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CONSTRUCTION OF LINEAR CODES WITH VARIOUS HERMITIAN HULL DIMENSIONS AND RELATED EAQECCS

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Abstract. The hull of a linear code is the intersection of the code and its dual code, which is effective for determining parameters of entanglement-assisted quantum error-correcting codes (EAQECCs). There are few constructions of linear codes with various Hermitian hull dimensions, aside from Hermitian LCD and self-orthogonal codes. The object of this paper is to introduce a building-up construction for constructing linear $[n+2, k+1]$ codes with ℓ or $(\ell+1)$-dimensional Hermitian hull from a given linear $[n, k]$ code with ℓ-dimension Hermitian hull of a smaller length. This construction includes the converse of the famous shortening technique as a special case. Using this method, we construct optimal quaternary linear codes of lengths up to 13 with Hermitian hull dimensions 2-5. As an application, we construct many EAQECCs, which improve the parameters of EAQECCs of Grassl's code table.

1. **Introduction.** In 1990, Assmus and Key [1] introduced the concept of the hull, which is helpful for classifying finite projective planes. The hull determined the complexity of certain algorithms, such as computing the automorphism group of a linear code and checking permutation equivalence of two linear codes [28, 39, 40]. Furthermore, they also have applications in quantum communication [17]. Therefore, characterizing and classifying optimal linear codes with various hulls is interesting and necessary.

Linear complementary dual (LCD) codes are linear codes with the smallest hull. Massey [37] introduced LCD codes in order to give a solution in information theory. LCD codes drew much attention recently due to their use in cryptography [11]. Further, they are helpful in constructing maximal-entanglement EAQECCs [27, 34]. LCD codes were extensively studied due to these applications. A non-exhaustive list is [12, 15, 18, 21, 31, 43, 45].

On the other hand, a code C is self-orthogonal if the hull of C is the code C. Since the beginning of coding theory, studying self-orthogonal codes has been an active research problem [38]. They are used in many fields, such as t-design theory, group theory, lattice theory and modular forms [2, 3, 17, 19, 41]. Besides interesting combinatorial structures, they were also used to construct quantum error-correcting

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codes (QECCs) [9, 10, 32, 33], where QECCs have important applications in entanglement distillation protocols [4]. For more research on self-orthogonal codes, one can refer to [23, 24, 26, 42, 44].

However, self-orthogonal codes are usually not readily available. To solve this problem, Brun, Devetak and Hsieh [8] proposed the concept of EAQECCs, which include QECCs as a special case and can be obtained from linear codes without self-orthogonality. Recently, there has been increasing interest in exploring linear codes with different hull dimensions. Particularly, Chen [13, 14] and Luo [35] studied the variation of the hull of linear codes up to monomial equivalence and constructed EAQECCs from known QECCs. Their results indicated that it is non-trivial to study binary linear codes with various Euclidean hulls and quaternary linear codes with various Hermitian hulls. Kim [22] proposed a method to construct binary linear codes with various Euclidean hulls. Therefore, the study of quaternary linear codes is open.

In fact, the Hermitian inner product has greater potential than the Euclidean inner product in constructing EAQECCs [17]. There are few constructions of linear codes with various Hermitian hull dimensions except for Hermitian LCD and self-orthogonal codes [25]. Therefore, a topic is to find an efficient method for constructing linear codes with different Hermitian hull dimensions. In this work, a systematic method is introduced to construct linear codes with various Hermitian hull dimensions. As a result, many optimal linear codes are obtained. As an application, we construct many binary EAQECCs, which improve the parameters of EAQECCs of Grassl's code table [16].

The paper is arranged as follows. Section 2 gives some preliminaries. Section 3 gives a building-up construction for linear codes with various Hermitian hull dimensions. Section 4 gives some numerical examples. As an application, some optimal or new EAQECCs are constructed. Section 5 concludes the paper.

2. Preliminaries.

2.1. Linear codes.

Let $\mathbb{F}_q$ denote the finite field with q elements. The *(Hamming) weight* $\mathrm{wt}(\mathbf{x})$ of $\mathbf{x} \in \mathbb{F}_q^n$ is defined by the number of nonzero components of $\mathbf{x}$. A *linear* $[n,k,d]_q$ *code* C is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum distance d, where d is the minimum weight of all nonzero codewords of C. For any $k \times n$ matrix G, we call G a *generator matrix* of a linear code C if rows of G form a basis of C. We define the *Hermitian dual code* of a linear $[n,k]_{q^2}$ code C as $C^{\perp_{\mathrm{H}}} = \{\mathbf{y} \in \mathbb{F}_{q^2}^n \mid \langle \mathbf{x},\mathbf{y}\rangle_{\mathrm{H}} = 0 \text{ for all } \mathbf{x} \in C\}$, $\langle \mathbf{x},\mathbf{y}\rangle_{\mathrm{H}} = \sum_{i=1}^n x_i\overline{y_i} = \sum_{i=1}^n x_i y_i^q$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{F}_{q^2}^n$ If H is a generator matrix of the dual code $C^{\perp_{\mathrm{H}}}$, then we call H a *parity-check matrix* of C. The *Hermitian hull* of a linear $[n,k]_{q^2}$ code C is defined by $\mathrm{Hull}_{\mathrm{H}}(C) = C \cap C^{\perp_{\mathrm{H}}}$. A linear $[n,k]_{q^2}^{\ell}$ code is a linear $[n,k]_{q^2}$ code with ℓ-dimensional Hermitian hull. A linear code C is called Hermitian *self-orthogonal* if $C \subseteq C^{\perp_{\mathrm{H}}}$, and Hermitian *linear complementary dual* (LCD) if $C \cap C^{\perp_{\mathrm{H}}} = \{\mathbf{0}\}$.

Definition 2.1. Suppose that n, k and ℓ are three integers. Let

$$D_{q^2}(n,k) := \max\{d \mid \text{there is a linear } [n,k,d]_{q^2} \text{ code}\},$$
$$D_{q^2}^H(n,k,\ell) := \max\{d \mid \text{there is a linear } [n,k,d]_{q^2}^{\ell} \text{ code}\}.$$

A linear $[n,k]_{q^2}^{\ell}$ code C is called *h_ℓ-optimal* if C has the minimum distance $D_{q^2}^H(n,k,\ell)$.

It is well-known that the *Griesmer bound* [20] on a linear $[n,k,d]_q$ code is given by $n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil$, where $\lceil a \rceil$ is the least integer greater than or equal to a real number a. It is clear that $D_{q^2}^H(n,k,\ell) \leq D_{q^2}(n,k)$. Hence their minimum distances are bounded by the Griesmer bound for any ℓ.

2.2. **Characterization of Hermitian hull.** For a matrix $A = (a_{ij})$, we define the transpose A^T and the conjugate matrix $\overline{A}$ of A as (a_{ji}) and $(\overline{a_{ij}})$, respectively.

Proposition 2.2. [17, Proposition 3.2] *Let C be a linear $[n,k]_{q^2}$ code with a generator matrix G. Then* $\dim(\mathrm{Hull}_{\mathrm{H}}(C)) = \dim(\mathrm{Hull}_{\mathrm{H}}(C^{\perp_{\mathrm{H}}})) = k - \mathrm{rank}(G\overline{G}^T)$.

The puncturing and shortening techniques are two extremely vital means of constructing new codes from old ones. For simplicity, we use the $[n]$ for the set $\{1,2,\ldots,n\}$, where n is a given positive integer. Let $S \subset [n]$ and C be a linear $[n,k,d]_{q^2}$ code. We delete the coordinates from S in each codeword in C to get the *punctured code* C^S. Consider the set $C(S)$ of codewords which are 0 on S, and *shortened code* C_S can be obtained through puncturing $C(S)$ on S.

Lemma 2.3. [36, Lemma 2] *Assume that C is a linear $[n,k,d]_{q^2}$ code. Let $S \subset [n]$ be such that $|S| = s$. Then*

(1) $(C^{\perp_{\mathrm{H}}})_S = (C^S)^{\perp_{\mathrm{H}}}$, $(C^{\perp_{\mathrm{H}}})^S = (C_S)^{\perp_{\mathrm{H}}}$.
(2) *If $s < d$, then* $\dim(C^S) = k$ *and* $\dim((C^{\perp_{\mathrm{H}}})_S) = n - s - k$.

Theorem 2.4. [36, Theorem 4] *Assume that C is a linear $[n,k,d]_{q^2}^{\ell}$ code. Let $S \subseteq [n]$ be such that $|S| = s$. Then we have the following statements.*

(1) $(\mathrm{Hull}_{\mathrm{H}}(C))_S \subseteq \mathrm{Hull}_{\mathrm{H}}(C^S)$ *and* $(\mathrm{Hull}_{\mathrm{H}}(C))_S \subseteq \mathrm{Hull}_{\mathrm{H}}(C_S)$
(2) *If S is a subset of information set of* $\mathrm{Hull}_{\mathrm{H}}(C)$, *then*

$$\mathrm{Hull}_{\mathrm{H}}(C^S) = \mathrm{Hull}_{\mathrm{H}}(C_S) = (\mathrm{Hull}_{\mathrm{H}}(C))_S \text{ and } \dim(\mathrm{Hull}_{\mathrm{H}}(C^S)) = \ell - s.$$

Proposition 2.5. *Suppose that $0 \leq s \leq \ell \leq k \leq n-1$. Then*

(1) $D_{q^2}^H(n,k,\ell) \leq D_{q^2}^H(n-s,k-s,\ell-s)$.
(2) $D_{q^2}^H(n,k,\ell) \leq D_{q^2}^H(n-s,k,\ell-s) + s$ *if* $D_{q^2}^H(n,k,\ell) \geq s+1$.

Proof. Let C be a linear $[n,k,d]_{q^2}^{\ell}$ code.

(1) By Theorem 2.4, there is a set S such that the shortened code C_S has parameters $[n-s,k-s]_{q^2}^{\ell-s}$. It turns out that $d(C_S) \geq d$ by the definition of the shortened code. Hence $D_{q^2}^H(n,k,\ell) \leq D_{q^2}^H(n-s,k-s,\ell-s)$.

(2) By Theorem 2.4 and Lemma 2.3, there is a set S such that the punctured code C^S has parameters $[n-s,k]_{q^2}^{\ell-s}$ if $D_{q^2}^H(n,k,\ell) \geq s+1$. It turns out that $d(C^S) \geq d-s$ by the definition of the punctured code. Hence $D_{q^2}^H(n,k,\ell) \leq D_{q^2}^H(n-s,k,\ell-s)+s$. □

3. Building-up construction for linear codes with various Hermitian hull dimensions.

In this section, we propose a method to obtain linear $[n+2,k+1]_{q^2}^{\ell}$ and $[n+2,k+1]_{q^2}^{\ell+1}$ codes from a given linear $[n,k]_{q^2}^{\ell}$ code.

3.1. **Construction I.** First, we extend the building-up construction method [25] for Hermitian self-dual codes to linear codes with prescribed Hermitian hull dimension.

Theorem 3.1. *Let C be a linear $[n,k]_{q^2}^{\ell}$ code with generator matrix G and parity-check matrix H. Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{F}_{q^2}^n$ with $\langle\mathbf{x},\mathbf{x}\rangle_{\mathrm{H}}=-1$ and $c\in\mathbb{F}_{q^2}$ with $c\overline{c}=-1$. Suppose that $\overline{y_i}=\langle\mathbf{x},\mathbf{r}_i\rangle_{\mathrm{H}}$ and $\overline{z_i}=\langle\mathbf{x},\mathbf{s}_i\rangle_{\mathrm{H}}$, where $\mathbf{r}_i$ and $\mathbf{s}_i$ are the i-th rows of G and H, respectively. Then*

$$G_1=\left[\begin{array}{cc|c} 1 & 0 & x_1\ x_2\ \cdots\ x_n \\ \hline -y_1 & cy_1 & \mathbf{r}_1 \\ -y_2 & cy_2 & \mathbf{r}_2 \\ \vdots & \vdots & \vdots \\ -y_k & cy_k & \mathbf{r}_k \end{array}\right].$$

generates a linear $[n+2,k+1]_{q^2}^{\ell+1}$ code C_1. We call it **Construction I**. *Moreover, C_1 has the following parity-check matrix*

$$H_1=\left[\begin{array}{cc|c} 1 & 0 & x_1\ x_2\ \cdots\ x_n \\ \hline -z_1 & cz_1 & \mathbf{s}_1 \\ -z_2 & cz_2 & \mathbf{s}_2 \\ \vdots & \vdots & \vdots \\ -z_{n-k} & cz_{n-k} & \mathbf{s}_{n-k} \end{array}\right].$$

Proof. Let $g_{i,j}$ denote the element in the i-th row and j-th column of G_1. If the first row of G_1 can be obtained by a linear combination of other rows, then $g_{1,2}=-cg_{1,1}\neq 0$, which contradicts the fact that $g_{1,2}=0$. Therefore, $\mathrm{rank}(G_1)=k+1$. Similarly, $\mathrm{rank}(H_1)=n-k+1$. It can be checked that

$$G_1\overline{G_1}^T=\begin{bmatrix} 0 & 0\ 0\ \cdots\ 0 \\ 0 & \\ \vdots & G\overline{G}^T \\ 0 & \end{bmatrix} \text{ and } G_1\overline{H_1}^T=\begin{bmatrix} 0 & 0\ 0\ \cdots\ 0 \\ 0 & \\ \vdots & G\overline{H}^T \\ 0 & \end{bmatrix}=O,$$

where O is the suitable zero matrix. According to Proposition 2.2, we have

$$\begin{aligned}\dim(\mathrm{Hull}_{\mathrm{H}}(C_1)) &= (k+1)-\mathrm{rank}(G_1\overline{G_1}^T)\\ &= (k+1)-\mathrm{rank}(G\overline{G}^T)\\ &= (k+1)-(k-\dim(\mathrm{Hull}_{\mathrm{H}}(C)))\\ &= (k+1)-(k-\ell)\\ &= \ell+1.\end{aligned}$$

Further, $\mathrm{rank}(H_1)=n-k+1=\dim(C_1^{\perp_{\mathrm{H}}})$. Hence C_1 has a parity-check matrix H_1. □

Example 1. Let $\mathbb{F}_4^*=\langle\omega\rangle$ and $c=\omega$. We start from a linear $[7,4,3]_4^1$ code. By applying Construction I, we construct a linear $[9,5,4]_4^2$ code C and the generator matrix

$$G = \left[\begin{array}{cc|ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 & \omega & 1 \\ \hline \omega^2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \omega^2 & 1 & 0 & 1 & 0 & 0 & \omega & \omega & 1 \\ 1 & \omega & 0 & 0 & 1 & 0 & \omega & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

Moreover, C has the following parity-check matrix

$$H = \left[\begin{array}{cc|ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 & \omega & 1 \\ \hline \omega^2 & 1 & 1 & 0 & 0 & 1 & \omega^2 & 0 & \omega \\ \omega^2 & 1 & 0 & 1 & 0 & \omega^2 & \omega & \omega^2 & \omega \\ 1 & \omega & 0 & 0 & 1 & \omega^2 & 1 & \omega^2 & 1 \end{array}\right].$$

Furthermore, the converse of Construction I has also been verified to be true, as shown below.

Theorem 3.2. *Suppose that $\ell \geq 2$. Then any linear $[n,k,d>2]_{q^2}^{\ell}$ code C can be constructed from a linear $[n-2,k-1]_{q^2}^{\ell-1}$ code C_0 using Construction I.*

Proof. Without loss of generality, we can assume that $[I_\ell \; A]$ is a generator matrix of $\mathrm{Hull}_{\mathrm{H}}(C)$. Then we can define a $k \times n$ generator matrix G of C as follows

$$G = \begin{bmatrix} I_\ell & A \\ O & B \end{bmatrix} = \begin{bmatrix} 1 \quad 0 & 0\cdots 0 & \mathbf{a}_1 \\ 0 \quad 1 & 0\cdots 0 & \mathbf{a}_2 \\ \multirow{2}{*}{O} & I_{\ell-2} & \widetilde{A} \\ & O & B \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 \quad -c & 0\cdots 0 & \mathbf{a}_1 - c\mathbf{a}_2 \\ 0 \quad -c & 0\cdots 0 & -c\mathbf{a}_2 \\ \multirow{2}{*}{O} & I_{\ell-2} & \widetilde{A} \\ & O & B \end{bmatrix} := G',$$

where O is the suitable zero matrix. Consider the code C_0 with the following generator matrix

$$G_0 = \begin{bmatrix} 0\cdots 0 & \mathbf{a}_1 - c\mathbf{a}_2 \\ I_{\ell-2} & \widetilde{A} \\ O & B \end{bmatrix}.$$

Obviously, C_0 has parameters $[n-2,k-1]_{q^2}$. Since the set $\{1,2\}$ is a subset of an information set of $\mathrm{Hull}_{\mathrm{H}}(C)$, it follows from (2) of Theorem 2.4 that the matrix $\begin{bmatrix} I_{\ell-2} & \widetilde{A} \\ O & B \end{bmatrix}$ generates a linear $[n-2,k-2]_{q^2}^{\ell-2}$ code C_2. Let $\mathbf{y} = (0,\cdots,0,\mathbf{a}_1 - c\mathbf{a}_2)$. Further, $\langle \mathbf{y},\mathbf{y}\rangle_{\mathrm{H}} = 0$ and $\mathbf{y} \in C_2^{\perp_{\mathrm{H}}}$. Hence C_0 is a linear $[n-2,k-1]_{q^2}^{\ell-1}$ code.

Assume that $\mathbf{x} = (0,\cdots,0,\mathbf{a}_1)$. It can be checked that $\langle \mathbf{x},\mathbf{x}\rangle_{\mathrm{H}} = -1$. Using the vector $\mathbf{x}$ and C_0, we can construct a linear $[n,k]_{q^2}$ code C_1 with the following matrix G_1 by theorem 3.1, where

$$G_1 = \left[\begin{array}{cc|cc} 1 & 0 & 0\cdots 0 & \mathbf{a}_1 \\ \hline 1 & -c & 0\cdots 0 & \mathbf{a}_1 - c\mathbf{a}_2 \\ \multicolumn{2}{c|}{\multirow{2}{*}{O}} & I_{\ell-2} & \widetilde{A} \\ & & O & B \end{array}\right],$$

and G_1 is row equivalent to G. Thus, C is equivalent to C_1. This completes the proof. $\square$

Corollary 3.3. *Let C be a linear $[n,k]_{q^2}^{\ell}$ code with generator matrix G and parity-check matrix H. Let $\mathbf{r}_i$ be the i-th row of G. If $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in C^{\perp_{\mathrm{H}}}$ and $\langle \mathbf{x},\mathbf{x}\rangle_{\mathrm{H}}=-1$. For $1\le j\le n-k$, suppose that $\overline{z_j}=\langle \mathbf{x},\mathbf{s}_j\rangle_{\mathrm{H}}$ where $\mathbf{s}_j$ is the j-th row of H. Then*

$$G_1'=\left[\begin{array}{c|c} 1 & x_1\ x_2\ \cdots\ x_n \\ \hline 0 & \mathbf{r}_1 \\ 0 & \mathbf{r}_2 \\ \vdots & \vdots \\ 0 & \mathbf{r}_k \end{array}\right]$$

generates a linear $[n+1,k+1]_{q^2}^{\ell+1}$ code C_1'. We call it **Construction I$'$**. *Moreover, C_1' has the following parity-check matrix*

$$H_1'=\left[\begin{array}{c|c} -z_1 & \mathbf{s}_1 \\ -z_2 & \mathbf{s}_2 \\ \vdots & \vdots \\ -z_{n-k} & \mathbf{s}_{n-k} \end{array}\right].$$

Proof. By Theorem 3.1, one can construct a linear $[n+2,k+1]_{q^2}^{\ell+1}$ code C_1. Since $\mathbf{x}\in C^{\perp_{\mathrm{H}}}$, $y_i=0$ for $1\le i\le k$. Thus, by puncturing C_1 on the second coordinate, we know that the following matrix

$$\left[\begin{array}{c|c} 1 & x_1\ x_2\ \cdots\ x_n \\ \hline 0 & \mathbf{r}_1 \\ 0 & \mathbf{r}_2 \\ \vdots & \vdots \\ 0 & \mathbf{r}_k \end{array}\right]$$

generates a linear $[n+1,k+1]_{q^2}^{\ell+1}$ code. The remaining proof is similar to that of Theorem 3.1, we omit it. □

Example 2. Let $\mathbb{F}_4^*=\langle\omega\rangle$. We start from a linear $[9,3,6]_4^3$ code. By applying Construction I$'$, one constructs a linear $[10,4,4]_4^4$ code C with the generator matrix

$$G=\left[\begin{array}{c|ccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 & \omega^2 & \omega^2 & 1 & \omega^2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & \omega & \omega & \omega^2 & \omega^2 \end{array}\right].$$

Moreover, C has the following parity-check matrix

$$H=\left[\begin{array}{c|ccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \omega^2 & 0 & 1 & 0 & 0 & 0 & 0 & \omega & \omega^2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & \omega^2 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \omega^2 & 0 & 0 & 0 & 0 & 1 & 0 & \omega & 1 & \omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \omega^2 & \omega^2 \end{array}\right].$$

Furthermore, the converse of Construction I$'$ has also been verified to be true, as shown below.

Theorem 3.4. *Suppose that $\ell\ge 1$. Then any linear $[n,k]_{q^2}^{\ell}$ code C can be obtained from a linear $[n-1,k-1]_{q^2}^{\ell-1}$ code C_0 using Construction I$'$.*

Proof. We can assume that $\mathrm{Hull}_{\mathrm{H}}(C)$ has a generator matrix $[I_\ell \;\; A]$.Then we can define a $k \times n$ generator matrix G for C in the form of

$$G = \begin{bmatrix} I_\ell & A \\ O & B \end{bmatrix},$$

where O is a suitable zero matrix. Consider $S = \{i\} \subset [\ell]$. Then C_S with a generator matrix

$$G_S = \begin{bmatrix} I_{\ell-1} & A_i \\ O & B \end{bmatrix},$$

where A_i is a matrix that removes the i-th row from matrix A, is a linear $[n-1, k-1]_{q^2}^{\ell-1}$ code by (2) of Theorem 2.4.

Assume that $\mathbf{x} = (0, \dots, 0, \mathbf{a}_i)$, where $\mathbf{a}_i$ is the i-th row of A. Using the vector $\mathbf{x}$ and C_0, we can construct a linear $[n, k]_{q^2}$ code C_1 with the following matrix G_1 by Theorem 3.1, where

$$G_1 = \left[\begin{array}{c|cc} 1 & 0\cdots 0 & \mathbf{a}_i \\ \hline 0 & & \\ \vdots & I_{\ell-1} & A_i \\ & O & B \\ 0 & & \end{array}\right] \sim \begin{bmatrix} I_\ell & A \\ O & B \end{bmatrix} = G.$$

This completes the proof. □

3.2. **Construction II.** Recently, Harada [18] proposed an interesting construction method for quaternary Hermitian LCD codes. By modifying Harada's construction, we give construction II as follows.

Theorem 3.5. *Let C be a linear $[n, k]_{q^2}^{\ell}$ code with generator matrix G and parity-check matrix H. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{F}_{q^2}^n$ with $\langle \mathbf{x}, \mathbf{x}\rangle_{\mathrm{H}} = 0$ and $c \in \mathbb{F}_{q^2}$ with $c\overline{c} = -1$. Suppose that $\overline{y_i} = \langle \mathbf{x}, \mathbf{r}_i\rangle_{\mathrm{H}}$ and $\overline{z_i} = \langle \mathbf{x}, \mathbf{s}_i\rangle_{\mathrm{H}}$ where $\mathbf{r}_i$ and $\mathbf{s}_i$ are the i-th rows of G and H, respectively. Then*

$$G_2 = \left[\begin{array}{cc|c} 1 & 0 & x_1 \; x_2 \; \cdots \; x_n \\ \hline -y_1 & cy_1 & \mathbf{r}_1 \\ -y_2 & cy_2 & \mathbf{r}_2 \\ \vdots & \vdots & \vdots \\ -y_k & cy_k & \mathbf{r}_k \end{array}\right]$$

generates a linear $[n+2, k+1]_{q^2}^{\ell}$ code C_2. We call it **Construction II**. *Moreover, C_2 has the following parity-check matrix*

$$H_2 = \left[\begin{array}{cc|c} 0 & c & x_1 \; x_2 \; \cdots \; x_n \\ \hline -z_1 & cz_1 & \mathbf{s}_1 \\ -z_2 & cz_2 & \mathbf{s}_2 \\ \vdots & \vdots & \vdots \\ -z_{n-k} & cz_{n-k} & \mathbf{s}_{n-k} \end{array}\right].$$

Proof. Similar to the proof of Theorem 3.1, it can be checked that $\mathrm{rank}(G_2) = k+1$, $\mathrm{rank}(H_2) = n - k + 1$ and

$$G_2\overline{G_2}^T = \begin{bmatrix} 1 & 0\;0\;\cdots\;0 \\ 0 & \\ \vdots & G\overline{G}^T \\ 0 & \end{bmatrix} \text{ and } G_1\overline{H_1}^T = \begin{bmatrix} 0 & 0\;0\;\cdots\;0 \\ 0 & \\ \vdots & G\overline{H}^T \\ 0 & \end{bmatrix} = O,$$

where O is the appropriate zero matrix. According to Proposition 2.2, we have

$$\begin{aligned}\dim(\mathrm{Hull_H}(C_2)) &= (k+1) - \mathrm{rank}(G_2\overline{G_2}^T)\\ &= (k+1) - \mathrm{rank}(G\overline{G}^T + 1)\\ &= (k+1) - (k - \dim(\mathrm{Hull_H}(C)) + 1)\\ &= (k+1) - (k - \ell + 1)\\ &= \ell.\end{aligned}$$

Further, $\mathrm{rank}(H_2) = n - k + 1 = \dim(C_2^{\perp_{\mathrm{H}}})$. Hence C_2 has a parity-check matrix H_2. □

Example 3. Let $\mathbb{F}_4^* = \langle\omega\rangle$ and $c = w$. We start from a linear $[7,4,3]_4^1$ code. By applying Construction II, we construct a linear $[9,5,4]_4^1$ code generated by the following matrix

$$G = \left[\begin{array}{cc|ccccccc} 1 & 0 & \omega & 0 & \omega^2 & 1 & \omega & 1 & 1 \\ \hline 1 & \omega & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \omega & \omega^2 & 0 & 1 & 0 & 0 & \omega & \omega & 1 \\ \omega^2 & \omega^3 & 0 & 0 & 1 & 0 & \omega & 1 & 1 \\ \omega & \omega^2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

Moreover, C has the following parity-check matrix

$$H = \left[\begin{array}{cc|ccccccc} 0 & \omega & \omega & 0 & \omega^2 & 1 & \omega & 1 & 1 \\ \hline \omega & \omega^2 & 1 & 0 & 0 & 1 & \omega^2 & 0 & \omega \\ \omega^2 & 1 & 0 & 1 & 0 & \omega^2 & \omega & \omega^2 & \omega \\ 0 & 0 & 0 & 0 & 1 & \omega^2 & 1 & \omega^2 & 1 \end{array}\right].$$

Next, we give **Construction II′**, which contains [30, Theorem 3.5 (3)] as a special case.

Corollary 3.6. *Let C be a linear $[n,k]_{q^2}^{\ell}$ code with generator matrix G and parity-check matrix H. Let $\mathbf{r}_i$ be the i-th row of G. If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in C^{\perp_{\mathrm{H}}}$ and $\langle \mathbf{x}, \mathbf{x}\rangle_{\mathrm{H}} \neq -1$. For $1 \leq j \leq n-k$, suppose that $\overline{z_j} = \langle \mathbf{x}, \mathbf{s}_j\rangle_{\mathrm{H}}$ where $\mathbf{s}_j$ is the j-th row of H. Then*

$$G_2' = \left[\begin{array}{c|c} 1 & x_1\ x_2\ \cdots\ x_n \\ \hline 0 & \mathbf{r}_1 \\ 0 & \mathbf{r}_2 \\ \vdots & \vdots \\ 0 & \mathbf{r}_k \end{array}\right]$$

generates a linear $[n+1, k+1]_{q^2}^{\ell}$ code C_2'. We call it **Construction II′**. *Moreover, C_2' has the following parity-check matrix*

$$H_2' = \left[\begin{array}{c|c} -z_1 & \mathbf{s}_1 \\ -z_2 & \mathbf{s}_2 \\ \vdots & \vdots \\ -z_{n-k} & \mathbf{s}_{n-k} \end{array}\right].$$

Proof. This proof is similar to the proof of Corollary 3.3, and it is omitted here. □

Example 4. Let $\mathbb{F}_4^* = \langle\omega\rangle$. We start from a linear $[6,2,4]_4^2$ code. By applying Construction II′, we construct a linear $[7,3,4]_4^2$ code C generated by the following matrix

$$G = \left[\begin{array}{c|cccccc} 1 & 0 & \omega^2 & 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & \omega^2 & \omega^2 \\ 0 & 0 & 1 & 0 & \omega^2 & \omega^2 & 1 \end{array}\right].$$

Moreover, C has the following parity-check matrix

$$H = \left[\begin{array}{c|cccccc} \omega^2 & 1 & 0 & 0 & 0 & 1 & \omega \\ \omega & 0 & 1 & 0 & 0 & \omega & \omega \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \omega^2 & 0 & 0 & 0 & 1 & \omega & 1 \end{array}\right].$$

Remark 3.7. Assume that C is a linear $[n,k,d]_{q^2}^\ell$ code. Let $S \subseteq [n]$ be such that $|S| = s$. Luo *et al.* [36, Theorem 4] showed that $\dim(\mathrm{Hull_H}(C_S)) = \ell - s$ if S is a subset of information set of $\mathrm{Hull_H}(C)$. Following this theorem, we proved that the converses of Constructions I and I′ are valid. However, we do not determine the dimension of $\mathrm{Hull_H}(C_S)$ for other cases. Therefore, we do not prove that the converses of Constructions II and II′ are valid.

4. Numerical examples and their application to EAQECCs.

The parameters $[[n,k,d;\gamma]]_q$ denote a q-ary EAQECC that encodes k information qubits into n channel qubits with the help of γ pre-shared entanglement pairs. A method to construct EAQECCs from linear codes is as follows.

Proposition 4.1. [17] *If there is a linear $[n,k,d]_{q^2}^\ell$ code, then there exists an $[[n,k-\ell,d;n-k-\ell]]_q$ EAQECC.*

Example 5. Let $\mathbb{F}_9^* = \langle\omega\rangle$ and $c = \omega$. We begin with a Hermitian LCD $[5,2,4]_9$ and use Theorem 3.1 to construct a linear $[7,3,5]_9^1$ code C generated by the following matrix

$$G = \left[\begin{array}{cc|ccccc} 1 & 0 & \omega^7 & \omega^7 & 1 & \omega^3 & 1 \\ \hline \omega^2 & \omega^7 & 1 & 0 & \omega & \omega & \omega^2 \\ 1 & \omega^5 & 0 & 1 & \omega^5 & \omega^3 & \omega^5 \end{array}\right].$$

On the other hand, $C^{\perp_H}$ is a linear $[7,4,4]_9^1$ code, which is h_1-optimal. As an application, we construct $[[7,2,5;3]]_3$ and $[[7,3,4;2]]_3$ EAQECCs, where the $[[7,3,4;2]]_3$ EAQECC is optimal with respect to [16]. Additionally, the $[[7,2,5;3]]_3$ EAQECC requires less entanglement than the best-known ternary $[[7,2,5;4]]_3$ EAQECC in [16].

By [13, Corollary 2.1] and [35, Theorem 7], there are LCD $[7,3,5]_9$ and $[7,4,4]_9$ codes. As an application, we construct $[[7,3,5;4]]_3$ and $[[7,4,4;3]]_3$ EAQECCs, both of which are optimal with respect to [16].

4.1. h_ℓ-optimal quaternary linear codes and related EAQECCs.

We construct several h_i-optimal quaternary linear codes of lengths up to 13 by applying Theorem 2.4, Constructions I, I′, II and II′. All computations were done by Magma [5]. Tables 1, 3, 5 and 7 list the best minimum distances for quaternary linear codes and Tables 2, 4, 6 and 8 display their related EAQECCs. The upper bounds in the tables come from Proposition 2.5 and Grassl's table [16]. Recently, Li, Shi and Liu [29] characterized h_1-optimal quaternary linear codes for $n \leq 12$ using a different method. There are some important classification in [6], which is helpful to characterize h_ℓ-optimal quaternary linear codes. For example, there exists a unique linear $[12,6,6]_4$ code, which is Hermitian LCD. Hence $D_4^H(12,6,\ell) \le 5$ for $1 \le \ell \le 6$. Similarly, we have $D_4^H(11,5,\ell) \le 5$ for $1 \le \ell \le 5$ and $D_4^H(11,6,\ell) \le 4$ for $1 \le \ell \le 6$.

In the following, we use $G_{n,k,d}^{\ell}$ to denote a generator matrix of a linear $[n,k,d]_4^{\ell}$ code $C_{n,k,d}^{\ell}$. To reduce redundancy, the codes in Tables 1, 3, 5 and 7 can be obtained from https://coding-theory.github.io/code2/. In this paper, we list generator matrices for $k = 5,6,7,8$ and $n = 13$.

Example 6. The dimension of hull is fixed at 1. We construct some h_1-optimal quaternary linear codes by applying Theorem 2.4, Constructions I, I′, II and II′. By BKLC database [5], there is a linear $[16,5,9]_4^4$ code $C_{16,5,9}^4$. By applying Theorem 2.4 to the code $C_{16,5,9}^4$, puncturing the code $C_{16,5,9}^4$ on the set $\{1,2,3\}$, a linear $[13,5,6]_4^1$ code is obtained.

- $n = 13$, $k \in \{3,4,5,6,7,8\}$ and $h = 1$

$$G_{13,3,9}^1 = \begin{bmatrix} 1\,0\,0\,\bar{\omega}\,\omega\,1\,\omega\,\bar{\omega}\,0\,\omega\,\bar{\omega}\,\bar{\omega}\,1 \\ 0\,1\,0\,0\,0\,1\,1\,1\,1\,1\,1\,1\,1 \\ 0\,0\,1\,1\,1\,0\,0\,1\,1\,\omega\,\omega\,\bar{\omega}\,\bar{\omega} \end{bmatrix},\ G_{13,4,7}^1 = \begin{bmatrix} 1\,0\,0\,0\,0\,1\,1\,\omega\,0\,\omega\,\bar{\omega}\,\bar{\omega}\,1 \\ 0\,1\,0\,0\,0\,1\,0\,\bar{\omega}\,\omega\,\bar{\omega}\,1\,0\,\bar{\omega} \\ 0\,0\,1\,0\,0\,\bar{\omega}\,\omega\,1\,\bar{\omega}\,0\,1\,\bar{\omega}\,\omega \\ 0\,0\,0\,1\,0\,1\,\omega\,0\,1\,\bar{\omega}\,\bar{\omega}\,\omega\,0 \end{bmatrix},$$

$$G_{13,5,6}^1 = \begin{bmatrix} 1\,0\,0\,0\,0\,\omega\,\omega\,1\,1\,1\,\omega\,1\,\omega \\ 0\,1\,0\,0\,0\,\bar{\omega}\,\omega\,1\,0\,\omega\,\bar{\omega}\,\omega\,\omega \\ 0\,0\,1\,0\,0\,\omega\,\omega\,\bar{\omega}\,\omega\,\omega\,0\,1\,1 \\ 0\,0\,0\,1\,0\,0\,0\,1\,1\,\omega\,1\,1\,1 \\ 0\,0\,0\,0\,1\,0\,\omega\,1\,1\,0\,\bar{\omega}\,\omega\,\bar{\omega} \end{bmatrix},\ G_{13,6,6}^1 = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,\bar{\omega}\,0\,0\,1\,\bar{\omega}\,\omega\,\omega \\ 0\,1\,0\,0\,0\,0\,0\,\omega\,\bar{\omega}\,\omega\,\omega\,\bar{\omega}\,\omega \\ 0\,0\,1\,0\,0\,0\,1\,\omega\,\omega\,\omega\,\omega\,1\,\bar{\omega} \\ 0\,0\,0\,1\,0\,0\,\omega\,0\,\omega\,0\,1\,\omega\,\bar{\omega} \\ 0\,0\,0\,0\,1\,0\,1\,\omega\,0\,\bar{\omega}\,\omega\,\bar{\omega}\,0 \\ 0\,0\,0\,0\,0\,1\,1\,0\,\omega\,\omega\,0\,\omega\,\omega \end{bmatrix},$$

$$G_{13,7,5}^1 = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,1\,0\,0\,1\,1\,\omega \\ 0\,1\,0\,0\,0\,0\,0\,1\,\bar{\omega}\,1\,0\,\bar{\omega}\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,1\,\bar{\omega}\,1\,0\,\bar{\omega} \\ 0\,0\,0\,1\,0\,0\,0\,\omega\,\bar{\omega}\,\bar{\omega}\,\bar{\omega}\,\omega\,\omega \\ 0\,0\,0\,0\,1\,0\,0\,1\,0\,\omega\,\bar{\omega}\,1\,\bar{\omega} \\ 0\,0\,0\,0\,0\,1\,0\,\omega\,\omega\,\omega\,\omega\,0\,\bar{\omega} \\ 0\,0\,0\,0\,0\,0\,1\,\omega\,1\,0\,\omega\,1\,\omega \end{bmatrix},\ G_{13,8,4}^1 = \begin{bmatrix} 1\,0\,\bar{\omega}\,1\,0\,1\,1\,1\,1\,1\,1\,1\,1 \\ \omega\,\bar{\omega}\,1\,0\,0\,0\,0\,0\,0\,\bar{\omega}\,1\,1\,\bar{\omega} \\ \bar{\omega}\,1\,0\,1\,0\,0\,0\,0\,0\,\bar{\omega}\,0\,\bar{\omega}\,\omega \\ \bar{\omega}\,1\,0\,0\,1\,0\,0\,0\,0\,\omega\,1\,1\,1 \\ 1\,\omega\,0\,0\,0\,1\,0\,0\,0\,1\,\bar{\omega}\,\omega\,0 \\ 1\,\omega\,0\,0\,0\,0\,1\,0\,0\,0\,1\,\bar{\omega}\,\omega \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,\omega\,\omega\,0\,1 \\ \bar{\omega}\,1\,0\,0\,0\,0\,0\,0\,1\,1\,\bar{\omega}\,1\,1 \end{bmatrix}.$$

Example 7. The dimension of hull is fixed at 2. We construct some h_2-optimal quaternary linear codes by applying Theorem 2.4, Constructions I, I′, II and II′. By [29, Table 5], $D_4^H(8,2,1) = 5$. It follows from Proposition 2.5 that $D_4^H(9,3,2) \le D_4^H(8,2,1) = 5$. By adding the zero column to $C_{8,3,5}^2$, we know that $D_4^H(9,3,2) \ge D_4^H(8,3,2) = 5$. Hence we have $D_4^H(9,3,2) = 5$.

- $n = 13$, $k \in \{5,6,7,8\}$ and $h = 2$

$$G_{13,5,6}^2 = \begin{bmatrix} 1\,0\,0\,0\,1\,0\,0\,\omega\,0\,\omega\,1\,1\,0 \\ 0\,1\,0\,0\,1\,0\,0\,0\,1\,\bar{\omega}\,1\,0\,\bar{\omega} \\ 0\,0\,1\,0\,\omega\,0\,0\,1\,\bar{\omega}\,1\,0\,\omega\,\omega \\ 0\,0\,0\,1\,\omega\,0\,0\,\bar{\omega}\,\bar{\omega}\,\omega\,\omega\,1\,1 \\ 0\,0\,0\,0\,0\,1\,0\,0\,\omega\,\omega\,1\,\bar{\omega}\,1 \end{bmatrix},\ G_{13,6,6}^2 = \begin{bmatrix} 1\,0\,0\,0\,0\,\omega\,0\,\omega\,\bar{\omega}\,1\,0\,0\,\omega \\ 0\,1\,0\,0\,0\,1\,0\,0\,\bar{\omega}\,\bar{\omega}\,\bar{\omega}\,0\,\bar{\omega} \\ 0\,0\,1\,0\,0\,\bar{\omega}\,0\,0\,\omega\,\bar{\omega}\,0\,\omega\,\bar{\omega} \\ 0\,0\,0\,1\,0\,\bar{\omega}\,0\,\omega\,\bar{\omega}\,\omega\,0\,\bar{\omega}\,0 \\ 0\,0\,0\,0\,1\,1\,0\,\omega\,0\,\bar{\omega}\,\omega\,\bar{\omega}\,\bar{\omega} \\ 0\,0\,0\,0\,0\,0\,1\,0\,\omega\,\omega\,1\,\bar{\omega}\,1 \end{bmatrix},$$

$$G^2_{13,7,5} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&1&\bar{\omega}&1&\bar{\omega} \\ 0&1&0&0&0&0&\omega&0&1&0&\bar{\omega}&0&\omega \\ 0&0&1&0&0&0&0&0&1&0&1&1&\omega \\ 0&0&0&1&0&0&\bar{\omega}&0&0&\omega&1&\bar{\omega}&\omega \\ 0&0&0&0&1&0&\omega&0&\bar{\omega}&\omega&\omega&0&\omega \\ 0&0&0&0&0&1&1&0&\bar{\omega}&\omega&1&1&0 \\ 0&0&0&0&0&0&0&1&\omega&1&\omega&1&\omega \end{bmatrix}, \quad G^2_{13,8,4} = \begin{bmatrix} 1&0&\bar{\omega}&\bar{\omega}&\omega&\omega&1&1&1&1&1&1&1 \\ \omega&\bar{\omega}&1&0&0&0&0&0&0&\bar{\omega}&1&1&\bar{\omega} \\ 0&0&0&1&0&0&0&0&0&\bar{\omega}&0&\bar{\omega}&\omega \\ 0&0&0&0&1&0&0&0&0&\omega&1&1&1 \\ \bar{\omega}&1&0&0&0&1&0&0&0&1&\bar{\omega}&\omega&0 \\ 1&\omega&0&0&0&0&1&0&0&0&1&\bar{\omega}&\omega \\ 0&0&0&0&0&0&0&1&0&\omega&\bar{\omega}&0&1 \\ \bar{\omega}&1&0&0&0&0&0&0&1&1&\bar{\omega}&1&1 \end{bmatrix}.$$

The EAQECCs related to Table 1 are displayed in Table 2, where "$*$" denotes the corresponding code is the best-known EAQECCs compared with [16]. The parameters in bold indicate that the corresponding code have new or better parameters compared with the best-known EAQECCs [16]. For example, we can construct a $[9, 4, 5]_4^2$ code. As an application, one can obtain a $[[9, 2, 5; 3]]_2$ EAQECC, which requires less entanglement than the best-known $[[9, 2, 5; 4]]_2$ EAQECC in [16].

Table 1: $D_4^H(n, k, 2)$ for $n \leq 13$

$n\backslash k$	2	3	4	5	6	7	8	9	10	11
4	2									
5	4	3								
6	4	3	2							
7	4	4	3	2						
8	6	5	4	3	2					
9	6	5	5	4	3	2				
10	8	6	6	5	4	3	2			
11	8	7	6	5	4	4	3	2		
12	8	8	≥ 6	6	5	4	4	3	2	
13	10	9	8	≥ 6	6	5	4	4	3	2

Table 2: The related $[[n, k, d; \gamma]]_2$ EAQECCs with $[d; \gamma]$ based on Table 1

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
4	$[2; 0]^*$									
5	$[4; 1]^*$	$[3; 0]^*$								
6	$[4; 2]$	$[3; 1]$	$[2; 0]^*$							
7	$[4; 3]$	$[4; 2]$	$[3; 1]^*$	$[2; 0]^*$						
8	$[6; 4]^*$	$[5; 3]$	$[4; 2]^*$	$[3; 1]$	$[2; 0]^*$					
9	$[6; 5]$	$[\mathbf{5}; \mathbf{4}]$	$[\mathbf{5}; \mathbf{3}]$	$[\mathbf{4}; \mathbf{2}]$	$[3; 1]^*$	$[2; 0]^*$				
10	$[\mathbf{8}; \mathbf{6}]$	$[6; 5]$	$[\mathbf{6}; \mathbf{4}]$	$[\mathbf{5}; \mathbf{3}]$	$[4; 2]^*$	$[3; 1]^*$	$[2; 0]^*$			
11	$[8; 7]^*$	$[7; 6]$	$[6; 5]^*$	$[5; 4]^*$	$[4; 3]$	$[\mathbf{4}; \mathbf{2}]$	$[3; 1]^*$	$[2; 0]$		
12	$[8; 8]$	$[8; 7]$	$[6; 6]$	$[6; 5]^*$	$[5; 4]^*$	$[4; 3]$	$[4; 2]^*$	$[3; 1]^*$	$[2; 0]^*$	
13	$[\mathbf{10}; \mathbf{9}]$	$[9; 8]^*$	$[\mathbf{8}; \mathbf{7}]^*$	$[6; 6]$	$[6; 5]^*$	$[\mathbf{5}; \mathbf{4}]$	$[4; 3]$	$[\mathbf{4}; \mathbf{2}]$	$[3; 1]^*$	$[2; 0]^*$

Example 8. The hull dimension is fixed at 3. We construct some h_3-optimal quaternary linear codes by applying Theorem 2.4, Constructions I, I′, II and II′. By Proposition 2.5, $D_4^H(10, 4, 3) \leq D_4^H(9, 3, 2) = 5$. The EAQECCs related to Table 3 are displayed in Table 4.

- $n = 13$, $k \in \{5, 6, 7, 8\}$ and $h = 3$

$$G^3_{13,5,6} = \begin{bmatrix} 1&0&0&\bar{\omega}&0&\omega&1&1&1&1&1&1&1 \\ 0&0&1&0&0&0&\omega&\omega&1&\omega&\omega&\bar{\omega}&\omega \\ \bar{\omega}&1&0&1&0&0&\omega&\bar{\omega}&\omega&\bar{\omega}&\omega&\omega&1 \\ 1&\omega&0&0&1&0&\omega&\bar{\omega}&1&0&\bar{\omega}&\omega&0 \\ \omega&\bar{\omega}&0&0&0&1&0&1&\bar{\omega}&\bar{\omega}&\bar{\omega}&0&\bar{\omega} \end{bmatrix}, \quad G^3_{13,6,6} = \begin{bmatrix} 1&0&0&0&0&\bar{\omega}&0&1&0&\bar{\omega}&\omega&\omega&0 \\ 0&1&0&0&0&1&0&\omega&\omega&\bar{\omega}&1&0&\bar{\omega} \\ 0&0&1&0&0&\omega&0&0&1&\omega&\omega&\omega&\omega \\ 0&0&0&1&0&\omega&0&\omega&\bar{\omega}&\omega&\bar{\omega}&1&\bar{\omega} \\ 0&0&0&0&1&\bar{\omega}&0&\bar{\omega}&\bar{\omega}&0&1&1&\bar{\omega} \\ 0&0&0&0&0&0&1&\omega&1&\omega&1&\omega&\bar{\omega} \end{bmatrix},$$

$$G^3_{13,7,5} = \begin{bmatrix} 1&0&0&0&0&0&0&\bar{\omega}&\bar{\omega}&\bar{\omega}&0&0&1 \\ 0&1&0&0&0&0&0&0&1&0&\bar{\omega}&1&\bar{\omega} \\ 0&0&1&0&0&0&0&1&0&1&1&\omega&0 \\ 0&0&0&1&0&0&0&\bar{\omega}&\bar{\omega}&0&0&\omega&\bar{\omega} \\ 0&0&0&0&1&0&0&0&1&\bar{\omega}&1&0&\bar{\omega} \\ 0&0&0&0&0&1&0&1&\omega&\bar{\omega}&1&\bar{\omega}&0 \\ 0&0&0&0&0&0&1&1&\bar{\omega}&\omega&0&1&1 \end{bmatrix}, \quad G^3_{13,8,4} = \begin{bmatrix} 1&0&0&0&0&0&0&0&\omega&\omega&0&\omega&0 \\ 0&1&0&0&0&0&0&0&\omega&0&\bar{\omega}&\omega&0 \\ 0&0&1&0&0&0&0&0&\bar{\omega}&\omega&\bar{\omega}&\omega&0 \\ 0&0&0&1&0&0&0&0&0&\bar{\omega}&\omega&1&0 \\ 0&0&0&0&1&0&0&0&\omega&\omega&1&1&0 \\ 0&0&0&0&0&1&0&0&1&1&\omega&\omega&0 \\ 0&0&0&0&0&0&1&0&0&1&1&1&0 \\ 0&0&0&0&0&0&0&1&1&\omega&\omega&\bar{\omega}&0 \end{bmatrix}.$$

Table 3: $D_4^H(n, k, 3)$ for $n \leq 13$

$n \backslash k$	3	4	5	6	7	8	9	10
6	4							
7	4	3						
8	4	4	3					
9	6	4	4	3				
10	6	5	4	3	3			
11	6	6	5	4	4	3		
12	8	7	6	5	4	4	3	
13	8	≥ 7	≥ 6	6	5	4	4	3

Table 4: The related $[[n, k, d; \gamma]]_2$ EAQECCs with $[d; \gamma]$ based on Table 3

$n \backslash k$	0	1	2	3	4	5	6	7
6	$[4; 0]^*$							
7	$[4; 1]^*$	$[3; 0]^*$						
8	$[4; 2]$	$[\mathbf{4}; \mathbf{1}]$	$[3; 0]^*$					
9	$[6; 3]^*$	$[\mathbf{4}; \mathbf{2}]$	$[4; 1]^*$	$[3; 0]^*$				
10	$[6; 4]$	$[\mathbf{5}; \mathbf{3}]$	$[4; 2]$	$[3; 1]$	$[3; 0]^*$			
11	$[6; 5]$	$[\mathbf{6}; \mathbf{4}]$	$[\mathbf{5}; \mathbf{3}]$	$[4; 2]$	$[4; 1]^*$	$[3, 0]^*$		
12	$[8; 6]^*$	$[\mathbf{7}; \mathbf{5}]$	$[6; 4]$	$[5; 3]^*$	$[4; 2]$	$[\mathbf{4}; \mathbf{1}]$	$[3; 0]^*$	
13	$[8; 7]$	$[\mathbf{7}; \mathbf{6}]$	$[6; 5]$	$[\mathbf{6}; \mathbf{4}]$	$[5; 3]^*$	$[4; 2]$	$[4; 1]^*$	$[3; 0]^*$

Example 9. The dimension of hull is fixed at 4. We construct h_4-optimal linear codes by applying Theorem 2.4, Constructions I, I′, II and II′. By Proposition 2.5, $D_4^H(10, 5, 4) \leq D_4^H(9, 4, 3) = 4$. The EAQECCs related to Table 5 are displayed in Table 6.

- $n = 13$, $k \in \{5, 6, 7, 8\}$ and $h = 4$

$$G^4_{13,5,7} = \begin{bmatrix} 1\,0\,1\,\bar{\omega}\,1\,0\,\bar{\omega}\,0\,1\,1\,1\,1\,1 \\ 1\,\omega\,1\,0\,0\,0\,1\,\omega\,\bar{\omega}\,\omega\,1\,\bar{\omega}\,1 \\ 1\,\omega\,0\,1\,0\,0\,1\,1\,1\,1\,\omega\,\bar{\omega}\,0 \\ \bar{\omega}\,1\,0\,0\,1\,0\,1\,\bar{\omega}\,\bar{\omega}\,1\,0\,1\,\bar{\omega} \\ 1\,\omega\,0\,0\,0\,1\,\bar{\omega}\,\bar{\omega}\,1\,0\,1\,\bar{\omega}\,\bar{\omega} \end{bmatrix},\ G^4_{13,6,6} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,1\,0\,\omega\,\bar{\omega}\,1\,\omega\,\omega \\ 0\,1\,0\,0\,0\,0\,\bar{\omega}\,\omega\,0\,1\,1\,\bar{\omega}\,0 \\ 0\,0\,1\,0\,0\,0\,\bar{\omega}\,\bar{\omega}\,\bar{\omega}\,\omega\,\omega\,0\,\omega \\ 0\,0\,0\,1\,0\,0\,0\,\omega\,0\,1\,\bar{\omega}\,1\,\omega \\ 0\,0\,0\,0\,1\,0\,\bar{\omega}\,\omega\,\bar{\omega}\,0\,\bar{\omega}\,1\,0 \\ 0\,0\,0\,0\,0\,1\,\bar{\omega}\,\omega\,\omega\,\bar{\omega}\,\bar{\omega}\,\omega\,\bar{\omega} \end{bmatrix},$$

$$G^4_{13,7,5} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,\omega\,0\,\bar{\omega}\,1\,0\,\omega\,1 \\ 0\,1\,0\,0\,0\,0\,\bar{\omega}\,0\,0\,0\,\omega\,\omega\,\omega \\ 0\,0\,1\,0\,0\,0\,\omega\,0\,1\,\bar{\omega}\,1\,\omega\,0 \\ 0\,0\,0\,1\,0\,0\,\omega\,0\,\omega\,1\,\bar{\omega}\,0\,\omega \\ 0\,0\,0\,0\,1\,0\,\omega\,0\,1\,\omega\,\omega\,0\,\bar{\omega} \\ 0\,0\,0\,0\,0\,1\,0\,0\,\bar{\omega}\,0\,1\,\bar{\omega}\,\omega \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,\bar{\omega}\,\omega\,\bar{\omega}\,1 \end{bmatrix},\ G^4_{13,8,4} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,\omega\,0\,1\,\omega\,0 \\ 0\,1\,0\,0\,0\,0\,0\,0\,\omega\,0\,\bar{\omega}\,\bar{\omega}\,1 \\ 0\,0\,1\,0\,0\,0\,0\,0\,1\,1\,\omega\,\bar{\omega}\,\omega \\ 0\,0\,0\,1\,0\,0\,0\,0\,1\,\omega\,\bar{\omega}\,1\,1 \\ 0\,0\,0\,0\,1\,0\,0\,0\,\omega\,1\,\bar{\omega}\,0\,1 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,\omega\,1\,\bar{\omega}\,1 \\ 0\,0\,0\,0\,0\,0\,1\,0\,\omega\,\bar{\omega}\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,1\,\omega\,1\,1\,\bar{\omega} \end{bmatrix}.$$

Table 5: $D_4^H(n, k, 4)$ for $n \leq 13$

$n\backslash k$	4	5	6	7	8	9
8	4					
9	4	3				
10	4	4	3			
11	6	5	4	3		
12	6	≥ 5	≥ 4	4	4	
13	8	7	6	5	4	3

Table 6: The related $[[n, k, d; \gamma]]_2$ EAQECCs with $[d; \gamma]$ based on Table 5

$n\backslash k$	0	1	2	3	4	5
8	$[4; 0]^*$					
9	$[4; 1]$	$[3; 0]^*$				
10	$[4; 2]$	$[4; 1]$	$[3; 0]$			
11	$[6; 3]$	$[5; 2]$	$[4; 1]$	$[3; 0]^*$		
12	$[6; 4]$	$[5; 3]$	$[4; 2]$	$[4; 1]$	$[4; 0]^*$	
13	$[8; 5]^*$	$[\mathbf{7}; \mathbf{4}]$	$[6; 3]$	$[5; 2]^*$	$[4; 1]$	$[3; 0]^*$

Example 10. The dimension of hull is fixed at 5. We construct some h_5-optimal linear $[n, k]_4$ codes by applying Theorem 2.4, Constructions I, I′, II and II′. The EAQECCs related to Table 7 are displayed in Table 8. For the optimality of $k = 5$, one can refer to [7].

- $n = 13$, $k \in \{5, 6, 7, 8\}$ and $h = 5$

$$G^5_{13,5,6} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,1 \\ 0\,1\,0\,0\,0\,\bar{\omega}\,\omega\,\omega\,0\,\omega\,\omega\,\bar{\omega}\,\bar{\omega} \\ 0\,0\,1\,0\,0\,1\,\bar{\omega}\,1\,1\,1\,\bar{\omega}\,0\,1 \\ 0\,0\,0\,1\,0\,\bar{\omega}\,1\,0\,\bar{\omega}\,\omega\,\bar{\omega}\,\omega\,1 \\ 0\,0\,0\,0\,1\,\bar{\omega}\,1\,\omega\,0\,0\,\omega\,1\,0 \end{bmatrix},\ G^5_{13,6,5} = \begin{bmatrix} 1\,0\,0\,0\,0\,\bar{\omega}\,0\,\omega\,0\,1\,1\,1\,0 \\ 0\,1\,0\,0\,0\,0\,0\,1\,\omega\,\omega\,0\,1\,0 \\ 0\,0\,1\,0\,0\,\omega\,0\,\bar{\omega}\,\omega\,\bar{\omega}\,0\,\omega\,1 \\ 0\,0\,0\,1\,0\,\bar{\omega}\,0\,\bar{\omega}\,\omega\,\omega\,1\,0\,\bar{\omega} \\ 0\,0\,0\,0\,1\,\bar{\omega}\,0\,\omega\,0\,\omega\,\omega\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,\omega\,\omega\,\bar{\omega}\,\omega\,0\,0 \end{bmatrix},$$

$$G_{13,7,5}^{5} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,\bar{\omega}\,1\,0\,1\,\omega \\ 0\,1\,0\,0\,0\,0\,0\,1\,\omega\,\bar{\omega}\,1\,\omega\,0 \\ 0\,0\,1\,0\,0\,0\,0\,1\,1\,0\,\omega\,\omega\,1 \\ 0\,0\,0\,1\,0\,0\,0\,\omega\,\omega\,1\,\omega\,\omega\,1 \\ 0\,0\,0\,0\,1\,0\,0\,\bar{\omega}\,0\,\bar{\omega}\,1\,\bar{\omega}\,0 \\ 0\,0\,0\,0\,0\,1\,0\,\omega\,\bar{\omega}\,\bar{\omega}\,1\,0\,1 \\ 0\,0\,0\,0\,0\,0\,1\,\omega\,\bar{\omega}\,\omega\,\omega\,1\,\omega \end{bmatrix}, G_{13,8,4}^{5} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0\,\bar{\omega}\,\omega\,\bar{\omega}\,1 \\ 0\,1\,0\,0\,0\,0\,0\,0\,\omega\,1\,\omega\,\bar{\omega}\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,\bar{\omega}\,0\,\bar{\omega}\,0\,1 \\ 0\,0\,0\,1\,0\,0\,0\,0\,1\,0\,\omega\,\bar{\omega}\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,\bar{\omega}\,\bar{\omega}\,1\,\omega\,\omega \\ 0\,0\,0\,0\,0\,1\,0\,0\,\omega\,0\,\bar{\omega}\,\bar{\omega}\,\bar{\omega} \\ 0\,0\,0\,0\,0\,0\,1\,0\,\omega\,1\,\bar{\omega}\,\bar{\omega}\,1 \\ 0\,0\,0\,0\,0\,0\,0\,1\,\bar{\omega}\,\bar{\omega}\,0\,1\,0 \end{bmatrix}.$$

Table 7: $D_4^H(n,k,5)$ for $n \leq 13$

$n\backslash k$	5	6	7	8
10	4			
11	4	3-4		
12	6	5	4	
13	6	5-6	5	4

Table 8: The related $[[n,k,d;\gamma]]_2$ EAQECCs with $[d;\gamma]$ based on Table 7

$n\backslash k$	0	1	2	3
10	$[4;0]^*$			
11	$[4;1]$	$[3;0]$		
12	$[6;2]$	$[5;1]$	$[4;0]^*$	
13	$[6;3]$	$[5;2]$	$[5;1]^*$	$[4;0]^*$

5. **Conclusion.** This paper has introduced an efficient method for constructing linear codes with various Hermitian hull dimensions. We have constructed optimal quaternary linear codes of lengths up to 13 with Hermitian hull dimensions 2-5. As an application, some new or improved EAQECCs have been constructed compared with Grassl's code table [16].

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