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Collection of formulas describing storage ring coherent instabilities. *

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Abstract

The formulas for coherent instabilities are summarized with the emphasis on realistic fill patterns.

1 Introduction

The analytic formulas for the traditional coherent instabilities in the high-energy storage rings are summarized to be used to study the proposed PEP-II upgrade. Most of the formulas are known and given with small modifications which may be useful to describe the beam stability for the realistic fill patterns. Some recent results are added. The formulas are given for the most important instabilities. That let us to simplify the formulas rather than to write them in the most general form.

2 Steady-state RF conditions

Here we follow P. Wilson [1] and G. Kraft [2].

Consider an rf cavity with the frequency of the fundamental mode ω_c, Q_0-factor, and shunt impedance R_0.

A cavity is excited by the wave coming from a klystron by some wave guide with the (real) wave impedance Z_w. The incoming wave is described by the current I_{in} and voltage V_{in}, $I_{in} = V_{in}/Z_w$.

Let us assume that the wave guide is coupled to the cavity through a transformer with the transformer ratio n. If there is no perfect matching, there is a reflected wave in the wave-guide with the amplitudes I_r and V_r, $I_r = V_r/Z_w$.

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The excitation current I_{ex} and the cavity voltage V_c are given by the transformer ratio

$$I_{ex} = \frac{1}{n}(I_{in} - I_r) = \frac{1}{n\,Z_w}(V_{in} - V_r)$$
$$V_c = n(V_{in} + V_r). \tag{1}$$

Excluding V_r,

$$V_r = \frac{V_c}{n} - V_{in}, \quad I_{ex} = \frac{1}{n\,Z_w}(2V_{in} - \frac{V_c}{n}). \tag{2}$$

The cavity is considered as an oscillator driven by the external current I_{ex} and the beam current I_B. Using the superposition principal, the variation of the cavity voltage $V_c(t)$ is described by the following equation:

$$(\frac{d^2}{dt^2} + \frac{\omega_c}{Q_0}\frac{d}{dt} + \omega_c^2)\,V_c(t) = \frac{\omega_c R_0}{Q_0}[\dot{I}_{ex} - \dot{I}_B]. \tag{3}$$

Using Eq. (2), this can be written as

$$(\frac{d^2}{dt^2} + \frac{\omega_c}{Q_L}\frac{d}{dt} + \omega_c^2)\,V_c(t) = \frac{\omega_c R_0}{Q_0}[\dot{I}_g - \dot{I}_B]. \tag{4}$$

where the loaded Q-factor

$$Q_L = \frac{Q_0}{1+\beta}, \tag{5}$$

is defined in terms of the rf coupling β,

$$\beta = \frac{R_0}{n^2 Z_w}, \tag{6}$$

and the generator current

$$I_g = \frac{2n\beta}{R_0}V_{in}. \tag{7}$$

Let us denote the complex amplitudes with hats as in

$$I_g(t) = \frac{1}{2}(\hat{I}_g e^{-i\omega_g t} + c.c.). \tag{8}$$

Here $\omega_g/2\pi$ is the rf (klystron) frequency. For a point-like bunch, the amplitude $\hat{I}_B = 2I_B^{dc}$, where I_B^{dc} is dc component of the beam current.

The average incoming power P_i, reflected power P_r, and power transferred to the cavity P_{tr} are

$$P_i = \frac{|\hat{V}_{in}|^2}{2Z_w} = \frac{R_0|\hat{I}_g|^2}{8\beta}, \quad P_r = \frac{|\hat{V}_r|^2}{2Z_w} = \frac{|\hat{V}_c|^2\beta}{2R_0}|1 - \frac{R_0\hat{I}_g}{2\beta\hat{V}_c}|^2.$$

$$P_{tr} = \frac{1}{2}Re[\hat{I}_{ex}\hat{V}_c^*] = Re[\frac{\hat{V}_c\hat{I}_g^*}{2}] - \frac{\beta|\hat{V}_c|^2}{2R_0}. \tag{9}$$

It is easy to check that $P_i = P_r + P_{tr}$.

As it follows from Eq. (4), the voltage in the resonance ($\omega_g = \omega_c$) is related to I_g, $\hat{V}_c = (\hat{I}_g - \hat{I}_B)R_L$ where the loaded shunt impedance $R_L = R_0/(1+\beta)$. Then,

$$P_{tr} = P_c + P_B, \tag{10}$$

where the power lost to the cavity walls P_c and power transferred to the beam P_B are

$$P_c = \frac{|\hat{V}_c|^2}{2R_0}, \quad P_B = Re[\frac{\hat{V}_c\hat{I}_B^*}{2}]. \tag{11}$$

Let us choose the phases in respect to real $\hat{I}_g$ and define the rf phase ϕ_s of the beam

$$I_B(t) = \frac{1}{2}(\hat{I}_B e^{-i\omega_g t} + c.c), \quad \hat{I}_B = |\hat{I}_B|e^{i(\phi_s - \phi_c)} = 2I_B^{dc}e^{i(\phi_s-\phi_c)}. \tag{12}$$

The phase ϕ_s is the rf phase defined by the energy loss per turn. With such definition, the voltage on a cavity and the beam current vary in time as

$$V_c(t) = |\hat{V}_c|\cos(\omega_g t + \phi_c), \quad I_B(t) = 2I_B^{dc}\cos(\omega_g t + \phi_c - \phi_s). \tag{13}$$

A particle crossing the cavity sees the voltage $V_{acc} = |\hat{V}_c|\cos(\phi_s)$. If particles shift toward the head of the beam train by $z > 0$, then ϕ_s is replaced in $I_B(t)$ by $\phi_s - \omega_g z/c_0$ and the synchrotron tune $\nu_s^2 \propto \sin(\phi_s)$. For stability, $\sin\phi_s > 0$.

Solution of Eq. (4) is

$$\hat{V}_c = Z_c(\omega_g)(\hat{I}_g - \hat{I}_B) \tag{14}$$

where the cavity impedance

$$Z_c(\omega) = -\frac{i(R_L/Q_L)}{\frac{\omega_c}{\omega} - \frac{\omega}{\omega_c} - \frac{i}{Q_L}}. \tag{15}$$

Defining the detuning angle ψ,

$$\tan(\psi) = Q_L(\frac{\omega_g}{\omega_c} - \frac{\omega_c}{\omega_g}), \tag{16}$$

the cavity impedance can be written as

$$Z_c(\omega_g) = R_L\, cos\psi\, e^{i\psi}. \tag{17}$$

Note that $|\psi| < \pi/2$.

Eq. (15) takes the form

$$\hat{V}_c = |\hat{V}_c| e^{-i\phi_c} = R_L\, cos\psi\, e^{i\psi}(\hat{I}_g - 2I_B^{dc} e^{i(\phi_s - \phi_c)}), \tag{18}$$

where the phase ϕ_c of the cavity voltage in respect to the generator current is introduced.

The accelerating voltage V_{ac} is related to the average power $P_B = I_B^{dc} V_{ac}$ transferred to the beam. Eq. (11) gives

$$P_B = \frac{1}{2}|\hat{V}_c||\hat{I}_B| \cos(\phi_s) = I_B^{dc} V_{ac}. \tag{19}$$

Then, $V_{ac} = |\hat{V}_c| \cos(\phi_s)$, and

$$V_{ac} = R_L\, cos\psi\, e^{i(\psi+\phi_c)}\, (\hat{I}_g - 2I_B^{dc} e^{i(\phi_s - \phi_c)}) \cos(\phi_s). \tag{20}$$

V_{ac} is real. Hence,

$$\hat{I}_g = 2I_B^{dc} \frac{\sin(\psi + \phi_s)}{\sin(\psi + \phi_c)}. \tag{21}$$

The cavity voltage and the power from klystron are

$$\begin{aligned} |\hat{V}_c| &= 2I_B^{dc} R_L \frac{\sin(\phi_s - \phi_c)\cos(\psi)}{\sin(\psi + \phi_c)}, \\ P_i &= \frac{R_0}{2\beta}(I_B^{dc})^2 [\frac{\sin(\psi + \phi_s)}{\sin(\psi + \phi_c)}]^2. \end{aligned} \tag{22}$$

For the beam stability, the accelerating voltage has to have negative slope, $dV_{acc}/d\phi_s < 0$ so that a particle having larger energy and, therefore, shifted above the transition energy to the tail $dz/dt = -\alpha\delta c_0$, sees the accelerating voltage $V_{acc} = |\hat{V}_c| \cos(\phi_s - \omega_g z/c_0)$ less than that for the equilibrium particle.

Let us define

$$Y = \frac{2R_0 I_B^{dc}}{|\hat{V}_c|}. \tag{23}$$

Here R_0 and $|\hat{V}_c|$ are shunt impedance and the amplitude of the voltage per cavity. The voltage Eq. (22) can be written as

$$\frac{1+\beta}{Y\cos\psi} = \frac{\sin(\phi_s - \phi_c)}{\sin(\psi + \phi_c)}. \tag{24}$$

Taking the derivative over ϕ_s, we keep I_B^{dc}, ψ, β, and I_g constant. Determining the derivative $d\phi_c/d\phi_s$ from Eq. (21), we get

$$\frac{d\phi_c}{d\phi_s} = \frac{\tan(\psi + \phi_c)}{\tan(\psi + \phi_s)}, \tag{25}$$

and

$$\frac{d}{d\phi_s}(\frac{1}{Y}) = \frac{\cos\psi}{(1+\beta)\sin(\psi+\phi_c)}[\cos(\phi_s - \phi_c) - \frac{\cos(\psi+\phi_s)}{\cos(\psi+\phi_c)}]. \tag{26}$$

Here, the phase ϕ_c can be determined from Eq. (24),

$$\tan(\phi_s - \phi_c) = \frac{(1+\beta)\sin(\psi+\phi_s)}{Y\cos\psi + (1+\beta)\cos(\psi+\phi_s)}. \tag{27}$$

Using this, Eq. (26) can be simplified,

$$\frac{d}{d\phi_s}(\frac{1}{Y}) = \frac{\sin(\psi+\phi_s)\cos\psi}{1+\beta+Y\cos\psi\ \cos(\psi+\phi_s)}. \tag{28}$$

The stability requires $(\cos\phi_s/Y)' < 0$, what can be written as

$$\frac{Y\sin(2\psi) - 2(1+\beta)\sin\phi_s}{1+\beta+Y\cos\psi\ \cos(\psi+\phi_s)} < 0. \tag{29}$$

The denominator is usually positive and Eq. (29) gives the Robinson criteria of stability

$$\pi/2 > \phi_s > 0; \quad \frac{Y\sin(2\psi)}{2(1+\beta)} < \sin\phi_s. \tag{30}$$

2.1 Optimum conditions

The "optimum conditions" correspond to cancellation of the reflected wave from the cavity, $V_r = 0$. In this case, Eqs. (2), (7) give

$$V_{ac} = |\hat{V}_c|\cos(\phi_s),$$
$$\hat{V}_c = \frac{R_0}{2\beta}\hat{I}_g. \tag{31}$$

Hence, the cavity phase $\phi_c = 0$. Separating the real and imaginary parts of Eq. (20) gives two equations. They give the following relations:

$$\tan\psi = \frac{\beta-1}{\beta+1}\tan\phi_s,$$
$$\beta = 1 + Y\cos\phi_s. \tag{32}$$

In this case, $V_{ac} = |\hat{V}_c| \cos\phi_s$, and

$$P_i = \frac{2(I_B^{dc})^2 R_0}{Y^2}(1 + Y\cos\phi_s) = \frac{|\hat{V}_c|^2}{2R_0}(1 + Y\cos\phi_s).). \tag{33}$$

2.2 Parked cavities

An idle cavity is excited only by the beam at the rf frequency ω_g and $I_g = 0$. Then,

$$P_B = -\frac{R_0}{2(1+\beta)}|\hat{I}_b|^2\cos^2\psi, \quad P_c = \frac{R_0}{2(1+\beta)^2}|\hat{I}_b|^2\cos^2\psi, \quad P_r = \frac{\beta R_0}{2(1+\beta)^2}|\hat{I}_b|^2\cos^2\psi, \tag{34}$$

and $P_c + P_r + P_B = 0$.

The impedance of the cavity is given by Eq. (15). Eq. (20) for the idle cavity takes the form

$$\hat{V}_c = |\hat{V}_c|e^{-i\phi_c} = -2I_B^{dc}R_L\, cos\psi\, e^{i(\psi+\phi_s)}. \tag{35}$$

To be consistent, the phases have to satisfy $\sin(\phi_c + \psi + \phi_s) = 0$, or $\phi_c = \pi - \psi - \phi_s$. The parked cavity contributes to acceleration $V_{ac} = -|\hat{V}_c|\cos\psi$ which can be minimized with large $\psi \simeq \pi/2$.

2.3 Low-level rf feedback

The low-level feedback is described by the feedback open-loop transfer function

$$G(\omega) = H(\omega)e^{i(\omega-\omega_g)\tau_{FB}}, \tag{36}$$

where H is the gain factor and $\tau_{FB}(\omega)$ is delay time.

The feedback detects a signal $\hat{V}_c$ from the cavity generating the current $\hat{I}_{FB} = \hat{V}_c/Z_{FB}$. The generated current is applied to the cavity (in parallel with the beam and generator currents) with the opposite polarity generating additional voltage

$$\delta\hat{V}_c = Z_c I_{FB} = \frac{Z_c}{Z_{FB}}\hat{V}_c. \tag{37}$$

By definition, $\delta\hat{V}_c = G(\omega)\hat{V}_c$, and

$$Z_{FB} = \frac{Z_c}{G(\omega)}. \tag{38}$$

The total voltage is the difference of the voltage cavity $\hat{V}_c = Z_c(\hat{I}_g - \hat{I}_B)$ and $\delta\hat{V}_c$. Hence, $\hat{V}_c = Z_c(\hat{I}_g - \hat{I}_B) - G\hat{V}_c$, i.e.

$$\hat{V}_c = Z_{tot}(\hat{I}_g - \hat{I}_B), \quad \frac{1}{Z_{tot}} = \frac{1}{Z_c} + \frac{1}{Z_{FB}}, \tag{39}$$

or

$$\hat{V}_c = \frac{Z_c}{1+G}(\hat{I}_g - \hat{I}_B). \tag{40}$$

The total impedance in Eq. (39) for small $\Delta\omega = \omega - \omega_c$ and for small delay time $\omega_g \tau_{FB}/Q_L << 1$ take the same form as the cavity impedance Eq. (15) but with R_L replaced by $R_H = R_L/(1+H)$, and Q_L by $Q_H = Q_L - \omega_c \tau_{FB}/[2(1+H)]$,

$$Z_{tot} = \frac{i(R_H/Q_H)(\omega_c/2)}{\omega - \omega_c + i(\omega_c/2Q_H)}. \tag{41}$$

2.4 Noise of the klystron

The generator current $I_g(t) = (1/2)\hat{I}_g e^{-i\omega_g t} + c.c.$ induces the cavity voltage

$$V_c(t) = \hat{I}_g R_L \cos\psi \cos(\omega_g t - \psi). \tag{42}$$

The noise of the klystron

$$\Delta I_g(t) = \int \frac{d\omega}{2\pi} \Delta I(\omega) e^{-i\omega t}, \tag{43}$$

induces fluctuations of the voltage on the cavity

$$\Delta V_c(t) = \int \frac{d\omega}{2\pi} Z_c(\omega) \Delta I(\omega) e^{-i\omega t}. \tag{44}$$

Additional averaged over time power $\Delta P =< \Delta I_g(t) \Delta V_g(t) >$ can be written in terms of the spectral density of the noise

$$< \Delta I(\omega) \Delta I(\omega') >= 2\pi < |\Delta I|^2_\omega > \delta(\omega - \omega') \tag{45}$$

as

$$\Delta P = \int \frac{d\omega}{2\pi} Re[Z_c(\omega)] \; < |\Delta I|^2_\omega > . \tag{46}$$

For the broad spectral density,

$$\Delta P = \omega_c (\frac{R_0}{Q_0}) < |\Delta I|^2_\omega > . \tag{47}$$

The induced voltage $\Delta V_c(t)$ on a localized cavity drives the synchrotron motion of a bunch

$$\frac{d^2z}{dt^2}+\gamma_d\frac{dz}{dt}+\omega_s^2 z(t)=-\frac{\alpha\omega_0 c}{2\pi E}e\Delta V_c(t)\sum_n e^{in\omega_0 t}, \tag{48}$$

where $\omega_0/2\pi$ is revolution frequency E is the particle energy, and γ_d is radiation damping. The induced fluctuations are

$$z(t)=-\frac{\alpha\omega_0 c}{2\pi E}\sum_n\int\frac{d\omega}{2\pi}\frac{eZ_c(\omega)\Delta I(\omega)e^{-i(\omega-n\omega_0)t}}{\omega_s^2-i\gamma_d(\omega-n\omega_0)-(\omega-n\omega_0)^2}. \tag{49}$$

The average $< z(t) >= 0$. The rms $< z^2(t) >$ is given in terms of the spectral density. If fluctuations are independent for n_c cavities in the ring, then

$$< z^2 >= n_c(\frac{\alpha\omega_0 c}{2\pi E})^2\sum_n\int\frac{d\omega}{2\pi}\frac{e^2|Z_c(\omega)|^2 < |\Delta I|^2_\omega >}{[\omega_s^2-(\omega-n\omega_0)^2]^2+\gamma_d^2(\omega-n\omega_0)^2]}. \tag{50}$$

For the PEP-II, the radiation damping $1/\gamma_d \simeq 18$ ms, and the width of the impedance $\Delta\omega=\omega_g/(2Q_L)>\omega_s>>\gamma_d$. Therefore, the main contribution is given by $n=\omega_g/\omega_0$ and

$$< z^2 >= n_c(\frac{\alpha\omega_0 c}{2\pi E})^2\frac{e^2|Z_c(\omega_g+\omega_s)|^2 < |\Delta I|^2_\omega >}{2\omega_s^2\gamma_d}. \tag{51}$$

In terms of the fluctuations ΔP per cavity, and neglecting the difference between $Z_c(\omega_c+\omega_s)$ and $Z_c(\omega_c)$, we get

$$< z^2 >= n_c(\frac{e\alpha\omega_0 c}{2\pi E})^2\frac{R_L Q_L}{2\omega_g\omega_s^2}(\frac{\Delta P}{\gamma_d})\cos^2\psi. \tag{52}$$

The estimate of the effect can be obtained from

$$\frac{< z^2 >}{\sigma_B^2}=\frac{1}{n_c}(\frac{\omega_s^4}{\omega_g^3\gamma_d})(\frac{Q_0\cos^2\psi}{4(1+\beta)^2\sin^2\phi_s(\alpha\delta)^2})(\frac{\Delta P}{P_c}), \tag{53}$$

where the power per cavity P_c is given by Eq. (11). The last factor in Eq (53) is the relative power of fluctuations per cavity and can be measured experimentally.

3 Longitudinal beam stability

Here we start discussion of the coherent instabilities. Some definitions are given in Appendix I.

3.1 Potential well distortion

The PWD bunch lengthening is the main effect below the threshold of the microwave instability.

Let us use the dimensionless variables $x = z/\sigma_B$, $p = -\delta/\delta_0$, where $z > 0$ is the shift in respect to the bunch centroid to the head of a bunch, σ_B and δ_0 are the zero-current rms of the bunch length and relative energy spread, respectively. The Fokker-Plank equation has the implicit steady-state Haisinskii (Boltzmann) solution for the distribution function

$$\rho(x,p) = \frac{1}{|N|} e^{-\{p^2/2+U_0(x)+\lambda_0 \int dx' \rho(x') S[\sigma_B(x'-x)]\}}. \tag{54}$$

Here $|N|$ is normalization constant defined by $\int dpdx\rho = 1$,

$$\begin{aligned} U_0(x) &= \frac{x^2}{2} - \frac{x^3}{6}(\frac{\omega_g\sigma_B}{c_0}) \cot\phi_s - \frac{x^4}{24}(\frac{\omega_g\sigma_B}{c_0})^2, \\ \lambda_0 &= \frac{N_B r_e}{2\pi R\gamma\alpha\delta_0^2}, \\ S(z) &= \int_0^z dz' W(z'), \end{aligned} \tag{55}$$

N_B is bunch population, r_e is classical electron radius, α is momentum compaction, $W(z)$ is the longitudinal wake (dimension V/pC or $1/cm$), $W(z) = 0$ for $z < 0$. In U_0 the rf nonlinearity of the lowest order are taken into account, the (zero current) synchrotron tune, $\nu_s^2 \propto sin\phi_s$, and the rf phase ϕ_s is defined to have $\sin\phi_s > 0$.

More convenient to rewrite the formulas in the following equivalent form minimizing contribution of small distances where wake can be a sharp function of z:

$$\rho(x,p) = \frac{1}{|N|} e^{-\{p^2/2+U_0(x)\}+\lambda_0 \int dx' \rho(x') \hat{S}[x'-x]\}}. \tag{56}$$

$$\hat{S}(x) = \int_{\sigma_B x}^{\infty} dz' W(z'), \tag{57}$$

3.2 Microwave instability

The single bunch microwave instability can be considered as extreme case of the mode-coupling instability where there are many modes become unstable and interact with each other. The onset of the instability can be expected when at least one of the modes (azimuthal or radial) is unstable. The history of the SLC damping ring serves as an example: the old vacuum chamber used to give the substantially inductive wake and the microwave instability was related to the unstable azimuthal sextupole mode ("strong" microwave instability). After installation of a new, smooth vacuum chamber, the wake became mostly resistive and the instability was related to the unstable radial modes

(”weak” microwave instability). Although the weak instability had lower threshold, the violent saw-tooth bunch dynamics was eliminated.

The threshold of the stability is usually estimated using Keil-Schnell criterion:

$$\frac{I_{bunch}^{peak}}{2\pi\alpha\delta_0^2(E/e)}|\frac{Z}{n}|_{eff} < 1, \tag{58}$$

where the peak bunch current,

$$I_{bunch}^{peak} = \sqrt{2\pi}(R/\sigma_B)I_{bunch}^{aver}, \quad I_{bunch}^{aver} = eN_Bf_0, \tag{59}$$

and the effective impedance

$$|\frac{Z}{n}|_{eff} = \frac{1}{\sum h_a(p\omega_0)}\sum_p h_a(p\omega_0)\frac{Z_l(p\omega_0)}{p} \tag{60}$$

is convolution of the longitudinal impedance with a-th momentum of the bunch spectrum

$$h_a(\omega) = (\frac{\omega\sigma_B}{c_0})^{2a}e^{-(\frac{\omega\sigma_B}{c_0})^2}. \tag{61}$$

For broad-band impedance, Eq. (60) can be written as

$$\frac{Z}{n}_{eff} = \frac{2x^{2a+1}}{\Gamma(a+1/2)}\int_0^\infty dpp^{2a-1}e^{-p^2x^2}\,Z(p\omega_0), \tag{62}$$

where $x = \omega_0\sigma_B/c_0$.

The threshold is, usually, given by the momentum $a = 1$ because $a = 0$ corresponds to perturbation of the bunch as a whole and terms $a > 1$ are small.

K. Oide has shown that the microwave instability is related to the anharmonicity of particle trajectories in the distorted potential. The analytic formulas which we use in the code for calculating the threshold of instability are given in Appendix 2.

3.3 Multibunch longitudinal stability

Results [3], [4] (F. Sacherer, J. M. Wang, B. Zotter) for the uniform fill of the ring are well known and can be found in the textbooks. The derivation is given below to clarify the implied approximations and to give formulas which can be used for arbitrary filling pattern and for the sake of completeness.

Let us consider a train of n_b bunches. In an equilibrium, the N-th bunch center is at the distance $s_N > 0$ from the head of the train which is at the location $s = c_0 t mod(2\pi R)$ around the ring. The distance s_N may include the shift of the rf phase due to the gap in the train. Position of the i-th particle in the N-th bunch is

$$s_{i,N}(t) = ct - s_N + z_{i,N}(t) \tag{63}$$

where $z > 0$ is displacement to the head of the bunch due to synchrotron motion,

$$z_{i,N}(t) = z_N^0 + a_{i,N} \sin(\omega_s t + \phi_{i,N}). \tag{64}$$

The first constant in time term is related to the equilibrium rf phase, $\omega_g z_N^0/c = \phi_{s,N}$ of the N-th bunch. In the last term, the amplitude $a_{i,n}$ may be itself a slow function of time. The offset z_N^0 is included in s_N.

The motion is described by the equation

$$\frac{d^2 z_{i,N}(t)}{dt^2} + [\omega_s^0(N)]^2 z_{i,N}(t) = \lambda_0 \frac{1}{< N_B >} \sum_{j,M} [W(t - t_{j,M}) - W(0)], \tag{65}$$

where $< N_B >$ is average bunch population,

$$\lambda_0 = \frac{\alpha c_0 < N_B > r_e \omega_0}{2\pi\gamma},$$
$$[\omega_s^0(N)]^2 = \frac{\alpha \omega_g c_0 e |\hat{V}_c| n_c}{2\pi R E} \sin\phi_s(N), \tag{66}$$

and $W(t)$ is longitudinal wake (with dimension V/pC or $1/cm$), $W(t < 0) = 0$. Here α is momentum compaction factor, N_B is bunch population, $\omega_g/2\pi$ is the rf frequency, n_c is number of cavities, and $n_c|\hat{V}_c|$ is the total maximum rf voltage per turn. The phases $\phi_s(N)$ are defined by the losses U per turn/per particle and, due to the train gap, are different for different bunches,

$$e n_c V_c \cos\phi_s(N) = U + e^2 \sum_M N_b(M)\, W(\frac{s_N - s_M}{c_0}). \tag{67}$$

The synchrotron damping is implied in Eq. (65). The longitudinal wake in Eq. (65) should not include the contribution of the fundamental mode which is already taken into account defining the steady-state V_c and ϕ_s.

Usually, the group velocity of the wakes excited in the ring is small and can be neglected. This is true for localized components such as rf cavities and also for the resistive wall provided the skin depth δ_0 at the revolution frequency is small compared to the bunch length σ_B, $\delta_0^2 << \sigma_B R$.

Then, the time $t_{j,M}$ when a particle j of the M-th bunch is at the same impedance generating element of the ring as the test particle i, N is

$$t_{j,M} = t - \frac{s_N - s_M}{c_0} + \frac{1}{c_0}[z_{i,N}(t) - z_{j,M}(t - \frac{s_N - s_M}{c_0})]. \tag{68}$$

This formula is correct for bunches in front of the bunch N with $s_M < s_N$. Otherwise, $s_N - s_M$ has to be replaced by $s_N - s_M + 2\pi R k$, where k is the number of a preceding turn. This rule is applicable also to the sum in Eqs. (67) and (65).

In the last equation we subtracted term $W(0) = \sum_M W(\frac{s_N - s_M}{c_0})$ already taken into account in Eq. (67).

In the frequency domain, the wake is given in terms of the longitudinal impedance,

$$W(t) = \int \frac{d\omega}{2\pi} Z(\omega) e^{-i\omega t}. \tag{69}$$

Substituting $t - t_{j,M}$ and averaging over incoherent synchrotron oscillations with amplitude a, we get

$$\frac{d^2 z_{i,N}(t)}{dt^2} + [\omega_s^0(N)]^2 z_{i,N}(t) = \frac{\lambda_0}{< N_B >} \sum_{j,M} \int \frac{d\omega}{2\pi} Z(\omega) e^{-i(\omega/c_0)(s_N - s_M)} e^{-(\omega a/c_0)^2}$$
$$\{e^{i(\omega/c_0)[z_{i,N}(t) - z_{j,M}(t-(s_N-s_M)/c_0)]} - 1\}. \tag{70}$$

The sum can be split over all bunches at a given turn plus over all preceding turns (neglecting the variation of the bunch populations with time),

$$\sum_M e^{-i(\omega/c_0)((s_N - s_M)} = \{\sum_{M=1}^{N} e^{-i(\omega/c_0)((s_N - s_M)} + \sum_{M=N+1}^{n_b} e^{-i(\omega/c_0)((2\pi R + s_N - s_M)}\} \sum_{k=0}^{\infty} e^{2\pi Rk}$$
$$= \{\sum_{M=1}^{N} e^{-i(\omega/c_0)((s_N - s_M)} + \sum_{M=N+1}^{n_b} e^{-i(\omega/c_0)((2\pi R + s_N - s_M)}\} \sum_{k=0}^{\infty} e^{-i\omega 2\pi (R/c_0)k}. \tag{71}$$

The last sum gives

$$\sum_{k=0}^{\infty} e^{-i\omega 2\pi (R/c_0)k} = \omega_0 \sum_k \delta(\omega - k\omega_0), \tag{72}$$

what allows to join two terms in the first sum:

$$\sum_M e^{-i(\omega/c_0)((s_N - s_M)} = \omega_0 \sum_{M=1}^{n_b} e^{-i(\omega/c_0)((s_N - s_M)} \sum_{k=0}^{\infty} \delta(\omega - k\omega_0). \tag{73}$$

Eq. (70) takes the form

$$\frac{d^2 z_{i,N}(t)}{dt^2} + [\omega_s^0(N)]^2 z_{i,N}(t) = \frac{\lambda_0 \omega_0}{2\pi < N_B >} \sum_{j,M=1}^{n_b} \sum_k Z(k\omega_0) e^{-ik(\omega_0/c_0)(s_N - s_M)}$$
$$e^{-(k\omega_0 a/c_0)^2} \{e^{i(k\omega_0/c_0)[z_{i,N}(t) - z_{j,M}(t-(s_N-s_M)/c_0)]} - 1\}. \tag{74}$$

Dipole oscillations of bunches can be described considering bunches as a macroparticle, $z_{i,N} = z_N$, and using the linear approximation over the amplitudes of coherent synchrotron oscillations. That gives

$$\frac{d^2 z_N(t)}{dt^2} + [\omega_s(N)]^2 z_n(t) = -\frac{i\lambda_0\omega_0^2}{2\pi c_0}\sum_{M=1}^{n_b}\frac{N_B(M)}{< N_B >}\sum_k kZ(k\omega_0)e^{-ik(\omega_0/c_0)(s_N-s_M)}$$
$$e^{-(k\omega_0 a/c_0)^2} z_{j,M}(t-(s_N-s_M)/c_0), \tag{75}$$

where the tune shift is included in $\omega_s(N)$,

$$\omega_s(N) = \omega_s^0(N) - \frac{i\lambda_0\omega_0^2}{4\pi\omega_s^0 c_0}\sum_M(\frac{N_B(M)}{< N_B >})\, kZ(k\omega_0)e^{-i(k\omega_0/c_0)(s_N-s_M)}e^{-(k\omega_0 a/c_0)^2}. \tag{76}$$

For the uniform distribution of equal bunches around the ring,

$$\sum_M e^{-i(k\omega_0/c_0)(s_N-s_M)} = n_b\sum_p \delta_{k,n_b p}. \tag{77}$$

In this case, the coherent shift is the same for all bunches. Eq. (76) is simplified,

$$\omega_s = \omega_s^0 - \frac{in_b\lambda_0\omega_0^2}{4\pi\omega_s^0 c_0}\sum_p (n_b p)Z(n_b p\omega_0)e^{-(n_b p\omega_0 a/c_0)^2}. \tag{78}$$

The solution of Eq. (75) can be found in the form

$$z_N(t) = \int\frac{d\Omega}{2\pi}a_N(\Omega)e^{-i\Omega t} + c.c. \tag{79}$$

.

If the coherent $\Omega << \omega_s$, $a_N(\omega)$ satisfy the following equation

$$\Omega a_N(\Omega) = \frac{i\lambda_0\omega_0^2}{4\pi\omega_s(N)c_0}\sum_{M=1}^{n_b}\frac{N_B(M)}{< N_B >}\sum_k kZ(k\omega_0)e^{-i(k\omega_0-\omega_s(N)-\Omega)(s_N-s_M)/c_0}$$
$$e^{-(k\omega_0 a/c_0)^2}a_M(\Omega+\omega_s(N)-\omega_s(M)). \tag{80}$$

If variation of the tune shift is small, $|\omega_s(N)-\omega_s(M)| << \Omega$, Eq. (80) is reduced to a linear matrix equation

$$\Omega a_N(\Omega) = \frac{i\lambda_0\omega_0^2}{4\pi\omega_s(N)c_0}\sum_{M=1}^{n_b}K_\Omega(N,M)a_M(\Omega), \tag{81}$$

where the matrix

$$K_\Omega(N,M) = \frac{N_B(M)}{< N_B >}\sum_k kZ(k\omega_0)e^{(i/c_0)(\omega_s(N)+\Omega-k\omega_0)(s_N-s_M)}e^{-(k\omega_0 a/c_0)^2}. \tag{82}$$

Then, $a_N(\Omega)$ is given by the superposition of eigen-vectors $X_M^\mu(\Omega)$, $\mu = 1,2,..,n_b$ of $K_\Omega(N,M)$,

$$K_\Omega(N,M)X_M^\mu(\Omega) = \kappa_\mu X_M^\mu(\Omega). \tag{83}$$

The spectrum of Ω is discrete, Ω_μ, $\mu = 1, 2, .., n_b$ has to be proportional one of the eigen-values κ_μ,

$$\Omega_\mu = \frac{i\lambda_0 \omega_0^2 \kappa_\mu}{4\pi\omega_s(N)c_0} = i\frac{\alpha I_{beam}\omega_0^2}{4\pi(E/e)\omega_s(N)}\frac{\kappa_\mu}{n_b}. \tag{84}$$

In the opposite case of $|\omega_s(N) - \omega_s(M)| > \Omega$ the instability is stabilized by the spread of the bunch-to-bunch synchrotron frequencies.

For equal uniformly distributed bunches, the eigen-vectors are

$$X_M^\mu(\Omega) = (1/\sqrt{n_b})e^{-2\pi i\mu M/n_b}, \tag{85}$$

and the eigenvalues are

$$\kappa_\mu = n_b \sum_p g_p Z(g_p\omega_0)e^{-g_p^2(a/R)^2}, \quad g_p = n_b p + \mu + \nu_s + \Omega_\mu/\omega_0, \quad \mu = 1, 2.., n_b. \tag{86}$$

The general formula for the uniform fill and $m = 1, 2..$ is given by Wang. It cited in ZAP manual [4] with some errors. The formula below is corrected version [3]:

$$\Omega_{\mu,m} = i\frac{\alpha I_{beam}\omega_0^2}{4\pi(E/e)\omega_s}\frac{(\sigma_B/2R)^{2(m-1)}}{m!(m-1)!}$$
$$\sum_{p=-\infty}^{\infty}(pn_b + \mu + m\nu_s)^{2m-1}e^{-(pn_B+\mu+m\nu_s)^2(\sigma_B/R)^2}\, Z[(n_b p + \mu + m\nu_s)\omega_0]. \tag{87}$$

The dipole oscillations considered above correspond to $m = 1$.

3.4 Robinson Instability

The Robinson instability usually is defined as corresponding to the dipole oscillations $m = 1$ of the $\mu = 0$ mode. It is defined by the contributions of the terms $p\omega_0 = \pm\omega_g$. $\Omega_{0,1}$ is proportional to the difference $Z(\omega_g + \omega_s) - Z^*(\omega_g - \omega_s)$. The beam stability requires $Re[Z(\omega_g + \omega_s)] < Re[Z(\omega_g - \omega_s)]$. For stability, the cavity has to be detuned down from the rf frequency, $\omega_c < \omega_g$.

4 Transverse Instabilities

The transverse motion of the i-th particle in the N-th bunch

$$y_{i,N}(t) = A_{i,N}(t)e^{-i\psi_{i,N}(t)} + c.c. \tag{88}$$

is coupled to the longitudinal motion through the energy dependence of the transverse tune $\omega_y(t) = d\psi(t)/dt$,

$$\psi_{i,N}(t) = \omega_y^0 t - \omega_\xi z_{i,N}(t)/c_0, \quad \omega_{i,N}^y = \frac{d\psi_{i,N}(t)}{dt}. \tag{89}$$

Here

$$\omega_\xi = \frac{\xi}{\alpha}\omega_y^0, \quad \xi = (1/\nu_y^0)(d\nu_y^0/d\delta) \tag{90}$$

are the chromatic frequency and the relative chromaticity, respectively. The complex amplitude $A_{i,N} = |A_{i,N}|e^{i\psi_{i,N}(0)}$ includes the initial betatron phase $\psi_{i,N}(0)$ of a particle.

The equation of motion

$$\begin{aligned} &\frac{d^2 y_{i,N}(t)}{dt^2} + [\omega_{i,N}^y(t)]^2 y_{i,N}(t) = RHS, \\ &RHS = \frac{r_e c_0^2}{2\pi R\gamma}\sum_{j,M} W_y(t - t_{j,M})\, y_{j,M}(t_{j,M}), \end{aligned} \tag{91}$$

where $W_y(t)$ is the transverse wake (dimension $V/pC/m$ or $1/cm^2$), $W(t) = 0$ for $t < 0$.

If the coherent tune shift is small compared to ω_y, Eq. (91) can be averaged over fast oscillations giving equation for the amplitudes

$$\begin{aligned} &\dot{A}_{i,N}(t) = i\lambda_y \sum_j W_y(t - t_{j,N}) y_{j,N}(t_{j,N})\, e^{i\psi_{i,N}(t)} \\ &-\lambda_y \sum_{j,M}\int\frac{d\omega}{2\pi}\int\frac{d\Omega'}{2\pi} Z_y(\omega) A_{j,M}(\Omega') e^{-i\Omega' t} \\ &e^{i(\omega-\Omega'-\omega_y^0)\tau_{N,M}}\; e^{(i/c_0)(\omega-\Omega'-\omega_y^0+\omega_\xi)[z_{i,N}(t)-z_{j,M}(t+\tau_{N,M})]}. \end{aligned} \tag{92}$$

Here,

$$\lambda_y = \frac{r_e c_0^2}{4\pi R\gamma\omega_y^0}, \quad \tau_{N,M} = \frac{s_M - s_N}{c_0}, \tag{93}$$

$A(\Omega)$ are Fourier harmonics of the amplitude $A(t)$,

$$A_N(t) = \int\frac{d\Omega}{2\pi} A_N(\Omega)\, e^{-i\Omega t}, \tag{94}$$

and the transverse impedance is introduced

$$W_y(t) = i\int\frac{d\omega}{2\pi} Z_y(\omega)\, e^{-i\omega t}. \tag{95}$$

With this definition, $Z_y(\omega)$ for ultra-relativistic case can have singularities only in the lower half-plane of ω.

The first term in the RHS of Eq. (92) is due to interaction of particles in the same bunch. Other terms describe interaction between bunches and, for bunch spacing large compared with the bunch rms length σ_B, bunches can be considered as point-like, $A_{j,M} = A_M$.

The bunch-by-bunch transverse feedback system damps the bunch centroid oscillations with the damping rate γ_{FB} adding to the right-hand-side of Eq. (92) the term

$$-\gamma_{FB}\, e^{i\psi_{i,N}(t)} \frac{1}{N_B(N)} \sum_{i=1}^{N_B(N)} A_{i,N} e^{-i\psi_{i,N}(t)}. \tag{96}$$

4.1 Head-tail instability

For a single bunch, Eq. (92) gives

$$\dot{A}_i(t) = i\lambda_y \sum_j W_y(t - t_j) y_j(t_j)\, e^{i\psi_i(t)}. \tag{97}$$

For the same bunch, we can average fast oscillating term $\propto e^{-2i\omega_y t}$, neglect the difference between t_j and t and, for moderate chromaticity, drop the chromatic shift in the bunch spectrum. That gives

$$\dot{A}_i(t) = i\lambda_y \sum_j W_y[z_j(t) - z_i(t)] A_j(t)\, e^{-i\omega_\xi [z_i(t) - z_j(t)]/c_0}. \tag{98}$$

In two-particle model (A. Chao), all particles are grouped into two macro-particles oscillating with the same frequency,

$$z_1 = a \sin(\omega_s + \phi_1), \quad z_2 = a \sin(\omega_s + \phi_2) \tag{99}$$

and phases $\phi_1 = \phi$, and $\phi_2 = \phi + \pi$. The sum over j gives $N_B/2$. Denoting

$$\kappa = 2(\xi/\alpha)(\omega_y^0/c_0)a, \quad \Lambda = \frac{\lambda_y N_B}{2\omega_s}, \tag{100}$$

Eq. (98) is reduced to two coupled equations

$$\begin{aligned} \dot{A}_1(t) &= i\Lambda\omega_s W_y[-2a \sin(\omega_s t + \phi)]\, A_2 e^{-i\kappa \sin(\omega_s t + \phi)}, \\ \dot{A}_2(t) &= i\Lambda\omega_s W_y[2a \sin(\omega_s t + \phi)]\, A_1 e^{i\kappa \sin(\omega_s t + \phi)}. \end{aligned} \tag{101}$$

Solution is described in the A.Chao textbook [3]: it is given by a map, first, from initial conditions at $t = 0$ to $t = T_s/2$, $T_s = 2\pi/\omega_s$, and then for the next half period of synchrotron oscillation. The eigen-values of the matrix describing the one-period transform are

$$\mu = 1 - \frac{1}{2}G^2 \pm \sqrt{[1 - \frac{1}{2}G^2]^2 - 1}, \tag{102}$$

where

$$G = \Lambda \int_0^\pi d\psi W_y(2a \sin\psi) e^{-i\kappa \sin\psi}. \tag{103}$$

Then, $A(t) \propto \mu^{t/T_s}$, and the growth rate $A(t) = e^{\Gamma t}$ is

$$\Gamma = Re[\frac{1}{T_s} \log \mu]. \tag{104}$$

The growth rate Eq. (104) includes the strong head-tail and chromatic head-tails effects.

More accurate consideration of the head-tail instability was given by Satoh and Chin [5]. The result is formulated as a matrix equation

$$|\delta_{h,l} + iKb_h(\lambda)\, M_{h,l}(\lambda)| \; = \; 0, \tag{105}$$

for the parameter $\lambda = \Omega/\omega_s$ where Ω is the coherent shift from the zero-current betatron tune $\nu_\perp$, and the instability takes place when the growth rate $Im[\Omega] > 0$. Here $M_{h,l}$, $h, l = 0, 1, 2, ..$ is matrix

$$\begin{aligned} M_{h,l}(\lambda) &= \sum_{p=-\infty}^{\infty} Z_\perp[(p + \nu_\perp + \lambda\nu_s)\omega_0] C_h[(p + \nu_\perp + \lambda\nu_s - \frac{\xi}{\alpha})\frac{\sigma_l}{R}] C_l[(p + \nu_\perp + \lambda\nu_s - \frac{\xi}{\alpha})\frac{\sigma_l}{R}], \\ C_h(x) &= \frac{1}{\sqrt{h!}}(\frac{x}{\sqrt{2}})^h \exp -\frac{x^2}{2}, \\ K &= \frac{I_{bunch}\beta_\perp}{4\pi(E/e)\nu_s}, \end{aligned} \tag{106}$$

$Z_\perp(\omega)$ is transverse impedance in Ohm/cm, $\xi = d\nu_\perp/d\delta$ is the absolute chromaticity, σ_l is the rms bunch length, $R = c0/\omega_0$ is the average machine radius, and α is the momentum compaction. The function

$$b_h(\lambda) = \sum_{k=0}^{[h/2]} (\frac{h!}{k!(h-k)!})\, \frac{\lambda}{\lambda^2 - (h-2k)^2} P[h,k], \tag{107}$$

where the upper limit is given by the integer part of $h/2$ and $P(h,k) = 1$ if $2k = h$ and $P(h,k) = 1$ otherwise. In calculations the matrix is truncated to a finite rank which is approximately equal to the number of azimuthal modes taken into account. Usually, the threshold of instability is given by the lowest modes. An example of calculations based on the Satoh-Chin formalism is given below.

4.2 Transverse coupled-bunch instability

First, let us study the multi-bunch transverse dipole instability describing oscillations of bunch centroids.

From Eq. (92), we get

$$\Omega A_{i,N}(\Omega) = -i\lambda_y \sum N_B(M) \int \frac{d\omega}{2\pi} \int \frac{d\Omega'}{2\pi} Z_y(\omega) A_{j,M}(\Omega')$$
$$\int dt e^{i(\Omega-\Omega')t} e^{-(\sigma_B/c_0)^2(\omega-\Omega'-\omega_y^0+\omega_\xi)^2} e^{(i/c_0)(\omega-\omega_y^0-\Omega')(s_M-s_N)}$$
$$< e^{(i/c_0)(\omega-\omega_y^0-\Omega'+\omega_\xi)[z_{i,N}(t)-z_{j,M}(t+(s_M-s_N)/c_0)]} >, \quad (108)$$

where $< .. >$ mean averaging over longitudinal motion. The integral over dt can be calculated substituting Eq. (64) for $z_N(t)$ and expanding

$$e^{\frac{i}{c_0}(\omega-\omega_y^0-\Omega'+\omega_\xi)[z_{i,N}(t)-z_{j,M}(t+(s_M-s_N)/c_0)]} = \sum_{m,m'} e^{im(\omega_s t+\phi_{i,N})-im'[\omega_s t+\omega_s(s_M-s_N)/c_0+\phi_{j,M}]}$$
$$J_m[\frac{a_N}{c_0}(\omega-\omega_y^0-\Omega'+\omega_\xi)] J_{m'}[\frac{a_M}{c_0}(\omega-\omega_y^0-\Omega'+\omega_\xi)]. \quad (109)$$

The sum over M can be reduced to the sum over one-turn as it was done in the longitudinal case, Eq. (108) takes the form

$$\Omega A_{i,N}[\Omega] = \sum_{m,m'} \sum_{M=1}^{n_b} K^{\Omega}_{m,m'}(N,M) \sum_{j=1}^{n_b} A_{j,M}[\Omega+(m-m')\omega_s], \quad (110)$$

where

$$K^{\Omega}_{m,m'}(N,M) = -i\frac{\lambda_y\omega_0}{2\pi} N_B(M) \sum_{k=-\infty}^{\infty} Z_y(k\omega_0+\omega_y^0+m\omega_s^0+\Omega)$$
$$e^{\frac{i}{c_0}k\omega_0(s_M-s_N)+im\phi_{i,N}-im'\phi_{j,M}}$$
$$J_m[\frac{a_N}{c_0}(k\omega_0+\omega_\xi+m'\omega_s)]\, J_{m'}[\frac{a_M}{c_0}(k\omega_0+\omega_\xi+m'\omega_s)]. \quad (111)$$

The argument of the Bessel functions for the long-range wakes is small. Therefore, the series over m, m' are series over a small parameter.

Averaging with the Gaussian distribution gives for $\omega\sigma_B/c_0 << 1$

$$\int J_m[\frac{a_N\omega}{c_0}]\, J_m[\frac{a_M\omega}{c_0}] \frac{ada}{\sigma_B^2} e^{-\frac{a^2}{2\sigma_b^2}} = I_m[(\frac{\omega\sigma_B}{c_0})^2]\, e^{-(\frac{\omega\sigma_B}{c_0})^2} \simeq \frac{(\frac{\omega\sigma_B}{c_0})^m}{2^m m!}\, e^{-(\frac{\omega\sigma_B}{c_0})^2}. \quad (112)$$

4.3 Transverse dipole coupled-bunch instability

In the lowest order, $m = m' = 0$, what corresponds to the coupled bunch transverse dipole oscillations of point-like bunches. In this case, we can drop the index numbering particles within a bunch replacing $A_{i,N}$ by A_N. Eq. (110) takes the form of the matrix equation

$$\Omega A_N(\Omega) = \sum_{M=1}^{n_b} K_0(N,M) A_M(\Omega), \tag{113}$$

where $K_0(N,M) = N_B(M) K^{\Omega}_{0,0}(N,M)$,

$$K_0(N,M) = -i\frac{\lambda_y \omega_0}{2\pi} N_B(M) \sum_{k=-\infty}^{\infty} Z_y(k\omega_0 + \omega_y^0 + \Omega)$$
$$e^{\frac{i}{c_0}k\omega_0(s_M - s_N)} e^{-(\sigma_B/c_0)^2(k\omega_0+\omega_\xi)^2}\, J_0[\frac{\sigma_{B,N}}{c_0}(k\omega_0 + \omega_\xi)]\, J_0[\frac{\sigma_{B,M}}{c_0}(k\omega_0 + \omega_\xi)]. \tag{114}$$

Coefficients $A_N(\Omega)$ are given by superposition of the eigen-vectors of the matrix $K_0(N,M)$. The spectrum of Ω is discrete, the coherent shift Ω has to be equal to one of the eigen-values of $K_0(N,M)$.

For the uniform fill, the eigen-vectors of $K_0(N,M)$ are $X^{\mu}_M = (1/\sqrt{n_b}) e^{-2\pi i \mu M/n_b}$. The spectrum of coupled-bunch motion is

$$\Omega_\mu = -i\lambda_y N_b n_b \frac{\omega_0}{2\pi} \sum_p Z_y(pn_b\omega_0 + \mu\omega_0 + \omega_y^0 + \Omega_\mu)\, e^{-(\sigma_B/c_0)^2(pn_b\omega_0+\mu\omega_0+\omega_\xi)^2}$$
$$J_0^2[\frac{\sigma_B}{c_0}(pn_b\omega_0 + \mu\omega_0 + \omega_\xi)], \quad \mu = 0, 1, ..., n_b, \tag{115}$$

where a_s is amplitude of the coherent synchrotron oscillations.

Eq. (115) is slightly different from the Wang's result [4]: $(pn_b\omega_0 + \mu\omega_0 + \omega_\xi)$ in the exponent of Eq. (115) is replaced in his formula by $(pn_b\omega_0 + \mu\omega_0 + \omega_\xi - \omega_y^0)$. The difference appears to come from neglecting the time delay in $y_{jM}(t_{jM})$ in Wang's formalism.

4.4 Transverse quadrupole coupled-bunch instability

For small amplitude of synchrotron oscillations $a_s\omega_{HOM}/c_0 <, 1$, the amplitude $A_N(\Omega)$ can be expanded over a_s. Neglecting terms $\propto z^2$ and reducing sum over all turns to the sum over one turn, we get

$$\Omega A_{i,N}(\Omega) = \sum_M^{n_B} \{G^{(0)}_{0,0}(N,M) < A_M(\Omega) >$$
$$+G^{(1)}_{1,0}(N,M)\hat{z}_{i,N} < A_{j,M}(\Omega+\omega_s) > -G^{(1)}_{0,-1}(N,M) < \hat{z}_{j,M} A_{j,M}(\Omega+\omega_s) >$$
$$-G^{(1)}_{-1,0}(N,M)\hat{z}^*_{i,N} < A_{j,M}(\Omega-\omega_s) > +G^{(1)}_{0,1}(N,M) < \hat{z}^*_{j,M} A_{j,M}(\Omega-\omega_s) >$$
$$+G^{(2)}_{1,1}(N,M)\hat{z}_{i,N} < \hat{z}^*_{j,M} A_{j,M}(\Omega) > +G^{(2)}_{-1,-1}(N,M)\hat{z}^*_{i,N} < \hat{z}_{j,M} A_{j,M}(\Omega) >\}. \tag{116}$$

Here $\hat{z}_{j,M} = a_{j,M} e^{i\phi_{j,M}}$, angular brackets mean averaging as in

$$< \hat{z}_{j,M} A_{j,M} >= \frac{1}{N_B(M)} \sum_j \hat{z}_{j,M} A_{j,M}, \tag{117}$$

where

$$G^{(m)}_{l,l'}(N,M)[\Omega] = -i\frac{\lambda_y \omega_0}{2\pi} N_B(M) \sum_{k=-\infty}^{\infty} (\frac{k\omega_0 + \omega_\xi + l'\omega_s}{2c_0})^m Z_y(k\omega_0 + \omega^0_y + \Omega + l\omega_s) e^{\frac{i}{c_0}k\omega_0(s_M - s_N)}. \tag{118}$$

The solution can be found in the form

$$A_{i,N}(\Omega) = A^{(0)}_N(\Omega) + \hat{z}_{i,N} A^{(+)}_N(\Omega) + \hat{z}^*_{i,N} A^{(-)}_{i,N}(\Omega). \tag{119}$$

The $A^{(\pm)}$ describe the $y - z$ correlation. Then, neglecting again terms $\propto < z^2 >$, we get

$$\begin{aligned} &A^{(0)}_N(\Omega) =< A_{i,N}(\Omega) >, \\ &< \hat{z}_{j,M} A_{j,M}(\Omega) >= \hat{d}_N A^{(0)}_N(\Omega) + < |z_{j,M}|^2 > A^{(-)}_N(\Omega), \\ &< \hat{z}^*_{j,M} A_{j,M}(\Omega) >= \hat{d}^*_N A^{(0)}_N(\Omega) + < |z_{j,M}|^2 > A^{(+)}_N(\Omega), \end{aligned} \tag{120}$$

where $d_M =< \hat{z}_{j,M} >$ is coherent longitudinal shift of the N-th bunch centroid and, for a Gaussian bunch, $< |z_{j,M}|^2 >= 2\sigma^2_{M,B}$.

Eq. (116) takes the form (we drop Ω in the arguments $G^{(m)}_{l,l'}(N,M)[\Omega]$):

$$\begin{aligned} \Omega A^{(0)}_N(\Omega) &= \sum_M^{n_B} G^{(0)}_{0,0}(N,M)[A^{(0)}_M(\Omega) + d_M A^{(+)}_M(\Omega) + d^*_M A^{(-)}_M(\Omega)] \\ &- \sum_M^{n_B} \{G^{(1)}_{0,-1}(N,M)[d_M A^{(0)}_M(\Omega + \omega_s) + 2\sigma^2_{M,B} A^{(-)}_M(\Omega + \omega_s)] \\ &+ \sum_M^{n_B} \{G^{(1)}_{0,1}(N,M)[d^*_M A^{(0)}_M(\Omega - \omega_s) + 2\sigma^2_{M,B} A^{(+)}_M(\Omega - \omega_s)]. \end{aligned} \tag{121}$$

$$\begin{aligned} \Omega A^{(+)}_N(\Omega) &= \sum_M^{n_B} G^{(1)}_{1,0}(N,M)[A^{(0)}_M(\Omega + \omega_s) + d_M A^{(+)}_M(\Omega + \omega_s) + d^*_M A^{(-)}_M(\Omega + \omega_s)] \\ &+ \sum_M^{n_B} \{G^{(2)}_{1,1}(N,M)[d^*_M A^{(0)}_M(\Omega) + 2\sigma^2_{M,B} A^{(+)}_M(\Omega)] \\ &- \sum_M^{n_B} \{G^{(2)}_{1,-1}(N,M)[d_M A^{(0)}_M(\Omega + 2\omega_s) + 2\sigma^2_{M,B} A^{(-)}_M(\Omega + 2\omega_s)]. \end{aligned} \tag{122}$$

$$\Omega A_N^{(-)}(\Omega) = -\sum_M^{n_B} G_{-1,0}^{(1)}(N,M)[A_M^{(0)}(\Omega-\omega_s) + d_M A_M^{(+)}(\Omega-\omega_s) + d_M^* A_M^{(-)}(\Omega-\omega_s)]$$
$$+\sum_M^{n_B}\{G_{-1,-1}^{(2)}(N,M)[d_M A_M^{(0)}(\Omega) + 2\sigma_{M,B}^2 A_M^{(-)}(\Omega)]$$
$$-\sum_M^{n_B}\{G_{-1,1}^{(2)}(N,M)[d_M^* A_M^{(0)}(\Omega-2\omega_s) + 2\sigma_{M,B}^2 A_M^{(+)}(\Omega-2\omega_s)]. \quad (123)$$

The bunch-by-bunch feedback system adds damping to each bunch proportional to the bunch centroid velocity $< \dot{y}_N >= (1/N_B)\sum_i dy_{i,n}/dt$. The FB can be described replacing $d^2y_{i,N}/dt^2 + \omega_b^2 y_{i,N}$ in the equation of motion by $d^2y_{i,N}/dt^2 + 2\gamma_{FB} < dy_N/dt > +\omega_b^2 y_{i,N}$. Eqs. 122)-(123) are then modified by adding

$$-i\frac{\gamma_{FB}}{2\pi}N_B(N)A_N^{(0)}(\Omega)\,\{1;\frac{\omega_\xi}{2\pi};-\frac{\omega_\xi}{2\pi}\} \quad (124)$$

to the right-hand-sides, respectively.

Let us consider the case where the longitudinal motion of the bunch centroid is initially not excited, $d^{\pm}=0$.

The amplitude $A^{\pm}$ are excited in this case by the vertical motion of the bunch centroid. Neglecting effect of these modes on the motion of the bunch centroid, we get from the first Eq. (121) the dispersion equation for the transverse dipole coupled bunch oscillations

$$\Omega A_N^{(0)}(\Omega) = \sum_M^{n_B} G_{0,0}^{(0)}(N,M)A_M^{(0)}(\Omega) \quad (125)$$

obtained already above, see Eq. (113).

Eqs. (122), (123) in the case $d=0$ give

$$\Omega A_N^{(+)}(\Omega) = \sum_M^{n_B} G_{1,0}^{(1)}(N,M)A_M^{(0)}(\Omega+\omega_s)$$
$$+\sum_M^{n_B}\{2\sigma_{M,B}^2 G_{1,1}^{(2)}(N,M)A_M^{(+)}(\Omega) - 2\sigma_{M,B}^2 G_{1,-1}^{(2)}(N,M)A_M^{(-)}(\Omega+2\omega_s)\}. \quad (126)$$

$$\Omega A_N^{(-)}(\Omega) = -\sum_M^{n_B} G_{-1,0}^{(1)}(N,M)A_M^{(0)}(\Omega-\omega_s)$$
$$+\sum_M^{n_B}\{2\sigma_{M,B}^2 G_{-1,-1}^{(2)}(N,M)A_M^{(-)}(\Omega) - 2\sigma_{M,B}^2 G_{-1,1}^{(2)}(N,M)A_M^{(+)}(\Omega-2\omega_s)\}. \quad (127)$$

For the uniform fill, the vectors Eq. (85) are the eigen-vectors of the matrices $G_{l,l'}^{(m)}(N,M)$,

$$\sum_M G^{(m)}_{l,l'}(N,M)[\Omega]X^\mu_M = \kappa^{(m)}_\mu(l,l')[\Omega]X^\mu_N, \tag{128}$$

where

$$\kappa^{(m)}_\mu(l,l')[\Omega] = -i\frac{\lambda_y\omega_0}{2\pi}N_B n_b \sum_{p=-\infty}^{\infty}\left(\frac{(pn_b+\mu)\omega_0+\omega_\xi+l'\omega_s}{2c_0}\right)^m Z_y((pn_b+\mu)\omega_0+\omega^0_y+\Omega+l\omega_s). \tag{129}$$

$A^{(\pm)}_N$ is a superposition of X^μ_N,

$$A^{(0)}_N[\Omega] = \sum_\mu g^{(0)}_\mu X^\mu_N, \quad A^{(\pm)}_N[\Omega\pm\omega_s] = \sum_\mu g^{(\pm)}_\mu X^\mu_N. \tag{130}$$

Note that

$$\sum_M G^{(m')}_{l,l'}(N,M)[\Omega']A^{(m)}_M[\Omega] = \sum_\mu g^{(m)}_\mu X^\mu_N \kappa^{(m')}_\mu(l,l')[\Omega'] \tag{131}$$

where m, m' and Ω, Ω' are not necessarily the same.

Hence, for the uniform bunch pattern, Eqs. (126), (127) give

$$\begin{aligned}
&[(\Omega-\omega_s) - 2\sigma^2\kappa^{(2)}_{1,1}(\Omega-\omega_s)]g^{(+)}_\mu + 2\sigma^2\kappa^{(2)}_{1,-1}(\Omega-\omega_s)g^{(-)}_\mu = \kappa^{(1)}_{1,0}(\Omega-\omega_s)g^{(0)}_\mu,\\
&[(\Omega+\omega_s) - 2\sigma^2\kappa^{(2)}_{-1,-1}(\Omega+\omega_s)]g^{(-)}_\mu\\
&+2\sigma^2\kappa^{(2)}_{-1,-1}(\Omega+\omega_s)g^{(+)}_\mu = -\kappa^{(1)}_{-1,0}(\Omega+\omega_s)]g^{(0)}_\mu.
\end{aligned} \tag{132}$$

The response to the excitation by the bunch centroid is infinite at the eigen-frequencies Ω of the matrix in the left-hand-side of Eqs. (132). The dispersion relations can be simplified using by the relations $\kappa^{(2)}_{1,1}(\Omega-\omega_s) = \kappa^{(2)}_{-1,1}(\Omega+\omega_s)$, and $\kappa^{(2)}_{-1,-1}(\Omega+\omega_s) = \kappa^{(2)}_{1,-1}(\Omega-\omega_s)$. That gives the (complex) coherent frequency shift of the μ-th quadrupole coupled bunch mode (correlated $y-z$ motion), $\mu = 0, 1, .., n_b$,

$$\Omega_\mu = \pm\omega_s + 2\sigma^2_B\kappa^{(2)}_\mu(\pm1,\pm1)[0]. \tag{133}$$

Note

$$\frac{\lambda_y\omega_0}{2\pi}N_b n_b = \frac{c_0\, I^{dc}_{beam}}{4\pi(E/e)\nu_y}. \tag{134}$$

More accurately would be to replace $1/\nu_y$ in the last formula by β_y/R at the location of the impedance generating element and take into account the bunch density factor $e^{-(\frac{\sigma_B}{c_0})^2(pn_b\omega_0+\mu\omega_0+\omega_\xi)^2}$.

4.5 Mode coupling in multibunch system (CBCM instability)

Now we can take into account effect of the quadrupole modes on the motion of the bunch centroids retaining terms $A^{(\pm)}(\Omega \pm \omega_s)$ in Eq. (116).

The coupling terms in Eq. (121)

$$-2\sum_{M}^{n_B}\sigma_{M,B}^2 G_{0,-1}^{(1)}(N,M)A_M^{(-)}(\Omega+\omega_s) + 2\sum_{M}^{n_B} 2\sigma_{M,B}^2 G_{0,1}^{(1)}(N,M)A_M^{(+)}(\Omega-\omega_s) \qquad (135)$$

according to Eqs. (132) are proportional to $A^{(0)}$. Generally speaking, they are small and can be taken into account by iterations. The nontrivial situation arises when the coherent tune shift is of the order of ω_s. Then modes can not be considered separately and coupling can lead to new Coupled-Bunch-Coupled-Mode (CBCM) instability in the multibunch system [6].

The system of equations Eq. (121) and Eqs. (132) takes the form :

$$\Omega A_N^{(0)}(\Omega) - \sum_{M}^{n_B} G_{0,0}^{(0)}(N,M)A_M^{(0)}(\Omega) =$$
$$2\sum_{M}^{n_B}\sigma_{M,B}^2\{-G_{0,-1}^{(1)}(N,M)A_M^{(-)}(\Omega+\omega_s) + G_{0,1}^{(1)}(N,M)A_M^{(+)}(\Omega-\omega_s)\}. \qquad (136)$$

$$(\Omega-\omega_s)A_N^{(+)}(\Omega-\omega_s) - 2\sum_{M}^{n_B}\sigma_{M,B}^2 G_{1,1}^{(2)}(N,M)|_{\Omega->\Omega-\omega_s}A_M^{(+)}(\Omega-\omega_s)$$
$$= \sum_{M}^{n_B}\{G_{1,0}^{(1)}(N,M)A_M^{(0)}(\Omega) - 2\sigma_{M,B}^2 G_{1,-1}^{(2)}(N,M)A_M^{(-)}(\Omega+2\omega_s)\}|_{\Omega->\Omega-\omega_s}, \quad (137)$$

$$(\Omega+\omega_s)A_N^{(-)}(\Omega+\omega_s) - 2\sum_{M}^{n_B}\sigma_{M,B}^2 G_{-1,-1}^{(2)}(N,M)|_{\Omega->\Omega+\omega_s}A_M^{(-)}(\Omega+\omega_s)$$
$$= -\sum_{M}^{n_B}\{G_{-1,0}^{(1)}(N,M)A_M^{(0)}(\Omega) + 2\sigma_{M,B}^2 G_{-1,1}^{(2)}(N,M)A_M^{(+)}(\Omega-2\omega_s)\}|_{\Omega->\Omega+\omega_s}. (138)$$

For the uniform fill, the vectors Eq. (85) are the eigen-vectors of the matrix $G_{l,l'}^{(m)}(N,M)$. Expanding

$$A_N^{(0)}[\Omega] = \sum_\mu g_\mu^{(0)} X_N^\mu, \quad A_N^{(\pm)}[\Omega\pm\omega_s] = \sum_\mu g_\mu^{(\pm)} X_N^\mu, \qquad (139)$$

we reduce Eqs. (136-138) to the system of algebraic equations for the amplitudes $g_\mu^{(0)}$ and $g_\mu^{(\pm)}$ for each coupled-bunch mode:

$$
\begin{aligned}
&\{\Omega - \kappa_\mu^{(0)}(0,0)[\Omega]\}\, g_\mu^{(0)} + 2\sigma_B^2\{\kappa_\mu^{(1)}(0,-1)[\Omega]\, g_\mu^{(-)} - \kappa_\mu^{(1)}(0,1)[\Omega]\, g_\mu^{(+)}\} = 0,\\
&\kappa_\mu^{(1)}(1,0)[\Omega-\omega_s]\, g_\mu^{(0)} - \{\Omega - \omega_s - 2\sigma_B^2\kappa_\mu^{(2)}(1,1)[\Omega-\omega_s]\}\, g_\mu^{(+)}\\
&-2\sigma_B^2\kappa_\mu^{(2)}(1,-1)[\Omega-\omega_s] g_\mu^{(-)} = 0,\\
&\kappa_\mu^{(1)}(-1,0)[\Omega+\omega_s]\, g_\mu^{(0)} + \{\Omega + \omega_s - 2\sigma_B^2\kappa_\mu^{(2)}(-1,-1)[\Omega+\omega_s]\}\, g_\mu^{(-)} +\\
&2\sigma_B^2\kappa_\mu^{(2)}(-1,1)[\Omega+\omega_s] g_\mu^{(+)} = 0.
\end{aligned} \tag{140}
$$

Eqs. (140) give the system of linear equations $M(\Omega)V = 0$ where the vector $V = \{g_\mu^{(0)}, g_\mu^{(+)}, g_\mu^{(-)}\}$. The system has a nontrivial solution at frequencies Ω given by the zeros of the determinant of the matrix $M(\Omega)$.

Let us apply these results to a single bunch putting $n_b = 1$, $\mu = 0$. Neglecting terms proportional to $\kappa_\mu^{(2)}$, using identities

$$
\begin{aligned}
&\kappa_0^{(1)}(1,0)[\Omega-\omega_s] = \kappa_0^{(1)}(-1,0)[\Omega+\omega_s] = \kappa_0^{(1)}(0,0)[\Omega],\\
&\kappa_0^{(1)}(0,-1)[\Omega] + \kappa_0^{(1)}(0,1)[\Omega] = 2\kappa_0^{(1)}(0,0)[\Omega]
\end{aligned} \tag{141}
$$

and notation k_0,

$$
\kappa_0^{(1)}(0,-1)[\Omega] - \kappa_0^{(1)}(0,1)[\Omega] = -(\frac{\omega_s}{c_0})k_0, \tag{142}
$$

we can determine the coherent shift Ω from the equation $|M| = 0$ or

$$
(k_0-\Omega)(\omega_s^2-\Omega^2) - 2\frac{(\omega_s\sigma)^2}{c_0}k_0\kappa_0^{(1)}(0,0)[\Omega] - 4\sigma^2\Omega(\kappa_0^{(1)}(0,0)[\Omega])^2 = 0. \tag{143}
$$

The explicit form of the coefficients here is

$$
\begin{aligned}
&k_0 = -i\frac{\lambda_y\omega_0}{2\pi}N_B\sum_{p=-\infty}^{\infty} Z_y(p\omega_0+\omega_y^0+\Omega),\\
&\kappa_0^{(1)}(0,0)[\Omega] = -i\frac{\lambda_y\omega_0}{2\pi}N_B\sum_{p=-\infty}^{\infty}(\frac{p\omega_0+\omega_\xi}{2c_0})Z_y(p\omega_0+\omega_y^0+\Omega).
\end{aligned} \tag{144}
$$

The coefficients are proportional to the components of the matrix $M_{h,l}(\lambda)$ in the Satoh-Chin formalism:

$$
k_0 = iK\omega_s M_{0,0}(\lambda), \quad \kappa_0^{(1)}(0,0)[\Omega] = iK\frac{\omega_s}{\sigma_l\sqrt{2}}M_{1,0}(\lambda), \tag{145}
$$

where $\lambda = -\Omega/\omega_s$.

Therefore, Eq. (141) can be written in the form

$$(1-\lambda^2)(iKM_{0,0}+\lambda)-2\lambda K^2M_{1,0}^2+(\frac{\sqrt{2}\sigma_l\omega_s}{c_0})K^2M_{0,0}M_{1,0}=0. \quad (146)$$

The ratio of the last term to the second one is of the order of ω_s/ω_{HOM} times where ω_{HOM} is the frequency where impedance $Z_\perp(\omega)$ start to roll off. Neglecting this term, we get the same equation that is given by the Satoh-Chin formalism with the rank of the truncated matrix equal two, see Fig. where we compare the Satoh-Chin formalism with the CBCM formalism applied for one bunch. The single narrow-band impedance is taken as an example with the shunt impedance $R_s = 0.68$ $MOhm/m$, $Q = 1$ and $\omega_{HOM}/2\pi = 1.30$ GHz. Other parameters[5] are: $\beta_\perp = 160$ m, $\omega 0/2\pi = 136$ KHz, $\sigma_l = 2.0$ cm, $E = 14.5$ GeV, $\alpha = 1.3\,10^{-3}$, $Q_s = 0.044$. The Satoh-Chin matrix is truncated to the rank two and the formalism gives the growth rate for two modes (shown in red). The CBCM formalism gives the growth rate only for the lowest mode (shown in blue). However, the thresholds of the coupled-mode instability in both cases agree very well.

4.6 CBCM with dipole motion

So far considering CBCM instability we neglected in Eqs. (121)-(123) terms proportional to the dipole momentum. Now we want to take them into account. For simplicity, we study here the uniform bunch pattern. Assuming that the longitudinal motion is dominated by a single unstable mode μ_0 with the amplitude d_0, we define

$$d_M = \sqrt{n_b}d_0X_M^\mu, \quad X_M^\mu = \frac{1}{\sqrt{n_b}}e^{2\pi i\mu M/n_b}. \quad (147)$$

Note that with this definition the amplitude of oscillations $z \simeq \sigma_l$ corresponds to $d_0 \simeq \sigma_l$.
For the uniform fill Eqs. (121)-(123) take the form

$$\Omega g^{(0)}_\mu(\Omega) = \kappa^{(0)}_\mu(0,0)[\Omega]\{g^{(0)}_\mu(\Omega) + d_0 g^{(+)}_{\mu-\mu_0}(\Omega) + d^*_0 g^{(-)}_{\mu+\mu_0}(\Omega)\}$$
$$-\{\kappa^{(1)}_\mu(0,-1)[\Omega]\,\{2\sigma^2_B g^{(-)}_\mu(\Omega+\omega_s) + d_0 g^{(0)}_{\mu-\mu_0}(\Omega+\omega_s)]\}$$
$$+\kappa^{(1)}_\mu(0,1)[\Omega]\,\{2\sigma^2_B g^{(+)}_\mu(\Omega-\omega_s) + d^*_0 g^{(0)}_{\mu+\mu_0}(\Omega-\omega_s)\}. \quad (148)$$

$$\Omega g^{(+)}_\mu(\Omega) = \kappa^{(1)}_\mu(1,0)[\Omega]\,\{g^{(0)}_\mu(\Omega+\omega_s) + d_0 g^{(+)}_{\mu-\mu_0}(\Omega+\omega_s) + d^*_0 g^{(-)}_{\mu+\mu_0}(\Omega+\omega_s)\}$$
$$+\kappa^{(2)}_\mu(1,1)[\Omega]\,\{d^*_0 g^{(0)}_{\mu+\mu_0}(\Omega) + 2\sigma^2_B g^{(+)}_\mu(\Omega)\}$$
$$-\kappa^{(2)}_\mu(1,-1)[\Omega]\{d_0 g^{(0)}_{\mu-\mu_0}(\Omega+2\omega_s) + 2\sigma^2_B g^{(-)}_\mu(\Omega+2\omega_s)\}. \quad (149)$$

$$\Omega g^{(-)}_\mu(\Omega) = -\kappa^{(1)}_\mu(-1,0)[\Omega]\,\{g^{(0)}_\mu(\Omega-\omega_s) + d_0 g^{(+)}_{\mu-\mu_0}(\Omega-\omega_s) + d^*_0 g^{(-)}_{\mu+\mu_0}(\Omega-\omega_s)\}$$
$$+\kappa^{(2)}_\mu(-1,-1)[\Omega]\,\{d_0 g^{(0)}_{\mu-\mu_0}(\Omega) + 2\sigma^2_B g^{(-)}_\mu(\Omega)\}$$
$$-\kappa^{(2)}_\mu(-1,1)[\Omega]\{d^*_0 g^{(0)}_{\mu+\mu_0}(\Omega-2\omega_s) + 2\sigma^2_B g^{(+)}_\mu(\Omega-2\omega_s)\}. \quad (150)$$

In the case $d_0 = 0$ Eqs. (121)-(123) are identical with Eq. (140) where only $g^{(0)}_\mu(\Omega)$ and $g^{(\pm)}_\mu(\Omega \mp \omega_s)$ are not equal to zero. If $d_0 \neq 0$, these terms induce components $g^{(\pm)}_\mu(\Omega)$,

$$\{\Omega - 2\sigma^2_B\kappa^{(2)}_\mu(1,1)[\Omega]\}\,g^{(+)}_\mu(\Omega) = d^*_0\,\{\kappa^{(2)}_\mu(1,1)[\Omega]g^{(0)}_{\mu+\mu_0}(\Omega) + \kappa^{(1)}_\mu(1,0)[\Omega]g^{(-)}_{\mu+\mu_0}(\Omega+\omega_s)\}, \quad (151)$$
$$\{\Omega - 2\sigma^2_B\kappa^{(2)}_\mu(-1,-1)[\Omega]\}\,g^{(-)}_\mu(\Omega) = d_0\,\{\kappa^{(2)}_\mu(-1,-1)[\Omega]g^{(0)}_{\mu-\mu_0}(\Omega) - \kappa^{(1)}_\mu(-1,0)[\Omega]g^{(+)}_{\mu-\mu_0}(\Omega-\omega_s)\}.$$

Taking into account these terms transforms Eq. (140) for the components ($g^{(0)} = g^{(0)}_\mu(\Omega)$, $g^{\pm} = g^{(\pm)}_\mu(\Omega \mp \omega_s)$) to

$$\{\Omega - \kappa^{(0)}_\mu(0,0)[\Omega]\}\,g^{(0)} + 2\sigma^2_B\{\kappa^{(1)}_\mu(0,-1)[\Omega]\,g^{(-)} - \kappa^{(1)}_\mu(0,1)[\Omega]\,g^{(+)}\,\} =$$
$$|d_0|^2\kappa^{(0)}_\mu(0,0)[\Omega]\,\{\,\frac{\kappa^{(2)}_{\mu-\mu_0}(1,1)[\Omega]g^{(0)} + \kappa^{(1)}_{\mu-\mu_0}(1,0)[\Omega]g^{(-)}}{\Omega - 2\sigma^2_B\kappa^{(2)}_{\mu-\mu_0}(1,1)[\Omega]}$$
$$+\frac{\kappa^{(2)}_{\mu+\mu_0}(-1,-1)[\Omega]g^{(0)} - \kappa^{(1)}_{\mu+\mu_0}(-1,0)[\Omega]g^{(+)}}{\Omega - 2\sigma^2_B\kappa^{(2)}_{\mu+\mu_0}(-1,-1)[\Omega]}\,\}. \quad (152)$$

$$\{\Omega - \omega_s - 2\sigma^2_B\kappa^{(2)}_\mu(1,1)[\Omega-\omega_s]\}\,g^{(+)} =$$

$$\kappa_\mu^{(1)}(1,0)[\Omega-\omega_s]\,g^{(0)} - 2\sigma_B^2\kappa_\mu^{(2)}(1,-1)[\Omega-\omega_s]g(-)$$
$$+|d_0|^2\kappa_\mu^{(1)}(1,0)[\Omega-\omega_s]\,\{\,\frac{\kappa^{(2)}_{\mu-\mu_0}(1,1)[\Omega]\,g^{(0)}+\kappa^{(1)}_{\mu-\mu_0}(1,0)[\Omega]\,g^{(-)}}{\Omega-2\sigma_B^2\kappa^{(2)}_{\mu-\mu_0}(1,1)[\Omega]}$$
$$+\frac{\kappa^{(2)}_{\mu+\mu_0}(-1,-1)[\Omega]\,g^{(0)}-\kappa^{(1)}_{\mu+\mu_0}(-1,0)[\Omega]\,g^{(+)}}{\Omega-2\sigma_B^2\kappa^{(2)}_{\mu+\mu_0}(-1,-1)[\Omega]}\,\}. \qquad (153)$$

$$\{\Omega+\omega_s-2\sigma_B^2\kappa_\mu^{(2)}(-1,-1)[\Omega+\omega_s]\}\,g^{(-)} =$$
$$-\kappa_\mu^{(1)}(-1,0)[\Omega+\omega_s]\,g^{(0)} - 2\sigma_B^2\kappa_\mu^{(2)}(-1,1)[\Omega+\omega_s]g(+)$$
$$-|d_0|^2\kappa_\mu^{(1)}(-1,0)[\Omega+\omega_s]\,\{\,\frac{\kappa^{(2)}_{\mu-\mu_0}(1,1)[\Omega]\,g^{(0)}+\kappa^{(1)}_{\mu-\mu_0}(1,0)[\Omega]\,g^{(-)}}{\Omega-2\sigma_B^2\kappa^{(2)}_{\mu-\mu_0}(1,1)[\Omega]}$$
$$+\frac{\kappa^{(2)}_{\mu+\mu_0}(-1,-1)[\Omega]\,g^{(0)}-\kappa^{(1)}_{\mu+\mu_0}(-1,0)[\Omega]\,g^{(+)}}{\Omega-2\sigma_B^2\kappa^{(2)}_{\mu+\mu_0}(-1,-1)[\Omega]}\,\}. \qquad (154)$$

5 Instability of the closed orbit

Recently [7] it was noticed that the resistive wall impedance Z_{RW} may lead to the closed orbit instability. At the low frequencies, where the skin depth δ is larger than the beam pipe wall thickness d, $\delta^2(\omega) >= bd$, the resistive wall transverse impedance per unit length of a round beam pipe with the radius b is

$$Z_{RW}(\omega) = -i\frac{Z_0}{\pi b^2}\,\frac{g}{1-i\omega/\omega_c}, \qquad (155)$$

where L is the length of the beam pipe,

$$g=1/2, \quad \omega_c=\frac{\omega\delta^2(\omega)}{bd}, \quad (\mu=0),$$
$$g=1, \quad \omega_c=\frac{\omega\delta^2(\omega)}{2bd}, \quad (\mu>>1). \qquad (156)$$

The impedance Eq. (156) is written in two cases: for the vacuum ($\mu=0$) and a magnetic material ($\mu >> 1$) outside of the beam pipe.
For a uniform distribution of bunches in the ring, the coherent frequency Ω is given by the standard formula. Let us consider a single bunch. Then

$$\Omega = -i\lambda\sum_p Z_y(p\omega_0+\omega_\beta+\Omega), \quad \lambda=\frac{I_{beam}c^2}{4(E/e)\omega_\beta}. \qquad (157)$$

As it will be clear later, only one term gives the main contribution providing the CB coherent frequency shift is small compared with $\tilde{\nu}\omega_0$ where $\tilde{\nu}$ is the fractional part of the tune. Neglecting all others terms, one get equation for Ω,

$$\Omega[1 - i\frac{\Omega + n\omega_0 + \omega_\beta}{\omega_c}] + \lambda\frac{Z_0 g}{\pi b^2} = 0. \tag{158}$$

The beam is unstable if $Im[\Omega] > 0$. The threshold of instability is

$$I_{th} = \frac{2\pi\tilde{n}u_y\nu_y}{gZ_0}(\frac{b}{R})^2\,(E/e). \tag{159}$$

Here $\tilde{\nu}$ is fractional part of the betatron tune, $Z_0 = 120\pi$ Ohm. The most dangerous mode is for n equal to the integer part of the tune.

6 Summary

6.1 Steady-state fundamental rf

$$Q_L = \frac{Q_0}{1+\beta}, \tag{160}$$

$$\cos[\phi_s] = U_{turn}/eV_{tot} \tag{161}$$

$$\tan(\psi) = Q_L(\frac{\omega_g}{\omega_c} - \frac{\omega_c}{\omega_g}), \tag{162}$$

$$Z_c(\omega_g) = R_L\,cos\psi\,e^{i\psi}. \tag{163}$$

$$Y = \frac{2R_0 I_B^{dc}}{|\hat{V}_c|}. \tag{164}$$

$$\frac{1+\beta}{Y} = \frac{\tan\phi_s - \tan\phi_c}{\tan\psi + \tan\phi_c}\cos(\phi_s). \tag{165}$$

$$P_i = \frac{R_0}{2\beta}(I_B^{dc})^2[\frac{\sin(\psi+\phi_s)}{\sin(\psi+\phi_c)}]^2. \tag{166}$$

6.1.1 The Robinson criteria of stability

$$\sin\phi_s > 0; \quad \frac{Y\sin(2\psi)}{2(1+\beta)} < \sin\phi_s. \tag{167}$$

6.1.2 Optimum conditions

$$\begin{aligned} \tan\psi &= \frac{\beta-1}{\beta+1}\tan\phi_s, \\ \beta &= 1 + Y\cos\phi_s. \end{aligned} \tag{168}$$

$$P_i = \frac{2(I_B^{dc})^2 R_0}{Y^2}(1+Y\cos\phi_s) = \frac{|\hat{V}_c|^2}{2R_0}(1+Y\cos\phi_s).). \tag{169}$$

6.2 Potential well distortion

$$\rho(x,p) = \frac{1}{|N|}e^{-(p^2/2+U_0(x)+\lambda_0\int dx'\rho(x')S[\sigma_B(x'-x)]}. \tag{170}$$

$|N|$ is normalization constant defined by $\int dpdx\rho = 1$,

$$\begin{aligned} U_0(x) &= \frac{x^2}{2} - \frac{x^3}{6}\left(\frac{\omega_g\sigma_B}{c_0}\right)\cot\phi_s - \frac{x^4}{24}\left(\frac{\omega_g\sigma_B}{c_0}\right)^2, \\ \lambda_0 &= \frac{N_B r_e}{2\pi R\gamma\alpha\delta_0^2}, \\ S(z) &= \int_0^z dz' W(z'), \end{aligned} \tag{171}$$

6.3 Multibunch longitudinal stability

Notations:

$$\begin{aligned} \lambda_0 &= \frac{\alpha c_0 < N_B > r_e\omega_0}{2\pi\gamma}, \\ [\omega_s^0(N)]^2 &= \frac{\alpha\omega_g c_0 e|\hat{V}_c|n_c}{2\pi RE}\sin\phi_s(N), \end{aligned} \tag{172}$$

$$en_c V_c\cos\phi_s(N) = U + e^2\sum_M N_b(M)\,W\left(\frac{s_N - s_M}{c_0}\right). \tag{173}$$

6.3.1 The synchrotron tune including wake-field effect

$$\omega_s(N) = \omega_s^0(N) - \frac{i\lambda_0\omega_0^2}{4\pi\omega_s^0 c_0}\sum_M (\frac{N_B(M)}{< N_B >})\, kZ(k\omega_0)e^{-i(k\omega_0/c_0)(s_N - s_M)}e^{-(k\omega_0 a/c_0)^2}. \qquad (174)$$

For the uniform fill,

$$\omega_s = \omega_s^0 - \frac{in_b\lambda_0\omega_0^2}{4\pi\omega_s^0 c_0}\sum_p (n_b p)Z(n_b p\omega_0)e^{-(n_b p\omega_0 a/c_0)^2}. \qquad (175)$$

6.3.2 The coherent shift

is given by the eigen values of the system

$$\Omega a_N(\Omega) = \frac{i\lambda_0\omega_0^2}{4\pi\omega_s(N)c_0}\sum_{M=1}^{n_b} K_\Omega(N,M)a_M(\Omega), \qquad (176)$$

where

$$K_\Omega(N,M) = \frac{N_B(M)}{< N_B >}\sum_k kZ(k\omega_0)e^{(i/c_0)(\omega_s(N)+\Omega-k\omega_0)(s_N-s_M)}e^{-(k\omega_0 a/c_0)^2}. \qquad (177)$$

For the uniform fill,

$$\Omega_\mu = \frac{i\lambda_0\omega_0^2\kappa_\mu}{4\pi\omega_s(N)c_0} = i\frac{\alpha I_{beam}\omega_0^2}{4\pi(E/e)\omega_s(N)}\frac{\kappa_\mu}{n_b}. \qquad (178)$$

where

$$\kappa_\mu = n_b\sum_p g_p Z(g_p\omega_0)e^{-g_p^2(a/R)^2}, \quad g_p = n_b p + \mu + \nu_s + \Omega_\mu/\omega_0, \quad \mu = 1, 2.., n_b. \qquad (179)$$

6.3.3 Quadrupole ($m = 2$) longitudinal coherent shifts

for the uniform fill are:

$$\Omega_{\mu,m} = i\frac{\alpha I_{beam}\omega_0^2}{2\pi(E/e)\omega_s}\frac{(\sigma_B/R)^{m-1}}{2^m(m-1)!}$$
$$\sum_{p=-\infty}^{\infty}(pn_b+\mu)^{2m}e^{-(pn_B+\mu)^2(\sigma_B/R)^2}\,\frac{Z[(n_b p + \mu + m\mu_s)\omega_0]}{n_b p + \mu + m\nu_s}. \qquad (180)$$

6.4 Transverse Instabilities

Notations:

$$\omega_\xi = \frac{\xi}{\alpha}\omega_y^0,$$
$$\xi = (1/\nu_y^0)(d\nu_y^0/d\delta) \tag{181}$$

$$\lambda_y = \frac{r_e c_0^2}{4\pi R\gamma\omega_y^0},$$
$$\tau_{N,M} = \frac{s_M - s_N}{c_0}, \tag{182}$$

6.4.1 Head-Tail

$$\mu = 1 - \frac{1}{2}G^2 \pm \sqrt{[1 - \frac{1}{2}G^2]^2 - 1}, \tag{183}$$

where

$$G = \Lambda \int_0^\pi d\psi W(2a\sin\psi)e^{-i\kappa\sin\psi}. \tag{184}$$

The growth rate $A(t) = e^{\Gamma t}$, where

$$\Gamma = Re[\frac{1}{T_s}\log\mu]. \tag{185}$$

6.4.2 Transverse dipole coupled-bunch instability:

The coherent shift is given by the eigen values of the system

$$\Omega A_N(\Omega) = \sum_{M=1}^{n_b} K_0(N,M)A_M(\Omega), \tag{186}$$

where

$$K_0(N,M) = i\frac{\lambda_y\omega_0}{2\pi}N_B(M)\sum_{k=-\infty}^{\infty} Z_y(k\omega_0 - \omega_y^0 + \Omega)$$
$$e^{\frac{i}{c_0}k\omega_0(s_M - s_N)}e^{-(\sigma_B/c_0)^2(k\omega_0+\omega_\xi)^2}\, J_0[\frac{\sigma_{B,N}}{c_0}(k\omega_0+\omega_\xi)]\, J_0[\frac{\sigma_{B,M}}{c_0}(k\omega_0+\omega_\xi)]. \tag{187}$$

For the uniform fill,

$$\Omega_\mu = i\lambda_y N_b n_b \frac{\omega_0}{2\pi} \sum_p Z_y(pn_b\omega_0 + \mu\omega_0 - \omega_y^0 + \Omega_\mu)\, e^{-(\sigma_B/c_0)^2(pn_b\omega_0+\mu\omega_0+\omega_\xi)^2}$$

$$J_0^2[\frac{\sigma_B}{c_0}(pn_b\omega_0 + \mu\omega_0 + \omega_\xi)], \quad \mu = 0, 1, ..., n_b, \tag{188}$$

6.4.3 Quadrupole ($m = 1$) coupled-bunch coherent shift

Define

$$G^{(m)}_{l,l'}(N,M) = i\frac{\lambda_y\omega_0}{2\pi} N_B(M) \sum_{k=-\infty}^{\infty} (\frac{k\omega_0 + \omega_\xi + l'\omega_s}{2c_0})^m Z_y(k\omega_0 - \omega_y^0 + \Omega + l\omega_s) e^{\frac{i}{c_0}k\omega_0(s_M - s_N)}. \tag{189}$$

The coherent shift is given by the eigen-values Ω of the system:

$$\Omega A_N^{(+)}(\Omega) - 2\sum_M^{n_B} \sigma^2_{M,B} G^{(2)}_{1,1}(N,M) A_M^{(+)}(\Omega) = 0,$$

$$\Omega A_N^{(-)}(\Omega) - 2\sum_M^{n_B} \sigma^2_{M,B} G^{(2)}_{-1,-1}(N,M) A_M^{(-)}(\Omega) = 0. \tag{190}$$

For the uniform fill,

$$\Omega_\mu = 2\sigma_B^2 \kappa_\mu^{(2)}(\pm1, \pm1)[\Omega_\mu]. \tag{191}$$

$$\kappa_\mu^{(m)}(l,l')[\Omega] = i\frac{\lambda_y\omega_0}{2\pi} N_B n_b \sum_{p=-\infty}^{\infty} (\frac{(pn_b + \mu)\omega_0 + \omega_\xi + l'\omega_s}{2c_0})^m Z_y((pn_b+\mu)\omega_0 - \omega_y^0 + \Omega + l\omega_s). \tag{192}$$

6.4.4 Coupled-bunch Mode coupling

For the uniform fill, the coherent shift is given by the eigen-values Ω of the system:

$$\{\Omega - \kappa_\mu^{(0)}(0,0)[\Omega]\}\, g_\mu^{(0)} = 2\sigma_B^2\{-\kappa_\mu^{(1)}(0,-1)[\Omega]\, g_\mu^{(-)} + \kappa_\mu^{(1)}(0,1)[\Omega]\, g_\mu^{(+)}\},$$

$$\{\Omega - \omega_s - 2\sigma_B^2\kappa_\mu^{(2)}(1,1)[\Omega - \omega_s]\}\, g_\mu^{(+)} = \kappa_\mu^{(1)}(1,0)[\Omega - \omega_s]\, g_\mu^{(0)}$$

$$-2\sigma_B^2\kappa_\mu^{(2)}(1,-1)[\Omega - \omega_s]\}\, g_\mu^{(-)},$$

$$\{\Omega + \omega_s - 2\sigma_B^2\kappa_\mu^{(2)}(-1,-1)[\Omega + \omega_s]\}\, g_\mu^{(-)} = -\kappa_\mu^{(1)}(-1,0)[\Omega + \omega_s]\, g_\mu^{(0)}$$

$$-2\sigma_B^2\kappa_\mu^{(2)}(-1,1)[\Omega + \omega_s]\}\, g_\mu^{(+)}. \tag{193}$$

6.5 Instability of the closed orbit

The threshold beam current

$$I_{th} = \frac{2\pi(E/e)\nu_y(\nu_y - n)}{gZ_0}(\frac{b}{R})^2, \tag{194}$$

where n = integer part of the betatron tune ν_y, $g = 1$ or $g = 1/2$, see Eq. (126).

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8 APPENDIX 1. Definitions of impedances and wakes

We use impedance $Z(\omega)$ analytic in the upper-half plane of ω and the wake $W(z)$ is zero at $z < 0$. Positive longitudinal wake means energy loss, that is the change of energy of the trailing particle following at the distance $z > 0$ behind the leading particle, changes by $\Delta E = -N_b e^2 W_l(z)$,

$$W_l(z) = \int \frac{d\omega}{2\pi} Z_l(\omega)\, e^{-i\omega z/c_0}. \tag{195}$$

Similarly, the transverse wake W_t defines $c_0 \Delta p_t = +N_b e^2 W_t(z)$. It is related to the transverse impedance as

$$W_t(z) = i \int \frac{d\omega}{2\pi} Z_t(\omega)\, e^{-i\omega z/c_0}. \tag{196}$$

The transverse impedance Z_t is related to Z_l by Panofsky-Wentzel theorem,

$$Z_r(\omega) = \frac{1}{\omega r_l/c_0} \frac{\partial Z_l}{\partial r_t}, \tag{197}$$

where r_l and r_t are offsets of the leading and trailing particles.
With these definitions,

$$Z_l(-\omega)* = Z_l(\omega^*), \quad Z_t(-\omega)* = -Z_t(\omega^*). \tag{198}$$

The narrow-band longitudinal impedance can be written as

$$\begin{aligned} Z_l(\omega) &= \frac{R_H}{1 - iQ_H(\omega/\omega_H - \omega_h/\omega)} \\ &\simeq i\frac{\omega_H}{2} \frac{R_H}{Q_H} [\frac{1}{\omega - \omega_H + i\omega_h/2Q_H} + \frac{1}{\omega + \omega_H + i\omega_h/2Q_H}]. \end{aligned} \tag{199}$$

Such impedance is inductive at small $\omega << \omega_H$, $Z_l(\omega) \to -iL\omega/c^2$, $L = R_H/(\omega_H Q_H)$.
The standard form of the resistive wall impedance is

$$\frac{Z(n)}{n} = Z_0(\frac{1-i}{2}) \frac{\delta(\omega)}{b}, \tag{200}$$

where $n = \omega/\omega_0$, and $\delta(\omega)$ is the skin depth.
The narrow-band transverse impedance is

$$\begin{aligned} Z_t(\omega) &= \frac{\omega_H}{\omega} \frac{R_H}{1 - iQ_H(\omega/\omega_H - \omega_h/\omega)} \\ &\simeq i\frac{\omega_H}{2} \frac{R_H}{Q_H} [\frac{1}{\omega - \omega_H + i\omega_h/2Q_H} - \frac{1}{\omega + \omega_H + i\omega_h/2Q_H}]. \end{aligned} \tag{201}$$

Dimensions of the wakes W_l and W_t (wakes per turn) are $V/pC \simeq 1/cm$ and $V/pC/m$, respectively. Impedances Z_l and Z_t (per turn) have dimensions $Ohm \propto 1/c_0$ and Ohm/m, respectively.

9 APPENDIX 2. The threshold of the microwave instability

We also can use the following simple formalism. Consider the Hamiltonian $H(x,p,s)$ describing synchrotron motion in the linear rf potential,

$$H(x,p,s) = \frac{p^2+x^2}{2} + \lambda \int dx'dp' \rho(x',p',s) S(x'-x), \quad S(x) = \int_0^{\sigma x} dz W_l(z), \tag{202}$$

where $x = z/\sigma_0$, $p = -\delta/\delta_0$ are dimensionless position of a particle ($z > 0$ is in the head of a bunch) and energy off-set, $s = \omega_{0,s}t$ is time in synchrotron periods, and the distribution function is normalized $\rho(x,p,s)dpdx = 1$. In this units, $S(x)$ is dimensionless, $S(x) = 0$ at $x < 0$. The zero-current synchrotron frequency is equal to one and

$$\lambda = \frac{N_b r_0}{2\pi R\gamma\alpha\delta_0^2}. \tag{203}$$

Below the threshold of instability, the distribution function and the Hamiltonian are time independent. It is always possible in this case to go to new angle-action variables I,ψ to make the Hamiltonian $H(I,\psi) = H_0(I)$ and $\rho(I,\psi) = \rho_0(I)$ independent of phases ψ. Above the threshold, there are unstable azimuthal harmonics and

$$\begin{aligned} H(I,\psi,s) &= H_0(I,s) + \lambda \sum_{m\neq 0} V_m(I,s)\, e^{im\psi}, \\ \rho(I,\psi,s) &= \rho_0(I,s) + \lambda \sum_{m\neq 0} \rho_m(I,s)\, e^{im\psi}. \end{aligned} \tag{204}$$

Actually, there is correction to the zero harmonics ρ_0 but, as it is shown below, the correction is small.

To transform Eq. (202) to the form Eq. (204) let us expand $S(x'-x)$ in azimuthal harmonics

$$S(x'-x) = \sum_m S_{m,m'}(I,I')\, e^{im\psi - im'\psi'}. \tag{205}$$

$S_{m,m'}$ is given in terms of the longitudinal impedance $Z(\omega)$,

$$\begin{aligned} S_{m,m'}(I,I') &= c_0 \int \frac{d\omega}{2\pi i} \frac{Z(\omega)}{\omega} \left[\delta_{m,0}\delta_{m',0} - C_m(\omega,I)\, C^*_{m'}(\omega,I')\right], \\ C_m(\omega,I) &= \int \frac{d\psi}{2\pi} e^{-im\psi}\, e^{i\frac{\omega\sigma}{c_0}x(I,\psi)}. \end{aligned} \tag{206}$$

Note, $S^*_{m,m'}(I,I') = S_{-m,-m'}(I,I')$. Therefore, $S_{0,0}(I,I')$ is real.
The term proportional to $\delta_{m,0}\delta_{m',0}$ is a constant independent of I,and I'. The Hamiltonian Eq. (202) can be written as

$$H(I,\psi,s) = \mathcal{H}_0 + \lambda \sum_{m'\neq 0} \int dI'd\psi' S_{0,m'}(I,I')\rho_{m'}(I',s)$$
$$+\lambda \sum_{(m,m')\neq 0} e^{im\psi} \int dI'd\psi' S_{m,m'}(I,I')\rho_{m'}(I',s), \tag{207}$$
$$\mathcal{H}_0 = \frac{p^2+x^2}{2} + \lambda \int dI'd\psi' S_{0,0}(I,I')\rho_0(I',s) + \lambda \sum_{m\neq 0} e^{im\psi} \int dI'd\psi' S_{m,0}(I,I')\rho_0(I',s).$$

Comparing Eq. (204) and Eq. (207), we define

$$V_m(I,s) = \sum_{m'\neq 0} \int dI'd\psi' S_{m,m'}(I,I')\rho_{m'}(I',s), \quad m \neq 0. \tag{208}$$

In the case where non-zero harmonics $\rho_m = 0$, $m \neq 0$, the Hamiltonian has to be independent of I, $H_0 = H_0(I)$. That is enough to define the transform from x,p to I,ψ.
To find the transform, it is convenient, first, to introduce canonical variables J,ϕ defining $x = \sqrt{2J}\sin\phi$, $p = \sqrt{2J}\cos\phi$, $(p^2+x^2)/2 = J$. The transform from J,ϕ to I,ψ has to be chosen to cancel all azimuthal harmonics in H_0 of Eq. (207),

$$J = I - \lambda \sum_{m\neq 0} e^{im\psi} \int dI'd\psi' S_{m,0}(I,I')\rho_0(I',s). \tag{209}$$

Then,

$$H_0(I,s) = I + \lambda \int dI'd\psi' S_{0,0}(I,I')\rho_0(I',s)$$
$$+\lambda \sum_{m\neq 0} \int dI'd\psi' S_{0,m}(I,I')\rho_m(I',s). \tag{210}$$

The bunch stability can be, as usual, studied using the Vlasov equation for azimuthal harmonics

$$\frac{\partial \rho_m(I,s)}{\partial s} + im\omega(I,s)\,\rho_m(I,s) - i\lambda \sum_{m'\neq 0} [m' \frac{\partial \rho_{m-m'}(I,s)}{\partial I} V_{m'} - (m-m')\rho_{m-m'} \frac{\partial V_{m'}(I,s)}{\partial I}] = 0, \tag{211}$$

where $\omega(I,s) = \partial H_0(I,s)/\partial I$.
The linearized Vlasov equation for azimuthal harmonics retains the non-zero harmonics only linearly,

$$\frac{\partial \rho_m(I,s)}{\partial s} + im\omega(I,s)\,\rho_m(I,s) - i\lambda m\,\frac{\partial \rho_0(I,s)}{\partial I}\,V_m(I,s) = 0, \qquad (212)$$

where in the linear approximation over non-zero azimuthal harmonics

$$\omega(I,s) = 1 + \lambda\frac{\partial V_0(I,s)}{\partial I}, \quad V_0(I,s) = \int dI' d\psi' S_{0,0}(I,I')\rho_0(I',s). \qquad (213)$$

Harmonics $\rho_0(I,s)$ satisfies

$$\frac{\partial \rho_0(I,s)}{\partial s} - i\lambda\frac{\partial}{\partial I}\sum_{m\neq 0}[m\rho_{-m}V_m(I,s)]. \qquad (214)$$

Therefore, in the linear approximation over ρ_m, $m \neq 0$, $\rho_0(I)$ and $\omega(I)$ are independent of s .

Eq. (212) can be reduced to the linear eigenvalue problem by substituting $\rho_m(I,s) = A_m(I)e^{-i\mu s}$ with some amplitudes $A(I)$. On a finite discrete mesh of I, I', Eq. (212) gives the matrix equation $\sum_{I'} M(I,I')A(I') = \mu A(I)$ which can be solved numerically. Then, the growth rate is equal to the imaginary part of the coherent shift μ which is the eigenvalue of the matrix $M(I,I')$,

$$M(I,I') = m\omega(I)\delta(I-I') - 2\pi\lambda m\,\frac{\partial \rho_0(I)}{\partial I}\,S_{m,m'}(I,I'), \quad (m,m') \neq 0. \qquad (215)$$

The matrix $M(I,I')$ can be symmetrized choosing new amplitudes $a(I)$, $A(I) = \sqrt{\frac{\partial \rho_0(I)}{\partial I}}a(I)$ provided $\rho(I)$ is monotonic in I. Then, as it was noticed by K.Oide, the matrix becomes real and symmetric in the linear approximation in λ. The eigenvalues in this case are real and there is no instability. Therefore, V_m has to be defined taking into account terms of the order of λ. That can be done as follows.

To be canonical variables, the Poisson bracket of J, ϕ has to be equal to one, $\{J,\phi\}_{I,\psi} = 1$. That and Eq. (209) defines ϕ. Neglecting terms of the order of λ^2,

$$\phi = \psi - i\lambda\sum_{m\neq 0}\frac{e^{im\psi}}{m}\,\frac{\partial}{\partial I}\int dI' d\psi' S_{m,0}(I,I')\rho_0(I',s) + o(\lambda^2). \qquad (216)$$

With the same accuracy, $x = x^{(0)} + \Delta x$,

$$x^{(0)}(I,\psi) = \sqrt{2I}\sin\psi, \quad \Delta x(I,\psi) = \sum_m{}' \,\Delta x_m(I)\,e^{im\psi},$$

$$\Delta x_m(I) = \frac{i\lambda}{2\sqrt{2I}}\int dI' d\psi' \rho_0(I')\{S_{m-1,0}(I,I') - S_{m+1,0}(I,I')$$

$$-2I\frac{\partial}{\partial I}[\frac{S_{m-1,0}(I,I')}{m-1} + \frac{S_{m+1,0}(I,I')}{m+1}]\}. \qquad (217)$$

The prime in the summation indicates that terms $S_{0,0}$ has to be omitted.

We can now expand $S_{m,m'}$ in series over λ:

$$S_{m,m'}(I,I') = S^{(0)}_{m,m'} + \lambda S^{(1)}_{m,m'} + ..., \tag{218}$$

where

$$S^{(0)}_{m,m'}(I,I') = -c_0 \int \frac{d\omega}{2\pi i}\, \frac{Z(\omega)}{\omega}\, J_m(\frac{\omega\sigma}{c_0}\sqrt{2I})\, J_{m'}(\frac{\omega\sigma}{c_0}\sqrt{2I'}), \tag{219}$$

$$S^{(1)}_{m,m'}(I,I') = \sigma \sum_k \int \frac{d\omega}{2\pi}\, Z(\omega)\, \int \frac{d\psi d\psi'}{(2\pi)^2}\, e^{i(m'\psi'-m\psi)}$$
$$e^{i\frac{\omega\sigma}{c_0}[x^{(0)}(I,\psi)-x^{(0)}(I',\psi')]}\, [\Delta x_k(I')\, e^{ik\psi'} - \Delta x_k(I) e^{ik\psi}]. \tag{220}$$

To calculate $M_{m,m'}$ taking into account terms of the order of λ^2, it suffices to neglect terms of the order of λ^2 in $S_{m,m'}$. That can be done substituting $S^0_{m,m'}(I,I')$ instead of $S_{m,m'}(I,I')$ in Δx_m, Eq. (217). Using the properties of the Bessel functions

$$J_{k-1}(z) - J_{k+1}(z) - z\,\frac{d}{dz}[\frac{J_{k-1}(z)}{k-1} + \frac{J_{k+1}(z)}{k+1}] = \frac{2z}{k^2-1}\, J_k(z), \tag{221}$$

Eq. (217) can be simplified and takes the form

$$\Delta x_m(I) = -\frac{\lambda\sigma}{2}(\frac{2}{m^2-1}) \int dI' d\psi' \rho_0(I')$$
$$\int \frac{d\omega}{2\pi} Z(\omega) J_0(\frac{\omega\sigma}{c_0}\sqrt{2I'}) J_m(\frac{\omega\sigma}{c_0}\sqrt{2I}). \tag{222}$$

Approximating $\rho_0(I) = (1/2\pi)e^{-I}$ and carrying out the integration

$$\int dI' d\psi' \rho_0(I')\, J_0(\frac{\omega\sigma}{c_0}\sqrt{2I'}) = e^{-\frac{1}{2}((\frac{\omega\sigma}{c_0})^2}, \tag{223}$$

we get further

$$\Delta x_m(I) = -\frac{\lambda\sigma}{2}(\frac{2}{m^2-1})\, \int \frac{d\omega}{2\pi} Z(\omega)\, J_m(\frac{\omega\sigma}{c_0}\sqrt{2I})\, e^{-\frac{1}{2}(\frac{\omega\sigma}{c_0})^2}. \tag{224}$$

Eq. (220) takes the form

$$S^{(1)}_{m,m'}(I,I') = \frac{\lambda}{2}\sigma^2[\sum_k{}' \frac{1}{k-1} - \sum_k{}' \frac{1}{k+1}] \int \frac{d\omega}{2\pi}\, Z(\omega) \int \frac{d\omega'}{2\pi}\, Z(\omega')\, e^{-\frac{1}{2}(\frac{\omega'\sigma}{c_0})^2}$$
$$\{J_k(\frac{\omega'\sigma}{c_0}\sqrt{2I}) J_{m'}(\frac{\omega\sigma}{c_0}\sqrt{2I'}) J_{m-k}(\frac{\omega\sigma}{c_0}\sqrt{2I})$$
$$-J_k(\frac{\omega'\sigma}{c_0}\sqrt{2I'}) J_m(\frac{\omega\sigma}{c_0}\sqrt{2I}) J_{m'+k}(\frac{\omega\sigma}{c_0}\sqrt{2I'})\}. \quad (225)$$

Here, in the first and the second sums terms $k = 1$ and $k = -1$ are excluded, respectively. The frequency $\omega(I)$ is given by the derivative $\partial S_{00} \partial I$. Therefore, the term proportional to $\delta_{m0}\delta_{m'0}$ in $S_{0,0}$ can be omitted. After that, $\omega(I)$ can be defined using Eqs. (219) and (225).

Finally, $M(I,I')$ is defined taking into account terms of the order of λ^2 by Eq. (215) and Eqs. (218),(219), and (225).