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New rotating regular black hole solution

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We briefly present new rotating regular black hole solution by converting the static, spherically symmetric Berej-Matyjasek-Tryniecki-Woronowicz spacetime which is ascociated with the general relativity coupled to the nonlinear electrodynamics by using the Newman-Janis algorithm.

I. INTRODUCTION

It is well known that exact solutions of the Einstein equations have one of the "mysterious" properties of the black hole that is called singularity. Singularity has been considered one of defects of the general relativity because the explanation of singularity cannot be made by can be created in order to eliminate singularity from the spacetime metric.

We know that there are three types of regular black hole solutions: (i) solutions that are continuous throughout spacetime; (ii) solutions with two simple regions, solutions that have boundary surfaces joining the two regions; and (iii) solutions with two separated regions, the solutions that have a surface layer, thin shell, joining the two regions.

There are two kinds of singularity: the coordinate singularity (event horizon) and the curvature singularity. We know that at the singularity the curvature of the manifold is becoming infinite. In the case of coordinate singularity, the g_{rr} component of the metric tensor goes to infinity. One can eliminate coordinate singularity by making transformations to the more fortunate coordinate system. Usually, by changing coordinates from the Boyer-Lindquist coordinates to the Eddington-Finkelstein ones, one can remove coordinate singularity from the spacetime metric. Eddington-Finkelstein coordinates are based on the freely falling photons. On the other hand, in the curvature singularity, the Riemann tensor components of the spacetime metric diverge. It is impossible to eliminate curvature singularity from the spacetime metric by coordinate transformations.

According to [1–4], the Kerr-like spacetime metrics can be derived from the Schwarzschild-like ones by using the Newman-Janis algorithm. The derivation of the Kerr spacetime metric from the Schwarzschild one has been given in several works [5–8]. Moreover, in the paper [8], the Kerr-Newman solution has been derived from the Reissner-Nordström spacetime metric. The Newman-Janis algorithm has been used to derive the radiating Kerr-Newman black hole in $f(R)$ gravity [9]. The exact nonstatic charged BTZ (Bañados-Teitelboim-Zanelli)-like solutions, in (N+1)-dimensional Einstein gravity, have been found in [10] in the presence of the negative cosmological constant. The Lovelock gravity in the critical spacetime dimension has been studied in Ref. [11].

In order to convert the static, spherically symmetric black hole spacetime metric into a rotational one [if this spacetime metric is given in the Boyer-Lindquist coordinates $(t,\ r,\ \theta,\ \phi)$] one has to proceed with the following five steps of the Newman-Janis algorithm: (i) a transition from the Boyer-Lindquist coordinates into the advanced Eddington-Filkenstein ones $(u,\ r,\ \theta,\ \phi)$ has to be performed; (ii) a null tetrad ($\mathbf{l}$, $\mathbf{n}$, $\mathbf{m}$, and $\bar{\mathbf{m}}$) (Newman-Penrose tetrad) for a produced metric have to be found; (iii) a complex coordinate transformations has to be applied; (iv) reverse coordinate transformations into the Boyer-Lindquist ones have to be done; and, (v) finally, unknown terms of the transformations have to be found based on the reality condition.

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Here, we convert the static, spherically symmetric Berej-Matyjasek-Tryniecki-Woronowicz (BMTW) regular black hole spacetime [12, 13] into the rotational one by using the Newman-Janis algorithm [1–3] and by studying some of its basic properties.

II. NEW ROTATING REGULAR BLACK HOLE SOLUTION

In this section, we describe the Newman-Janis algorithm that is used for converting the spherically symmetric static black hole spacetime metric into a rotational one. The Berej-Matyjasek-Trynieki-Woronowicz (BMTW) spacetime metric of the regular spherically symmetric black hole is given as [12]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{1}$$

where the lapse function $f(r)$ reads

$$f(r) = 1 - \frac{2M}{r}[1 - \tanh(\frac{Q^2}{2Mr})] \ , \tag{2}$$

M and Q are the total mass and charge of the black hole.

As can be seen from the lapse function (2), the spacetime metric (1) has only the coordinate singularity. This is why in order to remove this singularity one has to write the spacetime metric (1) in the advanced Eddington-Finkelstein coordinates. To do this, we make the following transformation for the incoming photon (or ray):

$$v = t - r^*, \tag{3}$$

and for the outgoing photon (or ray),

$$u = t + r^*, \tag{4}$$

where

$$r^* = \int \frac{dr}{f(r)}. \tag{5}$$

Hereafter, we consider only the outgoing photon (4) case. Then the spacetime metric (1) in the advanced Eddington-Finkelstein coordinates takes the form

$$ds^2 = -f(r)du^2 - 2dudr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{6}$$

The Newman-Penrose tetrad consists of four isotropic vectors $\mathbf{l}$, $\mathbf{n}$, $\mathbf{m}$, and $\bar{\mathbf{m}}$. $\mathbf{l}$ and $\mathbf{n}$ are real vectors, and $\mathbf{m}$ and $\bar{\mathbf{m}}$ are mutually complex conjugate vectors [8].

Newman-Penrose tetrads satisfy the orthogonality condition:

$$l^\mu \cdot m_\mu = l^\mu \cdot \bar{m}_\mu = n^\mu \cdot m_\mu = n^\mu \cdot \bar{m}_\mu = 0, \tag{7}$$

and also the isotropic condition:

$$l^\mu \cdot l_\mu = n^\mu \cdot n_\mu = m^\mu \cdot m_\mu = \bar{m}^\mu \cdot \bar{m}_\mu = 0. \tag{8}$$

Moreover, the basis vectors usually impose the following normalization condition:

$$l^\mu \cdot n_\mu = 1, \ \ m^\mu \cdot \bar{m}_\mu = -1, \tag{9}$$

where $\bar{m}^\mu$ is the complex conjugate of m^μ.

The contravariant components of the metric tensor of the spacetime metric (6) are

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & f(r) & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}. \tag{10}$$

We can rewrite (10) with the help of the Newman-Penrose tetrad as

$$g^{\mu\nu} = -l^{\mu} \cdot n^{\nu} - l^{\nu} \cdot n^{\mu} + m^{\mu} \cdot \bar{m}^{\nu} + m^{\nu} \cdot \bar{m}^{\mu}, \tag{11}$$

where the components of the null tetrad vectors are

$$\begin{aligned} l^{\mu} &= [0,1,0,0], n^{\mu} = [1, -\frac{1}{2}f(r), 0, 0], \\ m^{\mu} &= \frac{1}{\sqrt{2}r}[0,0,1,\frac{i}{\sin\theta}], \bar{m}^{\mu} = \frac{1}{\sqrt{2}r}[0,0,1,-\frac{i}{\sin\theta}]. \end{aligned} \tag{12}$$

As the next step, we make the following complex coordinate transformations:

$$\begin{aligned} \tilde{r} = r + ia\cos\theta, &\quad \tilde{u} = u - ia\cos\theta, \\ \tilde{\theta} = \theta, &\quad \tilde{\phi} = \phi. \end{aligned} \tag{13}$$

As a result of these transformations, the components of the null tetrad vectors take the form [5]

$$\begin{aligned} \tilde{l}^{\mu} &= [0,1,0,0], \tilde{n}^{\mu} = [1, -\frac{1}{2}\tilde{f}(r), 0, 0], \\ \tilde{m}^{\mu} &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}[ia\sin\theta, -ia\sin\theta, 1, \frac{i}{\sin\theta}], \\ \tilde{\bar{m}}^{\mu} &= \frac{1}{\sqrt{2}(r - ia\cos\theta)}[-ia\sin\theta, ia\sin\theta, 1, -\frac{i}{\sin\theta}], \end{aligned} \tag{14}$$

where the function

$$\tilde{f}(r) = 1 - \frac{2Mr}{\Sigma}[1 - tanh(\frac{Q^2 r}{2M\Sigma})] \tag{15}$$

is the new form of the lapse function (13) and
$\Sigma = r^2 + a^2\cos^2\theta$.

Then the metric tensor $g^{\mu\nu}$ takes new $\tilde{g}^{\mu\nu}$ form,

$$\tilde{g}^{\mu\nu} = -\tilde{l}^{\mu} \cdot \tilde{n}^{\nu} - \tilde{l}^{\nu} \cdot \tilde{n}^{\mu} + \tilde{m}^{\mu} \cdot \tilde{\bar{m}}^{\nu} + \tilde{m}^{\nu} \cdot \tilde{\bar{m}}^{\mu}, \tag{16}$$

or

$$\tilde{g}^{\mu\nu} = \begin{pmatrix} \frac{a^2\sin^2\theta}{\Sigma} & -1 - \frac{a^2\sin^2\theta}{\Sigma} & 0 & \frac{a}{\Sigma} \\ -1 - \frac{a^2\sin^2\theta}{\Sigma} & \tilde{f}(r) + \frac{a^2\sin^2\theta}{\Sigma} & 0 & -\frac{a}{\Sigma} \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ \frac{a}{\Sigma} & -\frac{a}{\Sigma} & 0 & \frac{1}{\Sigma\sin^2\theta} \end{pmatrix}. \tag{17}$$

The covariant components of the metric tensor (17) are

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -\tilde{f}(r) & -1 & 0 & a(\tilde{f}(r) - 1)\sin^2\theta \\ -1 & 0 & 0 & a\sin^2\theta \\ 0 & 0 & \Sigma & 0 \\ a(\tilde{f}(r) - 1)\sin^2\theta & a\sin^2\theta & 0 & \sin^2\theta[\Sigma - a^2(\tilde{f}(r) - 2)\sin^2\theta] \end{pmatrix}, \tag{18}$$

and the spacetime element can be written as

$$d\tilde{s}^2 = g_{uu}du^2 + 2g_{ur}dudr + 2g_{u\phi}dud\phi + 2g_{r\phi}drd\phi + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \tag{19}$$

At the last step of the Newman-Janis algorithm one has to perform the transition from the Eddington-Finkelstein coordinates to the Boyer-Lindquist one. To do this we had chosen (20) coordinate transformations:

$$\begin{aligned} du &= dt + \lambda(r)dr + \xi(\theta)d\theta, \\ d\phi &= d\phi + \chi(r)dr + \zeta(\theta)d\theta. \end{aligned} \tag{20}$$

By putting (20) into (19) we have the spacetime metric with several nondiagonal components. In reality the spacetime metric of the rotational black hole has only g_{03} (and g_{30}) nondiagonal component of the metric tensor. Based on the this condition we equalize the all nondiagonal components except from g_{03} to zero

$$\begin{aligned}
&g_{u\phi}\zeta(\theta) + g_{uu}\xi(\theta) = 0, \\
&g_{u\phi}\chi(r) + g_{uu}\lambda(r) + g_{ur} = 0, \\
&g_{r\phi} + g_{\phi\phi}\chi(r) + g_{u\phi}\lambda(r) = 0, \\
&\zeta(\theta)[g_{r\phi} + g_{\phi\phi}\chi(r) + g_{u\phi}\lambda(r)] + \xi(\theta)[g_{ur} + g_{u\phi}\chi(r) + g_{uu}\lambda(r)] = 0, \\
&g_{\phi\phi}\zeta(\theta) + g_{u\phi}\xi(\theta) = 0.
\end{aligned} \tag{21}$$

By solving equations (21) with respect to unknown transformation functions $\lambda(r)$, $\chi(r)$, $\xi(\theta)$ and $\zeta(\theta)$ and using (18) we will have the expressions of the transformation functions as following:

$$\begin{aligned}
\lambda(r) &= -\frac{\Sigma + a^2\sin^2\theta}{\Sigma\tilde{f}(r) + a^2\sin^2\theta} = -\frac{r^2 + a^2}{\Delta_r + 2Mr\tanh(\frac{Q^2r}{2M\Sigma})}, \\
\chi(r) &= -\frac{a}{\Sigma\tilde{f}(r) + a^2\sin^2\theta} = -\frac{a}{\Delta_r + 2Mr\tanh(\frac{Q^2r}{2M\Sigma})}, \\
\xi(\theta) &= 0, \\
\zeta(\theta) &= 0.
\end{aligned} \tag{22}$$

where $\Delta_r = r^2 + a^2 - 2Mr$, $\Sigma = r^2 + a^2\cos^2\theta$.

Finally, the spacetime metric (19) can be expressed in the Boyer-Lindquist coordinates as

$$d\tilde{s}^2 = -[1 - \frac{2Mr}{\Sigma}(1 - \tanh(\frac{Q^2r}{2M\Sigma}))]dt^2 + \frac{\Sigma}{r^2 + a^2 - 2Mr(1 - \tanh(\frac{Q^2r}{2M\Sigma}))}dr^2 \tag{23}$$

$$-2\frac{2Mr}{\Sigma}a\sin^2\theta(1 - \tanh(\frac{Q^2r}{2M\Sigma}))d\phi dt + \Sigma d\theta^2 + [r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}(1 - \tanh(\frac{Q^2r}{2M\Sigma}))]\sin^2\theta d\phi^2,$$

or

$$\begin{aligned}
d\tilde{s}^2 &= -\frac{\Delta_r + 2Mr\tanh(\frac{Q^2r}{2M\Sigma})}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\Sigma}[(r^2 + a^2)d\phi - adt]^2 \\
&+ \frac{\Sigma}{\Delta_r + 2Mr\tanh(\frac{Q^2r}{2M\Sigma})}dr^2 + \Sigma d\theta^2,
\end{aligned} \tag{24}$$

where $\tilde{f}(r)$ reads as

$$\tilde{f}(r) = 1 - \frac{2Mr}{\Sigma}[1 - \tanh(\frac{Q^2r}{2M\Sigma})] \tag{25}$$

If we do not take into account the charge of the black hole ($Q = 0$), the lapse function (2) takes the same form with one of the Schwarzschild spacetime metric and new spacetime metric (22) and (24) converts into the Kerr one, namely,

$$d\tilde{s}^2 = -\frac{\Delta_r}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\sin^2\theta}{\Sigma}[(r^2 + a^2)d\phi - adt]^2 + \Sigma d\theta^2. \tag{26}$$

The event horizon of the new rotating black hole is determined by solving the following equation:

$$\Delta_r + 2Mr\tanh(\frac{Q^2r}{2M\Sigma}) = 0\ , \tag{27}$$

while the timelike static limit, so-called apparent horizon of the black hole is found by solving the following equation:

$$\Sigma - 2Mr(1 - \tanh(\frac{Q^2r}{2M\Sigma})) = 0\ . \tag{28}$$

Since both equations (27) and (28) cannot be solved analytically, we present the solution of these equations in Fig. 1 for different values of charge and rotation parameters of the black hole. Where the outer lines represent the static limit radius, while the inner curves correspond to the event horizon of the black hole. One can see from Fig. 1 that with increasing the value of the charge (rotation) parameter, radii of the event horizon and static limit decrease, but the area of the ergoregion (the region between event horizon and static limit) increases. Further the properties of the current new black hole solution will be analyzed in our future works.

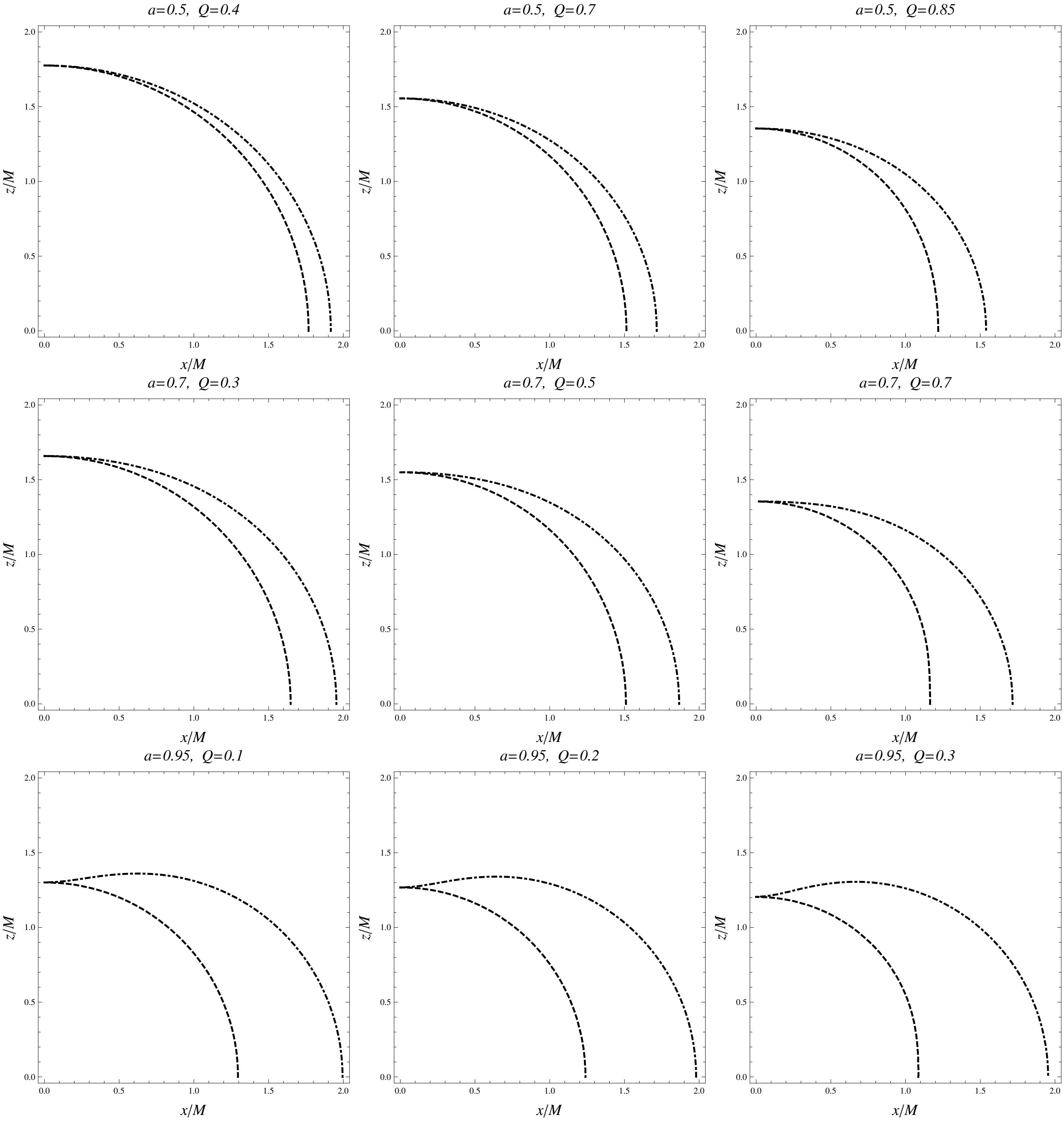

FIG. 1: Shape and size of the ergosphere for the different values of the rotation parameter a and charge Q. Dashed and dot-dashed lines represent event horizon and static limit.

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