MURA NOTES

L. Jackson Laslett
1 February 1955

APPROXIMATION OF EIGENVALUES, AND EIGENFUNCTIONS, BY VARIATIONAL METHODS

1. Motivation

In consideration of various accelerator designs employing the alternate-gradient principle, one is often faced with the problem of determining the values of the design parameters at the limits of stability. If one knows the general character of the solution to the differential equation at such points one may substitute a suitable simple trial function (or simple trial functions), containing adjustable parameters, into the associated variation problem and readily determine the eigenvalues with considerable accuracy.[1] It is the purpose of the present note (i) to illustrate the use of variational methods in a simple boundary-value problem where the dependent variable is fixed at the boundaries, (ii) to apply a similar technique to the Mathieu equation, for which the eigensolutions are periodic, and finally (iii) to point out the applicability of the method to a problem arising in connection with the analysis of a Mk. V FFAG accelerator.

2. Example Concerning a Boundary-Value Problem in which the Dependent Variable is Fixed at the Boundaries

We consider the differential equation

$$y'' + \lambda y = 0, \quad \text{with} \quad y(\pm 1) = 0.$$

The simplest solution to this problem is known to be of the form

$$y_1 = \cos\frac{\pi}{2}x \quad \text{and is obtained when} \quad \lambda = \frac{\pi^2}{4}.$$

The above problem is equivalent to the isoperimetric variation problem in which we seek a function, such that $y(\pm 1) = 0$, for which

$$\delta \int_{-1}^{1} y'^2\,dx = 0, \quad \text{subject to} \int_{-1}^{1} y^2\,dx = \text{const. (say 1)};$$

that is, introducing the Lagrange multiplier $-\lambda$, Euler's equation for

$$\delta \int_{-1}^{1} (y'^2 - \lambda y^2)\,dx = 0$$

is our original differential equation $y'' + \lambda y = 0$.

A trial solution (even in x), satisfying the boundary conditions, may be taken of the form

$$y = (1 - x^2)(a_1 + a_2 x^2),$$

for which $\int_{-1}^{1} (y'^2 - \lambda y^2)dx = (\frac{8}{3} - \frac{16}{15}\lambda)a_1^2 + (\frac{16}{15} - \frac{32}{105}\lambda)a_1 a_2 + (\frac{88}{105} - \frac{16}{315}\lambda)a_2^2.$

The latter expression will be stationary when

$$(\frac{16}{3} - \frac{32}{15}\lambda)a_1 + (\frac{16}{15} - \frac{32}{105}\lambda)a_2 = 0$$

$$(\frac{16}{15} - \frac{32}{105}\lambda)a_1 + (\frac{176}{105} - \frac{32}{315}\lambda)a_2 = 0.$$

We accordingly find that λ must be given by

$$\lambda^2 - 28\lambda + 63 = 0,$$

of which the lesser root is $\lambda = 14 - (133)^{1/2} = 2.4674_{37}$ and $\frac{a_2}{a_1} = -0.22075$.

This value of λ may be compared with $\frac{\pi^2}{4} = 2.467\,401\,100\,022\cdots$; the inclusion of additional parameters in the trial function would permit further improvement of the estimated value. [The use of three constants (a_1, a_2, a_3) has been reported (Buck) to give $\lambda = 2.467\,401\,10_8$.]

It may be noted that with the trial solution normalized so that our auxiliary integral (in this case $\int_{-1}^{1} y^2\,dx$) is unity, the value obtained for $\int_{-1}^{1} y'^2 dx$ may be shown to be our value of λ and will be greater than the exact eigenvalue.

The equivalent variation problems for other differential equations with other types of boundary conditions are presented in Courant-Hilbert[2], Ch. IV, Sect. 5, esp. p. 182. One may further note that, in particular, with J of the form

$$J = \int_{x_1}^{x_2} F(x, y, y')\,dx,$$

$$\delta J = \frac{\partial F}{\partial y'}\,\delta y\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2}\Big[\frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big) - \frac{\partial F}{\partial y}\Big]\,\delta y\,dx;$$

accordingly if $\frac{\partial F}{\partial y'}$ is independent of x or periodic (period $x_2 - x_1$) in x, boundary conditions requiring y to be periodic (period $x_2 - x_1$) result in the variation problem again reducing to the problem governed by Euler's equation

$$\frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big) - \frac{\partial F}{\partial y} = 0.$$

3. <u>Character of the Eigensolutions of the Mathieu Equation</u>

At the stability boundaries for the Mathieu equation,

$$\frac{d^2y}{dx^2} + (a + 16q\cos 2x)\,y = 0,$$

the characteristic bounded solutions are periodic, with period π or 2π. When $q = 0$, the periodic solutions are, of course,

$$\begin{matrix} 1 & \cos x & \cos 2x & \cos 3x & \cdots \\ & \sin x & \sin 2x & \sin 3x & \cdots; \end{matrix}$$

the Mathieu functions which reduce to these forms when $q \to 0$ are designated (notation of Whittaker and Watson[3])

$$\begin{matrix} ce_0(x,q) & ce_1(x,q) & ce_2(x,q) & ce_3(x,q) & \cdots \\ & se_1(x,q) & se_2(x,q) & se_3(x,q) & \cdots, \end{matrix}$$

the functions in the first line being even functions of x and those in the second line odd functions.

The stability boundaries are given in series form for the first few cases[3] and are also listed in tables[4]; coefficients for the Fourier expansion of the eigenfunctions are likewise available[3,4]. The stability boundaries are graphed in Fig. 1 and the character of the solutions illustrated in the accompanying Table I. The solutions of the table are arranged in the same order as the quantities appear on the graph. The quatities be_0, bo_1, be_1, ... are tabulated as functions of $s = 2|16q|$ in ref.[4], as are the coefficients in Fourier expansions of the even functions Se_0, Se_1, ... and of the odd functions So_1, So_2, ...; for $q < 0$, these functions give the desired solutions if we set $s = 2(-16q)$, while, for $q > 0$, we set $s = 2(16q)$ and replace the argument by $x \mp \pi/2$.

4. Approximation, by Variational Methods, of Eigenvalues and Eigenfunctions for Mathieu Equation

The first eigenvalues of Mathieu's equation (given by be_0, bo_1, be_1, bo_2) may be approximated by a procedure paralleling that employed in the example of Section 2. We consider, in this connection, the variation problem for the form of the eigenfunctions

$$\delta \int_0^{2\pi} (y'^2 - ay^2 - 16\,q\,y^2 \cos 2x)\,dx = 0$$

into which we introduce periodic trial solutions.

(i) For the first stability boundary we employ trial solutions, even in x and of period π, of the form

$$A_0 + A_1 \cos 2x + A_2 \cos 4x + \cdots .$$

If only two terms are retained, the integral becomes

$$2\pi \left[-a A_0^2 - 16\,q A_0 A_1 + (2 - \tfrac{a}{2}) A_1^2 \right].$$

This expression is stationary if

$$-2a\,A_0 - 16q\,A_1 = 0$$
$$-16q\,A_0 + (4 - a)\,A_1 = 0 \;;$$

and gives us a relation from which one obtains a good first estimate of the first stability boundary:

$$(16\,q)^2 = 2a\,(a - 4).$$

We thus obtain, for the first stability boundary,

$$a \doteq -2\left[(1 + 32\,q^2)^{1/2} - 1\right], \quad A_1/A_0 \doteq \frac{(1 + 32\,q^2)^{1/2} - 1}{4q} \quad \text{and,}$$

by way of example, if $16\,q = \pm 4$,

$$a \doteq -1.464\,, \quad A_1/A_0 \doteq \pm 0.732 \text{ (the sign being that of } q\text{)}.$$

[The second root for "a" is $2[(1 + 32q^2)^{1/2} + 1]$ or, in this example, 5.464 with $A_1/A_0 \doteq \mp 2.732$.]

Fig. 1

Stability Boundaries and Eigen solutions for Mathieu Equation

$$\frac{d^2y}{dx^2} + (a + 16q\cos 2x)\,y = 0 .$$

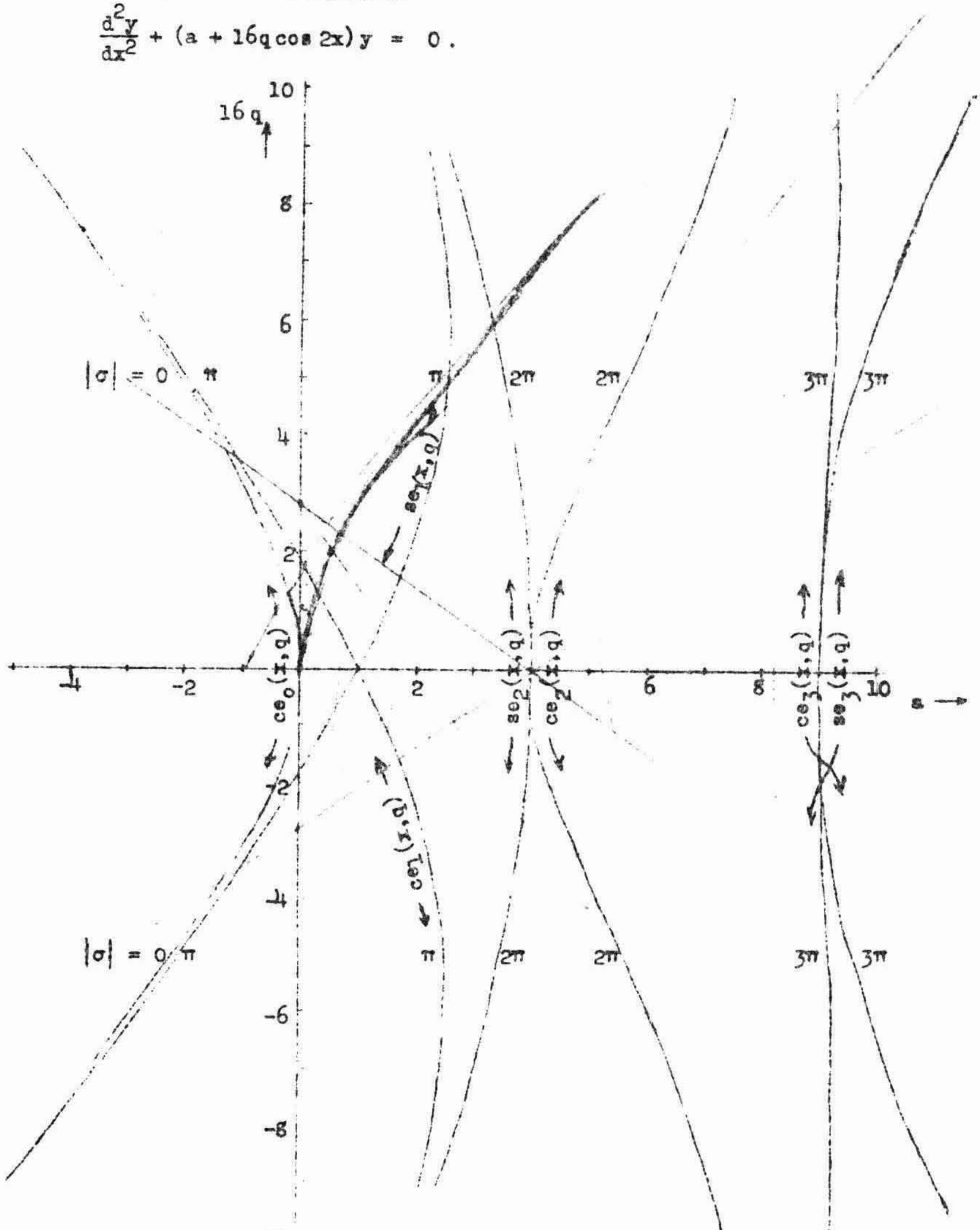

TABLE I

CHARACTER OF EIGENSOLUTIONS TO MATHIEU EQUATION

σ	0	π	π	2π	2π	3π	3π
a	$be_0(32q) - 16q$	$bo_1(32q) - 16q$	$be_1(32q) - 16q$	$bo_2(32q) - 16q$	$be_2(32q) - 16q$	$bo_3(32q) - 16q$	$be_3(32q) - 16q$
16q	ce_0	ce_1	se_1	se_2	ce_2	ce_3	se_3
4.0	1.0 + 0.7570 cos 2x + 0.0870 cos 4x	cos x + 0.1953 cos 3x + 0.0148 cos 5x	sin x + 0.3104 sin 3x + 0.0275 sin 5x	sin 2x + 0.1639 sin 4x + 0.0102 sin 6x	- 0.3866 + cos 2x + 0.1870 cos 4x	- 0.1972 cos x + cos 3x + 0.1269 cos 5x	- 0.3140 sin x + sin 3x + 0.1288 sin 5x
1.367	1.0 + 0.3260 cos 2x + 0.0138 cos 4x	cos x + 0.0785 cos 3x + 0.0022 cos 5x	sin x + 0.0930 sin 3x + 0.0027 sin 5x				
0	1	cos x	sin x	sin 2x	cos 2x	cos 3x	sin 3x
16q	ce_0 (From Se_0)	se_1 (From So_1)	ce_1 (From Se_1)	se_2 (From Se_2)	ce_2 (From Se_2)	se_3 (From So_3)	ce_3 (From Se_3)
0	1	sin x	cos x	sin 2x	cos 2x	sin 3x	cos 3x
-1.367	1.0 - 0.3260 cos 2x + 0.0138 cos 4x	sin x - 0.0785 sin 3x + 0.0022 sin 5x	cos x - 0.0930 cos 3x + 0.0027 cos 5x				
-4.0	1.0 - 0.7570 cos 2x + 0.0870 cos 4x	sin x - 0.1953 sin 3x + 0.0148 sin 5x	cos x - 0.3104 cos 3x + 0.0275 cos 5x	sin 2x - 0.1639 sin 4x + 0.0102 sin 6x	0.3866 + cos 2x - 0.1870 cos 4x	0.1972 sin x + sin 3x - 0.1269 sin 5x	0.3140 cos x + cos x - 0.1288 cos 5x
a	$be_0(\lvert 32q \rvert) - \lvert 16q \rvert$	$bo_1(\lvert 32q \rvert) - \lvert 16q \rvert$	$be_1(\lvert 32q \rvert) - \lvert 16q \rvert$	$bo_2(\lvert 32q \rvert) - \lvert 16q \rvert$	$be_2(\lvert 32q \rvert) - \lvert 16q \rvert$	$bo_3(\lvert 32q \rvert) - \lvert 16q \rvert$	$be_3(\lvert 32q \rvert) - \lvert 16q \rvert$
Period of Eigenfcn.	π	2π	2π	π	π	2π	2π

If we refine our approximation by including three parameters in the trial function, the integral becomes

$$2\pi[-aA_0^2 - 16qA_0A_1 + (2-\tfrac{a}{2})A_1^2 - 8qA_1A_2 + (8-\tfrac{a}{2})A_2^2].$$

We thus obtain the simultaneous equations

$$\begin{aligned} -2aA_0 \quad - 16qA_1 \qquad\qquad &= 0 \\ -16qA_0 + (4-a)A_1 - 8qA_2 &= 0 \\ -8qA_1 + (16-a)A_2 &= 0\,; \end{aligned}$$

we thus obtain for the first stability boundary (for $16q = \pm 4$)

$$a = -1.5136_5, \qquad A_1/A_0 = \pm 0.7568_2, \qquad A_2/A_0 = +0.0864$$

and a second solution

$$a \doteq +5.176, \qquad A_1/A_0 \doteq \mp 2.588, \qquad A_2/A_1 \doteq \pm 0.1848.$$

We thus are obtaining what appears to be a good approximation to the first eigenvalue and its associated solution as well as a reasonable estimate[5] of the value and solution corresponding to ce_2. The correct values are[4]

First solution: $a = -1.51396$, $A_1/A_0 = \pm 0.7570$, $A_2/A_0 = +0.0870$;
Second solution: $a = 5.17266$, $A_1/A_0 = \mp 2.5863$, $A_2/A_1 = \pm 0.1870$.

The first stability limit may, of course, be alternatively estimated by use of the smooth approximation;[7] in this way we find $a = -32q^2$, which represents a good approximation to the correct value when q is small (as is seen by expansion of our first result or by reference to the series given on p.411 of ref.3) and gives the numerical value -2 for $16q = \pm 4$.

(ii) One may proceed similarly to locate the second stability boundary and to examine the character of the associated eigensolutions. In this case (when $q > 0$) we employ trial solutions (with period 2π) of the form

$$B_1 \cos x + B_2 \cos 3x + B_3 \cos 5x + \cdots.$$

Retaining three terms, the integral becomes

$$2\pi[(\tfrac{1}{2} - 4q - \tfrac{a}{2})B_1^2 - 8qB_1B_2 + (\tfrac{9}{2} - \tfrac{a}{2})B_2^2 - 8qB_2B_3 + (\tfrac{25}{2} - \tfrac{a}{2})B_3^2]$$

and leads to the equations

$$\begin{aligned} (1-8q-a)B_1 - 8qB_2 \qquad\qquad &= 0 \\ -8qB_1 + (9-a)B_2 - 8qB_3 &= 0 \\ -8qB_2 + (25-a)B_3 &= 0. \end{aligned}$$

The location of the first stability boundary of the present type is then estimated to be, when $16q = 4$,

$$a = -1.39066, \quad \text{with } B_2/B_1 = 0.1953_3 \text{ and } B_3/B_1 = 0.0148_{03}.$$

The correct values are[4]

$$a = -1.39068, \quad \text{and } B_2/B_1 = 0.1953_4, \quad B_3/B_1 = 0.014848.$$

(iii) Proceeding to the next stability limit, one assumes trial solutions (again of period 2π) of the form

$$C_1 \sin x + C_2 \sin 3x + C_3 \sin 5x + \cdots .$$

Since we are concerned only with this problem as an illustration, we keep merely two terms here to obtain

$$2\pi \left[(\tfrac{1}{2} + 4q - \tfrac{a}{2}) C_1^{\,2} - 8q C_1 C_2 + (\tfrac{9}{2} - \tfrac{a}{2}) C_2^{\,2} \right]$$

for the integral.

We then obtain the equations

$$(1 + 8q - a)C_1 - 8q\, C_2 = 0$$
$$- 8q\, C_1 + (9 - a)C_2 = 0 ,$$

with the solution of interest, for $16\,q = 4$,

$$a = 6 - (13)^{1/2} = 2.3944 , \quad C_2/C_1 = 0.3028 .$$

The correct values are[4]

$$a = 2.3792 , \qquad C_2/C_1 = 0.3104 .$$

(iv) The fourth type of stability limit is investigated by aid of the trial function (period π)

$$D_1 \sin 2x + D_2 \sin 4x + D_3 \sin 6x + \cdots .$$

Again we retain only two terms to obtain

$$2\pi \left[(2 - \tfrac{a}{2}) D_1^{\,2} - 8q D_1 D_2 + (8 - \tfrac{a}{2}) D_2^{\,2} \right]$$

for the integral.

We then obtain the equations

$$(4 - a)D_1 - 8q\, D_2 = 0$$
$$- 8q\, D_1 + (16 - a)D_2 = 0 ,$$

with the solution of interest, for $16\,q = 4$,

$$a = 10 - (40)^{1/2} = 3.6754 , \quad D_2/D_1 = 0.1623 .$$

The correct values are[4]

$$a = 3.6722 , \qquad D_2/D_1 = 0.1639 .$$

5. Application of Variation Methods to a Problem arising in a Mk. V FFAG

In the analysis[8] of the oscillations about a scalloped orbit, as for a Mk. V FFAG accelerator, one obtains differential equations of the form

$$\frac{d^2y}{dt^2} + \left\{a + \left[b\cos 2t + c\cos(4t+\delta)\right]\right\} y = 0.$$

or

$$\frac{d^2y}{dt^2} + \left\{a + \left[b\cos 2t + c\cos\delta\,\cos 4t - c\sin\delta\,\sin 4t\right]\right\} y = 0.$$

In a typical case,

$$b = \pm 1.3672, \quad c = \pm 0.2462, \quad \text{and} \quad \delta = 0.0331 \text{ radian.}$$

It is desired to determine values of the parameter "a" at those stability limits which lie near zero.

(i) Since δ is small it may be expected that a good estimate of the stability boundaries may, in fact, be obtainable by setting $\delta = 0$ and using trial solutions

$$A_0 + A_2\cos 2t + A_4\cos 4t \quad \text{for one boundary}$$

and $A_1\cos t + A_3\cos 3t$ (in the case the upper sign for "b" is taken) at the other boundary.

In these respective cases, proceding by methods similar to those used before, one finds the determanental equations

$$\begin{vmatrix} -2a & -b & -c \\ -b & 4-a-\frac{c}{2} & -\frac{b}{2} \\ -c & -\frac{b}{2} & 16-a \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 1-a-\frac{b}{2} & -\frac{b}{2}-\frac{c}{2} \\ -\frac{b}{2}-\frac{c}{2} & 9-a \end{vmatrix} = 0.$$

The first determanental equation leads to the first boundary location (when b and c have the values indicated)

$$a = -0.23429 \quad \text{for} \quad c > 0,$$

$$a = -0.21545 \quad \text{for} \quad c < 0.$$

The values obtained in general from this first determanental equation approach, when b and c are small, the value given by the smooth approximation[7]:

$$a \cong -\left(\frac{b^2}{8} + \frac{c^2}{32}\right)$$

but, in third order (order of b^2c), appear to permit a slightly more negative value of "a" when the maximum positive excursions of the $\cos 2t$ and $\cos 4t$ terms add in phase. For the values of b and c assumed here the smooth approximation gives $a \cong -0.2355$.

The second determanental equation leads to the second stability boundary estimated to be given by

$$a \doteq 0.2421 \quad \text{for} \quad c > 0,$$

$$a \doteq 0.2804 \quad \text{for} \quad c < 0.$$

(ii) If we do not neglect δ in the given problem, it then appears appropriate to take trial functions of a more general form, although the determanental equation will be found to factor into two equations, corresponding to eigensolutions of periods π and 2π.

We accordingly take as a trial function

$$\begin{aligned} y = A_0 &+ A_1\cos t + A_2\cos 2t + A_3\cos 3t + A_4\cos 4t \\ &+ B_1\sin t + B_2\sin 2t + B_3\sin 3t + B_4\sin 4t \, , \end{aligned}$$

for which the integral which is to take on a stationary value is

$$\begin{aligned} 2\pi\Big[-aA_0^2 &- bA_0A_2 - c\cos\delta\, A_0A_4 + c\sin\delta\, A_0B_4 \\ &+ \left(\tfrac{1}{2} - \tfrac{a}{2} - \tfrac{b}{4}\right)A_1^2 + \left(-\tfrac{b}{2} - \tfrac{c}{2}\cos\delta\right)A_1A_3 + \tfrac{c}{2}\sin\delta\, A_1B_3 \\ &+ \left(2 - \tfrac{a}{2} - \tfrac{c}{4}\cos\delta\right)A_2^2 - \tfrac{b}{2}A_2A_4 + \tfrac{c}{2}\sin\delta\, A_2B_2 \\ &+ \left(\tfrac{9}{2} - \tfrac{a}{2}\right)A_3^2 + \tfrac{c}{2}\sin\delta\, A_3B_1 + \left(8 - \tfrac{a}{2}\right)A_4^2 \\ &+ \left(\tfrac{1}{2} - \tfrac{a}{2} + \tfrac{b}{4}\right)B_1^2 + \left(-\tfrac{b}{2} + \tfrac{c}{2}\cos\delta\right)B_1B_3 \\ &+ \left(2 - \tfrac{a}{2} + \tfrac{c}{4}\cos\delta\right)B_2^2 - \tfrac{b}{2}B_2B_4 \\ &+ \left(\tfrac{9}{2} - \tfrac{a}{2}\right)B_3^2 + \left(8 - \tfrac{a}{2}\right)B_4^2 \Big] . \end{aligned}$$

The resulting determanental equation may be factored to read

$$\begin{vmatrix} -2a & -b & -c\cos\delta & 0 & c\sin\delta \\ -b & 4 - a - \frac{c}{2}\cos\delta & -b/2 & \frac{c}{2}\sin\delta & 0 \\ -c\cos\delta & -b/2 & 16 - a & 0 & 0 \\ 0 & \frac{c}{2}\sin\delta & 0 & 4 - a + \frac{c}{2}\cos\delta & -b/2 \\ c\sin\delta & 0 & 0 & -b/2 & 16 - a \end{vmatrix} \times$$

$$\begin{vmatrix} 1 - a - \frac{b}{2} & -\frac{b}{2} - \frac{c}{2}\cos\delta & 0 & \frac{c}{2}\sin\delta \\ -\frac{b}{2} - \frac{c}{2}\cos\delta & 9 - a & \frac{c}{2}\sin\delta & 0 \\ 0 & \frac{c}{2}\cos\delta & 1 - a + \frac{b}{2} & -\frac{b}{2} + \frac{c}{2}\cos\delta \\ \frac{c}{2}\sin\delta & 0 & -\frac{b}{2} + \frac{c}{2}\cos\delta & 9 - a \end{vmatrix} = 0 .$$

and is seen to reduce to the previous result if δ is set equal to zero. Vanishing of the first determanent would permit one to obtain ratios of non-vanishing coefficients A_0, A_2, A_4, B_2, B_4, corresponding to a solution of period π, and the vanishing of the second permit an independent similar determination of A_1, A_3, B_1, B_3, corresponding to a solution of period 2π.

With regard to the 5×5 determanent, it has been noted that it will factor when $\delta = 0$ to give the earlier result. If $\delta \neq 0$, the determanent may be expanded as a sum of 3×3 minors and their associated 2×2 cofactors to give a correction of order $c^2\delta^2$ to the original 3×3 determanent. In addition, it is to be noted that the original 3×3 determanent is itself modified by a term of order $c\delta^2$; a rough numerical check seems to indicate that this latter effect is somewhat the greater and would result (as might be expected) in bringing together the estimates of the first stability boundary for the two cases $b \gtrless 0$. With the present value of δ, however, the change of "a" is believed to be small -- perhaps of the order of ± 0.003 -- and a direct revaluation has not been undertaken.

With regard to the 4×4 determanent associated with the next stability limit a similar situation is seen to apply. Expansion in a series of products of 2×2 determanents and adjustment of the original 2×2 determanant to take account of $\cos\delta \neq 1$ is seen once again to introduce corrections of the order of δ^2.

6. References

1. W. Ritz, J. für reine u. angew. Math. CXXXV, 1-61 (1909). Cited in Courant-Hilbert[2], I, 157.

2. R. Courant u. D. Hilbert, Meth. der Math. Physik I (Springer, Berlin, 1924).

3. E. T. Whittaker and E. N. Watson, Modern Analysis (Cambridge University Press, 1927), Sect. 19.3.

4. f. ex. NBS Computation Laboratory "Tables Relating to Mathieu Functions" (Columbia University Press, N.Y., 1951).

5. For further discussion of these general methods see N. Kryloff, Mem. des Sciences Math. 49. See also comments and further references in Bateman's[6] introduction.

6. H. Bateman, Partial Differential Equations (Cambridge University Press, 1932; Dover, N.Y., 1944).

7. K. R. Symon, KRS(MURA)-1, -4 (1954).

8. Work in progress, summarized in letters to Dr. Kerst dated 21, 23, 25 January 1955, and based on the original model proposed by Kerst [cf. MURA-DWK/KMT/LWJ/KRS - 3 and MURA-DWK-7].

MU-RA "NOTES" [Treble Clef]