DERIVATIVE MOUFANG TRANSFORMATIONS

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A Moufang loop [1,2] is a quasigroup G with the two-sided identity element e in which the Moufang identity

$$(ag)(ha) = a(gh)a$$

holds. The (first) derivative loop G'_a of G is defined [2] as the Moufang loop with the derivative multiplication

$$(gh)'_a := (g\hat{a})(ah) ,$$

where $\hat{a}$ denotes the inverse element of a in G. Let $Tr(X)$ be the transformation group of a set X and let the identity transformation of X be denoted as E. A pair (S,T) of the mappings $g \to S_g$, $g \to T_g$ of G into $Tr(X)$ is said [3] to be an action of G on X if

$$1) \quad S_e = T_e = E$$

and

$$2) \quad S_g T_g S_h = S_{gh} T_g ,$$

$$3) \quad S_g T_g T_h = T_{hg} S_g$$

are satisfied for all g,h of G. The pair (S,T) is called also a birepresentation of G (in $Tr(X)$). The transformations

$$x \to gx := S_g x \quad , \quad x \to xg := T_g x$$

$(x \in X;\ g \in G)$ are called G-transformations of X.

For a fixed element a of G, the (first) derivative $(S,T)'_a$ of a birepresentation (S,T) of G can be defined as the pair of the mappings

$$g \to (S_g)'_a := T_a S_g T_{\hat{a}} \quad , \quad g \to (T_g)'_a := S_{\hat{a}} T_g S_a$$

of G into $Tr(X)$. The transformations

$$x \to (gx)'_a := (S_g)'_a\, x \quad , \quad x \to (xg)'_a := (T_g)'_a\, x$$

$(x \in X;\ a,g \in G)$ are called the derivatives of G-transformations of X.

1° The derivatives $(gx)'_a$ and $(xg)'_a$ of gx and xg $(x \in X;\ a,g \in G)$ can be redefined by

$$(gx)'_a = (g\hat{a})(ax) \quad \text{and} \quad (xg)'_a = (x\hat{a})(ag)\ ,$$

respectively.

2° The derivatives $(gx)'_a$ and $(xg)'_a$ $(x \in X;\ a,g \in G)$ obey the identities

$$(ag)x = a(gx)'_a \quad , \quad (ax)g = a(xg)'_a\ ,$$

$$x(ga) = (xg)'_{\hat{a}}a \quad , \quad g(xa) = (gx)'_{\hat{a}}a\ .$$

3° (S,T) is closed under the double derivation:

$$((gx)'_a)'_b = (gx)'_{ab} \quad \text{and} \quad ((xg)'_a)'_b = (xg)'_{ab}$$

for all x in X and g,a,b in G. This property can be formally expressed as

$$((S,T)'_a)'_b = (S,T)'_{ab}\ .$$

4° The derivative $(S,T)'_a$ of a birepresentation (S,T) of G turns out to be a birepresentation of the derivative Moufang loop G'_a of G.

5° Every birepresentation of the derivative Moufang loop G'_a of G turns out to be the derivative of some birepresentation of G.

In view of 4° and 5°, the derivatives of birepresentations of G are natural to call its derivative birepresentations. The properties 1° - 5° of derivative G-transformations are in good accordance with the ideas of V.D.Belousov [2].

REFERENCES

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