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A study of relativistic fluids with applications to cosmology: A variational approach

THESIS PRESENTED IN FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN APPLIED MATHEMATICS

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Declaration

The work presented in this thesis is partly based on collaborations with Celestine Wafo Soh (Mathematics Department, Jackson State University, USA), Ebrahim Fredericks and Bob Osano (Gravitation and Cosmology Group, Mathematics and Applied Mathematics at University of Cape Town). Subsections, sections and/or paragraphs (including the last two appendices) in this thesis are partially based on the listed publications, article in preparation and article to be submitted soon for peer-review below:

1. T. Oreta, E. Fredericks, C. W. Soh and B. Osano, *Variational symmetries of tensor Lagrangians*, In preparation.

2. T. Oreta and B. Osano, *Post-inflationary evolution of inflation-generated, cosmological magnetic fields*, to be submitted soon to a journal.

3. B. Osano and T. Oreta, *A transient phase in cosmological evolution: A multi-fluid approximation for a quasi-thermodynamical equilibrium*, Gen. Rel. Grav. Volume **52**, (2020).

4. B. Osano and T. Oreta, *Multi-fluid theory and cosmology: A convective variational approach to interacting dark-sector*, Int. Jour. of Mod. Phys. D. Volume **28**, (2019).

Therefore, I hereby declare that either this thesis or a portion of it has not been submitted either in the same form or different form to any other university or institution for either a degree or any other qualification and that it represents my own work. This is both in concept and execution.

Timothy Oreta

Tuesday, 21st September 2021

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Cape Town, South Africa

Timothy Oreta

Tuesday, 21^{st} September 2021

Abstract

This thesis examines relativistic fluids. We have used the *variational* approach to develop tools for studying the dynamics of relativistic fluids to apply this to cosmological modelling. Studies like these go beyond the standard model in cosmology. Researchers believe that such extensions to the standard cosmological model are pivotal to resolving some of the long-standing cosmological problems. An example of such problems is the *origin*, *growth* (from quantum electromagnetic fluctuations to large-scale magnetic fields during inflation) and *evolution* of cosmological magnetic fields that exhibit as large-scale (cosmological) magnetic fields in late time. One other example is the *coincidence problem*. The standard approach in such studies is to use modelling in the form of the single-fluid formalism. As an alternative one can consider the single-fluid and multi-fluid formalisms that incorporate aspects of electrodynamics and thermodynamics, respectively in the context of the variational approach. This might help us make progress in trying to either resolve some of these problems or at least open up new ways of addressing them. In this regard, we have extended the well-known Müeller-Israel-Stewart (hereafter MIS) formalism to allow us to examine the effect on fluid flow in which the components of the multi-species fluids interact thermodynamically. We use the extension to the MIS theory in the context of interacting species to study the growth of dark matter and dark energy, and find that either interaction or *entrainment* involving dark energy and dark matter suggests a mutual relative modulation of the growth behaviour of the two densities. This may aid in resolving the coincidence problem. Our examination of inflation-generated, large-scale magnetic fields reveals a *super-adiabatically evolving mode* from the beginning of the radiation-dominated epoch to either much later during the epoch or probably extending far into the era of matter domination which may account for late time, large-scale magnetic fields.

Keywords: Cosmology; relativistic fluids; variational approach; single-fluid; multi-fluid; dark energy; dark-matter; thermodynamics; interaction; entrainment; inflation; magnetic

fields.

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Conventions and assumptions

The following conventions, abbreviations and assumptions are used throughout the thesis, unless otherwise specified. We adopt the units $c = \kappa = \hbar = 8\pi G = \frac{8\pi G}{c^4} = 1$. The Latin indices $a, b, c, ...$ run from 0 to 3, whereas the symbols $\mathcal{I}, \mathcal{J}, ...$ run from 1 to 3. The symbols ∇_a and ; represent the usual covariant derivative, whereas ∂_a and , represent partial derivatives, and $\tilde{\nabla}$ represents the spatial covariant derivative. Overdots represent differentiation with respect to cosmic time. We use the $(-+++)$ to represent the space-time signature unless otherwise specified. The terms multi-species fluid system, multiple species multi-fluid system or multi-fluid system are used interchangeably in the thesis as they all refer to the same system.

Chapter 1

Thesis introduction: Cosmology

1.1 General relativity

Cosmology can be defined as a study of the universe as a whole. This involves the investigation of its origin, evolution and final fate. Cosmology is probably as old as human civilisation. It was of philosophical interest until the 20^{th} century when reliable cosmological observations became available and thus it became an exact science. Einstein's theory of general relativity, which was published in the fall of 1915 [1, 2], led to the development of modern physical cosmology. General relativity can be formulated from the variation of the Einstein Hilbert action

$$S_{\mathcal{EH}} = \frac{1}{2}\int_{All\ space-time} d^4x\sqrt{-g}(R + 2\mathcal{L}_m), \tag{1.1}$$

where R represents the Ricci scalar, $\mathcal{L}_m$ represents the Lagrangian of the matter field and g represents the determinant of the space-time metric denoted by g_{ab}. One can show after varying the action (1.1) that the gravity side of the Einstein field equations takes the form

$$\mathcal{G}_{ab} = R_{ab} - \frac{1}{2}g_{ab}R. \tag{1.2}$$

R_{ab} represents the Ricci tensor and the complete Einstein equations are of the form

$$\mathcal{G}_{ab} = T_{ab}, \tag{1.3}$$

where the right-hand side is the matter side. T_{ab} represents the stress-momentum tensor that is sourced by matter in space-time [3]. In equations (1.1) and (1.3) we have set $\frac{8\pi G}{c^4} = 1$.

In his formulation, Einstein was able to show the connection between matter and gravity. He described gravity to be the effect of the bending of space-time around a massive object.

Einstein's formulation of gravity for curved space-time replaces Newtonian gravity for flat space-time. With this explanation of gravity, a test particle moving with four-vector velocity denoted by u^a, moves along the geometry of space with a geodesic path expressed as

$$u^a \nabla_a u^b \ = \ 0, \tag{1.4}$$

where ∇_a represents a covariant derivative. The four-vector velocity u^a can be expressed as

$$u^a \ = \ \frac{dx^a}{d\tau}, \tag{1.5}$$

where τ represents proper time along a geodesic path [4]. An explanation that the geometry of space-time should be curved due to matter on it, proved to be a novel perception of space and time. The explanation is that the space-time dynamical continuum evolves in the form of energy and momentum. This (general relativity) has been thoroughly investigated to the extent that it has become part of the conceptual foundations which has led to developments and advancements of (modern) physical cosmology.

In 1917 [5], Einstein provided a promising model solution to equation (1.3). To obtain this solution, Einstein assumed that the universe was neither expanding nor contracting [6]. This assumption was based on a theoretical simplification rather than on firm observational data. Einstein aimed to balance the self-attraction of matter in the known universe at the time by adding a term denoted by Λ_0 (which has become known as the cosmological constant) in the equations. This leads to equation (1.2) being expressed as

$$\mathcal{G}_{ab} + \Lambda_0 g_{ab} \ = \ T_{ab}, \tag{1.6}$$

and therefore keeping the universe static [7,8]. This formulation was modified by Friedmann in the year 1922 [9]. Friedmann did not try to balance the self-attraction of matter in the known universe at the time. Instead, he allowed a possibility of either a contracting or an expanding universe. This meant a non-static universe. Building on this idea, Lema$\hat{\imath}$tre's 1927 [10, 11] prediction of the redshift of receding galaxies meant an expanding universe. This was later confirmed by observations done by Hubble in 1929 [12]. The work that was done by Friedmann and Lema$\hat{\imath}$tre was further developed in the 1920s and the 1930s [13–17]. The development led to the formulation that describes the late universe. The formulation was developed by Friedmann [9], Lema$\hat{\imath}$tre [10], Robertson [13] and Walker [17] and it is called the Friedmann, Lema$\hat{\imath}$tre, Robertson and Walker cosmological model (denoted by $FLRW$ model). The $FLRW$ model starts with the assumption of homogeneity[1] and isotropy[2] of

[1]Defined in the next section.

[2]Defined in the next section.

space-time also referred to as the cosmological principle.

1.2 Cosmological principle

The evolutionary history of the universe is rooted in the $FLRW$ [13–18] cosmological model previously discussed. The model is very successful in explaining the evolving universe as revealed by observations and is the foundation of the standard model of cosmology[3]. The model is succesful in describing how the universe evolved from the beginning to current time.

Isotropy and homogeneity are important features of the large-scale structure of the universe [18]. Isotropy refers to uniformity in all orientations of the large-scale structure of the universe while homogeneity implies that the large-scale structure of the universe appears the same at every point. Isotropy ensures that observations made from a single point of view can be representative of the whole universe and can therefore be used to test models on all cosmological scales. An isotropic and homogeneous view of the universe was only an assumption for most of the 20^{th} century until later confirmed via cosmological observations. It is known as the cosmological principle. It dates to one of the earlier works of Einstein [9–11, 13–17]. His use of the cosmological principle was not based on observations [5,6]. This assumption allows one to simplify the mathematical analysis that is used to develop cosmological models.

In the 20^{th} century, observational data from the cosmic microwave background radiation (hereafter $CMBR$)[4] confirmed the large-scale homogeneity [as demonstrated in figure 1.1 which shows the 2 degree field ($2dF$) galaxy redshift survey [19] and the 3 dimensional (3-D) rendering of the $2dF$ survey is given in figure 1.2] and isotropy[5] of the observable universe. Uniformity of the temperature of the $CMBR$ still remains the best evidence for the isotropy of the observed universe (as seen in figure 1.3). The description based on the cosmological principle represents the simplest possible interpretation of the universe's large-scale structure [18]. The expansion anisotropy of the universe has also been revealed as a temperature anisotropy in the $CMBR$. Similarly, density inhomogeneities[6] of the universe also led to anisotropies in temperature implying that the $CMBR$ should be a very sensitive probe, that

[3]This model also called the ΛCDM model assumes that the universe was created in the big-bang from pure energy, and is now composed of 5% baryonic matter (denoted by $\mathcal{Y}''$ in this thesis), 27% dark matter (denoted by $\mathcal{Y}'$ in this thesis), and 68% dark energy (denoted by $\mathcal{Z}$ in this thesis).

[4]$CMBR$ in standard cosmology is the remnant of electromagnetic radiation from the early stage of the universe.

[5]A picture of the $CMBR$ observation of the Wilkinson Microwave Anisotropy Probe (denoted by $WMAP$) is presented in figure 1.3. The picture shows to a large extent that the universe is isotropic.

[6]This means that the mass of the universe is not homogeneously distributed [20].

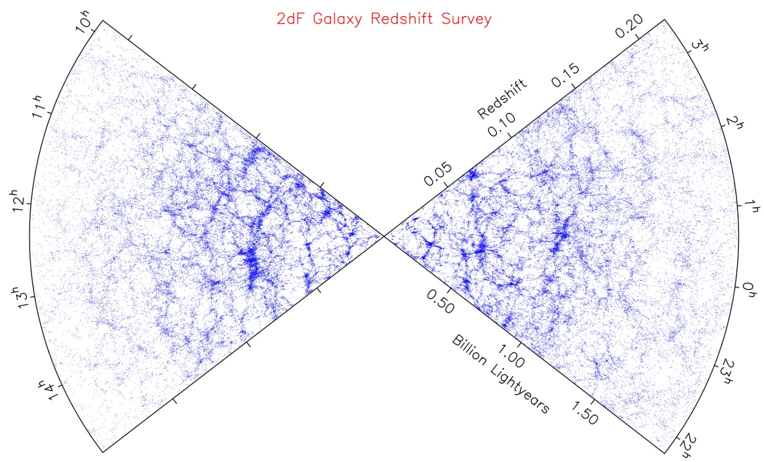

Figure 1.1: Figure showing a large-scale structure of the universe from a $2dF$ galaxy redshift survey [19, 21]. This is proof of the large-scale homogeneity of the universe.

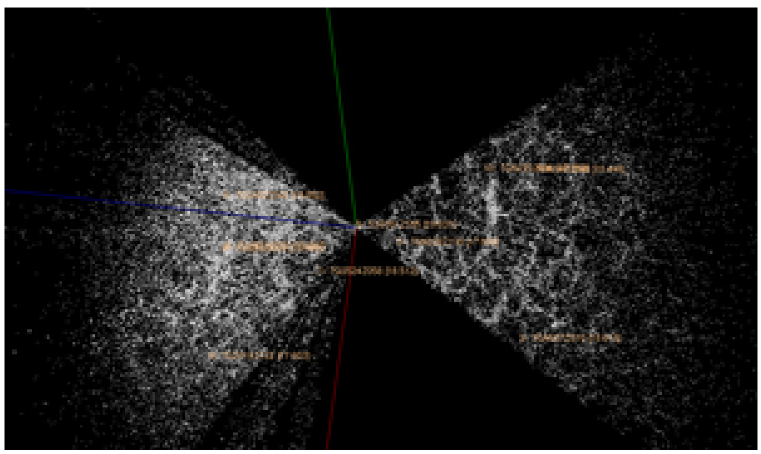

Figure 1.2: 3-dimensional map of figure 1.1 [22].

is sensitive to density inhomogeneities up to scales that are beyond the present time Hubble volume. The uniformity of the $CMBR$ shows that at the epoch of last scattering[7], the universe was to a very precise degree both homogeneous and isotropic. This shows that the cosmological principle holds provided that the universe is observed at sufficiently large scales.

1.2.1 FLRW metric

The large-scale structure of the universe is assumed to be homogeneous and isotropic. This enables one to describe our local Hubble volume. The metric for the spatial sections of such

[7]This period occurred 380000 years after the big-bang. Recombination took place at this time.

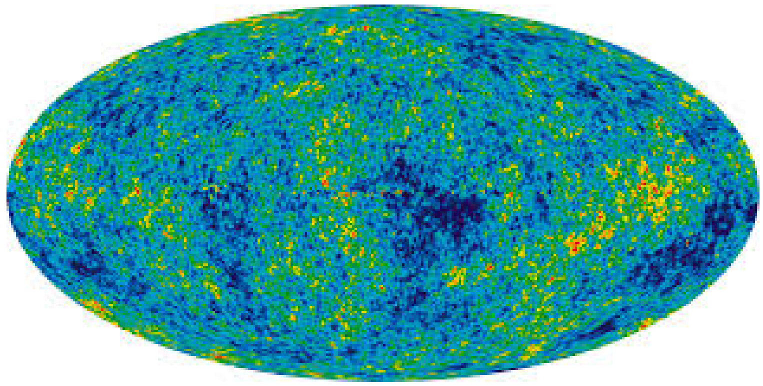

Figure 1.3: This figure shows the temperature difference in the $CMBR$ ranging from -200μK to +200μK. The figure confirms the large-scale isotropy of the universe due to uniformity of radiation distribution as shown [23].

a universe is given by

$$ds^2 \;=\; -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \tag{1.7}$$

where the co-moving coordinates t represents proper time measured by an observer at rest in a co-moving frame, r represents distance from the observer (in a spherical polar coordinate system), θ represents the angle from the polar direction, ϕ represents the azimuthal angle, and the variable $a(t)$ represents the cosmic scale factor and κ represents spatial curvature. The co-moving coordinates can be set in such a way that $\kappa = +1$ represents constant (spatial) positive curvature, $\kappa = -1$ represents constant (spatial) negative curvature and $\kappa = 0$ represents zero spatial curvature. The r is dimensionless. This means that the scale factor $a(t)$ has dimensions of length implying that relative ratios to it should be physical. The range of the r is $0 \leq r \leq 1$ for $\kappa = +1$. Given t which represents proper time measured by an observer at rest in a co-moving frame, implies that the coordinates (r, θ, ϕ) are constants. Observers initially moving relatively to a co-moving frame will eventually come to rest in it. Therefore, an introduction of an isotropic and homogeneous fluid that is initially at rest in a co-moving frame will result in constant t hypersurfaces being orthogonal to the flow of the fluid. This will always coincide with hypersurfaces that are of both constant fluid density and spatial homogeneity.

1.2.2 The stress-energy-momentum tensor

To be consistent with symmetries of the metric tensor g_{ab}, spatial components must be equal by isotropy and the stress-energy-momentum tensor denoted by $T^a_{\ b}$ must be diagonal [18]. $T^a_{\ b}$ of a perfect fluid is the simplest one can consider. It is characterised by a time-dependent energy density and pressure denoted by $\rho(t)$ and $p(t)$, respectively such that

$$T^a_{\ b} = (\rho + p)u^a u_b + p\delta^a_{\ b} = diag(-\rho, p, p, p), \tag{1.8}$$

and $u^a = (1, 0, 0, 0)$ for a co-moving coordinate system. u^a represents the 4-vector velocity field of a fluid. Relative to the co-moving coordinates, the fluid remains at rest. In general, the matter content is supplemented by an equation of state. For this, a barotropic fluid is often assumed. This fluid has pressure that is dependent only on its density, *i.e.*, $p = p(\rho)$. If one considers a relationship that is linear between density ρ and pressure p of the form

$$w = \frac{p}{\rho}, \tag{1.9}$$

such that w represents an equation of state parameter, one can build toy models of cosmological fluids. Occasionally, more exotic equations of state are examined. For example, either radiation (or relativistic particles) has a $T^a_{\ b}$ that is traceless, which implies an equation of state of the form

$$\frac{p_{\mathcal{X}}}{\rho_{\mathcal{X}}} = \frac{1}{3}, \tag{1.10}$$

where $\mathcal{X}$ represents radiation. Therefore, $w_{\mathcal{X}} = \frac{1}{3}$. For physical (gravitating) matter, one will often require $\rho > 0$ and $p > 0$, thus implying $w > 0$. At the very least, that is for $w \neq -\frac{1}{3}$ and $w > -\frac{1}{3}$, we have $w = \frac{p}{\rho} > -\frac{1}{3}$, thus implying $(\rho + 3p) > 0$. On the other hand, a cosmological constant denoted by Λ_0, corresponds to the distribution of matter which is of $w_{\Lambda_0} = -1$. However, this is not in agreement with either $\rho > 0$ or $(\rho + 3p) > 0$ as $w > -1$ for such cases.

Equation (1.7) can be expressed as

$$ds^2 = -dt^2 + a^2(t)\tilde{g}_{\mathcal{I}\mathcal{J}}dx^{\mathcal{I}}dx^{\mathcal{J}}, \tag{1.11}$$

such that $\mathcal{I}$ and $\mathcal{J} = 1, 2, 3$, and $\tilde{g}_{\mathcal{I}\mathcal{J}} = \frac{g_{\mathcal{I}\mathcal{J}}}{a^2}$ [18]. Note that objects with a tilde will represent 3-dimensional quantities which are calculated using $\tilde{g}_{\mathcal{I}\mathcal{J}}$ in this subsection and the next section.

The law of conservation of $T^a_{\ b}$ [18] is

$$\nabla_a T^{ab} = 0, \tag{1.12}$$

where a and b run from 0 to 3. Spatial components of this conservation law can be expressed as

$$\nabla_a T^{a\mathcal{I}} \;=\; 0, \tag{1.13}$$

where $\mathcal{I} = 1, 2, 3$.

Christoffel symbols are denoted by

$$\Gamma^a{}_{bc} \;=\; \frac{1}{2}g^{ad}\left(\frac{\partial g_{db}}{\partial x^c} + \frac{\partial g_{dc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d}\right) = \frac{1}{2}g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d}). \tag{1.14}$$

It is assumed (along with the connection to be torsion free) that $\nabla_a g_{ab} = 0$ for the Riemann manifold. This leads to the definition of the connection coefficients (1.14). Based on the metric (1.11) the symbols can be calculated in terms of the $a(t)$ and $\tilde{\Gamma}^{\mathcal{I}}{}_{\mathcal{J}\mathcal{K}}$; the non-vanishing components are

$$\Gamma^{\mathcal{I}}{}_{\mathcal{J}\mathcal{K}} \;=\; \tilde{\Gamma}^{\mathcal{I}}{}_{\mathcal{J}\mathcal{K}},\ \Gamma^{\mathcal{I}}{}_{\mathcal{J}0} = \frac{\dot{a}}{a}\delta^{\mathcal{I}}{}_{\mathcal{J}},\ \Gamma^0{}_{\mathcal{I}\mathcal{J}} = \frac{\dot{a}}{a}g_{\mathcal{I}\mathcal{J}} = \dot{a}a\tilde{g}_{\mathcal{I}\mathcal{J}}. \tag{1.15}$$

A conservation law for the time-component of equation (1.12) is

$$\nabla_a T^{a0} \;=\; \partial_a T^{a0} + \Gamma^a{}_{ab}T^{b0} + \Gamma^0{}_{ab}T^{ab} = 0, \tag{1.16}$$

which for a perfect fluid yields

$$\dot{\rho} + \Gamma^a{}_{a0}\rho + \Gamma^0{}_{00}\rho + \Gamma^0{}_{\mathcal{I}\mathcal{J}}T^{\mathcal{I}\mathcal{J}} \;=\; 0, \tag{1.17}$$

Based on the metric (1.7), $\Gamma^0{}_{00} = 0$ and $\Gamma^a{}_{a0} = 3H$. Then based on the same metric and equation (1.11), equation (1.17) can be recast into

$$\dot{\rho} + 3H(\rho + p) \;=\; 0. \tag{1.18}$$

1.3 The first Friedmann equation

When one allows for the presence of Λ_0, equation (1.3) is modified to

$$\mathcal{G}_{ab} + \Lambda_0 g_{ab} \;=\; T_{ab}. \tag{1.19}$$

[18]. One can then re-write the equation in a more convenient form, as below:

$$R_{ab} = \left(T_{ab} - \frac{1}{2} g_{ab} T^c{}_c\right) + \Lambda_0 g_{ab}. \quad (1.20)$$

Due to isotropy, there are only two equations that are independent. These are time and any of the spatial components. Therefore, the relevant components of the Riemann tensor

$$R^c{}_{eab} = \partial_a \Gamma^c{}_{eb} - \partial_b \Gamma^c{}_{ea} + \Gamma^c{}_{ad}\Gamma^d{}_{be} - \Gamma^c{}_{bd}\Gamma^d{}_{ae}, \quad (1.21)$$

for the metric (1.11) are

$$R^{\mathcal{I}}{}_{0\mathcal{J}0} = -\frac{\ddot{a}}{a}\delta^{\mathcal{I}}{}_{\mathcal{J}},\ R^0{}_{\mathcal{I}0\mathcal{J}} = \ddot{a}a\tilde{g}_{\mathcal{I}\mathcal{J}},\ R^{\mathcal{K}}{}_{\mathcal{I}\mathcal{K}\mathcal{J}} = \tilde{R}_{\mathcal{I}\mathcal{J}} + 2\dot{a}^2\tilde{g}_{\mathcal{I}\mathcal{J}}. \quad (1.22)$$

Due to the maximal symmetry of $\tilde{g}_{\mathcal{I}\mathcal{J}}$, one can use $\tilde{R}_{\mathcal{I}\mathcal{J}} = 2\kappa\tilde{g}_{\mathcal{I}\mathcal{J}}$ to calculate R_{ab} [18] where $R_{ab} = g^{cd}R_{cadb}$ and $R = g^{ab}R_{ab} = R^a{}_a$. As a result, the non-zero components are

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a}, \\ R_{\mathcal{I}\mathcal{J}} &= (a\ddot{a} + 2\dot{a}^2 + 2\kappa)\tilde{g}_{\mathcal{I}\mathcal{J}} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2\kappa}{a^2}\right) g_{\mathcal{I}\mathcal{J}}. \end{aligned} \quad (1.23)$$

The Einstein equations reduce to

$$\begin{aligned} -3\frac{\ddot{a}}{a} &= \frac{1}{2}(\rho + 3p) - \Lambda_0, \\ \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2\kappa}{a^2} &= \frac{1}{2}(\rho - p) + \Lambda_0. \end{aligned} \quad (1.24)$$

$\ddot{a}$ from the 2^{nd} equation of (1.24) can be replaced using the 1^{st} equation of (1.24), which yields

$$H^2 + \frac{\kappa}{a^2} = \frac{\rho}{3} + \frac{\Lambda_0}{3}. \quad (1.25)$$

This is the first Friedmann equation [18]. It is supplemented by the conservation equation (1.16). As stated in subsection (1.2.2) a cosmological constant denoted by Λ_0 which corresponds to matter distribution of $w = -1$, violates the energy conditions, *i.e.*, $\rho > 0$ or $\rho + 3p > 0$. This hints at the existence of either exotic type of matter or energy.

1.4 Dark-sector

In the early 1990s, it was argued that the universe could have enough energy density to cease its expansion and then collapse again [24]. In this case gravity would slow down the expansion with the passage of time. The force due to gravity pulls together all matter in the universe. On the other hand, the universe might have very little energy density to the extent that it would never cease expanding. In 1998, observations of (very) distant supernovae by the Hubble space telescope showed that the universe is expanding faster than it was in the (distant) past. This implies the universe to be accelerating and is a puzzle.

Three types of explanations seem plausible [24]. These are:

1. The late time universe acceleration is due to the Λ_0 in the long-discarded version of gravity theory by Einstein.
2. An assumption that a strange kind of energy-fluid filled space and,
3. Probably Einstein's theory of gravity is somehow incorrect and a new theory should include a type of scalar field[8] that creates the late time universe acceleration.

Researchers still have not come up with a definitive explanation [24]. However, they have a name for the unknown source of late time acceleration. It is coined dark energy.

1.4.1 Dark energy

Despite significant advances in cosmology, knowledge of the whole observable universe is very scanty [24]. Owing to knowledge of how dark energy affects expansion of the universe (as illustrated by figure 1.4), it is possible to estimate its amount. It is estimated that about 68% and 27% of the universe is made up of dark energy and dark matter, respectively. Everything else constitutes just less than 5% of the universe. That is, all ordinary matter is quite insignificant compared to dark energy and dark matter.

A possible explanation for the mysterious dark energy is that it could be one of space-time properties [24]. Einstein was the first person to indicate that properties of space-time could be quantified. The analysis of his equations showed the possibility of space-time expansion. This is the first property. One version of his theory predicts that space-time can possess its own energy. This version has Λ_0. Since energy is inherent to the fabric of space-time itself, it would not diminish even if space-time increased in size. As the universe expands, its energy density remains constant. As a result, the expansion rate of the universe (filled with energy inherent to the fabric of space-time itself), remains constant. This energy (which is inherent

[8] A scalar field is a field that associates a scalar value to every point in space-time. This field is invariant under any Lorentz transformation.

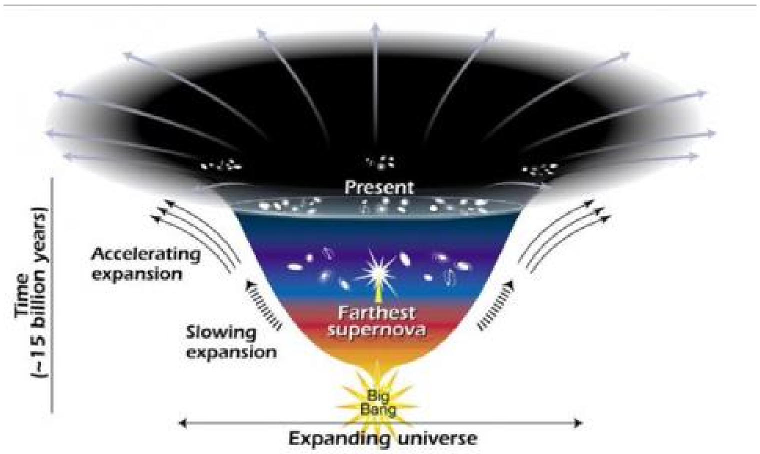

Figure 1.4: The figure above shows the swift expansion of the universe due to a driving force provided by dark energy [24]. Image courtesy of $NASA/STSci/AnnFeild$.

to the fabric of space-time itself) is thought to cause the universe to accelerate. However, cosmologists are puzzled as to why Λ_0 is in equation (1.6) or (1.19). It is even more puzzling as to why it is just the right value needed to account for the observed late time acceleration of the universe. Quantum mechanics tries to provide an explanation as to how energy is acquired by space-time. In the theory, it is assumed that space-time is full of temporary (or virtual) particles that continually disappear and appear. Physicists calculated the amount of energy space could acquire using quantum mechanics and they found a value which is 10^{120} times larger than that in the classical theory of general relativity [24, 25].

A further assumption for this mysterious dark energy is that it is a new kind of dynamical energy fluid or field; something that fills space-time. It is called quintessence. Most forms of energy such as matter or radiation slow down the expansion of the universe due to the attractive force of gravity [26]. For quintessence, however, the gravitational force is repulsive, and this causes the expansion of the universe to accelerate [26] . We do not know what it is or how it interacts with ordinary matter or energy.

There is one possibility. It is the idea that the gravity theory by Einstein might not be correct [24]. Researchers are trying to find out if the solution to the dark energy problem can be found via a new theory of gravity [27]. Further refined observations on how galaxies come together to form clusters could be used to test the new theory. If it does turn out that a new gravity theory is needed, then this merits the question regarding the type of theory proposed. Such a theory would have to correctly describe the motion of objects in our solar system, just like Einstein's gravity theory does and still make different predictions of the universe. A number of such theories have been formulated [27]. However, some of them seem to be quite compelling.

1.4.2 Dark matter

Fitting a theoretical model to the composition of the universe has led researchers to estimate that the universe is comprised of 27% dark matter [24]. Very little is known about dark matter. However, researchers are certain of what it is not. One view is that dark matter is not matter that is made up of either particles or baryons. This view comes from the knowledge that radiation passing through clouds of baryonic matter is absorbed, allowing the detection of baryons. The view that dark matter is not baryonic at all is more popular. In this case it is thought that dark matter is made up of some unusual axions[9], that are either particle-like or simply, weakly interacting massive particles (hereafter $WIMPS$). It is not anti-matter[10] as the unique gamma rays that are produced when anti-matter annihilates with matter aren't observed. From the results of gravitational lensing experiments, one can conclude through observations that large-galaxy-sized black holes should also be ruled out. Light from objects that are far away (or extra-terrestrial objects) is bent by high concentrations of matter and this light passes close to the large-galaxy-sized black holes. However, one cannot observe sufficient lensing events to propose that such objects make up the needed 27% contribution from dark matter. Another view is that baryonic matter could make up dark matter if it were all tied up in either brown dwarfs or in small dense chunks of heavy elements [28]. These are called massive compact halo objects (hereafter $MACHOS$) [28].

1.5 The early universe

The early, adolescent and adult universe are radiation-dominated, matter-dominated and Λ_0 dominated, respectively [18]. Inflation occurred in the very early period of the expansion of the universe. Vacuum energy is the main contributor to the $T^a_{\ b}$ during this brief period of inflation. It can be modelled using a de Sitter space (a solution of the Einstein's static model). In the model, the universe expands exponentially and accelerates. The de Sitter space-time is devoid of ordinary matter and is spatially flat. The de Sitter universe applies to a time (denoted by t) of about $t = 10^{-35}s$ after the big-bang. In this model, the dynamics of the universe are dominated by Λ_0 which is believed to correspond to an inflaton field[11] in the very early universe. This is thought to be preceded by a big-bang singularity[12] [29–31].

Λ_0 affects the expansion rate of the universe. The expansion rate, denoted by H, is

[9] An axion is an hypothetical particle that includes the so-called Peccei-Quinn mechanism and has a mass of 10^{-5} to $10^{-3}eV/c^2$, decay width of 10^9 to $10^{12}eV/c^2$, zero spin and no electric charge.

[10] Anti-matter is matter that is composed of anti-particles of the corresponding particles of ordinary matter.

[11] This field is a scalar field whose evolution leads to an inflationary expansion of the universe. The field is characterised by a negative pressure that yields a tremendous repulsive gravity during a brief lapse of time.

[12] A singularity is a region of space-time where the density of matter, or the curvature of space-time, becomes infinite.

proportional to Λ_0 (cosmological constant) via

$$H \propto \sqrt{\Lambda_0}. \tag{1.26}$$

The larger the Λ_0 the higher the expansion rate H. In order to model the complete evolutionary history of the universe, the de Sitter model which describes the very early universe needs to be either augmented or patched by a model which describes the later part of the universe.

As shown by observations on the distance scales for which the cosmological principle applies, the $FLRW$ metric is quite a good approximation for space-time within the Hubble volume on large-scales. Such an approximation allows one to investigate a number of early universe phenomena.

1.6 Cosmological horizons

A cosmological horizon can be defined as a measure of the distance from which one could possibly retrieve information. It sets the size and scale of the observable universe. Now, a fundamental question in cosmology that one may need to consider is, what proportion of the universe is in causal contact? A more precise question would be, what values of the coordinates (r, θ, ϕ) would a light signal emitted at time $t = 0$ reach the co-moving observer with coordinates (r_1, θ_1, ϕ_1) either before or at t? It can be shown that this can be evaluated in terms of the $FLRW$ metric [18]. Consider the null geodesic equation $ds^2 = 0$ satisfied by a light signal, due to space-time homogeneity, one may choose $r_1 = 0$ without loss of generality. The geodesics via $r = 0$ are lines that are of constant ϕ and θ. They are like great circles that pass through the poles of a 2-sphere and are also of unchanging θ (implying longitudinal circles), $d\theta = 0$ and $d\phi = 0$. The choice of direction (θ_1, ϕ_1) is irrelevant due to space isotropy. If $r_{\mathcal{H}}$ is a coordinate, then a signal of light emitted from the coordinate position of $(r_{\mathcal{H}}, \theta_1, \phi_1)$ at $t = 0$ will arrive at $r_1 = 0$ at time t deduced by

$$\int_0^t \frac{dt'}{a(t')} \quad = \quad \int_0^{r_{\mathcal{H}}} \frac{dr'}{\sqrt{1 - \kappa r'^2}}. \tag{1.27}$$

The proper distance to the horizon denoted by $R_{\mathcal{H}}$ and measured at t is

$$R_{\mathcal{H}}(t) \quad = \quad a(t) \int_0^t \frac{dt'}{a(t')} = a(t) \int_0^a \frac{da'}{a'} \frac{1}{a' H(a')} = a(t) \int_0^{r_{\mathcal{H}}} \frac{dr'}{\sqrt{1 - \kappa r'^2}}, \tag{1.28}$$

where H represents the Hubble parameter. If $R_{\mathcal{H}}$ is finite, the particle horizon is the boundary between the visible universe and the part of the universe from which light signals have not reached us.

In the theories of special and general relativity, a light cone is the path that a flash of light, emanating from a single event and travelling in all directions, would take through space-time. If one were to imagine that the light is confined to a two-dimensional plane, the light from the flash spreads out in a circle after the event (a point at the exact position and time) denoted by E_0. If we graph the growing circle where the vertical axis of the graph represents time, the result is a cone (called a future light cone). The past light cone behaves like the future light cone in reverse, that is, a circle which contracts in radius at the speed of light until it converges to E_0. Our past light cone is then limited if $R_{\mathcal{H}}$ is finite [18].

Now, the behaviour of the $a(t)$ near a singularity determines whether the particle horizon is finite or not. For $R_{\mathcal{H}} \approx t$ in the model of standard cosmology, the horizon is finite. The Hubble horizon expressed as

$$\frac{1}{H} = \frac{a}{\dot{a}}, \tag{1.29}$$

is not the particle horizon. The Hubble horizon represents the distance over which particles can travel in the course of one expansion time. This time is the period in which the $a(t)$ doubles. The distance can be expressed in the form

$$dt \sim \left(\frac{da}{a}\right) H^{-1}. \tag{1.30}$$

Hence, the Hubble horizon represents an alternative approach to measuring whether or not particles are causally connected. Two particles that are separated by distances which are beyond the Hubble horizon cannot currently communicate.

One can define the distance of the co-moving particle horizon in the form [18]

$$\tau_{\mathcal{H}} = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da'}{H(a')a'^2} = \int_0^a d\ln a' \left(\frac{1}{Ha'}\right). \tag{1.31}$$

The co-moving horizon is then the logarithmic integral of co-moving $(aH)^{-1}$. There is a conceptual difference between $\tau_{\mathcal{H}}$ and $(aH)^{-1}$. If two objects were separated by co-moving distances and are greater than $\tau_{\mathcal{H}}$, then it is impossible that the objects could have communicated with each other. In contrast, the event horizon is the largest distance one can explore in the infinite future. Its expression is given as

$$R_{\mathcal{E}}(t) \quad = \quad a(t)\int_t^{\infty}\frac{dt'}{a(t')}. \tag{1.32}$$

If the expansion of the universe is in the form of the a such that $a \sim t^{\mathcal{N}}$ where $\mathcal{N} < 1$, then equation (1.32) will be infinite.

The idea of an event horizon is very useful in mathematical relativity. However, it is not directly useful in a number of physical contexts ranging from theories of quantum gravity to cosmology[13]. This was recognised early on [32]. There are other types of horizons in cosmology, for example, the apparent horizon found in the study of black-holes. An apparent horizon is a surface that is the boundary between light rays that are directed outwards and moving outwards, and those directed outwards but moving inward. Apparent horizons are trapped regions [32, 33].

Hawking's idea of an outer trapped surface is simply a compact and space-like 2-dimensional sub-manifold in space-time such that the expansion of the outgoing null normal to the surface is non-positive [34]. Hence, a trapped region in the space-like 3-surface is then the set of all points in the surface through which there passes the outer trapped surface, lying entirely in the space-like 3-surface.

1.7 Cosmology and fluid theory

Cosmology is usually examined using a single-fluid model [35]. For this approximation, modelling of the universe is based on a world-line of a single observer. Studies of inflation [36, 37], the radiation-dominated epoch [23], the matter-dominated epoch [38] and dark energy [39] are carried out using the single-fluid approximation. Combining the different epochs allows one to build a model of an evolving universe. Predictions made by this so-called standard cosmological model agree with observations to a large extent. However, the analyses of the $CMBR$ and the discovery of its anisotropy [40] implies that not all predictions of cosmological theories are confirmed. The interplay between observations and theories have not always been smooth. This has resulted in a number of theoretical questions not being answered. Some findings of cosmological observations have led to questions that demand a re-examination of underlying theories. Some examples of these cosmological observations are the so-called *axis of evil* in the cosmic microwave background [41] and the late time acceleration [42, 43].

[13] The idea is useful for black-holes though not entirely (physical) as space-time black-holes are embedded in $FLRW$.

Modelling of transitions between epochs is uncommon in literature. This kind of modelling could be a way of easing the tension between observation and theory. This approach is proposed as various attempts which involve tweaking of either a current theory or improving technology to provide an explanation of an anomalous observation, have resulted in limited success. This has forced several researchers to infer that there is need for a complete overhaul of underlying theories, either of gravity or of material content in the universe. The modelling of transition between epochs can be applied to studies of cosmological eras. An era can be defined as an event that signifies when a change has taken place to an extent that it marks the beginning of a new epoch. The dynamics of the universe is dominated by material of one kind in a given era. There is a transition from one domination to the next. Note that dynamics of the flow is still impacted if the transition does not involve a switch in the material that is dominant, but a freeze-out[14]. Assuming that transitions occur gradually allows a transient period not to be fully dominated by one of the fluid species. This could lead to the resolution of some of the disparities between theory and observation. Considering the above statements, modelling a transition between the different epochs and analysing such periods is important.

Under a suitable continuum hypothesis, any non-rigid multi-bodied state can be described as a fluid which follows certain equations of motion. This could be said of the dark matter particles. Motion of a many-body system is modelled using fluid dynamics. Now, considering the well-known assumption that dark energy is a new kind of energy fluid, examination of multi-fluid and (a possible) entrainment[15](or tilted momenta) effect that involve interaction of dark matter and dark energy is a possibility.

Now, although the single-fluid approximation[16] has successfully played a significant role in cosmological modelling, it is our contention that a multi-fluid formalism is more suitable for modelling. A single-fluid approximation is the limit of the multi-fluid approximation [44]. The success of the single-fluid approach lies in the fact that different species making up the cosmological fluid may start off evolving differently. Eventually, all species become locked-in, hence, rendering one species dominant at a given epoch. Therefore, the fundamental observer world-line is defined by the world-line of the dominant species. The mechanism that allows this to happen is yet to be formulated. Therefore, a re-examination of relativistic, multi-fluid theory in particular, and its application to cosmology in general is needed.

[14]Freeze-out represents a scenario when interaction rate of a particle is smaller than the Hubble rate. This means that reactions stop and population of these particles as a whole changes only due to space expansion.

[15]This can be defined as the transport of fluid across an interface between either two or more bodies of fluid by a shear-induced turbulent flux.

[16]An approximation that will be utilised to examine magnetic fields later in this work.

A generic model of dark energy that interacts with dark matter for illustrative purposes can be assumed. Since the interactions between dark matter and dark energy can fit well with the observational data, and could potentially provide new physics, they have been a subject of much inquiry. The observational fact that the present values of the densities of dark matter and dark energy are of the same order of magnitude, seems to indicate that we are currently living in a very special period of the cosmic history. Within the standard cosmological model, a density ratio of the order of one just at the present epoch can be seen as coincidental since it requires very special initial conditions in the early universe. The corresponding *why now* question constitutes the *coincidence problem* [45]. The interactions between dark matter and dark energy might help resolve the coincidence problem. They might provide a possible explanation as to why the present values of the densities of dark energy and dark matter are of the same order of magnitude, something that would require very special initial conditions in the early universe. Unlike the well-behaved, non-interacting models with constant w (given a barotropic equation of state, where w is the proportionality parameter), an interacting model can manifest instabilities in the perturbations of the dark-sector at early times, resulting in new physics (or new phenomena). Examples of this are in interacting dark-sector models that have been studied in [46–52].

1.8 Cosmological magnetic fields

Magnetic fields are found everywhere in the universe [53]. Different objects and structures such as planets, stars, galaxies and clusters of galaxies are known to carry magnetic fields that are large and extensive. The fields are all-pervasive, they play a vital role in controlling how celestial sources form, evolve and die out. They could have played a significant role in structure formation [53]. They may have affected a number of relevant processes which took place in the early universe, as well as the formation of the universe geometry itself. One can now see that understanding the universe is impossible without understanding magnetic fields. They fill interstellar space, affect the evolution of galaxies and galaxy clusters, contribute significantly to the total pressure of interstellar gas, are essential for the onset of star formation, and control the density and distribution of cosmic rays in the interstellar medium.

The origin and evolution of primordial magnetic fields are tied to the evolution of the universe. Therefore, searching for their origin and evolution is tantamount to searching for information about the universe in its infant or early state. Success in this could provide us with a *snapshot* of the early universe. However, their *origin*, *growth* and *evolution* is a mystery.

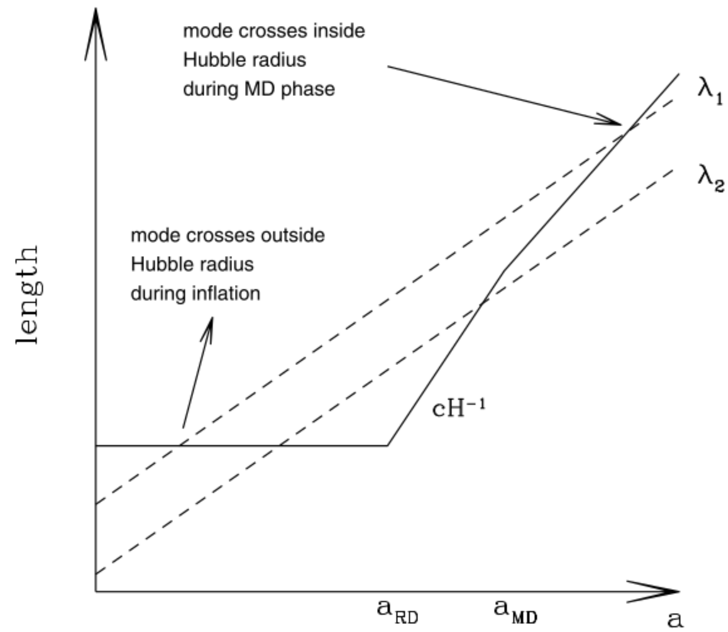

Figure 1.5: This figure shows the evolution of inflation-produced, large-scale magnetic fields. RD, MD and a represent the radiation-dominated epoch, matter-dominated epoch and scale factor, respectively. λ represent the magnetic modes. cH^{-1} represents the Hubble horizon. Now, as magnetic fields cross the horizon or evolve below the horizon during the inflationary phase, they evolve in an oscillatory pattern while well above the horizon they evolve in a power-law form pattern. λ_1 and λ_2 are two modes that cross the Hubble horizon at different times during inflation, evolve above horizon scales and re-enter inside the Hubble radius at different times as shown in the figure. The modes are both generated during inflation. However, we are not examining the hypothesis represented by the figure. We introduce the figure to aid the reader see the contradiction as discussed in the paragraphs. That is the contradiction of magnetic fields evolving in either power law form below the Hubble horizon or from the beginning of de Sitter phase and instead of evolving in an oscillatory form until well above the horizon. The figure clearly shows the evolution of magnetic modes both below and well above the Hubble horizon [56, 57]

Several cosmological and astrophysical theories for generating the galactic magnetic fields have been devised. Mechanisms for primordial magnetic fields generation, growth and evolution have been widely studied. The proposed mechanisms include magnetogenesis and growth of magnetic fields during the inflationary or de Sitter phase [54, 55], and magnetic amplification after the inflationary epoch. However, these proposed mechanisms are not without challenges. One example is a mechanism known as the galactic dynamo which is based on the conversion of kinetic energy of turbulence motion of the conductive interstellar medium into magnetic energy. The efficiency of such a kind of magneto-hydro-dynamic (hereafter MHD) engine has led to it being questioned due to both improved theoretical modelling and new (or latest) observations of cosmic magnetic fields in high redshift galaxies. It is also understood that magnetic fields in the inter-cluster medium cannot form from

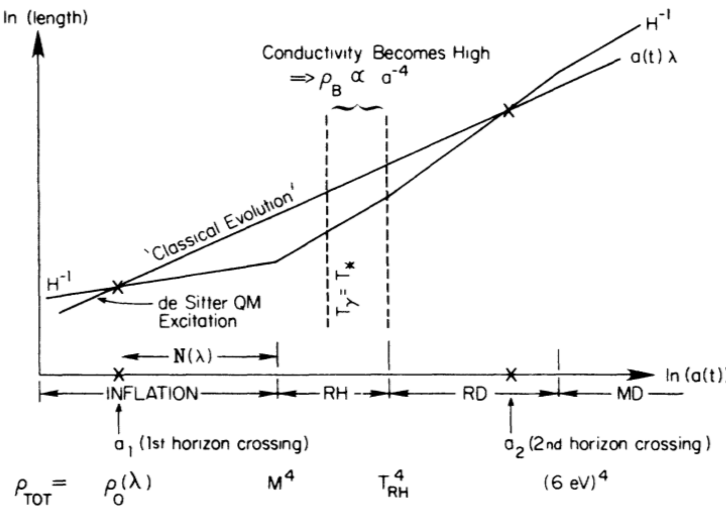

Figure 1.6: This figure shows the evolution of inflation-produced, large-scale magnetic fields. RH, RD, MD, ρ_{TOT}, λ, $\rho_0(\lambda)$, T_{RH}, QM and ρ_B represent the reheating, radiation-dominated epoch, matter-dominated epoch, total density, magnetic mode, density at the beginning of the inflationary epoch, temperature of the reheating epoch, quantum mechanics and magnetic field density, respectively. H^{-1} represents the Hubble horizon. The first cross between the path of the magnetic mode λ and the horizon is the first horizon crossing of magnetic field perturbations while the second cross is the second horizon crossing of the fields. The mode shown crosses the Hubble horizon during inflation, evolves above horizon scales and re-enters the Hubble radius much later (probably during either radiation-domination or matter-domination) as the universe expands. However, as before in figure 1.6, we are not examining the hypothesis represented by the figure. We introduce the figure in support of figure 1.6. That is, we introduce it to aid the reader see the contradiction as discussed in the paragraphs. That is the contradiction of magnetic fields evolving in either power law form below the Hubble horizon or from the beginning of de Sitter phase and instead of evolving in an oscillatory form until well above the horizon. The figure clearly shows the evolution of magnetic modes both below and well above the Hubble horizon [54, 56]

ejection of the galactic fields [53]. Therefore, a continued effort in searching for a mechanism responsible for the origin, growth and evolution of cosmological (or large-scale) magnetic fields has to be carried out by looking back into the remote past (of the inflationary phase and more reasons follow in the next paragraph) and devising a more realistic mechanism.

We will examine the growth and evolution of magnetic fields which are generated during inflation. Note that gravitational waves are also generated during inflation [58] as a counterpart. The energy transfer between the two is of interest and has been considered elsewhere [59–61]. In this work, gravitational waves will be ignored. Inflation can generate and grow magnetic fields (naturally) up to either cosmological horizon or super-horizon scales. This implies that inflation can easily generate primordial magnetic fields. Magnetic fields generated after inflation are either far too small in scale (or of sub-horizon scales) [56].

In order to examine if there is a link between the magnetic fields generated during inflation and those which are observed in the current universe, it is crucial to understand how they evolved either after de Sitter phase or the inflationary epoch until present time. Here, we will examine their evolution on a flat $FLRW$ universe.

In most literature on the evolution of magnetic fields, it is found that they evolve adiabatically for the flat $FLRW$ model [56]. This leads to rapid decay and is considered as the main obstacle of magnetic fields generated during either a de Sitter phase or the inflationary epoch to survive to be either the seeds for fields observed now or hence seeding the galactic dynamo. Magnetic fields that are generated during inflation are believed to begin their adiabatic evolution from the moment they cross the Hubble horizon (for the first time) and thereby lead to astrophysically irrelevant magnetic field strength at current time [56]. Surprisingly, the strengths detected in inter-galactic fields are around $10^{-16}G$, which is much stronger than expected.

One way to understand this is through *super-adiabatic amplification* of magnetic fields. It means slower magnetic decay rates than the adiabatic decay rate. In this work, we will use the *variational* approach to develop a single-fluid model and with that we will derive equations that will enable us to examine the evolution of magnetic fields in the single-fluid model(s) (of radiation-dominated or/ and matter-dominated epochs), and hence show that super-adiabatic amplification of magnetic fields is possible. In other words we will use the variational formalism to derive equations of motion that will enable us to examine the evolution of inflation-generated, cosmological magnetic fields in single-fluid models of either the radiation-dominated or matter-dominated epochs (or both). With this, we study relativistic fluids with applications to cosmology in that we analyse the behaviour of evolution of inflation-generated, cosmological magnetic fields in single-fluid models of either the radiation-dominated or matter-dominated epochs. That is, we investigate how the behaviour of evolution of the large-scale magnetic fields is either influenced or affected as they evolve in the single-fluid models of either the radiation-dominated or matter-dominated epochs. Considering the above statements, it seems that the current time amplitude of magnetic fields, arising from inflationary magnetogenesis can actually be much larger than what has been claimed in most previous studies. With this model, we will also be able to resolve a contradiction in [56] where an equation that represents the evolution of inflation-generated, cosmological magnetic fields well above the Hubble radius or horizon also represents the evolution of the magnetic fields from the beginning of the inflationary phase [56]. Before magnetic fields evolve on scales well above the horizon, they follow an oscillatory pattern form of evolution (actually from the beginning of the inflationary phase). On scales well above the horizon they follow a power law form pattern of evolution. This is clearly problematic. This can be seen more clearly in figures (1.5) and (1.6), and will be addressed again in chapter 6.

1.9 Thesis plan

The thesis plan is as follows:

1. In chapter 2, relativistic fluids and the M*ü*eller-Israel-Stewart (hereafter MIS) formalisms of both standard, and extended MIS theories are examined.

2. In chapter 3, the variational formalisms for relativistic fluids is investigated. The formalism of a point particle is illustrated, and the pull-back formalism for a single-fluid approximation and then finally a convective variational approach are examined.

3. In chapter 4, the application of a modified convective variational formalism to interacting multi-fluid models is investigated.

4. In chapter 5, the thermodynamics of relativistic, dissipative and interacting multi-fluid systems is examined, thus enabling an examination of the entrainment effect of the interacting dark-sector, using the (slightly) modified convective variational formalism.

5. In chapter 6, the generation (or origin), growth and evolution of inflation-generated magnetic fields are investigated in single-fluid models. We use equations that are derived using the variational formalism to examine the evolution of inflation-generated, cosmological magnetic fields in single-fluid models of the radiation-dominated epoch and the matter-dominated epoch.

6. Finally, in chapter 7, the thesis is summarised and recommendations for future investigations are stated.

Chapter 2

Relativistic fluids and the Mueller-Israel-Stewart (MIS) theories

2.1 Relativistic fluids theory

Understanding the evolution of a many-bodied system such as the evolving or expanding universe is a very important problem in modern physics [62]. Relativistic fluid mechanics provides a mechanism to determine the macroscopic motion of the system. Before we proceed further, we will first define a fluid. Traditionally, it is defined as a substance that flows to fill space and does not support shear[1] stress. This definition though somewhat lacking presents the idea that fluids flow and distort. Under a suitable continuum hypothesis, any non-rigid multi-bodied state can be described as a fluid which follows certain equations of motion and can be constructed in such a way that they incorporate special relativity as well as curved space-time effects.

The motion of a many-body system is approximated using fluid dynamics [62]. In principle, a true description of the evolution of a fluid would need to account for the motion of each individual particle. This description however, is impractical and of no substantial worth when modelling sufficiently large systems such as the expanding or evolving universe. Hence, provided that the desired level of accuracy is much higher than the continuum approximation, it is acceptable to consider the system as a fluid. Such an approximation has been applied to relativistic fluids and such applications are varied. They have been applied to many different domains from plasma physics, astrophysics and cosmology [63].

Relativistic fluids are relevant to physics in that many-particle astrophysical and cosmological systems are the best sources of detectable effects which are associated with general relativity.

[1] This seems to be a secondary consideration. It distinguishes fluids from solids.

The theory of relativistic fluids has continued to receive considerable attention since the seminal work of Landau and Lifschitz[2] who set the foundation for the present day study of relativistic fluids in [64]. Interest in the theory of relativistic fluids is largely driven by its potential use in studies of astrophysics and cosmology. The theory is explored by using mathematical models. Though mathematical modelling processes present several challenges, these processes are carried out in the studies of relativistic fluids in the context of astrophysics and cosmology where approximations and assumptions are made hence alleviating mathematical modelling challenges. They take into account physically, plausible scenarios.

The challenges presented by mathematical modelling processes are either theoretical or conceptual [44]. Problems classified under conceptual category involve difficulty in identifying specific and measurable variables that give rise to a framework for characterising relativistic fluids. The theoretical-related problems involve the foundational theories, which include the theories of fluids, general relativity and thermodynamics, as applied to multi-species environments. The problems can be resolved by considering a unified framework of how single-species and multi-species fluids are treated. A study of multi-species fluids requires a way of approximating aggregated fluid properties. For example, one may use the multi-fluid approach. Multi-fluid models allow species to interact; entrainment is an example.

Assuming the components interact thermodynamically, ways of examining thermodynamics may be required. Carrying out such a study requires the linking of theories which are constructed by using dynamics of single-fluids [65–68] to those which are constructed by using either multi-fluids or multi-species fluids that are relativistic [69–71]. One may also need to go beyond perfect fluids; consider fluids that exhibit dissipation. For example, those in which bulk viscosity plays a role can be considered [72]. Dissipation manifests in the lab environment, in astrophysics and predictably in cosmology. Particularly, it manifests in flows involving heat flow in the presence of thermal resistance, in fluid flows with viscosity, diffusion, chemical reactions and electric current that flow in resistive media. Dissipation has been incorporated and examined in the modelling of both non-relativistic [65, 73, 74] and relativistic fluids. However, there are open questions regarding the incorporation and examination of dissipation in the modelling of relativistic fluids [65, 73, 74].

Dissipation is usually largely ignored in cosmology [75, 76]. One would ask why dissipation should be incorporated in relativistic fluids, and how would it be done? The authors of [77] are motivated by the need to develop a formalism that could be used to study gravitational radiations which emanate from compact objects, particularly neutron stars.

[2]They also defined super-fluidity and established the theory of super-fluidity.

In some of these astrophysical objects, radiative processes are known to be influenced by dissipation. Additionally, there are processes that occur during structure formation and during reheating epochs in the early universe that suggest that dissipation could have played a role. Therefore, ideally dissipation should be taken into account. The same can be said of the dynamics of dark matter [78, 79] and of heat flow in general [80–82]. To account for these, one is supposed to formulate a formalism that incorporates dissipation.

The most interesting development in classical relativistic fluid dynamics is the consideration of multi-fluid systems that are composed of elements whose collective dynamics can involve a super fluid/superconductor, heat flow or the treatment of electromagnetic charge as a dynamical variable [73, 80, 83]. This is pointed out in [84]. Such systems are being used to study a wider range of relevant phenomena. For example, the nature of interactions between the different species which may affect how the mixture flows, effects that may only be captured in the multi-fluid treatment, and not in the single-fluid theory. These include dissipation and entrainment, of which the latter is much less known or studied, particularly in the sub-class of relativistic fluids. However, the developments have been patchy, and a general theory remains incomplete in at least two different respects. On one hand, they require the coupling of dissipation to electromagnetism, while on the other hand they require the inclusion of dissipation. These developments need examination. They hold the key to the greater applicability of the fluid theory in both astrophysics and cosmology.

In fluid dynamics one may need to build fluid dynamics theories in which components of the fluids are assumed to interact thermodynamically with each other, and manifest dissipation and bulk viscosity. This would need one to consider fluids that are not perfect. Therefore, to examine such theories, the MIS formalism is used. This formalism will be reviewed in the next section. Additionally, in order to understand relativistic fluids, it is important to develop the mathematical tools that can look at curves in a curved space-time [62]. One can start with a line-interval denoted by ds^2. This can be used to find length of an infinitesimally small line-element in four-space and is given as

$$ds^2 \;=\; g_{ab}dx^a dx^b, \tag{2.1}$$

where space-time indices denoted by a and b run as (0, 1, 2, 3). This is equivalent to (t, x, y, z) in Cartesian space-time where t represents cosmic time. The line element in equation (2.1) is invariant of the chosen coordinate system. This implies that it is a scalar. The metric of space-time geometry g_{ab} serves the role of a weighting function, used in defining the length of a path.

Placement of indices is very important in curved space-time. The metric g_{ab} is used to lower and raise indices. That is

$$\mathcal{V}^a = g^{ab}\mathcal{V}_b, \tag{2.2}$$

or

$$\mathcal{V}_a = g_{ab}\mathcal{V}^b. \tag{2.3}$$

Furthermore

$$\mathcal{V}^a = g^{ab}g_{bc}\mathcal{V}^c, \tag{2.4}$$

implying that

$$g^{ab}g_{bc} = \delta^a{}_c. \tag{2.5}$$

$\delta^a{}_c$ represents the Kronecker delta function.

The infinitesimal length of the ds^2 divides up into three regimes that are different from each other [62]. These are:

1. *Time-like.* The curve ds^2 is said to be time-like if $ds^2 < 0$. Events which occur at the same location at different times on a rest frame are said to be time-like separate.

2. *Null.* The curve ds^2 is said to be null if there does not exist a rest frame. This implies that $ds^2 = 0$.

3. *Space-like.* The curve ds^2 is said to be space-like if $ds^2 > 0$. Events which occur at the same time at different locations on a rest frame are said to be space-like separated.

It turns out that light moves along null paths and that all matter travels along time-like curves [62]. Therefore, for time-like curves, the proper time denoted as τ represents time measured in an observer's rest-frame and it can be expressed as

$$d\tau^2 = -ds^2. \tag{2.6}$$

Hence, using this definition, the 4-vector velocity is defined as in equation (1.5). This tensor calculus is introduced in order to allow for the derivation of physical laws, independent of a particular coordinate system. This means that a tensor will inevitably obey certain transformation laws. The tranformation relation for a vector, with higher-order tensors transforming in a consistent manner is

$$\overline{\mathcal{V}}_a = \frac{\partial x^b}{\partial \overline{x}^a}\mathcal{V}_b, \tag{2.7}$$

or

$$\overline{\mathcal{V}}^a = \frac{\partial \overline{x}^a}{\partial x^b}\mathcal{V}^b. \tag{2.8}$$

2.1.1 Material and covariant derivatives

Knowledge of how to find the derivative at a given point of a vector field is important [62]. The rate of change of a vector field denoted by $\mathcal{V}^b$ in a particular direction x^a can be found simply by taking the partial derivative in a flat space-time. In curved space-time, the derivative is not so easy to define. Illustrating this, we consider a vector denoted by $\vec{\mathcal{V}}$ such that $\vec{\mathcal{V}} = \mathcal{V}^a\vec{e}_a$ where $\vec{e}_a$ represents a basis vector at a point. The basis vectors are constant in flat space-time when Minkowski coordinates are used while in curved space-time, they are not. An example are spherical polar coordinates where basis vectors are not constant, and can change point-to-point. Then

$$\partial_a(\mathcal{V}^b\vec{e}_b) = (\partial_a\mathcal{V}^b)\vec{e}_b, \tag{2.9}$$

for flat space-time where Minkowski coordinates are used and

$$\partial_a(\mathcal{V}^b\vec{e}_b) = (\partial_a\mathcal{V}^b)\vec{e}_b + \mathcal{V}^b\partial_a\vec{e}_b, \tag{2.10}$$

for curved space-time.

This example shows that there are two main issues to resolve when defining the derivative in curved space-time [62]. The first one is, how does one find a limit in curved space-time? The second is, how does one ensure that the derivative transforms correctly? These issues can be resolved by defining the covariant differential operator as

$$\nabla_a\mathcal{V}^b = \partial_a\mathcal{V}^b + \Gamma^b{}_{ac}\mathcal{V}^c, \tag{2.11}$$

or

$$\nabla_a\mathcal{V}_b = \partial_a\mathcal{V}_b - \Gamma^c{}_{ba}\mathcal{V}_c. \tag{2.12}$$

For a rank-2 tensor, the expression is

$$\nabla_c T^{ab} = \partial_c T^{ab} + \Gamma^a{}_{dc} T^{db} + \Gamma^b{}_{dc} T^{ad}, \tag{2.13}$$

or

$$\nabla_c T_{ab} = \partial_c T_{ab} - \Gamma^d{}_{ca} T_{db} - \Gamma^d{}_{cb} T_{ad}. \tag{2.14}$$

The material derivative is expressed as

$$\frac{D}{D\tau}\mathcal{V}^a[x_a(\tau)] = \frac{dx^a}{d\tau}\nabla_a \mathcal{V}^b, \tag{2.15}$$

leading to

$$\frac{D}{D\tau}\mathcal{V}^a[x_a(\tau)] = u^a \nabla_a \mathcal{V}^b. \tag{2.16}$$

Note that either $\mathcal{V}^a$ or $\mathcal{V}^b$ is meant to represent a tangent vector along a curve. For time-like curves, and after using equation (2.6)

$$u^a u_a = -1, \tag{2.17}$$

implying equation (1.4).

2.1.2 Stress-momentum tensor interms of internal energy density per unit mass

Different systems have different stress-energy-momentum tensors $T^{ab}s$ [62]. Hence, considering the perfect fluid T^{ab} is necessary. In a general frame of an observer it can be written as

$$T^{ab} = (e+p)u^a u^b + p g^{ab}, \tag{2.18}$$

where p represents pressure, e represents the total energy density and u^a represents the local four-velocity of the fluid. The e can be expressed as

$$e = \rho(\epsilon + 1), \tag{2.19}$$

with ρ representing the rest frame mass energy density and ϵ representing the internal energy density per unit mass. The continuity equation can be written out as

$$\nabla_a(\rho u^a) = 0, \tag{2.20}$$

which ensures conservation of mass. A more general conservation energy equation for this system can be derived. This is done by projecting (1.12) onto u_a. Using the identities of equations (2.17) and (1.4) yields

$$u^a \nabla_a e = -(e+p)\nabla_a u^a, \tag{2.21}$$

Projecting equation (1.12) into its space-like components using

$$h_{ab} = g_{ab} + u_a u_b, \tag{2.22}$$

and utilising the identities in equations (2.17) and (1.4) [62], yields

$$(e+p)u^a \nabla_a u^b = -h^{ba}\nabla_a p. \tag{2.23}$$

2.1.3 Relativistic Euler equations

The aim here is to write a system of equations which can be used to solve for the flow of a fluid [62]. At this point, equations (2.21) and (2.23) which are the conservation of energy and conservation of momentum equations are given. To help understand what these equations mean, it is insightful to compare the equations with their classical counterparts. This is done by expanding equation (2.23). Then

$$(e+p)u^a \nabla_a u_b = -\nabla_b p - u_b u^a \nabla_a p, \tag{2.24}$$

from which one can write out the spatial components as

$$(e+p)\frac{D\vec{u}}{D\tau} = -\nabla p - \vec{u}\frac{Dp}{D\tau}, \tag{2.25}$$

and

$$u^a \nabla_a e = -(e+p)\nabla_a u^a, \tag{2.26}$$

where $\frac{D}{D\tau} = u^a \nabla_a$ and $\vec{u}$ represents three-velocity. Equations (2.25) and (2.26) are the momentum and continuity equations, respectively.

2.1.4 Thermodynamics and a single-fluid model

Although there are several ways to derive the fundamental equations that form the basis for single-fluid modelling, a particularly instructive approach is where the classical Einstein equations are derived from thermodynamical considerations. This is given in reference [85]. In the derivation of Einstein equations, this approach uses the heat relation $\delta Q = TdS$, where T represents temperature and S represents entropy [44]. It can be shown that the heat equation is analogous to the equation of state, provided that the local equilibrium conditions exists. A length scale for which the conditions are assumed to hold with no threat of the emergence of transient thermodynamics does exist [86]. Entropy is connected to the causal horizon which holds information [87] that could potentially be decoded, but the tools for doing this are yet to be developed. The link between the equation of state and entropy can be shown. As an example, assuming that the entropy function is known and that it is given in terms of total internal energy denoted by E, volume denoted by V, number density denoted by N, temperature denoted by T, pressure denoted by p and chemical potential denoted by μ, from the first law of thermodynamics, it follows that

$$dQ \;=\; dE + pdV - \mu dN, \tag{2.27}$$

where $\delta Q = TdS$. From this equation, one can conclude that

$$\frac{\partial S}{\partial E} \;=\; T^{-1},\; T\frac{\partial S}{\partial N} = \mu,\; T\frac{\partial S}{\partial V} = p. \tag{2.28}$$

The equation of state is denoted by the function $S = S(E, N, V)$. T in reference [88] represents the Unruh temperature as measured by a uniformly accelerated observer (across a causal horizon), while heat is energy flux across a causal horizon that can be felt via the gravitational field it generates. There exists a limit where the ratio of Unruh temperature and energy flux both remain finite, although acceleration diverges as the observer world-line approaches the horizon. Thermodynamics is examined in this limit in [88]. Their analysis is done for a single observer world-line which corresponds to single-fluid modelling of fluid flow. We would like to construct a complementary argument for multi-observer world-lines. In references [35, 89], initial examination of thermodynamics in multi-fluid theory is provided where it is shown that entropy always increases as expected. An essential starting point that will hopefully provide a far greater understanding of thermodynamics in a multi-fluid environment is provided later in these studies.

The Lagrangian density of the form

$$\Lambda(\rho, \vec{u}) \;=\; \frac{1}{2}\rho\vec{u}^2 - \mathcal{E}(\rho), \tag{2.29}$$

with ρ representing density, $\vec{u}$ representing velocity and E representing internal energy density of fluid, can be used as a starting point in studies of single-fluid hydrodynamics [44]. An equation of state which is barotropic is assumed in the above equation (2.29). Therefore, Λ

can be connected to the pressure p and chemical potential μ in the usual way. Information that is related to the underlying flow lines, the local geometry and physics of the fluid is concealed by the formulation above of equation (2.29). Assuming that entropy is conserved, a more nuanced formulation given in terms of mass m and number density denoted by n, takes the form

$$\Lambda(m,n) \;=\; -mn_n - \mathcal{E}(n^2{}_n). \tag{2.30}$$

One can then write the $T^a_{\ b}$ in the usual form

$$T^a_{\ b} \;=\; \psi\delta^a{}_b + n^a\mu_b, \tag{2.31}$$

where

$$\psi \;=\; \Lambda - n^c\mu_c,\ \mu_b = g_{ba}\beta n^a,\ \beta = -2\frac{\partial\Lambda}{\partial n^2{}_n}, \tag{2.32}$$

and ψ represents pressure. The usefulness of presenting the Lagrangian in terms of the n and m will become clear in the multi-fluid model.

2.2 The (Standard) MIS theory

Early theories on single-fluid irreversible thermodynamics [90] predicted instantaneous propagation of viscous and thermal effects due to the parabolic nature of the resultant differential equations. One example of the early theories on single-fluid irreversible thermodynamics is the Eckart theory. This theory is unstable in the sense that small spatially bounded departures of fluids from equilibrium at one instant of time will diverge exponentially with time. The time scales for the instabilities are very minute. These results provide overwhelming motivation for abandoning the Eckart theory in favour of the MIS theories which are free of these problems [91]. After analysis of dynamics of small departures of the fluids (in Eckart theory) from their equilibrium states, it is shown that the Eckart theory predicts rapid evolution away from equilibrium. This makes the theory very unstable and consequently unacceptable as a reasonable physical theory. In a series of papers, the Eckart theory was contrasted with the relativistic kinetic theory [88, 92, 93]. Following early work on the theories and connecting with Grad's 14-moment kinetic theory description, it was concluded that a satisfactory formalism had to be *second order* in the various fields [94, 95]. This approach is known to admit stable equilibrium states, and fluctuations about equilibrium are known to propagate causally via hyperbolic differential equations [96]. Hence, the formalism is an attractive alternative to the simpler but pathological Eckart theory[3].

The consequence of the early theories is that they were predictive only for slowly varying systems. The problem was traced to the non-perturbative [97] truncation procedure which led to the dropping of quadratic terms from the heat and viscous stresses in the expression for the entropy four-vector. This was unsuitable for fast varying systems and therefore, a new theory was required. Furthermore, to analyse the dynamics of fluids in the Eckart theory, linearised equations which govern the evolution of small perturbations about equilibrium are derived. The Eckart theory is then studied. The equations which govern the evolution of the perturbations are shown to be non-hyperbolic [91]. This led to the development of the MIS theory. In this section, we will consider the theory. It is based on the single-fluid formalism.

The properties of the fluids can be described by a number flux denoted by N^a, an energy-momentum tensor denoted by T^{ab} and an entropy flux denoted by S^a. Given an arbitrary reference velocity denoted by u^a, which satifies the condition in (2.17) and the projection tensor denoted by $h^{ab} = g^{ab}+u^a u^b$, the state tensors (or non-arbitrary parameters) denoted by N^a, S^a and T^{ab}, can be decomposed as

$$N^a = \mathbf{n} u^a + n^a, \tag{2.33}$$

$$S^a = s u^a + s^a, \tag{2.34}$$

and

$$T^{ab} = \rho u^a u^b + P h^{ab} + 2u^{(a} q^{b)} + \pi^{ab}. \tag{2.35}$$

Here $\mathbf{n} = -N^a u_a$ represents particle equilibrium density, n^a represents diffusion current (where $n^a u_a = 0$), s represents entropy density, s^a represents entropy flux relative to u^a (such that $s_a u^a = 0$), ρ represents energy density, $\pi^{ba} = \pi^{ab}$ represents anisotropic pressure, q^a represents heat flux vector and P represents aggregate pressure of equilibrium (hereafter p) and bulk viscosity (hereafter Π) pressures. The ρ is given by

$$\rho = u_a u_b T^{ab}. \tag{2.36}$$

The quantities, q^a, π^{ab} and Π are dissipative quantities.

[3]Please also check the appendix $A.1$ for a brief description of the Eckart theory (and why it violates causality) under the heading, *Brief notes on the Eckart theory*.

A vector denoted by $\dot{\mathcal{V}}^a$ is defined as

$$\dot{\mathcal{V}}^a = u^b \nabla_b \mathcal{V}^a. \tag{2.37}$$

The gradient of u^a is decomposed as

$$\nabla_b u_a = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} - \dot{u}_a u_b, \tag{2.38}$$

where $\omega_{ab} = \omega_{[ab]}$ (representing vorticity tensor such that $\omega_{ab}u^b = 0$) can be expressed as

$$\omega_{ab} \equiv 2\nabla_{[a}\mu_{b]} = \nabla_a \mu_b - \nabla_b \mu_a, \tag{2.39}$$

$\sigma_{ab} = \sigma_{(ab)}$ (representing trace-free shear tensor such that $\sigma_{ab}u^b = \sigma_a{}^a = 0$) can be expressed as

$$\sigma_{ab} = \frac{1}{2}(\perp^c{}_b \nabla_c u_a + \perp^c{}_a \nabla_c u_b) - \frac{1}{3} \perp_{ab} \theta. \tag{2.40}$$

$\theta = \nabla_a u^a$ represents expansion scalar, $\mu_b = \mu u_b$ of which μ represents chemical potential, $\perp^a{}_b = \delta^a{}_b + u^a u_b$ (note that $u^a u_a = -1$), $\delta^a{}_b$ represents the Kronecker delta function and $\dot{u}^a$ represents the acceleration vector. Using these definitions, the laws of conservation of N^a, T^{ab}, the second law of thermodynamics, and the law of conservation of S^a can be deduced as follows [93]

$$\nabla_a N^a = \dot{\mathbf{n}} + \mathbf{n}\theta + \nabla_a n^a = 0, \tag{2.41}$$

$$-u_a \nabla_b T^{ab} = \dot{\rho} + (\rho + P)\theta + \dot{u}_a q^a + \nabla_a q^a + \sigma_{ab}\pi^{ab} = 0, \tag{2.42}$$

$$h_{ab}\nabla_c T^{bc} = (\rho + P)\dot{u}_a + h_a{}^b(\nabla_b P + \dot{q}_b + \nabla_c \pi_b{}^c) + \left(\omega_{ab} + \sigma_{ab} + \frac{4}{3}\theta h_{ab}\right) q^b = 0, \tag{2.43}$$

and

$$\nabla_a S^a = \dot{s} + s\theta + \nabla_a s^a \geqslant 0. \tag{2.44}$$

Equations (2.42) and (2.43) imply that the state parameter T^{ab} is subject to the conservation law (1.12). In an equilibrium state, the entropy flux S^a is subject to the conservation law

$$\nabla_a S^a = 0. \tag{2.45}$$

By allowing key parameters such as entropy and T^{ab} to be functions of a broader number of properties, over and above the standard volume and internal energy [90] leads to an appropriate starting point in the formulation of a theory of irreversible thermodynamics. Such an extension can be formalised by giving the properties a generic scalar denoted by f, vector denoted by f^a and tensor denoted by f^{ab}, in which case the entropy flux density denoted by s is $s = s(f, f^a, f^{ab})$. The tensors can be clearly defined as to have the physical meaning discussed in [35,98,99]. They represent both bulk and surface terms. The extended description will then include surface entropy. This is of great significance. A total derivative can then be derived and this leads to

$$ds \equiv \frac{\partial s}{\partial f}df + \frac{\partial s}{\partial f^a}df^a + \frac{\partial s}{\partial f^{ab}}df^{ab}. \tag{2.46}$$

Here the rank two tensor is an extension and leads to a generalised Gibbs relation. More than one of the intrinsic properties can be characterised by a scalar, a vector or a tensor. An example is where heat is a vector, and both internal energy and volume are scalars. Given that $\frac{\partial s}{\partial E} = \frac{1}{T}$ for $T = T(f, f^a, f^{ab})$ representing a non-linear temperature and E representing internal energy, the coefficients in equation (2.46) can be treated normally. By restricting equation (2.46) to scalars denoted by μ and ν, $P(\mu, \nu)$ can represent a thermodynamical potential with

$$\nu dP = \mathbf{n}d\mu - (\rho + P)d\nu, \tag{2.47}$$

and

$$s = (\rho + P)\nu - \mathbf{n}\mu. \tag{2.48}$$

These equations imply the Gibbs relation

$$ds = \nu d\rho - \mu d\mathbf{n}. \tag{2.49}$$

Here $T = \nu^{-1}$ represents the temperature and $\kappa = \frac{\mu}{\nu}$ represents the relativistic chemical potential. It can then be postulated that

$$S^a = su^a + \frac{1}{T}q^a - \mathcal{Q}^a, \tag{2.50}$$

for the standard set of properties. Here $\mathcal{Q}^a$ represents a collection of second-order terms and is expressed explicitly as

$$\mathcal{Q}^a = \frac{u^a}{2T}(\beta_0\Pi^2 + \beta_1 q_a q^a + \beta_2 \pi^{ab}\pi_{ab}) - \frac{1}{T}(\alpha_0 \Pi q^a + \alpha_1 \pi^{ab} q_b + \mathcal{F}). \tag{2.51}$$

Here $\mathcal{F}$ represents a function of energy density, isotropic pressure, energy flux and the symmetric shear tensor. $\beta_0, \beta_1, \beta_2, \alpha_0$ and α_1 represent coefficients. Note that β_0, β_1 and β_2 are dependent on T while α_0 and α_1 are dependent on n, T, s, ρ and p. It follows that the T^{ab} for such non-perfect fluids takes the form in equation (2.35). The possible particle drift leading to the particle flux takes the form N^a in equation (2.33).

2.3 The extended MIS formalism

In this section we consider a formalism for the thermodynamics of relativistic and dissipative systems of multi-fluids [90]. In this regard, we examine the extended MIS theory with the assumption that the MIS formalism is the standard model and its extension must necessarily recover it when subjected to physically motivated constraints or conditions that impact the nature of fluid approximation. It's extension to the multi-fluid theory is considered in [90]. After examining the single-fluid formalism, one can see that several properties that are established in the model suddenly lose clarity [90]. The definitions of a universal temperature [100, 101], entropy and heat are some of the properties when considering thermodynamics. Then this lack of clarity affects the thermodynamic laws. Therefore, this will require scrutiny.

The problem of the definition of the universal temperature was first encountered in the non-equilibrium thermodynamics models in the single-fluid formalism [90]. At the centre of this problem is the seemingly non-existence of a Lorentzian type of transformation between reference frames that readily recover the black-body temperature, provided the necessary constraints [100]. Although some progress has been made regarding this issue [101], a common agreement has not yet been reached. It is known that when solving gravitational field equations, the standard method takes into account the bulk effects and does not consider the surface effects when the definition of entropy is being considered. However, when surface terms are evaluated at the horizon they give the entropy of the horizon [102]. This implies that the body and surface terms are to be included when considering entropy.

The characterisation of work and heat are also not straightforward [103]. Heat can be defined as a property which is endowed with a certain microscopic degree of freedom and is capable of manifesting phenomena that is thermal, and this creates what is known as a micro-structure [90]. One can consider the converse of this and hence ask how the microstructures affect the microproperties in space-time dynamics. It is this idea of micro-structure that motivates the development of the extended MIS theory.

Furthermore, the standard concepts of heat and work are often frame dependent [90]. This implies that the chosen frame of reference will determine the notion of the two concepts. It has been shown that the concept of volume is always[4] frame dependent. This leads to disparities in the estimation of fundamental quantities. This lack of clarity becomes worse when one considers the multi-fluid system. However, one can develop global parameters that are linked to the local frame and allow for ease of physical interpretation. This can be shown in the extended MIS formalism.

In this extension, the species number flux current denoted by ${N^a}_{(\mathcal{I})}$, the entropy flux vector denoted by ${S^a}_{(\mathcal{I})}$ and the stress-energy-momentum tensor denoted by $T^{ab}_{(\mathcal{I})}$ are taken to be the primary extensive parameters where $\mathcal{I} = \mathcal{Y}', \mathcal{Z},\ \mathcal{Y}'\mathcal{Z}$, and $\mathcal{I} = \mathcal{Y}'\mathcal{Z}$ represents non-standard interactions resulting from $\mathcal{Y}'$ interacting with $\mathcal{Z}$ [35]. Note that $\mathcal{Y}'$ and $\mathcal{Z}$ represent dark matter and dark energy, respectively.

We now discuss briefly the conservation properties in a multi-fluid environment. Understanding the conservation properties in a multi-fluid environment requires distinguishing between species interactions with dissipative properties:

1. Without chemical reactions (or interactions) and other non-standard reactions (or interactions) and,
2. With chemical reactions (or interactions) [35].

One can now consider generic rank 2 tensors denoted by $f^{ab}_{\mathcal{Y}'}$ and $f^{ab}_{\mathcal{Z}}$ as tensor properties for the two species $\mathcal{Y}'$ and $\mathcal{Z}$, respectively. For case 1, individual species obey their own cumulative conservation laws. This means that for $f^{ab}_{\mathcal{Y}'}$ and $f^{ab}_{\mathcal{Z}}$, $\nabla_a f^{ab}_{\mathcal{Y}'} = 0$ and $\nabla_a f^{ab}_{\mathcal{Z}} = 0$, respectively. Note that ∇_a represents a covariant derivative. Then

$$\nabla_a(f^{ab}_{\mathcal{Y}'} + f^{ab}_{\mathcal{Z}}) \;=\; \nabla_a f^{ab}_{\mathcal{Y}'} + \nabla_a f^{ab}_{\mathcal{Z}}, \tag{2.52}$$

which leads to

$$\nabla_a(f^{ab}_{\mathcal{Y}'} + f^{ab}_{\mathcal{Z}}) \;=\; 0, \tag{2.53}$$

for case 1. For case 2, which involves interaction of $\mathcal{Y}'$ and $\mathcal{Z}$ (like entrainment)

$$\nabla_a(f^{ab}_{\mathcal{Y}'} + f^{ab}_{\mathcal{Z}} + f^{ab}_{\mathcal{Y}'\mathcal{Z}}) \;=\; 0, \tag{2.54}$$

where

$$\nabla_a(f^{ab}_{\mathcal{Y}'} + f^{ab}_{\mathcal{Z}}) \neq 0, \tag{2.55}$$

[4] There is no unique definition of time and consequently no unique definition of space.

and the last term in equation (2.54) encodes chemical reactions and other non-standard interaction properties (like entrainment) [35].

Considering the interaction components [${N^a}_{\mathcal{Y}'\mathcal{Z}}$ (representing number flux of interacting components), $T^{ab}_{\mathcal{Y}'\mathcal{Z}}$ (representing energy-momentum tensor of interacting components) and ${S^a}_{\mathcal{Y}'\mathcal{Z}}$ (representing entropy flux of interacting components)] in the multi-fluid formulation leads to

$$\nabla_a \sum_{\mathcal{I}} {N^a}_{\mathcal{I}} \;\; = \;\; 0 = \nabla_a \sum_{\mathcal{I}} T^{ab}_{\mathcal{I}},\; \nabla_a \sum_{\mathcal{I}} {S^a}_{\mathcal{I}} > 0, \tag{2.56}$$

such that $\mathcal{I} = \mathcal{Y}', \mathcal{Z}, \mathcal{Y}'\mathcal{Z}$ where $\mathcal{I} = \mathcal{Y}'\mathcal{Z}$ represent non-standard interaction resulting from $\mathcal{Y}'$ interacting with $\mathcal{Z}$. Note that in the equilibrium state

$$\nabla_a \sum_{\mathcal{I}} {S^a}_{\mathcal{I}} \;\; = \;\; 0. \tag{2.57}$$

When two observers move with 4-velocities denoted by ${u^a}_{\mathcal{Y}'}$ and ${u^a}_{\mathcal{Z}}$, there will be non-identical (different) rest-frames and different projections on their respective frames [90]. The projections are represented by

$$h^{ab}_{\mathcal{Y}'} \;\; = \;\; g^{ab}_{\mathcal{Y}'} + {u^a}_{\mathcal{Y}'}{u^b}_{\mathcal{Y}'}, \tag{2.58}$$

and

$$h^{ab}_{\mathcal{Z}} \;\; = \;\; g^{ab}_{\mathcal{Z}} + {u^a}_{\mathcal{Z}}{u^b}_{\mathcal{Z}}, \tag{2.59}$$

with the case[5]

$$g^{ab}_{\mathcal{Y}'} \equiv g^{ab}_{\mathcal{Z}}, \tag{2.60}$$

where $\mathcal{Y}'$ and $\mathcal{Z}$ represent dark matter and dark energy, respectively. Note that either $g^{ab}_{\mathcal{Y}'}$ or $g^{ab}_{\mathcal{Z}}$ represents a metric that is of space-time geometry. One can then consider (or examine) one of the fluids (which is identical to examining the other fluid). We will consider the fluid of dark matter flowing with 4-velocity denoted by ${u^a}_{\mathcal{Y}'}$. The condition (2.17) implies the existence of the projection tensor

$$\mathcal{U}_{ab(\mathcal{Y}')} \;\; = \;\; -u_{a(\mathcal{Y}')}u_{b(\mathcal{Y}')}, \tag{2.61}$$

[5] This is reminiscent of the particle frame and energy frame occupying space with the same geometry.

where $\mathcal{Y}'$ represents dark matter. This obeys the condition

$$\mathcal{U}^a{}_{c(\mathcal{Y}')}\mathcal{U}^c{}_{b(\mathcal{Y}')} = \mathcal{U}^a{}_{b(\mathcal{Y}')}, \tag{2.62}$$

which projects onto the tangent space of the dark matter $\mathcal{Y}'$ worldline. Given that $\hat{u}^a$ and $\hat{u}^a{}_E$ represent 4-velocity of a particle frame and an energy frame, respectively, then demanding that an energy and particle frame of the unified approach satisfy

$$|\hat{u}^a - \hat{u}^a{}_E| \ll 1, \tag{2.63}$$

one can define a resultant four-velocity as

$$\hat{u}^a = f^a(u^a{}_{\mathcal{Y}'}, u^a{}_{\mathcal{Z}}), \tag{2.64}$$

and the corresponding projection tensor as

$$\hat{h}^a{}_b = \hat{g}^a{}_b + \hat{u}^a\hat{u}_b. \tag{2.65}$$

This projects onto the rest frame of the fluid mixture such that

$$\hat{h}^b{}_a\hat{u}_b = 0. \tag{2.66}$$

In a non-relativistic case, we choose the four-velocity $\hat{u}^a$ as ξ-vector densities denoted by $\vec{\rho}$ where $\xi \in$(1, 2, 3, 4,......). Then the requirement of concavity demands that energy density denoted by $\hat{\rho}$, be a concave function of the variables $\vec{\rho}$, meaning $\frac{\partial^2\hat{\rho}}{\partial\vec{\rho}\partial\vec{\rho}}$ ∼negative definite. In the relativistic case we choose the four-velocity $\hat{u}^a$ as the densities denoted by $\rho_\xi = \rho^a\xi_a$ in a generic Lorentz frame that moves with the four-velocity denoted by $c\xi_a$ (c represents the speed of light) with respect to the observer. Note that $\xi^a\xi_a = 1$, $\xi^0 > 0$ and ρ^a represent ξ four-fluxes. We cannot be certain that in all these frames energy density is concave as a function of ρ_ξ. Hence, we assume that there is at least one ξ_a denoted by $\xi'{}_a$ such that $\hat{\rho}_{\xi'}$ is concave with respect to $\rho_{\xi'} = \rho^a\xi_a$, meaning $\frac{\partial^2\hat{\rho}_{\xi'}}{\partial\rho_{\xi'}\partial\rho_{\xi'}}$ ∼negative definite. The co-vector denoted by $\xi'{}_a$ can then be chosen in such a way that the concavity of $\hat{\rho}_{\xi'}$ implies symmetric hyperbolicity of the field equations denoted by $\rho_{\xi'}$ [104].

Now, assuming that the observer with $\hat{u}^a$ is not accelerating, the velocity fields are chosen in such a way that they satisfy the concavity requirement [104]. Given the 4-velocity $\hat{u}^a$, the observer with this velocity will record the $\hat{\rho}$ and the particle flux denoted by

$$N^d = f^d(N^d_{\mathcal{Y}'}, N^d_{\mathcal{Z}}). \tag{2.67}$$

It follows that the T^{ab} can be expressed as

$$T^{ab} \;=\; \sum_{\mathcal{I}} T^{ab}_{\mathcal{I}}. \tag{2.68}$$

Then, the decompositions below are possible

$$\begin{aligned} N^a{}_{(\mathcal{I})} &= \mathbf{n}_{(\mathcal{I})}u^a + n^a{}_{(\mathcal{I})}, \\ T^{ab}_{(\mathcal{I})} &= \rho_{(\mathcal{I})}u^a u^b + p_{(\mathcal{I})}h^{ab} + 2u^{(a}{}_{(\mathcal{I})}q^{b)}{}_{(\mathcal{I})} + \pi^{ab}{}_{(\mathcal{I})}. \end{aligned} \tag{2.69}$$

In the decomposition of $N^a{}_{\mathcal{I}}$, one will have $\mathbf{n}_{(\mathcal{I})}$ and $n^a{}_{\mathcal{I}}$ which represent species number density and diffusion current, respectively while in the decomposition of $T^{ab}_{\mathcal{I}}$, one will have $h^{ab} = g^{ab} + u^a u^b$, $\rho_{(\mathcal{I})} = u_a u_b T^{ba}$, $p_{(\mathcal{I})}$, $\pi^{ba}_{\mathcal{I}}$ and $q^a{}_{(\mathcal{I})}$ which represent projection tensor, energy density, isotropic pressure, anisotropic pressure and heat flux vector, respectively. h^{ab} represents a projection tensor to a rest-frame of the various fluid species. It can only be used when species separate as shown in equations (2.58) and (2.59). In comparisons of either energy and particle frames in either reference [70] or rest and boosted frames in reference [105] of the single-fluid formalism, different velocities have been considered.

Thermodynamics and statistical mechanics offer two formal definitions for entropy [90]. As we are interested in fluid dynamics theories in which components of the fluids are assumed to interact thermodynamically, we will consider the thermodynamic viewpoint. Then for a system that is composed of constituents, in classical thermodynamics theory, the state of the system is found by taking the averages of thermodynamic properties of the constituents. This would mean observing the cumulative behaviour. The initial development of the concept considered such averages for a system that was in thermodynamical equilibrium via statistical mechanics. The latter development of the theory considered extending the theory by incorporating aspects that allowed for non-equilibrium thermodynamics via the kinetic theory. The treatment of statistical thermodynamics [106] is based on assumptions that are given in terms of the behaviour of simple systems. The systems are microscopically homogeneous, isotropic and devoid of electric charge, chemical reactions and electric force fields or surface effects. In a multi-fluid system where some of the properties just mentioned before cannot be ignored, it is essential that one goes beyond the simple system assumptions. Therefore, quadratic terms are incorporated [70, 93, 104] in the heat flux and viscous stresses in the expression for the entropy four-vector, thus yielding a generalised theory that is able to describe brief non-equilibrium thermodynamics and satisfies the causality condition. Then the entropy vector ${S^a}_{(\mathcal{I})}$ can take the form

$$ {S^a}_{(\mathcal{I})} = s_{(\mathcal{I})}u^a + {s^a}_{(\mathcal{I})}, \tag{2.70} $$

where $s_{(\mathcal{I})}$ and ${s^a}_{(\mathcal{I})}$ represent entropy density and entropy flux, respectively. Equation (2.70) in an explicit form can be expressed as

$$ \begin{aligned} {S^a}_{(\mathcal{I})} &= s_{(\mathcal{I})}{u^a}_{(\mathcal{I})} + \frac{q^a}{T} - (\beta_{(\mathcal{I})}\Pi^2 + \beta_{1(\mathcal{I})}q_{(\mathcal{I})b}{q^b}_{(\mathcal{I})} + \beta_{2(\mathcal{I})}\pi_{(\mathcal{I})cd}{\pi^{cd}}_{(\mathcal{I})})\frac{u^a}{2T} \\ &+ (\alpha_{0(\mathcal{I})}\Pi {q^a}_{(\mathcal{I})} + \alpha_{1(\mathcal{I})}{\pi^{ab}}_{(\mathcal{I})}q_{(\mathcal{I})b})\frac{1}{T}, \end{aligned} \tag{2.71} $$

where $s_{(\mathcal{I})}$ represents entropy density and ${s^a}_{\mathcal{I}}$ represents entropy flux with respect to u^a where $s_{a(\mathcal{I})}u^a = 0$ (Note that $\mathcal{I} = \mathcal{Y}', \mathcal{Z}, \mathcal{Y}'\mathcal{Z}$ where $\mathcal{I} = \mathcal{Y}'\mathcal{Z}$ represents non-standard interaction resulting from $\mathcal{Y}'$ interacting with $\mathcal{Z}$). $\Pi_{(\mathcal{I})}$ represents bulk viscosity and $\beta_{(\mathcal{I})}, \beta_{1(\mathcal{I})}, \beta_{2(\mathcal{I})}, \alpha_{0(\mathcal{I})}$, and $\alpha_{1(\mathcal{I})}$ are coefficients that are simply the generalised case of the counterparts in the standard MIS formalism. Though complex detailed interactions are very difficult to measure, they are tractable as shown in [90].

The total entropy vector S^a can then take the phenomenological form [35]

$$ S^a = {S^a}_{(\mathcal{I})} + \bar{S}^a. \tag{2.72} $$

The last term with a bar in the above equation (2.72) represents interaction effect. It is of importance to reflect on the dynamics and changes that take place in this approximation. One can assume that there is a gradual change that sees the terms with the bars being important at time t_{qe} [where q and e each represent a (different) species] to having no effect at time $t > t_{qe}$. This means that transiting to truly separate fluids requires a full multi-fluid approximation.

2.4 Chapter summary

We have reviewed relativistic fluids, the (standard) MIS and the extended MIS theories. We have examined the reasons why the theory of relativistic fluids was developed. We have examined the mathematical details of the theory. We have looked at the material and covariant derivatives where one has to find a derivative of a given point for a vector field. We have examined the relativistic Euler equations which is a system of equations used to solve for fluid flow. We have examined thermodynamics and the single-fluid model. Its Lagrangian is presented in terms of n and m. We have examined both standard and extended MIS theories. For the (standard) MIS formalism, a relation for the entropy flux S^a is postulated.

Similarly, for the extended MIS theory, a relation for the entropy vector ${S^a}_{\mathcal{I}}$ is postulated.

Chapter 3

Variational formalisms for relativistic fluids

3.1 Variational principle

The principles of the variational formalism have a long and distinguished history in physics. Apart from global derivations of physical principles, equivalent to local differential equations, they are useful for approximating problems too difficult for analytical solutions. In recent years, there has been some progress in using classical variational action principles to approximate the motion of classical systems. This demands for a review of the recent developments in this area of classical mechanics. In classical mechanics, variational principles are usually referred to as least action principles [107]. This is because the quantity subject to variations is the action. We will consider the calculus of variations as it underlies the variational principle.

The integral of the variational principle was first championed by Gottfried Leibniz. During the 18^{th} century, Bernoulli, a student of Leibniz at the time, formulated the field of variational calculus [107]. It underlies the integral variational formalism to mechanics. Bernoulli solved the brachistochrone puzzle which involves finding the path for which the transit time between two points is the shortest. Later on during the 18^{th} century, the pre-eminent Swiss mathematician and a student of Bernoulli, developed the calculus of variations with full mathematical rigor. A student of Euler by the name of Lagrange $(1736 - 1813)$ culminated the development of the Lagrangian variational approach to classical mechanics. The Euler-Lagrange formalism to classical mechanics stems from a deep philosophical belief that the laws of nature are based on a principle of economy. This implies that the physical universe follows paths through space and time that are based on extrema principles. This brings one to the Hamilton's least action principle which states that, for a true trajectory of a system, Hamilton's action denoted by $\mathcal{A}$ is stationary for trajectories which run from the fixed initial space-time point denoted by q_i point to the fixed final space-time point denoted by q_f such that

$$(\delta \mathcal{A})_{\triangle t} = 0, \tag{3.1}$$

where $\triangle t = t_2 - t_1$ and t_1 represents initial time while t_2 represents final time.

One can illustrate this by considering motion in one direction. The illustration involves a simple physics problem of a point particle which has always served as a guide to deep principles used in much more complex problems in physics [73]. An action most suitable for the free point particle can be expressed as [73]

$$\mathcal{A} = \int_{t_1}^{t_2} \mathcal{T} dt, \tag{3.2}$$

where $\mathcal{T}$ represents kinetic energy, t represents time, t_1 and t_2 represent initial and final time, respectively. Kinetic energy can be expressed as

$$\mathcal{T} = \frac{m[\dot{x}(t)]^2}{2}, \tag{3.3}$$

where m represents mass, $x(t)$ represents path taken by the point particle and $\dot{x}$ represents speed of the point particle. Hamilton's principle states that the actual path taken by the point particle is the one for which the action in equation (3.2) is stationary. Then the Euler-Lagrange equations given by

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{x}} - \frac{\partial \mathcal{T}}{\partial x} = 0, \tag{3.4}$$

lead to

$$\frac{d}{dt}(m\dot{x}) = m\ddot{x} = 0. \tag{3.5}$$

Alternatively, a variation of $\mathcal{A}$ with respect to $x(t)$ leads to

$$\delta \mathcal{A} = -\int_{t_1}^{t_2} dt(m\ddot{x})\delta x + (m\dot{x}\delta x)|_{t_1}^{t_2}. \tag{3.6}$$

In the absence of forces on the particle, d'Alembert's principle of least action can be applied [73, 108]. This principle states that the paths that make the $\mathcal{A}$ stationary (implying $\delta\mathcal{A} = 0$) are those that yield the true motion. One can see from the varied action that the $x(t)$ that satisfy the boundary conditions

$$\delta x(t_1) \quad = \quad 0 = \delta x(t_2), \tag{3.7}$$

and the equation of motion (3.5) lead to

$$\delta \mathcal{A} \quad = \quad 0. \tag{3.8}$$

This same principle (or reasoning) can be applied to much more complex problems in physics.

The standard Lagrangian denoted by $\mathcal{L}$ is defined as the difference between the $\mathcal{T}$ and potential energy. This means that

$$\mathcal{L}(x, \dot{x}; t) \quad = \quad \mathcal{T} - \mathcal{V}(x, t), \tag{3.9}$$

where $\mathcal{V}(x, t)$ represents potential energy. The laws of classical mechanics can be expressed in terms of the Hamiltonian variational principle. This principle states that the motion of the system between the t_1 and the t_2 follows a path that minimises the $\mathcal{A}$, and defined as the time integral of the Lagrangian. This is expressed as

$$\mathcal{A}(x) \quad = \quad \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}; t) dt. \tag{3.10}$$

This variational formalism is both elegant and beautiful. It provides the mathematics required to determine the path that minimises the action integral. It has withstood the rigors of experimental confirmation. It is an exceedingly powerful alternative formalism to the intuitive Newtonian formalism to mechanics, and it is also recognised to be more fundamental than Newton's laws of motion. The variational formalism to mechanics is the only formalism that can handle the theory of relativity. One can use the principle to derive equations of motion and stress-energy-momentum tensors denoted by $T^a_{\ b}s$. This formalism differs from the usual text-book derivation of the equations of motion from the divergence of the $T^a_{\ b}$, in that one clearly obtains the relativistic Euler equation as an integrability condition on the relativistic vorticity [73]. This means that the results obtained depend on the whole path, and not just the initial and final points. That is, it is different from a total derivative which can be integrated, and thus depends on only the lower and upper limits of the integration. This implies that results obtained using the variational formalism should be more accurate than those that depend on the initial and final points only.

We consider equation (3.10). Hamilton's principle states that the actual path taken by a point particle which is subjected to the conservative forces $\mathcal{V}(x, t)$ is the one for which the action in equation (3.10) is stationary [107]. Therefore, the Euler-Lagrange equations for the action given by

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}, \tag{3.11}$$

yields

$$\frac{d}{dt}(m\dot{x}) + \frac{\partial \mathcal{V}}{\partial x} = 0. \tag{3.12}$$

This can be rewritten as

$$m\ddot{x} + \frac{\partial \mathcal{V}}{\partial x} = 0, \tag{3.13}$$

or

$$m\ddot{x} - \mathcal{F}_x = 0, \tag{3.14}$$

where $\mathcal{F}_x = -\frac{\partial \mathcal{V}}{\partial x}$. In other words, for boundary conditions that make the boundary term in $\delta\mathcal{A}$ zero, Hamilton's principle suggests Newton's second law

$$\mathcal{F}_x = m\ddot{x}, \tag{3.15}$$

where

$$\mathcal{F}_x = -\frac{\partial \mathcal{V}(x,t)}{\partial x}. \tag{3.16}$$

Notice that the potential energy is allowed to be time-dependent. However, if the potential energy is time-dependent, then the Lagrangian will have no explicit dependence on time t. This means that there is a first-integral of the Euler-Lagrange equations.

One can work on a system that requires the incorporation of all other forces (non-conservative and/or external forces) other than the conservative ones. Note that these forces (to be incorporated) cannot be put in action principles. However, from Newton's second law, it is possible to have the expression

$$m\ddot{x} + \frac{\partial \mathcal{V}}{\partial x} = \mathcal{F}_{\mathcal{T}}, \tag{3.17}$$

where $\mathcal{F}_{\mathcal{T}}$ represents all other forces, and both conservative and all other forces act. Analysis of equation (3.17) shows that the kinetic and conservative forces which enter the left-hand side of equation (3.17) follow from the Euler-Lagrange equations for the action in equation

(3.10). This implies that

$$\frac{\delta \mathcal{A}}{\delta x} = -\left(m\ddot{x} + \frac{\partial \mathcal{V}}{\partial x}\right). \tag{3.18}$$

Then in the absence of all other forces $\mathcal{F}_{\mathcal{T}}$, the action principle leads to equations of motion. In the presence of all other forces $\mathcal{F}_{\mathcal{T}}$, the action defines the kinetic and conservative force terms that are to be balanced by all other forces $\mathcal{F}_{\mathcal{T}}$. This also defines momentum. Note that the main effect of all other forces $\mathcal{F}_{\mathcal{T}}$ can be to draw away kinetic energy from the system [73]. One can now discuss the more complex variational formalisms which are relevant to our work.

3.2 The pull-back approach for a single-fluid model

We consider an embedding

$$\Phi : \Sigma \to \mathcal{M}, \tag{3.19}$$

where Σ represents a hyper-surface and $\mathcal{M}$ represents an ambient space or manifold. Then a pull-back refers to the operation of restricting (or pulling back) tensors on a manifold denoted by $\mathcal{M}$ to those on Σ [109]. The simplest of this kind of formalism is the pull-back of a scalar function denoted by f on $\mathcal{M}$ such that

$$f : \mathcal{M} \to \mathcal{R}, \tag{3.20}$$

and $\mathcal{M}$ to a function on Σ where $\mathcal{R}$ represents a set of real numbers. The restriction to Σ defines a scalar on Σ such that

$$f|_{\Sigma} : \Sigma \to \mathcal{R}. \tag{3.21}$$

When one is considering the embedding Φ, this can be phrased as the statement that the embedding map Φ can be used to pull-back the function f on $\mathcal{M}$ to a function $\Phi^* f$ on Σ defined by

$$\Phi^* f : \Sigma \to \mathcal{R}, \tag{3.22}$$

and

$$(\Phi^* f)(y) = f[\Phi(y)], \tag{3.23}$$

where y represents a scalar.

One can now consider vectors and co-vectors. When one considers co-vectors as linear functions on vectors, then upon restriction of a co-vector field on $\mathcal{M}$ to Σ, one obtains a co-vector field on Σ [109]. This is possible because its action on any vector at $x \in \Sigma \subset \mathcal{M}$ where x represents a scalar, is well-defined. Particularly, its the action on vectors tangent to Σ and this is all that is required to make the action a well-defined co-vector on Σ. For equations this implies that if $\mathcal{U}_a$ is a co-vector field on $\mathcal{M}$, then it can be pulled-back to a co-vector field u_b on Σ via

$$u_b = (\Phi^* \mathcal{U})_b = \frac{\partial x^a}{\partial y^b} \mathcal{U}_a, \tag{3.24}$$

where a and b represent space-time indices. This can be expressed as

$$u_b = \mathcal{E}^a{}_b \mathcal{U}_a, \tag{3.25}$$

where

$$\mathcal{E}^a{}_b = \frac{\partial x^a}{\partial y^b}. \tag{3.26}$$

One can see that this is a co-vector field on Σ. This formulation can also be understood in terms of the differentials denoted by dx^a and the pull-back of the generally covariant object $\mathcal{U}_a dx^a$. One can pull-back dx^a to Σ leading to

$$\mathcal{U}_a dx^a|_\Sigma = \mathcal{U}_a \mathcal{E}^a{}_b dy^b = u_b dy^b. \tag{3.27}$$

In the same way one can restrict higher-rank covariant tensor fields denoted by $\mathcal{U}_{a.....c}$ on $\mathcal{M}$ to Σ such that

$$(\Phi^* \mathcal{U})_{d.....f} = \mathcal{E}^a{}_d .....\mathcal{E}^c{}_f \mathcal{U}_{a.....c}. \tag{3.28}$$

This restriction formalism [110–112] is used to construct a Lagrangian displacement of the number density flux current denoted by $n^a{}_X$ where a represents a space-time index and X represents the single-fluid approximation being considered, that is X represents the radiation-dominated epoch or the matter-dominated epoch. Let $n^X{}_{bcd}$ represent a three-form that is dual to the $n^a{}_X$ such that

$$n^X{}_{bcd} = \epsilon_{bcda} n^a{}_X, \ n^a{}_X = \frac{1}{3!} \epsilon^{abcd} n^X{}_{bcd}, \tag{3.29}$$

and

$$n^2{}_X = \frac{1}{3!} n^X{}_{bcd} n^{bcd}{}_X. \tag{3.30}$$

If the convention for transforming between the two dual forms is

$$\epsilon^{abcd}\epsilon_{eabc} = 3!\delta^d{}_e, \tag{3.31}$$

then one can use a well defined restriction; $\mathcal{C}^A{}_X$. It pulls n^X_{bcd} into the matter space where it takes the identity n^X_{BCD} $(B, C, D, etc = 1, 2, 3)$. This implies that

$$n^X{}_{bcd} = \frac{\partial \mathcal{C}^{[B}{}_X}{\partial x^b} \frac{\partial \mathcal{C}^C{}_X}{\partial x^c} \frac{\partial \mathcal{C}^{D]}{}_X}{\partial x^d} n^X{}_{BCD}. \tag{3.32}$$

Similarly, this can be done for the chemical potential denoted by μ^{bcd}_X, where in this case $\mathcal{C}^A{}_X$ represents a push-forward. $\mathcal{C}^A{}_X$ is Lie-dragged along individual fluid world-lines leading to its conservation. Particularly

$$\frac{d\mathcal{C}^A{}_X}{d\tau_X} = u^a{}_X \nabla_a \mathcal{C}^A{}_X = 0, \tag{3.33}$$

where τ_X represents proper time. $\mathcal{C}^A{}_X$ is an unconstrained scalar. It can be subjected to the variational principle with the hope of obtaining field equations for the fluxes. One can then use the Lagrangian displacement denoted by $\xi^a{}_X$ [73] to link variations of matter space variables to space-time variables. A relativistic Lagrangian variation which is associated with the $\xi^a{}_X$ can be defined as

$$\triangle_X \equiv \delta + \mathcal{L}_{\xi_X}, \tag{3.34}$$

where the first term on the right-hand side of this (3.34) represents Euler's variation and the second term is the Lie derivative. In terms of this variation

$$\triangle_X \mathcal{C}^A{}_X = \delta \mathcal{C}^A{}_X + \mathcal{L}_{\xi_X} \mathcal{C}^A{}_X = 0. \tag{3.35}$$

This means that

$$\delta \mathcal{C}^A{}_X = -(\nabla_a \mathcal{C}^A{}_X) \xi^a{}_X. \tag{3.36}$$

Using this, one can show that

$$\triangle_X n^X{}_{bcd} = 0. \tag{3.37}$$

This implies that

$$\delta n^a{}_X = -\mathcal{L}_{\xi_X} n^a{}_X - n^a{}_X \left(\nabla_e \xi^e{}_X + \frac{1}{2} g^{ef} \delta g_{ef} \right). \tag{3.38}$$

This constraint guarantees that flux $n^a{}_X$ is conserved in the equations of motion. This is because the variations of the matter Lagrangians are (or can be) expressed in terms of the $\xi^a{}_X$.

3.3 The convective variational approach

We consider the convective variational formalism (as it utilises as much as possible the tools of the trade of relativistic fields). This formalism was constructed in order to incorporate dissipative effects (reactivity for example) in a relativistic fluid description [73]. After introduction of additional dynamical fields, the formalism respects causality and it does not yield serious stability problems. Since it is based on making maximal use of variational principle arguments, the construction of this formalism is less phenomenological. Owing to its extreme generality, the construction is useful in more complicated cases like for example, in multi-fluid dynamics. Its technical mathematical approach is very good and it draws attention to the importance of new kinds of variables. Its mathematics is also convenient, elegant and physically intuitive. It has no essential physical limitations relative to the formalisms before it. It unites the physical adaptability of the formalisms before it with at least some of the mathematical advantages of the divergence approach and other formalisms. Additional parameters required for this formalism are in principle calculable. However, they are quite difficult (in practise) to extract from arguments of micro-physics. Despite this, it is still worth considering as discussed above.

As mentioned before, the convective variational formalism was developed to cater for non-conservative media [113]. In this formalism, a single scalar function denoted by Λ say of relevant variables plays a crucial role. This quantity is known as the master function. It generalises the role of a Lagrangian for a conservative model, of the kind whose prototype is a standard Taub mass density function for a perfect fluid [114,115]. The master function Λ in explicit form is $\Lambda = \Lambda(n^a{}_\mathcal{I}, g_{ab})$ where $n^a{}_\mathcal{I}$ represents flux, a and b represent space-time indices, $\mathcal{I}$ represents a constituent index and g_{ab} represents the space-time metric. As mentioned before, it is the key quantity in the convective variational framework. Its most fundamental role is to determine a set of dynamically conjugate variables which implies a set of effective variables. Then $\mu^\mathcal{I}{}_a$ represent momenta that are conjugate to the fluxes $n^a{}_\mathcal{I}$. Given that the momenta $\mu^\mathcal{I}{}_a$ are related to the constituent $\mathcal{I}$ and $n^a{}_\mathcal{I} = n_\mathcal{I} u^a{}_\mathcal{I}$ where $u^a{}_\mathcal{I}$ represents four-velocity and a is a space-time index, then

$$\delta\Lambda = \sum_I \mu^I{}_a \delta n^a{}_I + \frac{\partial \Lambda}{\partial g^{ab}} \delta g^{ab}. \tag{3.39}$$

Then, according to the principle of least action, one will require that the action be extremal for arbitrary variations. This leads to equations of motion and stress-energy-momentum tensors $T^a_b s$. Now, Einstein field equations are expressed as (stated before)

$$\mathcal{G}^a{}_b = T^a{}_b, \tag{3.40}$$

where (stated before)

$$\mathcal{G}^a{}_b = R^a{}_b - \frac{1}{2} R g^a{}_b. \tag{3.41}$$

The Einstein field equations' or (or tensor's) covariant divergence vanishes identically meaning (stated before)

$$\nabla_a \mathcal{G}^a{}_b = 0. \tag{3.42}$$

This implies that

$$\nabla_a T^a{}_b = 0, \tag{3.43}$$

which is the conservation of the T^a_b (as stated before). We consider an Euler relation

$$\frac{E}{V} = \mu \frac{N}{V} - p + T \frac{S}{V}, \tag{3.44}$$

[44] where E represents internal energy, N represents number of particles, V represents volume, S represents entropy, μ represents chemical potential, p represents pressure and T represents temperature. One can let $\rho_{\mathcal{I}} = \frac{E}{V}$ represent total energy density, $s = \frac{S}{V}$ represent total entropy density and $n^{\mathcal{I}} = \frac{N}{V}$ represent total particle number density. The Euler relation can then be written in the form

$$\rho_{\mathcal{I}} + p = \sum_I \mu^I n^I + Ts. \tag{3.45}$$

One can now consider

$$d\rho_{\mathcal{I}} = \frac{\partial \rho_{\mathcal{I}}}{\partial n_{\mathcal{I}}} dn_{\mathcal{I}}. \tag{3.46}$$

Then from equation (3.45)

$$\frac{\partial \rho_{\mathcal{I}}}{\partial n_{\mathcal{I}}} = \mu^{\mathcal{I}}, \tag{3.47}$$

which implies

$$d\rho_{\mathcal{I}} = \mu^{\mathcal{I}} dn_{\mathcal{I}}. \tag{3.48}$$

Using the definitions $n^a{}_{\mathcal{I}} = n_{\mathcal{I}} u^a{}_{\mathcal{I}}$ and $\mu^{\mathcal{I}}{}_a = \mu^{\mathcal{I}} u^{\mathcal{I}}{}_a$ with the condition $u_a du^a = 0$ (because $u^a{}_{\mathcal{I}} u^{\mathcal{I}}{}_a = -1$), leads to

$$-d\rho_{\mathcal{I}} = -\mu^{\mathcal{I}} dn_{\mathcal{I}} = u^a{}_{\mathcal{I}} u^{\mathcal{I}}{}_a \mu^{\mathcal{I}} dn_{\mathcal{I}} = \mu^{\mathcal{I}}{}_a dn^a{}_{\mathcal{I}}. \tag{3.49}$$

Then

$$d\rho_{\mathcal{I}} = -\mu^{I}{}_a dn^a{}_{I}. \tag{3.50}$$

A variationally defined T^a_b is given by

$$T^a_b = \psi g^a{}_b + \sum_{\mathcal{I}} n^a{}_{\mathcal{I}} \mu^{\mathcal{I}}{}_b, \tag{3.51}$$

where

$$\psi = \Lambda + \sum_{\mathcal{I}} h_{\mathcal{I}}, \tag{3.52}$$

and

$$h_{\mathcal{I}} = -\mu^{\mathcal{I}}{}_a n^a_{\mathcal{I}}. \tag{3.53}$$

The crucial Noether identity takes the form

$$\nabla_a T^a_b = \sum_{\mathcal{I}} \mathcal{F}^{\mathcal{I}}_b, \tag{3.54}$$

where $\mathcal{F}^{\mathcal{I}}_b$ represents the generalised force density. ψ represents a generalised pressure function (which is the dynamical conjugate of Λ) and $h_{\mathcal{I}}$ represent simple contractions for enthalpy contributions. The identity in equation (3.54) enables the condition for a fluid medium to be free from external influence, as expressed by the covariant conservation (which in a curved space-time background is strictly only a pseudo-conservation law), requiring the vanishing of the $\nabla_a T^a_b$ be re-formulated as a Newtonian-type force balance equation. $\mathcal{F}^{\mathcal{I}}_b$

represents the generalised force density which needs to be worked out [including equation (3.51)] in the following paragraphs. It is associated with the contribution in the momenta ${\mu^{\mathcal{I}}}_a$ and it is also related to the divergence of the variationally defined T^a_b [as shown in equation (3.54)] in equation (3.51).

The Noether identity in the preceding paragraph is based on the postulate that the algebraic functional dependence of Λ be fully covariant [113]. This implies that apart from the independent matter field variable ${n^a}_{\mathcal{I}}$, the Λ should depend only on the symmetric metric g_{ab} that is used for raising and lowering space-time indices. A fully covariant formalism is used as it is mathematically useful to consider the effect of arbitrary variations subject only to the symmetry condition

$$g_{ab} \;\; = \;\; g_{(ab)}. \tag{3.55}$$

Given the conditions above, the most general infinitesimal variation of the Λ that can be conceived will have the form of equation (3.39). For this case, ${n^a}_{\mathcal{I}}$ represents a set of vectors which in turn represent diverse currents of entropy and whatever kinds of neutral or charged (not necessarily conserved) particles with the specification of the partial derivatives completed in view of equation (3.55) by the appropriate symmetry condition

$$\frac{\delta\Lambda}{\delta g_{ab}} \;\; = \;\; \frac{\delta\Lambda}{\delta g_{ba}}. \tag{3.56}$$

A variation of the master function Λ can be shown to be

$$\delta\Lambda \;\; = \;\; \xi\mathcal{L}\Lambda = \mathcal{L}_\xi\Lambda = \xi^a\nabla_a\Lambda, \tag{3.57}$$

where ξ^a represents an arbitrary infinitesimal displacement vector field. The infinitesimal variations of the variables appearing in equation (3.39) are given by the corresponding Lie derivatives and these are

$$\delta {n^a}_{\mathcal{I}} \;\; = \;\; \xi\mathcal{L}{n^a}_{\mathcal{I}} = \mathcal{L}_\xi {n^a}_{\mathcal{I}} = \xi^b\nabla_b {n^a}_{\mathcal{I}} - {n^b}_{\mathcal{I}}\nabla_b\xi^a, \tag{3.58}$$

and

$$\delta g_{ab} \;\; = \;\; \xi\mathcal{L}g_{ab} = \mathcal{L}_\xi g_{ab} = 2\left[\frac{1}{2}(\nabla_a\xi_b + \nabla_b\xi_a)\right] = 2\nabla_{(a}\xi_{b)}. \tag{3.59}$$

One can then substitute equations (3.57), (3.58) and (3.59) in equation (3.39). This leads to

$$\xi^a \nabla_a \Lambda = \sum_{\mathcal{I}} \mu^{\mathcal{I}}{}_a (\xi^b \nabla_b n^a{}_{\mathcal{I}} - n^b{}_{\mathcal{I}} \nabla_b \xi^a) + \frac{\partial \Lambda}{\partial g_{ba}} \left\{ 2 \left[\frac{1}{2} (\nabla_b \xi_a + \nabla_a \xi_b) \right] \right\}, \tag{3.60}$$

where ∇_a represents the operator of covariant differentiation with respect to the metric g_{ab}. Note that equation (3.60) pulls out the symmetric pieces because the metric is symmetric. Then equation (3.60) can be re-written as

$$\xi^a \nabla_a \Lambda - \xi^b (\nabla_b n^a{}_{\mathcal{I}}) \sum_{\mathcal{I}} \mu^{\mathcal{I}}{}_a = -\sum_{\mathcal{I}} \mu^{\mathcal{I}}{}_a n^b{}_{\mathcal{I}} (\nabla_b \xi^a) + 2 \frac{\partial \Lambda}{\partial g_{ba}} \nabla_a \xi_b. \tag{3.61}$$

For the second term on the right-hand side of equation (3.61), the indices are swapped. The first term on the right-hand side of (3.61) is also considered. a is raised and lowered on $\mu^{\mathcal{I}}{}_a$ and ξ^a, respectively without losing generality. Using a symmetry condition denoted by $\nabla_a \xi_b = \nabla_b \xi_a$ and multiplying throughout by -1 yields

$$(-\nabla_a \Lambda + \sum_{\mathcal{I}} \mu^{\mathcal{I}}{}_b \nabla_a n^b{}_{\mathcal{I}}) \xi^a = \left(\sum_{\mathcal{I}} \mu^{\mathcal{I}a} n^b{}_{\mathcal{I}} - 2 \frac{\partial \Lambda}{\partial g_{ba}} \right) \nabla_a \xi_b, \tag{3.62}$$

such that

$$\left(\sum_{\mathcal{I}} \mu^a{}_{\mathcal{I}} n^b{}_{\mathcal{I}} - 2 \frac{\partial \Lambda}{\partial g_{ba}} \right) \nabla_a \xi_b = (\sum_{\mathcal{I}} \mu^{\mathcal{I}}{}_b \nabla_a n^b{}_{\mathcal{I}} - \nabla_a \Lambda) \xi^a. \tag{3.63}$$

The coefficients of $\nabla_a \xi_b$ and ξ^a vanish if $\nabla_a \xi_b$ and ξ^a are abitrary. This yields the Noether identities below

$$\nabla_a \Lambda = \sum_{\mathcal{I}} \mu^{\mathcal{I}}{}_b \nabla_a n^b{}_{\mathcal{I}}, \tag{3.64}$$

jointly with a less trivial relation

$$\frac{2 \partial \Lambda}{\partial g_{ba}} = \sum_{\mathcal{I}} \mu^{\mathcal{I}a} n^b{}_{\mathcal{I}}. \tag{3.65}$$

For both equations (3.64) and (3.65), the symmetry on the left-hand side implies a corresponding identical symmetry on the right-hand side. This means

$$\sum_{\mathcal{I}} \mu^{\mathcal{I}[a} n^{b]}{}_{\mathcal{I}} \equiv 0. \tag{3.66}$$

The construction of a covariant master function $\Lambda(n^a{}_{\mathcal{I}}, g_{ab})$ determines a set of canonically associated generalised momentum tensors $\mu^{\mathcal{I}}{}_a$ [113]. It can also be used to specify a corresponding set of generalised force density $\mathcal{F}^{\mathcal{I}}{}_a$ jointly with a canonically associated T^{ab}. The $\nabla_b T^b{}_a$ will be equal to the force $\sum_{\mathcal{I}} \mathcal{F}^{\mathcal{I}}{}_a$ in accordance with the Noether identity (3.54). With fluid world-line variational principles, one can consider the restriction of the general variation (3.39) of the Λ that is generated by a corresponding set of arbitrary displacement vectors $\xi^a{}_{\mathcal{I}}$ acting on the contravariant tensor densities that are metricly associated with the corresponding independent variables. In this case, the variable is $||g||^{\frac{1}{2}} n^a{}_{\mathcal{I}}$. At the same time the effect of an arbitrary Eulerian metric variation is

$$\delta g_{ab} \;=\; h_{ab},\ \delta||g||^{\frac{1}{2}} = \frac{1}{2}||g||^{\frac{1}{2}} h^a{}_a. \tag{3.67}$$

Evaluation of the resulting variation of the corresponding purely tensorial quantities can be expressed as

$$\delta n^a{}_{\mathcal{I}} \;=\; \xi_{\mathcal{I}} \mathcal{L} n^a{}_{\mathcal{I}} + n^a{}_{\mathcal{I}} \left(\nabla_b \xi^b{}_{\mathcal{I}} - \frac{1}{2} h^b{}_b \right). \tag{3.68}$$

The required force density $\mathcal{F}^{\mathcal{I}}_a$and the relevant T^{ab} can be derived by writing the resulting variation in the scalar density denoted by $\Lambda||g||^{\frac{1}{2}}$ in the form

$$||g||^{-\frac{1}{2}} \delta(||g||^{\frac{1}{2}}\Lambda) + \nabla_a R^a \equiv \sum_{\mathcal{I}} \mathcal{F}^{\mathcal{I}}{}_a \xi^a{}_{\mathcal{I}} + \frac{1}{2} T^{ab} h_{ab}. \tag{3.69}$$

The variation of the integral of Λ does not receive any contributions from the remainder denoted by R^a. This is because $\delta n^a{}_{\mathcal{I}} = \delta g^{ab} = 0$.

One can now examine equation (3.54) to see if it still holds as an identity in the general case independently of any field equations that might be imposed [113]. The volume integral of the variation given by equation (3.69) vanishes if all the displacement vector fields are taken to have a common value ξ^a and if the metric variation is generated [according to equation (3.59)] by the same displacement whose net result is then effectively just that of a mere coordinate transformation in which case equation (3.69) reduces to an identity relation of the simple form

$$\nabla_a(\Lambda \xi^a - T^a_b \xi^b + R^b) \;=\; \xi^a (\sum_{\mathcal{I}} \mathcal{F}^{\mathcal{I}}{}_a - \nabla_b T^b{}_a). \tag{3.70}$$

We consider the integral on the left-hand side of equation (3.70) over a region of displacement that is bounded. After applying Green's theorem, the integral vanishes for an arbitrary ξ^a. It follows that the force-balance equation (3.54) holds as an identity that is independent of whatever conservative or dissipative field equations that may be suggested.

On working out the terms in equation (3.69), one finds that the remainder R^a in the divergence contribution will be expressed as [113]

$$R^a \quad = \quad 2\sum_{\mathcal{I}} \mu^{\mathcal{I}}{}_b n^{[a}{}_{\mathcal{I}} \xi^{b]}{}_{\mathcal{I}}. \tag{3.71}$$

The force density is given for the particle and entropy fluxes which can be expressed as

$$\mathcal{F}^{\mathcal{I}}_{\ a} \quad = \quad \nabla_b (n^b{}_{\mathcal{I}} \mu^{\mathcal{I}}{}_a) - n^b{}_{\mathcal{I}} \nabla_a \mu^{\mathcal{I}}{}_b. \tag{3.72}$$

This expression can be re-expressed as

$$\mathcal{F}^{\mathcal{I}}_{\ a} \quad = \quad \mu^{\mathcal{I}}{}_a \nabla_b n^b{}_{\mathcal{I}} + n^b{}_{\mathcal{I}} \nabla_b \mu^{\mathcal{I}}{}_a - n^b{}_{\mathcal{I}} \nabla_a \mu^{\mathcal{I}}{}_b, \tag{3.73}$$

which in turn leads to

$$\mathcal{F}^{\mathcal{I}}_{\ a} \quad = \quad \mu^{\mathcal{I}}{}_a \nabla_b n^b{}_{\mathcal{I}} + n^b{}_{\mathcal{I}} (\nabla_b \mu^{\mathcal{I}}{}_a - \nabla_a \mu^{\mathcal{I}}{}_b). \tag{3.74}$$

In a more compact form, equation (3.74) can be expressed as

$$\mathcal{F}^{\mathcal{I}}_{\ a} \quad = \quad \mu^{\mathcal{I}}{}_a \nabla_b n^b{}_{\mathcal{I}} + n^b{}_{\mathcal{I}} \nabla_{[b} \mu^{\mathcal{I}}{}_{a]}. \tag{3.75}$$

The formula of equation (3.51) for the corresponding canonically associated $T^a_{\ b}$ can be derived from equation (3.69) using equation (3.65). The symmetry of the canonically constructed T^{ab} is not manifest in equation (3.51). However, the symmetry property $T^{ab} = T^{(ab)}$ can be seen to follow directly from the Noether identity of equation (3.66).

One can now introduce a set of convection vectors denoted by $\beta^a{}_{\mathcal{I}}$ such that $h_{\mathcal{I}} \beta^a{}_{\mathcal{I}} = n^a{}_{\mathcal{I}}$ and $\mu^{\mathcal{I}}{}_a \beta^a{}_{\mathcal{I}} = -1$ [113]. These vectors are proportional to the fluxes in the case where they are associated with the currents themselves. Combining momenta $\mu^{\mathcal{I}}{}_a$ with flux current denoted by $n^a{}_{\mathcal{I}}$ leads to $h_{\mathcal{I}} \mu^{\mathcal{I}}{}_a \beta^a{}_{\mathcal{I}} = n^a{}_{\mathcal{I}} \mu^{\mathcal{I}}{}_a$. This implies that $-h_{\mathcal{I}} = \mu^{\mathcal{I}}{}_a n^a{}_{\mathcal{I}}$ meaning that $h_{\mathcal{I}} = -\mu^{\mathcal{I}}{}_a n^a{}_{\mathcal{I}}$. With these definitions in mind, a projection operator can be introduced as follows

$$\perp^{ab}_{\mathcal{I}} \quad = \quad g^{ab} + \mu^a{}_{\mathcal{I}} \beta^b{}_{\mathcal{I}}. \tag{3.76}$$

Then combining the projection operator with the convection vectors yields

$$\perp^a{}_{\mathcal{I}b}\, \beta^b{}_{\mathcal{I}} \quad = \quad (g^a{}_b + \mu^a{}_{\mathcal{I}} \beta_{\mathcal{I}b}) \beta^b{}_{\mathcal{I}}. \tag{3.77}$$

Expanding the right-hand side leads to

$$\perp^{a}{}_{\mathcal{I}b}\,\beta^{b}{}_{\mathcal{I}} \;=\; g^{a}{}_{b}\beta^{b}{}_{\mathcal{I}} + \mu^{a}{}_{\mathcal{I}}\beta^{b}{}_{\mathcal{I}}\beta_{\mathcal{I}b}. \tag{3.78}$$

b is a dummy index on the second term on the right-hand side of (3.78). Hence, equation (3.78) simplifies to

$$\perp^{a}{}_{\mathcal{I}b}\,\beta^{b}{}_{\mathcal{I}} \;=\; \beta^{a}{}_{\mathcal{I}} - \beta^{a}{}_{\mathcal{I}} = 0. \tag{3.79}$$

Similarly, combining the projection operator and the momenta yields

$$\perp^{ab}_{\mathcal{I}}\,\mu^{\mathcal{I}}{}_{b} \;=\; (g^{ab} + \mu^{a}{}_{\mathcal{I}}\beta^{b}{}_{\mathcal{I}})\mu^{\mathcal{I}}{}_{b}. \tag{3.80}$$

Using $\beta^{b}{}_{\mathcal{I}}\mu^{\mathcal{I}}{}_{b} = -1$ leads to

$$\perp^{ab}_{\mathcal{I}}\,\mu^{\mathcal{I}}{}_{b} \;=\; \mu^{\mathcal{I}a} - \mu^{\mathcal{I}a} = 0. \tag{3.81}$$

This then implies that

$$\perp^{a}{}_{\mathcal{I}b}\,\beta_{a\mathcal{I}} \;=\; \perp^{ab}_{\mathcal{I}}\,\mu^{\mathcal{I}}{}_{b}. \tag{3.82}$$

One can now consider the force density given by

$$\mathcal{F}^{\mathcal{I}}_{a} \;=\; \mu^{\mathcal{I}}{}_{a}\nabla_{b}n^{b}{}_{\mathcal{I}} + n^{b}{}_{\mathcal{I}}(\nabla_{b}\mu_{a} - \nabla_{a}\mu_{b}). \tag{3.83}$$

Combining equation (3.83) with $\beta^{a}{}_{\mathcal{I}}$ yields

$$\beta^{a}{}_{\mathcal{I}}\mathcal{F}^{\mathcal{I}}_{a} \;=\; \beta^{a}{}_{\mathcal{I}}\mu^{\mathcal{I}}{}_{a}\nabla_{b}n^{b}{}_{\mathcal{I}} + \beta^{a}{}_{\mathcal{I}}n^{b}{}_{\mathcal{I}}(\nabla_{b}\mu_{a} - \nabla_{a}\mu_{b}). \tag{3.84}$$

Then using the condition $\mu^{\mathcal{I}}{}_{a}\beta^{a}{}_{\mathcal{I}} = -1$ in equation (3.84) leads to

$$\beta^{a}{}_{\mathcal{I}}\mathcal{F}^{\mathcal{I}}_{a} \;=\; -\nabla_{b}n^{b}{}_{\mathcal{I}} + \beta^{a}{}_{\mathcal{I}}n^{b}{}_{\mathcal{I}}\nabla_{b}\mu_{a} - \beta^{a}{}_{\mathcal{I}}n^{b}{}_{\mathcal{I}}\nabla_{a}\mu_{b}. \tag{3.85}$$

The product rule demands the equation

$$\beta^{a}{}_{\mathcal{I}}\mathcal{F}^{\mathcal{I}}_{a} \;=\; -\nabla_{b}n^{b}{}_{\mathcal{I}} + n^{b}{}_{\mathcal{I}}\nabla_{b}(\beta^{a}{}_{\mathcal{I}}\mu_{a}) - n^{b}{}_{\mathcal{I}}\nabla_{a}(\beta^{a}{}_{\mathcal{I}}\mu_{b}) - n^{b}{}_{\mathcal{I}}\mu_{a}\nabla_{b}\beta^{a}{}_{\mathcal{I}} + n^{b}{}_{\mathcal{I}}\mu_{b}\nabla_{a}\beta^{a}{}_{\mathcal{I}}. \tag{3.86}$$

Using the metric $g^a{}_b$ yields

$$\beta^a{}_{\mathcal{I}}\mathcal{F}^{\mathcal{I}}{}_a = -\nabla_b n^b{}_{\mathcal{I}} + n^b{}_{\mathcal{I}}\nabla_b(\beta^a{}_{\mathcal{I}}\mu_a) - n^b{}_{\mathcal{I}}\nabla_b(\beta^a{}_{\mathcal{I}}\mu_a) - n^b{}_{\mathcal{I}}\mu_a\nabla_b\beta^a{}_{\mathcal{I}} + n^b{}_{\mathcal{I}}\mu_a\nabla_b\beta^a{}_{\mathcal{I}}, \tag{3.87}$$

which in turn clearly leads to

$$\beta^a{}_{\mathcal{I}}\mathcal{F}^{\mathcal{I}}{}_a = -\nabla_b n^b{}_{\mathcal{I}}. \tag{3.88}$$

Equation (3.88) can be expressed as

$$\nabla_a n^a{}_{\mathcal{I}} = -\beta^a{}_{\mathcal{I}}\mathcal{F}^{\mathcal{I}}{}_a. \tag{3.89}$$

Proceeding further

$$\mathcal{L}_{\mathcal{I}}\mu_{\mathcal{I}a} = \beta^b{}_{\mathcal{I}}\nabla_b\mu_a + \mu_b\nabla_a\beta^b{}_{\mathcal{I}}. \tag{3.90}$$

Combining equation (3.90) with $h_{\mathcal{I}} = -\mu^{\mathcal{I}}{}_a n^a{}_{\mathcal{I}}$ leads to

$$h_{\mathcal{I}}\mathcal{L}_{\mathcal{I}}\mu^{\mathcal{I}}{}_a = -\mu^{\mathcal{I}}{}_c n^c{}_{\mathcal{I}}\beta^b{}_{\mathcal{I}}\nabla_b\mu_a - \mu^{\mathcal{I}}{}_c n^c{}_{\mathcal{I}}\mu_b\nabla_a\beta^b{}_{\mathcal{I}}. \tag{3.91}$$

Then using equations (3.74) and (3.76) lead to

$$\perp^{\mathcal{I}be}\mathcal{F}^{\mathcal{I}}{}_e = g^{be}[\mu_{e\mathcal{I}}\nabla_f n^f{}_{\mathcal{I}} + n^f{}_{\mathcal{I}}(\nabla_f\mu^{\mathcal{I}}{}_e - \nabla_e\mu^{\mathcal{I}}{}_f)] + \mu^{\mathcal{I}b}\beta^{\mathcal{I}e}[\mu_{\mathcal{I}e}\nabla_f n^f{}_{\mathcal{I}} + n^f{}_{\mathcal{I}}(\nabla_f\mu^{\mathcal{I}}{}_e - \nabla_e\mu^{\mathcal{I}}{}_f)]. \tag{3.92}$$

Expanding further yields

$$\perp^{\mathcal{I}be}\mathcal{F}^{\mathcal{I}}{}_e = \mu^b{}_{\mathcal{I}}\nabla_f n^f{}_{\mathcal{I}} + n^f{}_{\mathcal{I}}(\nabla_f\mu^{\mathcal{I}b} - \nabla^b\mu^{\mathcal{I}}{}_f) + \mu^{\mathcal{I}b}\beta^{\mathcal{I}e}\mu_{\mathcal{I}e}\nabla_f n^f{}_{\mathcal{I}} + \mu^{\mathcal{I}b}\beta^{\mathcal{I}e}n^f{}_{\mathcal{I}}(\nabla_f\mu^{\mathcal{I}}{}_e - \nabla_e\mu^{\mathcal{I}}{}_f), \tag{3.93}$$

which in turn leads to

$$\perp^{\mathcal{I}be}\mathcal{F}^{\mathcal{I}}{}_e = \mu^b{}_{\mathcal{I}}\nabla_f n^f{}_{\mathcal{I}} + n^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}b} - n^f{}_{\mathcal{I}}\nabla^b\mu^{\mathcal{I}}{}_f + \mu^{\mathcal{I}b}\beta^{\mathcal{I}e}\mu_{\mathcal{I}e}\nabla_f n^f{}_{\mathcal{I}} + \mu^{\mathcal{I}b}\beta^{\mathcal{I}e}n^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}}{}_e - \mu^{\mathcal{I}b}\beta^{\mathcal{I}e}n^f{}_{\mathcal{I}}\nabla_e\mu^{\mathcal{I}}{}_f. \tag{3.94}$$

Using the condition $\mu^{\mathcal{I}}{}_a\beta^a{}_{\mathcal{I}} = -1$ in equation (3.94) yields

$$\perp^{\mathcal{I}be}\mathcal{F}^{\mathcal{I}}{}_e = \mu^{b\mathcal{I}}\nabla_f n^f{}_{\mathcal{I}} + n^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}b} - n^f{}_{\mathcal{I}}\nabla^b\mu^{\mathcal{I}}{}_f - \mu^{b\mathcal{I}}\nabla_f n^f{}_{\mathcal{I}} + \mu^{\mathcal{I}b}\beta^{\mathcal{I}e}n^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}}{}_e - \mu^{\mathcal{I}b}\beta^{\mathcal{I}e}n^f{}_{\mathcal{I}}\nabla_e\mu^{\mathcal{I}}{}_f. \tag{3.95}$$

Simplifying equation (3.95) leads to

$$\perp^{\mathcal{I}be}\,\mathcal{F}^{\mathcal{I}}_{\ e} \;=\; n^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}b} - n^f{}_{\mathcal{I}}\nabla^b\mu^{\mathcal{I}}{}_f. \tag{3.96}$$

Substituting in (or using) $h_{\mathcal{I}}\beta^a{}_{\mathcal{I}} = n^a{}_{\mathcal{I}}$ leads to

$$\perp^{\mathcal{I}be}\,\mathcal{F}^{\mathcal{I}}_{\ e} \;=\; h_{\mathcal{I}}\beta^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}b} - h_{\mathcal{I}}\beta^f{}_{\mathcal{I}}\nabla^b\mu^{\mathcal{I}}{}_f, \tag{3.97}$$

which can be expressed as

$$\perp^{\mathcal{I}be}\,\mathcal{F}^{\mathcal{I}}_{\ e} \;=\; h_{\mathcal{I}}\beta^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}b} - h_{\mathcal{I}}\nabla^b(\beta^f{}_{\mathcal{I}}\mu^{\mathcal{I}}{}_f) + h_{\mathcal{I}}\mu^{\mathcal{I}}{}_f\nabla^b\beta^f{}_{\mathcal{I}}. \tag{3.98}$$

Using $\mu^{\mathcal{I}}{}_a\beta^a{}_{\mathcal{I}} = -1$ in equation (3.98) leads to

$$\perp^{\mathcal{I}be}\,\mathcal{F}^{\mathcal{I}}_{\ e} \;=\; h_{\mathcal{I}}\beta^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}b} + h_{\mathcal{I}}\mu^{\mathcal{I}}{}_f\nabla^b\beta^f{}_{\mathcal{I}}. \tag{3.99}$$

Substituting for $h_{\mathcal{I}}$ and using $h_{\mathcal{I}} = -\mu^{\mathcal{I}}{}_a n^a{}_{\mathcal{I}}$ in equation (3.99) yields

$$\perp^{\mathcal{I}be}\,\mathcal{F}^{\mathcal{I}}_{\ e} \;=\; -\mu^{\mathcal{I}}{}_c n^c{}_{\mathcal{I}}\beta^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}b} - \mu^{\mathcal{I}}{}_c n^c{}_{\mathcal{I}}\mu^{\mathcal{I}}{}_f\nabla^b\beta^f{}_{\mathcal{I}}, \tag{3.100}$$

which can be expressed as

$$\perp^{\mathcal{I}be}\,\mathcal{F}^{\mathcal{I}}_{\ e} \;=\; -\mu^{\mathcal{I}}{}_c n^c{}_{\mathcal{I}}\beta^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}b} - \mu^{\mathcal{I}}{}_c n^c{}_{\mathcal{I}}\mu^{\mathcal{I}}\nabla^b\beta^f{}_{\mathcal{I}}. \tag{3.101}$$

This can further be expressed as

$$\perp^{\mathcal{I}}{}_{be}\,\mathcal{F}^e{}_{\mathcal{I}} \;=\; -\mu^{\mathcal{I}}{}_c n^c{}_{\mathcal{I}}(\beta^f{}_{\mathcal{I}}\nabla_f\mu^{\mathcal{I}}{}_b + \mu^{\mathcal{I}}{}_f\nabla_b\beta^f{}_{\mathcal{I}}). \tag{3.102}$$

Expanding out equation (3.102) leads to

$$\perp^{\mathcal{I}}{}_{ae}\,\mathcal{F}^e{}_{\mathcal{I}} \;=\; -\mu^{\mathcal{I}}{}_c n^c{}_{\mathcal{I}}\beta^b{}_{\mathcal{I}}\nabla_b\mu_a - \mu^{\mathcal{I}}{}_c n^c{}_{\mathcal{I}}\mu_b\nabla_a\beta^b{}_{\mathcal{I}}. \tag{3.103}$$

Changing the index e to b (without loss of generality or change of meaning), using $h_{\mathcal{I}} = -\mu^{\mathcal{I}}{}_a n^a{}_{\mathcal{I}}$ and then comparing with equation (3.91) leads to the conclusion that

$$\perp^{\mathcal{I}}{}_{ab}\,\mathcal{F}^b{}_{\mathcal{I}} \;=\; h_{\mathcal{I}}\mathcal{L}_{\mathcal{I}}\mu^{\mathcal{I}}{}_a. \tag{3.104}$$

Suppose that one chose to work in a given frame that is moving at four velocity u^a and having unit normalisation with $\perp^a{}_b$ which represents an associated (symmetric) orthogonal projection tensor where $u_a u^a = -1$, then this implies that a natural convection vector can be broken down into the form [73]

$$\beta^a{}_{\mathcal{I}} = \beta_{\mathcal{I}}(u^a + v^a{}_{\mathcal{I}}), \tag{3.105}$$

such that the relative velocity vector is restricted in a manner that allows it to satisfy the orthogonality condition

$$u_a v^a{}_{\mathcal{I}} = 0. \tag{3.106}$$

The scalar factor denoted by $\beta_{\mathcal{I}}$ has an inverse that is represented by

$$\mu_{\mathcal{I}} = \beta^{-1}_{\mathcal{I}}, \tag{3.107}$$

where $\mu_{\mathcal{I}}$ are the effective chemical type potentials with respect to the chosen frame of reference and $v^a{}_{\mathcal{I}}$ is a velocity vector.

One can now make contact with physics. The definitions in the preceding paragraphs have been introduced in such a way that the reference frame component of the Noether identity in equation (3.54) works out in the suggestive form given below [73]

$$u^a \nabla_b T^b{}_a + \sum_{\mathcal{I}} (\mu^{\mathcal{I}} \nabla_a n^a{}_{\mathcal{I}} + v^a{}_{\mathcal{I}} \mathcal{F}^{\mathcal{I}}{}_a) = 0. \tag{3.108}$$

The exploitation of this equation is a guide to the appropriate form of the dynamic equations that should be chosen in setting up specific phenomenological models for particular purposes. To see how things work out in the convective variational approach, one can single out the entropy fluid (with index s) by defining $s^a = n^a{}_s$ and $T = \mu_s$. One can then assume that the remaining species are governed by conservation laws of the form given below

$$\nabla_a n^a{}_{\mathcal{I}} = \Gamma_{\mathcal{I}}, \tag{3.109}$$

subject to the constraint of total baryon conservation

$$\sum_{\mathcal{I} \neq s} \Gamma_{\mathcal{I}} \equiv 0. \tag{3.110}$$

This simplifies equation (3.108). With the just mentioned (above) conditions and the fact that the $\nabla_a T^a_b = 0$ leads to

$$T \nabla_a s^a \equiv -\sum_{\mathcal{I} \neq s} \mu^{\mathcal{I}} \Gamma_{\mathcal{I}} - \sum_{\mathcal{I}} v^a{}_{\mathcal{I}} \mathcal{F}^{\mathcal{I}}{}_a. \tag{3.111}$$

The first term on the right-hand side of equation (3.111) represents the entropy increase due to chemical reactions. The simplest way to ensure that the second law of thermodynamics is satisfied, is to make the term positive definite. To complete the formalism, one can assume the term just mentioned to be linear. This would imply expansion of each $\Gamma_{\mathcal{I}}$ according to

$$\Gamma_{\mathcal{I}} \;=\; -\sum_{\mathcal{J}\neq s} \mathcal{C}_{\mathcal{I}\mathcal{J}} \mu^{\mathcal{J}}, \tag{3.112}$$

where $\mathcal{C}_{\mathcal{I}\mathcal{J}}$ represents a positive definite (or indefinite) matrix composed of the various reaction rates.

3.4 Chapter summary

We have reviewed a simple illustration of the variational principle, the pull-back approach for single-fluid models and Carter's canonical framework (or the convective variational approach) for multi-fluids systems. We used the simple physics problem for a point particle to make the variational principle more comprehensible. The simple physics problem for a point particle has served as a guide to deep principles for problems which are much harder in the field of physics. The application of the variational principle was then examined. In the pull-back approach for single-fluid models, it is shown that $n^a{}_X$ is conserved for equations of motion after using a variational proposition where X represents the radiation-dominated or matter-dominated epochs. The convective variational approach is developed and shown to be suitable for dissipative multi-fluid systems.

Chapter 4

Convective variational formalism to interacting multi-fluid systems

4.1 Prelude to convective variational formalism to interacting multi-fluid systems

Under a suitable continuum hypothesis, any non-rigid multi-bodied state can be described as a fluid which follows certain equations of motion. This could be said of the dark matter particles. Motion of a many-body system is modelled using fluid dynamics. Now, considering the well-known assumption that dark energy is a new kind of (dynamical) energy fluid, then examination of multi-fluid and (a possible) entrainment effect that involve interaction of dark matter and dark energy is a possibility. This implies the use of a multi-fluid approach to examine (the entrainment effect of) the interaction of dark matter and dark energy. This also means that the multi-fluid approach is applied to cosmology.

In this chapter, we examine the properties of a multi-fluid system that will be appropriate to cosmology [44]. Before we proceed further, we first define the term *multi-fluid*. A multi-fluid is a fluid mixture that is made up of many species where each species is physically treated as separate, and which uniquely contributes to the continuum properties of the mixture. One can consider an example where there is intra-species heat flow due to the different species having different temperatures. When the equations of energy for the individual species are included in the system of equations in the multi-fluid formalism, then the ensuing artifact that became apparent in the treatment of such a fluid is captured. The prospect of separate, mean velocities is also raised due to different temperatures. In the multi-fluid environment, these velocities are obtained from their respective momentum equations. It suffices to say that the system of fluid equations comprises both momentum equations for the whole multi-fluid system and transport equations for individual species[1]. This implies that there are several fundamental observers, and as a consequence there are anisotropic models such as the Bianchi type I model [116, 117].

Current observations show that the universe is isotropic and homogeneous on large-scales. This is against the anisotropic models [44]. The question of whether the universe transitioned either into the state of homogeneity and isotropisation or started out as isotropic and homogeneous is still a subject of intense scrutiny [118]. For any anisotropic model (such as those resulting from a multi-fluid approximation) to connect with observations the models become isotropic in late times. This suggests that a possible extension to a general multi-fluid model is reasonable as the subject of isotropisation has been considered [84, 119] where it was found that isotropisation occurs in a two-fluid model. In fact, there exists a Bianchi type I epoch where the matter flux dominates and eventually evolves to a $FLRW$ model. This is effectively a single-fluid model as shown in [84].

Generally, the multi-fluid equations are developed on the basis of the intuitive concept of mutually penetrating continuums [120]. The derivations of the multi-fluid equations is based on the method of Chapman-Enskog which involves the Boltzmann equation and which incorporates particle collisions in a gas mixture [44]. The development in [44] addresses conservation of number, momentum and energy density of each species. It also addresses fluid approximation.

The development in this chapter builds on the convective variational approach [77,84,113]. It is different from the convective variational formalism [44] and it is a complementary formalism to that in reference [120]. The multi-fluid formalism allows for the incorporation of convective terms via the momentum equations and these terms are usually absent in the transport equations. One can now consider the multi-fluid formalism and thermodynamics in the next section.

4.2 Thermodynamics and a multi-fluid model

We consider two-fluid species of dark matter denoted by $\mathcal{Y}'$ and dark energy denoted by $\mathcal{Z}$ (rather than one) that occupy a shared volume [44]. We have Λ that encodes contributions from both fluid species rather than the individual $\Lambda_{(\mathcal{Y}')} = \Lambda(m_{(\mathcal{Y}')}, {n^2}_{n_{(\mathcal{Y}')}}, {n^2}_{S_{(\mathcal{Y}')}})$ and $\Lambda_{(\mathcal{Z})} =$

[1]The formalism developed by Carter (the convective variational formalism) has a momentum equation for each independent fluid. This means that the fluid four velocities are dynamically independent for each fluid species. Nevertheless, one can logically assume that all the velocities can be added and an average velocity found for the whole multi-fluid mixture. Hence, the system of fluid equations would include both momentum equations for the whole multi-fluid system and the transport equations for the individual species.

$\Lambda(m_{(\mathcal{Z})}, n^2{}_{n_{(\mathcal{Z})}}, n^2{}_{S_{(\mathcal{Z})}})$, respectively. Let $n_{(\mathcal{Y}')}$ and $n_{(\mathcal{Z})}$ represent the two number densities of the two species, and $u^a{}_{(\mathcal{Y}')}$ and $u^b{}_{(\mathcal{Z})}$ represent their corresponding velocities where a and b represent space-time indices. Assuming that there are no chemical interactions or reactions between the two species of dark matter and dark energy, the fluxes for the individual species are given by $n^a{}_{\mathcal{Y}'} = n_{\mathcal{Y}'} u^a{}_{\mathcal{Y}'}$ and $n^b{}_{\mathcal{Z}} = n_{\mathcal{Z}} u^b{}_{\mathcal{Z}}$, respectively. The number density for each species is separately conserved, that is

$$\nabla_a n^a{}_{\mathcal{Y}'} \;=\; \nabla_a n^a{}_{\mathcal{Z}} = 0. \tag{4.1}$$

One can derive associated co-moving densities associated with the given flux by taking the flux $n^a{}_{\mathcal{Y}'}$ for each component as a fundamental field. Let $n^a{}_{\mathcal{Y}'}$ and $n^b{}_{\mathcal{Y}'}$ be the two fluxes for the fluid of type $\mathcal{Y}'$ (implying the same species) that are endowed with the space-time indices a and b, respectively. It follows that one will have the product $n^a{}_{\mathcal{Y}'} n^b{}_{\mathcal{Y}'} = n^2{}_{\mathcal{Y}'} u^a{}_{\mathcal{Y}'} u^b{}_{\mathcal{Y}'}$. The co-moving density is then obtained as follows

$$g_{ab} n^a{}_{\mathcal{Y}'} n^b{}_{\mathcal{Y}'} \;=\; g_{ab} n^2{}_{\mathcal{Y}'} u^a{}_{\mathcal{Y}'} u^b{}_{\mathcal{Y}'}, \tag{4.2}$$

which can be expressed as

$$g_{ab} n^a{}_{\mathcal{Y}'} n^b{}_{\mathcal{Y}'} \;=\; n^2{}_{\mathcal{Y}'} (g_{ab} u^a{}_{\mathcal{Y}'} u^b{}_{\mathcal{Y}'}), \tag{4.3}$$

and then lowering the index b on the four velocity $u^b{}_{\mathcal{Y}'}$ by using the metric g_{ab} and using $u^{\mathcal{Y}'}_a u^a{}_{\mathcal{Y}'} = -1$ leads to

$$g_{ab} n^a{}_{\mathcal{Y}'} n^b{}_{\mathcal{Y}'} \;=\; -n^2{}_{\mathcal{Y}'}. \tag{4.4}$$

This can be extended to a two-fluid model such that

$$g_{ab} n^a{}_{\mathcal{Y}'} n^b{}_{\mathcal{Z}} \;=\; g_{ab} n_{\mathcal{Y}'} u^a{}_{\mathcal{Y}'} n_{\mathcal{Z}} u^b{}_{\mathcal{Z}}, \tag{4.5}$$

which can be expressed as

$$g_{ab} n^a{}_{\mathcal{Y}'} n^b{}_{\mathcal{Z}} \;=\; n^2{}_{\mathcal{Y}'\mathcal{Z}} (g_{ab} u^a{}_{\mathcal{Y}'} u^b{}_{\mathcal{Z}}), \tag{4.6}$$

where $n_{\mathcal{Y}'} n_{\mathcal{Z}} = n^2{}_{\mathcal{Y}'\mathcal{Z}}$. Lowering the index b on $u^b{}_{\mathcal{Z}}$ by using the metric g_{ab} leads to

$$g_{ab} n^a{}_{\mathcal{Y}'} n^b{}_{\mathcal{Z}} \;=\; n^2{}_{\mathcal{Y}'\mathcal{Z}} (u^a{}_{\mathcal{Y}'} u_{a(\mathcal{Z})}). \tag{4.7}$$

Given that $u^a{}_{\mathcal{Y}'} u_{a(\mathcal{Z})} = -1$, (4.7) yields

$$g_{ab} n^a{}_{\mathcal{Y}'} n^b{}_{\mathcal{Z}} \;=\; -n^2{}_{\mathcal{Y}'\mathcal{Z}}. \tag{4.8}$$

If there are no chemical interactions and taken in totality, the formulation just presented suggests that the energy density denoted by Λ can be expressed as a function of energy density scalars such that

$$\Lambda \;=\; \Lambda(m_{\mathcal{Y}'}, m_{\mathcal{Z}}, n^2{}_{n_{(\mathcal{Y}')}}, n^2{}_{S_{\mathcal{Y}'}}, n^2{}_{n_{\mathcal{Z}}}, n^2{}_{S_{\mathcal{Z}}}). \tag{4.9}$$

If a chemical interaction occurs, the expression below can be suggested

$$\Lambda \;=\; \Lambda(m_{\mathcal{Y}'}, m_{\mathcal{Z}}, n^2{}_{n_{\mathcal{Y}'}}, n^2{}_{n_{\mathcal{Z}}}, n^2{}_{S_{\mathcal{Y}'}}, n^2{}_{S_{\mathcal{Z}}}, n^2{}_{n_{\mathcal{Y}'\mathcal{Z}}}, n^2{}_{S_{\mathcal{Y}'\mathcal{Z}}}). \tag{4.10}$$

Varying (4.10) yields

$$\delta\Lambda \;=\; \sum_{\mathcal{Y}'} \frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'}} \delta n^2{}_{\mathcal{Y}'} + \sum_{\mathcal{Y}'\mathcal{Z}} \frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}} \delta n^2{}_{\mathcal{Y}'\mathcal{Z}} + \frac{\partial \Lambda}{\partial g^{ab}} \delta g^{ab}. \tag{4.11}$$

Introducing $u^{\mathcal{Y}'}_a u^a{}_{\mathcal{Y}'} = -1$ on the first term on the right-hand side of equation (4.11) leads to

$$\delta\Lambda \;=\; \sum_{\mathcal{Y}'} -2\frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'}} \frac{u^{\mathcal{Y}'}_b u^b{}_{\mathcal{Y}'}}{2} \delta n^2{}_{\mathcal{Y}'} + \sum_{\mathcal{Y}'\mathcal{Z}} \frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}} \delta n^2{}_{\mathcal{Y}'\mathcal{Z}} + \frac{\partial \Lambda}{\partial g^{ab}} \delta g^{ab}. \tag{4.12}$$

Introducing $n^a{}_{\mathcal{Y}'} = n_{\mathcal{Y}'} u^a{}_{\mathcal{Y}'}$ on the first term of the right-hand side of equation (4.12) yields

$$\delta\Lambda \;=\; \sum_{\mathcal{Y}'} -2\frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'}} \frac{1}{2} \delta(n^{\mathcal{Y}'}_b n^b{}_{\mathcal{Y}'}) + \sum_{\mathcal{Y}'\mathcal{Z}} \frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}} \delta n^2{}_{\mathcal{Y}'\mathcal{Z}} + \frac{\partial \Lambda}{\partial g^{ab}} \delta g^{ab}. \tag{4.13}$$

Varying $n^{\mathcal{Y}'}_b n^b{}_{\mathcal{Y}'}$ in (4.13) leads to

$$\delta\Lambda \;=\; \sum_{\mathcal{Y}'} -2\frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'}} n_{b\mathcal{Y}'} \delta n^b{}_{\mathcal{Y}'} + \sum_{\mathcal{Y}'\mathcal{Z}} \frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}} \delta n^2{}_{\mathcal{Y}'\mathcal{Z}} + \frac{\partial \Lambda}{\partial g^{ab}} \delta g^{ab}. \tag{4.14}$$

Introducing g_{ab} and $u^a{}_{\mathcal{Z}} u_{a(\mathcal{Y}')} = -1$ on the first and second terms, respectively on the right-hand side of equation (4.14) yields

$$\delta\Lambda \;=\; -\frac{2\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'}} g_{ba} n^a{}_{\mathcal{Y}'} \delta n^b{}_{\mathcal{Y}'} - \frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}} u^a{}_{\mathcal{Z}} u_{a(\mathcal{Y}')} \delta n^2{}_{\mathcal{Y}'\mathcal{Z}} + \frac{\partial \Lambda}{\partial g^{ab}} \delta g^{ab}. \tag{4.15}$$

Introducing g_{ab} on the second term of the right-hand side of equation (4.15) leads to

$$\delta\Lambda \;=\; -2\frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'}} g_{ba} n^a{}_{\mathcal{Y}'} \delta n^b{}_{\mathcal{Y}'} - \frac{\partial \Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}} g_{ab} u^a{}_{\mathcal{Z}} u^b{}_{\mathcal{Y}'} \delta n^2{}_{\mathcal{Y}'\mathcal{Z}} + \frac{\partial \Lambda}{\partial g^{ab}} \delta g^{ab}. \tag{4.16}$$

Using the definition for flux current on the term yields

$$\delta\Lambda = -2g_{ba}\frac{\partial\Lambda}{\partial n^2{}_{\mathcal{Y}'}}n^a{}_{\mathcal{Y}'}\delta n^b{}_{\mathcal{Y}'} - g_{ba}\frac{\partial\Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}}\delta(n^b{}_{\mathcal{Y}'}n^a{}_{\mathcal{Z}}) + \frac{\partial\Lambda}{\partial g^{ab}}\delta g^{ab}. \tag{4.17}$$

Varying one of the flux currents on the second term of the right-hand side of equation (4.17) leads to

$$\delta\Lambda = -2g_{ab}\frac{\partial\Lambda}{\partial n^2{}_{\mathcal{Y}'}}n^a{}_{\mathcal{Y}'}\delta n^b{}_{\mathcal{Y}'} - g_{ba}\frac{\partial\Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}}n^a{}_{\mathcal{Z}}\delta n^b{}_{\mathcal{Y}'} + \frac{\partial\Lambda}{\partial g^{ab}}\delta g^{ab}. \tag{4.18}$$

Writing (4.18) in suitable form yields

$$\delta\Lambda = g_{ba}\Bigg(-2\frac{\partial\Lambda}{\partial n^2{}_{\mathcal{Y}'}}n^a{}_{\mathcal{Y}'} - \frac{\partial\Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}}n^a{}_{\mathcal{Z}}\Bigg)\delta n^b{}_{\mathcal{Y}'} + \frac{\partial\Lambda}{\partial g^{ab}}\delta g^{ab}. \tag{4.19}$$

Similarly, for the case where the constituent indices are interchanged leads to

$$\delta\Lambda = -2g_{ba}\frac{\partial\Lambda}{\partial n^2{}_{\mathcal{Z}}}n^a{}_{\mathcal{Z}}\delta n^b{}_{\mathcal{Z}} - g_{ba}\frac{\partial\Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}}n^a{}_{\mathcal{Y}'}\delta n^b{}_{\mathcal{Z}} + \frac{\partial\Lambda}{\partial g^{ab}}\delta g^{ab}. \tag{4.20}$$

Expressing this equation in suitable form yields

$$\delta\Lambda = g_{ba}\Bigg(-2\frac{\partial\Lambda}{\partial n^2{}_{\mathcal{Z}}}n^a{}_{\mathcal{Z}} - \frac{\partial\Lambda}{\partial n^2{}_{\mathcal{Y}'\mathcal{Z}}}n^a{}_{\mathcal{Y}'}\Bigg)\delta n^b{}_{\mathcal{Z}} + \frac{\partial\Lambda}{\partial g^{ab}}\delta g^{ab}. \tag{4.21}$$

Then equation (4.19) leads to

$$\delta\Lambda = \mu^{\mathcal{Y}'}_{\ b}\delta n^b{}_{\mathcal{Y}'} + \frac{\partial\Lambda}{\partial g^{ab}}\delta g^{ab}, \tag{4.22}$$

and equation (4.21) yields

$$\delta\Lambda = \mu^{\mathcal{Z}}_{\ b}\delta n^b{}_{\mathcal{Z}} + \frac{\partial\Lambda}{\partial g^{ab}}\delta g^{ab}. \tag{4.23}$$

The momentum conjugates will then take the form

$$\mu^{\mathcal{Y}'}_{\ b} = g_{ba}(\mathcal{B}^{\mathcal{Y}'}n^a{}_{\mathcal{Y}'} + \mathcal{A}^{\mathcal{Y}'\mathcal{Z}}n^a{}_{\mathcal{Z}}), \tag{4.24}$$

and

$$\mu^{\mathcal{Z}}_{\ b} = g_{ba}(\mathcal{B}^{\mathcal{Z}}n^a{}_{\mathcal{Z}} + \mathcal{A}^{\mathcal{Z}\mathcal{Y}'}n^a{}_{\mathcal{Y}'}), \tag{4.25}$$

where

$$\mathcal{A}^{\mathcal{Y}'\mathcal{Z}} \;=\; \mathcal{A}^{\mathcal{Z}\mathcal{Y}'} = -\frac{\partial\Lambda}{\partial {n^2}_{\mathcal{Y}'\mathcal{Z}}},\ \mathcal{Y}' \neq \mathcal{Z}, \tag{4.26}$$

and

$$\mathcal{B}^{\mathcal{Y}'} \;=\; -2\frac{\partial\Lambda}{\partial {n^2}_{\mathcal{Y}'}},\ \mathcal{B}^{\mathcal{Z}} = -2\frac{\partial\Lambda}{\partial {n^2}_{\mathcal{Z}}}. \tag{4.27}$$

The expressions in equation (4.27) remind us that these are bulk-fluid effects which are present regardless of the number of fluids and constituents present. The last terms in equations (4.24) and (4.25) encapsulates the entrainment effect. This raises serious questions about how one might define local thermodynamics equilibrium. The local thermodynamic energy is recovered in the limit when all fluxes are parallel. The terms ${n^2}_{n(\mathcal{Y}'\mathcal{Z})}$ and ${n^2}_{S(\mathcal{Y}'\mathcal{Z})}$ become meaningful when dissipative fluids are considered. These terms will be considered in a moment. First, one can consider the idea of the Λ that has more than two species. Mathematically, this would mean that the Λ could easily have the product ${n^4}_{n(W\mathcal{Y}'\mathcal{Z}Z)} = {n^2}_{n(WZ)}{n^2}_{n(\mathcal{Y}'\mathcal{Z})}$ as one of its entries where W and Z each represent a different fluid species. This comes from the mathematical fact $g_{ab}{n^a}_W {n^b}_{\mathcal{Y}'} g_{bc} {n^b}_{\mathcal{Z}} {n^c}_Z$. This implies that the product of entrainment involves four species of fluids. The physics of such products are somewhat unclear. Some of the couplings that may make the modelling process difficult are:

1. Matter-matter (fluxes, flow-lines and entrainment),

2. Matter-space-time (fluxes and metric-stress-energy-momentum tensor),

3. Matter-electromagnetism (fluxes and current) and,

4. Electromagnetism-space-time (curvature and potential).

The last two couplings are closely associated with couplings in general relativity. This is because space-time curvature is at the behest of matter distribution. The matter-matter coupling is straight-forward. The state of matter involved in the couplings may be determined thermodynamically, in principle [121] where only few parameters are monitored. This happens because the fluid changes and other associated or dependent parameters are recovered via the equation of state. Note that this does not suggest that space-time is a fluid[2].

Then one needs to only monitor the truly independent variables where the equation of state is known, but this raises the question of whether it is possible to determine or constrain the equation of state if the relationships between primary variables are all known. The question is not significant, given that we would like to apply the formalism to a multi-fluid environment

[2]Space-time curvature is at the behest of matter distribution and not dynamics of matter.

that includes dark matter whose equation of state is not yet established. In light of the above statements, one can choose to express the Λ for a two-fluid model involving two entrained species as follows

$$\begin{aligned}\Lambda &= \Lambda_1(m_{(\mathcal{Y}')}, m_{(\mathcal{Z})}, n_{n_{(\mathcal{Y}')}}, n_{n_{(\mathcal{Z})}}, n_{S_{(\mathcal{Y}')}}, n_{S_{(\mathcal{Z})}}) \\ &+ \Lambda_2({n^2}_{n_{(\mathcal{Y}')}}, {n^2}_{n_{(\mathcal{Z})}}, {n^2}_{S_{(\mathcal{Y}')}}, {n^2}_{S_{(\mathcal{Z})}}, {n^2}_{n_{(\mathcal{Y}'\mathcal{Z})}}, {n^2}_{S_{(\mathcal{Y}'\mathcal{Z})}}).\end{aligned} \tag{4.28}$$

Since we are interested in entrainment, we will focus on Λ_2 as it encodes entrainment. One can generalise a Λ for two interacting fluid species whose variation leads to the multi-fluid equations that obey conservation laws. This is

$$\Lambda = \frac{1}{2} g_{ab} \sum_{\mathcal{I}} \left(\frac{m_{\mathcal{I}} {n^a}_{\mathcal{I}} {n^b}_{\mathcal{I}}}{n_{\mathcal{I}}} \right) - \mathcal{E}(n_{\mathcal{I}}, {n^a}_{\mathcal{I}}), \tag{4.29}$$

where $\mathcal{I} = \mathcal{Y}',\ \mathcal{Z},\ \mathcal{Y}'\mathcal{Z}$. One can now examine how corresponding equations of motion are derived in the next section.

4.3 Equations of motion

In this section we use the Newtonian context in our analysis. Nevertheless, in the end we obtain the T^a_b which is composed of the conservative section denoted by $T^a_{b(Cons)}$ (and obtained by varying a Lagrangian density with respect to g^{ab}) and $\mathcal{T}^a_{b(Disp)}$ which is present due to dissipative effects. This is all in the realm of relativistic fluids. Now, considering commensurate conservation laws, one will be able to obtain equations of motion for the multi-fluid environment [44]. We will then need mass, both linear and angular momentum, and energy conservation laws. If we let $m^{\mathcal{Z}}$ represent the particle mass of species $\mathcal{Z}$ where $\mathcal{Z}$ represents dark energy, then the corresponding $\rho_{\mathcal{Z}}$ is given by the product $\rho_{\mathcal{Z}} = m^{\mathcal{Z}} n_{\mathcal{Z}}$. The total (local) mass density, denoted by ρ, of the system is $\rho = \sum_{\mathcal{Z}} \rho_{\mathcal{Z}} = \sum_{\mathcal{Z}} m^{\mathcal{Z}} n_{\mathcal{Z}}$. The sum ${\rho^{\mathcal{I}}}_{\mathcal{Z}} = \sum_{\mathcal{Z}} m^{\mathcal{Z}} {n^{\mathcal{I}}}_{\mathcal{Z}}$ is the total mass-density where $\mathcal{I} = 1,\ 2,\ 3$. The mass m of a fluid can be found by integrating ρ over the control volume denoted by V. This implies that

$$m = \int_V \rho dV. \tag{4.30}$$

${\rho^{\mathcal{I}}}_{\mathcal{Z}}$ is integrated over ∂V representing an infinitesimal volume, where an amount of m leaves or enters the volume. Overall conservation of m implies

$$\frac{d}{dt} \int_V \rho dV = - \int_{\partial V} \rho^{\mathcal{I}} \eta_{\mathcal{I}} dA = - \int_V \nabla_{\mathcal{I}} \rho^{\mathcal{I}} dV, \tag{4.31}$$

where $\eta_{\mathcal{I}}$ represents a unit vector. This is after using the divergence theorem. For the local condition, this equation will yield

$$\partial_t \rho \;\; = \;\; -\nabla_{\mathcal{I}} \rho^{\mathcal{I}}. \tag{4.32}$$

If $\Gamma_{\mathcal{Z}}$ is the rate at which particle $\mathcal{Z}$ is created per unit of V, then it follows that

$$\partial_t n_{\mathcal{Z}} + \nabla_{\mathcal{I}} n^{\mathcal{I}}{}_{\mathcal{Z}} \;\; = \;\; \Gamma_{\mathcal{Z}}, \tag{4.33}$$

where $\sum_{\mathcal{Z}} m_{\mathcal{Z}} \Gamma_{\mathcal{Z}} = 0$.

One can let the local linear momentum per particle and density of the $\mathcal{Z}$ be denoted by $P^{\mathcal{Z}}_b$ and π_b, respectively such that they both conjugate the $n_{\mathcal{Z}}$ [44]. Then it follows that the set of total density, total local and global linear momentum for control V is given by

$$\{\pi_b, p_b, P_b\} \;\; = \;\; \left\{ \sum_{\mathcal{Z}} n_{\mathcal{Z}} p^{\mathcal{Z}}_{\ b}, \sum_{\mathcal{Z}} p^{\mathcal{Z}}_{\ b}, \int_V \pi_b dV \right\}. \tag{4.34}$$

where $p^{\mathcal{Z}}_{\ b} = m^{\mathcal{Z}} u^{\mathcal{Z}}_{\ b}$. In equation (4.34) the last entry in the set allows for the investigation of conservation of linear momentum. For species $\mathcal{Z}$, the time derivative of its linear momentum is given by

$$\frac{dP^{\mathcal{Z}}_{\ b}}{dt} \;\; = \;\; \frac{d}{dt}\int_V \pi^{\mathcal{Z}}_{\ b} dV = -\int_{\partial V} T^{\mathcal{Z}a}_{\ b} \eta_a dA + \int_V \mathcal{F}^{\mathcal{Z}}_{\ b} = \int_V (\mathcal{F}^{\mathcal{Z}}_{\ b} - \nabla_a T^{\mathcal{Z}a}_{\ b}) dV. \tag{4.35}$$

where $T^{\mathcal{Z}a}_{\ b}$ represents the b^{th} component of the linear momentum of the $\mathcal{Z}$ in the direction orthogonal to a, and $\mathcal{F}^{\mathcal{Z}}_{\ b}$ represents the total external force density of $\mathcal{Z}$ acting on the control V. Note that η_a represents a unit vector. In equation (4.35), after the second equal to sign, we used the divergence theorem on the first integral. We deduced the last term on the last integral. Then, from the same equation (4.35)

$$\frac{d}{dt}\pi^{\mathcal{Z}}_{\ b} + \nabla_a T^{\mathcal{Z}a}_{\ b} \;\; = \;\; \mathcal{F}^{\mathcal{Z}}_{\ b}, \tag{4.36}$$

and hence, for all species

$$\frac{d}{dt}\sum_{\mathcal{Z}} \pi^{\mathcal{Z}}_{\ b} + \nabla_a \sum_{\mathcal{Z}} T^{\mathcal{Z}a}_{\ b} \;\; = \;\; \sum_{\mathcal{Z}} \mathcal{F}^{\mathcal{Z}}_{\ b}. \tag{4.37}$$

However, this equation (4.37) does not include internal forces. Given a control V and let E represent total energy, let $\mathcal{E}$ represent total energy per unit V, and let $\mathcal{Q}^b$ represent energy transfer per unit time t and unit area, then using the same analogy as in equation (4.35) leads to

$$E = \int_V \mathcal{E} dV, \tag{4.38}$$

and its time derivative is

$$\frac{dE}{dt} = \frac{d}{dt}\int_V \mathcal{E} dV = -\int_{\partial V} \mathcal{Q}^b \eta_b dA + \int_V \mathcal{G} dV = \int_V (\mathcal{G} - \nabla_b \mathcal{Q}^b) dV, \tag{4.39}$$

where $\mathcal{G}$ represents total external force. Note that we used the divergence theorem on the first integral after the second equal to sign leading to the last term in the last integral. This yields

$$\frac{d\mathcal{E}}{dt} + \nabla_b \mathcal{Q}^b = \mathcal{G}, \tag{4.40}$$

which is the expected energy conservation law. In this approach $\mathcal{E} = \mathcal{E}(n_{\mathcal{Z}}, n^b{}_{\mathcal{Z}})$ and requires the specification of the equation of state. Additionally, the extraction of the equations of motion (4.36) will require the complete specification of $\mathcal{E}$, $\mathcal{Q}^b$, $\mathcal{F}^{(\mathcal{Z})b}$, $T_a^{(\mathcal{Z})b}$ and $T^b{}_a$. Note that the T^a_b can be split into two. This means that

$$T^b{}_a = T^b{}_{a(Con)} + \mathcal{T}^b{}_{a(Disp)}, \tag{4.41}$$

where $T^b{}_{a(Con)}$ represents the conservative section of $T^b{}_a$ which is obtained by varying the Lagrangian density with respect to the metric g^{ab} while $\mathcal{T}^b{}_{a(Disp)}$ is present due to dissipative effects. This can be considered in the interacting multi-fluid dark-sector. This could lead to insights on the entrainment effect of the interacting dark-sector.

4.4 Chapter summary

In this chapter, we have examined:

1. Multi-fluid models for relativistic fluids from the flux point of view and,
2. The formulation of a modified convective variational approach for interacting multi-fluid systems.

One will see that this can be applied to the interacting multi-fluid system of the dark-sector after examining the following chapter which is chapter 5.

Chapter 5

Thermodynamics of relativistic and dissipative multi-fluid systems

5.1 Prelude to thermodynamics of multi-fluid systems

Our goal in this chapter is to use the multi-fluid approach to examine the entrainment effect of the interaction between dark matter and dark energy. For this to be really possible, we analyse the (generalised) second law of thermodynamics[1] and determine if it holds in interacting multi-fluid systems using either a more (or most) accurate approach for this task.

The attempt in [69] to match the convective variational model with the (standard) MIS theory of dissipative fluids found that the two theories are not equivalent to all orders, but are members of a set of related theories. It was found that the two formalisms lead to the same causal connections when subjected to perturbations about a thermodynamic equilibrium. It follows that in the thermal equilibrium limit, the two formalisms manifest similar characteristic surfaces and causality properties. Due to these similarities, one can choose to analyse the second law of thermodynamics (for multi-fluid systems) in the extended MIS theory for multi-fluids. This means that one should examine $\nabla_a S^a$, where ∇_a is defined with respect to a rest frame of an observer moving with a merged four-velocity u^a. This velocity is merged in the sense that there are three particle species being considered with a single-observer world-line, and hence the cosmic time $t \equiv u^a \nabla_a$ where u^a represents a common four-velocity. The velocity u^a is the determinant of a world-line of a fiducial frame of reference. This is single-fluid approximation and the metric is equation (1.7).

[1]The generalised second law of thermodynamics asserts that the sum of black-hole entropy, and the entropy of the entity (that is dark matter and baryonic matter as one entity) of dark matter and baryonic matter, and radiation fields in the black-hole exterior region never decreases. We use both terms of generalised second law of thermodynamics and second law of thermodynamics interchangeably as they both establish the concept of entropy as a physical property of a thermodynamic system. Furthermore, the generalised second law of thermodynamics is a curious parallel of the second law of thermodynamics [122].

In this chapter, we examine thermodynamics of relativistic and dissipative multi-fluid systems. We let the fluid species of radiation, dark matter and baryonic matter as one entity and dark energy be denoted by $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively [35]. Then energy density ρ in the first Friedmann equation takes the form below

$$\rho \;=\; \rho_{\mathcal{X}} + \rho_{\mathcal{Y}} + \rho_{\mathcal{Z}}, \tag{5.1}$$

where $\rho_{\mathcal{X}}$ represents energy density of radiation, $\mathcal{Y}$ represents energy density of dark matter and baryonic matter as one entity, and $\rho_{\mathcal{Z}}$ represents energy density of dark energy. We assume that $8\pi G = 1 = c^2$. Then the derivative of the first Friedmann equation is

$$\frac{d}{dt}\left(H^2 + \frac{\kappa}{a^2}\right) \;=\; \frac{\dot{\rho}}{3} = \frac{1}{3}\left(\dot{\rho}_{\mathcal{X}} + \dot{\rho}_{\mathcal{Y}} + \dot{\rho}_{\mathcal{Z}}\right). \tag{5.2}$$

This is formulated at a thermodynamical equilibrium state. It expresses the evolution of material content of the universe in terms of either hydrodynamic or single-fluid approximation. Our interest is in the transitions between epochs, where the content of the universe could be described as being in thermal quasi-equilibrium, and where hydrodynamical approximation begins to break down. Some transitions such as those which occured before matter-radiation decoupling can be analysed using hydrodynamic approximation which implies using a single-fluid approach in fluid thermal equilibrium. Other transitions will require a multi-species single-fluid approximation, while others will require either a multi-species multi-fluid approximation or simply a multi-fluid approximation. An example of this is a transition that involves a species being frozen-out when timescales become comparable to the timescale of the cosmic expansion, thus leading to the species breaking from the equilibrium. For example, the radiation-dominated epoch, just before matter-radiation decoupling, but after nucleosynthesis where the material content is made up of plasma of nucleons, electrons and photons (all in thermal equilibrium in which some of the content interact via radiative processes like Thomson scattering) and ends with baryons and electrons becoming separate fluids after breaking away from the thermal equilibrium. As a result, only photon gas is left as a remnant of the earlier cosmic plasma. The initial nucleons, electrons and photon fluid can be treated as a single-fluid, while baryons and electrons are best treated or modelled using the multi-species multi-fluid approximation.

The work we would like to embark on is to model the break-away behaviour (for example, the break-away behaviour of baryons and electrons as mentioned before). For the individual species energy-densities, their conservation equations (from the fluid equation) are given by

$$\begin{aligned} \dot{\rho}_{\mathcal{Z}} + 3H(\rho_{\mathcal{Z}} + p_{\mathcal{Z}}) &= -\mathcal{Q}, \\ \dot{\rho}_{\mathcal{Y}} + 3H(\rho_{\mathcal{Y}} + p_{\mathcal{Y}}) &= \mathcal{Q}, \\ \dot{\rho}_{\mathcal{X}} + 3H(\rho_{\mathcal{X}} + p_{\mathcal{X}}) &= 0, \end{aligned} \tag{5.3}$$

where $\mathcal{Q}$ [89] represents an interaction term. For the above conservation equations, the assumption considered is that the species are in thermal equilibrium. Therefore, the species are evolving together and hence the single-fluid approximation applies. Now, what if one of the species breaks away from the system? Particularly, let species $\mathcal{Y}$ be made of two sub-species $\mathcal{Y}'$ which represents dark matter and $\mathcal{Y}''$ which represents baryonic matter where only $\mathcal{Y}'$ interacts with the species $\mathcal{Z}$, and where the sub-species $\mathcal{Y}''$ is able to break away from equilibrium. This can be presented as follows

$$\begin{aligned} \dot{\rho}_{\mathcal{Z}} + 3H(\rho_{\mathcal{Z}} + p_{\mathcal{Z}}) &= -\mathcal{Q}, \\ \dot{\rho}_{\mathcal{Y}'} + 3H(\rho_{\mathcal{Y}'} + p_{\mathcal{Y}'}) &= \mathcal{Q}, \\ \dot{\rho}_{\mathcal{X}} + 3H(\rho_{\mathcal{X}} + p_{\mathcal{X}}) &= 0, \end{aligned} \tag{5.4}$$

which are in an adjusted or new equilibrium and

$$\dot{\rho}_{\mathcal{Y}''} + 3H(\rho_{\mathcal{Y}''} + p_{\mathcal{Y}''}) = 0, \tag{5.5}$$

which is out of equilibrium with the system in equation (5.4) or the first three. The equilibrium experienced by the system in equation (5.4) is different to that experienced by the system in equation (5.3). This change to a new equilibria is a process that is preceded by a quasi-equilibrium period that requires an irreversible theory for the investigation of the multi-fluid dynamics. As far as we are concerned, no complete version of such a theory exists and this is what we will develop. The starting point for such a development is the extended irreversible thermodynamics theory or the extended MIS formalism. However, this is pursued elsewhere [90]. For this study, an approximation of such a development will be adequate. We now apply the extended MIS formalism to a multi-fluid system in the next section.

5.2 Extended MIS formalism to a multi-fluid system

We will examine fluid species of dark energy denoted by $\mathcal{Z}$, baryonic matter and dark matter as one entity denoted by $\mathcal{Y}$ and radiation denoted by $\mathcal{X}$. This system of fluids is made up of the three species as mentioned. This implies that $\mathcal{I} = \mathcal{X},\ \mathcal{Y},\ \mathcal{Z}$. Note that the interaction involves the dark-sector components only, and though the focus is on the fluids mentioned, this study illustrates the application of the extended MIS formalism to either multi-species multi-fluid systems or multi-fluid systems in general. Perfect fluids which are in equilibrium state do not generate entropy or heat due to friction. This is because their dynamics is devoid of dissipation and is irreversible. However, for most astrophysical and cosmological processes, perfect fluid models are inadequate for modelling such processes. More physically motivated realistic fluids which exhibit irreversible properties are the most suitable environments for modelling most astrophysical and cosmological processes. A relativistic theory of dissipative fluids [72] is required, as some processes in astrophysics and cosmology can only be understood as dissipative processes. For single-fluid approximation, it was shown that irreversible thermodynamics implies that the entropy is no longer conserved but grows per the second law of thermodynamics. Our work here is to find out if this law holds in a multi-fluid environment.

As the fluid evolves, we consider the limit where one or more of the species just begins to freeze-out. Just before this process of freezing-out begins, a dynamical apparent horizon might be a possibility [35]. Now, if the break-away species were to exhibit a uniform acceleration or deceleration in comparison to the remaining species, then one might have a Rindler horizon[2].

We would like to apply the extended MIS formalism in the freeze-out transient period. The MIS theory was initially developed for short-range interactions. It is a speculative theory and it was applied to the holographic[3] Rindler horizon [123]. Knowledge of this regime is scanty. Therefore, it is open for speculation. In the context before the freeze-out, we have the dynamical apparent horizon which evolves into the Rindler-like horizon after the freeze-out period. Therefore, it is most logical to consider the dynamical apparent horizon first. Using spherical symmetry, the metric [$FLRW$ metric in (1.7)] in single-fluid approximation can be written as

$$ds^2 = \gamma_{ab} dx^a dx^b + \tilde{r}^2 d\Omega^2{}_3, \tag{5.6}$$

where $\tilde{r} = a(t)r$, $x^0 = t$ (implying $c = 1$), $x^1 = r$, $\gamma_{ab} = diag\left(-1, \frac{a^2}{1-\kappa r^2}\right)$[4] and $d\Omega^2{}_3 = d\theta + \sin^2\theta d\phi^2$. The relation $\gamma^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0$ determines the dynamical apparent horizon. This

[2]A Rindler coordinate chart has a coordinate singularity where a metric tensor that is expressed in the Rindler coordinates has a vanishing determinant. In certain scenarios, the Rindler horizon can be considered simply as the boundary of the Rindler coordinates.

[3]Holography is a concept that states that any quantum gravity theory should have a description in terms of a quantum field theory which does not contain gravity in one dimension [124].

[4]Note that for this case a and b range over 0 and 1 only.

implies that a vector denoted by $\nabla\tilde{r}$ is null on the apparent horizon surface. In deriving an apparent horizon radius for $FLRW$ [125], the following steps are given

$$\gamma^{ab}\partial_a\tilde{r}\partial_b\tilde{r} \;=\; 0, \tag{5.7}$$

which leads to

$$\gamma^{00}\partial_0\tilde{r}\partial_0\tilde{r} + \gamma^{11}\partial_1\tilde{r}\partial_1\tilde{r} \;=\; 0, \tag{5.8}$$

for $a = 0$, $b = 0$, $a = 1$ and $b = 1$. Using γ^{ab} (which is the inverse of γ_{ab}) and finding the derivatives yields

$$-(\dot{a}r + a\dot{r})^2 + \frac{(1-\kappa r^2)}{a^2}a^2 \;=\; 0, \tag{5.9}$$

which after simplifying leads to

$$\kappa r^2 + \dot{a}^2 r^2 \;=\; 1. \tag{5.10}$$

Factoring out r^2 yields

$$r^2\left(\kappa + a^2\frac{\dot{a}^2}{a^2}\right) \;=\; 1. \tag{5.11}$$

a^2 is factored out and then using $H = \frac{\dot{a}}{a}$ leads to

$$r^2 a^2\left(\frac{\kappa}{a^2} + H^2\right) \;=\; 1. \tag{5.12}$$

Using $\tilde{r} = ar$ yields

$$\tilde{r}_A^2\left(H^2 + \frac{\kappa}{a^2}\right) \;=\; 1, \tag{5.13}$$

which in turn leads to

$$\tilde{r}_A \;=\; \frac{1}{\sqrt{H^2 + \frac{\kappa}{a^2}}}. \tag{5.14}$$

In [126–128] the horizon just before is interpreted as a causal horizon.

Following equations (5.1) and (5.14) yields

$$H^2 + \frac{\kappa}{a^2} = \frac{1}{\tilde{r}_A^2} = \frac{1}{3}(\rho_{\mathcal{X}} + \rho_{\mathcal{Y}} + \rho_{\mathcal{Z}}). \tag{5.15}$$

The evolution with respect to proper time denoted by t yields

$$\frac{d}{dt}\left(H^2 + \frac{\kappa}{a^2}\right) = -2\frac{\dot{\tilde{r}}_A}{\tilde{r}_A^3} = \frac{1}{3}(\dot{\rho}_{\mathcal{Z}} + \dot{\rho}_{\mathcal{X}} + \dot{\rho}_{\mathcal{Y}'}). \tag{5.16}$$

where $\mathcal{Y}'$ represents the remnant of $\mathcal{Y}$ (this remnant is dark matter). This then yields

$$\dot{\tilde{r}}_A = -\frac{\tilde{r}_A^3}{6}(\dot{\rho}_{\mathcal{Z}} + \dot{\rho}_{\mathcal{X}} + \dot{\rho}_{\mathcal{Y}'}), \tag{5.17}$$

for species in the adjusted equilibrium just after the period of freeze-out. Using equation (5.4) (or the fluid equation) in equation (5.17) leads to

$$\dot{\tilde{r}}_A = -\frac{\tilde{r}_A^3}{6}[-3H(\rho_{\mathcal{Z}} + p_{\mathcal{Z}}) - 3H(\rho_{\mathcal{X}} + p_{\mathcal{X}}) - 3H(\rho_{\mathcal{Y}'} + p_{\mathcal{Y}'})], \tag{5.18}$$

which yields

$$\dot{\tilde{r}}_A = \frac{H\tilde{r}_A^3}{2}\sum_{\mathcal{I}}(\rho_{\mathcal{I}} + p_{\mathcal{I}}), \tag{5.19}$$

where $\mathcal{I} = \mathcal{X},\ \mathcal{Y}',\ \mathcal{Z}$. It follows that the individual entropy differential is

$$dS_{\mathcal{I}} = \frac{1}{T_{(\mathcal{I})}}(p_{\mathcal{I}}dV + dE_{\mathcal{I}} - \mu_{\mathcal{I}}dN_{\mathcal{I}}), \tag{5.20}$$

at almost thermal-equilibrium. The total entropy differential is

$$dS = \sum_{\mathcal{I}}\frac{1}{T_{\mathcal{I}}}(p_{\mathcal{I}}dV + dE_{\mathcal{I}} - \mu_{\mathcal{I}}dN_{\mathcal{I}}), \tag{5.21}$$

which can be expressed as

$$dS \simeq \frac{1}{T}\sum_{\mathcal{I}}(p_{\mathcal{I}}dV + dE_{\mathcal{I}} - \mu_{\mathcal{I}}dN_{\mathcal{I}}), \tag{5.22}$$

at almost thermal equilibrium where quasi-equilibrium is defined by demanding the species temperature difference to be insignificant. Hence, $T_{(\mathcal{I})} = T$. The analysis of the full system can still be performed without considering the just mentioned assumption. The full detailed analysis of the complete system is considered elsewhere. For a quasi-equilibrium description that is mediated by the different species contributions, we investigate how total entropy evolves in time in the description of quasi-equilibrium. Given a volume denoted by $V = \frac{4\pi\tilde{r}_A^3}{3}$, individual entropy evolution can be expressed as

$$\dot{S}_{\mathcal{I}} = \frac{1}{T}(4\pi\tilde{r}_A^2 p_{\mathcal{I}}\dot{\tilde{r}}_A + \dot{E}_{\mathcal{I}} - \mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}). \tag{5.23}$$

Total entropy S expressed as $S = \sum_{\mathcal{I}} S_{\mathcal{I}}$ can be considered as

$$S = S(S_{\mathcal{X}}, S_{\mathcal{Y}}, S_{\mathcal{Z}}). \tag{5.24}$$

Horizon temperature denoted by T_h is taken to be equal to a multi-fluid temperature that is denoted by T. This is mediated by the geometry of the universe. Almost all theories of gravity are known to have both bulk and surface expressions. In these theories, the surface terms are usually ignored when determining a field equation, yet when these surface terms are evaluated on the horizon, they yield entropy of the horizon. As shown in [129–133], there exists a connection that is holographic between the surface and bulk terms. It exists indirectly between thermodynamics of the horizon and space-time dynamics. One may now need time evolution of volume. Let volume be denoted by V, internal energy be denoted by E and number density be denoted by N. Now, given V such that $V = \frac{4\pi\tilde{r}_A^3}{3}$, then its time evolution should be $\dot{V} = 4\pi\tilde{r}_A^2\dot{\tilde{r}}_A$. These are important in that they connect the thermodynamical quantities such as energies denoted by E and pressures denoted by P with cosmological quantities, energy densities denoted by ρ and pressures denoted by p. The internal energy for the three species being considered (for illustrative purposes) are

$$E_{\mathcal{I}} = \frac{4\pi}{3}\tilde{r}_A^3\rho_{\mathcal{I}}, \tag{5.25}$$

where again $\mathcal{I} = \mathcal{X},\ \mathcal{Y},\ \mathcal{Z}$. Using Euler's relation we have

$$p_{\mathcal{I}} = p(\rho_{\mathcal{I}}, s_{\mathcal{I}}, n_{\mathcal{I}}), \tag{5.26}$$

where $\rho_{\mathcal{I}} = \frac{E_{\mathcal{I}}}{V}$, $s_{\mathcal{I}} = \frac{S_{\mathcal{I}}}{V}$ and $n_{\mathcal{I}} = \frac{N_{\mathcal{I}}}{V}$. One can assume an equation of state that is barotropic and consistent with adiabatic pressure provided one was to ignore transfer of energy due to internal degrees of freedom but one. However, our interest is in a much broader categorisation of fluid species. The restriction will therefore not be implemented in our analysis. The horizon temperature is related to its radius when thermodynamics of a black hole is extended to cosmology [125, 132–135]. This is such that

$$T_h = \frac{1}{2\pi\tilde{r}_A}, \tag{5.27}$$

where T_h represents horizon temperature. Horizon entropy is defined as $S_h = \frac{4\pi\tilde{r}_A^2}{4G} = 8\pi^2\tilde{r}_A^2$

where $8\pi G = 1$. Hence, the total entropy can be expressed as

$$S_{Tot} \simeq \sum_{\mathcal{I}} S_{\mathcal{I}} + S_h. \tag{5.28}$$

Now, from equation (5.25) it follows that

$$\dot{E}_{\mathcal{I}} = \frac{4\pi}{3} 3\tilde{r}_A^2 \dot{\tilde{r}}_A \rho_{\mathcal{I}} + \frac{4\pi}{3}\tilde{r}_A^3 \dot{\rho}_{\mathcal{I}}. \tag{5.29}$$

Substituting the fluid equation in equation (5.29) leads to

$$\dot{E}_{\mathcal{I}} = 4\pi\tilde{r}_A^2 \dot{\tilde{r}}_A \rho_{\mathcal{I}} + \frac{4\pi}{3}\tilde{r}_A^3[-3H(\rho_{\mathcal{I}} + p_{\mathcal{I}})], \tag{5.30}$$

of which after simplifying yields

$$\dot{E}_{\mathcal{I}} = 4\pi\tilde{r}_A^2 \dot{\tilde{r}}_A \rho_{\mathcal{I}} - 4\pi\tilde{r}_A^3 H(\rho_{\mathcal{I}} + p_{\mathcal{I}}). \tag{5.31}$$

Substituting this equation (5.31) in equation (5.23) leads to

$$\dot{S}_{\mathcal{I}} = \frac{1}{T}[4\pi\tilde{r}_A^2 p_{\mathcal{I}} \dot{\tilde{r}}_A + 4\pi\tilde{r}_A^2 \dot{\tilde{r}}_A \rho_{\mathcal{I}} - 4\pi\tilde{r}_A^3 H(\rho_{\mathcal{I}} + p_{\mathcal{I}}) - \mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}]. \tag{5.32}$$

Expanding and then factoring out $\frac{4\pi\tilde{r}_A^2}{T}$ in the 1^{st} four terms on the right-hand side yields

$$\dot{S}_{\mathcal{I}} = \frac{4\pi\tilde{r}_A^2}{T}[p_{\mathcal{I}}\dot{\tilde{r}}_A + \dot{\tilde{r}}_A\rho_{\mathcal{I}} - \tilde{r}_A H(\rho_{\mathcal{I}} + p_{\mathcal{I}})] - \frac{1}{T}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}. \tag{5.33}$$

Factoring out $\dot{\tilde{r}}_A$ on the 1^{st} two terms of the right-hand side leads to

$$\dot{S}_{\mathcal{I}} = \frac{4\pi\tilde{r}_A^2}{T}[(\rho_{\mathcal{I}} + p_{\mathcal{I}})\dot{\tilde{r}}_A - (\rho_{\mathcal{I}} + p_{\mathcal{I}})\tilde{r}_A H] - \frac{1}{T}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}. \tag{5.34}$$

Factorising out $(\rho_{\mathcal{I}} + p_{\mathcal{I}})$ leads to

$$\dot{S}_{\mathcal{I}} = \frac{4\pi\tilde{r}_A^2}{T}(\rho_{\mathcal{I}} + p_{\mathcal{I}})(\dot{\tilde{r}}_A - \tilde{r}_A H) - \frac{1}{T}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}. \tag{5.35}$$

Time evolution of total entropy takes the form

$$\dot{S}_{Tot} \simeq \dot{S}_{\mathcal{X}} + \dot{S}_{\mathcal{Y}'} + \dot{S}_{\mathcal{Z}} + \dot{S}_h, \tag{5.36}$$

where an overdot represents a derivative with respect to cosmic time t and $\dot{S}_{\mathcal{X}},\ \dot{S}_{\mathcal{Y}'},\ \dot{S}_{\mathcal{Z}},\ \dot{S}_h$ represent the evolution of the radiation, dark matter, dark energy and horizon entropies, respectively. From both equations (5.35) and (5.36)

$$\begin{aligned} \dot{S}_{\mathcal{Z}} &= \frac{4\pi\tilde{r}_A^2}{T}\left[(\rho_{\mathcal{Z}}+p_{\mathcal{Z}})(\dot{\tilde{r}_A}-\tilde{r}_AH)-\frac{\tilde{r}_A}{3}\mathcal{Q}\right]-\frac{1}{T}\mu_{\mathcal{Z}}\dot{N}_{\mathcal{Z}}, \\ \dot{S}_{\mathcal{Y}'} &= \frac{4\pi\tilde{r}_A^2}{T}\left[(\rho_{\mathcal{Y}'}+p_{\mathcal{Y}'})(\dot{\tilde{r}_A}-\tilde{r}_AH)+\frac{\tilde{r}_A}{3}\mathcal{Q}\right]-\frac{1}{T}\mu_{\mathcal{Y}'}\dot{N}_{\mathcal{Y}'}, \\ \dot{S}_{\mathcal{X}} &= \frac{4\pi\tilde{r}_A^2}{T}[(\rho_{\mathcal{X}}+p_{\mathcal{X}})(\dot{\tilde{r}_A}-\tilde{r}_AH)]-\frac{1}{T}\mu_{\mathcal{X}}\dot{N}_{\mathcal{X}}, \end{aligned} \tag{5.37}$$

where $\mathcal{Q}$ represents the gravitational interaction. The horizon entropy S_h evolves as $\dot{S}_h = 16\pi^2\tilde{r}_A\dot{\tilde{r}_A}$. Hence, the total entropy obeys the evolution equation

$$\dot{S}_{Tot} \simeq \frac{4\pi\tilde{r}_A^2}{T}[(\rho_{\mathcal{I}}+p_{\mathcal{I}})(\dot{\tilde{r}_A}-\tilde{r}_AH)]-\frac{1}{T}\sum_{\mathcal{I}}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}+16\pi^2\tilde{r}_A\dot{\tilde{r}_A}. \tag{5.38}$$

Substituting the horizon temperature in T leads to

$$\dot{S}_{Tot} \simeq 8\pi^2\tilde{r}_A^3\sum_{\mathcal{I}}[(\rho_{\mathcal{I}}+p_{\mathcal{I}})(\dot{\tilde{r}_A}-\tilde{r}_AH)]-\sum_{\mathcal{I}}2\pi\tilde{r}_A\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}+16\pi^2\tilde{r}_A\dot{\tilde{r}_A}. \tag{5.39}$$

Substituting equation (5.19) in equation (5.39) yields

$$\dot{S}_{Tot} \simeq 8\pi^2\tilde{r}_A^3\sum_{\mathcal{I}}\left\{[\rho_{\mathcal{I}}+p_{\mathcal{I}}]\left[\frac{H\tilde{r}_A^3}{2}(\rho_{\mathcal{I}}+p_{\mathcal{I}})-\tilde{r}_AH\right]\right\}+16\pi^2\tilde{r}_A\frac{H\tilde{r}_A^3}{2}\sum_{\mathcal{I}}(\rho_{\mathcal{I}}+p_{\mathcal{I}})-2\pi\tilde{r}_A\sum_{\mathcal{I}}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}. \tag{5.40}$$

Expanding out the terms in curly brackets and simplifying the terms on the second summation sign of the right-hand side leads to

$$\dot{S}_{Tot} \simeq 8\pi^2\tilde{r}_A^3\left[\frac{H\tilde{r}_A^3}{2}(\sum_{\mathcal{I}}\rho_{\mathcal{I}}+p_{\mathcal{I}})^2-\tilde{r}_AH\sum_{\mathcal{I}}(\rho_{\mathcal{I}}+p_{\mathcal{I}})\right]+8\pi^2\tilde{r}_A^4H\sum_{\mathcal{I}}(\rho_{\mathcal{I}}+p_{\mathcal{I}})-2\pi\tilde{r}_A\sum_{\mathcal{I}}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}. \tag{5.41}$$

Expanding out the terms in square brackets yields

$$\dot{S}_{Tot} \simeq 4\pi^2\tilde{r}_A^6H[\sum_{\mathcal{I}}(\rho_{\mathcal{I}}+p_{\mathcal{I}})]^2-8\pi^2\tilde{r}_A^4H\sum_{\mathcal{I}}(\rho_{\mathcal{I}}+p_{\mathcal{I}})+8\pi^2\tilde{r}_A^4H\sum_{\mathcal{I}}(\rho_{\mathcal{I}}+p_{\mathcal{I}})-2\pi\tilde{r}_A\sum_{\mathcal{I}}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}. \tag{5.42}$$

Cancellation of the terms on the second and third summation signs leads to

$$\dot{S}_{Tot} \simeq 4\pi^2\tilde{r}_A^6H[\sum_{\mathcal{I}}(\rho_{\mathcal{I}}+p_{\mathcal{I}})]^2-2\pi\tilde{r}_A\sum_{\mathcal{I}}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}. \tag{5.43}$$

It is comprehensible that the result holds regardless of the nature of $\mathcal{Q}$. The finding amends results found in [89] where a new equation of state for one of the species (*e.g.* dark energy) emerges for a critical $\dot{S}_{Tot} = 0$ thus implying that

$$[\sum_{\mathcal{I}}(\rho_{\mathcal{I}} + p_{\mathcal{I}})]^2 = \frac{1}{2\pi\tilde{r}_A^5 H}\sum_{\mathcal{I}}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}. \tag{5.44}$$

Expanding the left-hand side leads to

$$p_{\mathcal{Z}} + \rho_{\mathcal{Z}} + \rho_{\mathcal{X}} + p_{\mathcal{X}} + \rho_{\mathcal{Y}'} + p_{\mathcal{Y}'} = \sqrt{\frac{1}{2\pi\tilde{r}_A^5 H}\sum_{\mathcal{I}}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}}. \tag{5.45}$$

Dividing through by $\rho_{\mathcal{Z}}$ yields

$$\frac{p_{\mathcal{Z}}}{\rho_{\mathcal{Z}}} + 1 + \frac{1}{\rho_{\mathcal{Z}}}(\rho_{\mathcal{X}} + p_{\mathcal{X}}) + \frac{1}{\rho_{\mathcal{Z}}}(\rho_{\mathcal{Y}'} + p_{\mathcal{Y}'}) = \frac{1}{\rho_{\mathcal{Z}}}\sqrt{\frac{1}{2\pi\tilde{r}_A^5 H}\sum_{\mathcal{I}}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}}. \tag{5.46}$$

Re-arranging this equation where $\frac{p_{\mathcal{Z}}}{\rho_{\mathcal{Z}}}$ would be the only term left on the left-hand side leads to

$$\begin{aligned}\frac{p_{\mathcal{Z}}}{\rho_{\mathcal{Z}}} &= -1 - (\rho_{\mathcal{Y}'} + p_{\mathcal{Y}'})\frac{1}{\rho_{\mathcal{Z}}} - (\rho_{\mathcal{X}} + p_{\mathcal{X}})\frac{1}{\rho_{\mathcal{Z}}} \\ &+ \frac{1}{\rho_{\mathcal{Z}}}\sqrt{\frac{1}{2\pi\tilde{r}_A^5 H}\sum_{\mathcal{I}}\mu_{\mathcal{I}}\dot{N}_{\mathcal{I}}},\end{aligned} \tag{5.47}$$

such that $\omega_{\mathcal{Z}} = \frac{p_{\mathcal{Z}}}{\rho_{\mathcal{Z}}}$. For a case that is extreme such that either dark energy or dark matter or radiation was to freeze-out, a similar analysis can be carried out.

5.3 Explanation of findings

We examined equation (5.39). By definition $\tilde{r}_A, H, p_{\mathcal{I}}$ and $\rho_{\mathcal{I}}$ are positive definite. Equations (5.17) and (5.19) ensure $\dot{\tilde{r}}_A > 0$. A summation result is positive as the radius of the horizon is greater than the Hubble parameter. One can confirm this by setting

$$(\rho_{\mathcal{I}} + p_{\mathcal{I}})(\dot{\tilde{r}}_A - \tilde{r}_A H) > 0, \tag{5.48}$$

and this infers that

$$\frac{\dot{\tilde{r}}_A}{\tilde{r}_A} > H = \frac{\dot{a}}{a}. \tag{5.49}$$

This is expected for a case where a surface term is ignored. Given that $\sum_{\mathcal{I}} \mu_{\mathcal{I}} \dot{N}_{\mathcal{I}} < 0$, then definitely $\dot{S}_{Tot} > 0$. This is expected for Gibbs free energy for negative chemical potentials. One can label the Gibbs free energy as E_G. Then, $\dot{E}_G < 0$ infers that $\dot{S}_G > 0$. This establishes the generalised second law of thermodynamics for the interacting dark-sector. This occurs at the onset of a freeze-out. When a chemical interaction is included in a multi-fluid approximation, the second law of thermodynamics is conserved. Effects of a non-zero chemical potential on the equation of state of dark energy in the single-fluid approximation was investigated in reference [136]. It was found that the equation of state depends heavily on the magnitude and sign of the chemical potential. Equation (5.47) amends that result and has potential to lift $\omega_{\mathcal{Z}}$ into the non-phantom state. Providing an estimate would be tricky as this would require an accurate estimation of $\tilde{r}_A$ and $\sum_{\mathcal{I}} \mu_{\mathcal{I}} \dot{N}_{\mathcal{I}}$ in the state of quasi-equilibrium. In light of the above statements, we have examined a cosmological scenario that involves three particle species. Two of these interact both gravitationally and chemically upon using the multi-fluid approach.

5.4 Multi-fluid formalism application to the interacting dark-sector model

Having established the conservation of the second law of thermodynamics for interacting multi-fluid systems, specifically the interacting dark-sector, one can choose to apply the slightly modified convective variational formalism to the sector. Predictions of existence of dark energy denoted by $\mathcal{Z}$ and dark matter denoted by $\mathcal{Y}'$ come from cosmological observations [44]. After fitting a model that is theoretical to the composition of the universe, and given an amalgam of different observations of cosmological scales, the universe is shown to comprise of 68% dark energy, 27% dark matter and 5% baryonic matter denoted by $\mathcal{Y}''$. However, model fitting apart from being predictive, does not give physics of constituent particles. Surprisingly, two species that we know very little about are the very ones that have profound effects on the dynamics or evolution of the universe hence leading to structure formation in the early universe and an accelerated expansion in the late universe. On one hand, knowledge scarcity of dark matter and dark energy begs the question whether the single-fluid model is partly liable and on the other, could the multi-fluid formalism shed any light on the puzzle? The very foundation of the Copernican principle and the more precise cosmological principle are touched upon by these questions [137, 138]. From the single-fluid model, it appears that there is a need for modification of the cosmological principle in a given cosmological epoch. However, our main interest is not examination of the cosmological principle.

For dark matter, one can rule out a number of several possible candidates although we do not know what it is made of. These include stars, planets, baryons, anti-matter and large galaxy-sized black holes. There are a few viable dark matter possibilities. Baryonic matter tied-up in brown dwarfs or heavy elements is thought to make up dark matter and they are known as $MACHOs$. However, the most popular view is that dark matter is made up of $WIMPS$. In other words, dark matter is not baryonic at all. This implies that any analysis involving dark matter requires assumptions about its nature. At this stage, one can ask, can any of these candidates allow for entrainment? In other words, can one use entrainment to distinguish characteristics between the candidates? For example, one can consider a case where dark energy should be modelled as scalar fields that are light and is only gravitationally coupled to dark matter where other interactions are assumed to be insignificant. Couplings that exhibit conformal and disconformal transformations of the dark-sector geometry, particularly that of dark matter are considered. For such a theory, the Lagrangian density is expressed as

$$d\mathcal{S} = \sqrt{-g}\left[\frac{\mathcal{K}(g)}{2\kappa} + \mathcal{L}_{(SM)} + \mathcal{L}_{(\mathcal{Z})} + \mathcal{L}_{(\mathcal{Z}-\mathcal{Y}')}\right] + \sqrt{\tilde{g}}\tilde{\mathcal{L}}_{(\mathcal{Y}')}, \tag{5.50}$$

such that SM represents non dark-sector given by model particles that are standard, $\mathcal{K}(g)$ represents a function of g where g is a determinant of g_{ab}, $\mathcal{L}_{(\mathcal{Z})}$ represents a scalar field, *e.g.* quintessence, field of the form

$$\mathcal{L}_{(\mathcal{Z})} = \frac{\nabla^c\phi\nabla_c\phi}{2} - V(\phi), \tag{5.51}$$

and

$$\tilde{\mathcal{L}}_{(\mathcal{Y}')} = \tilde{\mathcal{L}}_{(\mathcal{Y}')}(\tilde{g}_{cd}, \varphi), \tag{5.52}$$

where φ represents dark matter potential. Note that $\kappa^2 = 8\pi G$. The gravitational coupling is conciliated via

$$\tilde{g}_{ab} = c_1(\phi)g_{ab} + c_2(\phi)\phi_{,a}\phi_{,b}, \tag{5.53}$$

where c_1 and c_2 represent conformal and disconformal factors, respectively. Now, let background space-time be $FLRW$ such that

$$ds^2 = g_{ab}dx^a dx^b = a^2(\eta)(-d\eta^2 + \delta_{\mathcal{I}\mathcal{J}}dx^{\mathcal{I}}dx^{\mathcal{J}}). \tag{5.54}$$

Note that the disconformal metric arises from

$$d\tilde{s}^2 = \tilde{g}_{ab}dx^a dx^b = c_1(\phi)a^2(\eta)(-\gamma^2 d\eta^2 + \delta_{\mathcal{I}\mathcal{J}}dx^{\mathcal{I}}dx^{\mathcal{J}}), \tag{5.55}$$

where the disconformal scalar is given by

$$\gamma^2 = 1 + \left(\frac{c_2}{c_1}\right) g^{ab}\phi_{,a}\phi_{,b}. \tag{5.56}$$

Then, in this theory, components that play a role are a massless relativistic component or radiative component, a baryonic component, a dark matter component and a scalar field component. An assumption is made that neutrinos have no mass. The illustration is restricted to a case where the baryons and the relativistic components are not coupled to the scalar field [139]. The fundamental relation below

$$\rho_X + p_X = \mu_X n_X + f_X(T, S), \tag{5.57}$$

is obeyed by each species X. The form below

$$\sum_X(\rho_X + p_X) = \sum_X \mu_X n_X + f(T, S), \tag{5.58}$$

should be obeyed by the system as a whole such that $f(T, S)$ is a function of T and S. This expresses thermodynamical coupling of the components. This should not be muddled up with either gravitational coupling or any other couplings that may exhibit in the system, particularly entrainment. For one to examine the other couplings, one can express the conservation equations in terms of parameters that would allow one to examine the couplings. Evolution of number densities obey conservaton equations

$$\begin{aligned} \mu_{\mathcal{Z}}\dot{n}_{\mathcal{Z}} + 3H\mu_{\mathcal{Z}}n_{\mathcal{Z}} &= -\mathcal{Q}, \\ \mu_{\mathcal{Y}'}\dot{n}_{\mathcal{Y}'} + 3H\mu_{\mathcal{Y}'}n_{\mathcal{Y}'} &= \mathcal{Q}', \\ \mu_X\dot{n}_X + 3H\mu_X n_X &= \mathcal{Q}' - \mathcal{Q}, \end{aligned} \tag{5.59}$$

such that $\mathcal{Q}$ and $\mathcal{Q}'$ represent interaction terms. The negative sign represents an interchange of energies between species. The case $\mathcal{Q} = \mathcal{Q}'$ is analogous to a model where dark energy interacts only with dark matter. For *e.g.* in [139], X represents gravitationally uncoupled radiative and baryonic components. $\mathcal{Q}$ is expressed as $\mathcal{Q} = \dot{f} + 3Hf$ and it represents interaction between dark-sector constituents while H represents the Hubble parameter. It is most advantageous to think of $\mathcal{Q} = \mathcal{Q}_{grav} + \mathcal{Q}_{chem}$ where $\mathcal{Q}_{grav}$ represents gravitational interaction and $\mathcal{Q}_{chem}$ represents a chemical interaction. Interaction between the dark-sector and baryons is assumed to be insignificant. One can clearly determine the form of $\mathcal{Q}_{grav}$ as is done in reference [139] for gravitational coupling if the scalar field follows the Klein-Gordon equation and couples to dark matter in such a way that

$$\ddot{\phi} + 2\dot{\phi}H + \frac{dV}{d\phi}a^2 \quad = \quad \mathcal{Q}_{grav}a^2. \tag{5.60}$$

Particularly

$$\mathcal{Q}_{grav} \quad = \quad -\frac{a^2{c'}_1 - 2c_2(3H\dot{\phi} + a^2V') + \left(\frac{{c'}_1}{c_1} - \frac{{c'}_2}{2c_2}\right)\dot{\phi}^2}{2[a^2c_1 + c_2(a^2\rho_{\mathcal{Y}'} - \dot{\phi}^2)]}\rho_{\mathcal{Y}'}. \tag{5.61}$$

It is of significance to note that in that study, exchange of energy is between dark energy and dark matter which are only gravitationally coupled. Then, stress-energy-momentum tensor $T^a_{\ b}s$ for either dark matter or dark energy are not conserved individually. A Friedmann equation in the form

$$H^2 \quad = \quad \frac{\kappa^2}{3}a^2\left(\sum_{\mathcal{I}}\mu_{\mathcal{I}}n_{\mathcal{I}} + f\right), \tag{5.62}$$

can be suggested after considering all couplings that are of a gravitational and chemical nature, and $\mathcal{I}$ represents either all species or constituents. Note also that f represents an entropy temperature function. Now, by specifying the constants c_1 and c_2, potential and of great significance an entropy temperature function f, one can proceed to analyse dynamics. The interaction constant can then be established. This is phenomenology. It adds to the repertoire of the dark-sector interactions, and it is over and above the traditional case(s) of gravitation.

5.5 Chapter summary

The potential use of the multi-fluid approach to model flow of fluid where one or more of the fluid species suffers freeze-out hence separating from the rest of the flow has been investigated. We examined the cosmological case involving the interacting dark-sector and radiation for illustrative purposes. The application is speculative. Such speculation is justified as our knowledge of the dark-sector is scanty. Therefore, we assumed the universe to be a thermodynamical system that is enclosed by the dynamical apparent horizon. We then calculated the separate entropy variation for each of the fluid species. Total entropy in the universe is given by the sum of the entropy variations together with that of the common horizon. The generalised second law of thermodynamics is found to be valid. Take note that we used the dynamical apparent horizon(s) only. This means cases involving other types of horizons were not considered. Though it has been shown that the second law of thermodynamics holds for the interacting dark-sector in the presence of radiation, more examination is needed to make the results applicable to either quantitative or numerical analysis of a cosmological nature.

Now, the second law of thermodynamics holds in interacting multi-fluid environments. For the interacting dark-sector specifically, we examined the entrainment effect of the interaction between dark matter and dark energy, and we found that the entrainment effect of the interaction of dark matter and dark energy suggests a mutual relative modulation of the growth behaviour of the two densities. This might aid in resolving the coincidence problem. Chemical coupling and gravitational coupling of the dark-sector were constructed, of which we considered an example of conformal-disconformal coupling of a gravitational nature as a contrasting example. The convective variational approach for interacting multi-fluid systems could be useful in distinguishing cosmological features of the couplings. This is something that can be probed by present time cosmological observations. One would then invoke restrictions on the nature of the interaction examined. Extending the formalism developed to a general formalism to examine the growth of dark matter perturbations in the presence of interactions between dark matter and dark energy would be interesting. The study of the signature of such interactions on the temperature anisotropies of the large-scale cosmic microwave background would then be possible. It was found that [50] the effect of such interactions has a significant signature on both the growth of dark matter structure and the late integrated Sachs Wolfe effect in the single-fluid approximation. However, how would this change, given the multi-fluid formalism that has been considered, should be a topic that can (or will) be examined in the near future.

Chapter 6

Evolutionary history of cosmological magnetic fields

6.1 Prelude to history of magnetic fields

In this chapter we consider the evolution of inflation-generated, cosmological magnetic fields. We use the single-fluid model in the context of the variational formalism to derive equations that will enable us to examine the evolution of inflation-generated, cosmological magnetic fields from around the beginning of the radiation-dominated epoch to current time. The single-fluid models that will be used are the radiation-dominated epoch and the matter-dominated epoch. These models are in the context of the variational formalism. In other words we use the variational approach to derive equations of motion that will enable us to examine the evolution of inflation-generated, cosmological magnetic fields during the single-fluid models of the radiation-dominated and matter-dominated epochs. With this, we study relativistic fluids with applications to cosmology in that we analyse the behaviour of evolution of inflation-generated, cosmological magnetic fields in single-fluid models of the radiation-dominated or matter-dominated epochs. That is, we investigate how the behaviour of evolution of the cosmological magnetic fields is either influenced or affected as they evolve in the single-fluid models of the radiation-dominated or matter-dominated epochs. Before we begin our examination, we review the history on inflation-generated, cosmological magnetic fields and we examine their evolution up to around the beginning of the radiation-dominated epoch.

It is believed that inflation [54, 140–143] is a prime candidate for the production of primeval magnetic fields. Inflation provides the kinematic means of producing modes that would result in very-long-wavelength effects at early times (such as at the beginning of the radiation-dominated epoch) through micro-physical processes that operated on scales that are less than the Hubble radius. An electromagnetic wave with physical wavelength denoted by λ_{phys} is such that $\lambda_{phys} \gtrsim H^{-1}$ has the appearance of static electric and magnetic fields denoted as $\vec{E}$ and $\vec{B}$, respectively (note that H represents the expansion rate of the universe). This implies that the long-wavelength photons ($\lambda_{phys} \gg H^{-1}$) can yield large-scale magnetic fields (which evolve super-adiabatically on those scales). Inflation provides the dynamical means of exciting electromagnetic waves: de Sitter-generated quantum-mechanical fluctuations excite modes with $\lambda_{phys} < H^{-1}$. The energy density in the mode with $\lambda_{phys} \simeq H^{-1}$ is $\frac{d\rho}{dk} \sim H^3$ where ρ represents density and k represents wavenumber. During inflation, the universe is devoid of plasma[1] and is not a good conductor. Therefore, the magnetic flux is not necessarily conserved (even though magnetic fields evolve adiabatically throughout de Sitter phase [56]) and the primeval magnetic flux strength can increase.

In this work, we will also consider the evolution of magnetic fields due to the direct coupling of the gravitational and electromagnetic fields as magnetic fields evolve much later in the expansion of the universe (that is much later during either the radiation-dominated epoch or the matter-dominated epoch). We then review additional terms in the Lagrangian either of the form RA_aA^a or RA^2 or $R_{ab}A^aA^b$ where R represents the curvature scalar, R_{ab} represents the Ricci tensor and A^b represents the electromagnetic four-vector potential. A photon will have an effective, time-dependent mass because of the additional terms. At first glance, this is quite undesired as charge conservation is broken [54]. However, the terms do not lead to any effects which contradict either present-day observations or experiments. Because of the terms, the photon mass that arises is $m_\gamma \sim R^{\frac{1}{2}}$ where $R^{\frac{1}{2}} \sim H$. Note that charge non-conservation would only manifest itself on scales of either the horizon or larger (anyway, conductivity is very low on those scales hence it would be reasonable to assume that there is no charge). However, again, this has no observable consequences. Hence, they would not affect the propagation of photons outside massive bodies. One might worry about corrections the terms would introduce to the equation of state in the radiation-dominated universe. The corrections are negligible for temperatures where the evolution of the universe is relatively understood (*e.g.*, during recombination) and they are of the order $\frac{H^2}{T^2} \sim \frac{T^2}{m^2{}_{pl}}$ where T represents temperature and m_{pl} represents Planck mass. Hence, these terms cannot spoil successful predictions which are made by using the standard Maxwell equations.

Furthermore, standard Maxwell's equations cannot describe a homogeneous and isotropic universe with a uniformly distributed net charge because the electromagnetic field tensor in such a universe must be vanishing everywhere. One can call this the type I problem [144]. For a closed universe with non-zero net charge, standard Maxwell's equations always fail

[1] Plasma is one of the four fundamental states of matter, and consists of a gas of ions and free electrons. It is the most abundant form of ordinary matter in the universe [145].

regardless of the space-time symmetry and charge distribution. One can call this a type II problem. The simplest way to resolve the problems is by introducing either the term RA_aA^a or RA^2 in the Lagrangian [144]. The electromagnetic field equations that arise as a result of the introduction of the term RA^2 can naturally arise from spontaneous symmetry breaking, and the Higgs mechanism in quantum field theory where photons acquire a mass by devouring massless Goldstone bosons. However, photons lose their mass again when the symmetry is restored. Hence, the problems reappear. The other way of resolving the problems is by introducing the term $R_{ab}A^aA^b$ in the Lagrangian. The electromagnetic field equations that arise due to the term $R_{ab}A^aA^b$ do not introduce a new dimensional parameter and so return to Maxwell's equations in either a flat or Ricci-flat space-time. This implies that in a Ricci-flat space-time, gauge invariance is restored.

6.2 Inflationary magnetic fields

Since electromagnetic fields permeate the inflationary universe [54], one can consider equations [144]

$$\nabla_b F^{ab} - \phi_1 R A^a - \phi_2 R^a{}_b A^b = J^a, \tag{6.1}$$

and

$$\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0, \tag{6.2}$$

during the inflationary phase where ϕ_1 and ϕ_2 represent coupling constants, J^a represents a 4-vector current, and F_{ab} represents an electromagnetic field tensor. Symmetry of Christoffel symbols denoted by $\Gamma^d{}_{ab} = \Gamma^d{}_{ba}$ enables one to replace the conventional derivatives instead of covariant ones. Then equation (6.2) implies that

$$\partial_a F_{bc} - \Gamma^d{}_{ab} F_{dc} - \Gamma^d{}_{ac} F_{bd} + \partial_b F_{ca} - \Gamma^d{}_{bc} F_{da} - \Gamma^d{}_{ba} F_{cd} + \partial_c F_{ab} - \Gamma^d{}_{ca} F_{db} - \Gamma^d{}_{cb} F_{ad} = 0. \tag{6.3}$$

One can then write equation (6.3) in a more suitable form. The equation is given below

$$\partial_a F_{bc} - \Gamma^d{}_{ab} F_{dc} - \Gamma^d{}_{ac} F_{bd} + \partial_b F_{ca} - \Gamma^d{}_{bc} F_{da} + \Gamma^d{}_{ba} F_{dc} + \partial_c F_{ab} + \Gamma^d{}_{ca} F_{bd} + \Gamma^d{}_{cb} F_{da} = 0. \tag{6.4}$$

This then yields

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0. \tag{6.5}$$

All spatial derivatives are with respect to co-moving coordinates [64].

During inflation, the universe is free from plasma. Therefore, its electrical conductivity is very poor. In other words, there is no net charge in the inflationary universe. Equation (6.1) will then be reduced to

$$\nabla_b F^{ab} \;=\; 0. \tag{6.6}$$

This implies that for source-free fields, equation (6.6) is obtained [57,146]. To study equations (6.5) and (6.6), one can write them in terms of $\vec{E}$ which represents an electric field and $\vec{B}$ which represents a magnetic field, using [54]

$$F_{ab} = \begin{bmatrix} 0 & -a^2E_x & -a^2E_y & -a^2E_z \\ a^2E_x & 0 & a^2B_z & -a^2B_y \\ a^2E_y & -a^2B_z & 0 & a^2B_x \\ a^2E_z & a^2B_y & -a^2B_x & 0 \end{bmatrix} \tag{6.7}$$

One can then use F_{ab} which is represented by the matrix (6.7). Equations (6.5) and (6.6) can then be recast into the form

$$\frac{1}{a^2}\frac{\partial(a^2\vec{B})}{\partial\eta} + \nabla\times\vec{E} \;=\; 0, \tag{6.8}$$

and

$$\frac{1}{a^2}\frac{\partial(a^2\vec{E})}{\partial\eta} - \nabla\times\vec{B} \;=\; 0. \tag{6.9}$$

where η represents conformal time. Equations (6.8) and (6.9) are expressed as

$$\frac{\partial(a^2\vec{E})}{\partial\eta} - \nabla\times a^2\vec{B} \;=\; 0, \tag{6.10}$$

and

$$\frac{\partial(a^2\vec{B})}{\partial\eta} + \nabla\times a^2\vec{E} \;=\; 0, \tag{6.11}$$

respectively. Due to conformal invariance, the electric and magnetic fields can be re-scaled as

$$\tilde{E} \equiv a^2\vec{E}, \tag{6.12}$$

and

$$\tilde{B} \equiv a^2 \vec{B}, \tag{6.13}$$

respectively [147]. This implies that equations (6.10) and (6.11) can be written out as

$$\frac{\partial(\tilde{E})}{\partial\eta} - \nabla \times \tilde{B} \equiv 0, \tag{6.14}$$

and

$$\frac{\partial(\tilde{B})}{\partial\eta} + \nabla \times \tilde{E} \equiv 0. \tag{6.15}$$

Note that

$$\nabla \times \nabla \times \tilde{B} \;\; = \;\; \nabla(\nabla \cdot \tilde{B}) - \nabla^2 \tilde{B}, \tag{6.16}$$

and upon using $\nabla a^2 = 0$, yields

$$\nabla \times \nabla \times \tilde{B} \;\; = \;\; a^2 \nabla(\nabla \cdot \vec{B}) - \nabla^2 \tilde{B}. \tag{6.17}$$

Given that $\nabla \cdot \vec{B} = 0$, equation (6.17) leads to

$$\nabla \times \nabla \times \tilde{B} \;\; = \;\; 0 - \nabla^2 \tilde{B}, \tag{6.18}$$

which in turn yields

$$\nabla \times \nabla \times \tilde{B} \;\; = \;\; -\nabla^2 \tilde{B}. \tag{6.19}$$

Taking the curl of equation (6.14), using equation (6.19) and then making use of equation (6.15) to eliminate $\tilde{E}$ leads to

$$\frac{\partial^2 \tilde{B}}{\partial \eta^2} - \nabla^2 \tilde{B} \;\; = \;\; 0. \tag{6.20}$$

This can be expressed as

$$\tilde{B}'' - \nabla^2 \tilde{B} \;\; = \;\; 0, \tag{6.21}$$

where primes represent derivatives with respect to conformal time. The evolution of inflation-generated magnetic fields on scales that are not well-above the Hubble horizon and in environments of poor conductivity are represented by equation (6.21).

To solve equation (6.21), one can introduce the harmonic splitting of

$$\tilde{B} = \sum_k \tilde{B}_k Q^k, \tag{6.22}$$

where $\tilde{B}_k$ represents a k^{th} inflation-generated magnetic mode [56,148,149]. This is also shown in arXiv:1403.5505, an article by B. Osano. Q^k represent vector harmonics that are pure and satisfy the conditions

$$\nabla Q^k = Q'^k, \tag{6.23}$$

and the Laplace-Beltrami equation of a particular version given by [149]

$$\nabla^2 Q^k = -k^2 Q^k. \tag{6.24}$$

Applying the above decomposition (6.24) to (6.21), yields the decoupling of the harmonics [149]. Either the wave formula or equation (6.21) of the k^{th} inflation-generated magnetic mode is expressed as

$$\tilde{B}''_k + k^2 \tilde{B}_k = 0. \tag{6.25}$$

The characteristic equation of (6.25) is

$$m^2{}_o + k^2 = 0. \tag{6.26}$$

The roots of this equation are complex and they are represented by

$$m_{o1} = \alpha_1 + i\alpha_2, \tag{6.27}$$

and

$$m_{o2} = \alpha_1 - i\alpha_2. \tag{6.28}$$

Solving equation (6.26) leads to

$$m_{o1} = 0 + ik, \tag{6.29}$$

and

$$m_{o2} = 0 - ik, \tag{6.30}$$

implying that $\alpha_1 = 0$ and $\alpha_2 = k$. One arrives at the characteristic equation by assuming that all solutions to the differential equation (6.25) is in the form

$$\tilde{B}_k(\eta) \quad = \quad e^{m\eta}. \tag{6.31}$$

Substituting m_{o1} and m_{o2} in equation (6.31) yields

$$\tilde{B}_{k1}(\eta) \quad = \quad e^{ik\eta}, \tag{6.32}$$

and

$$\tilde{B}_{k2}(\eta) \quad = \quad e^{-ik\eta}. \tag{6.33}$$

Using Euler's relation, equations (6.32) and (6.33) can be expressed as

$$\tilde{B}_{k1}(\eta) \quad = \quad \cos k\eta + i \sin k\eta, \tag{6.34}$$

and

$$\tilde{B}_{k2}(\eta) \quad = \quad \cos k\eta - i \sin k\eta. \tag{6.35}$$

One can let

$$\tilde{B}_k(\eta) \quad = \quad c_i \tilde{B}_{k1}(\eta) + c_{ii} \tilde{B}_{k2}(\eta), \tag{6.36}$$

be also a solution where c_i and c_{ii} are constants. Writing equation (6.36) explicitly implies

$$\tilde{B}_k(\eta) \quad = \quad c_i e^{ik\eta} + c_{ii} e^{-ik\eta}. \tag{6.37}$$

This leads to

$$\tilde{B}_k(\eta) \quad = \quad c_i(\cos k\eta + i \sin k\eta) + c_{ii}(\cos k\eta - i \sin k\eta), \tag{6.38}$$

and this in turn yields

$$\tilde{B}_k(\eta) \quad = \quad (c_i + c_{ii}) \cos k\eta + i(c_i - c_{ii}) \sin k\eta. \tag{6.39}$$

Taking $c_i + c_{ii} = C_1$ and $i(c_i - c_{ii}) = C_2$ leads to the solution

$$\tilde{B}_k(\eta) \quad = \quad C_1 \cos k\eta + C_2 \sin k\eta, \tag{6.40}$$

where C_1 and C_2 are constants. This gives all solutions (either real or complex) of the differential equation (6.25). The solutions are real when the constants C_1 and C_2 are real (in this work, they are real). Equation (6.40) can then be expressed as an oscillatory solution

$$a^2 B_k = C_1 \cos k\eta + C_2 \sin k\eta. \tag{6.41}$$

The equation for the actual magnetic field is $B_k = \frac{\tilde{B}_k}{a^2}$ without loss of generality. Note that $|\vec{B}_k| \equiv B_k$. One can use B_k for more economical writing and without loss of generality. Now, we have that

$$k\eta = \frac{\lambda_H}{\lambda_k}, \tag{6.42}$$

where $\lambda_H = \frac{1}{H}$ and $\lambda_k = \frac{a}{k}$ [56]. Equation (6.41) applies to inflationary magnetic fields as they cross the horizon at either the end of de Sitter phase or just before its end [56]. Note that for sub-horizon scales, the area around horizon scales and well above horizon scales, $|k\eta| \gg 1$, $|k\eta| \sim 1$ and $|k\eta| \ll 1$, respectively [150]. During de Sitter phase, the Hubble horizon is constant. After de Sitter phase the Hubble horizon expands. Since the background space-time is spatially flat, equation (6.41) $\left[\text{which can be recast as } B_k = (C_1 \cos k\eta + C_2 \sin k\eta)\left(\frac{a_0}{a}\right)^2\right]$ leads to adiabatic decay even though electrical conductivity is very poor [57] on either super-horizon scales (or sub-horizon scales during inflation). This implies that equation (6.41) can apply to the evolution of inflationary magnetic fields that are above, but near the horizon after de Sitter. During this time, magnetic fields are scale independent[2] (hence there is no large growth of magnetic fields on these scales). The equation also applies to the evolution of magnetic fields during inflation on sub-horizon scales up to the time that they cross the horizon just before either the end of de Sitter phase or at its end. They can evolve in this manner probably up to either just or around the beginning of the radiation-dominated epoch when $\eta \ll 1$ where η represents conformal time (Explanation in a moment). The ratio in equation (6.42) measures the physical size of the magnetic field $\left(\lambda_k = \frac{a}{k}\right)$ relative to the Hubble horizon $\left(\lambda_H = \frac{1}{H}\right)$ [56]. At the beginning of the radiation-dominated epoch, the universe is still in its early stages and therefore again $\eta \ll 1$ at this time of expansion of the universe.

At the moment inflation-generated, large-scale magnetic fields cross the Hubble horizon at

[2] This means that the ratio of the energy density in the (quantum mechanical) fluctuation relative to the (space-time) background density, is the same at the time when the mode reenters the Hubble horizon (radius) as when it crossed outside the Hubble horizon at either the end of de Sitter phase or just before the end of the phase.

the end of de Sitter phase [56], they obey either the wavelike equation (6.21) or equation (6.41). When electromagnetic fields permeate the universe from inflation to the era of the radiation-dominated epoch including either up to the cosmological horizon or slightly beyond horizon scales (implying super-horizon scales that are not well above the horizon), then what would happen? Well, either equation (6.21) or equation (6.41) will represent a wave-like equation that in turn will represent the evolution of inflation-generated, magnetic fields that are near the horizon most probable up to the beginning of the radiation-dominated epoch. This is possible because:

1. Magnetic fields are scale independent [57] after crossing the Hubble horizon for the first time [hence, no large growths of (electro-)magnetic fields on super-horizon scales].

2. During de Sitter phase, the Hubble horizon is constant (does not either expand or grow or contract). However, after the phase, the horizon expands or grows. Given that the distance to the Hubble horizon denoted by r_H is $r_H = cH^{-1}$ where c represents speed of light, the recessional velocity of the Hubble horizon denoted as v_r will be $v_r = Hr_H = HcH^{-1} = c$ which represents the speed of light. Objects beyond the Hubble horizon recede at speeds faster than the speed of light denoted as c [2, 151–160] (Uses of proper distance [2, 154–160]). Therefore, it is possible that super-horizon-sized perturbations of magnetic fields might have receded with a speed that would allow the perturbations to be well above the Hubble horizon at around the beginning of the radiation-dominated epoch and not before that.

3. Super-horizon-sized perturbations of magnetic fields still have oscillation periods that are longer than the age of the universe by around the beginning of the radiation-dominated epoch and this is probable since (conformal) time denoted by η is $\eta \ll 1$ at this time of the universe expansion. This implies that the perturbations have not yet started to oscillate properly by around the beginning of the epoch. Hence, the magnetic-mode oscillation will not have reached its first wave-crest by either the (or around the) beginning of the epoch (more details later). Note that before magnetic field perturbations reach their first wave-crest, they will be evolving well above the horizon by around the beginning of the radiation-dominated epoch, and not before that time. Therefore, magnetic fields will evolve in the form of either equation (6.41) or equation (6.21) either up to (or around) the beginning of the radiation-dominated epoch and then in the form of equation (6.59) afterwards until they cross the horizon for a second time. Note that a magnetic field mode reaches its maximum length (beyond the Hubble horizon; actually well above the horizon) as an oscillating field at around the beginning of the radiation-dominated epoch. After this, magnetic fields will be decaying gradually until current time. Additionally, magnetic field strength (and length) at the beginning of the inflationary epoch is approximately zero (0). Therefore, it is most

logical to assume that (super-adiabatic) evolution of magnetic fields begins at around the beginning of the radiation-dominated epoch (more reasons are mentioned later). This is reasonable as η is still $\eta \ll 1$ at this time of the universe expansion.

Electromagnetic fields are evolving rapidly away from the Hubble horizon and not growing. This implies that magnetic fields are evolving rapidly away from the Hubble horizon in the form of equation (6.41) and not growing since magnetic fields are scale independent on super-horizon scales. When electromagnetic fields permeate the whole universe to well above horizon scales, then by around the beginning of the radiation-dominated epoch, magnetic fields would be evolving well above the horizon and this can be shown to be probable since (conformal) time denoted by η is either still much less than unity or $\eta \ll 1$ by this time of the universe expansion (to be shown later).

One can derive equations of motion by using the variational formalism for a single-fluid model. The variational approach enables one to develop equations of motion that will allow one to examine the evolution of inflation-generated, large-scale magnetic fields in more realistic fluids as opposed to the idealised fluids. In the context of the formalism, the single-fluid model we will first consider for the derivation of the equations of motion is the radiation-dominated epoch epoch. We will derive equations of motion up to cosmological scales that are slightly above the Hubble horizon. Note that this approach of a single-fluid model will account for the coupling of a fluid to dynamical space-time [84]. We will then use the equations of motion to examine the evolution of inflation-generated, large-scale magnetic fields. The evolution will also extend to the matter-dominated epoch until current time. Therefore, in the context of the variational approach we will use the single-fluid formalism to derive the equations of motion that will also be useful in the modelling of the evolution of magnetic fields during either the radiation-dominated or matter-dominated epoch (or from the time magnetic fields crossed the horizon for a second time) until current time.

To derive the first set of the equations of motion, a single-non conducting fluid action principle is set up [84]. Using the pull-back approach to set up variations of A_a leads to equations of motion being derived [84]. In the action, there is a piece of the anti-symmetric Faraday tensor denoted by F_{ab} [84] and defined as

$$F_{ab} = \nabla_a A_b - \nabla_b A_a. \tag{6.43}$$

It satisfies the Bianchi identity (6.2). The action is composed of the fluid and Maxwell actions, and a coupling term between the fluid and four-potential A_a. Varying the action with respect to A_a yields pieces of action. An energy functional denoted by Λ [84] is the Lagrangian of the fluid action denoted as S_M. Λ can also be referred to as a master function.

To derive the second set of the equations of motion, a single-conducting fluid action principle is set up [84]. Using the pull-back approach to set up variations of A_a leads to equations of motion being derived [84]. A_a couples charged fluids to the electromagnetic field and vice versa. In the action, there will be a coupling term represented by ${J^a}_X A_a$ where ${J^a}_X$ represents flux current of the single-conducting fluid being considered. In this case, the single-conducting fluid being considered is denoted by X where X represents either the radiation-dominated epoch or matter-dominated epoch. All the pieces of the action in the preceding paragraph are considered here. Additionally, one can consider the coupling of $A_a A^a$ to R and $A^a A^b$ to R_{ab}. The couplings are of the form: Either $\phi RA^2 + \phi_0 R_{ab} A^a A^b$ or ϕRA^2 or $\phi_0 R_{ab} A^a A^b$ where ϕ and ϕ_0 represent coupling constants. These are included in the action. Then with all this one can vary the action with respect to A_a, thus yielding pieces of an action. With this, one can consider all the pieces of the (total) actions for both single non-conducting and conducting fluid scenarios. We will now use the variational formalism to derive equations of motion that will enable us to examine the evolution of inflation-generated, cosmological magnetic fields in the single-fluid model of the radiation-dominated epoch. Here, we will examine the evolution of magnetic fields from around the beginning of the radiation-dominated epoch to the time these fields cross the Hubble horizon for a second time either during the radiation-dominated epoch or the matter-dominated epoch.

6.3 Single non-conducting fluid formalism action principles

To either prove or show the possibility of existence of inflation-generated, large-scale magnetic fields during the radiation-dominated epoch represented by X (especially specifically at or around the beginning of the radiation-dominated era though applicable throughout the epoch), we use the single non-conducting fluid formalism in the context of the *variational* approach to derive equation (6.41) while neglecting current. This will allow us to derive equations that possibly represent the evolution of inflation-generated, cosmological magnetic fields on scales slightly above the horizon (since conductivity is very poor on such scales). In other words we use the variational approach to derive equations of motion [equation (6.6)] that will enable us to examine the evolution of inflation-generated, cosmological magnetic fields during the single-fluid model of the radiation-dominated epoch. Equation (6.6) will be used to derive equation (6.41). On sub-horizon scales, during the radiation-dominated epoch there is current. Magnetic fields generated after inflation are either too small in scale or are of sub-horizon scales. Hence, the equations that will be derived either will or should represent the evolution of inflation-generated, large-scale magnetic fields. Therefore, we consider the single non-conducting fluid action principle. Due to the variational formalism context of the single-fluid model being considered, it is possible to extend it to account for electromagnetism. Hence, we will consider the fluid and Maxwell actions in this section. The fluid action denoted by S_F has as Lagrangian, the master function denoted by Λ which depends on $n^2_X = -n^X_a n^a{}_X$ and the metric denoted as g_{ab}, and note that $n^a{}_X$ represents either number density four current or flux. Also note that X represents the radiation-dominated epoch. The magnitude of $n^a{}_X$ denoted by n_X represents particle number density [73]. Varying S_F yields [84]

$$\delta S_F = \delta\left(\int_{\mathcal{M}} d^4x\sqrt{-g}\Lambda\right) = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\mu^X_a \delta n^a{}_X + \frac{1}{2}(\Lambda g^{ab} + n^a{}_X \mu^b{}_X)\delta g_{ab}\right], \quad (6.44)$$

where $\mathcal{M}$ represents either a manifold or hypersurface, g represents the determinant of g_{ab} and μ^X_a represents the canonically conjugate momenta to $n^a{}_X$. Then letting [84]

$$\mathcal{B}^X = -2\frac{\partial\Lambda}{\partial n^2{}_X}, \quad (6.45)$$

leads to [84]

$$\mu^X_a = g_{ab}\mathcal{B}^X n^b{}_X. \quad (6.46)$$

The use of $\mathcal{B}^X$ is to remind one that this is a bulk fluid effect which is present regardless of the number of fluids and constituents.

We now consider the Maxwell action given by [84]

$$S_{Max} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}F_{ab}F^{ab}. \quad (6.47)$$

Varying this action with respect to A_a and g_{ab} yields [84]

$$\delta S_{Max} = \frac{1}{4\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}(\nabla_a F^{ab})\delta A_b - \frac{1}{32\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}(F_{cd}F^{cd}g^{ab} - 4F^{ac}F^b{}_c)\delta g_{ab}. \quad (6.48)$$

Thus

$$\delta S = \delta S_F + \delta S_{Max}, \quad (6.49)$$

which leads to

$$\delta S = \int_{\mathcal{M}} d^4x\sqrt{-g}\left\{\mu^X_{\ a}\delta n^a_{\ X} + \frac{1}{4\pi}\nabla_b F^{ba}\delta A_a + \frac{1}{2}\left[\Lambda g^{ab} + n^a_{\ X}\mu^b_{\ X} - \frac{1}{16\pi}(F_{cd}F^{cd}g^{ab} - 4F^{ac}F^b_{\ c})\right]\delta g_{ab}\right\}. \tag{6.50}$$

If the variation of the four-current was left unconstrained, the equations of motion for the fluid derived from the varied action (6.50) would require incorrectly that the momentum denoted as $\mu^X_{\ a}$ should vanish in all cases. This means that the variation of the conserved four-current must be constrained. This implies that not all components of $n^a_{\ X}$ can be treated as independent. In terms of the constrained Lagrangian displacement of equation (3.38), the variation (6.49) leads to

$$\begin{aligned} \delta S = & \int_{\mathcal{M}} d^4x\sqrt{-g}\left\{\frac{1}{4\pi}\nabla_b F^{ba}\delta A_a + \frac{1}{2}\left[(\psi\delta^a_{\ c} + n^a_{\ X}\mu^X_{\ c})g^{cb}\right.\right. \\ & \left.\left. - \frac{1}{16\pi}(F_{cd}F^{cd}g^{ab} - 4F^{ac}F^b_{\ c})\right]\delta g_{ab} - \mathcal{F}^X_{\ b}\xi^b_{\ X}\right\}, \end{aligned} \tag{6.51}$$

where

$$\mathcal{F}^X_{\ b} = 2n^a_{\ X}\mathcal{W}^X_{\ ab} = 0, \tag{6.52}$$

$$\psi \equiv \Lambda - n^a_{\ X}\mu^X_{\ a}, \tag{6.53}$$

and [73]

$$\mathcal{W}^X_{\ ab} = 2\nabla_{[a}\mu^X_{\ b]}. \tag{6.54}$$

This leads to the definition

$$\mathcal{W}^X_{\ ab} = \nabla_a\mu^X_{\ b} - \nabla_b\mu^X_{\ a}, \tag{6.55}$$

One can ignore the total divergence term since it does not contribute to either the field equations or the stress-energy-momentum tensor $T^a_{\ b}$ (the divergence theorem implies that the total divergence term becomes a boundary term in the action). This is for the case where the universe is permeated by electromagnetic fields in the absence of currents. One can now derive the field equations and the corresponding $T^a_{\ b}s$.

The change in δS in equation (6.51) must vanish for all δA if the action is at an extremum.

For the photon field, this demands equation (6.6), and also including equation (6.2). This can lead to equation (6.41) after using the matrix (6.7). Similarly, the change in δS in equation (6.51) must also vanish for all δg_{ab} if the action is an extremum. This demands that

$$T^{ab} = \frac{1}{2}\left[(\psi\delta^a{}_c + n^a{}_X\mu^X{}_c)g^{cb} - \frac{1}{16\pi}(F_{cd}F^{cd}g^{ab} - 4F^{ac}F^b{}_c)\right], \tag{6.56}$$

where ψ represents equation (6.53). Note that the Lorentz gauge condition is represented as

$$\nabla_a A^a = 0. \tag{6.57}$$

Due to diffeomorphism invariance, the conservation of the stress-energy-momentum tensor T^a_b can be established. This implies $\nabla_a T^a_b = 0$. It is a possibility that inflation-generated, magnetic fields could be either well above or well outside the Hubble radius just at the beginning of the radiation-dominated epoch. This means that

$$\frac{\lambda_H}{\lambda_k} \ll 1, \tag{6.58}$$

which implies that $k\eta \ll 1$ in conformal-time terms. Then a simple Taylor expansion of equation (6.41) leads to [56]

$$a^2 B_k = C_1 + C_2 k\eta, \tag{6.59}$$

where $a = a(\eta)$. Equations (6.41), (6.42), (6.58) and (6.59) can help one see the concept of transition from oscillation to power-law growth at either the Hubble threshold or near the threshold. This type of transition has been considered in cosmological perturbation theory [56]. For example, during the radiation-dominated epoch, the type of transition just being considered happens to linear density perturbations. The transition reflects the fact that super-horizon-sized perturbations have not yet started to oscillate properly because they have oscillation periods that are longer than the age of the universe by the beginning of the radiation-dominated epoch. To illustrate, let [56]

$$\lambda_H \equiv \frac{1}{H} \simeq t_{\mathcal{U}}, \tag{6.60}$$

and

$$\lambda_k \equiv t_k, \tag{6.61}$$

where $t_{\mathcal{U}}$ and t_k represent the age of the universe and the period of the magnetic-mode oscillation, respectively. Note that $dt = a(t)d\eta$ where t represents either cosmic or proper time and η represents conformal time. Then equation (6.58) suggests that $t_k \gg t_{\mathcal{U}}$ which implies that the magnetic-mode oscillation has not yet reached its first wave-crest.

Once well above the Hubble horizon and as long as they stay there, the magnetic fields remain causally disconnected and their evolution is only affected by the background expansion [56]. Currents exist by this time of the universe expansion, but due to causality they only exist on sub-horizon scales. This implies that the ideal-MHD limit cannot be applied to super-horizon scales [56]. Since causal physics can never affect super-Hubble length perturbations, the process of magnetic flux freezing (which is causal physics) should be causal.

In other words, causality implies that the time required for the freezing in information to travel the whole length of a super-horizon magnetic field is larger than the age of the universe at the time [56]. Hence, the magnetic fields cannot readjust themselves to the new environment and freeze in until they have crossed back inside the horizon. Instead, as long as they are well above the Hubble radius, the magnetic fields are not affected by causal physics and retain only the memory of their distant past [56]. This means that the magnetic fields evolution is governed by equation (6.59) which is derived from equation (6.21). Note that it is not the first time that the source free approach is applied to the study of large-scale cosmological magnetic fields ([56] and references therein). One can now examine super-adiabatic magnetic amplification.

6.4 Evolution of magnetic fields on length scales well above the Hubble radius

The presence of the second term on the right-hand side of equation (6.59) does not necessarily guarantee the adiabatic decay of magnetic fields [56]. Therefore, one can focus on this term. Note that it is not negligible even though $k\eta$ is $k\eta \ll 1$ on well above horizon scales. When the initial conditions at the beginning of the radiation-dominated epoch are such that $C_2 \gg C_1$, then the second term on the right-hand side of equation (6.59) can lead to super-adiabatic evolution of magnetic fields as shown in figure (6.1).

As the universe expands, the conformal time denoted as η increases and the product $k\eta$ becomes larger than unity. This means that magnetic fields have reentered the Hubble radius and are obeying $\rho_B \propto a^{-4}$. This implies that equation (6.59) is no longer valid. Note that inflation-generated, large-scale magnetic fields decayed adiabatically throughout de Sitter phase [56] despite the absence of currents.

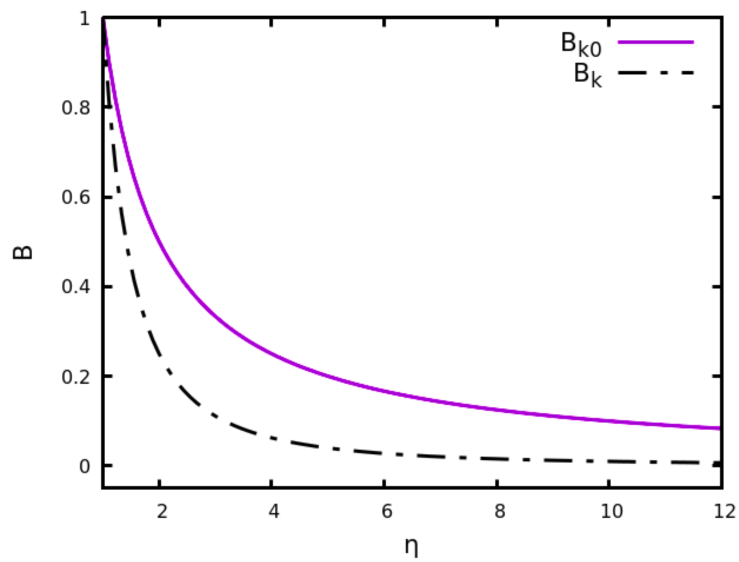

Figure 6.1: This schematic figure shows the evolution of magnetic fields from the beginning of the radiation-dominated epoch on super-horizon scales well above the horizon. B represents magnetic field strength while η represents time. B_{k0} represents super-adiabatic evolution of magnetic fields, while B_k represents adiabatic evolution of magnetic fields. This figure shows that super-adiabatic decay of magnetic fields is significantly slower than adiabatic decay of magnetic fields. The super-adiabatic decay corresponds to the scenario when magnetic fields are evolving at scales that are well above the Hubble horizon

One can examine the evolution of inflation-generated magnetic fields on scales that are well above the Hubble radius. We consider equation (6.59). One can express it as

$$B = C_1 a^{-2} + C_2 k\eta a^{-2}. \tag{6.62}$$

This can be written as

$$B_0 = C_1 a_0^{-2} + C_2 k\eta_0 a_0^{-2}, \tag{6.63}$$

where the subscript 0 represents the beginning of the radiation-dominated era. Note that this is a given initial time. In [56] the subcript 0 represents the beginning of de Sitter phase. However, this is problematic as inflation-generated magnetic fields evolve as equation (6.41) as they cross the cosmological Hubble horizon (for the first time) during de Sitter phase [56] and evolve as equation (6.59) when the magnetic fields are either well above the horizon or Hubble radius. In [56], magnetic fields are shown to evolve as equation (6.59) from the beginning of exponential expansion of the inflationary epoch. When one considers this carefully, one will see that it is problematic. Equation (6.59) does not represent the evolution of magnetic fields either at/or from the beginning of exponential expansion of de Sitter phase. Examining figures 1.5 and 1.6 might aid in illustrating this. Our work tries to resolve the problem. Furthermore, we assume instantaneous reheating [161]. This refers to the ideal case where after the inflationary epoch the universe enters directly in the radiation-dominated epoch [161]. In this case, the energy density at just the beginning of the radiation-dominated epoch which is the same as that at the end of the reheating epoch, is equal to the energy density at the end of the inflationary epoch [161]. Therefore, it is logical that the subscript 0 either should (or can) represent the beginning of the radiation-dominated epoch. Moreover, if the subscript 0 were to represent the beginning of inflation, then we should expect astrophysically irrelevant magnetic field strength in current time as either magnetic field strength or size at the beginning of inflation was approximately zero $(0)G$ [3]. Now, differentiating equation (6.63) with respect to conformal time denoted as η yields

$$B'_0 = -2C_1 a_0^{-3} a'_0 + (-2)C_2 k\eta_0 a_0^{-3} a'_0 + C_2 k a_0^{-2}. \tag{6.64}$$

Using the relation $H = \frac{a'}{a^2}$ in equation (6.64) leads to

$$B'_0 = -\frac{2C_1 H_0}{a_0} - \frac{2C_2 k\eta_0 H_0}{a_0} + \frac{C_2 k}{a^2_0}. \tag{6.65}$$

Multiplying equation (6.63) throughout by $2H_0B_0$ and then adding it to equation (6.65) yields

$$2a_0 H_0 B_0 + B'_0 = \frac{C_2 k}{a^2_0}, \tag{6.66}$$

which in turn leads to

$$C_2 = \frac{\eta_0(2a_0H_0B_0 + B'_0)a^2_0}{k\eta_0}. \tag{6.67}$$

Substituting equation (6.67) in equation (6.63) yields

$$C_1 a_0^{-2} = B_0 - \frac{(2a_0H_0B_0 + B'_0)k\eta_0 a_0^{-2} a^2_0}{k}, \tag{6.68}$$

which leads to [56]

$$C_1 = [B_0 - \eta_0(2a_0H_0B_0 + B'_0)]a^2_0. \tag{6.69}$$

Substituting equations (6.67) and (6.69) in equation (6.62) yields

$$B = [B_0 - \eta_0(2a_0H_0B_0 + B'_0)]\left(\frac{a_0}{a}\right)^2 + \eta_0(2a_0H_0B_0 + B'_0)\left(\frac{a_0}{a}\right)^2\left(\frac{\eta}{\eta_0}\right). \tag{6.70}$$

[3] This can be shown to be true by equation (6.78) later. Note that $\eta_0 B'_0 = \xi B_0$ where $\xi \in \mathbb{R}$ [56].

a and η are related by

$$a = a_0 \left(\frac{\eta}{\eta_0}\right)^{\frac{2}{1+3w}}, \tag{6.71}$$

where $w \neq -\frac{1}{3}$ and w represents the barotropic index of matter. Then

$$a' = \frac{2a_0}{(1+3w)(\eta_0)^{\frac{2}{1+3w}}}(\eta)^{\frac{1-3w}{1+3w}}. \tag{6.72}$$

Using the relation $H = \frac{a'}{a^2}$ leads to

$$H = \frac{\frac{2a_0(\eta)^{\frac{1-3w}{1+3w}}}{(1+3w)(\eta_0)^{\frac{2}{1+3w}}}}{{a^2}_0\left(\frac{\eta}{\eta_0}\right)^{\frac{4}{1+3w}}}, \tag{6.73}$$

which in turn yields

$$H = \frac{2}{a_0(1+3w)}\eta^{-\frac{3(1+w)}{1+3w}}\eta_0^{\frac{2}{1+3w}}. \tag{6.74}$$

This simplifies to

$$H = \frac{2}{a_0(1+3w)\eta}\left(\frac{\eta}{\eta_0}\right)^{\frac{2}{1+3w}}. \tag{6.75}$$

Using the relation (6.71) in equation (6.75) leads to

$$H = \frac{2}{(1+3w)a\eta}. \tag{6.76}$$

The relation (6.76) and (6.71) are used in (6.70), and this yields

$$\begin{aligned} B &= \left\{B_0 - \eta_0\left[2a_0\frac{2}{(1+3w)a_0\eta_0}B_0 + {B'}_0\right]\right\}\left(\frac{a_0}{a}\right)^2 \\ &+ \eta_0\left[2a_0\frac{2}{(1+3w)a_0\eta_0}B_0 + {B'}_0\right]\left(\frac{a_0}{a}\right)^2\left(\frac{a_0}{a}\right)^{-\frac{(1+3w)}{2}}, \end{aligned} \tag{6.77}$$

which in turn leads to [56]

$$B = -\left[\left(\frac{4}{1+3w}-1\right)B_0+\eta_0 {B'}_0\right]\left(\frac{a_0}{a}\right)^2+\left(\frac{4B_0}{1+3w}+\eta_0 {B'}_0\right)\left(\frac{a_0}{a}\right)^{\frac{3(1-w)}{2}}. \quad (6.78)$$

Equation (6.78) monitors the linear evolution of super-horizon-sized magnetic fields on spatially flat $FLRW$ space-time backgrounds that are filled with a single barotropic medium and permeated with electromagnetic fields. The barotropic index of the matter denoted as w is treated as a constant. This equation applies only to the periods of the radiation-dominated and matter-dominated epochs during which w is constant and $w \neq -\frac{1}{3}$. For the radiation-dominated epoch and matter-dominated epoch, $w = \frac{1}{3}$ and $w = 0$, respectively.

After examining equation (6.78), one will notice that the first of the two magnetic modes on the right-hand side always either decay or evolve adiabatically [56]. However, the rate of the second mode is not a priori fixed, but depends on the equation of state of the cosmic medium. It also determines the relation between $a(t)$ and η. Particularly, as long as w =constant and $w > -\frac{1}{3}$, the second mode on the right-hand side of equation (6.78) decays at a rate slower than the adiabatic. Hence, when dealing with conventional matter, super-horizon-sized magnetic fields on spatially flat $FLRW$ space-time backgrounds are super-adiabatically amplified. This is possible when the initial conditions allow the second mode in equation (6.78) to survive and dominate.

One can now take a closer look at the post-inflationary magnetic evolution (particularly, after the beginning of the radiation-dominated epoch). During the epoch of radiation-domination, $w = \frac{1}{3}$ which implies that $a \propto \eta$ and $H = \frac{1}{a\eta}$. Then, throughout the period of the epoch, equation (6.78) takes the form [56]

$$B = -(B_0+\eta_0 {B'}_0)\left(\frac{a_0}{a}\right)^2+(2B_0+\eta_0 {B'}_0)\left(\frac{a_0}{a}\right), \quad (6.79)$$

ensuring that large-scale magnetic fields drop as $B \propto a^{-1}$ when radiation dominates the energy density of the universe.

During either the matter-dominated or dust era, $w = 0$, $a \propto \eta^2$ and $H = \frac{2}{a\eta}$. Then equation (6.78) reduces to [56]

$$B = -(3B_0+\eta_0 {B'}_0)\left(\frac{a_0}{a}\right)^2+(4B_0+\eta_0 {B'}_0)\left(\frac{a_0}{a}\right)^{\frac{3}{2}}. \quad (6.80)$$

This ensures that large-scale magnetic fields drop as $B \propto a^{-\frac{3}{2}}$ when matter dominates the energy density of the universe. One can see that after the reheating epoch, large-scale magnetic fields on spatially flat $FLRW$ space-time backgrounds obey solutions which always contain modes with decay rates that are slower than the adiabatic [56]. These slowly decaying magnetic modes depend on their associated coefficients. When the adiabatic magnetic decaying modes are roughly the same order of magnitude as the slowly decaying magnetic modes, the latter quickly takes over and dictates the subsequent evolution of the magnetic fields. By examining equations (6.79) and (6.80), one can see that for whatever value of $\eta_0 {B'}_0$ the adiabatic magnetic decaying modes are always roughly the same order of magnitude as the slowly decaying magnetic modes.

Therefore, (there is a possibility that) conventional large-scale magnetic fields are (or can be) super-adiabatically amplified from (around) the beginning of the radiation-dominated epoch of a flat $FLRW$ universe until they cross the horizon for a second time much later, as the universe expands probably during either the radiation-dominated epoch or during the matter-dominated epoch. This means that the residual strength of magnetic fields could be considerably larger than expected. The overall amplification depends on the scale of the magnetic mode in question, and this determines the time of horizon entry [56]. Once inside the horizon, the magnetic flux remains conserved, and the magnetic fields decay adiabatically until current time. This happens because the ideal-MHD limit takes over on sub-horizon scales where the electric currents take over, eliminate the electric fields and freeze their magnetic counterparts into the highly conductive single-fluid medium. We will now use the variational approach to derive equations of motion that will enable us to examine the evolution of inflation-generated, cosmological magnetic fields after second Hubble horizon crossing either during the single-fluid model of the radiation-dominated epoch or the matter-dominated epoch.

6.5 Single-conducting fluid formalism action principles

We now consider the single-conducting fluid model. Due to the variational formalism context of the single-fluid formalism, it is possible to extend it to account for electromagnetism, either charged fluids (or components) and either coupling of R with $A^a A_a$ or coupling of R_{ab} with $A^a A^b$ or the sum of the couplings of R with $A^a A_a$ and R_{ab} with $A^a A^b$. For this fluid, the minimal coupling of the Maxwell field to the charge current density is obtained from [84]

$$S_C = \int_{\mathcal{M}} d^4x \sqrt{-g} {J^a}_X A_a, \qquad (6.81)$$

where $J^a{}_X = e_X n^a{}_X$ represents flux current and X represents the evening of the radiation-dominated epoch or the matter-dominated epoch [54]. Varying this action with respect to $n^a{}_X$, A_a and g_{ab} yields [84]

$$\delta S_C = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[J^a{}_X\delta A_a + e_X A_a\delta n^a{}_X + \frac{1}{2}J^a{}_X A_a g^{bc}\delta g_{bc}\right]. \tag{6.82}$$

Thus [84]

$$\begin{aligned}\delta S_F + \delta S_{Max} + \delta S_C &= \int_{\mathcal{M}} d^4x\sqrt{-g}\Bigg\{[\mu^X_a + e_X A_a]\delta n^a{}_X + \frac{1}{4\pi}[\nabla_b F^{ba} + 4\pi J^a{}_X]\delta A_a + \frac{1}{2}\Bigg[\Lambda g^{ab} \\ &+ n^a{}_X\mu^b{}_X + J^c{}_X A_c g^{ab} - \frac{1}{16\pi}(F_{cd}F^{cd}g^{ab} - 4F^{ac}F^b{}_c)\Bigg]\delta g_{ab}\Bigg\}. \end{aligned} \tag{6.83}$$

One can now consider the coupling term actions

$$S_\phi = \int_{\mathcal{M}} d^4x\sqrt{-g}\phi R A^2, \tag{6.84}$$

and

$$S_{\phi_0} = \int_{\mathcal{M}} d^4x\sqrt{-g}\phi_0 R_{ab}A^a A^b, \tag{6.85}$$

where $S_{\phi_T} = S_\phi + S_{\phi_0}$. Varying S_{ϕ_T} yields

$$\begin{aligned}\delta S_{\phi_T} &= \int_{\mathcal{M}} d^4x\sqrt{-g}(2\phi R A^a + 2\phi_0 R^a{}_b A^b)\delta A_a - \int_{\mathcal{M}} d^4x\sqrt{-g}\phi g^{ha}g^{fb}R_{fh}A^2\delta g_{ab} \\ &+ 2\int_{\mathcal{M}} d^4x\sqrt{-g}\phi g^{ab}A^2\Bigg\{\frac{1}{2}\nabla_c[g^{cd}(\nabla_a\delta g_{db} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ba})] \\ &+ \frac{1}{2}g_{fh}\nabla_b\nabla_a\delta g^{fh}\Bigg\} + \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}g^{ab}(\phi R A^2 + \phi_0 R_{cd}A^c A^d)\delta g_{ab}. \end{aligned} \tag{6.86}$$

The total action takes the form

$$\delta S = \delta S_M + \delta S_{Max} + \delta S_C + \delta S_{\phi_T}. \tag{6.87}$$

Thus

$$\begin{aligned}\delta S &= \int_{\mathcal{M}} d^4x\sqrt{-g}\Bigg\{[\mu^X_{\ a} + e_X A_a]\delta n^a_{\ X} + \frac{1}{4\pi}[\nabla_b F^{ba} + 4\pi J^a_{\ X} + 8\pi\phi R A^a + 8\pi\phi_0 R^a_{\ b}A^b]\delta A_a + \frac{1}{2}\Bigg[\Lambda g^{ab} \\ &+ n^a_{\ X}\mu^b_{\ X} + J^c_{\ X}A_c g^{ab}) - \frac{1}{16\pi}(F_{cd}F^{cd}g^{ab} - 4F^{ac}F^b_{\ c}) - \phi g^{ha}g^{fb}R_{fh}A^2\Bigg]\delta g_{ab} \\ &+ 2\phi g^{ab}A^2\left[\frac{1}{2}\nabla_c(g^{cd}(\nabla_a\delta g_{db} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ba})) + \frac{1}{2}g_{fh}\nabla_b\nabla_a\delta g^{fh}\right] \\ &+ \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}g^{ab}(\phi R A^2 + \phi_0 R_{cd}A^cA^d)\delta g_{ab}\Bigg\}. \end{aligned} \tag{6.88}$$

The minimal coupling of the Maxwell field to the charge current density of the single-fluid model denoted as X yields a modification of the conjugate momentum. Thus [84]

$$\tilde{\mu}^X_{\ a} = \mu^X_{\ a} + e_X A_a. \tag{6.89}$$

The term which is proportional to $\delta n^a_{\ X}$ implies that the momentum denoted by $\tilde{\mu}^X_{\ a}$ must vanish [84]. Note that the fluid momenta changes from $\mu^X_{\ a}$ to $\tilde{\mu}^X_{\ a}$ [84]. This fact has to be incorporated. This implies that a pull-back formalism for a single-fluid approximation is required. For a general relativistic fluid, the field equations are obtained from an action principle. This forms the foundation for the variations of the fundamental fluid variables in the action principle. Then the field equations can be derived from the action principle after using the pull-back formalism [84]. In terms of the constrained Lagrangian displacement of equation (3.38), the variation (6.87) takes the form

$$\begin{aligned}\delta S &= \int_{\mathcal{M}} d^4x\sqrt{-g}\Bigg\{\frac{1}{4\pi}[\nabla_b F^{ba} + 4\pi J^a_{\ X} + 8\pi\phi R A^a + 8\pi\phi_0 R^a_{\ b}A^b]\delta A_a + \frac{1}{2}\Bigg[(\psi\delta^a_{\ c} \\ &+ n^a_{\ X}\mu^X_{\ c})g^{cb} - \frac{1}{16\pi}(F_{cd}F^{cd}g^{ab} - 4F^{ac}F^b_{\ c}) - \phi g^{ah}g^{fb}R_{fh}A^2\Bigg]\delta g_{ab} + 2\phi g^{ab}A^2\Bigg[\frac{1}{2}\nabla_c(g^{cd}(\nabla_a\delta g_{db} \\ &+ \nabla_b\delta g_{ad} - \nabla_d\delta g_{ba})) + \frac{1}{2}g_{fh}\nabla_b\nabla_a\delta g^{fh}\Bigg] + \frac{1}{2}g^{ab}[\phi R A^2 + \phi_0 R_{cd}A^cA^d]\delta g_{ab} - \mathcal{F}^X_{\ b}\xi^b_{\ X}\Bigg\}, \end{aligned} \tag{6.90}$$

where $\mathcal{F}^X_{\ b}$ represents force density given by [73]

$$\mathcal{F}^X_{\ b} = 2n^a_{\ X}\tilde{\mathcal{W}}^X_{\ ab} + F^X_{\ b}. \tag{6.91}$$

$\tilde{\mathcal{W}}^X_{\ ab}$ is defined as [73]

$$\tilde{\mathcal{W}}^X_{\ ab} = 2\nabla_{[a}\tilde{\mu}^X_{\ b]}. \tag{6.92}$$

This leads to the definition

$$\tilde{\mathcal{W}}^X_{ab} \;=\; \nabla_a \tilde{\mu}^X_{\ b} - \nabla_b \tilde{\mu}^X_{\ a}, \tag{6.93}$$

and ψ is defined as [73]

$$\psi \equiv \Lambda - n^a{}_X \mu^X_{\ a}. \tag{6.94}$$

$F^X_{\ b}$ represents the dissipative force due to entropy increase from conductivity. One can ignore the total divergence term since it does not contribute to either the field equations or $T^a_{\ b}$ (the divergence theorem implies that the total divergence term becomes a boundary term in the action).

The change δS in equation (6.90) must vanish for all δA if the action is at an extremum. This demands that

$$\nabla_b F^{ab} - 8\pi\phi R A^a - 8\pi\phi_0 R^a{}_b A^b \;=\; 4\pi J^a{}_X, \tag{6.95}$$

also including equation (6.2) and one can assume the simple Ohm's law for current [54]. Similarly, the change in δS in equation (6.90) must also vanish for all δg_{ab} if the action is an extremum. This demands that

$$T^{ab} \;=\; \frac{1}{2}\left[(\psi\delta^a{}_c + n^a{}_X\mu^X{}_c)g^{cb} - \frac{1}{16\pi}(F_{cd}F^{cd}g^{ab} - 4F^{ac}F^b{}_c) - \phi g^{ah}g^{fb}R_{fh}A^2 + \frac{1}{2}g^{ab}(\phi R A^2 + \phi_0 R_{cd}A^c A^d)\right]. \tag{6.96}$$

Equation (6.95) can be expressed in the form

$$\nabla_b F^{ab} - 8\pi\phi R A^a \;=\; 4\pi J^a{}_X + 8\pi\phi_0 R^a{}_b A^b, \tag{6.97}$$

which in turn leads to

$$\nabla_b F^{ab} - 8\pi\phi R A^a \;=\; 4\pi \mathcal{J}^a{}_{eff}, \tag{6.98}$$

where

$$\mathcal{J}^a{}_{eff} \;=\; J^a{}_X + 2\phi_0 R^a{}_b A^b. \tag{6.99}$$

From the identity

$$\nabla_a \nabla_b F^{ab} \;=\; 0, \tag{6.100}$$

the field equation (6.98) implies that

$$\nabla_a A^a \quad = \quad \frac{1}{2\phi R}\nabla_a \mathcal{J}^a_{\ eff}. \tag{6.101}$$

When $\phi R \neq 0$, then

$$\nabla_a \mathcal{J}^a_{\ eff} \quad = \quad 0, \tag{6.102}$$

if and only if

$$\nabla_a A^a \quad = \quad 0, \tag{6.103}$$

which is just the Lorentz gauge condition. For $\phi_0 \neq 0$ equation (6.102) implies that the gauge condition

$$\nabla_a (R^a_{\ b} A^b) \quad = \quad 0, \tag{6.104}$$

has to be satisfied and therefore

$$\nabla_a J^a_{\ X} \quad = \quad 0, \tag{6.105}$$

which is the law of conservation of charge.

For the case where $\phi_0 = 0$, the identity (6.100) leads to [144]

$$\nabla_a A^a \quad = \quad \frac{1}{2\phi R}\nabla_a J^a_{\ X}. \tag{6.106}$$

When $\phi R \neq 0$, equation (6.105) is obtained if and only if equation (6.103) is obeyed.

For the case where $\phi = 0$, the identity (6.100) yields equation (6.102) where $\mathcal{J}^a_{\ eff}$ is equation (6.99). Equation (6.102) implies the gauge condition (6.104) and hence, the conservation of charge equation (6.105).

For T^a_b of equations (6.96), its conservation can be established by considering diffeomorphism invariance. Then, $\nabla_a T^a_b = 0$. Note that an alternative way of showing conservation of quantities for the Lagrangian systems in this work is shown in an upcoming article by me and other authors. The title of the article is *Variational symmetries of tensor Lagrangians*. This is described briefly in the appendix A.3. One can now examine the evolution of magnetic fields after crossing the Hubble horizon for a second time.

6.6 Evolution of magnetic fields after second horizon crossing

To study equations (6.2) and (6.95) together, one can write them in terms of the electric and magnetic fields denoted as $\vec{E}$ and $\vec{B}$, respectively where one can use the matrix represented by F_{ab} in (6.7). We consider spatially flat $FLRW$ cosmologies where the line element is given by

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{6.107}$$

This can be expressed as

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2), \tag{6.108}$$

in conformal-time coordinates. Note that $t(\eta)$ represents clock (conformal) time. One can then derive R and $R^{\mathcal{I}}{}_{\mathcal{I}}$ leading to

$$R^{\mathcal{I}}{}_{\mathcal{I}} \equiv \frac{\ddot{a}}{a^3} + \left(\frac{\dot{a}}{a^2}\right), \tag{6.109}$$

and

$$R \equiv 6\frac{\ddot{a}}{a}. \tag{6.110}$$

Using R and $R^{\mathcal{I}}{}_{\mathcal{I}}$ leads to equations (6.95) and (6.2) being re-cast into the forms

$$\frac{1}{a^2}\frac{\partial(a^2\vec{E})}{\partial\eta} - \nabla\times\vec{B} - \frac{n}{\eta}\frac{\vec{A}}{a^2} \equiv \sigma_c\vec{E}, \tag{6.111}$$

and

$$\frac{1}{a^2}\frac{\partial(a^2\vec{B})}{\partial\eta} + \nabla\times\vec{E} = 0, \tag{6.112}$$

respectively, where

$$n \equiv \eta^2\left\{8\pi\left[6\phi\frac{\ddot{a}}{a} + \phi_0\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\right]\right\}. \tag{6.113}$$

σ_c represents conductivity of the single-conducting fluid denoted by X and under consideration now. In equation (6.113) n represents a constant whenever $a(\eta)$ varies as a power of η. Taking the curl of equation (6.111) and using equation (6.112) to eliminate $\vec{E}$

yields

$$\frac{1}{a^2}\frac{\partial^2(a^2\vec{B})}{\partial\eta^2} - \nabla^2\vec{B} + \frac{n}{\eta^2}\vec{B} \equiv -\sigma_c\frac{1}{a^2}\frac{\partial(a^2\vec{B})}{\partial\eta}. \tag{6.114}$$

This equation can be expanded in terms of its Fourier components as it is linear in $\vec{B}$. With the definition

$$\vec{F}_k(\eta) \equiv a^2\int d^3x e^{i\vec{k}\cdot\vec{x}}\vec{B}(\vec{x},\eta), \tag{6.115}$$

equation (6.114) can be recast into the form

$$\ddot{\vec{F}}_k + k^2\vec{F}_k + \frac{n}{\eta^2}\vec{F}_k \equiv -\sigma_c a\dot{\vec{F}}_k, \tag{6.116}$$

where k represents a wavenumber. Associated with the co-moving scale, $\lambda \sim k^{-1}$ and the quantity $\vec{F}_k$ represents a measure of magnetic flux. Given

$$\rho_B(k) \equiv k\frac{d\rho_B}{dk}, \tag{6.117}$$

the energy density in the k^{th} mode of the magnetic field is

$$\rho_B = \frac{|\vec{F}|^2}{a^4}, \tag{6.118}$$

which means that for $\sigma_c \gg \frac{1}{\eta a} \sim H$ implies $\frac{\partial\vec{F}}{\partial\eta} \to 0$ hence, $\vec{F}$ ~constant. This means that $\rho_B \propto a^{-4}$ which implies conservation of magnetic flux in the evolution of magnetic fields either much later during the radiation-dominated epoch or during the matter-dominated epoch until current time on sub-horizon scales (due to causality [56]). This is applicable for the case where either $\phi = 0$ or $\phi_0 = 0$. This confirms adiabatic evolution of cosmological magnetic fields from second horizon crossing until current time.

In other words, one can consider the simple Ohm's law denoted by $\vec{J}_X = \sigma_c\vec{E}$ where $\vec{J}_X$ represents 3-current in the conducting fluid denoted by X, σ_c represents conductivity of X and $\vec{E}$ represents electric field [162, 163]. At the ideal-MHD limit, the conductivity is very high. That is $\sigma_c \to \infty$. This implies that the electric field vanishes and the currents keep the magnetic field frozen-in with X [164] on sub-horizon scales (due to causality [56]). This implies that $\rho_B \propto a^{-4}$ which means conservation of magnetic flux. When $\sigma_c \to 0$, the currents vanish despite the presence of an electric field [164].

6.7 Approximate evaluation of left-over magnetic field strength in current time

Inflation-generated magnetic fields decay adiabatically up to the beginning of the radiation-dominated epoch. This is from the beginning of inflation. They first cross the horizon at either the end or just before the end of a de Sitter phase. They evolve adiabatically on sub-horizon scales after crossing the Hubble horizon for the second time. Therefore, super-adiabatic amplification should occur from the beginning of the radiation-dominated epoch up to second horizon crossing. If $\rho_B = B^2$ represents magnetic energy density and ρ represents density of a dominant matter component, then this would mean that at the end of de Sitter phase [56]

$$\left(\frac{\rho_B}{\rho}\right)_{DS} \simeq 10^{-94}\left(\frac{M}{10^{17}}\right)^{\frac{4}{3}}\left(\frac{T_{RH}}{10^{10}}\right)^{-\frac{4}{3}}\lambda_B^{-4}, \tag{6.119}$$

where DS represents de Sitter phase, M represents scale of inflation, T_{RH} represents reheating epoch temperature (both measured in GeV) and λ_B represents a current physical scale (measured in Mpc) of the inflation-generated magnetic mode being considered. Assuming instantaneous reheating we have $\rho_B \propto a^{-4}$ and $\rho \simeq \rho_X \propto a^{-4}$ in the epoch of reheating [161]. Hence, throughout the reheating epoch, the dimensionless ratio denoted by $\frac{\rho_B}{\rho}$ stays constant and this infers

$$\left(\frac{\rho_B}{\rho}\right)_{DS} \simeq \left(\frac{\rho_B}{\rho}\right)_{RH} \simeq \left(\frac{\rho_B}{\rho_{X_i}}\right), \tag{6.120}$$

where RH represents reheating and X_i represents at just the beginning of the radiation-dominated epoch. Once into the radiation-dominated era, $\rho_B \propto a^{-2}$ and $\rho \simeq \rho_X \propto a^{-4}$ where ρ_X represents energy density of a radiative component. Therefore, for an inflation-generated magnetic mode that crosses inside the Hubble horizon before equipartition [56]

$$\left(\frac{\rho_B}{\rho_X}\right) \simeq \left(\frac{\rho_B}{\rho}\right)_{DS}\left(\frac{T_{RH}}{T_{HC}}\right)^2 \simeq 10^{-94}\left(\frac{M}{10^{17}}\right)^{\frac{4}{3}}\left(\frac{T_{RH}}{10^{10}}\right)^{-\frac{4}{3}}\left(\frac{T_{RH}}{T_{HC}}\right)^2\lambda_B^{-4}. \tag{6.121}$$

T_{HC} represents temperature at the second horizon crossing where HC represents second horizon crossing.

After the second horizon crossing, $\rho_B \propto a^{-4}$ and $\rho_X \propto a^{-4}$. This ensures that their ratio remains constant until current time. This means that [56]

$$\left(\frac{\rho_B}{\rho_X}\right)_{\dagger} \simeq 10^{-94}\left(\frac{M}{10^{17}}\right)^{\frac{4}{3}}\left(\frac{T_{RH}}{10^{10}}\right)^{-\frac{4}{3}}\left(\frac{T_{RH}}{T_{HC}}\right)^{2}\lambda_B^{-4}, \tag{6.122}$$

in current time. The suffix † represents current time. Inflation-generated magnetic fields with co-moving current size and close to $10kpc$ reenters the Hubble horizon at $T_{HC} \simeq 3\times10^{-6}GeV$ [56]. Now, $(\rho_X)_\dagger \simeq 10^{-15}GeV^4$ [56]. Substituting this value and the value for T_{HC} in equation (4.122) leads to [56]

$$B_\dagger \simeq 10^{-33}\left(\frac{M}{10^{17}}\right)^{\frac{2}{3}}\left(\frac{T_{RH}}{10^{10}}\right)^{\frac{1}{3}}G. \tag{6.123}$$

This implies that when $M \simeq 10^{17}GeV$ and $T_{RH} \simeq 10^{10}GeV$ the magnitude of inflation-generated cosmological magnetic fields with current physical size of approximately $10kpc$ should be roughly $10^{-33}G$ which is quite higher than $10^{-53}G$ [56]. Therefore, this means that by appealing to causality, the final strength of conventional inflation-generated magnetic seeds can be increased significantly. The work done here refines and adds to the repertoire of existing theories on the generation, growth and evolutionary history of inflation-generated, cosmological magnetic fields.

6.8 Chapter summary

Adiabatic decay of magnetic fields on flat $FLRW$ space-time backgrounds translates into magnetic strengths below $10^{-50}G$ in current time [149] where the wavelength (denoted as λ_B) of magnetic fields is set to $\lambda_B \simeq 10Mpc$. This is the minimum required for the dynamo to work [56]. Such fields can never neither seed the galactic dynamo nor can they affect the dynamics of the universe. In the context of the *variational* approach, we used both the single-conducting and single non-conducting fluid formalisms to derive both the modified Maxwell's tensors and standard Maxwell's tensors (for a photon field). In other words we used the variational formalism to derive equations of motion that enabled us to examine the evolution of inflation-generated, cosmological magnetic fields during the single-fluid models of the radiation-dominated epoch and the matter-dominated epoch. We then used the equations to examine the evolution of inflation-generated, cosmological magnetic fields. Now, in single-fluid environments where $\mathcal{B}^X = -2\frac{\partial\Lambda}{\partial n^2{}_X}$ and X represents either the radiation-dominated epoch or matter-dominated epoch, we find that the magnetic decay rate slows down as $B \propto a^{-1}$ during the radiation-dominated epoch and $B \propto a^{-\frac{3}{2}}$ during the matter-dominated epoch until their re-entry inside the horizon when the magnetic-decay rate would be adiabatic until current time. With this, we studied relativistic fluids with applications to cosmology in that we analysed the behaviour of evolution of inflation-generated, cosmological magnetic fields in single-fluid models of either the radiation-dominated or matter-dominated epochs. That is, we investigated how the behaviour of evolution of the cosmological magnetic fields is either influenced or affected as they evolve in the single-fluid models of either the radiation-dominated or matter-dominated epochs. With all this, the suggestion is that there is a possible existence of a super-adiabatic evolving mode from either the beginning (or around the beginning) of the radiation-dominated epoch to much later in the expansion history of the universe (much later during the radiation-dominated epoch) or probably extending far into the matter-dominated epoch. This may account for late time, large-scale magnetic fields (for example, inter-galactic fields, with strengths of around $10^{-16}G$ [56]). This work also tries to resolve the problem in [56] where inflation-generated magnetic fields evolve as equation (6.59) [instead of either equation (6.21) or (6.41)] either at (or from) the beginning of exponential expansion of de Sitter phase or either at (or from) the beginning of the inflationary epoch. We approximately evaluated the magnetic field strength for current time. We found that by appealing to causality, the final strength of conventional inflationary magnetic seeds can be increased significantly. Now, what would the magnetic field strength be if super-horizon-sized magnetic fields crossed the horizon a second time after equipartition, that is, some time much later during the matter-dominated epoch? This, we will pursue in the near future.

Chapter 7

Thesis summary

In this thesis (or work), we set out to examine relativistic fluids in the context of cosmology. Studies of this nature are usually done by using the formalism of single-fluids. Examples of such studies that are examined using the single-fluid approximation are studies concerned with the *origin*, *growth* and *evolution* of inflation-generated magnetic fields, and the *coincidence problem*. The origin, growth and evolution of cosmological magnetic fields remains an importantly open question despite the established widespread presence of magnetic fields in the universe [165–168]. In such studies, it is usually found that cosmological magnetic fields evolve adiabatically throughout their post-inflationary evolutionary history until present time. As a result, this would lead to magnetic field strength that will never seed the galactic dynamo and neither can they affect the dynamics of the universe. It was also found that the current values of the densities of dark matter and dark energy are of the identical order of magnitude. This would need very special initial conditions in the early universe. This leads to the coincidence problem.

To resolve such issues, we used the variational formalism to develop tools for examining the dynamics of relativistic fluids in the context of cosmology. Attempts have been made to resolve the problems using the formalism of single-fluids. In [56], problems of generation, growth and evolution of inflation-generated magnetic fields were investigated by utilising the single-fluid formalism, and one of the findings was that inflation-generated magnetic fields evolved in a super-adiabatic form from either the re-heating era (or the era of inflation) until the second cosmological horizon crossing much later during either the epoch of radiation-domination or the epoch of matter-domination.

More fitting mechanisms can be obtained by using the *variational* approach. The results obtained using the variational formalism depend on the complete path taken, and not only on initial and final points of a path. Particularly, in the context of the variational approach, we used the single-fluid and multi-fluid formalisms that incorporate aspects of electrodynamics and thermodynamics, respectively. In the same context, we used the single-fluid approach to derive equations of motion that allowed us to examine the evolution of inflation-generated magnetic fields from the beginning of the radiation-dominated era to the present time. We built the single-fluid model in the context of the variational formalism. We used this formalism to derive equations of motion that allowed us to examine the evolution of inflation-generated magnetic fields after crossing the cosmological horizon for a second time. We extended the MIS theory to allow us to examine the effect on fluid flow in which the components of the multi-species fluids interact thermodynamically. That is, we used the variational multi-fluid formalism particularly, a slightly modified convective variational approach to examine the entrainment effect of an interacting multi-fluid system of the dark-sector.

In chapter 4, we examined the application of the multi-fluid formalism to interacting multi-fluid systems. In particular, we used the slightly modified convective *variational* approach. In this model, we considered two-fluid species (rather than one); that is dark matter and dark energy that occupy a shared volume [44]. We considered the master function Λ that encodes contributions from both species rather than a single species. We examined the case where a chemical interaction occurs between the species. This suggests equation (4.10). Varying the equation with respect to the individual species fluxes, the interacting species flux and the metric led to equations (4.22) and (4.23). We then derived the momentum conjugates (4.24) and (4.25). We expressed the master function Λ in the form of equation (4.28) for a model of two fluids that involved the entrainment of two species. However, since we are interested in entrainment, the focus was on Λ_2 as it encodes entrainment. We then generalised the Lagrangian density Λ for two interacting fluid species whose variation leads to the multi-fluid equations that obey the laws of conservation. Considering commensurate conservation laws, the equations of motion for the multi-fluids environment are developed.

The multi-fluid approach can be used to examine the entrainment effect of the interaction between dark matter and dark energy. For this to be really possible, we analyse the (generalised) second law of thermodynamics and determine if it holds in interacting multi-fluid systems using a more (or most) accurate approach for this task.

An attempt in reference [69] to match the convective *variational* formalism and the (standard) MIS approach of dissipative fluids found that the two formalisms are not equivalent to all orders, but are members of a set of related theories. It was found that the two models led to the same causal connections when subjected to perturbations about a thermodynamic equilibrium. It followed that in the thermal equilibrium limit, the two models manifested similar characteristic surfaces and causality properties. Due to these similarities, we chose to analyse the second law of thermodynamics by utilising the extended MIS formalism for multi-fluids. This implies that one should examine $\nabla_a S^a$, where ∇_a is defined relative to the frame of rest of an observer in motion with a merged u^a. This velocity is merged in the sense that there are three particle species being considered with a single-observer world-line and hence the cosmic time expressed as $t \equiv u^a \nabla_a$ is such that u^a represents the common four-velocity.

In chapter 5, we examined the thermodynamics of relativistic multi-fluid models that are dissipative. The fluid species we considered were radiation, dark matter and baryonic matter as one entity and dark energy. Our interest was in a transition between eras where the content of the universe could be described as being in thermal quasi-equilibrium and where single-fluid or hydrodynamical approximation began to break down. An example of this is a transition involving species being frozen-out when timescales become (almost) equivalent to the timescale of the cosmic expansion leading to the species disintegrating away from the equilibrium. The task we pursued was to model the break-away behaviour. To that, we used the extended MIS theory to examine the multi-species multi-fluid environments. We examined dark energy, baryonic matter and non-baryonic dark matter as one entity and radiation, thus implying that the system was made up of the three fluid species. The interaction involved the dark-sector components only. Though the focus is on the species which are mentioned, this study illustrates the application of the extended MIS formalism to either multi-species multi-fluid or multi-species or multi-fluid environments, in general. We examined if the second law of thermodynamics holds in such an environment.

Now, just before the process of freezing-out began one could have the dynamical apparent horizon. If the break-away species were to manifest itself as either a uniform acceleration or deceleration (almost) equivalent to the remaining species, then one might encounter the Rindler horizon. Our task was to apply the extended MIS formalism in the freeze-out transient period. Before the freeze-out, we had the dynamical apparent horizon which evolved into the Rindler-like horizon after the freeze-out period. Hence, we considered the dynamical apparent horizon first. We considered the radius of the apparent horizon for $FLRW$ which is expressed as in equation (5.14). Further analysis led to equation (5.47). We then examined equation (5.39). Our examination of the equation led to the conclusion that the generalised second law of thermodynamics holds for a multi-fluid system of an interacting dark-sector. This is at the beginning of the freeze-out period. Including a chemical interaction in an environment of multi-fluids (for example, the interacting dark-sector) leads to the second law of thermodynamics being conserved. Having established the conservation of the second law of thermodynamics for interacting multi-fluid environments, and specifically the interacting dark-sector, we applied the slightly modified convective *variational* approach to the sector. Our examination of the *entrainment* effect for the interacting dark-sector suggests a mutual relative modulation of the growth behaviour of the two densities of dark matter and dark energy. With this, the *coincidence problem* might be resolved.

It is found that [50] the effect of the interactions of dark matter and dark energy have a significant signature on the development of the dark matter structure along with the late integrated Sachs Wolfe effect in a formalism of single-fluids. However, how would this change given the model of multi-fluids considered, is an issue that is worth considering and will be studied in the near future.

In chapter 6, we examined the evolution of inflation-generated magnetic fields until present time. First, we considered the evolution of inflation-generated magnetic fields during either the era of inflation or de Sitter phase. Magnetic fields could have crossed the Hubble horizon at either the end of de Sitter phase or just before its end. The universe either expanded exponentially or accelerated during de Sitter phase. After inflation-generated magnetic fields crossed the cosmological Hubble horizon for the first time at either the end or just before the end of de Sitter phase, they continued evolving adiabatically on scales slightly above the horizon. Magnetic fields were scale independent by this time of the universe expansion. This implies that there was no large-scale growth of magnetic fields after inflation-generated magnetic fields crossed the cosmological horizon for the first time. Magnetic fields then evolved adiabatically until the beginning of the radiation-dominated era. We used the single non-conducting fluid approach in the context of the variational formalism to derive equations of motion that led to either equations or expressions in terms of magnetic field symbols (usually used in cosmological modelling of magnetic fields). This was done for the radiation-dominated era. It was shown that there was a possibility of having inflation-generated magnetic fields evolve adiabatically either at or around the beginning of the radiation-dominated era (as shown by expressions of magnetic field symbols). We used the expressions of magnetic field symbols to examine the evolution of inflation-generated magnetic fields from the beginning of the radiation-dominated era until second horizon crossing of magnetic fields. The single-conducting fluid formalism in the context of the variational approach was then used to derive equations of motion that in turn would lead to equations that represented the evolution of inflation-generated magnetic fields. This was after magnetic fields had crossed the cosmological horizon for a second time and evolved adiabatically until present time.

During de Sitter phase, conductivity was extremely low [54, 56]. Hence, there are no currents [54, 56]. This implies that the modified Maxwell equation (6.1) reduces to equation

(6.6). Using the matrix (6.7), equations (6.5) and (6.6) are re-cast in the form of equation (6.21). This equation represents the evolution of inflation-generated magnetic fields as they cross the cosmological Hubble horizon just either before or at the end of de Sitter phase. Equation (6.21) was then expressed in the form of equation (6.25). Solving this equation yields equation (6.41). On slightly above horizon scales, the background space-time is spatially flat. On sub-horizon scales, the background space-time is spatially flat too. Equation (6.41) will then lead to an adiabatic decay of magnetic fields even though conductivity is very poor [57] on scales slightly above the horizon. This means that equation (6.41) represents evolution of the inflation-generated magnetic fields after de Sitter phase on scales which are slightly above the horizon. After de Sitter phase, the universe expands normally (not either expanding exponentially or accelerating). Given that inflation-generated magnetic fields were scale independent [57] after crossing the Hubble horizon for the first time, no large-scale growth of magnetic fields occurred. Instead, the fields could have evolved as equation (6.41) up to the beginning of the radiation-dominated epoch when conformal time denoted as η was still $\eta \ll 1$. By then magnetic fields could have been evolving well above the horizon and this was shown to be possible since $\eta \ll 1$ at that time of the universe expansion. This implies that equation (6.41) can be reduced to equation (6.59).

To prove the existence of inflation-generated magnetic fields during the radiation-dominated era (specifically either at or around the beginning of the radiation-dominated epoch), we used the single non-conducting formalism in the context of the *variational* approach during the epoch, which led to the derivation of equation (6.41). This equation was reduced to equation (6.59) for scales that were well above the horizon, and this occurs either at or around the beginning of the radiation-dominated epoch when conformal time denoted by η is such that $\eta \ll 1$. We then used equation (6.59) to examine the *evolution* of inflation-generated magnetic fields from the beginning of the radiation-dominated era to the time they crossed the cosmological horizon for a second time. In other words we used the variational formalism to derive equations of motion that enabled us to examine the evolution of inflation-generated, cosmological magnetic fields in single-fluid models of the radiation-dominated and matter-dominated epochs. Our examination suggests a possible existence of the super-adiabatically *evolving* mode from the beginning of the radiation-dominated era until second horizon crossing of magnetic fields much later during either the radiation-dominated era or during the epoch of matter-domination. The single-conducting fluid formalism in the context of the *variational* approach was then used to derive equations of motion. We then derived equations describing the evolution of inflation-generated magnetic fields on sub-horizon scales. The equations show that magnetic fields *decay* adiabatically from second horizon crossing (probably far into either the radiation-dominated era or the epoch of matter-domination) until present time. In light of the above statements, the residual strength of magnetic fields could be considerably larger than expected. This may account for late time, large-scale magnetic fields detected in recent times. For example, inter-galactic fields with strengths of around $10^{-16}G$ [56]. In [56], it was assumed that inflation-generated magnetic fields started evolving from the beginning of either the epoch of inflation or at the very beginning of the exponential expansion of the de Sitter phase. However, after careful analysis of the equations in [56], one will see that magnetic fields do not start evolving at the beginning of the exponential expansion of the de Sitter phase. Instead, they start evolving at the time when magnetic fields are well above the Hubble horizon. This is a contradiction. This thesis tries to resolve the issue. We make an ansatz that inflation-generated magnetic fields crossed the cosmological Hubble horizon either just before or at the end of the de Sitter phase which is still during de Sitter phase [56]. Magnetic fields then evolved adiabatically on super-horizon scales which were slightly above the horizon until the beginning of the epoch of radiation-domination. From the beginning of the epoch of radiation-domination, magnetic fields evolved (super-adiabatically) on scales well above the Hubble horizon until they crossed the cosmological Hubble horizon for a second time. In light of (just) the above statements, the contradiction in [56] is resolved.

Appendix A

A few detailed concepts, calculations or derivations

A.1 Brief notes on the Eckart theory

In Eckart theory, the energy density denoted by ρ and the particle number density denoted by n can in principle be measured by an observer moving along either the vector field or four-velocity of a fluid denoted by u^a [91]. The entropy per particle denoted by s can be defined by the equilibrium equation of state for the fluid:

$$s = s(\rho, n). \tag{A.1}$$

All other thermodynamic variables can then be defined by using the first law of thermodynamics. Particularly, the temperature denoted by T and the pressure denoted by p are expressed as:

$$T^{-1} = n\left(\frac{\partial s}{\partial \rho}\right)_n, \tag{A.2}$$

and

$$p = -\rho - n^2 T\left(\frac{\partial s}{\partial n}\right)_\rho, \tag{A.3}$$

respectively.

We examine the transverse perturbations of an Eckart fluid. The differential equations for the perturbations in transverse variables can be decoupled into a single second-order equation for the transverse components of the perturbed velocity expressed as

$$\kappa T{\partial^2}_t \delta u^{\mathcal{I}} - (\rho + p)\partial_t \delta u^{\mathcal{I}} + \eta {\partial^2}_x \delta u^{\mathcal{I}} \;\; = \;\; 0, \tag{A.4}$$

where ρ represents density, p represents pressure, $u^a\partial_a = \partial_t$, κ represents the Boltzmann constant, T represents temperature, $u^{\mathcal{I}}$ represents temperature as a function of space and time (such that $\mathcal{I} = x, y, z$) and η represents conformal time [169]. This equation is *elliptic* for $\delta u^{\mathcal{I}}$. One can clearly see that the solution to equation $(A.4)$ violates any reasonable definition of causality. To elaborate the non-causal behaviour, we integrate the equation to determine the *evolution* of a perturbation which at $t = 0$ is a simple δ function; that is $\delta u^{\mathcal{I}}(x, 0) = \delta x$. The evolution of these *initial data* can be determined by the usual Fourier transform methods. The solution of equation $(A.4)$ with the initial condition is expressed as

$$\delta u^{\mathcal{I}}(x,t) \;\; = \;\; \frac{(\rho+p)t}{2\pi\kappa t}\left(\frac{\eta t^2}{\kappa T} + x^2\right)^{-\frac{1}{2}} K_0(z)\left[\frac{\rho+p}{2\sqrt{(\kappa T\eta)}}\left(\frac{\eta t^2}{\kappa T} + x^2\right)^{\frac{1}{2}}\right] exp\left[\frac{(\rho+p)t}{2\kappa T}\right], \tag{A.5}$$

where $K_0(z)$ represents a modified Bessel function. In the limit that $t^2 \gg \frac{\kappa T x^2}{\eta}$ and $t \gg \frac{\kappa T}{\rho+p}$, equation $(A.5)$ reduces to the standard classical expression for the diffusion of shear stresses and the expression is

$$\delta u^{\mathcal{I}}(x,t) \;\; = \;\; \left(\frac{\rho+p}{4\pi\eta t}\right)^{\frac{1}{2}} exp\left[-\frac{x^2(\rho+p)}{4\eta t}\right]. \tag{A.6}$$

This implies that the classical expression is valid inside a future cone that is determined by the velocity given as $\left(\frac{\eta c^4}{\kappa T}\right)^{\frac{1}{2}}$ where c represents speed of light. For a normal laboratory fluid, this velocity is very large. The characteristic time represented by $\frac{\kappa T}{[(\rho c^2+p)c^2]}$ is very short for normal fluids. Hence, the classical expression for the diffusion of shear stresses is valid in a region which includes and extends outside a future light cone of a plane $(x, t) = (0, 0)$ where the initial disturbance in $\delta u^{\mathcal{I}}$ occurs. Therefore, equation $(A.5)$, the Green's function for the evolution of transverse perturbations in the fully relativistic Eckart theory, violates causality. Furthermore, information can be transmitted faster than c in this theory. This means that the theory cannot be an acceptable relativistic theory.

A.2 Detailed variation of the actions (6.84) and (6.85)

We consider the actions (6.84) and (6.85). Varying action (6.84) leads to

$$\delta S = \delta \int_{\mathcal{M}} d^4x\sqrt{-g}\phi R A^2. \tag{A.7}$$

Introducing a metric yields

$$\delta S = \delta \int_{\mathcal{M}} d^4x\sqrt{-g}\phi g^{ab} R_{ab} A_c A^c, \tag{A.8}$$

which in turn leads to

$$\begin{aligned}\delta S = & \int_{\mathcal{M}} d^4x\sqrt{-g}\delta\phi g^{ab} R_{ab} A^c A_c + \int_{\mathcal{M}} d^4x\sqrt{-g} g^{ab}\delta R_{ab}\phi A^c A_c + 2\int_{\mathcal{M}} d^4x\sqrt{-g}\phi g^{ab} R_{ab} A^c \delta A_c \\ & + \int_{\mathcal{M}} d^4x\delta\sqrt{-g}\phi g^{ab} R_{ab} A^c A_c + \int_{\mathcal{M}} d^4x\sqrt{-g}\phi\delta g^{ab} R_{ab} A_c A^c. \end{aligned} \tag{A.9}$$

Varying action (6.85) yields

$$\delta S_{\phi_0} = \delta \int_{\mathcal{M}} d^4x\sqrt{-g}\phi R_{ab} A^a A^b, \tag{A.10}$$

which in turn leads to

$$\begin{aligned}\delta S_{\phi_0} = & \int_{\mathcal{M}} d^4x\sqrt{-g}\delta\phi R_{ab} A^a A^b + \int_{\mathcal{M}} d^4x\sqrt{-g}\delta R_{ab}\phi A^a A^b + \int_{\mathcal{M}} d^4x\sqrt{-g}\phi R_{ab}\delta A^a A^b \\ & + \int_{\mathcal{M}} d^4x\sqrt{-g}\phi R_{ab} A^a \delta A^b + \int_{\mathcal{M}} d^4x\delta\sqrt{-g}\phi R_{ab} A^a A^b. \end{aligned} \tag{A.11}$$

Varying with respect to ϕ and $\sqrt{-g}$ yields 0 and

$$\delta\sqrt{-g} = \frac{1}{2} g^{ab}\delta g_{ab}, \tag{A.12}$$

respectively. Varying g^{ab} leads to

$$\delta g^{ab} = \delta g^{ba}, \tag{A.13}$$

which in turn yields

$$\delta g^{ab} = -g^{bf} g^{ah}\delta g_{fh}, \tag{A.14}$$

and finally leads to

$$\delta g^{ab} = -g^{ba} g^{ab}\delta g_{ab}. \tag{A.15}$$

The Ricci tensor is defined by

$$R_{ab} \;=\; R^{c}{}_{acb} = \partial_c \Gamma^{c}{}_{ab} - \partial_b \Gamma^{c}{}_{ac} + \Gamma^{c}{}_{cd}\Gamma^{d}{}_{ba} - \Gamma^{c}{}_{bd}\Gamma^{d}{}_{ac}. \qquad (A.16)$$

Varying the Ricci tensor yields [144]

$$\delta R_{ab} \;=\; \partial_c \delta\Gamma^{c}{}_{ab} - \partial_b \delta\Gamma^{c}{}_{ac} + \Gamma^{d}{}_{ba}\delta\Gamma^{c}{}_{cd} + \Gamma^{c}{}_{cd}\delta\Gamma^{d}{}_{ba} - \Gamma^{d}{}_{ac}\delta\Gamma^{c}{}_{bd} - \Gamma^{c}{}_{bd}\delta\Gamma^{d}{}_{ac}, \quad (A.17)$$

which in turn leads to

$$\begin{aligned}\delta R_{ab} \;&=\; \partial_c \delta\Gamma^{c}{}_{ab} + \Gamma^{c}{}_{cd}\delta\Gamma^{d}{}_{ba} - \Gamma^{d}{}_{ac}\delta\Gamma^{c}{}_{bd} - \Gamma^{d}{}_{bc}\delta\Gamma^{c}{}_{ad} \\ &-\; (\partial_b \delta\Gamma^{c}{}_{ac} + \Gamma^{c}{}_{bd}\delta\Gamma^{d}{}_{ac} - \Gamma^{d}{}_{ba}\delta\Gamma^{c}{}_{cd} - \Gamma^{d}{}_{bc}\delta\Gamma^{c}{}_{ad}. \qquad (A.18)\end{aligned}$$

We use the covariant derivative in the two pieces of $(A.18)$ such that

$$\nabla_c \delta\Gamma^{c}{}_{ab} \;=\; \partial_c \delta\Gamma^{c}{}_{ab} + \Gamma^{c}{}_{cd}\delta\Gamma^{d}{}_{ba} - \Gamma^{d}{}_{ac}\delta\Gamma^{c}{}_{bd} - \Gamma^{d}{}_{bc}\delta\Gamma^{c}{}_{ad}, \qquad (A.19)$$

and

$$\nabla_b \delta\Gamma^{c}{}_{ac} \;=\; \partial_b \delta\Gamma^{c}{}_{ac} + \Gamma^{c}{}_{bd}\delta\Gamma^{d}{}_{ac} - \Gamma^{d}{}_{ba}\delta\Gamma^{c}{}_{cd} - \Gamma^{d}{}_{bc}\delta\Gamma^{c}{}_{ad}. \qquad (A.20)$$

One can then express $(A.18)$ as

$$\delta R_{ab} \;=\; \nabla_c \delta\Gamma^{c}{}_{ab} - \nabla_b \delta\Gamma^{c}{}_{ac}. \qquad (A.21)$$

Then

$$\delta\Gamma^{c}{}_{ab} \;=\; \frac{1}{2}\delta g^{cd}(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}) + \frac{1}{2} g^{cd}(\partial_a \delta g_{db} + \partial_b \delta g_{da} - \partial_d \delta g_{ab}). \qquad (A.22)$$

One can have the expression

$$\nabla_d \delta g_{ba} \;=\; \partial_d \delta g_{ba} - \Gamma^{c}{}_{db}\delta g_{ca} - \Gamma^{c}{}_{da}\delta g_{bc}. \qquad (A.23)$$

Then using the symmetry of the Christoffel symbols denoted by $\Gamma^{c}{}_{ab} = \Gamma^{c}{}_{ba}$, one can re-write $(A.22)$ as

$$\begin{aligned}\delta\Gamma^{c}{}_{ab} \;&=\; \frac{1}{2}\delta g^{cd}(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}) + \frac{1}{2} g^{cd}(\nabla_a \delta g_{db} + \Gamma^{e}{}_{ad}\delta g_{cb} + \Gamma^{e}{}_{ab}\delta g_{dc} + \nabla_b \delta g_{ad} + \Gamma^{e}{}_{bd}\delta g_{ca} \\ &+\; \Gamma^{e}{}_{ba}\delta g_{dc} - \nabla_d \delta g_{ba} - \Gamma^{e}{}_{db}\delta g_{ca} - \Gamma^{e}{}_{da}\delta g_{bc}). \qquad (A.24)\end{aligned}$$

Simplifying $(A.24)$ yields

$$\begin{aligned}\delta\Gamma^{c}{}_{ab} &= \frac{1}{2}\delta g^{cd}(\partial_a g_{db}+\partial_b g_{da}-\partial_d g_{ab})+g^{cd}\Gamma^{e}{}_{ba}\delta g_{de} \\ &+ \frac{1}{2}g^{cd}(\nabla_a\delta g_{db}+\nabla_b\delta g_{ad}-\nabla_d\delta g_{ba}).\end{aligned} \tag{A.25}$$

One can write the variation of the metric denoted by g_{de} as

$$\delta g_{de} = -g_{df}g_{eh}\delta g^{fh}. \tag{A.26}$$

Using this in $(A.25)$ leads to

$$\begin{aligned}\delta\Gamma^{c}{}_{ab} &= \frac{1}{2}\delta g^{cd}(\partial_a g_{db}+\partial_b g_{da}-\partial_d g_{ab})-\delta g^{fh}g_{df}g_{eh}g^{cd}\Gamma^{e}{}_{ba} \\ &+ \frac{1}{2}g^{cd}(\nabla_a\delta g_{db}+\nabla_b\delta g_{ad}-\nabla_d\delta g_{ba}).\end{aligned} \tag{A.27}$$

One can rewrite $(A.27)$ as

$$\delta\Gamma^{c}{}_{ab} = \delta g^{ch}g_{eh}\Gamma^{e}{}_{ab}-\delta g^{fh}\delta^{c}_{f}g_{eh}\Gamma^{e}{}_{ba}+\frac{1}{2}g^{cd}(\nabla_a\delta g_{db}+\nabla_b\delta g_{ad}-\nabla_d\delta g_{ba}), \tag{A.28}$$

which yields

$$\delta\Gamma^{c}{}_{ab} = \delta g^{ch}g_{eh}\Gamma^{e}{}_{ab}-\delta g^{ch}g_{eh}\Gamma^{e}{}_{ba}+\frac{1}{2}g^{cd}(\nabla_a\delta g_{db}+\nabla_b\delta g_{ad}-\nabla_d\delta g_{ba}). \tag{A.29}$$

Simplifying this as much as possible yields

$$\delta\Gamma^{c}{}_{ab} = \frac{1}{2}g^{cd}(\nabla_a\delta g_{db}+\nabla_b\delta g_{ad}-\nabla_d\delta g_{ba}). \tag{A.30}$$

Similarly for $\delta\Gamma^{c}{}_{ac}$, one will obtain

$$\delta\Gamma^{c}{}_{ac} = \frac{1}{2}g^{cd}\nabla_a\delta g_{dc}+\frac{1}{2}g^{dc}\nabla_d\delta g_{ac}-\frac{1}{2}g^{cd}\nabla_d\delta g_{ca}, \tag{A.31}$$

and simplifying leads to

$$\delta\Gamma^{c}{}_{ac} = \frac{1}{2}g^{cd}\nabla_a\delta g_{dc}. \tag{A.32}$$

Using $\delta g_{db} = -g_{bf}g_{dh}\delta g^{fh}$, $\delta g_{ad} = -g_{af}g_{dh}\delta g^{fh}$ and $\delta g_{ba} = -g_{af}g_{bh}\delta g^{fh}$ in $(A.30)$ yields

$$\delta\Gamma^{c}{}_{ab} = \frac{1}{2}g^{cd}[\nabla_a(-g_{bf}g_{dh}\delta g^{fh})+\nabla_b(-g_{af}g_{dh}\delta g^{fh})-\nabla_d(-g_{af}g_{bh}\delta g^{fh})], \tag{A.33}$$

which leads to

$$\delta\Gamma^{c}{}_{ab} = -\frac{1}{2}g^{cd}[g_{bf}g_{dh}\nabla_a(\delta g^{fh}) + g_{af}g_{dh}\nabla_b(\delta g^{fh}) - g_{af}g_{bh}\nabla_d(\delta g^{fh})]. \quad (A.34)$$

Simplifying yields

$$\delta\Gamma^{c}{}_{ab} = -\frac{1}{2}[\delta^c_h g_{bf}\nabla_a(\delta g^{fh}) + \delta^c_h g_{af}\nabla_b(\delta g^{fh}) - g_{af}g_{bh}g^{dc}\nabla_d(\delta g^{fh})]. \quad (A.35)$$

Relabelling indices on the first and second terms of (A.35), and then simplifying leads to

$$\delta\Gamma^{c}{}_{ab} = -\frac{1}{2}[\delta^c_c g_{bd}\nabla_a(\delta g^{cd}) + \delta^c_c g_{ad}\nabla_b(\delta g^{dc}) - g_{af}g_{bh}\nabla^c(\delta g^{fh})], \quad (A.36)$$

where $\nabla^c = g^{dc}\nabla_d$. Further simplification yields

$$\delta\Gamma^{c}{}_{ab} = -\frac{1}{2}[g_{bd}\nabla_a(\delta g^{cd}) + g_{ad}\nabla_b(\delta g^{dc}) - g_{af}g_{bh}\nabla^c(\delta g^{fh})]. \quad (A.37)$$

Using $\delta g_{dc} = -g_{df}g_{ch}\delta g^{fh}$ in (A.32) leads to

$$\delta\Gamma^{c}{}_{ac} = -\frac{1}{2}g^{cd}g_{df}g_{ch}\nabla_a\delta g^{fh}, \quad (A.38)$$

which in turn yields

$$\delta\Gamma^{c}{}_{ac} = -\frac{1}{2}\delta^c_f g_{ch}\nabla_a\delta g^{fh}. \quad (A.39)$$

Simplifying leads to

$$\delta\Gamma^{c}{}_{ac} = -\frac{1}{2}g_{fh}\nabla_a\delta g^{fh}. \quad (A.40)$$

One can have the expression

$$\nabla_c\delta\Gamma^{c}{}_{ab} = \frac{1}{2}\nabla_c[g^{cd}(\nabla_a\delta g_{db} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ba})], \quad (A.41)$$

which can be simplified to

$$\nabla_c\delta\Gamma^{c}_{ab} = \frac{1}{2}g^{cd}\nabla_c(\nabla_a\delta g_{db} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ba}). \quad (A.42)$$

The covariant derivative of (A.40) is

$$\nabla_b\delta\Gamma^{c}{}_{ac} = -\frac{1}{2}\nabla_b(g_{fh}\nabla_a\delta g^{fh}), \quad (A.43)$$

which can be simplified to

$$\nabla_b \delta \Gamma^c{}_{ac} = -\frac{1}{2} g_{fh} \nabla_b \nabla_a \delta g^{fh}. \tag{A.44}$$

$(A.42)$ and $(A.44)$ can then be used in $(A.21)$. This will lead to the evaluation of $(A.9)$ and $(A.11)$. With this, one can derive action (6.86) which leads to action (6.88).

A.3 Alternative way of showing conservation of quantities for Lagrangian systems in chapter 6

Given a physical system denoted by $\mathcal{L}$, the conservation laws characterising $\mathcal{L}$ are intimately related to the symmetry properties of $\mathcal{L}$. A precise description of this relation is given by Noether's theorem [170–174]. It states that a physical property of $\mathcal{L}$ is conserved for one and all continous symmetries of $\mathcal{L}$. This means that there is a constant of motion denoted by I^a and expressed as

$$I^a(x^a, A_a, A_{a,b}) \equiv f^a(x^a, A_a) - \left[\xi^a \mathcal{L} + (\eta^{a,bc} - A_{b,c}\xi^a) \frac{\partial \mathcal{L}}{\partial A_{b,c}} \right], \tag{A.45}$$

such that

$$\frac{\partial I^a}{\partial x^a} \equiv 0, \tag{A.46}$$

where $f^a \approx 0$ or negligible [175]. Space-time indices are denoted by a, b and c, A_a represents an electromagnetic field potential vector, f^a represents a gauge function, and ξ^a and η^a are functions of x^a and A_a. Our task is to show that the systems we are considering in chapter 6 obey equation $(A.46)$. We will first consider the system that is composed of the fluid, Maxwell, Coulomb and the coupling term actions. Then, $\mathcal{L}$ should be expressed in the form

$$\mathcal{L} = \Lambda + \frac{1}{4} F_{ab} F^{ab} + J^a{}_X A_a + \phi R A^2 + \phi_0 R_{ab} A^a A^b, \tag{A.47}$$

which can also be written as

$$\mathcal{L} = \Lambda + \frac{1}{4} g^{ca} g^{db} (A_{b,a} A_{d,c} - A_{b,a} A_{c,d} - A_{a,b} A_{d,c} + A_{a,b} A_{c,d}) + J^a{}_X A_a + \phi R A_a A^a + \phi_0 R_{ab} A^a A^b. \tag{A.48}$$

To make our work easier, we adopt the assumption $4\pi \sim 1$ without loss of generality. We then work out

$$X^{[1]}\mathcal{L} + \mathcal{L}D_a\xi^a \equiv D_a f^a, \tag{A.49}$$

where

$$X^{[1]} = \xi^a\partial_a + \eta^a\partial_{A_a} + \eta^{a,b}\partial_{A_{a,b}}, \tag{A.50}$$

and

$$D_a \equiv \partial_a + A_{b,a}\frac{\partial}{\partial A_b}. \tag{A.51}$$

We evaluate equation $(A.49)$ in detail. We consider $X^{[1]}\mathcal{L}$. We then consider $\xi^a\partial_a\mathcal{L}$ first. This leads to

$$\begin{aligned}\xi^e\partial_e\mathcal{L} &= \xi^e\Lambda_{,e} + \xi^e(g^{ca}g^{db})_{,e}(A_{b,a}A_{d,c} - A_{b,a}A_{c,d} - A_{a,b}A_{d,c} + A_{a,b}A_{c,d}) + \xi^e g^{ca}g^{db}(A_{b,ae}A_{d,c} \\ &+ A_{b,a}A_{d,ce} - A_{b,ae}A_{c,d} - A_{b,a}A_{c,de} - A_{a,be}A_{d,c} - A_{a,b}A_{d,ce} + A_{a,be}A_{c,d} + A_{a,b}A_{c,de}) \\ &+ \xi^e {J^a}_{X,e}A_a + {J^a}_X A_{a,e}\xi^e + \xi^e(\phi R)_{,e}A_aA^a + \xi^e\phi RA_{a,e}A^a + \xi^e\phi RA_a g^{ba}A_{b,e} + \xi^e(\phi_0 R_{ab})_{,e}A^aA^b \\ &+ \xi^e\phi_0 R_{ab}g^{ca}A_{c,e}A^b + \xi^e\phi_0 R_{ab}A^a g^{cb}A_{c,e}.\end{aligned} \tag{A.52}$$

Evaluation of $\eta^a\partial_{A_a}\mathcal{L}$ yields

$$\eta_e\frac{\partial\mathcal{L}}{\partial A_e} = \left[{J^a}_X\frac{\partial A_a}{\partial A_e} + \phi R\left(\frac{\partial A_a}{\partial A_e}A^a + A_a\frac{\partial A^a}{\partial A_e}\right) + \phi_0 R_{ab}\left(\frac{\partial A^a}{\partial A_e}A^b + A^a\frac{\partial A^b}{\partial A_e}\right)\right]\eta_e, \tag{A.53}$$

which in turn leads to

$$\eta_e\frac{\partial\mathcal{L}}{\partial A_e} = ({J^e}_X + 2\phi RA^e + 2\phi_0 {R^e}_a A^a)\eta_e. \tag{A.54}$$

We then evaluate $\eta^{a,b}\partial_{A_{a,b}}\mathcal{L}$. This yields

$$\begin{aligned}\eta^{e,f}\frac{\partial\mathcal{L}}{\partial A_{e,f}} &= \eta^{e,f}\Bigg\{g^{ca}g^{db}\Bigg[\frac{\partial(\partial_a A_b)A_{d,c}}{\partial(\partial_f A_e)} + A_{b,a}\frac{\partial(\partial_c A_d)}{\partial(\partial_f A_e)} - \frac{\partial(\partial_a A_b)A_{c,d}}{\partial(\partial_f A_e)} - A_{b,a}\frac{\partial(\partial_d A_c)}{\partial(\partial_f A_e)} \\ &- \frac{\partial(\partial_b A_a)A_{d,c}}{\partial(\partial_f A_e)} - A_{a,b}\frac{\partial(\partial_c A_d)}{\partial(\partial_f A_e)} + \frac{\partial(\partial_b A_a)A_{c,d}}{\partial(\partial_f A_e)} + A_{a,b}\frac{\partial(\partial_d A_c)}{\partial(\partial_f A_e)}\Bigg]\Bigg\}.\end{aligned} \tag{A.55}$$

We consider the expression

$$\eta^{a,b} \;=\; D_b(\eta^a) - A_{a,c}D_b(\xi^c). \tag{A.56}$$

When evaluated it leads to

$$\eta^{a,b} \;=\; \eta^a{}_{,b} - A_{a,c}\xi^c{}_{,b} + A_{c,b}\frac{\partial\eta^a}{\partial A_c} - A_{a,c}A_{d,b}\frac{\partial\xi^c}{\partial A_d}. \tag{A.57}$$

Substituting this equation in equation $(A.55)$ and then simplifying further yields

$$\begin{aligned}\eta^{e,\,f}\frac{\partial\mathcal{L}}{\partial A_{e,f}} \;=&\; 4A^{e,f}\eta^e{}_{,f} - 4A^{e,f}A_{e,g}\xi^g{}_{,f} + 4A^{e,f}A_{g,f}\frac{\partial\eta^e}{\partial A_g} - 4A^{e,f}A_{e,g}A_{h,f}\frac{\xi^g}{\partial A_h}\\ -&\; 4A^{f,e}\eta^e{}_{,f} + 4A^{f,e}A_{e,g}\xi^g{}_{,f} - 4A^{f,e}A_{g,f}\frac{\partial\eta^e}{\partial A_g} + 4A^{f,e}A_{e,g}A_{h,f}\frac{\partial\xi^g}{\partial A_h}.\end{aligned} \tag{A.58}$$

The evaluation of $\mathcal{L}D_a\xi^a$ leads to

$$\begin{aligned}\mathcal{L}D_e\xi^e \;=&\; \Lambda\partial_e\xi^e + g^{ca}g^{db}(A_{b,a}A_{d,c}\partial_e\xi^e - A_{b,a}A_{c,d}\partial_e\xi^e - A_{a,b}A_{d,c}\partial_e\xi^e + A_{a,b}A_{c,d}\partial_e\xi^e) + J^a{}_X A_a\partial_e\xi^e\\ +&\; \phi RA_aA^a\partial_e\xi^e + \phi_0R_{ab}A^aA^b\partial_e\xi^e + \Lambda A_{e,f}\frac{\partial\xi^e}{\partial A_f} + g^{ca}g^{db}\Bigg(A_{b,a}A_{d,c}A_{f,e}\frac{\partial\xi^e}{\partial A_f} - A_{b,a}A_{c,d}A_{f,e}\frac{\partial\xi^e}{\partial A_f}\\ -&\; A_{a,b}A_{d,c}A_{f,e}\frac{\partial\xi^e}{\partial A_f} + A_{a,b}A_{c,d}A_{f,e}\frac{\partial\xi^e}{\partial A_f}\Bigg) + J^a{}_X A_aA_{f,e}\frac{\partial\xi^e}{\partial A_f}\\ +&\; \phi RA_aA^aA_{f,e}\frac{\partial\xi^e}{\partial A_f} + \phi_0R_{ab}A^aA^bA_{f,e}\frac{\partial\xi^e}{\partial A_f}.\end{aligned} \tag{A.59}$$

We finally evaluate D_af^a which yields

$$D_ef^e \equiv \partial_ef^e + A_{f,e}\frac{\partial f^e}{\partial A_f}. \tag{A.60}$$

We then substitute everything evaluated in equation $(A.49)$ and this leads to

$$\begin{aligned}
\partial_e f^e + A_{f,e}\frac{\partial f^e}{\partial A_f} &= \xi^e\Lambda_{,e} + \xi^e(g^{ca}g^{db})_{,e}(A_{b,a}A_{d,c} - A_{b,c}A_{c,d} - A_{a,b}A_{d,c} + A_{a,b}A_{c,d}) + \xi^e g^{ca}g^{db}(A_{b,ae}A_{d,c} \\
&+ A_{b,a}A_{d,ce} - A_{b,ae}A_{c,d} - A_{b,a}A_{c,de} - A_{a,be}A_{d,c} - A_{a,b}A_{d,ce} + A_{a,be}A_{c,d} + A_{a,b}A_{c,de}) \\
&+ \xi^e({J^a}_X)_{,\,e}A_a + {J^a}_X A_{a,e}\xi^e + \xi^e(\phi R)_{,e}A_aA^a + \xi^e\phi R A_{a,e}A^a + \xi^e\phi R A_a g^{ba}A_{b,e} \\
&+ \xi^e(\phi_0 R_{ab})_{,e}A^aA^b + \xi^e\phi_0 R_{ab}g^{ca}A_{c,e}A^b + \xi^e\phi_0 R_{ab}A^a g^{cb}A_{c,e} + \eta_e {J^e}_X + \eta_e 2\phi R A^e \\
&+ \eta_e 2\phi {R^e}_a A^a + 4{\eta^e}_{,f}g^{cf}g^{de}A_{d,c} - 4A_{e,g}{\xi^g}_{,f}g^{cf}g^{de}A_{d,c} + 4A_{g,f}\frac{\partial\eta^e}{\partial A_g}g^{cf}g^{de}A_{d,c} \\
&- 4A_{e,g}A_{h,f}\frac{\partial\xi^g}{\partial A_h}g^{cf}g^{de}A_{d,c} - 4{\eta^e}_{,f}g^{ce}g^{df}A_{d,c} + 4A_{e,g}{\xi^g}_{,f}g^{ce}g^{df}A_{d,c} - 4A_{g,f}\frac{\partial\eta^e}{\partial A_g}g^{ce}g^{df}A_{d,c} \\
&+ 4A_{e,g}A_{h,f}\frac{\partial\xi^g}{\partial A_h}g^{ce}g^{df}A_{d,c} + \Lambda\partial_e\xi^e + g^{ca}g^{db}(A_{b,a}A_{d,c}\partial_e\xi^e - A_{b,a}A_{c,d}\partial_e\xi^e - A_{a,b}A_{d,c}\partial_e\xi^e \\
&+ A_{a,b}A_{c,d}\partial_e\xi^e) + {J^a}_X A_a\partial_e\xi^e + \phi R A_aA^a\partial_e\xi^e + \phi_0 R_{ab}A^aA^b\partial_e\xi^e + \Lambda A_{e,f}\frac{\partial\xi^e}{\partial A_f} \\
&+ g^{ca}g^{db}\left(A_{b,a}A_{d,c}A_{f,e}\frac{\partial\xi^e}{\partial A_f} - A_{b,a}A_{c,d}A_{f,e}\frac{\partial\xi^e}{\partial A_f} - A_{a,b}A_{d,c}A_{f,e}\frac{\partial\xi^e}{\partial A_f} + A_{a,b}A_{c,d}A_{f,e}\frac{\partial\xi^e}{\partial A_f}\right) \\
&+ {J^a}_X A_aA_{f,e}\frac{\partial\xi^e}{\partial A_f} + \phi R A_aA^aA_{f,e}\frac{\partial\xi^e}{\partial A_f} + \phi_0 R_{ab}A^aA^bA_{f,e}\frac{\partial\xi^e}{\partial A_f}. \qquad (A.61)
\end{aligned}$$

Given η^a, ξ^a, and f^a which depend on $(x^a,\ A_a)$ only, the only way in which equation $(A.61)$ can be identically satisfied is that either the coefficient of each of the products of $A_{a,b}$ or products of the derivatives of A_a on each side of the equation coincides (we are not using the equations of motion; here x^a, A_a and $A_{a,b}$ are independent). We consider one such product. Equating coefficients of $A_{b,a}A_{d,c}A_{f,e}$ on both sides of equation $(A.61)$ yields

$$g^{ca}g^{db}\frac{\partial\xi^e}{\partial A_f} = 0. \qquad (A.62)$$

Solving for ξ^e, then

$$g^{ca}g_{ca}g^{db}g_{db}\frac{\partial\xi^e}{\partial A_f} = 0, \qquad (A.63)$$

which leads to

$${\delta^a}_a{\delta^b}_b\frac{\partial\xi^e}{\partial A_f} = 0. \qquad (A.64)$$

This then yields

$$16\frac{\partial\xi^e}{\partial A_f} = 0, \qquad (A.65)$$

which leads to

$$\xi^e = \xi^e(x^a). \tag{A.66}$$

This shows that $\xi^a \neq 0$ which implies that $I^a \neq 0$ if either $f^a \approx 0$ (or a trivial constant). This can lead to equation $(A.46)$. This is applicable for the case where either $\phi = 0$ or $\phi_0 = 0$ or $\phi = \phi_0 = 0$ or/and ${J^a}_X \sim 0$. One will arrive at the same conclusion when one considers a system denoted by $\mathcal{L}(x^a, n^a, {n^a}_{,b})$.

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