Survey on the Improvement and Application of HHL Algorithm

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Abstract. Quantum computing is a new computing mode that follows the laws of quantum mechanics to control quantum information units for computation. In terms of computational efficiency, due to the existence of quantum mechanical superposition, some known quantum algorithms can process problems faster than traditional general-purpose computers. HHL algorithm is an algorithm for solving linear system problems. Compared with classical algorithms in solving linear equations, it has an exponential acceleration effect in certain cases and as a sub-module, it is widely used in some machine learning algorithms to form quantum machines learning algorithms. However, there are some limiting factors in the use of this algorithm, which affect the overall effect of the algorithm. How to improve it to make the algorithm perform better has become an important issue in the field of quantum computing. This paper summarizes the optimization and improvement of HHL algorithm since it was proposed, and the application of HHL algorithm in machine learning, and discusses some possible future improvements of some subroutines in HHL algorithm.
Keywords. Quantum Computing, HHL Algorithm, LinearSystems, Quantum Machine-Learning.

1. Introduction

In recent years, quantum computing has developed very rapidly. Due to the unique superposition and entanglement of quantum, quantum computing has parallel computing capabilities and information carrying capacity that are unmatched by classical computing technology. The development bottleneck of traditional computers has reached, due to the failure of Moore's Law. So the development of quantum computing has become the mainstream today. In 1982, Benioff [1] proposed the quantum mechanical model of Turing machine and Feynman [2] proposed the idea of using quantum system for information processing. In 1985, Deutsch [3] first proposed the quantum Turing machine model, and designed the first algorithm specially designed for quantum computers using quantum properties. In 1994, Shor [4] proposed a large number prime factorization algorithm based on Fourier transform, which can quickly decompose the prime factors of large numbers by using the parallelism of quantum computing. In 1996, Grover [5] proposed a quantum search algorithm, which can perform quadratic acceleration on unstructured data. In 2009, Harrow et al. [6] proposed a quantum algorithm for solving linear equations, often referred to as the HHL algorithm by researchers. These algorithms have made a significant contribution to demonstrating the advantages of quantum over classical quantum, bringing real attention to quantum computing in the scientific community. Linear systems are the core of many fields of science, engineering and optimization problems. Many fields today rely heavily on the solution of linear equation problems. Compared with classical algorithms, HHL algorithm has exponential performance in certain cases. The acceleration effect is of great significance in the fields of weather forecasting, economics, biological engineering, and computational science. On the other hand, in order to achieve computing advantages, quantum computing has been introduced into the field of machine learning, replacing the more complex part of the machine learning algorithm with the corresponding quantum version for calculation, thereby reducing the time and space complexity of the algorithm. In the direction of quantum machine learning, the implementation of some quantum machine learning algorithms is mainly to use the HHL algorithm to replace the computationally complex part of traditional machine learning and the structure of the entire algorithm is mostly the structure of traditional machine learning algorithms, but due to the algorithm Some of the computationally complex parts of the system are solved by quantum algorithms, thereby improving the overall efficiency. This paper will focus on the in-depth discussion of the HHL algorithm as the core and introduce the improvement and application of the HHL algorithm since its proposal.

2. Simulation and Improvement of HHL Algorithm

2.1. Implementation of HHL Algorithm

Linear systems are the core of many scientific and engineering fields. Because the HHL algorithm achieves the exponential acceleration effect of classical algorithms under certain conditions, it can be widely used in data processing, machine learning, numerical computing and other scenarios in the future. In 2009, Harrow, Hassidim and Lloyd first proposed a quantum algorithm for solving systems of linear equations using techniques such as Hamiltonian simulation and phase estimation. In the literature [7], Danial et al. introduced the HHL algorithm and its improved optimization in detail, such as quantum phase estimation and amplitude amplification and error analysis. The three subroutines of the HHL algorithm are: Phase estimation, Control rotation, Inverse phase estimation. The circuit diagram is shown in figure 1. The general steps of the algorithm are:

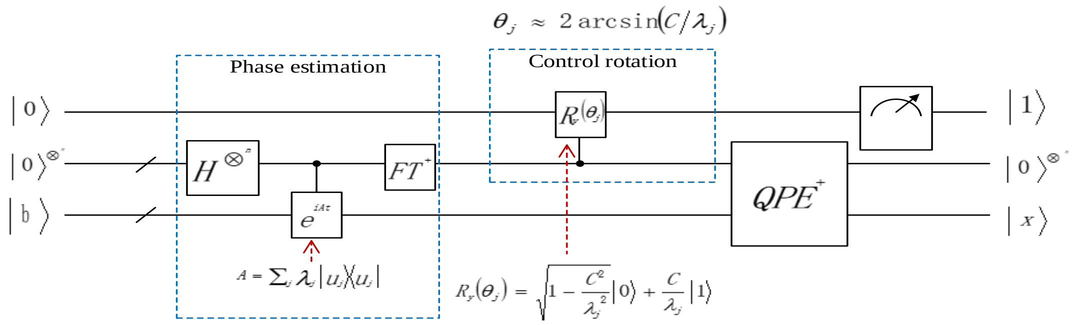

Figure 1. HHL algorithm circuit diagram.

1. A phase estimation subroutine is applied to the initial bits, which is a general step for decomposing a quantum state on a particular basis. Initialize quantum state $|b\rangle$, The initial state of the whole system is$|0\rangle^{\otimes n}|b\rangle$. When the phase estimation subroutine ends, the state of the entire system is $\sum_j \beta_j\, |\lambda_j\rangle|\mu_j\rangle$.

2. Perform a controlled rotation operation, which is intended to turn the$|\lambda_j\rangle$ to $\lambda_j^{-1}|\lambda_j\rangle$.

3. Using inverse quantum phase estimation to restore$|\lambda_j\rangle$ to $|0\rangle$, the state of the entire system is:

$$\sum_{\mathrm{j}} \left(\sqrt{1-\frac{c^2}{\lambda_j{}^2}}|0\rangle + \frac{c}{\lambda_j}|1\rangle \right) \beta_j\, |0\rangle\, |\mu_j\rangle \tag{1}$$

The most important step in the process of solving linear equations is the quantum phase estimation algorithm that estimates the eigenvalues of Hermitian and unitary matrices. At this stage, an important step is to perform a Hamiltonian simulation of some Hermitian matrix A to prepare it as unitary operator U=e^{iAt}. Hamiltonian simulation is method of finding an efficient algorithm to realize the time evolution of quantum states under given conditions, assuming that the input state is $|E_j>$, after Hamiltonian simulation, the state of this form is obtained $(1/\sqrt{M})\sum_{l=0}^{m-1} e^{i\lambda_j l} |l > E_j$, then perform an inverse quantum Fourier transform on $|l >$ to recover λ_j[8].

Most of the current experiments are performed to solve 2 × 2 linear equations and 4 × 4 linear equations for various input vectors. The HHL algorithm experiments were realized in the nuclear magnetic resonance system [9], the optical quantum system [10] and superconducting quantum computing system [11]. These experiments demonstrate the solution of binary linear equations by using 4 qubits and logic gate operations respectively and prove the feasibility of this algorithm. Reference [12] takes the HHL quantum algorithm as an example and gives the quantum circuits of the HHL quantum algorithm corresponding to 4 qubits and 7 qubits. The experimental results show that when the input matrix is a second-order matrix, the fidelity is high, but when the input matrix is a fourth-order matrix, the probability of the result being close to the true value is high only when the matrix is sparse. Reference [13] uses four qubits and four controlled logic gates to implement each required subroutine, demonstrating how the algorithm works. For different input vectors, the quantum computer gave solutions to linear equations with fairly high accuracy, with fidelities ranging from 0.825 to 0.993. Figure 2 presents the fidelity of experimental results for different quantum systems:

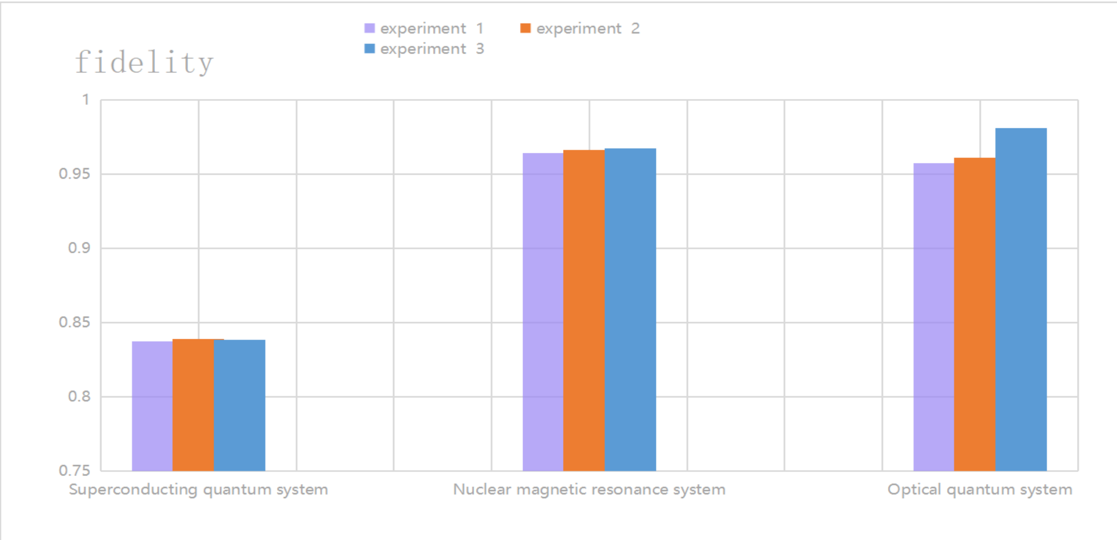

Figure 2. Fidelity of experimental results in quantum system.

2.2. Improvement of HHL Algorithm

For the HHL algorithm to solve the linear equation system, when the coefficient matrix A is a sparse well-conditioned matrix, the quantum algorithm achieves an exponential speedup in the dimension of the matrix compared to the classical algorithm. However, the algorithm has some limitations that make it difficult to achieve the ideal exponential speedup. The requirements for achieving exponential acceleration are [14]:

1. Matrix A must be sparse or can be efficiently decomposed into sparse form.
2. The condition number of A must be scaled to polylog N, where N is the size of the linear system.
3. The elements of A can be efficiently computed by a black-box *Oracle*.

In response to these constraints, scholars have also proposed different improvements, and after avoiding these constraints, the performance of the algorithm has also been improved. In 2010, Ambainis [15] and others proposed an improved version of the algorithm using variable time amplitude amplification technology. The improved version reduces the dependence of the algorithm complexity on the matrix condition number while keeping other factors in the algorithm unchanged. In 2017, Wossnig [16] used the quantum singular value estimation algorithm to reduce the dependence on matrix sparsity in computational complexity and realized that the quantum linear system algorithm can also achieve exponential speedup when dealing with dense matrices. The Hamiltonian is an operator related to the total energy of a quantum system, which can be used to describe the evolution of a quantum system over time, and the evolution of quantum states can be achieved by manipulating the Hamiltonian. The process of processing A using Hamiltonian simulation in the HHL algorithm can be called a black box. Literature [17] extensively studies the Hamiltonian simulation problem in quantum circuits for solving linear equations, and how to simulate e^{iAt} in time. Sparse Hamiltonian simulations play a big role in quantum algorithms [18, 19]. Reference [20] uses the group leader optimization algorithm (GLOA) to process the Hamiltonian simulation steps: implementing controlled unit $U = e^{iAt}$, Through this algorithm, the U-gate is approximately decomposed into basic quantum gates. When the U-gate is decomposed into a series of basic gates, its quantum circuit can be easily constructed. In order to achieve relatively high accuracy of the HHL algorithm, in the Hamiltonian simulation operation stage, low error is required to simulate the Hamiltonian and the rotation step is controlled to obtain accurate results. The algorithm proposed in [21, 22] helps decomposing Hamiltonian simulation unitary operators to arbitrary precision, Hamiltonian simulation is an important method for dealing with matrices. In table 1 we summarize several different Hamiltonian simulation methods and their complexity.

Table 1. Different Hamiltonian simulation methods.

Hamilton Simulation	Complexity
Error containing simulation	$O(max\|H_i\|^2\, t^2/\epsilon)$
Trotter-Suzuki simulation	$O((t^2 max(H_1, H_2)^2)/\epsilon)$
Based on quantum walk simulation	$O\left(s\|H_{max}\|\, t/\sqrt{\epsilon}\right)$

In the process of using the HHL algorithm to solve the linear equation problem, the subroutines in the HHL algorithm will also affect the accuracy of the calculation results, so that the effect of the HHL algorithm is limited. Among the three subroutines of the algorithm, the phase estimation stage is prone to precision errors. Phase estimation is achieved in two stages: In the first stage, the phase of the eigenvalue is extracted and placed in the probability radiation of the quantum state. In the second stage, the phase in the probability amplitude is extracted and placed in the ground state of the quantum state. The final output is the estimated phase, which can be used to further find the eigenvalues of the input matrix. At this stage, for the input matrix, it is first converted to a unitary operator by Hamiltonian simulation e^{iAt}, then the phase information of the unitary operator is stored on the qubit. In this process, the eigenvalue information of the unitary operator is represented by the qubit. The accuracy of the solution depends on the eigenvalue represented by the qubit assigned to the phase estimation and the higher the number of qubits, the higher the accuracy of the phase estimation. The accuracy of the phase estimation algorithm is therefore limited by the number of qubits that represent the eigenvalues of the matrix. Over the years, different scholars have accelerated and improved the accuracy of the algorithm by improving subroutines in the HHL algorithm. In 2015, Childs et al. [23] improved the HHL algorithm by using techniques such as unitary matrix linear combination. Compared with the original HHL algorithm, the algorithm has an exponential acceleration effect for both dimension parameters and precision parameters and reduces the computational cost, in dependence on precision in complexity. Literature [24, 25] discussed some caveats proposed by scholars for the HHL algorithm: the condition number k of matrix A , Hamiltonian simulation e^{iAt}, quantum state preparation $|b\rangle$, the solution of the HHL algorithm is quantum state$|x\rangle$, However, the influence of these factors on the calculation results of the HHL algorithm is also acceptable. The biggest problem is that the singular value of matrix *A* needs to be located between $1/k$ and 1. Although this problem can be solved by scaling, it is often difficult to achieve the desired scaling. The problem arises in the eigenvalue stage of the matrix that is processed by quantum phase estimation. Reference [26] analyzes the limiting factors of the accuracy of the HHL algorithm solution, and then proposes an iterative improvement method for HHL, and uses the iterative improvement method to verify the 4×4 linear equation system. Through the iterative improvement method, the optimal solution is obtained, and the accuracy of the improved algorithm can exceed the accuracy limit brought by the number of qubits in the phase estimation stage of the original algorithm. In 2018, Lee [27] proposed a hybrid quantum algorithm for solving linear equations based on the HHL algorithm. The algorithm is mainly composed of phase estimation algorithm, classical calculation and simplified HHL algorithm. In 2018, Lee [27] proposed a hybrid quantum algorithm for solving linear equations based on the HHL algorithm. The algorithm is mainly composed of a phase estimation algorithm, a classical calculation and a simplified HHL algorithm. First, the phase estimation is repeated. Obtain the k-bit classical eigenvalue information and then use the classical computer to analyze the measurement results of the first step. According to the analyzed data, a simplified circuit of the controlled rotation part is realized [28] and finally the reduced controlled rotation part is used to replace the original one. The controlled rotation section executes the HHL algorithm. The paper summarizes the algorithm to calculate the probability distribution of different eigenvalues, as shown in figure 3. It can be seen that the accuracy of the improved algorithm in solving the linear equation problem is higher than the original algorithm.

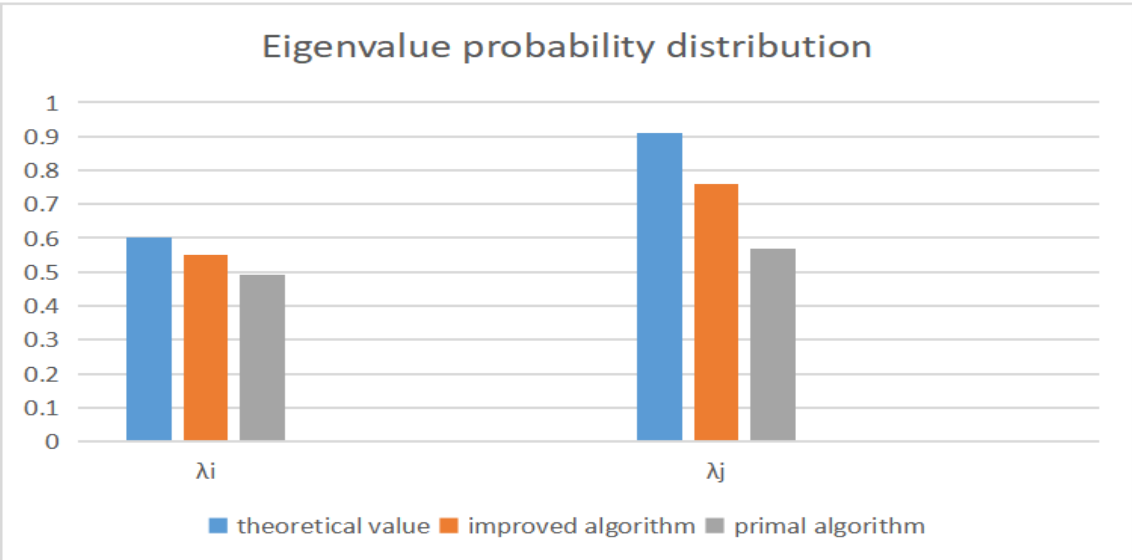

Figure 3. Eigenvalue estimation probability distribution.

In this hybrid algorithm, the quantum phase estimation part needs a classical information feedforward to reduce the circuit depth of the HHL algorithm and the accurate performance in solving a specific linear equation system is higher than that of the HHL algorithm. The controlled rotation operation in the HHL algorithm realizes the proportional extraction of the reciprocal of the ground state value to the probability amplitude of the corresponding ground state through an additional qubit, that is, $|\lambda_j\rangle$ to $\lambda_j^{-1}|\lambda_j\rangle$, Based on quantum phase estimation, literature [29] proposes a modular approach to implement an arbitrary controlled quantum rotation algorithm, and uses numerical simulations to illustrate the effect of controlled rotation on the fidelity of the implementation, proving that The controlled rotation method makes the quantum algorithm have high fidelity without affecting the acceleration of the original algorithm. Its structure is shown in figure 4:

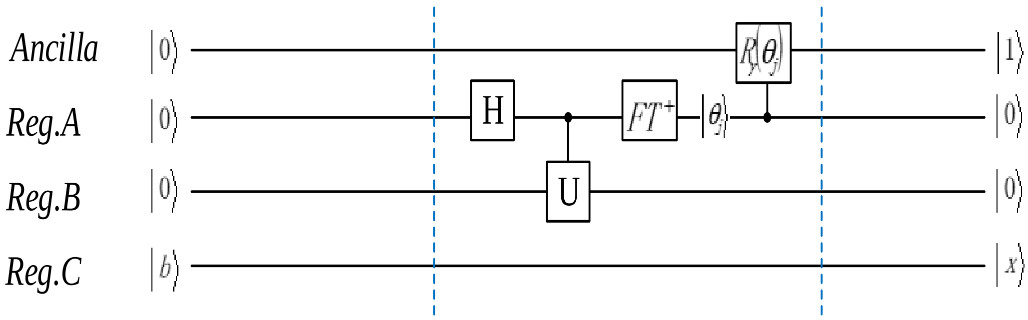

Figure 4. Improved controlled rotation.

Gao et al. [30] systematically studied the performance of the HHL algorithm with different precisions and different redundant qubits and proposed a new quantum-classical hybrid algorithm for the case of incomplete phase estimation, which can be used without sacrificing Reduce the demand for qubit resources under the premise of accuracy. n order to intuitively compare the improvement effect brought by the improvement of the HHL algorithm by scholars, table 2 lists the improved results of different scholars after improving the algorithm:

Table 2. HHL algorithm improvement comparison.

Algorithm	Before Improvement	After Improvement
[15]	$O(k^2 \, log\, N)$	$O(k(log^3 k)\, log\, N)$
[16]	$O(k^2 nploy\, n/\epsilon)$	$O\left(k^2 \sqrt{n} ploy\, n/\epsilon\right)$
[25]	$O\left(ploy(1/\epsilon)\right)$	$O(ploy\, log(1/\epsilon))$
[31]	$O(log(N)\, s^2\, k^2/\epsilon)$	$O(log(N)\, \lceil n \rceil s^2\, k^2/\epsilon)$

3. Quantum Machine Learning

3.1. Quantum Regression Algorithm

Because many traditional machine learning problems are ultimately related to the solution of optimization problems, which often involve the solution of linear equations, the HHL algorithm is exponentially faster than the classical algorithm for solving linear equations with sparse Hermitian matrices, so The HHL algorithm can help speed up the optimization steps in classical machine learning and scholars apply quantum algorithms to traditional machine learning algorithms to improve the algorithms, thus forming the research direction of quantum machine learning. Quantum machine learning uses the high parallelism of quantum computing to further optimize traditional machine learning and the proposal of the HHL algorithm directly promotes the development of the entire quantum machine learning direction. Some quantum machine learning algorithms based on linear system problems use some of the techniques in the HHL algorithm to a greater or lesser extent. For the improvement of machine learning algorithms into quantum machine learning algorithms, it is not only through quantum algorithms to deal with complex parts of classical algorithms, the main problem is how to efficiently map the classical data in the original machine learning algorithm into quantum states. The current mainstream method is based on the physical realization of QRAM and uses QRAM to realize the preparation of any quantum state, and then perform subsequent quantum state calculations. Reference [31] summarizes the quantum machine learning algorithms in the past ten years, and at the same time, compares and analyzes the differences and connections between quantum machine learning algorithms and traditional machine learning algorithms.

Linear regression is one of the most important tasks in data mining and machine learning. The linear regression problem is equivalent to solving a linear system $A^{\dagger}Ax$=$A^{\dagger}b$, where A is the data matrix, b is the given vector and the prediction of the new data ω is equivalent to calculating the inner product $\omega\cdot x$. Therefore, through the HHL algorithm, we can effectively find $|x\rangle$ then when we prepare the quantum state$|\omega\rangle$ of ω, we can effectively calculate $\langle\omega|x\rangle$, linear regression is all about finding the best fit parameters and using it to predict new data. Reference [32] reviewed the important progress of quantum regression algorithms in recent years, including quantum linear regression and quantum ridge regression algorithms, and proposed a quantum logistic regression algorithm based on gradient descent. Compared with classical algorithms, these quantum algorithms have exponential acceleration effect under reasonable assumptions, showing the unique advantages of quantum computing. In 2012, Wiebe et al. first proposed a quantum linear regression algorithm based on the HHL algorithm [33]. When the data matrix is sparse and has a very low condition number, the algorithm has an exponential acceleration effect compared to the classical algorithm. The purpose of the quantum linear regression algorithm is to use the quantum algorithm to solve the optimal fitting parameter ω:

$$\omega = X^{+}y = (X^{\dagger}X)^{-1}X^{\dagger}y \tag{2}$$

The calculation process is divided into two parts: $y' = X^{\dagger}y$ and $\omega = (X^{\dagger}X)^{-1}X^{+}y = (X^{\dagger}X)^{-1}y'$, for $y' = X^{\dagger}y$, First, the matrix y is loaded on the quantum state through probability amplitude encoding, and the matrix X is converted into Hermitian matrix $I(X) = \begin{pmatrix} 0 & X \\ X^{\dagger} & 0 \end{pmatrix}$, $y' = X^{\dagger}y$ to $|y'\rangle = I(X)\,|y\rangle$, $\omega = (X^{\dagger}X)^{-1}X^{\dagger}y$,the problem can be transformed into solving $|\omega\rangle = I((X^{\dagger}X)^{-1})|y'\rangle$, when converted to solving quantum states$|\omega\rangle$ and $|y'\rangle$, at this point, the HHL algorithm can be used to solve it, then the quantum state $|\omega\rangle$ of the fitting parameters can be obtained. Reference [34] details the solution process of quantum linear regression. In 2019, Zhao et al. used the HHL algorithm to design a quantum Gaussian process regression algorithm for the prediction stage of Gaussian process regression [35]. This algorithm has an exponential acceleration effect compared to the classical algorithm. Reference [36] proposed the first quantum ridge regression algorithm, but the data matrix processed by this algorithm needs to be a low-rank matrix. In 2017, Yu [37] et al. proposed a quantum ridge regression algorithm whose data matrix is a low-rank matrix. The steps are similar to the HHL algorithm, except that the dense low-rank matrix simulation technology is used in the quantum simulation part [38]. In 2019, Yu [39] proposed an improved quantum ridge regression algorithm. Compared with the previous algorithm, it can only be effective when the matrix is sparse. The improved algorithm can handle non-sparse matrices well. These algorithms all cleverly use the HHL algorithm skills to solve the problem. Reference [40] designed a quantum circuit for multiple linear regression, and proposed to use the HHL algorithm to solve the multiple linear regression problem, because any multiple regression problem can be transformed into an equivalent linear equation problem or a quantum linear system problem. The result proves that the circuit of HHL algorithm can be used for multiple linear regression problem after fine-tuning and preprocessing. The table 3 gives a comparison of the complexity of each algorithm in the HHL-based quantum regression algorithm:

Table 3. Quantum regression algorithm based on HHL.

Algorithm	Quantum Complexity
[33]	$O(log(N)s^3\,\kappa^6/\epsilon)$
[35]	$\tilde{O}(n)$
[36]	$O(log(N+M)\,s^2\,\kappa^3/\epsilon^2)$
[39]	$O(ploylog(N+M)\,\kappa^5/\epsilon^4)$
[41]	$O(log(N)\kappa^2\epsilon^2)$

3.2. Quantum Classification Algorithms

Traditional machine learning can be mainly divided into three categories: supervised learning, unsupervised learning and reinforcement learning. Quantum computing facilitates the study of supervised classification and unsupervised clustering problems. Reference [42] provides supervised and unsupervised quantum machine learning algorithms. Classification problem is an important class in machine learning. In 2014, Rebentrost et al. [43] used the HHL algorithm to design a quantum support vector machine (QSVM), its core idea is to use the HHL algorithm to solve the inner product operation problem of the training data, which has an exponential acceleration effect compared to the classic SVM algorithm. The QSVM algorithm first encodes the eigenvectors into the quantum state by means of probability amplitude encoding:

$$|x_i\rangle = |x_i|^{-1} \cdot \sum_{j=1}^{m} x_{ij}\, |j\rangle \tag{3}$$

where *m* is the feature dimension, $|x_i|^{-1}$is the normalized vector and x_{ij} , $x_{ij}$is the *j*th feature of the *i*th feature vector. Next, prepare the quantum state of the training set:

$$|\chi\rangle = \left(\sqrt{N_\chi}\right)^{-1} \cdot \sum_{i=1}^{n} (|x_i|\, |i\rangle |x_i\rangle) \tag{4}$$

$N_\chi = \sum_{i=1}^{n} |{x_i}^2|$, x_i is the *i*th training sample. The inner product operation $K_{ij} = x_i \cdot x_j$ of the training data can be obtained by solving the partial trace of the density matrix$|\chi\rangle\langle\chi|$ to the normalized kernel matrix:

$$tr_2(|\chi\rangle\langle\chi|) = \frac{1}{N_\chi} \sum_{i,j=1}^{M} |x_i||x_j| \langle x_i|x_j\rangle |i\rangle|j\rangle = \frac{K}{trK} \tag{5}$$

$x_i \cdot x_j = |x_i||x_j|\langle x_i|x_j\rangle$, through this method, the quantum system is connected with the kernel matrix of traditional machine learning and the parallelism of the evolution operation between quantum states can be used to complete the acceleration of the corresponding kernel matrix calculation in traditional machine learning and then a quantum version is proposed. The least squares support vector machine can use the quantum HHL algorithm to realize the accelerated solution of the linear equation system in the least squares support vector machine algorithm. Reference [44] uses 4 quantum bits to realize the recognition of the most basic handwritten digits 6 and 9 on the nuclear magnetic platform and the accuracy of the result is as high as 99%, which shows the feasibility of the QSVM algorithm. In the same year, the literature [45] proposed the quantum principal component analysis algorithm (QPCA). When the rank of the data covariance matrix is very low, the algorithm can generate principal components in the form of quantum states in polynomial time. Compared with classical principal component analysis, it has the advantages of Exponential acceleration effect. The algorithm uses multiple copies of the unknown density matrix to construct the eigenvector corresponding to the largest eigenvalue, which can be applied to the discrimination and assignment of quantum states. Among many mathematical models of machine learning, there is a core module called singular value threshold (SVT), which is widely used to solve problems based on multi-kernel norm minimization. In order to speed up the processing speed of SVT, quantum singular value threshold (QSVT) Algorithms are proposed that can execute SVT operators at exponential speed. In the literature [46], it was discussed that the QSVT algorithm can be composed of two core subroutines based on the HHL algorithm, namely phase estimation and controlled rotation and its circuit structure is similar to that of the HHL algorithm. After that, Duan [47] designed the quantum the circuit provides the possibility to realize the algorithm on a quantum computer. The design of the control rotation part of the circuit has an inspirational effect on the circuit design of the HHL algorithm. In addition, there are many quantum machine learning algorithms based on HHL, including quantum recommendation systems [48] and quantum neural networks [49]. Most of these categories of algorithms use the HHL algorithm as a sub-module to process raw. The computationally complex part of the algorithm greatly promotes the acceleration of the algorithm. Table 4 shows the complexity comparison between quantum machine learning algorithms and traditional machine learning algorithms:

Table 4. Quantum machine learning algorithm complexity.

Algorithm	Quantum Complexity	Classical Complexity
QSVM	$O(log(MN))$	$O(MN)$
QSVT	$O(\log_2 MN)$	$O(poly(MN))$
QPCA	Secondary to accelerate	—
QRR	$O\big(poly(k)polylog(MN)\big)$	$O(MN)$

4. Summary

This paper mainly summarizes some improvements made by scholars for the HHL algorithm since it was proposed, which has improved the performance of the algorithm. However, when the HHL algorithm is applied, we still have to consider some limiting factors in the algorithm, as in the text What we propose, whether it is the preparation of quantum states or the simulation of matrices, these factors are unavoidable and must be dealt with in our use. How to ensure that these factors can be perfectly handled when using the HHL algorithm or use other Methods to replace some of these subroutines will be the focus of future work.

In addition, in view of the influence of the number of qubits in the HHL algorithm on the accuracy of the neutron program, such as the number of qubits in the phase estimation program, we mentioned the structure of the quantum-classical hybrid algorithm. The ideal data from the next step is used as the input to improve the accuracy of its results. In the future, our work can try to simulate the phase estimation subroutine on a high-performance computing platform to obtain higher-precision eigenvalue information, because the time and space required for quantum computing simulations on traditional classical computers are limited. The overhead increases exponentially with the number of simulated bits. Even if we obtain the result of the phase estimation in a classical way, the accuracy cannot be higher and the high-performance computing platform has become a powerful tool for quantum simulation. Its huge memory and data processing speed allow us to simulate algorithms with a larger number of qubits, so that the calculation results can achieve higher precision. Therefore, it is of great significance to establish a hybrid quantum-classical hybrid computing based on hybrid architecture. Reference [50] uses a supercomputer to simulate large-scale quantum Fourier transform and realizes the simulation and optimization of 46-qubit QFT algorithm. The author of Reference [51] uses high-performance heterogeneous cluster technology to achieve multi-bit quantum computing simulation. Therefore, simulation on a classical computer can not only provide a reliable verification platform for quantum algorithms and quantum circuits [52-54] but also help us understand the boundary between classical computing and quantum computing [55]. Hybrid computing models will play a bigger role in the future.

On the other hand, some quantum machine learning algorithms based on the HHL algorithm are based on the existing classical machine learning algorithms and replace the more complex parts with quantum computing for calculation, thereby improving their computational efficiency. In this process, we need to What is considered is not only how to use the HHL algorithm to solve some of them, but also how to effectively prepare the data in the classical algorithm into a quantum state, that is, the quantization of classical information. However, the time complexity of building QRAM in this process is still high, which affects the acceleration brought by quantum algorithms. Therefore, the technology for mapping classical data into quantum states still needs to be improved.

5. Conclusions

The improvement and application of HHL algorithm are reviewed in this paper. In terms of HHL algorithm improvement and optimization, the limitations of the algorithm, phase estimation subroutines, Hamiltonian simulation, controlled rotation and other improvements are introduced respectively. In terms of application, the application of HHL algorithm in machine learning is mainly introduced, which briefly describe the regression problem and the classification problem.

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