a value of x_{typ} can be used that is sufficiently small to allow the leading coefficients to be accurately evaluated. Used in Eq. (17) they express what I have called a difference map. It can be seen that this particular interpolated map and the difference map differ only in the value of x_{typ} used.

7. NUMERICAL RESULTS

The various maps have been iterated for various swing amplitudes; the results are plotted in the following figures. In every case the deviation, approximate minus

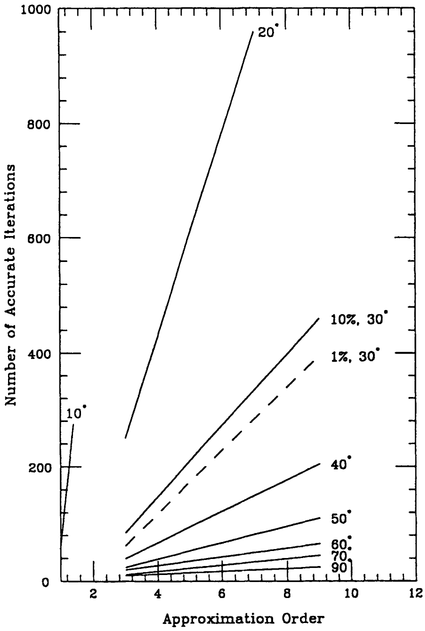


FIGURE 1 Dependence of the number of accurate iterations on the approximation amplitude, for various pendulum swing amplitudes.

exact, $\sqrt{(x_{\rm ap}-x_{\rm ex})^2+(p_{\rm ap}-p_{\rm ex})^2}$, is calculated. Since the exact phase point remains on the unit circle, this is both the absolute value and the fractional absolute value of the vector-phase-space displacement of the approximate solution from the exact solution; we take it as the latter, evaluating it as a percent error. Rather than plotting this error as a function of iteration number one can note the iteration number at which the error first exceeds some value, such as 10%.

Data of that sort are plotted in Fig. 1. For these data, and all other plots in this paper, the map period v in Eq. (13) has been taken to be $1.6\times\pi$; that corresponds to an evolution time interval equal to 0.8 of the small amplitude period. As a result, the number of iterations and the number of periods have comparable magnitudes. If we define "accurate" iterations to be those for which the error is less than 10%, Fig. 1 is a plot of number of accurate iterations versus the approximation order for various pendulum swing amplitudes. Clearly long-term accuracy degrades as the amplitude increases. For one amplitude, 30 degrees, the 1% error data are shown; it shows no striking qualitative difference from the 10% data. What is striking is that the number of accurate iterations varies linearly with the approximation order, at least over the range studied. The data of Fig. 1 can be described semi-quantitatively by the following crude and simple formula:

$$\text{number of accurate iterations} \approx 10\,\frac{\text{approximation order}}{\text{amplitude}^2}, \tag{19}$$

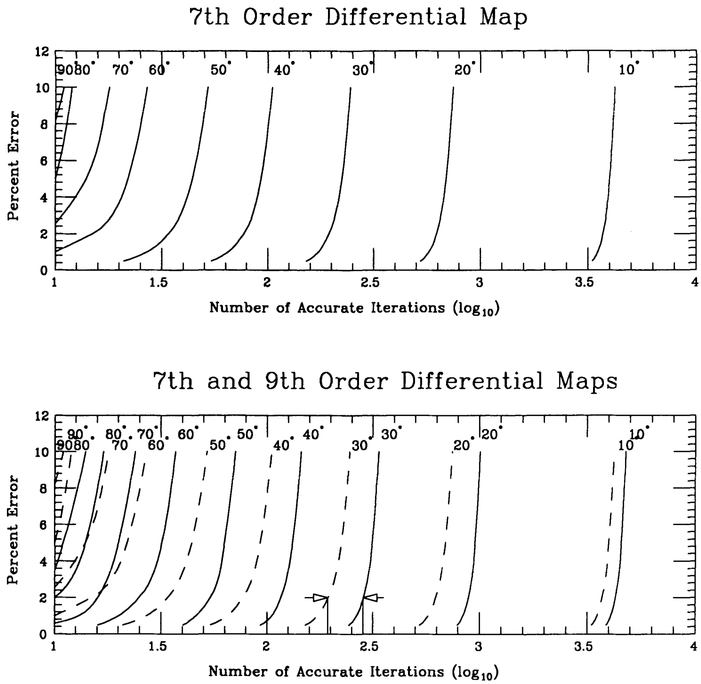


FIGURE 2 Comparison of results of iterating seventh- and ninth-order differential maps. As explained in the text, this and all following figures have a similar format; comparison is facilitated by the dashed-line over plotting, in the lower graph, of the upper-graph curves.

where "amplitude" is the maximum pendulum angle in radians and "order" is one for linear transfer matrices and increases by one for each power of x or p.

The remaining figures compare the abilities of different maps to predict the motion over long times. The figures all have the same format; the solid curves in the upper and lower plots are what is being compared; for convenience the upper data are replotted in the lower graph, joined by broken curves. In every case the solid curves in the lower graph come from a differential map. Figure 2 compares seventh- and ninth-order differential maps. An example comparison is indicated. For 30 degree amplitude, if 2% accuracy is required, then the number of accurate iterations with a seventh-order map is $\log_{10}^{-1} 2.29 = 195$ turns. Going to ninth order yields $\log_{10}^{-1} 2.45 = 282$ accurate iterations.

The purpose of Fig. 3 is to compare an interpolated and a differential map of the same (ninth) order. For interpolated maps the parameter x_{typ} must be specified. In this paper it is always chosen so that, in the particular order, the largest grid point is $x = 1.0$; that is, $x_{\mathrm{typ}} = 2/(\text{order} - 1)$. Somewhat smaller values in the range 0.9 to 1.0 gave greater precision, but they have not been used since the improvement was not great and the optimum depends on pendulum amplitude. Depending on how the comparison is made, the interpolated map can be said to be either somewhat more accurate, or far more accurate, than the differential map. Again working on the 30 degree case, and demanding 10%

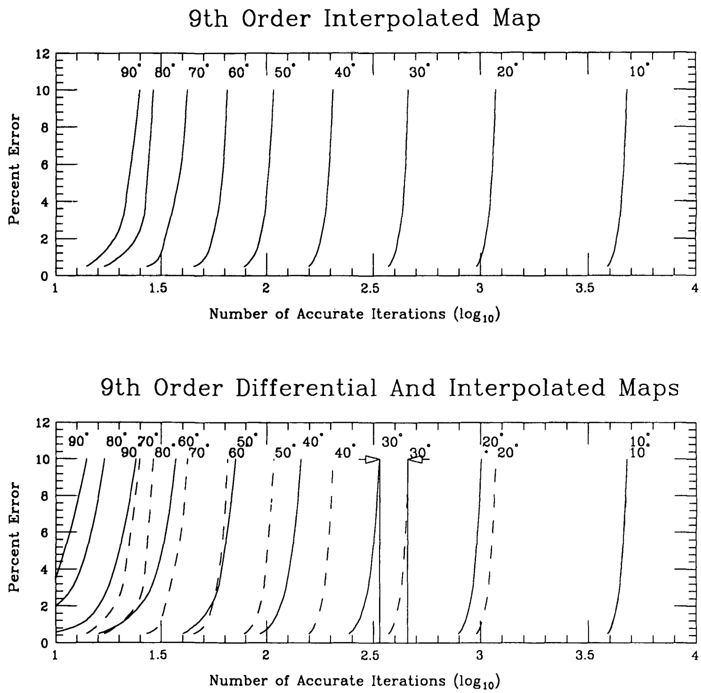


FIGURE 3 Comparison of results of iterating a differential map and an interpolated map, both of ninth order.

accuracy, the differential map yields 335 accurate iterations. After this many iterations the interpolated map is still yielding accuracy much better than 1%. On the other hand, after 457 turns, the interpolated-map inaccuracy exceeds 10%; this is not that significant an improvement over the 335 turns of accurate differential-map tracking.

The remaining figures compare differential and difference maps. For the difference maps, x_{typ} was chosen according to the formula $x_{typ} = 0.2/(\text{order} - 1)$. For seventh-order maps the agreement is very good, as shown in Fig. 4. The overplotting in the lower graph of difference-map results is barely visible. (The absence of a 10-degree difference curve is due to the occurrence of a computer overflow during its evaluation.) Going to ninth order yields Fig. 5, which shows that the difference map has broken down and is inferior to the seventh-order map; that is presumably due to roundoff error in the numerical differentiation employed in evaluating the difference map. To confirm this, Fig. 6 was generated, using extended precision in the computer. The precision was "real*16", meaning that a floating-point number is represented by a 128-bit binary number in the computer; such numbers are accurate to about 33 decimal digits. For the previous calculation, 64-bit numbers, accurate to about 16 decimal digits, were used. With the extended precision the difference and differential maps are indistinguishable graphically. Numerically the greatest deviation in the logarithm of the number of accurate iterations is 2.5276 instead of 2.5302.

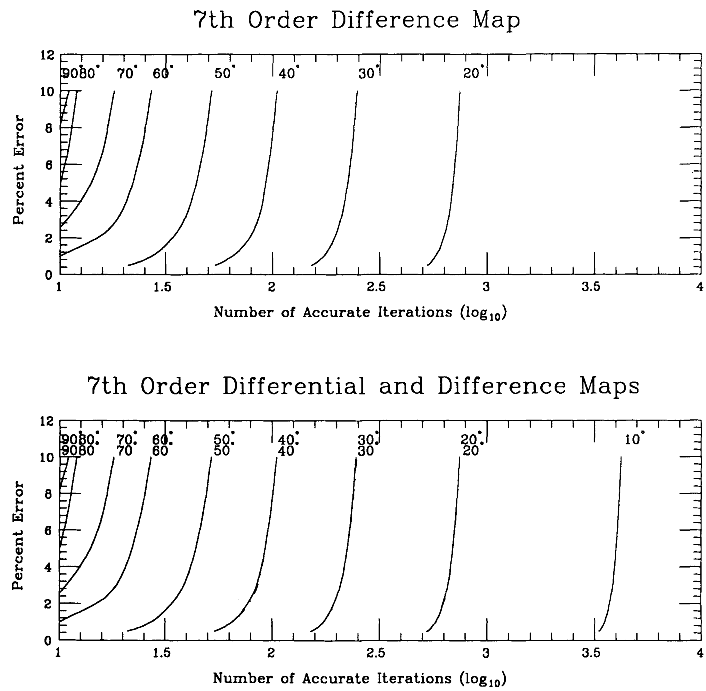


FIGURE 4 Comparison of results of iterating a differential map and a difference map, both of seventh order—"lower-precision" computation.

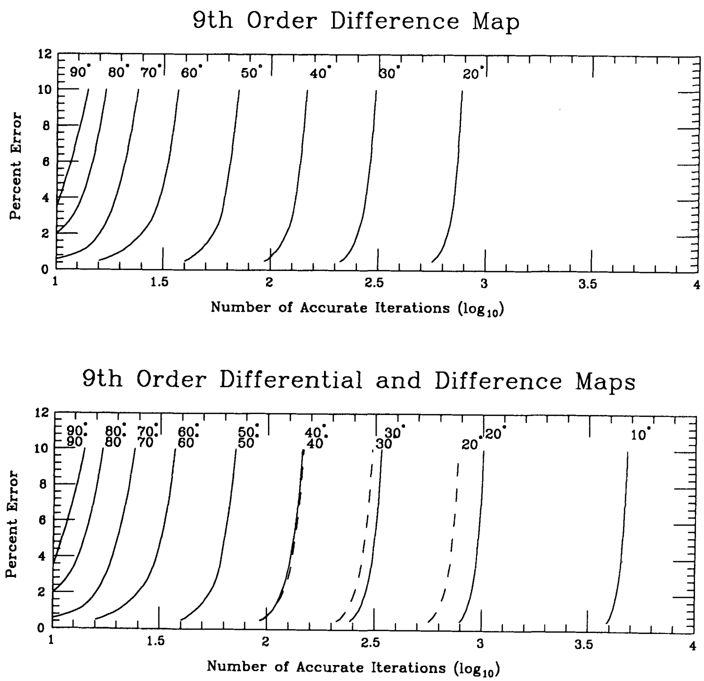
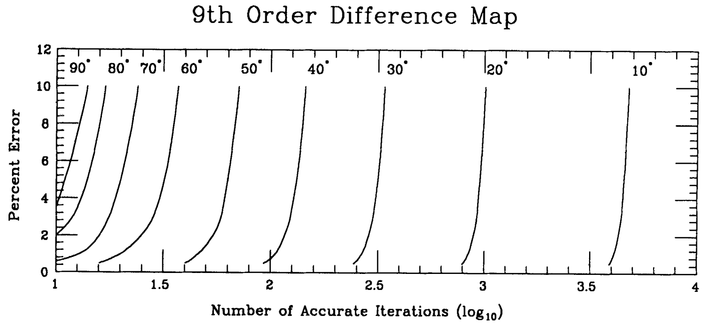


FIGURE 5 Comparison of results of iterating a differential map and a difference map, both of ninth order—"lower-precision" computation.



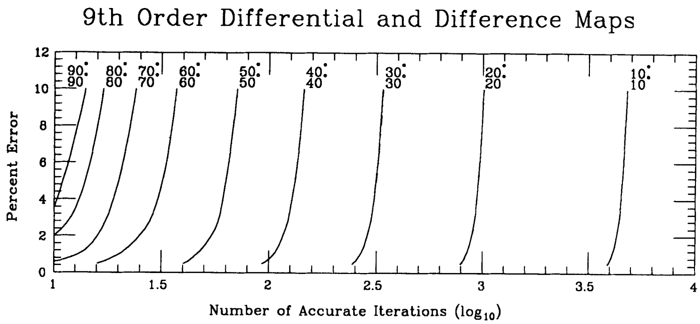


FIGURE 6 Comparison of results of iterating a differential map and a difference map, both of ninth order—"higher-precision" computation.

8. CONCLUSIONS

Quantitative results have been given for the accuracy achievable in predicting the long-term motion of a pendulum by various procedures. For this particular simple system, it is shown that, when the nonlinearity is "appreciable," iteration of approximate maps yields accurate prediction for only some hundreds of periods of oscillation. This depends on the approximation order and swing amplitude, as has been explained in detail. The number of accurate iterations increases roughly in proportion to the approximation order. This leads eventually to a diminishing return, since the factor by which the number of map coefficients increases, as the order is increased by one, is roughly equal to the number of degrees of freedom.

At least in the case of the simple pendulum, these results suggest that map iteration is not promising for long-term prediction. It is, however, possible, for another system such as an accelerator containing nonlinear elements, that maps could be useful for long-term prediction. To be persuasive in that case, it would be desirable to show ways in which the pendulum system is atypical and gives misleading results and to show the characteristics of nonlinear systems that allow useful application of map-prediction techniques.

It has also been shown that greater accuracy can be achieved using interpolated maps than using differential maps. This is not at all surprising since the differential map amounts to extrapolation from the origin using Taylor series; this should be expected to be less accurate than interpolation based on a grid of exact values. By using interpolation formulas of sufficient sophistication, one supposes that maps almost as accurate as the grid-point maps can be obtained, but that has not been investigated here.

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