Rotating Black Holes in Chern-Simons Modified Gravity

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Abstract

The Chern-Simons modification of general relativity requires the modification of the Kerr solution for a rotating black hole. We present approximate rotating black hole solutions in Chern-Simons modified gravity.

1 Introduction

Chern-Simons (CS) modified gravity is a theory in which the Einstein-Hilbert action is modified by the CS term [1]. It is interesting to note that the CS modified gravity can be obtained from several approaches to quantum gravity [2]. The remarkable characteristic of the CS term is to violate parity symmetry. This characteristic leads to the result that solutions for a rotating black hole in the CS modified gravity inevitably has a different form from that of the Kerr solution. However, the Schwarzschild solution still holds in the CS modified gravity. The latter fact ensures that the CS modified gravity survives under observational constraints at present. In the present work, we provide approximate solutions for a rotating black hole in the CS modified gravity.

This paper is organized as follows. In Sec. 2, we briefly review two models of the CS modified gravity. In Sec. 3, we provide approximate solutions for a rotating black hole in the two models (see [3–5] in detail). Finally, we provide a summary in Sec. 4. Throughout the paper, we use geometrized units with $c = G = 1$.

2 CS modified gravity

2.1 Non-dynamical CS modified gravity

The action of non-dynamical CS modified gravity is provided by [1]

$$I = \int d^4x\sqrt{-g}\left[-\frac{R}{16\pi} + \frac{\ell}{64\pi}\vartheta\, {}^*R^\tau{}_\sigma{}^{\mu\nu}R^\sigma{}_{\tau\mu\nu} + \mathcal{L}_\mathrm{m}\right], \tag{1}$$

where g is the determinant of the metric $g_{\mu\nu}$, $R \equiv g^{\alpha\beta}R_{\alpha\beta}$ ($R_{\alpha\beta} \equiv R^\lambda{}_{\alpha\lambda\beta}$) is the Ricci scalar, $R^\tau{}_{\sigma\alpha\beta} \equiv \partial_\beta\Gamma^\tau_{\sigma\alpha} - \cdots$ is the Riemann tensor ($\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbols), ℓ is a coupling constant, and $\mathcal{L}_\mathrm{m}$ is the Lagrangian density for matter. The dual Riemann tensor is defined by ${}^*R^\tau{}_\sigma{}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R^\tau{}_{\sigma\alpha\beta}$, where $\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor with $\varepsilon^{0123} \equiv 1/\sqrt{-g}$. In this model, ϑ is an external scalar function. The term $\sqrt{-g}\,{}^*R^\tau{}_\sigma{}^{\mu\nu}R^\sigma{}_{\tau\mu\nu}$ is mathematically called Chern-Pontryagin density. The Chern-Pontryagin density can be written as a total derivative of the CS topological current K^μ defined by

$$K^\mu = \varepsilon^{\mu\alpha\beta\gamma}\left[\Gamma^\sigma_{\alpha\tau}\partial_\beta\Gamma^\tau_{\gamma\sigma} + \frac{2}{3}\Gamma^\sigma_{\alpha\tau}\Gamma^\tau_{\beta\eta}\Gamma^\eta_{\gamma\sigma}\right], \tag{2}$$

Thus we have

$$\partial_\alpha\left(\sqrt{-g}K^\alpha\right) = \frac{1}{2}\sqrt{-g}\,{}^*R^\tau{}_\sigma{}^{\mu\nu}R^\sigma{}_{\tau\mu\nu}. \tag{3}$$

Hence the action (1) can be written in terms of the CS topological current,

$$I = \int d^4x\sqrt{-g}\left[-\frac{R}{16\pi} - \frac{\ell}{32\pi}\left(\partial_\alpha\vartheta\right)K^\alpha + \mathcal{L}_\mathrm{m}\right]. \tag{4}$$

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Therefore the gravitational theory provided by Eq. (4), i.e., Eq. (1) with non-vanishing $\partial_\alpha \vartheta$, is called *CS modified gravity*.

From the variation in the action with respect to the metric $g_{\mu\nu}$, we obtain the field equation

$$G^{\mu\nu} + \ell C^{\mu\nu} = -8\pi T_{\rm m}{}^{\mu\nu}, \tag{5}$$

where $G^{\mu\nu}$ is the Einstein tensor, $T_{\rm m}^{\mu\nu}$ is the energy-momentum tensor, and $C^{\mu\nu}$ is the Cotton tensor defined by

$$C^{\mu\nu} \equiv -\frac{1}{2}\left[(\nabla_\sigma \vartheta)\left(\varepsilon^{\sigma\mu\alpha\beta}\nabla_\alpha R^\nu{}_\beta + \varepsilon^{\sigma\nu\alpha\beta}\nabla_\alpha R^\mu{}_\beta\right) + (\nabla_\sigma\nabla_\tau\vartheta)\left({}^*R^{\tau\mu\sigma\nu} + {}^*R^{\tau\nu\sigma\mu}\right)\right]. \tag{6}$$

The covariant divergence of Eq. (5) leads to the constraint [1]

$${}^*R^\tau{}_\sigma{}^{\mu\nu} R^\sigma{}_{\tau\mu\nu} = 0. \tag{7}$$

This equation is called the Chern-Pontryagin constraint. Therefore the non-dynamical CS modified gravity is governed by Eq. (5) with the constraint (7).

2.2 Dynamical CS modified gravity

In dynamical CS modified gravity, the scalar function ϑ is replaced with a scalar field which behaves as a dynamical variable in the gravitational system. The action is provided by

$$I = \int d^4x\sqrt{-g}\left[-\frac{R}{16\pi} + \frac{\ell}{64\pi}\vartheta\, {}^*R^\tau{}_\sigma{}^{\mu\nu} R^\sigma{}_{\tau\mu\nu} - \frac{1}{2}g^{\mu\nu}(\partial_\mu\vartheta)(\partial_\nu\vartheta) + \mathcal{L}_{\rm m}\right], \tag{8}$$

where the kinematic term for ϑ is added, but a potential for ϑ is ignored. Then the field equations are given by

$$G^{\mu\nu} + \ell C^{\mu\nu} = -8\pi\left(T_{\rm m}{}^{\mu\nu} + T_\vartheta{}^{\mu\nu}\right), \tag{9}$$

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\vartheta = -\frac{\ell}{64\pi}\, {}^*R^\tau{}_\sigma{}^{\mu\nu} R^\sigma{}_{\tau\mu\nu}. \tag{10}$$

The constraint (7) in the non-dynamical model is now replaced with the equation of motion (10) for the scalar field ϑ.

3 Slowly rotating black holes

We discuss slowly rotating black hole solutions in the two models of CS modified gravity. For this purpose, let us consider the perturbation of the Schwarzschild spacetime. In the non-dynamical model, the Schwarzschild metric satisfies both the field equation (5) and the constraint (7). Furthermore, the Schwarzschild metric satisfies the field equations (9) and (10) in the dynamical model under the condition in which the boundary condition $\vartheta \to 0$ at infinity is imposed. Thus we take the Schwarzschild metric as the background. To obtain slowly rotating black hole solutions, we consider the perturbed metric given by

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2M}{r}\right)(1 + h(r,\theta))\,dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}(1 + m(r,\theta))\,dr^2 \\ & + r^2(1 + k(r,\theta))\left[d\theta^2 + \sin^2\theta\,(d\phi - \omega(r,\theta)\,dt)^2\right], \end{aligned} \tag{11}$$

where M is the mass of a black hole, the functions $h(r,\theta)$, $m(r,\theta)$, $k(r,\theta)$ and $\omega(r,\theta)$ are of the first order in $\epsilon \sim J/M^2$ (J is the angular momentum of the black hole). Here ϵ is considered to be a small parameter for the perturbation. Hereafter, we take account of equations up to the first order in ϵ. By substituting Eq. (11) into the field equations (5) and (7), or (9) and (10) and solving those equations, we can obtain the metric solutions.

3.1 Solutions in the non-dynamical model

We discuss a slowly rotating black hole in the non-dynamical model. From the Chern-Pontryagin constraint (7), we derive

$$(\nabla^{\nu}\vartheta)\,\frac{3M}{r^3}\sin\theta\;\left(\omega_{,r\theta}+2\cot\theta\,\omega_{,r}\right)=0, \tag{12}$$

where a subscript comma denotes the partial differentiation with respect to the coordinates. It should be noted that the function $\omega\,(r,\theta)$ only appears in Eq. (12). As the solution for Eq. (12), we find

$$\omega\,(r,\theta)=\frac{\varpi(r)}{\sin^2\theta}, \tag{13}$$

where ϖ is a function of r only. Equation (13) means that the function $\omega\,(r,\theta)$ is singular on the rotational axis ($\theta=0$ and π), irrespective of the choice of the scalar function ϑ. Thus $g_{t\phi}$ does not vanish on the rotational axis unless $\varpi(r)$ is identically zero, and the shift vector $N_i\equiv g_{ti}$ ($i=r,\theta,\phi$) is also singular on the rotational axis.

When we adopt $\vartheta=r\cos\theta/\mu_0$ (μ_0 is a constant), we can find the metric solution for a slowly rotating black hole which is given by [3]

$$\begin{aligned} ds^2 \;&=\; -\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\\ &\quad -2\left[C_1\left(1-\frac{2M}{r}\right)+\frac{C_2}{r}\left\{r^2-2Mr-4M^2+4M(r-2M)\ln\left(r-2M\right)\right\}\right]dtd\phi, \end{aligned} \tag{14}$$

where C_1 and C_2 are constants. Although this solution has the singularity on the rotational axis, this spacetime can interestingly provide the flat rotation curves at a large distance from the black hole [4]. (See [6, 7] for other solutions in the non-dynamical model.)

3.2 Solutions in the dynamical model

Next we consider a slowly rotating black hole in the dynamical model. To obtain the solution, we have to remind that ϑ is a dynamical variable. The scalar field ϑ is assumed to be expanded as

$$\vartheta\,(r,\theta)=\vartheta^{(1)}\,(r,\theta)+O\left(\epsilon^2\right), \tag{15}$$

where $\vartheta^{(1)}\,(r,\theta)\sim O(\epsilon)$. Furthermore, we consider the coupling constant ℓ to be a small parameter and expand the field equations in a power series of ℓ. By solving the field equations (9) and (10), we can find the solution [5]

$$\vartheta \;=\; -\ell\frac{J}{128\pi M^2 r^4}\left(5r^2+10Mr+18M^2\right)\cos\theta, \tag{16}$$

$$\begin{aligned} ds^2 \;&=\; -\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\\ &\quad -\frac{4J}{r}\left[1-\ell^2\frac{1}{3584\pi M r^5}\left(70r^2+120Mr+189M^2\right)\right]\sin^2\theta dtd\phi, \end{aligned} \tag{17}$$

where J is the angular momentum of the black hole. In deriving Eqs. (16) and (17), we assumed the solutions of $\vartheta^{(1)}=0$ and $\omega=-2J/r^3$ at order $\ell^0\epsilon$, where ω coincides with the first order approximation of the Kerr black hole. (The same solution was also obtained independently by Yunes and Pretorius [7].) From the $(t\phi)$-component of the metric in Eq. (17), we find that the frame-dragging effect is suppressed by the CS correction, because the second term of order ℓ^2 in the bracket is negative for any value of r. (See [5, 7] for astrophysical implications.)

4 Summary

We have discussed slowly rotating black hole solutions in the two models of CS modified gravity. In the non-dynamical model, the singularity like a spinning cosmic string appears on the rotational axis owing to the Chern-Pontryagin constraint. However, the solution for a slowly rotating black hole interestingly provides the flat rotation curves far away from the black hole. On the other hand, such a singularity does not appear in the dynamical model. The solution in the dynamical model reduces to the first order approximation of the Kerr solution when the coupling constant vanishes. To impose a severe observational constraint for the CS coupling constant using astrophysical objects, we need the solution for a rapidly rotating black hole, which will be obtained in the future work.

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