CONTINUOUS AND DISCRETE SYMMETRY FROM CONFORMAL FIELD THEORY

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ABSTRACT

Locality is used to give an explicit construction of twisted conformal field theory. This fixes both the continuous and discrete symmetries of the theory. In general, all weight one conformal fields close to form an affine Kac-Moody algebra, whose zero modes generate the continuous symmetry group. For the Z_2-twisted bosonic theory associated with the Leech lattice, there is no continuous symmetry and the discrete symmetry is the largest finite simple group, the Monster F_1. For Z_N-twisted fermion conformal fields, the weight one-half and a subset of the weight one fields form a twisted super Kac-Moody algebra, whose semidirect product with the super Virasoro algebra has zero mode commutators equivalent to those of the untwisted super Kac-Moody algebra.

1. Introduction

Both continuous and discrete symmetries of the dynamics of particle interactions are fixed by conformal field theory (CFT), when string theory is used to describe nature. In this talk, the concise framework of consistent twisted conformal field theory is reviewed[1]. The explicit construction of the Z_2-bosonic theory associated with a d-dimensional momentum lattice, provides in the case of the Leech lattice, the natural module[2] of the Monster group[3]. It's triality element is identified, and is seen to be a generic feature of twisted conformal field theory[4]. The construction of the vertex operators, i.e. the conformal fields, for all the states is then extended to Z_2-twisted fermionic conformal field theory. The weight one-half states and a subset of all the weight one states in a Z_N-twisted CFT form a twisted super Kac-Moody algebra, whose semidirect product with the super Virasoro algebra is shown to have zero mode commutators *identical* to those of the untwisted super algebra. This follows from the fact that, in general, in the Z_N-twisted sectors, the intertwining relations of the vertex operators requires a shift in the definition of not only the Virasoro generator (which is familiar from the Ramond, i.e. Z_2-twisted sector), but also the super Virasoro generator and the Kac-Moody generators associated with the Cartan subalgebra[5].

This analysis is useful in studying the detailed properties of the vertex operators in superconformal field theory. In particular, a viable low-energy phenomenology predicted by Type II superstrings in four-dimensions would be extremely economical, and thereby have a good chance to offer a precise connection[6−10] between string theory and the standard model. The spontaneous breakdown of space-time supersymmetry and the associated non-vanishing vacuum expectation value of the dilaton field provides a possible resolution of the presently[11] "missing quark doublet" in Type II. Even without supersymmetry breaking, a more thorough investigation and the explicit construction of the conformal fields in the context of a consistent local theory may well indicate that one should take the string more seriously, i.e. that the (supersymmetric) standard model is the ground state of a conformal field theory. In this case, the discrete symmetries of the CFT will be responsible for the absence of baryon and lepton violating interactions[12], since unlike the conventional standard model, it is known that its supersymmetric versions require additional symmetries to eliminate such interactions.

2. Twisted Bosonic CFT

The twisted conformal field theory $\widetilde{\mathcal{H}}(\Lambda)$ associated with a lattice Λ of dimension d is defined for a Z_2 reflection twist by keeping the $\theta = 1$ subset of the states created by integrally-moded bosonic operators a^j_m, $1 \le j \le d$, $m \in \mathbf{Z}$ from momentum states $|\lambda\rangle, \lambda \in \Lambda$. Here $\theta|\lambda\rangle = |-\lambda\rangle$ and $\theta a^j_m \theta^{-1} = -a^j_m$. To this we add in the $\theta = 1$ subspace of the space $\mathcal{H}_T(\Lambda)$ generated from an irreducible representation space $\mathcal{X}_0(\Lambda)$ for the gamma matrix algebra $\{\gamma_\lambda : \lambda \in \Lambda\}$ associated with Λ, by half-integrally moded oscillators c^j_r, $1 \le j \le d$, $r \in \mathbf{Z} + \frac{1}{2}$. In this case, the involution θ is defined by $\theta c_r^j \theta^{-1} = -c_r^j$. The oscillators satisfy the commutation relations $[a_m^i, a_n^j] = m\delta_{m,-n}\delta^{ij}$, $[c_r^i, c_s^j] = r\delta_{r,-s}\delta^{ij}$, and $[a_m^i, c_r^i] = 0$. From the locality requirement of Eq.(12), we find that the twisted CFT is bosonic and meromorphic provided that $d = \dim\Lambda$ is a multiple of eight and both $\sqrt{2}\Lambda^*$ and Λ are even, a condition implied by self-duality of the lattice. If Λ is the $d = 24$ Leech lattice, then $\widetilde{\mathcal{H}}(\Lambda)$ is the natural module for the Monster group.

In the twisted CFT, the untwisted and twisted sectors of $\widetilde{\mathcal{H}}(\Lambda)$ are the subspaces $\mathcal{H}^+(\Lambda)$ and $\mathcal{H}_T^+(\Lambda)$ on which $\theta = 1$. If $\psi \in \mathcal{H}^+(\Lambda)$, $\mathcal{V}(\psi, z)$ maps $\mathcal{H}^+(\Lambda) \to \mathcal{H}^+(\Lambda)$ and $\mathcal{H}_T^+(\Lambda) \to \mathcal{H}_T^+(\Lambda)$ whereas $\mathcal{V}(\chi, z)$ maps $\mathcal{H}^+(\Lambda) \to \mathcal{H}_T^+(\Lambda)$ and $\mathcal{H}_T^+(\Lambda) \to \mathcal{H}^+(\Lambda)$ if $\chi \in \mathcal{H}_T^+(\Lambda)$. Thus we can write these vertex operators in matrix form

$$\mathcal{V}(\psi, z) = \begin{pmatrix} V(\psi, z) & 0 \\ 0 & V_T(\psi, z) \end{pmatrix} \qquad (1)$$

$$\mathcal{V}(\chi, z) = \begin{pmatrix} 0 & \overline{W}(\chi, z) \\ W(\chi, z) & 0 \end{pmatrix} \qquad (2)$$

In this notation, the vertex operators of the twisted CFT $\widetilde{\mathcal{H}}(\Lambda)$ are given by, for the untwisted states: $\psi = \left(\prod_{a=1}^M a_{-m_a}^{j_a}\right)|\lambda\rangle$,

$$\begin{aligned} V(\psi, z) &= \sum_{\lambda' \in \Lambda} \langle\lambda'| : e^{F(-z)} : |\psi\rangle \sigma_{\lambda'} \qquad (3) \\ &=: \left(\prod_{a=1}^M \frac{i}{(m_a - 1)!} \frac{d^{m_a} X^{j_a}(z)}{dz^{m_a}}\right) \exp\{i\lambda \cdot X(z)\}\sigma_\lambda : \end{aligned}$$

and

$$\begin{aligned} V_T(\psi, z) &= V_T^0(e^{\Delta(z)}\psi, z) \\ &= \sum_{\lambda' \in \Lambda} \gamma_{\lambda'}\langle\lambda'| : e^{B(-z)} : e^{A(-z)}|\psi\rangle, \qquad (4) \end{aligned}$$

where

$$\begin{aligned} V_T^0(\psi, z) &= \sum_{\lambda' \in \Lambda} (4z)^{-\frac{1}{2}\lambda'^2} \gamma_{\lambda'}\langle\lambda'| : e^{B(-z)} : |\psi\rangle \\ &=: \left(\prod_{a=1}^M \frac{i}{(m_a - 1)!} \frac{d^{m_a} R^{j_a}(z)}{dz^{m_a}}\right) \exp\{i\lambda \cdot R(z)\} : \\ &\cdot (4z)^{-\frac{1}{2}\lambda^2}\gamma_\lambda \qquad (5) \end{aligned}$$

and

$$X^j(z) = q^j - ip^j \log z + i\sum_{n \neq 0} \frac{a_n^j}{n} z^{-n}, \qquad (6)$$

$$R(z) = i\sum_{r=-\infty}^{\infty} \frac{c_r}{r} z^{-r}; \qquad (7)$$

and for the twisted states: $\chi = \left(\prod_{a=1}^M c_{-m_a}^{j_a}\right)\mathcal{X}_0$, the analogue of the fermion emission operator is

$$W(\chi, z) = e^{zL_{-1}^c}\tilde{W}(\psi, z), \qquad (8)$$

where

$$\tilde{W}(\chi, z) = \sum_{\lambda \in \Lambda} \gamma_\lambda\langle\lambda| : e^{B(z)} : e^{A(z)}|\chi\rangle, \qquad (9)$$

and

$$\overline{W}(\chi, z) = z^{-2h_\chi} W(e^{z^* L_1^c}\overline{\chi}, 1/z^*)^\dagger. \qquad (10)$$

In the above expressions we define

$$\Delta(z) = \frac{1}{2} \sum_{\substack{m,n \geq 0 \\ (m,n) \neq (0,0)}} \binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n} \frac{z^{-m-n}}{m+n} a_m \cdot a_n. \qquad (11)$$

Expressions for A(z), B(z), and F(z) are also written as bilinears in oscillators and are given in Ref.[1,5]. Note that the special state ψ_L is given in these CFT's by $\frac{1}{2}a_{-1} \cdot a_{-1}|0\rangle$ and that its vertex operator is $L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} : a_m \cdot a_{n-m} :$ from Eq.(3), and $L_n^c = \frac{1}{2}\sum_{m=-\infty}^{\infty} : c_m \cdot c_{n-m} : + \frac{d}{16}$ from Eq.(4). The cocycle operators σ_λ and γ_λ on the untwisted and twisted sectors respectively are defined and discussed comprehensively in Ref. [1]. The locality relation satisfied by the vertex operators is:

$$\mathcal{V}(\psi, z)\mathcal{V}(\phi, \zeta) = \mathcal{V}(\phi, \zeta)\mathcal{V}(\psi, z). \qquad (12)$$

3. Twisted Fermionic CFT

The space of states for the Z_2-twisted fermionic theory, $\widetilde{\mathcal{H}}$, is obtained by starting with the states of the untwisted Neveu-Schwarz theory, $\mathcal{H}$, adding in a twisted Ramond sector, $\mathcal{H}_T$, and keeping only the subspace of each defined by $\theta = 1$, with $\theta^2 = 1$. The states of the untwisted theory are generated by the action of d infinite sets of half-integrally moded oscillators, b_s^j, $1 \leq j \leq d$, on the vacuum state, Ψ_0. The twisted sector is obtained from the action of d infinite sets of integrally moded oscillators, d_n^j, on the twisted ground states which form a $2^{d/2}$ irreducible representation, $\mathcal{X}_0$, of the gamma matrix Clifford algebra, $\{\gamma^j\}$. The involution θ is defined

$\theta b^i_s\theta^{-1} = -b^i_s$, and on the twisted space, $\mathcal{H}_T$, by $\theta|0\rangle^{\pm}_R = \pm|0\rangle_R$, $\theta d^i_n\theta^{-1} = -d^i_n$, where $\mathcal{X}_0 = |0\rangle^+_R + |0\rangle^-_R$ and we are assuming d is a multiple of 8, (which is necessary for the spectrum of L^d_0 to contain half-integral values).

This theory consists of fermionic and bosonic fields. As in the bosonic case, the conformal field theories discussed here are defined on the complex plane, or rather the Riemann sphere, and are *chiral*, *i.e.* holomorphic. In this case, the intertwining relation (12) is generally defined by

$$\mathcal{V}(\psi,z)\mathcal{V}(\phi,\zeta) = \epsilon_{\psi\phi}\mathcal{V}(\phi,\zeta)\mathcal{V}(\psi,z) \qquad (13)$$

in the sense of analytic continuation, where $\epsilon_{\psi\phi} = 1$ if either of the states ψ or ϕ are bosons, and $\epsilon_{\psi\phi} = -1$ if both of them are fermions. We will construct (in the F_1- picture) the vertex operators $\mathcal{V}(\psi,z)$, i.e. conformal fields which are in one-to-one correspondence with a basis of states for the theory:

$$\mathcal{V}(\psi,z)|0\rangle = e^{zL_{-1}}\psi . \qquad (14)$$

In the language of superconformal field theory, these vertex operators are the lower components of the superfields. Here $|0\rangle \equiv \Psi_0$ is the vacuum and L_{-1} one of the moments of the special vertex operator $V(\psi_L,z) = \sum_n L_n z^{-n-2}$, which satisfy the Virasoro algebra: $[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{24}m(m^2-1)\delta_{m,-n}$, where m,n run over the integers, $L^\dagger_n = L_{-n}$, and $L_n|0\rangle = 0$ for $n \geq -1$.

The oscillators satisfy the anti-commutation relations $\{b^i_r, b^j_s\} = \delta^{ij}\delta_{r,-s}$, $\{d^i_m, d^j_n\} = \delta^{ij}\delta_{m,-n}$, and $\{b^i_r, d^j_n\} = 0$, where $b^{j\,\dagger}_s = b^j_{-s}$, $b^j_s|0\rangle = 0$, $s > 0$, $d^{j\,\dagger}_n = d^j_{-n}$, $d^j_n|0\rangle_R = 0$, $n > 0$. In these theories, the special state ψ_L is given by $\frac{1}{2}b_{-\frac{3}{2}}\cdot b_{-\frac{1}{2}}|0\rangle$ and that its vertex operator is defined by $L_n = \frac{1}{2}\sum^{\infty}_{s=-\infty}(\frac{1}{2}n - s) : b_s\cdot b_{n-s} :$ in the untwisted sector from Eq. (15), and by $L^d_n = \frac{1}{2}\sum^{\infty}_{m=-\infty}(\frac{1}{2}n - m) : d_m\cdot d_{n-m} : + \frac{d}{16}\delta_{n0}$ in the twisted sector from Eq. (16).

The states in the Neveu-Schwarz sector are given by $\psi = \left(\prod^M_{a=1} b^{j_a}_{-s_a}\right)|0\rangle$, where each s_a is a positive half-odd integer, and the product is understood to be written down in a definite order, *e.g.* left to right, in order to avoid a sign ambiguity,, and each oscillator occurs at most once. The vertex operators for these states with $(m_a = s_a - \frac{1}{2})$ are given by

$$\begin{aligned} V(\psi,z) &= : \left(\prod^M_{a=1}\frac{1}{m_a!}\frac{d^{m_a}b^{j_a}(z)}{dz^{m_a}}\right) : \\ &= \langle 0'| : e^{F(-z)} : |\psi\rangle , \end{aligned} \qquad (15)$$

where we have introduced the Neveu-Schwarz fermion conformal fields $b^j(z) = \sum^{\infty}_{s=-\infty} b^j_s z^{-s-\frac{1}{2}}$ and

$$\begin{aligned} V_T(\psi,z) &= V^0_T(e^{\Delta(z)}\psi,z)) \\ &= \langle 0| : e^{B(-z)} : e^{A(-z)}|\psi\rangle , \end{aligned} \qquad (16)$$

where

$$\begin{aligned} V^0_T(\psi,z) &=: \left(\prod^M_{a=1}\frac{1}{(m_a)!}\frac{d^{m_a}d^{j_a}(z)}{dz^{m_a}}\right) : \\ &= \langle 0| : e^{B(-z)} : |\psi\rangle , \end{aligned} \qquad (17)$$

with the Ramond fermion fields defined as

$$d^j(z) = \sum^{\infty}_{n=-\infty} d^j_n z^{-n-\frac{1}{2}} . \qquad (18)$$

For the twisted states $\chi = \left(\prod^M_{a=1} d^{j_a}_{-m_a}\right)|0\rangle^{\pm}_R$, the fermion emission operator is

$$W(\chi,z) = e^{zL^d_{-1}}\tilde{W}(\psi,z) , \qquad (19)$$

where

$$\tilde{W}(\chi,z) = \langle 0| : e^{B(z)} : e^{A(z)}|\chi\rangle . \qquad (20)$$

In the above expressions, define $A(z) = \Delta(-z)$ where

$$\Delta(z) = \frac{1}{4}\sum_{r,s>0}\begin{pmatrix} -\frac{1}{2} \\ r-\frac{1}{2}\end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ s-\frac{1}{2}\end{pmatrix}\frac{r-s}{r+s}z^{-r-s}b_r\cdot b_s . \qquad (21)$$

Similar expressions for B(z) and F(z) are also written as bilinears in oscillators and are given in Ref.[5]. In general these "lower component" vertex operators are not meromorphic, for *eg.* $V_T(b^j_{-\frac{1}{2}}|0\rangle,z) = d^j(z) = \sum_n d^j_n z^{-n-\frac{1}{2}}$. Therefore although the intertwining relation is satisfied: $V_T(b^j_{-\frac{1}{2}}|0\rangle,z)W(|0\rangle^+_R,\zeta) = W(|0\rangle^+_R,\zeta)V(b^j_{-\frac{1}{2}}|0\rangle,z)$, the operator product expansion $\tilde{W}(|0\rangle^+_R,\zeta)V(b^j_{-\frac{1}{2}}|0\rangle$ is double valued.

4. Twisted Super Kac-Moody Algebra

In fermionic conformal field field theory, the weight one-half and a subset of the weight one fields may form the "lower" and "upper" components of massless superfields, which are the vertex operators for the massless states in the F_1 and F_2-pictures respectively. The moments of these conformal fields generate a super Kac-Moody algebra which forms a semi-direct product with the super Virasoro algebra. In the presence of Z_N-twisted fermionic fields, this set of operators will close to form a twisted super Kac-Moody algebra, whose semi-direct product with the super Virasoro algebra is shown to have zero mode commutators identical to those of the untwisted super algebra.[5]

In this case, the weight one-half fields are given by $h^i(e^{2\pi i}z) = e^{i\pi\omega}h^i(z)$ and $h^\alpha(e^{2\pi i}z) = e^{i\pi\omega}e^{-2\pi i\lambda\cdot\alpha}h^\alpha(z)$, where $\omega = 0,1$ for Neveu-Schwarz and Ramond sectors respectively; i, α label the Cartan subalgebra and the roots, respectively of a dimension d semi-simple Lie algebra g; and the Z_N-twist is labelled by the vector λ such that $-\frac{1}{2} \leq \lambda\cdot\alpha \leq \frac{1}{2}$. In analogy with Eq.(21) we find

$$\Delta(z) = \frac{1}{4}\sum_{\substack{r,s>0\\ i}} \begin{pmatrix} -\frac{1}{2} \\ r-\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ s-\frac{1}{2} \end{pmatrix} \frac{r-s}{r+s} z^{-r-s} b^i_r b^i_s$$
$$-\frac{1}{2}\sum_{\substack{r,s>0\\ \alpha}} (\lambda\cdot\alpha) \begin{pmatrix} \lambda\cdot\alpha-\frac{3}{2} \\ r-\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\lambda\cdot\alpha-\frac{1}{2} \\ s-\frac{1}{2} \end{pmatrix} \frac{z^{-r-s}}{r+s} b^\alpha_r b^{-\alpha}_s \tag{22}$$

A realization of the twisted super Kac-Moody and super Virasoro algebras is given by the vertex operators $H^i(z)$, $E^\alpha(z)$, $L(z)$, and $G(z)$ as follows. The necessity for the shift in the definition of $H^i(z)$ is seen from its identification as $\mathcal{V}(\frac{1}{2}\sum_\alpha \alpha^i b^\alpha_{-\frac{1}{2}} b^{-\alpha}_{-\frac{1}{2}}|0\rangle, z) \equiv H^i(z)$, and from (22) and (16), where $h(z)$ replaces $d(z)$. Here $N(\alpha,\gamma)$ are the structure constants of g in the Cartan-Weyl basis, and $\sum_\alpha \alpha^i\alpha^j = c_\psi\delta^{ij}$. For $\omega = 0$:

$$H^i(z) = \frac{1}{2}\sum_\alpha \alpha^i h^\alpha(z) h^{-\alpha}(z) - \frac{1}{2z}\sum_\alpha \alpha^i \alpha\cdot\lambda \tag{23}$$

$$E^\alpha(z) = \sum_j \alpha^j h^j(z) h^\alpha(z) + \frac{1}{2}\sum_{\substack{\gamma \\ \alpha-\gamma\, \mathrm{root}}} N(-\gamma,\alpha) h^\gamma(z) h^{\alpha-\gamma}(z) \tag{24}$$

$$L(z) = \frac{1}{2}\sum_i \frac{dh^i(z)}{dz} h^i(z) + \frac{1}{2}\sum_\alpha \frac{dh^\alpha(z)}{dz} h^{-\alpha}(z) + \frac{1}{4z^2}\sum_\alpha (\alpha\cdot\lambda)^2 \tag{25}$$

$$G(z) = \frac{1}{2\sqrt{c_\psi/2}}\Big(\sum_{i,\alpha} \alpha^i h^i(z) h^\alpha(z) h^{-\alpha}(z) + \frac{1}{3}\sum_{\substack{\alpha,\gamma \\ \alpha-\gamma\, \mathrm{root}}} N(-\gamma,\alpha) h^\gamma(z) h^{\alpha-\gamma}(z) h^{-\alpha}(z) - \frac{1}{z}\sum_{i,\alpha} \alpha^i \alpha\cdot\lambda h^i(z)\Big) \tag{26}$$

5. References

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