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Efficiency of AdS-RN Black Holes as Heat Engines in the Framework of Rastall Gravity

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Abstract. We compute the efficiencies of AdS-Reissner-Nordstrom black hole surrounded by quintessence, as heat engines in the framework of Rastall gravity. For this purpose, we construct a conventional heat engine system out of a black hole and tried to analyse the influence of various parameters of the theory on the efficiency of the engine. We also compared the obtained efficiencies with the carnot efficiency which is the maximum efficiency that is achievable for any heat engine. It was found that the parameters of the theory namely the charge q, rastall parameter β and structure constant Ns has drastic influence on increasing or decreasing the efficiencies of the black hole heat engines.

1. Introduction

At the core of black hole thermodynamics lies the fundamental relationship between the macroscopic thermodynamic quantities that characterise black holes (such temperature and entropy) and the quantum characteristics of particles close to the event horizon (like Hawking radiation). This relationship is a fundamental bridge connecting classical thermodynamics and quantum mechanics. The foundation of black hole thermodynamics can be traced back in the seminal work of Bekenstein, Carter and Hawking [1–6], where they proposed the laws of black hole thermodynamics. A thermodynamic system exhibits phase transitions which occur at critical values of thermodynamic quantities. Such a feature was also found to be exhibited by black holes in the 1983 paper of Hawking and Page [6]. Thus, it was firmly established that black holes are indeed treatable as a thermodynamic system with corresponding thermodynamic properties. However, the phase transition phenomena is observed for black holes in anti de-Sitter geometry [7]. The link between the classical bulk geometry of an AdS spacetime and a quantum conformal field theory existing on the boundary of anti-deSitter spacetime gained prominence after Maldacena's work in 1999 [8]. This AdS-CFT correspondance acted as a bridge between quantum and gravitational theory.

The thermodynamics of anti-de Sitter (AdS) black holes in extended phase space has attracted a lot of attention lately ([9–11] for a review). The cosmological constant has been perceived as a new thermodynamic variable, pressure given by the expression :

$$P = \frac{3}{8\pi l^2}, \tag{1}$$

where l represents the AdS radius. Its conjugate volume V can also be determined. Dolan first demonstrated the possibility of extraction of mechanical work from AdS black hole [12]. Joule-Thomson expansion [13,14], holographic heat engine in charged AdS black holes [15], Born-Infeld black holes [16], polytropic black holes [17] to name a few, are some of the recent realisations in this field.

In this work, we have considered a modified theory of gravity which is inspired from violation of energy-momentum conservation and first proposed in 1972 by Rastall and called the Rastall gravity theory [18]. Rastall theory is actively investigted and produces good match with various observations such as accelerated expansion of the universe, galactic rotation curves, gravitational lensing to name a few. Motivated by these works, we have decided to choose Rastall theory of gravity, which produces simpler field equations that are easy to deal with. Regarding the AdS consideration, we are aware of the accelerated expanding universe recently proved by observations. The most feasible and preliminary consideration in literature to explain this expansion theoretically is to consider a quintessential fluid providing the negative pressure for this expansion. In this work, we have computed the efficiency of heat engine constructed out of the AdS black hole in the framework of Rastall gravity. The black hole solution is taken from the Ref [10] and various thermodynamic parameters needed for efficiency calculations have been computed.

The plan of this paper is as follows. In the second section, we have introduced Rastall gravity and Ads black hole solution briefly. In the third section, we have discussed about working of the heat engine and then computed the efficiency of the heat engine. In the fourth section, we have presented the results of the work graphically and in the last section, we have conclusion with future directions.

2. Rastall gravity and AdS black hole solution

Rastall gravity (RG) could not draw much interest when it was first proposed but later a large amount of work has been carried out in this framework. Oliveira and his team studied various properties of neutron stars in RG [19]. Heydarzade and his team first worked out a complete solution of black hole in the framework of RG [20]. Heydarzade and Darabi also worked out black hole solutions with surrounding perfect fluids in RG framework [21]. Another recent work by Cai and team explores quasinormal modes of Schwarzschild black holes surrounded by cloud of strings in RG [22]. We now show some parts of the mathematical treatments and details can be inferred from references listed.

RG modifies the conservation condition of energy-momentum as given below:

$$\nabla_\nu T^{\mu\nu} = a_\mu. \tag{2}$$

For consistency with GR, we should have the following form of the four-vector:

$$a^\mu = \lambda \nabla^\mu R. \tag{3}$$

Here, λ is the free parameter. In the presence of cosmological constant, the field equations for RG takes the following form:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \beta g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{4}$$

where $G_{\mu\nu}$ represents the Einstein tensor and parameter $\beta = \kappa\lambda$ is the Rastall parameter. Λ is the cosmological constant parameter. Now, we present the final solution for the charged black hole surrounded by quintessence in RG framework as computed in Ref. [10] as shown below:

$$f(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2} + N_s r^{\frac{4\beta}{1-2\beta}} + \frac{8\pi P r^2}{3-12\beta}. \tag{5}$$

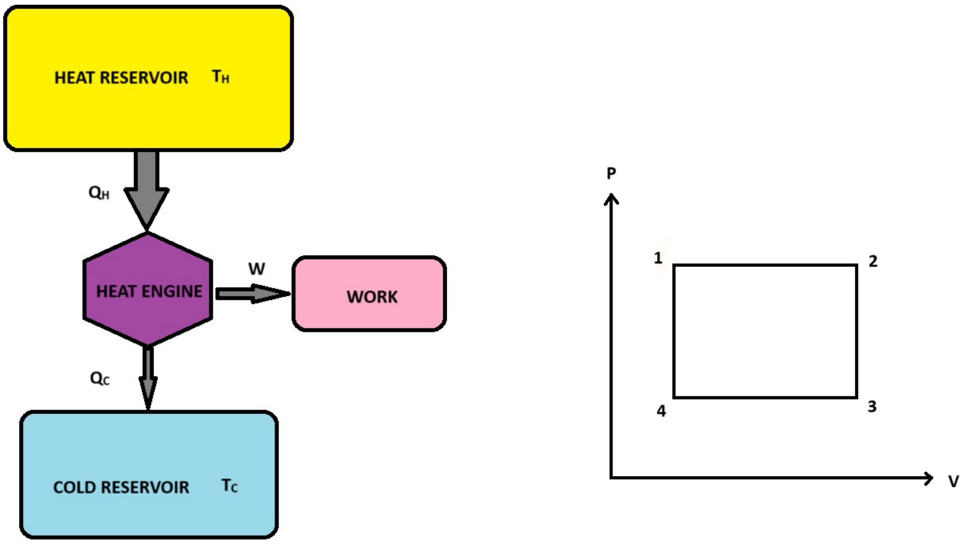

Figure 1. The operation of a Heat engine and the P-V cycle. Image similar to [7] used here.

Here, M and N_s represent black hole mass and structure constant and $\Lambda = -8\pi P$ where P represents pressure. Note that the mathematical aspects of this derivation has been already presented in Ref. [10] and we have considered the equation of state of quintessence $\omega = -\frac{1}{3}$.

3. Heat engine efficiency of the black hole

Heat engine is mathematically fabricated as a closed P-V graph in which heat is absorbed from the reservoir and from this heat Q_H, an amount of work W is extracted and Q_C amount of heat is expelled out. In that manner, the formula for efficiency of such a heat engine comes out to be [7] :

$$\eta = \frac{W}{Q_H}. \tag{6}$$

Carnot engine efficiency represents the maximum possible efficiency that can be obtained for any heat engine and is given by [7]:

$$\eta_c = 1 - \frac{Q_C}{Q_H} = 1 - \frac{T_C}{T_H}. \tag{7}$$

Here, T_C and T_H stands for temperatures of the sink where all the heat flows and the source from which heat is derived respectively. As shown in figure 1, the black hole absorbs Q_H amount of heat during isothermal expansion and Q_H heat is rejected during isothermal compression. We connect these two paths using adiabatic paths as in carnot engines. For simplicity, we have considered rectangular paths as shown in figure. The area covered by the rectangle in the P-V graph gives the effective work done by the black hole during the whole cycle $(1 \to 2 \to 3 \to 4 \to 1)$.

The net work done by the black hole in one complete cycle is given by:

$$W = W_{1\to 2} + W_{3\to 4}. \tag{8}$$

The heat absorbed by the black hole heat engine is given by:

$$Q_H = \int_{S_1}^{S_2} T dS = M_2 - M_1. \tag{9}$$

Before proceeding, we have to determine the thermodynamic quantities associated with the black hole. The temperature of the black hole is calculated to be:

$$T = \frac{f'(r)}{4\pi}\Big|_{r=r_H} = -\frac{\Big(8\beta^2+2\beta-1\Big)N_s r_H^{\frac{2}{1-2\beta}} + (2\beta-1)\Big(r_H^2\Big(-4\beta+8\pi P r_H^2+1\Big)+(4\beta-1)q^2\Big)}{4\pi\Big(8\beta^2-6\beta+1\Big)r_H^3}. \tag{10}$$

We can also obtain the temperature of the black hole using the first law of black hole thermodynamics $T_{BH} = \frac{dM}{dS}$ and obtain the same expression as (10).

Similarly, we can compute the volume parameter related to the black hole metric as [7, 10]:

$$V = \frac{dM}{dP} = \frac{4\pi r_H^3}{3-12\beta}. \tag{11}$$

Finally, the entropy S is calculated from the black hole metric as:

$$S = \pi r_H^2. \tag{12}$$

Now, we come back to our expression for work extracted from the black hole from equation (6):

$$W = \frac{4(P_1-P_4)(S_2^{\frac{3}{2}}-S_1^{\frac{3}{2}})}{\pi^{\frac{1}{2}}(3-12\beta)}. \tag{13}$$

And the expression for efficiency of the black hole is calculated as:

$$Q_H = \frac{\sqrt{S_2}-\sqrt{S_1}+\pi q^2(S_2^{-\frac{1}{2}}-S_1^{-\frac{1}{2}}) - N_s\pi^{1+\frac{1}{-1+2\beta}}(S_1^{\frac{1+2\beta}{2-4\beta}}-S_2^{\frac{1+2\beta}{2-4\beta}}) + \frac{8P_1(S_1^{\frac{3}{2}}-S_2^{\frac{3}{2}})}{-3+12\beta}}{2\sqrt{\pi}}. \tag{14}$$

From equation (7), we have the expression of efficiency of the heat engine as:

$$\eta = \frac{\frac{4(P_1-P_4)(S_2^{\frac{3}{2}}-S_1^{\frac{3}{2}})}{\pi^{\frac{1}{2}}(3-12\beta)}}{\frac{\sqrt{S_2}-\sqrt{S_1}+\pi q^2(S_2^{-\frac{1}{2}}-S_1^{-\frac{1}{2}}) - N_s\pi^{1+\frac{1}{-1+2\beta}}(S_1^{\frac{1+2\beta}{2-4\beta}}-S_2^{\frac{1+2\beta}{2-4\beta}}) + \frac{8P_1(S_1^{\frac{3}{2}}-S_2^{\frac{3}{2}})}{-3+12\beta}}{2\sqrt{\pi}}}. \tag{15}$$

Also, the expression for Carnot efficiency gives us:

$$\eta_c = 1 - \frac{S_2^{3/2}\Big[\pi^{\frac{1}{2\beta-1}}\left(8\beta^2+2\beta-1\right)N_s S_1^{\frac{1}{1-2\beta}} + (2\beta-1)\left(\frac{S_1(-4\beta+8P_4S_1+1)}{\pi}+(4\beta-1)q^2\right)\Big]}{S_1^{3/2}\Big[\pi^{\frac{1}{2\beta-1}}\left(8\beta^2+2\beta-1\right)N_s S_2^{\frac{1}{1-2\beta}} + (2\beta-1)\left(\frac{S_2(-4\beta+8P_2S_2+1)}{\pi}+(4\beta-1)q^2\right)\Big]}. \tag{16}$$

In the following, we show a tabulated data of efficiency and its dependence with the model parameters.

Table 1. Table showing the efficiency of the black hole heat engine with various values of the model parameters. Here, values of parameters $P_1 = 4$, $P_4 = 1$, $S_1 = 1$, $S_2 = 2$, $P_2 = 10$ has been chosen.

q	β	N_s	Efficiency (η)	$\frac{\eta}{\eta_c}$
0.5	0.3	1.0	0.7556	0.7032
2.0	0.3	1.0	0.7296	0.6642
5.0	0.3	1.0	0.6117	0.4908
2.0	0.1	1.5	0.8124	0.7974
2.0	0.4	1.5	0.6878	0.5986
1.0	0.1	0.5	0.7558	0.7087
1.0	0.1	2.0	0.7382	0.6860

4. Results and Conclusion

We have computed the expression for the efficiency as well as the carnot efficiency of the black hole heat engine and now we are in a position to show the dependence of various parameters of the theory on the efficiency and effective efficiency of the heat engine.

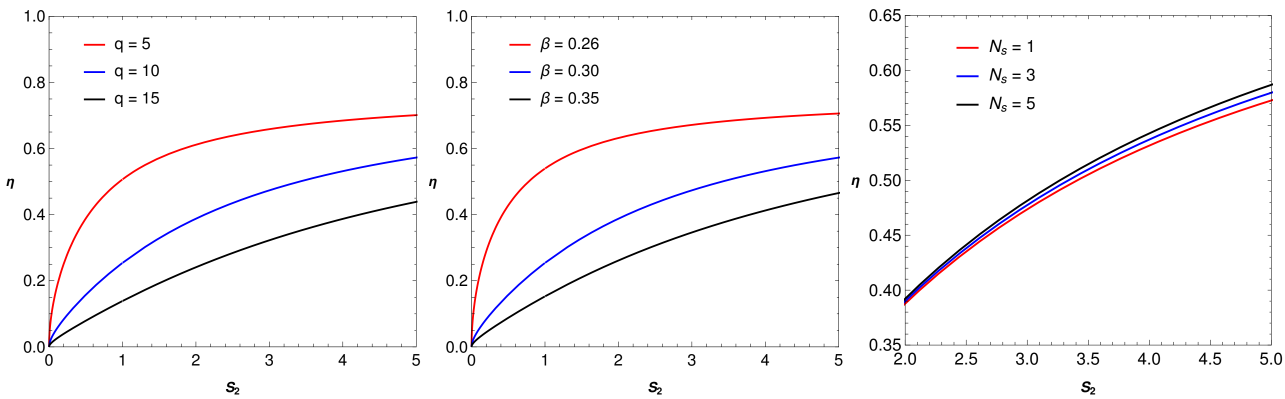

Figure 2. Efficiency of black hole versus entropy S_2 for different variations in model parameters. For the first plot, we use $P_1 = 4$, $P_4 = 1$, $S_1 = 1$, $\beta = 0.3$, $N_s = 1$. For the second plot, we use $P_1 = 4$, $P_4 = 1$, $S_1 = 1$, $q = 10$, $N_s = 1$ and for the third plot, we have $P_1 = 4$, $P_4 = 12$, $S_1 = 1$, $q = 10$, $\beta = 0.3$.

The figure 2 shows the variation of efficiency versus entropy S_2 for different values of the model parameters q, β and N_s for different mentioned values of the parameters. From the first plot, it is evident that efficiency increases more rapidly for smaller values of charge q but for higher entropy, it saturates in all cases. The second plot shows the efficiency versus entropy curve for different values of β parameter. It is clear that lower values of β favours higher efficiency. This trend reverses in case of structure parameter N_s where efficiency increases with increasing N_s. In all cases, efficiency of the black hole max out at approximately 70%. In the next figure 3, we show the relative efficiency of the black hole heat engine versus the entropy S_2 for mentioned values of parameters. Here we see similar trend as of the previous case. This implies that charge q and β take smaller values for higher relative efficiencies while opposite situation arise for N_s. One thing to note is that for higher values of charge q, we see an oscillating pattern which

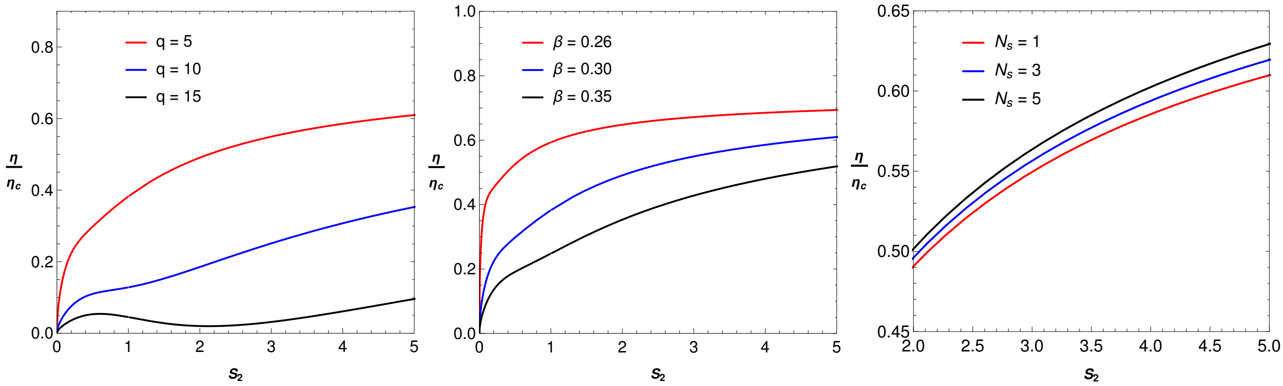

Figure 3. The relative efficiency of the heat engine versus entropy S_2 with various values of model parameters. The first plot uses $P_1 = 4$, $P_4 = 1$, $S_1 = 1$, $\beta = 0.3$, $N_s = 1$, $P_2 = 10$. The second plot uses $P_1 = 4$, $P_4 = 1$, $S_1 = 1$, $q = 10$, $N_s = 1$, $P_2 = 10$ and for the third plot, we use $P_1 = 4$, $P_4 = 12$, $S_1 = 1$, $q = 5$, $\beta = 0.3$, $P_2 = 10$.

flattens out for higher S_2. This is missing for smaller q plots.

In this work, we have constructed a mathematical framework of an AdS black hole with quintessential surrounding as a heat engine and calculated its efficiency. The aim of this work is to show the effect of the model parameters of the AdS black hole (surrounded by quintessence matter in RG) on the theoretical efficiency and carnot efficiency respectively. It is concluded that the model parameters have a sizeable influence on efficiency of the heat engine which has been mentioned in the earlier part of this section.

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