



Braneworld black holes and entropy bounds

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ABSTRACT

The Bousso's D -bound entropy for the various possible black hole solutions on a 4-dimensional brane is checked. It is found that the D -bound entropy here is apparently different from that of obtained for the 4-dimensional black hole solutions. This difference is interpreted as the extra loss of information, associated to the extra dimension, when an extra-dimensional black hole is moved outward the observer's cosmological horizon. Also, it is discussed that N -bound entropy is hold for the possible solutions here. Finally, by adopting the recent Bohr-like approach to black hole quantum physics for the excited black holes, the obtained results are written also in terms of the black hole excited states.

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1. Introduction

Black hole (BH) quantum physics started in the 70s of last century with the remarkable works of Bekenstein [1,2] and Hawking [3]. It is a general conviction that Hawking radiation [3] and Bekenstein–Hawking entropy [2,3] are the two most important provisions of a yet unknown theory of quantum gravity which will permit to unify Einstein's general theory of relativity (GTR) with quantum mechanics. In fact, researchers in quantum gravity think that BHs should be the fundamental bricks of quantum gravity in the same way that atoms are the fundamental bricks of quantum mechanics [4]. In this framework, a fundamental result again by Bekenstein was that BHs have the maximum entropy for given mass and size which is allowed by quantum theory and by the GTR [5]. This Bekenstein bound represents an upper limit on the entropy that can be contained within a given finite region of space having a finite amount of energy. In other words, it is the maximum amount of information required to perfectly describe a given physical system down to the quantum level [5]. Thus, assuming that the region of space and the energy of the system are finite, the information necessary to perfectly describe it, is also finite [5]. Bekenstein bound has also important consequences in the physics of information and in computer science because it is connected with the so-called Bremermann's Limit [6], a maximum information-processing rate for a physical system having finite size

and energy. Also, the field equations of the GTR can be also derived by assuming the correctness of the Bekenstein bound and of the laws of thermodynamics [7]. Today we have various arguments which show that some form of the bound must exist if one wants the laws of thermodynamics and the GTR to be mutually consistent [8]. A generalization of the Bekenstein bound was attempted by Bousso [9], who conjectured an entropy bound having statistical origin and to be valid in all space-times admitted by Einstein's equation. This is the covariant entropy bound and reduces to Bekenstein bound for systems with limited self-gravity [9]. Bousso also proposed a bound on the entropy of matter systems within the cosmological horizon called the “ D -bound” which claims that the total observable entropy is bounded by the inverse of the cosmological constant [10]. The dependance of D -bound on the cosmological constant and the area of initial horizon [10] turns out to be useful for at least one application [11], but its relationship to the flat space Bekenstein bound still remains obscure. If one implies the cosmological horizon in terms of gravitational radius rather than the energy of the system, the D -bound will be the same as the Bekenstein bound. Note that although a special limit is taken but the agreement is non-trivial, because the background geometry differs significantly from the flat space. So, the D -bound in its general form may be regarded as a de Sitter space equivalent of the flat space Bekenstein bound. Moreover, Bousso derived the Bekenstein bound from Geroch process to the higher dimensional spacetimes, i.e. $D > 4$. It has been done for both of the asymptotically de Sitter spaces (associated to the cosmological horizon) and for asymptotically flat spaces (associated to a black hole).

In this paper, we will discuss Bousso's D -bound in the framework of the braneworld BHs [12], see also [13] as a review on the

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braneworld black holes. These objects arise from the gravitational collapse of matter trapped on a brane [14]. In particular, Bousso's D -bound entropy will be checked for the various possible BH solutions on 4D brane. After that, it will be shown that the relation between the D -bound and Bekenstein entropy bound put restriction on the braneworld BH solutions.

The organization of the paper is as follows. In section 2, we review the entropy bounds. Then, in section 3, we introduce the general vacuum black hole solution and its subclasses on a 4D brane obtained in [12]. In the following sections, we investigate D -bound entropy for the these mentioned subclasses. The paper ends in section 9, with some concluding remarks.

2. The entropy bounds

Following Bousso [10] for the D -bound on matter entropy in de Sitter space, one can infer some physical results by considering the D -bound and its relationship with Bekenstein entropy bound. In this line, to derive D -bound, one may suppose a matter system within the apparent cosmological horizon of an observer. In such a situation, the observer is in a universe with a future de-Sitter asymptotic. By moving relative to the matter system toward the asymptotic region, this observer can be witness of a thermodynamical process by which the matter system is moved outward the cosmological horizon. Then, the observer will find himself in the space-time that has been converted to empty pure de-Sitter space. In this process, the initial thermodynamical system, the asymptotic de Sitter space including the matter system, has the total entropy

$$S = S_m + \frac{A_c}{4}, \quad (1)$$

where S_m is the entropy of the matter inside the cosmological horizon and $A_c/4$ is the Bekenstein–Hawking entropy associated to the enclosing apparent cosmological horizon. At the end of the process, the final entropy of the system will be $S_0 = A_0/4$ in which A_0 is the area of the cosmological horizon of the de Sitter space empty of any matter. Now, regarding the generalized second law, i.e. $S \leq S_0$, one arrives at [10]

$$S_m \leq \frac{1}{4}(A_0 - A_c). \quad (2)$$

This is the so-called D -bound on the matter systems in an asymptotically de Sitter space. For empty de Sitter space, we have $A_0 = A_c$ and consequently the D -bound vanishes, because there is no matter present. Using the fact $S_m \geq 0$, one realizes $A_c \leq A_0$. Then, a matter system enclosed by a cosmological horizon has smaller area than the horizon area of an empty de-Sitter space. Now, consider a BH, as the matter system, in an asymptotically de Sitter space. For this case, the matter entropy S_m is the BH's Bekenstein–Hawking entropy. Then, one can verify that this new configuration also satisfies the D -bound and consequently, the area of the cosmological horizon surrounding the black hole A_c will be smaller than A_0 , the cosmological horizon of the empty de Sitter space.

The metric for this case is given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_2^2, \quad (3)$$

where

$$f(r) = 1 - \frac{2M}{r} - \Lambda r^2. \quad (4)$$

Setting $f(r) = 0$ gives the locations of the BH horizon r_b and the cosmological horizon r_c in the presence of the BH, respectively. The

values of these horizon locations depends on the mass parameter M such that by setting $M = 0$, we arrive at the empty de Sitter space. Then, for the latter case, the only positive root of $f = 0$ is $r_c = r_0$ representing the radius of the cosmological horizon. But in the case of $M > 0$, there is another root r_b which represents the BH horizon ($r_b \approx 2M$ for small mass parameter M). By increasing the mass parameter M , the BH horizon r_b increases while the cosmological horizon r_c decreases. Remember that for the empty de Sitter space, the cosmological horizon has area $A_0 = 4\pi r_0^2$ while there exists a matter inside the system, the cosmological horizon has area $A_c = 4\pi r_c^2$. For $M = 0$, r_c has its maximum value as $r_c = r_0$ and decreases for $M > 0$. Then, for all range of M , we have $A_c < A_0$ and we can consider the D -bound.

The D -bound (2) states that for this case, the BH entropy i.e. $A_b/4 = \pi r_b^2$, is less than $\pi(r_0^2 - r_c^2)$. Thus, we can consider the total entropy of the system of matter, which is enclosed by the cosmological horizon (the entropy of Schwarzschild–de Sitter space), as

$$S = \pi(r_c^2 + r_b^2), \quad (5)$$

and we know that this entropy is less than the entropy of empty de-Sitter space

$$S_0 = \pi r_0^2. \quad (6)$$

By solving the cubic equation (4), i.e. $f(r) = 0$, finding its positive roots and putting it into (5), we can rewrite the system's total entropy S in the following form for small M parameter

$$S = \pi r_0^2 \left(1 - \frac{2M}{r_0}\right) + O(M^2). \quad (7)$$

Now, we can investigate the relation between this D -bound with the Bekenstein entropy bound. For the de Sitter space, the energy of the system is not well-defined because there is no suitable asymptotic region for this space. However, Birkhoff's theorem [15] implies that there exists some Schwarzschild–de Sitter solution for a spherical system such that its metric is the same as the metric at large radii. This large radii can be regarded as the cosmological horizon radius, i.e. r_c . Then, one can call this BH as the *system's equivalent BH*, and its radius is the gravitational radius r_g of the system.

In the flat space, the gravitational radius is exactly twice the mass-energy, i.e. $r_g = 2M$ [15], and one can express the Bekenstein bound in terms of both these quantities, i.e. mass-energy or gravitational radius. However, for an asymptotically de Sitter space, one can still define the gravitational radius r_g , while the mass-energy cannot be defined. In this case, one may define A_0 in the D -bound, the relation (2), in terms of r_c and r_g rather than r_0 .

Also, the mass parameter can be obtained in terms of the BH radius by the help of the equation (4) as

$$2M = r_b \left(1 - \frac{r_b^2}{r_0^2}\right). \quad (8)$$

By setting $r_b = r_g$, we can express A_0 in terms of the r_g and r_c . For this purpose, in the limit of small *equivalent BHs* ($r_g \ll r_c$) which corresponds to light matter systems, one finds

$$r_0^2 = r_c^2 \left(1 + \frac{r_g}{r_c}\right) + O\left[\left(\frac{r_g}{r_c}\right)^2\right]. \quad (9)$$

Then, by rewriting A_0 using this equation, the D -bound (2), to the first order in r_g reads as

$$S_m \leq \pi r_g r_c. \quad (10)$$

Now, recall the Bekenstein entropy bound defined in terms of gravitational radius $r_g = 2M$ as

$$S_m \leq \pi r_g R, \quad (11)$$

where R is the radius of the sphere enclosing the system. By comparing (10) with (11), we find that the D -bound coincides with Bekenstein bound, using the fact that in de Sitter space a stable system cannot be larger than $R = r_c$.

For the excited BHs, i.e. the BHs which emitted a large amount of Hawking quanta, the recent Bohr-like approach to BH quantum physics [4,16,17] permits to write the BH gravitational radius in function of the BH quantum level as [4,16,17]

$$(r_g)_n = 2M_n = 2\sqrt{M_I^2 - \frac{n}{2}}, \quad (12)$$

where n is the BH principal quantum number, M_n is the mass of the BH excited at the level n and M_I is the initial BH mass, i.e. the BH mass before that Hawking radiation starts to be emitted. In fact, in [4,16,17] the intuitive but general conviction that BHs result in highly excited states representing both the “Hydrogen atom” and the “quasi-thermal emission” in quantum gravity has been shown to be correct, because the Schwarzschild BH results somewhat similar to the historical semi-classical hydrogen atom introduced by Bohr in 1913 [18,19]. For the excited BHs, using the Eq. (12), the Eqs. from (8) to (11) become

$$2\sqrt{M_I^2 - \frac{n}{2}} = r_b(1 - \frac{r_b^2}{r_0^2}), \quad (13)$$

$$r_0^2 = r_c^2(1 + \frac{2\sqrt{M_I^2 - \frac{n}{2}}}{r_c}) + O\left(\frac{2M_I^2 - \frac{n}{2}}{r_c^2}\right), \quad (14)$$

$$S_m \leq 2\pi r_c \sqrt{M_I^2 - \frac{n}{2}}, \quad (15)$$

and finally

$$S_m \leq 2\pi R \sqrt{M_I^2 - \frac{n}{2}}. \quad (16)$$

Then, it is seen that for the excited BHs, we have tighter bound depending on the BH principal quantum number n .

Here, it is worth mentioning to a brief discussion on the N -bound entropy. The N -bound states that the observable entropy S in any universe which has a positive cosmological constant Λ is bounded by $N = \frac{3\pi}{\Lambda}$, regardless of its matter content [11]. For our purpose, one can write the N -bound as follows

$$S = S_m + S_c \leq N. \quad (17)$$

Here, using (7) and (17), we have

$$\pi r_0^2(1 - \frac{2M}{r_0}) \leq \frac{3\pi}{\Lambda} = \pi r_0^2, \quad (18)$$

which indicates that the N -bound is also satisfied.

In the following section, we introduce the general vacuum black hole solution and its subclasses on a 4D brane. Then, we discuss on the corresponding entropy bounds related to each of these subclasses.

3. Vacuum black hole solutions on the brane

In a braneworld model, the visible Universe with 3 spatial dimensions is considered as being restricted to a brane inside a

higher-dimensional space. If one assumes the additional dimensions to be compact (that is curled up in themselves and having their lengths of order of the Planck length), such dimensions are inevitably within the Universe. Instead, if one assumes the additional dimensions to be not compact, the higher-dimensional space is called *the bulk*. In that case, on one hand, other branes can move through the bulk. On the other hand, some extra dimensions can be extensive and even infinite. A first attempt to discuss a braneworld model was the pioneering work [20]. More than 15 years later, we find the works of Gogberashvili [21], Randall–Sundrum (RS) scenarios, i.e. RS1 [22] and RS2 [23], Arkani-Hamed–Dimopoulos–Dvali (ADD) model [24], Dvali–Gabadadze–Porrati (DGP) model [25], see [26] and [27] for a review on brane gravity. In this Section, we start from the black hole solution [12] in the most general braneworld model introduced in [28] and developed in [29–38], without giving any detail on this model and its applications. In this general model, there is no specific junction conditions or Z_2 symmetry and consequently, this model differs from the usual RS braneworld scenario where Z_2 symmetry is applied across a background 4D brane considered as a boundary embedded in an ambient bulk space. In this case, the extrinsic curvature of the background boundary is completely determined by the confined energy-momentum tensor on the brane using the Israel–Darmois–Lanczos (IDL) condition.

We consider a 4D brane spacetime (\mathcal{M}_4, g) embedded in a 5D bulk space $(\mathcal{M}_5, \mathcal{G})$. In order to obtain the vacuum solution on the brane, we assume that the 4D brane (\mathcal{M}_4, g) is devoid of matter fields and the 5D ambient bulk space $(\mathcal{M}_5, \mathcal{G})$ possesses a constant curvature. Then, using the Gauss–Codazzi equations [39], the following induced equations on the 4D brane can be obtained [12]

$$G_{\mu\nu} = Q_{\mu\nu}, \quad (19)$$

where

$$Q_{\mu\nu} = (K_{\mu}{}^{\gamma} K_{\gamma\nu} - K K_{\mu\nu}) - \frac{1}{2}(K \circ K - K^2)g_{\mu\nu}, \quad (20)$$

is a completely geometrical quantity resulted from the extrinsic curvature $K_{\mu\nu}$ of the 4D embedded brane where also we defined the terms $K \circ K \equiv K_{\mu\nu} K^{\mu\nu}$ and $K \equiv g^{\mu\nu} K_{\mu\nu}$.

It is clear that the right hand side of Eq. (19) appears as the modification to the vacuum field equations on the brane with respect to the standard Einstein field equations of the GTR. This modification has a geometric origin and is resulted from the extrinsic curvature of the 4D brane embedded within its nD ambient bulk space. From the spirit of $Q_{\mu\nu}$, one can verify that it is an independently conserved quantity, i.e. it possesses a null divergence as $\nabla_{\mu} Q^{\mu\nu} = 0$.

In order to find a general static spherically symmetric BH spacetimes, one can consider the following metric

$$ds^2 = -e^{\mu(r)} dt^2 + e^{\nu(r)} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2). \quad (21)$$

Then, using the Codazzi equation, the induced field equations (1) and the conservation equation $\nabla_{\mu} Q^{\mu\nu} = 0$, one can obtain the non-vanishing components of the extrinsic curvature tensor $K_{\mu\nu}$ as

$$K_{00}(r) = -\alpha e^{\mu(r)}, \quad (22)$$

$$K_{11}(r) = \alpha e^{\nu(r)}, \quad (23)$$

$$K_{22}(r) = \alpha r^2 + \beta r, \quad (24)$$

$$K_{33}(r, \theta) = \alpha r^2 \sin^2 \theta + \beta r \sin^2 \theta, \quad (25)$$

where α and β are integration constants. Using the components of the extrinsic curvature tensor, one can find the non-vanishing components of the $Q_{\mu\nu}$ tensor as

$$\begin{aligned} Q_{00} &= -\frac{g_{00}}{r^2} (3\alpha^2 r^2 + 4\alpha\beta r + \beta^2), \\ Q_{11} &= -\frac{g_{11}}{r^2} (3\alpha^2 r^2 + 4\alpha\beta r + \beta^2), \\ Q_{22} &= \frac{g_{22}}{r} (-3\alpha^2 r - 2\alpha\beta), \\ Q_{33} &= \frac{g_{33}}{r} (-3\alpha^2 r - 2\alpha\beta). \end{aligned} \quad (26)$$

Then, by solving the field equations (19), one can find the metric components as

$$e^{\mu(r)} = e^{-\nu(r)} = 1 - \frac{2M}{r} - \alpha^2 r^2 - 2\alpha\beta r - \beta^2, \quad (27)$$

where M is the central BH mass and α and β are integration constants resulted from the extrinsic curvature of the embedded brane which are playing the role of cosmological parameters. In this regard, the three modification terms in (27), relative to the familiar Schwarzschild solution [40], have geometric origins and are arising from the non-trivial extrinsic geometry of the brane within its higher dimensional ambient bulk space. Then, regarding the metric functions (27), one can distinguish the following distinct subclasses:

- the case of $\beta = 0$,
- the case of $\alpha^2 \simeq 0$,
- the case of $\alpha \neq 0$ and $\beta \neq 0$,
- the case of $\alpha^2 \simeq 0$ and $\beta^2 \simeq 0$,
- the case of $\alpha = \beta = 0$,
- the case of $\alpha = 0$ and $\beta \neq 0$,
- the case of $M = 0$,
- the case of $M = \beta = 0$,
- the case of $M = \alpha = 0$.

In the following sections, we investigate the D -bound and its relation with the Bekenstein entropy bound for each of these distinct solutions on brane in detail.

4. The entropy bounds for the case of $\beta = 0$

This solution corresponds to the metric functions

$$e^{\mu(r)} = e^{-\nu(r)} = 1 - \frac{2M}{r} - \alpha^2 r^2, \quad (28)$$

representing the Schwarzschild–de Sitter BH with positive cosmological constant, i.e. $\Lambda = \alpha^2$ [41]. Interestingly, in this case, the cosmological constant has a geometric origin, rather than its ad-hoc introduction to the field equations of GR and arises from the extrinsic curvature of the brane in a higher dimensional bulk. Then, the discussions here on the entropy bounds are the same as the section 2 and we avoid to repeat. The only point is that there is a geometric origin for the cosmological constant of the de-Sitter space.

5. The entropy bounds for the case of $\alpha^2 \simeq 0$

The corresponding solution is given by

$$e^{\mu(r)} = e^{-\nu(r)} = 1 - \frac{2M}{r} - 2\alpha\beta r - \beta^2. \quad (29)$$

Except the β^2 term, this solution looks like to the Kiselev BH [42] in GR, see also its generalization to the Rastall theory [43]. In this

solution, the BH is surrounded by a quintessence field with the field structure parameter $\sigma = 2\alpha\beta$ [42]. Then, this can be called as the Schwarzschild–quintessence-like BH on the brane. Now, let us find the solution of the equation

$$f(r) = 1 - \frac{2M}{r} - 2\alpha\beta r - \beta^2 = 0. \quad (30)$$

For $M = 0$ or equivalently in the BH absence, the Eq. (30) gives

$$r_0 = \frac{1 - \beta^2}{2\alpha\beta}, \quad (31)$$

which represents the cosmological horizon of an empty of matter (BH) space. In the presence of BH, for $\alpha\beta > 0$ and $1 - \beta^2 > 0$, there are two solutions for the Eq. (30) as

$$r_c = \frac{(1 - \beta^2) + \sqrt{(\beta^2 - 1)^2 - 16\alpha\beta M}}{4\alpha\beta}, \quad (32)$$

and

$$r_b = \frac{(1 - \beta^2) - \sqrt{(\beta^2 - 1)^2 - 16\alpha\beta M}}{4\alpha\beta}. \quad (33)$$

In the case of excited BHs, one can use the Eq. (12) and rewrite these solutions in terms of the BH quantum level and of the BH initial mass as

$$r_c = \frac{(1 - \beta^2) + \sqrt{(\beta^2 - 1)^2 - 16\alpha\beta \sqrt{M_l^2 - \frac{n}{2}}}}{4\alpha\beta}, \quad (34)$$

and

$$r_b = \frac{(1 - \beta^2) - \sqrt{(\beta^2 - 1)^2 - 16\alpha\beta \sqrt{M_l^2 - \frac{n}{2}}}}{4\alpha\beta}. \quad (35)$$

Then, for the next conveniences, we rewrite r_c^2 as

$$r_c^2 = \frac{(1 - \beta^2)r_b}{2\alpha\beta} \left(1 - \frac{2M}{1 - \beta^2} \right), \quad (36)$$

which becomes

$$r_c^2 = \frac{(1 - \beta^2)r_b}{2\alpha\beta} \left(1 - \frac{2\sqrt{M_l^2 - \frac{n}{2}}}{1 - \beta^2} \right), \quad (37)$$

for the excited BHs.

Interestingly, from the Eqs. (32) and (33), we find $r_b + r_c = r_0$. Now, let us consider the D -bound and its relationship with Bekenstein bound. To check the D -bound, suppose a matter system (BH) within the apparent cosmological horizon of an observer. This observer lies in a Universe which is going to be asymptotically quintessence-like Universe in the future. The observer can be witness of a thermodynamical process by which the matter system (BH) is dropped across the cosmological horizon. Then, he will be in the space–time that has been converted to empty quintessence-like space with the radius r_0 . In this process, the initial thermodynamical system has entropy given by the Eq. (1). In this way, S_m is the entropy of the matter (BH) inside the cosmological horizon and A_c is the area of cosmological horizon in the presence of matter system (BH). At the end of process, the final entropy of the system will be $S_0 = A_0/4$, using the fact that the quarter of the area of the apparent cosmological horizon is the Bekenstein–Hawking entropy. Here, A_0 is the area of the horizon of the empty quintessence-like

space. Then, using the generalized second law ($S \leq S_0$) leading to the D -bound of the Eq. (2), we find

$$S_m \leq \pi \left(\left(\frac{1-\beta^2}{2\alpha\beta} \right)^2 - \left(\frac{1-\beta^2}{2\alpha\beta} \right) r_c + \frac{Mr_c}{\alpha\beta} \right). \quad (38)$$

Using the Eq. (31), one can rewrite the Eq. (38) as

$$S_m \leq \pi \left(\left(\frac{1-\beta^2}{2\alpha\beta} \right) (r_0 - r_c) + \frac{Mr_c}{\alpha\beta} \right). \quad (39)$$

On the other hand, taking the limit of r_b for small β values, one gets

$$r_b = \frac{2M}{1-\beta^2}. \quad (40)$$

Now, if one recalls the approach in the section 2, one can take $r_b = r_g$. Then, we have

$$2M = r_g(1-\beta^2). \quad (41)$$

Putting the Eq. (41) and Eq. (31) in the Eq. (39) and using the relation $r_b + r_c = r_0$, one obtains

$$S_m \leq \pi r_g r_c (1+r_c) + r_g^2 (1+r_c). \quad (42)$$

Here, it is worth to discuss about the terms appeared in the RHS of (42) in comparison to the term in the RHS of (10). Actually, it turns out that the terms in the RHS of (42) are appeared because of the extra loss of information, associated to the extra dimension, when an extra-dimensional black hole is moved outward the observer's cosmological horizon.

For further discussion we compare this with the covariant entropy bound of a 4-dimensional black hole. The covariant entropy bound for this black hole leads to [9]

$$S_m \leq \frac{A}{4}, \quad (43)$$

where A is the area of black hole's horizon. So, here the covariant entropy bound becomes $S_m \leq \pi r_g^2$. This bound is tighter than the bound in inequality (42). This result is not surprising because even for a 4-dimensional black hole, the covariant entropy bound is tighter than D -bound [10].

Regarding the N -bound, we note that since $\alpha^2 \simeq 0$ which means $\Lambda \simeq 0$, then the N -bound becomes infinite and so this kind of black hole in a braneworld may be allowed.

6. The entropy bounds for the case of $\alpha \neq 0$ and $\beta \neq 0$

This is the most general solution on the brane where

$$\begin{aligned} f(r) &= 1 - \frac{2M}{r} - \alpha^2 r^2 - 2\alpha\beta r - \beta^2 \\ &= 1 - \frac{2M}{r} - (\alpha r + \beta)^2. \end{aligned} \quad (44)$$

In the BH absence, i.e. $M = 0$, the Eq. (44) gives

$$(\alpha r_0 + \beta)^2 = 1, \quad (45)$$

where r_0 is the radius of the cosmological horizon in the absence of any matter system (BH). Using the Eq. (45), one can rewrite the Eq. (44) as

$$f(r) = 1 - \frac{2M}{r} - \frac{(\alpha r + \beta)^2}{(\alpha r_0 + \beta)^2}. \quad (46)$$

Now, by solving $f(r) = 0$, one can find two roots r_c and r_b as the cosmological horizon in the presence of matter system (BH) and the BH horizon, respectively, as

$$1 - \frac{2M}{r_c} - \frac{(\alpha r_c + \beta)^2}{(\alpha r_0 + \beta)^2} = 0, \quad (47)$$

and

$$1 - \frac{2M}{r_b} - \frac{(\alpha r_b + \beta)^2}{(\alpha r_0 + \beta)^2} = 0. \quad (48)$$

If $2M \ll r_c$, using (47), we have

$$(\alpha r_0 + \beta)^2 = (\alpha r_c + \beta)^2 \left(1 + \frac{2M}{r_c} + \frac{4M^2}{r_c^2} \right) + O\left(\frac{2M}{r_c}\right)^3, \quad (49)$$

which leads to

$$r_0^2 = r_c^2 + 4M^2 + \frac{2\beta}{\alpha}(r_c - r_0) + 2Mr_c + \frac{2M\beta^2}{r_c\alpha^2} + \frac{4M\beta}{\alpha}. \quad (50)$$

Also, we can rewrite the equation (48) as

$$r_b = 2M \left(1 - \left(\frac{\alpha r_b + \beta}{\alpha r_0 + \beta} \right)^2 \right)^{-1}. \quad (51)$$

We note that the condition $2M \ll r_c$ will be, in principle, satisfied for the excited astrophysics BHs in the future, when a lot of their mass will be radiated in terms of Hawking quanta. In that case, if one uses again the Eq. (12), the Eqs. from (49) to (51) can be re-written in terms of the BH quantum level and the BH initial mass as

$$\begin{aligned} (\alpha r_0 + \beta)^2 &= (\alpha r_c + \beta)^2 \left(1 + \frac{2\sqrt{M_l^2 - \frac{n}{2}}}{r_c} + \frac{4(M_l^2 - \frac{n}{2})}{r_c^2} \right) \\ &\quad + O\left(\frac{2\sqrt{M_l^2 - \frac{n}{2}}}{r_c}\right)^3, \end{aligned} \quad (52)$$

and

$$\begin{aligned} r_0^2 &= r_c^2 + 4 \left(M_l^2 - \frac{n}{2} \right) + \frac{2\beta}{\alpha}(r_c - r_0) \\ &\quad + 2\sqrt{M_l^2 - \frac{n}{2}}r_c + \frac{2\sqrt{M_l^2 - \frac{n}{2}}\beta^2}{r_c\alpha^2} + \frac{4\sqrt{M_l^2 - \frac{n}{2}}\beta}{\alpha}, \end{aligned} \quad (53)$$

and

$$r_b = 2\sqrt{M_l^2 - \frac{n}{2}} \left(1 - \left(\frac{\alpha r_b + \beta}{\alpha r_0 + \beta} \right)^2 \right)^{-1}. \quad (54)$$

Now, we can investigate the relation between the D -bound and the Bekenstein bound for this type of BHs on the brane. To derive the D -bound, suppose a matter system within the apparent cosmological horizon of an observer. Regarding the Eq. (44), although the structure of the whole system is different than the usual Schwarzschild-de Sitter system, particularly for the mean distances, we find that here the observer is also in a Universe which is going to be asymptotically de Sitter in the future. The observer can be witness of a thermodynamical process by which the matter system is dropped across the cosmological horizon. Then, he will be in the space-time that has been converted to empty de Sitter-like space but not pure de Sitter. In this process the initial thermodynamical system has entropy, like as the equation (2), where S_m is the entropy of the matter inside the cosmological horizon and A_c is the area of the cosmological horizon. A quarter of the area of the apparent cosmological horizon is the Bekenstein-Hawking entropy. At the end of process, the final entropy of the system will be $S_0 = A_0/4$. Here, A_0 is the area of horizon of

the empty de Sitter-like space. Then, the generalized second law, $S \leq S_0$, leads to D -bound

$$S_m \leq 2\pi M \left(2M + r_c + \frac{\beta^2}{r_c \alpha^2} + \frac{2\beta}{\alpha} \right) + \frac{2\beta}{\alpha} (r_c - r_0), \quad (55)$$

where we used the Eq. (2). Using the Eq. (36), we can rewrite this relation as

$$S_m \leq \pi r_b \left(1 - \left(\frac{\alpha r_b + \beta}{\alpha r_0 + \beta} \right)^2 \right) \left(r_b \left(1 - \left(\frac{\alpha r_b + \beta}{\alpha r_0 + \beta} \right)^2 \right) + r_c + \frac{\beta^2}{r_c \alpha^2} + \frac{2\beta}{\alpha} \right) + \frac{2\beta}{\alpha} (r_c - r_0). \quad (56)$$

Replacing $r_b = r_g$, one gets

$$S_m \leq \pi r_g \left(1 - \left(\frac{\alpha r_g + \beta}{\alpha r_0 + \beta} \right)^2 \right) \left(r_g \left(1 - \left(\frac{\alpha r_g + \beta}{\alpha r_0 + \beta} \right)^2 \right) + r_c + \frac{\beta^2}{r_c \alpha^2} + \frac{2\beta}{\alpha} \right) + \frac{2\beta}{\alpha} (r_c - r_0), \quad (57)$$

which yields

$$S_m \leq \pi r_g r_c + \pi r_g \left(r_g \left(1 - \left(\frac{\alpha r_g + \beta}{\alpha r_0 + \beta} \right)^2 \right) + \frac{\beta^2}{r_c \alpha^2} + \frac{2\beta}{\alpha} \right) - \pi r_g \left(\frac{\alpha r_g + \beta}{\alpha r_0 + \beta} \right)^2 \left(r_g \left(1 - \left(\frac{\alpha r_g + \beta}{\alpha r_0 + \beta} \right)^2 \right) + r_c + \frac{\beta^2}{r_c \alpha^2} + \frac{2\beta}{\alpha} \right) + \frac{2\beta}{\alpha} (r_c - r_0). \quad (58)$$

By comparing this D -bound with the Bekenstein entropy bound (11) with $r_g = r_b \cong 2M$ for $r_b \ll r_0$, using the equation (6) and $r_c = R$, we find that there are two physical possibilities as

- The extra terms should vanish in order to maintain the Bekenstein bound for this type of black holes on the brane.
- The extra terms should possess total negative values, but small relative to $\pi r_c r_g$, in order to lead a D -bound tighter than the Bekenstein bound for this type of black holes on the brane.

Both of these two possibilities put restrictions on the geometric parameters α and β of the embedded brane within its ambient space.

If one sets $\beta = 0$, one finds

$$S_m \leq \pi r_c r_g \left(1 + \frac{r_g}{r_c} \right), \quad (59)$$

where in the limit of $r_g \ll r_c$, the relation $S_m \leq \pi r_c r_g$ in (10) can be recovered, as the obtained result in [10].

Also in this case, if one considers excited BHs, the equations from (55) to (59) can be written in terms of the BH quantum level and the BH initial mass through the Eq. (12). For this case, the Eq. (59) becomes

$$S_m \leq \pi r_c r_g \left(1 + \frac{2\sqrt{M_I^2 - \frac{n}{2}}}{r_c} \right). \quad (60)$$

Here, it is also seen that for the excited BH, we have tighter entropy bound relative to the initial their states.

We can discuss here about the N -bound for the solution (44). Because the cosmological constant term $\alpha^2 r^2$ in the spacetime of the metric (44) is still asymptotically dominant term, so it is expected that the N -bound will hold also for the solution (44) similar to the de-Sitter and Schwarzschild–de-Sitter spaces.

7. The entropy bounds for the case of $\alpha^2 \simeq 0$ and $\beta^2 \simeq 0$

The corresponding solution is given by

$$e^{\mu(r)} = e^{-\nu(r)} = 1 - \frac{2M}{r} - 2\alpha\beta r, \quad (61)$$

which is exactly the Schwarzschild BH in the quintessence field [42] with the quintessence structure parameter $\sigma = 2\alpha\beta$. Now, we need to find the solutions of

$$f(r) = 1 - \frac{2M}{r} - 2\alpha\beta r - \beta^2 = 0. \quad (62)$$

For $M = 0$ or equivalently in the BH absence, the Eq. (62) gives

$$r_0 = \frac{1}{2\alpha\beta}, \quad (63)$$

which represents the cosmological horizon of an empty of matter (BH) space. In the presence of BH, there are two solutions for (62) as

$$r_c = \frac{1 + \sqrt{1 - 16\alpha\beta M}}{4\alpha\beta}, \quad (64)$$

and

$$r_b = \frac{1 - \sqrt{1 - 16\alpha\beta M}}{4\alpha\beta}, \quad (65)$$

representing the cosmological horizon in the BH presence and the BH horizon, respectively. Here, it is useful to rewrite (64) as

$$r_c^2 = \frac{1}{2\alpha\beta} (r_c + 2M). \quad (66)$$

Then, by using $r_b + r_c = r_0$ and the generalized second law ($S \leq S_0$), we find the D -bound

$$\begin{aligned} S_m &\leq \pi r_0^2 - \pi r_c^2 \\ &= \pi ((r_c + r_g)^2 - r_0(r_c + 2M)) \\ &= \pi (r_0 r_g - 2r_0 M) \\ &= \pi (2r_c M + r_g^2 - 2r_0 M). \end{aligned} \quad (67)$$

Since $r_0 \gg r_g$ then the third term overcomes to the second, i.e. $r_g^2 - 2r_0 M < 0$, and this relation represents a tighter bound than the Bekenstein and covariant entropy bounds for this type of BHs on the brane. Then, these BHs can exist as the real physical BH solutions on the brane, if one regards the covariant bound as the basic physical entropy bound.

For this case, if one considers the excited BHs, the Eq. (12) permits to rewrite the Eq. (67) in terms of the BH excited state as

$$\begin{aligned} S_m &\leq \pi r_0^2 - \pi r_c^2 \\ &= \pi \left((r_c + r_g)^2 - r_0 \left(r_c + 2\sqrt{M_I^2 - \frac{n}{2}} \right) \right) \\ &= \pi \left(r_0 r_g - 2r_0 \sqrt{M_I^2 - \frac{n}{2}} \right) \\ &= \pi \left(2r_c \sqrt{M_I^2 - \frac{n}{2}} + r_g^2 - 2r_0 \sqrt{M_I^2 - \frac{n}{2}} \right), \end{aligned} \quad (68)$$

representing a tighter bound.

Because of $\alpha^2 \simeq 0$ which means $\Lambda \simeq 0$, the N -bound becomes infinite and so this kind of black hole in a braneworld may be allowed.

8. The remaining cases

8.1. The case of $\alpha = \beta = 0$

Here, the corresponding metric is the familiar Schwarzschild solution [15]

$$e^{\mu(r)} = e^{-\nu(r)} = 1 - \frac{2M}{r}. \quad (69)$$

In this case, there is no cosmological horizon. Thus, one cannot consider the thermodynamical process defined for obtaining the D -bound or N -bound. Therefore, that method cannot be applied for this case.

8.2. The case of $\alpha = 0$ and $\beta \neq 0$

In this case, we have no cosmological horizon and consequently, we can not define the thermodynamic process considered in [10] and [11] to obtain the D -bound or N -bound.

8.3. The case of $M = 0$

In this case, we have two horizons without BHs. If the inner horizon can be go out of the outer horizon, like the BH in the thermodynamical process in the studied cases in sections 3–6, it may be possible to find D -bound or N -bound. This is an issue that will be analyzed in our future work [44].

8.4. The case of $M = \beta = 0$

This case represents the pure de Sitter space where $A_c = A_0$ and the D -bound vanishes.

8.5. The case of $M = \alpha = 0$

There is no cosmological horizon for this case to define the mentioned thermodynamical process in [10].

9. Concluding remarks

In this paper, we have focused on the Bousso's D -bound entropy and on the Bekenstein's entropy bound. In particular, Bousso's D -bound entropy has been checked for the various possible extra dimensional black hole solutions. It turns out that the D -bound entropy here is apparently different from that of obtained for the 4-dimensional black hole solutions. This difference is interpreted as the extra loss of information, associated to the extra dimension, when an extra-dimensional black hole is moved outward the observer's cosmological horizon. We have also discussed briefly about the N -bound entropy for the possible black hole solutions on the braneworld, represented by the cases $\alpha^2 = 0$ and $\alpha^2 \neq 0$. It turns out that the N -bound holds for both cases. In addition, through the recent Bohr-like approach to black hole quantum physics for the excited black holes, it has been possible to rewrite the various obtained results also in function of the black hole quantum principal number, i.e. in function of the black hole quantum excited state. In this regard, we have tighter entropy bound for the excited black holes relative their initial states. We hope to further extend our analysis in a future paper [44].

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