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Solution of the Majorana equation and its physical interpretation

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Abstract This paper deals with the solution of the Majorana equation (ME) and its physical interpretation. A tutorial section on the standard Dirac equation presents the basic material required in the subsequent discussion of the ME. Its related purely imaginary gamma matrices and the associated generators of the Lorentz group are presented. Then, the solution of the real part of the ME is derived. Yet, the spinorial Lorentz transformation must also be considered and turns out to be the key for obtaining the complex spinor solution of the complete ME, in which the polarization matrix plays a prominent role. The associated complex spinor basis is derived, which encompasses the particle/antiparticle and spin-up/down duality. The symmetries of the Majorana equation are finally discussed together with the CPT theorem. In conclusion, the Majorana equation is fully equivalent to the usual Dirac equation. The physical consequences are discussed in the final section.

1 Introduction

In the early thirties of the past century, a purely real representation of the Dirac equation (DE) [1] was found by the young Italian physicist Majorana [2]. The physical meaning of it was then and later strongly debated [3], because that equation while being real seemed to offer mathematical solutions for the relativistic quantum kinetics of chargeless neutrinos. Yet, since then, fundamental theoretical questions remained unsettled, such as whether Majorana particles are their own antiparticles, and many new experimental results have been obtained.

Of course, for the physics student and seasoned researcher alike many of the relevant theoretical issues are dealt with in depth in textbooks of quantum-field theory (QFT) and on the standard model (SM) such as the ones by Kaku [4] and Peskin and Schroeder [5], or the more recent modern one by Schwartz [6]. In their special book, Fukugita and Yanagida [7] have treated the most important aspects of neutrino physics that were of main interest at the turn of the century. They discussed exhaustively the theory as well as applications in nuclear, particle and astrophysics.

The Dirac equation is fundamental in all of this; however, the physical nature of the neutrinos involved remains unclear. They were often described by the massless Weyl equations involving only two-component Pauli spinors. In his paper, Case [8] derived the Majorana equation completely on its own rather than as an afterthought when treating the Dirac equation, as he then phrased it, when he formulated more than seventy years ago his theory.

A more recent series of papers by Pal [9] and Marsch [10–12] obtained and discussed two-component Majorana equations as derived completely on their own rather than as spin-off of the DE. As reviewed by Aste [13], the Majorana formalism, which can describe also massive neutral fermions by the help of complex two-component spinors, was then argued to be of fundamental importance for the understanding of the mathematical aspects of supersymmetric and other extensions of the SM, which are expected to play an important role in the LHC era. In their comprehensive review, Elliot and Franz [14] describe also the role Majorana fermions play nowadays even in several fields of condensed-matter physics.

Since convincing empirical evidence for finite neutrino masses, most strikingly by means of neutrino oscillations, has been found in the past few decades, nowadays weakly massive neutrinos have to be accepted as physical reality. Neutrino oscillations [15] were discovered through the study of atmospheric neutrinos, which are produced as decay products in hadronic showers resulting from collisions of cosmic rays with atomic nuclei in the Earth's atmosphere. Electron and muon neutrinos are produced mainly by the decay chains of charged pions in particle accelerators.

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McDonald [16] provided a comprehensive review of the measurements of solar neutrinos. The experiments to date have provided a clear indication that solar neutrinos are undergoing flavor transformation and that the dominant mechanism for this is neutrino oscillation. These measurements also provide strong confirmation of the model calculations of the solar interior.

Neutrinos are very special particles [18]. The experimental and theoretical studies of them have played a crucial role in our understanding of elementary particles and their interactions. In the past century, there was no evidence that neutrinos have masses, and therefore, the SM of particle physics assumed at the outset that they are massless, like all the other fermions involved. However, quite relatively small yet nonzero masses of all three types of neutrinos have been discovered. These small neutrino masses have profound implications for particle physics and the evolution of the universe. A detailed comparison of best-fit values and uncertainties of various neutrino-mass measurements are provided in the recent paper of the KATRIN collaboration [17].

The review paper of Kajita [18] discussed in detail the discovery of the neutrino masses. It is now cogent that mass is included in the Majorana equation. Moreover, as the neutrinos in the SM carry also the weak charge, they interact with the Z boson and cannot be considered as neutral. Marsch and Narita recently suggested a mechanism other than the Higgs mechanism for how minimal masses can be acquired by the boson Z-field energy in the rest frame of any SM fermion. Even if neutrinos have zero electric charge and carry no strong charge, they must be considered as weakly charged massive fermions.

In this paper, we will concentrate once again on the mathematical solution of the Majorana equation (ME) in connection with the DE and show that it, when being fully evaluated, also describes just as the standard Dirac equation an elementary fermion with its four kinetic particle/antiparticle and spin-up/down degrees of freedom. Thus, we hope to give new convincing physical answers to some of the questions raised in the past and therewith clarify the physical significance and relevance of the Majorana version of the DE.

The content of this paper is organized in the following way. The physics of spin is presented briefly, and the important notion of spin helicity [19–21] is introduced. A more tutorial section on the Dirac equation in the Weyl basis presents material needed subsequently in the discussion of the Majorana equation. The related purely imaginary gamma matrices and the associated generators of the Lorentz group are presented. Then, the solution of the real Majorana equation is derived, and the importance of the Lorentz transformation is emphasized for obtaining the full complex spinor solution of the Majorana equation, in which the polarization matrix plays a key role. The complex spinor basis is derived. The Majorana spinor quantum field and the symmetries of the Majorana equation are finally discussed together with the CPT theorem. In the end, a short summary and the conclusions are given.

2 Spin one-half and its algebra

The spin is a fundamental quantity of quantum mechanics, and the spin of one-half a genuine trait of an elementary fermion in quantum-field theory. Spin may be generally considered as a kind of intrinsic rotation, which is described by the complex three-vector operator $\mathbf{S}$ having non-commuting components. It obeys the commutator relation (we use the units of $\hbar = c = 1$)

$$\mathbf{S} \times \mathbf{S} = \mathrm{i}\mathbf{S}. \tag{1}$$

The spin one-half was introduced by Pauli [22] into the Schrödinger equation in order to describe the magnetic moment of the electron. For that spin, we have $\mathbf{S} = 1/2\boldsymbol{\sigma}$ in terms of the Pauli matrix vector, which reads explicitly in standard form

$$\boldsymbol{\sigma} = (\sigma_\mathrm{x}, \sigma_\mathrm{y}, \sigma_\mathrm{z}) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \tag{2}$$

These matrices are the three generators of the Lie group $SU(2)$ that is isomorphic to the group $SO(3)$ of rotations in real Euclidean space $\mathbb{R}^3$. Their triple product yields $\sigma_\mathrm{x}\sigma_\mathrm{y}\sigma_\mathrm{z} = \mathrm{i}1_2$. Let $\mathbf{v}$ be any three-vector, then the quantity $(\boldsymbol{\sigma} \cdot \mathbf{v})$ is called its helicity. Since $(\boldsymbol{\sigma} \cdot \mathbf{v})^2 = 1_2\, \mathbf{v} \cdot \mathbf{v}$, this relation can be employed to replace the scalar vector product of $\mathbf{v}$ with itself by the square of the helicity matrix. This method, sometimes called spinor-helicity formalism [6], is exploiting the property of the spin one-half matrices, which define the algebra

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2 g_{ij} 1_2. \tag{3}$$

The symbol 1_n denotes the unit matrix of dimension n, and g_{ij} names here the related Euclidean metric in real space $\mathbb{R}^3$.

3 The Dirac equation and the spinor generators of the Lorentz Group

In their influential famous work, Wigner [23] and Bargman and Wigner [24] classified the relativistic states of particles with spin in terms of the two Casimir operators of the Poincaré and Lorentz group (LG) [25]. The first Casimir operator of a massive particle is its squared four-momentum that is equal to its mass squared, which is a relativistic invariant. The second is the first times the particle spin squared $\mathbf{S}^2$, which is a rotational invariant in real space. We then obtain in Fourier space with the four-momentum $p^{\mu} = (E, \mathbf{p})$ the result

$$p^{\mu} p_{\mu} \mathbf{S}^2 = m^2 s(s+1) 1_{2s+1}. \tag{4}$$

Here, s is the spin quantum number. For a fermion with spin $s = 1/2$ one gets [20], by help of the algebra (3) yielding the momentum helicity, the result

$$E^2 1_2 - (\boldsymbol{\sigma} \cdot \mathbf{p})^2 = m^2 1_2, \tag{5}$$

which can be reformulated by means of the Pauli matrices and put into the quadratic form

$$(\gamma_{\mu} p^{\mu})^2 = m^2, \tag{6}$$

where we introduced the famous gamma four-vector matrix, $\gamma^{\mu} = (\gamma_0, \boldsymbol{\gamma})$, of Dirac, reading in his and the Weyl basis as follows:

$$\begin{aligned} \gamma^{\mu}_{\mathrm{D}} &= (\sigma_{\mathrm{z}} \otimes 1_2, \mathrm{i}\sigma_{\mathrm{y}} \otimes \boldsymbol{\sigma}), \\ \gamma^{\mu}_{\mathrm{W}} &= (\sigma_{\mathrm{x}} \otimes 1_2, \mathrm{i}\sigma_{\mathrm{y}} \otimes \boldsymbol{\sigma}). \end{aligned} \tag{7}$$

Hereby, use was made of the so-called Kronecker or outer matrix product, which obeys the important product rule, $(A \otimes B)(C \otimes D) = (A\,B \otimes C\,D)$, and is used frequently in the following text. By permutation of the Pauli matrices on the left side of the Kronecker product, one can obtain four more forms of the gamma matrices [26], which yet are not in use, because the Weyl basis is most convenient.

We take now the square root of (6) and then replace the four-momentum p_{μ} by the related quantum mechanical operator $\hat{P}_{\mu} = \mathrm{i}\partial_{\mu} = \mathrm{i}(\partial_t, \partial_{\mathbf{x}})$. It acts on a four-component spinor field $\psi(x)$ with x being an abbreviation for $x^{\mu} = (t, \mathbf{x})$. Then, this yields the famous Dirac [1] equation

$$\gamma^{\mu} \mathrm{i}\partial_{\mu} \psi = m\psi. \tag{8}$$

The gamma matrices obey the Clifford algebra in the four-dimensional Minkowski spacetime, with the related metric $g^{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$, and anticommute in the following way

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} 1_4. \tag{9}$$

The gamma matrices in (7) reveal, on the right side of the Kronecker product by the appearance of the three-vector $\boldsymbol{\sigma}$, the important role played by the fermion spin. The sigma matrices on the left side are connected with the particle/antiparticle degrees of freedom. Through exploitation of the Clifford algebra, we obtain the spinorial rotation operator $\mathbf{J}$ and rapidity or boost operator $\mathbf{K}$. The three rotation vector components are defined as

$$J_{\mathrm{x}} = \frac{\mathrm{i}}{2} \gamma_{\mathrm{y}} \gamma_{\mathrm{z}}, \tag{10}$$

with cyclic permutation of the indices. The boost operator reads

$$\mathbf{K} = \frac{\mathrm{i}}{2} \gamma_0 \boldsymbol{\gamma}. \tag{11}$$

If we evaluate explicitly these expressions for the spinorial rotation and boost matrix vectors, we obtain for the Dirac equation in the Weyl basis the results

$$\mathbf{J} = \frac{1}{2} 1_2 \otimes \boldsymbol{\sigma}, \ \mathbf{K} = \frac{1}{2\mathrm{i}} \sigma_{\mathrm{z}} \otimes \boldsymbol{\sigma}. \tag{12}$$

These two matrix–vector operators are the spinorial generators (like their relatives [25] in Minkowski spacetime) of the spinor representation of the Lorentz Group. They obey the linked algebra associated with the LG as follows $\mathbf{J} \times \mathbf{J} = \mathrm{i}\mathbf{J}$, $\mathbf{J} \times \mathbf{K} = \mathrm{i}\mathbf{K}$, $\mathbf{K} \times \mathbf{J} = \mathrm{i}\mathbf{K}$, $\mathbf{K} \times \mathbf{K} = -\mathrm{i}\mathbf{J}$. With rotation angle vector $\boldsymbol{\theta}$ and the boost vector $\boldsymbol{\beta}$, we obtain the spinor Lorentz transformation

$$\Lambda = \exp(\mathrm{i}\boldsymbol{\theta} \cdot \mathbf{J} + \mathrm{i}\boldsymbol{\beta} \cdot \mathbf{K}). \tag{13}$$

The inverse transformation is given by $\Lambda^{-1} = \gamma_0 \Lambda^{\dagger} \gamma_0$. This is a consequence of the definition of the rotation and boost matrix vectors given in (10) and (11), which yield $[\gamma_0, \mathbf{J}] = 0$, and in contrast $\{\gamma_0, \mathbf{K}\} = 0$. Yet, both vectors do not commute with $\boldsymbol{\gamma}$!

This concise tutorial section on the standard Dirac equation and its gamma matrices illustrates that, because of their algebraic form and fully complex nature, they describe a particle/antiparticle pair and separately the spin one-half dependence on the matrix three-vector $\boldsymbol{\sigma}$.

4 The purely imaginary Majorana matrices

As mentioned already in Introduction, the purely imaginary Majorana [2] representation of the Dirac equation caused historically speaking quite a bit of excitement and irritation when it was detected a few years after Dirac established his novel relativistic equation for the electron.

We may write the Majorana gamma matrices after [2, 4, 6] in the Dirac and Weyl basis in the following forms

$$\gamma^{\mu}_{\rm MD} = (\sigma_{\rm z} \otimes \sigma_{\rm y}, 1_2 \otimes {\rm i}\sigma_{\rm z}, \sigma_{\rm y} \otimes {\rm i}\sigma_{\rm y}, -1_2 \otimes {\rm i}\sigma_{\rm x})$$
$$\gamma^{\mu}_{\rm MW} = (\sigma_{\rm x} \otimes \sigma_{\rm y}, 1_2 \otimes {\rm i}\sigma_{\rm z}, \sigma_{\rm y} \otimes {\rm i}\sigma_{\rm y}, -1_2 \otimes {\rm i}\sigma_{\rm x}). \tag{14}$$

These gamma matrices are obviously purely imaginary and their Clifford algebra follows readily from that of the Pauli matrices given in (3). They are connected to the standard gammas of (7) by a unitary transformation after Aste [13]. The chiral matrix γ_5 is conventionally defined as fourfold product $\gamma_5 = {\rm i}\gamma_0\gamma_{\rm x}\gamma_{\rm y}\gamma_{\rm z}$, and thus anticommutes with the other four. This multiplication results in $\gamma^5_{\rm MW} = \sigma_{\rm z} \otimes \sigma_{\rm y}$ in the Weyl and $\gamma^5_{\rm MD} = -\sigma_{\rm x} \otimes \sigma_{\rm y}$ in the Dirac representation. We continue by considering just the Majorana–Weyl representation and omit the MW subscript.

After the general equation (10), the spinorial rotation, or shortly the spin operator, in the Majorana–Weyl representation reads

$$\mathbf{J} = \frac{1}{2}(\sigma_{\rm y} \otimes \sigma_{\rm z}, 1_2 \otimes \sigma_{\rm y}, -\sigma_{\rm y} \otimes \sigma_{\rm x}). \tag{15}$$

After the general equation (10), the spinorial rapidity, or shortly the boost operator, in the Majorana–Weyl representation reads

$$\mathbf{K} = \frac{1}{2{\rm i}}(\sigma_{\rm x} \otimes \sigma_{\rm x}, \sigma_{\rm z} \otimes 1_2, \sigma_{\rm x} \otimes \sigma_{\rm z}). \tag{16}$$

In conclusion of this section, we find that all relevant 4×4 matrices in the Majorana representation are purely imaginary, in particular the spin vector $\mathbf{J}$ and boost vector $\mathbf{K}$. Therewith, also the Lorentz algebra consists of purely imaginary matrices and the resulting Lorentz transformation (13) is real, like it is for the spacetime coordinates x^{μ}. Note that the chiral operator matrix γ_5 commutes by definition with the spin and boost vector operators and thus with the Lorentz transformation, i.e., $[\Lambda, \gamma_5] = 0$, which does therefore not change the chirality of a spinor.

5 The real Majorana equation and its spinor-field solutions

We return now to the Dirac equation (14) and insert into it the Majorana gamma matrices in the Weyl representation. We write $\gamma^{\mu}_{\rm M} = {\rm i}\delta_{\mu}$ and then obtain the real delta matrices

$$\delta^{\mu} = (\delta_0, \boldsymbol{\delta}) = (\sigma_{\rm z} \otimes (-{\rm i}\sigma_{\rm y}), 1_2 \otimes \sigma_{\rm z}, \sigma_{\rm y} \otimes \sigma_{\rm y}, -1_2 \otimes \sigma_{\rm x}), \tag{17}$$

obeying the modified Clifford algebra

$$\delta^{\mu}\delta^{\nu} + \delta^{\nu}\delta^{\mu} = -2g^{\mu\nu}1_4, \tag{18}$$

which implies that $\delta_0^2 = -1_4$ and $\delta_j^2 = 1_4$. Using these delta matrices results in the real Dirac, i.e., the Majorana equation that reads

$$(\delta^{\mu}\partial_{\mu} + m1_4)\varphi = 0. \tag{19}$$

Concerning the spacetime dependence of the vector function φ we look for solutions that depend, with $x^{\mu} = (t, \mathbf{x})$, on the Lorentz-invariant phase function

$$\phi(t, \mathbf{x}) = x^{\mu}p_{\mu} = Et - \mathbf{p} \cdot \mathbf{x}. \tag{20}$$

Then, $\delta^{\mu}\partial_{\mu}\phi = \delta^{\mu}p_{\mu} = \delta_0 E - \boldsymbol{\delta} \cdot \mathbf{p}$. Since the Majorana equation is a real linear differential equation, we make a solution ansatz involving the sine and cosine trigonometric function of the argument ϕ. The real Majorana equation is a matrix equation in the real vector space $\mathbb{R}^4$ with the standard orthogonal basis (written here as transposed vectors): $\varphi_1^T = (1, 0, 0, 0)$, $\varphi_2^T = (0, 1, 0, 0)$, $\varphi_3^T = (0, 0, 1, 0)$, $\varphi_4^T = (0, 0, 0, 1)$. However, as inspection of the Lorentz transformation (13) involving the spin and boost matrices given in Appendix shows, this real basis turns out to be not appropriate as it is not Lorentz invariant. Thus, we require complex basis vectors, which are derived later in Sect. 7. With the proper ansatz ($a, b \in \mathbb{C}^4$)

$$\varphi(t, \mathbf{x}) = a\cos(\phi(t, \mathbf{x})) + b\sin(\phi(t, \mathbf{x})), \tag{21}$$

we obtain the two complex equations

$$\delta^{\mu}p_{\mu}a - m1_4 b = 0, \quad \delta^{\mu}p_{\mu}b + m1_4 a = 0. \tag{22}$$

Inserting b into the equation for a, we obtain as requirement for non-trivial solutions that $(\delta^\mu p_\mu)^2 + m^2 1_4 = 0$, which yields with $(\delta^\mu p_\mu)^2 = (\delta_0 E - \boldsymbol{\delta} \cdot \mathbf{p})^2 = (-E^2 + \mathbf{p}^2)1_4$ the wave dispersion relation

$$E(p) = \sqrt{m^2 + p^2}. \tag{23}$$

The non-physical negative root will be disregarded. Thus, we obtain, with the new phase function $\phi(t, \mathbf{x}) = x^\mu p_\mu = E(p)t - \mathbf{p} \cdot \mathbf{x}$, the two possible solutions

$$\varphi_a(t, \mathbf{x}) = \left(1_4 \cos(\phi(t, \mathbf{x})) + \frac{1}{m}\delta^\mu p_\mu \sin(\phi(t, \mathbf{x}))\right)a. \tag{24}$$

$$\varphi_b(t, \mathbf{x}) = \left(1_4 \sin(\phi(t, \mathbf{x})) - \frac{1}{m}\delta^\mu p_\mu \cos(\phi(t, \mathbf{x}))\right)b. \tag{25}$$

The standard inner vector product of v in the real space $\mathbb{R}^4$ is defined as $v \cdot v = v^T v = \sum_{j=1}^4 v_j v_j$. However, at this point we need to recall that Lorentz invariance requires to employ an appropriate inner product for the complex vectors. Following standard definitions [6], we define the conjugated vector as $\bar{a} = (\delta_0 a)^\dagger = a^\dagger \delta_0^\dagger = -a^\dagger \delta_0$. Subsequently, we make use of the fact that $(\delta_0 \delta^\mu a)^\dagger = a^\dagger (\delta^\mu)^T \delta_0^T = a^\dagger \delta_0 (\delta_0 (\delta^\mu)^T \delta_0) = -\bar{a}\delta^\mu$. Using this property of the delta matrices, one can show that the above solutions can in a Lorentz-invariant way be made orthogonal.

We emphasize that these solutions are rather general for arbitrary complex vectors a and b, a result which is a little irritating. Namely, choosing any of the four basis vectors, φ_j, we would obtain four solutions of the real Majorana equation, yet their individual physical meaning remains unclear. However, for the standard Dirac equation one obtains the duality of the particle/antiparticle and spin-up/down degrees of freedom, which should not get lost by the possible and permissible choice of purely imaginary gamma matrices. To see that this is indeed the case, we express the sine and cosine functions in (24) and (25) by the defining exponentials. Then, we obtain for the two fields $\varphi_{a,b}$ the similar result

$$\varphi_a(t, \mathbf{x}) = (P_+ \exp(-\mathrm{i}\phi(t, \mathbf{x}) + P_- \exp(\mathrm{i}\phi(t, \mathbf{x}))a. \tag{26}$$

$$\varphi_b(t, \mathbf{x}) = \mathrm{i}(P_+ \exp(-\mathrm{i}\phi(t, \mathbf{x}) - P_- \exp(\mathrm{i}\phi(t, \mathbf{x}))b. \tag{27}$$

Here, we introduced two idempotent complex projection matrices as follows:

$$P_\pm = \frac{1}{2}\left(1_4 \pm \frac{\mathrm{i}}{m}\delta^\mu p_\mu\right) = \frac{1}{2m}\left(m 1_4 \pm \gamma^\mu_{\mathrm{M}} p_\mu\right), \tag{28}$$

which obey the subsequent relations

$$P_\pm^2 = P_\pm;\ P_\pm^* = P_\mp;\ P_+ + P_- = 1_4;\ P_+ P_- = 0. \tag{29}$$

Note that both spinor functions (26) and (27) associated with a and b are complex. We can linearly add them up on an equal footing, reading $\varphi = \varphi_a + \varphi_b$. Then, we get

$$\varphi(t, \mathbf{x}) = P_+ c_+ \exp(-\mathrm{i}\phi(t, \mathbf{x})) + P_- c_- \exp(\mathrm{i}\phi(t, \mathbf{x})) \tag{30}$$

The two new complex vectors read $c_\pm = (a \pm \mathrm{i}b)$. Yet, we recall that $P_\pm^* = P_\mp$, and therefore, the solution (30) was still real if $c_\pm$ were complex-conjugated, i.e., $c_\pm^* = c_\mp$, implying that a and b are real. In the jargon used to classify the solutions of the standard Dirac equation in the Weyl basis, we then see that the solution (30) is a superposition of the usual particle and antiparticle type. However, we still have the freedom to choose the vectors a and b. What physical arguments determine their properties?

6 Helicity and polarization matrix

In the section on the standard Dirac equation, see the equations (7), (8) and (12), we have shown that the momentum spin helicity $\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}$, with $\hat{\mathbf{p}} = \mathbf{p}/p$ plays a key role in the classification of the particle's degrees of freedom. This is so because the momentum helicities of both $\mathbf{J} \cdot \hat{\mathbf{p}}$ and the rapidity $\boldsymbol{\gamma} \cdot \hat{\mathbf{p}}$ do commute, despite the fact that $\mathbf{J}$ and $\boldsymbol{\gamma}$ themselves do not. The comparison of equations (14) and (15) reveals that this general property of helicity is also true for the Majorana matrices, which have the form $\gamma^\mu = \mathrm{i}\delta^\mu$. However, in the case of the Majorana equation the momentum spin helicity is given by $\boldsymbol{\Sigma} \cdot \hat{\mathbf{p}}$, with the three 4×4 spin matrices presented in Appendix. We use the notation that $\hat{\mathbf{p}} = (x, y, z)$ with $x^2 + y^2 + z^2 = 1$. Then, the two helicities of interest here read in their matrix forms

$$h_s = \mathrm{i}\begin{pmatrix} 0 & -y & -x & z \\ y & 0 & z & x \\ x & -z & 0 & -y \\ -z & -x & y & 0 \end{pmatrix},\ h_\delta = \begin{pmatrix} x & -z & 0 & -y \\ -z & -x & y & 0 \\ 0 & y & x & -z \\ -y & 0 & -z & -x \end{pmatrix}. \tag{31}$$

Their squares are identical to the unit matrix: $h_s^2 = h_\delta^2 = 1_4$. Therefore, both have the eigenvalues ± 1 corresponding to two types of eigenfunctions. We shall continue with using the real kinetic helicity matrix $h_\delta = \boldsymbol{\delta} \cdot \hat{\mathbf{p}}$ that equals its transposed. Since we are in search for a real solution of the Majorana equation, we should not make use of the purely imaginary spin operator $\mathbf{J}$, because it can, according to the basic equation (1), not be constructed without use of the imaginary unit. Similarly, the boost operator $\mathbf{K}$ is purely imaginary as well, since δ_0 and $\boldsymbol{\delta}$ are real. Consequently, the Lorentz transformation (13) for the real vector functions $\varphi_{a,b}$ given in (24) and (25) is also real, like it is for simple coordinate transformation of $x^\mu = (t, \mathbf{x})$.

Given all these arguments, we would then prefer to use the kinetic helicity as it appears in the term $\delta^\mu p_\mu$. Thus, we obtain the eigenvalue equation

$$(\delta^\mu p_\mu)\chi(\hat{\mathbf{p}}) = (\delta_0 E(p) - p\, h_\delta)\chi(\hat{\mathbf{p}}) = P\chi(\hat{\mathbf{p}}). \tag{32}$$

The associated polarization matrix $P = P(E(p), \mathbf{p})$ is defined as

$$P = \begin{pmatrix} -p_x & p_z - E(p) & 0 & p_y \\ p_z + E(p) & p_x & -p_y & 0 \\ 0 & -p_y & -p_x & p_z + E(p) \\ p_y & 0 & p_z - E(p) & p_x \end{pmatrix}. \tag{33}$$

Squaring P yields $-m^2 1_4$, and thus $P^2/m^2 = -1_4$. Therefore, the eigenfunction of P cannot be real. We recall that $(\delta_0 \delta^\mu \delta_0)^T = \delta^\mu$. Therefore, we have $\delta_0 P^T \delta_0 = P$, a property going to be used for the Lorentz transformation of this matrix. We can then rewrite the previous solutions as

$$\varphi_a(t, \mathbf{x}) = \left(1_4 \cos(\phi(t, \mathbf{x})) + \frac{P}{m}\sin(\phi(t, \mathbf{x}))\right)a. \tag{34}$$

$$\varphi_b(t, \mathbf{x}) = \left(1_4 \sin(\phi(t, \mathbf{x})) - \frac{P}{m}\cos(\phi(t, \mathbf{x}))\right)b. \tag{35}$$

We stress that these two solutions are fully equivalent, as they only differ by a constant phase in the trigonometric functions, since $\cos(\phi - \pi/2) = \sin(\phi)$ and $\sin(\phi - \pi/2) = -\cos(\phi)$. So either solution a or b fully exhausts the set of possible Majorana spinor fields. We can also calculate the conjugated solutions as

$$\overline{\varphi_a}(t, \mathbf{x}) = \bar{a}\left(1_4 \cos(\phi(t, \mathbf{x})) - \frac{P}{m}\sin(\phi(t, \mathbf{x}))\right). \tag{36}$$

$$\overline{\varphi_b}(t, \mathbf{x}) = \bar{b}\left(1_4 \sin(\phi(t, \mathbf{x})) + \frac{P}{m}\cos(\phi(t, \mathbf{x}))\right). \tag{37}$$

Here, we used that $\bar{a} = -a^\dagger \delta_0$ and that $\overline{Pa} = a^\dagger P^T \delta_0^T = -\bar{a}(\delta_0 P^T \delta_0) = -\bar{a}P$. Therefore, $\overline{\varphi_a} \cdot \varphi_a = \bar{a} \cdot a$, and similarly for b. The solutions would be simplified, if one could chose a and b to be the eigenvectors of P, which we already named above $\chi(\mathbf{p})$. However, as $P^2 = -m^2 1_4$, its eigenvalues are imaginary, and the related eigenvectors are complex and thus not useful for our aim to find the most transparent complex solutions of the Majorana equation.

But it is clear that these cannot be real owing to the requirement of Lorentz invariance, although the spacetime-dependent matrix parts of the spinor fields (34) and (35) are real. They only depend on x^μ and p^μ and are the same for the Lorentz-invariant basis spinors a or b. Yet as shown below, they must be complex and are defined in the subsequent section.

7 The complex Majorana basis vectors

Inspection of the real Majorana equation (19) reveals that for $\mathbf{p} = 0$ in the particle's rest frame the only relevant gamma matrix is δ_0, which determines the energy in that frame as a function of the mass m. Although not being apparent, the fermion spin $\mathbf{J}$ must also be considered. Its orientation in the rest frame is arbitrary, but inspection of (15) suggests that the y component is most convenient. Therefore, we provide here the four eigenvectors of the spin component $J_y = \frac{1}{2}\Sigma_y$ and of $\gamma_0 = \mathrm{i}\delta_0$, which commute with each other. As $\Sigma_y = 1_2 \otimes \sigma_y$ and $\gamma_0 = \sigma_x \otimes \sigma_y$ in the Weyl basis, we need the eigenfunctions of the x and y components of $\boldsymbol{\sigma}$. One obtains

$$\sigma_y \phi_\pm = \pm\phi_\pm, \quad \phi_+ = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \quad \phi_- = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}. \tag{38}$$

$$\sigma_x \chi_\pm = \pm\chi_\pm, \quad \chi_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{39}$$

Using these vectors, we obtain with the definitions $w_1 = \chi_+ \otimes \phi_+$, $w_2 = \chi_+ \otimes \phi_-$, $w_3 = \chi_- \otimes \phi_+$, $w_4 = \chi_- \otimes \phi_-$, the results for the orthogonal complex eigenvectors in $\mathbb{C}^4$ in the following form

$$w_1 = \begin{pmatrix} 1 \\ \mathrm{i} \\ 1 \\ \mathrm{i} \end{pmatrix}, w_2 = \begin{pmatrix} \mathrm{i} \\ 1 \\ \mathrm{i} \\ 1 \end{pmatrix}, w_3 = \begin{pmatrix} 1 \\ \mathrm{i} \\ -1 \\ -\mathrm{i} \end{pmatrix}, w_4 = \begin{pmatrix} \mathrm{i} \\ 1 \\ -\mathrm{i} \\ -1 \end{pmatrix}. \tag{40}$$

Using these eigenfunctions, we obtain $\Sigma_y w_{1,3} = w_{1,3}$, and $\Sigma_y w_{2,4} = -w_{2,4}$, corresponding to spin-down and spin-up. Similarly, we obtain $\gamma_0 w_{1,4} = w_{1,4}$ and $\gamma_0 w_{2,3} = -w_{2,3}$, in association with the particle, respectively, the antiparticle, which is in accord with the usual definition of the Lorentz-invariant inner product, $\bar{w}_{1,4} \cdot w_{1,4} = 1$ for the particle and $\bar{w}_{2,3} \cdot w_{2,3} = -1$ for the antiparticle.

In terms of these four basis spinors (that are just complex vectors in $\mathbb{C}^4$), and considering now the solution a only, we obtain the general free Majorana spinor fields from the four basis states as follows:

$$\varphi^u_{\uparrow,\downarrow}(t,\mathbf{x}) = M(t,\mathbf{x})u_{\uparrow,\downarrow},\ \varphi^v_{\uparrow,\downarrow}(t,\mathbf{x}) = M(t,\mathbf{x})v_{\uparrow,\downarrow}, \tag{41}$$

with the same mixing or spacetime-dependent rotation matrix for particles and antiparticles, reading

$$M(t,\mathbf{x}) = \left(1_4 \cos(\phi(t,\mathbf{x})) + \frac{P}{m}\sin(\phi(t,\mathbf{x}))\right), \tag{42}$$

which involves the polarization matrix $P(p^\mu)$ according to equation (33). The conjugated matrix $\overline{M}$ is obtained by replacing P by $-P$ in (42) according to the result given in (36). Here, we used conventional notation for the four vectors (u for particle with spin-up and spin-down and v for antiparticle with spin-up and spin-down), which together form an orthonormal basis in $\mathbb{C}^4$. Their definitions are

$$u_\uparrow = w_1/2,\ u_\downarrow = w_4/2,\ v_\uparrow = w_3/2,\ v_\downarrow = w_2/2. \tag{43}$$

Their conjugate spinors are

$$\bar{u}_\uparrow = \bar{w}_1/2,\ \bar{u}_\downarrow = \bar{w}_4/2,\ \bar{v}_\uparrow = \bar{w}_3/2,\ \bar{v}_\downarrow = \bar{w}_2/2, \tag{44}$$

whereby the conjugate basis spinors are given by

$$\begin{aligned}\bar{w}_1 = (1, -\mathrm{i}, 1, -\mathrm{i}),\ \bar{w}_2 = (\mathrm{i}, -1, \mathrm{i}, -1),\\ \bar{w}_3 = (-1, \mathrm{i}, 1, -\mathrm{i}),\ \bar{w}_4 = (-\mathrm{i}, 1, \mathrm{i}, -1).\end{aligned} \tag{45}$$

The particle and antiparticle states are always orthogonal, as well as the spin-up and spin-down states. Both the adjectives mixing and rotation are appropriate to characterize P, as, for example, the inner product of $\chi = Mu_\uparrow$ and $\overline{\chi} = \overline{u_\uparrow}\overline{M}$ yields $\overline{u}_\uparrow \overline{M} M u_\uparrow = \overline{u}_\uparrow u_\uparrow = 1$. So the modulus of each spinor remains unchanged during propagation of the fermion, and all four basis states are in an equal fashion rotated yet not mixed among themselves.

Finally, it is interesting to calculate the expectation values of the polarization matrix P in terms of the basis spinors. We obtain then

$$\bar{u}_{\uparrow,\downarrow} P u_{\uparrow,\downarrow} = 0,\ \bar{v}_{\uparrow,\downarrow} P v_{\uparrow,\downarrow} = 0. \tag{46}$$

We may also now evaluate the expectation value of the four-momentum operator $\hat{P}_\mu = \mathrm{i}\partial_\mu = \mathrm{i}(\partial_t, \partial_\mathbf{x})$ in the Majorana field χ after (41), for example, for the spin-up particle, and then obtain

$$\left\langle \hat{P}_\mu \right\rangle = \overline{\chi}\mathrm{i}\partial_\mu \chi = \overline{u}_\uparrow \overline{M}\mathrm{i}\partial_\mu M u_\uparrow = \frac{p_\mu}{m}\overline{u}_\uparrow P u_\uparrow = 0. \tag{47}$$

Thus, the expectation value of the four-momentum of the Majorana spinor field is zero for any particle/antiparticle and spin-up/down state.

8 The Majorana spinor quantum field

So far we have discussed the Majorana spinor field. Here, we shall briefly address the quantum field as well. Essentially, we have already shown that the Majorana equation is fully equivalent to the standard Dirac equation that is mostly written in the convenient Weyl basis. Equations (41) and (42) are less appropriate for the present purpose, because they emphasize the real character of the spacetime part of the solution, whereas merely the spinor part is complex. Yet, we already reformulated that solution in terms of standard plane waves in (30).

For the transition to a quantum field we have to multiply them by amplitudes that become creation ($b^\dagger_{\mathbf{p},s}$) and destruction ($a_{\mathbf{p},s}$) operators in the quantum-field parlance for states with momentum $\mathbf{p}$ and spin s. We define the subsequent orthogonal polarization vectors by employing the eigenvectors of (43) and (44) with the spin index s and using the conventional u and v naming, whereby $u = c_+$, and $v = c_-$. Thus, we obtain the polarization spinors

$$U_s(p_\mu) = P_+(p_\mu)u_s,\ V_s(p_\mu) = P_-(p_\mu)v_s. \tag{48}$$

We recall that $\overline{u}_s u_{s'} = \delta_{s,s'}$, $\overline{v}_s v_{s'} = -\delta_{s,s'}$, and $\overline{u}_s v_{s'} = 0 = \overline{v}_s u_{s'}$, and that $P_+P_- = 0$. Moreover, $\overline{P_\pm} = P_\pm$. Therefore, the orthogonality relations are transferred to the capital U and V.

$$\varphi(t,\mathbf{x}) = \sum_{\mathbf{p},s}\left(a_{\mathbf{p},s} U_s \exp(-\mathrm{i}\phi(t,\mathbf{x})) + b^\dagger_{\mathbf{p},s} V_s \exp(\mathrm{i}\phi(t,\mathbf{x})\right). \tag{49}$$

Here, we summed over all possible spin and momentum states. This expression is nothing but the standard version of the spin one-half massive fermion quantum field of Dirac. The consequences, for example, for causality and stability of that spinor field φ, are lucidly discussed in the context of spin and statistics in the modern textbook of Schwartz [6].

9 Symmetries and CPT theorem

This short final section deals with the symmetries of the Majorana equation and the CPT theorem resulting from Lorentz invariance. We return to the real Majorana equation (19) for a massive fermion and repeat it here again

$$(\delta^{\mu}\partial_{\mu} + m\mathbb{1}_4)\varphi = 0. \tag{50}$$

Squaring it and using the Clifford algebra (18) of the deltas yields the Klein–Gordon (KG) equation that reads

$$(\delta^{\mu}\partial_{\mu})^2\varphi = (-(\frac{\partial}{\partial t})^2 + (\frac{\partial}{\partial \mathbf{x}})^2)\varphi = \frac{1}{\lambda^2}\varphi. \tag{51}$$

Here, the squared, positive definite Compton wavelength ($\lambda = \hbar/(mc)$) appears. We introduce the operator of complex conjugation, $\mathtt{C}z = z^*$, for any complex number $z = x + \mathrm{i}y$, with $z^* = x - \mathrm{i}y$. Moreover, we have the operators of space inversion, i.e., parity, $\mathtt{P}\mathbf{x} = -\mathbf{x}$ and time inversion, $\mathtt{T}t = -t$. Inspection of the KG equation shows, that it is invariant against $\mathtt{P}$ and $\mathtt{T}$, and trivially against $\mathtt{C}$, as long as φ is real.

However, the Majorana spinor equation (48), which is linear in t and $\mathbf{x}$, is apparently not invariant against spacetime inversions, and neither against complex conjugation, as the basis spinors (40) are complex. So one requires the delta matrices, which anticommute among each other, to compensate for the negative signs after spacetime inversion. An obvious and appropriate choice is to take the time inversion as $\mathcal{T} = \delta_0\delta_5\mathtt{T}$, the parity as $\mathcal{P} = \delta_0\mathtt{P}$, and the complex conjugation as $\mathcal{C} = \mathbb{1}_4\mathtt{C}$. The associated matrix inversions are $\mathtt{P}^{-1} = \delta^T = -\delta_0$, and $\mathtt{T}^{-1} = (\delta_0\delta_5)^T = \delta_5\delta_0$. We thus obtain the combined CPT symmetry operation

$$\mathcal{CPT} = -\delta_5\mathtt{CPT},\ \mathcal{CPT}\varphi(t,\mathbf{x}) = -\delta_5\varphi^*(-t,-\mathbf{x}). \tag{52}$$

Application of CPT on the spacetime part $M(t, \mathbf{x})$ (42) of the free Majorana spinor field (41) gives $[\mathcal{CPT}, M(t, \mathbf{x})] = 0$, because $\{P, \delta_5\} = 0$ according to the definition of P in (32). This CPT invariance is a consequence of the Majorana equation being real. However, CPT operates through δ_5 and complex conjugation $\mathtt{C}$ on the basis spinors (43) that appear in (41). They transform under $\delta_5\mathtt{C} = (-\mathrm{i}\sigma_\mathrm{z} \otimes \sigma_\mathrm{y})\mathtt{C}$ as follows:

$$u_{\uparrow} \mapsto v_{\downarrow},\ u_{\downarrow} \mapsto -v_{\uparrow},\ v_{\uparrow} \mapsto u_{\downarrow},\ v_{\downarrow} \mapsto -u_{\uparrow}. \tag{53}$$

In conclusion, the basis is reproduced (aside of the unimportant minus signs), whereby the spins are flipped and the particles changed into their antiparticles. These inversions are essentially caused by the fact that all Majorana spin and gamma matrices are purely imaginary.

10 Discussion and conclusion

In this paper, we have reconsidered the Majorana equation and derived its solution (42) in terms of a product with a real part in the form of what we called the spacetime rotation matrix $M(t, \mathbf{x})$ as given in (42) and a complex part that sums up the four basis spinors involved, which are the eigenvectors of the spin component Σ_y and the matrix $\gamma_0 = \mathrm{i}\delta_0$. Lorentz invariance requires complex spinors and thus purely real solutions are not possible. The real matrix $M(t, \mathbf{x})$ is a function of the Lorentz-invariant phase $\Phi(t, \mathbf{x}) = E(p)t - \mathbf{p}\cdot\mathbf{x}$, which appears as argument in the trigonometric sine and cosine function. The purely imaginary Majorana gamma matrices can be directly obtained after Aste [13] by a unitary transformation reading

$$U = U^{\dagger} = U^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_2 & \sigma_\mathrm{y} \\ \sigma_\mathrm{y} & -\mathbb{1}_2 \end{pmatrix}. \tag{54}$$

This transformation yields

$$\gamma^{\mu}_{\mathrm{MW}} = U\gamma^{\mu}_{\mathrm{D}}U,\ \gamma^{\mu}_{\mathrm{MD}} = U\gamma^{\mu}_{\mathrm{W}}U. \tag{55}$$

The corresponding gamma matrices were all given in equations (7) and (14). Application of U on the Dirac equation (8) then yields the connections between the spinor fields

$$\varphi_{\mathrm{MW}} = U\psi_{\mathrm{D}},\ \varphi_{\mathrm{MD}} = U\psi_{\mathrm{W}}. \tag{56}$$

Consequently, the Majorana spinor fields are fully equivalent to the standard Weyl and Dirac spinor fields. Thus as we concluded in the previous section on the quantum fields, all these versions are equivalent in their physical content, though they appear in various different shapes. Which one to use in practice is a matter of mathematical convenience, habit or tradition.

Let us consider also the complex-conjugated version of the Majorana equation in order to see whether a real version makes sense. For this purpose, we formally operate with the charge conjugation operator $\mathcal{C}$ on the Majorana spinor field (41), and thus, we obtain

$$(\varphi^{u}_{\uparrow,\downarrow}(t,\mathbf{x}))^{\mathcal{C}} = M(t,\mathbf{x})u^{*}_{\uparrow,\downarrow}, \; (\varphi^{v}_{\uparrow,\downarrow}(t,\mathbf{x}))^{\mathcal{C}} = M(t,\mathbf{x})v^{*}_{\uparrow,\downarrow}, \tag{57}$$

which are also solutions of the real Majorana equation (19). Thus, we obtain by adequate addition four purely real solutions (with $j = 1, 2, 3, 4$) given by

$$\Phi_j = \varphi_j + \varphi_j^{\mathcal{C}} = M(t,\mathbf{x})W_j = \Phi_j^{\mathcal{C}}, \tag{58}$$

with the four orthogonal real basis vectors

$$W_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, W_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, W_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, W_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{59}$$

What is the physical interpretation of these solutions? Using these eigenfunctions we obtain $\Sigma_y W_{1,2} = \pm \mathrm{i} W_{2,1}$, and $\Sigma_y W_{3,4} = \pm \mathrm{i} W_{4,3}$. Similarly, we obtain $\gamma_0 W_{1,2} = \pm \mathrm{i} W_{2,1}$ and $\gamma_0 W_{4,3} = \pm \mathrm{i} W_{3,4}$. Moreover, we obtain $\gamma_5 W_{1,2} = \pm \mathrm{i} W_{4,3}$ and $\gamma_5 W_{3,4} = \pm \mathrm{i} W_{2,1}$. Consequently, the real basis vectors of (59) are neither eigenvectors of γ_0 and γ_5 nor of the spin component Σ_y, which means that the spin (up or down) and particle nature (particle or antiparticle) and chirality all are not defined! The associated expectation values in these four states are therefore zero, and most importantly the normalization of W_j cannot be done in a Lorentz-invariant way.

So we conclude that there is no purely real spinor solution of the Majorana equation which is physically acceptable or meaningful. It is a matter of taste or convenience which version of the Dirac equation to use in practice. In applications, the Weyl version has already won the race, because it factorizes in a transparent way into the two chiral Weyl equations for fermions of zero mass. However, empirical evidence does not permit to do this for neutrinos, even if their measured masses are rather small, in the eV-range only for the first generation. Moreover, as they are known to carry the weak charge coupling them to the Z-boson gauge field, they cannot be described by a real equation but require the full complex DE that may come in whatever version is most appropriate for the physical problem under consideration.

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Spin matrices

We quote here once the resulting 4×4 matrices explicitly, also for the γ_0 matrix. The spin is $\mathbf{J} = \frac{1}{2}\mathbf{\Sigma}$. The associated purely imaginary but Hermitian sigma matrices read

$$\begin{aligned} \Sigma_x = \mathrm{i}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; \Sigma_y = \mathrm{i}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ \Sigma_z = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \gamma_0 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{A1}$$

All four matrices when being squared give the unit matrix 1_4, and thus, they have the real eigenvalues of ± 1. More importantly, the spin commutes with γ_0, which therefore can have common eigenfunctions with any one component of **J**. Their triple product yields $\Sigma_x \Sigma_y \Sigma_z = i 1_4$, which guarantees the standard commutation relation (1).

Boost matrices

We quote here also the 4×4 boost matrices explicitly, as well as that for the γ_5 matrix. The boost matrix vector **K** is given by the associated purely imaginary anti-Hermitian matrices that read

$$K_x = \frac{i}{2}\begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}, \; K_y = \frac{i}{2}\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix},$$
$$K_z = \frac{i}{2}\begin{pmatrix} 0&0&1&0 \\ 0&0&0&-1 \\ 1&0&0&0 \\ 0&-1&0&0 \end{pmatrix}, \; \gamma_5 = i\begin{pmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&-1&0 \end{pmatrix}. \tag{B2}$$

These three boost matrices when being squared give a quarter of the negative unit matrix, i.e., $-1_4/4$, and thus, they have the imaginary eigenvalues of $\pm i/2$. More importantly, the boost vector commutes with γ_5, which can therefore have common eigenfunctions with any one component of **K**, but the eigenvalues of the Hermitian γ_5 are ± 1. Their triple product yields $K_x K_y K_z = \frac{1}{8}\gamma_5$.

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