



# Anisotropic homogeneous string cosmology with two-loop corrections

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## Abstract

The two-loop (order  $\alpha'$ )  $\beta$ -function equations, which are equivalent to the equations of motion of  $\alpha'$ -corrected string effective action, are considered for anisotropic homogeneous space–times. These equations are solved for all Bianchi-type models in two schemes of effective action, namely  $R^2$  and Gauss–Bonnet schemes with zero cosmological constant and then the metric, dilaton and  $B$ -field are found at  $\alpha'$  perturbative corrections.

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## 1. Introduction

The application of low energy, tree-level string effective action for describing the evolution of early universe with a very low coupling,  $g = e^{-\phi}$ , and curvature is well accepted [1–4]. Solutions of the low energy string effective action have been found in several cosmological backgrounds. For example, the case of Friedmann–Robertson–Walker (FRW) backgrounds have been discussed in [5], homogeneous anisotropic space–times with contributing of the antisymmetric tensor field have been investigated in [6–8],  $(4 + 1)$  dimensional homogeneous anisotropic models have been studied in [9] and inhomogeneous models have been discussed in [10,11]. However, evolving the universe toward a strong coupling, dilaton and curvature will gain an inevitable ac-

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celerated growth [12,13]. But the singularity of curvature and coupling is widely believed to be regularized by the expansion of the leading order effective action and including higher-order corrections [14–16]. Generally, there are two types of expansions consisting of a quantum nature expansion in string coupling named string loop expansion and a stingy type  $\alpha'$  expansion, where the string-length is  $\lambda_s = \sqrt{\alpha'}$  [4]. The  $\alpha'$ -corrections are commonly considered for regularizing the curvature. Consideration of the  $\alpha'$ -corrections to the low energy effective action is important in the case of high curvature, where the loop-corrections are significant when the string coupling is strong enough. So, until the coupling is sufficiently small in the high curvature regime the loop corrections can be neglected and the  $\alpha'$ -corrections are included only [4].

Vanishing of the two-loop  $\beta$ -function equations of the corresponding  $\sigma$ -model which are the requirement of the conformal invariance of the theory up to two-loop, are equivalent to the background field equations of motion of the  $\alpha'$ -corrected string effective action in the string frame. The two-loop  $\beta$ -functions and the possible higher order  $\alpha'$ -correction terms to the string effective actions are investigated in [17,18]. These corrections are generically quadratic type terms and give higher order time derivative field equations. There is a renormalization scheme (RS) dependence in the two-loop order of  $\beta$ -functions and their  $\alpha'$ -corrected effective action, where the two Gauss–Bonnet and  $R^2$  schemes are distinguished [17]. The effective action of the former case has nice properties like being unitary, physical ghost free and having  $O(d, d)$  symmetry which is related to the T-duality [19–21].

The  $\alpha'$ -corrected field equations for massless string modes have been solved for several classes of backgrounds such as black hole [22–25] and cosmology [26–36]. The dynamics of the Bianchi IX and I universe in Gauss–Bonnet gravity from the 1-loop superstring effective action have been investigated in [37], [38]. There are some works on solving two-loop  $\beta$ -function equations, for instance, in the [36], where the antisymmetric  $B$ -field is set to zero and the metric and dilaton fields are considered only.

In fact, in the presence of the quadratic corrections the equations are not easy to solve. Furthermore, if the  $B$ -field is considered too, the equations become more nontrivial. There are some few classes of such solutions, for example [39,40], where the equations of motion of the effective field theory are considered. Here our goal is to investigate the solutions of two-loop  $\beta$ -function equations with a dilaton and a  $B$ -field contributions on homogeneous anisotropic space–times. Indeed, the RS dependence of the two-loop  $\beta$ -functions appears in the presence of an antisymmetric  $B$ -field. Therefore, including the  $B$ -field requires considering the RS dependence and the two useful schemes of  $R^2$  and Gauss–Bonnet are our interests.

The paper is organized as follows. In section 2 we recall the general form of two-loop  $\beta$ -functions of the two RS. In section 3 we derive the explicit forms of the two-loop  $\beta$ -function equations on the anisotropic homogeneous space–times in terms of Hubble coefficients of the string frame, dilaton and  $B$ -field in the  $R^2$  and Gauss–Bonnet schemes. In Section 4 we present the solutions of the  $\beta$ -function equations in all Bianchi-type models. Finally, the main results of this paper are finally summarized in Section 5.

## 2. Two-loop (order $\alpha'$ ) $\beta$ -functions and $\alpha'$ -corrected string effective action

Vanishing of the  $\beta$ -functions of  $\sigma$ -model is the requirement of conformal symmetry of the theory [41]. On the other hand, the  $\beta$ -functions are equivalent to the equations of motion of the effective action [17]. The two-loop  $\beta$ -functions, the  $\alpha'$ -corrected effective actions and the on-shell equivalence of the equations of motion of the effective action and the conformal invariance

conditions in  $\alpha'$  order have been considered in [17]. Generally, the  $\beta$ -functions of the  $\sigma$ -model with the backgrounds of dilaton  $\phi$ , antisymmetric  $B$ -field and metric  $G$  are given by [17]

$$\bar{\beta}_{\mu\nu}^G = \beta_{\mu\nu}^G + 2\alpha' \nabla_\mu \nabla_\nu \phi + \nabla_{(\mu} W_{\nu)}, \tag{1}$$

$$\bar{\beta}_{\mu\nu}^B = \beta_{\mu\nu}^B + \alpha' H_{\mu\nu}{}^\lambda \nabla_\lambda \phi + \frac{1}{2} H_{\mu\nu}{}^\lambda W_\lambda + \partial_{(\mu} L_{\nu)}, \tag{2}$$

$$\bar{\beta}^\phi = \beta^\phi + W_2^\mu \partial_\mu \phi, \tag{3}$$

where the  $\beta_{\mu\nu}^G$ ,  $\beta_{\mu\nu}^B$  and  $\beta^\phi$  are the standard RG  $\beta$ -functions. The field strength  $H$  is defined by  $H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}$  and the  $W_\mu$  and  $L_\mu$  are given by the renormalization matrix as follows [41]

$$W_\mu = \alpha'^2 (l_1 \nabla_\mu H^2 + l_2 H_{\mu\lambda\rho} \nabla_\sigma H^{\sigma\lambda\rho}), \tag{4}$$

$$L_\mu = \alpha'^2 l_3 H_{\mu\lambda\rho} \nabla_\sigma H^{\sigma\lambda\rho}. \tag{5}$$

There is a RS dependence in  $\beta$ -function equations at two-loop order in the presence of  $B$ -field. These equations are calculated in [17] where the  $R^2$  and Gauss–Bonnet schemes are pointed especially. By choosing any of RS’s, the resulting  $\beta$ -functions will be different, but transformation between the RS’s can be easily done by using the leading order of equations of motion and appropriate field redefinitions [17,18]. Solutions of the  $\beta$ -function equations of any RS will be different, but still equivalent because they belong to different definitions of physical metric, dilaton and  $B$ -field. The two-loop  $\beta$ -functions of the metric,  $B$ -field are as follows [17]

$$\begin{aligned} \frac{1}{\alpha'} \bar{\beta}_{\mu\nu}^G = & R_{\mu\nu} - \frac{1}{4} H_{\mu\nu}^2 - \nabla_\mu \nabla_\nu \phi + \frac{\alpha'}{2} [R_{\mu\alpha\beta\gamma} R_{\nu}{}^{\alpha\beta\gamma} - \frac{3}{2} R_{(\mu}{}^{\alpha\beta\gamma} H_{\nu)\alpha\lambda} H_{\beta\gamma}{}^\lambda \\ & - \frac{1}{2} R^{\alpha\beta\rho\sigma} H_{\mu\alpha\beta} H^{\nu\rho\sigma} + \frac{1}{8} (H^4)_{\mu\nu} \\ & + \frac{1}{4} \nabla_\lambda H_{\mu\alpha\beta} \nabla^\lambda H_{\nu}{}^{\alpha\beta} + \frac{1}{12} \nabla_\mu H_{\alpha\beta\gamma} \nabla_\nu H^{\alpha\beta\gamma} + \frac{1}{8} H_{\mu\alpha\lambda} H_{\nu\beta}{}^\lambda (H^2)^{\alpha\beta} \\ & - \frac{f}{2} (R_{\mu\alpha\beta\nu} (H^2)^{\alpha\beta} + 2 R_{(\mu}{}^{\alpha\beta\gamma} H_{\nu)\alpha\lambda} H_{\beta\gamma}{}^\lambda + R^{\alpha\beta\rho\sigma} H_{\mu\alpha\beta} H^{\nu\rho\sigma} \\ & - \nabla_\lambda H_{\mu\alpha\beta} \nabla^\lambda H_{\nu}{}^{\alpha\beta}) - \frac{1}{12} \nabla_\mu \nabla_\nu H^2], \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{1}{\alpha'} \beta_{\mu\nu}^B (f = 1) = & -\frac{1}{2} \nabla^\mu H_{\mu\nu\rho} + \frac{\alpha'}{4} (2R_{[\mu\gamma\alpha\beta} \nabla^\gamma H^{\alpha\beta}{}_{\nu]} + \nabla_\gamma H_{\alpha\beta[\mu} H_{\nu]\rho}{}^\alpha H^{\beta\gamma\rho} \\ & + 2\nabla_\beta H_{\alpha[\nu} H_{\mu]}{}^{\alpha\beta} + \frac{1}{2} H_{\alpha\beta}^2 \nabla^\alpha H^\beta{}_{\mu\nu}), \end{aligned} \tag{7}$$

$$\frac{1}{\alpha'} \beta_{\mu\nu}^B (f = -1) = \hat{R}_{[\mu\nu]} + \frac{\alpha'}{2} \left( \hat{R}^{\alpha\beta\gamma}{}_{[\nu} \hat{R}_{\mu]\alpha\beta\gamma} - \frac{1}{2} \hat{R}^{\beta\gamma\alpha}{}_{[\nu} \hat{R}_{\mu]\alpha\beta\gamma} + \frac{1}{2} \hat{R}_{\alpha[\mu\nu]\beta} H^{\alpha\beta} \right), \tag{8}$$

where  $H^4 = H_{\mu\nu\lambda} H^{\nu\rho\kappa} H_{\rho\sigma}{}^\lambda H^{\sigma\mu}{}_\kappa$ ,  $H_{\mu\nu}^2 = H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma}$  and the  $\hat{R}_{\mu\nu\sigma}^\rho$  is the Riemann tensor associated with the generalized connection for the  $\sigma$ -model torsion  $\hat{\Gamma}_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho - \frac{1}{2} H_{\mu\nu}^\rho$ .<sup>1</sup> The  $f$  parameter indicates the RS dependence and especially the  $f = 1$  and  $f = -1$  are correspond to the  $R^2$  and Gauss–Bonnet schemes, respectively.

<sup>1</sup>  $\hat{R}^\kappa{}_{\lambda\mu\nu} = R^\kappa{}_{\lambda\mu\nu} - \frac{1}{2} \nabla_\nu H_{\nu\lambda}^\kappa - \frac{1}{2} \nabla_\nu H_{\mu\lambda}^\kappa + \frac{1}{4} H_{\nu\lambda}^\gamma H_{\mu\gamma}^\kappa - \frac{1}{4} H_{\mu\lambda}^\gamma H^\kappa{}_{\nu\gamma}$ .

For the effective action, there are two parameterizations namely the  $\sigma$ -parameterization and the  $s$ -parameterization where their metrics are related with each other in 4 dimensional space-time by

$$\bar{G}_{\mu\nu}^{(\sigma)} = e^\phi G_{\mu\nu}^{(s)}. \tag{9}$$

Alternatively, these two parameterizations indicate two frames called respectively string frame,  $G_{\mu\nu}^{(\sigma)}$ , and Einstein frame,  $\bar{G}_{\mu\nu}^{(s)}$ . The  $\alpha'$ -corrected effective action in the  $\sigma$ -parameterization is given by [17]

$$S = \int d^4x \sqrt{G} e^\phi \left( R - \frac{1}{12} H^2 + (\nabla\phi)^2 - \Lambda + \frac{\alpha'}{4} (R_{\mu\nu\rho\lambda}^2 - \frac{1}{2} R^{\alpha\beta\rho\sigma} H_{\alpha\beta\lambda} H_{\rho\sigma}{}^\lambda + \frac{1}{24} H^4 - \frac{1}{8} (H_{\mu\nu}{}^2)^2 \right). \tag{10}$$

Its variation with respect to the dilaton gives the averaged  $\beta$ -function of dilaton,  $\tilde{\beta}^{(\phi)}$ , which can be written in terms of metric and dilaton  $\beta$ -functions as follows

$$\tilde{\beta}^\phi = \bar{\beta}^\phi - \frac{1}{4} \bar{\beta}_{\mu\nu}^G G^{\mu\nu}, \tag{11}$$

and is given by

$$\frac{1}{\alpha'} \tilde{\beta}^\phi = -R + \frac{1}{12} H^2 + 2\nabla_\mu \nabla^\mu \phi + (\partial_\mu \phi)^2 + \Lambda - \frac{\alpha'}{4} (R_{\mu\nu\rho\lambda}^2 - \frac{1}{2} R^{\alpha\beta\rho\sigma} H_{\alpha\beta\lambda} H_{\rho\sigma}{}^\lambda + \frac{1}{24} H^4 - \frac{1}{8} (H_{\mu\nu}{}^2)^2). \tag{12}$$

Furthermore, the Gauss–Bonnet effective action  $s$ -parameterization in 4-dimensions for bosonic string is given by [17]

$$S = \int d^4x \sqrt{G} \left( R - \frac{1}{12} e^{2\phi} H^2 - \frac{1}{2} (\nabla\phi)^2 - \Lambda e^\phi + \frac{\alpha' e^\phi}{4} [R_{\mu\nu\rho\lambda}^2 - 4R_{\mu\nu}^2 + R^2 + e^{2\phi} (-\frac{1}{2} R^{\alpha\beta\rho\sigma} H_{\alpha\beta\lambda} H_{\rho\sigma}{}^\lambda + \frac{1}{2} H_{\mu\nu}^2 \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{6} H^2 (\nabla\phi)^2) + e^{2\phi} (\frac{1}{24} H^4 + \frac{1}{8} (H_{\mu\nu}{}^2)^2 - \frac{5}{144} (H^2)^2) \right]. \tag{13}$$

In fact, the  $\alpha'$ -corrected effective action in its most general form is parametrized by 20 constants [42]. Since  $S$ -matrix is invariant under the field redefinitions of type [42]

$$\delta G_{\mu\nu} = \alpha' (b_1 R_{\mu\nu} + b_2 \partial_\mu \phi \partial_\nu \phi + b_3 H_{\mu\nu}^2 + G_{\mu\nu} (b_4 R + b_5 (\partial\phi)^2 + b_6 \nabla^2 \phi + b_7 H^2)), \tag{14}$$

$$\delta B_{\mu\nu} = \alpha' (b_8 \nabla^\lambda H_{\lambda\mu\nu} + b_9 H_{\mu\nu}{}^\lambda \partial_\lambda \phi), \tag{15}$$

$$\delta \phi = \alpha' (b_{10} R + b_{11} (\partial\phi)^2 + b_{12} \nabla^2 \phi + b_{13} H^2), \tag{16}$$

there is a field redefinition ambiguity and we have a class of physically equivalent effective actions [43]. The (14)–(16) can leave 3 constants together with 5 combinations of the 20 constants invariant. Thus, the  $\alpha'$ -corrected effective action is parametrized by 8 essential coefficients and it is easy to relate it to the Gauss–Bonnet effective action by an appropriate field redefinition

and using the leading order equations of motion [17]. Hence, in the  $\sigma$ -model context, if one works with the higher order corrections in effective field theory and the equations of motions, it is convenience to transform by the field redefinitions to the Gauss–Bonnet effective action which has no higher than second derivative in its field equations and is free of ghost. The 8 essential coefficients that parametrize the  $\alpha'$ -corrected effective action must be fixed by comparison with the  $S$ -matrix [17]. Also, the two-loop  $\beta$ -functions and the equations of motion of the effective action are identical if a field redefinition is applied [17]. Working with the field equation in the effective field theory, the field redefinitions may be considered, for instance as it has been done in [39]. In this work, we will not redefine the fields and maintain in the convention provided by the  $\beta$ -functions.

Moreover, it is worth mentioning that in a non-critical  $D$  dimensional bosonic theory, the constant  $\Lambda$  equals to  $\frac{2(D-26)}{3\alpha'}$  and is related to the central charge deficit of the original theory [44, 45]. From a cosmological point of view, the  $\Lambda$  is analogous to the non-vanishing cosmological constant in standard theory of gravity [46]. We will set  $\Lambda = 0$  in our analysis. This may be a good approximation at early times if compared with the  $\Lambda$  the curvature and/or kinetic energy are large,  $\Lambda \ll R, (\nabla\phi)^2, H^2$  [47]. However,  $\Lambda$  is required to be considered for expanding universe at late times.

### 3. Anisotropic homogeneous two-loop string cosmology

Anisotropic homogeneous space–times are achieved by relaxing the isotropy requirement of FRW-type models. These type of space–times have been revealed in string cosmology models when approaches to find special backgrounds were being investigated to describe the early evolution of the universe. Their general metric in string frame is written as the following form [48]

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = -g_{00}(t) dt^2 + e_\alpha^i(x) g_{ij}(t) e_\beta^j(x) dx^\alpha dx^\beta, \quad (17)$$

where  $\alpha$  and  $\beta$  are indices of spatial the submanifold and the  $i, j$  are indices of the bases of an isometry group  $G$ . The  $e_\alpha^i(x)$  denote vielbeins which for any Bianchi model are given in Appendix A. Non-diagonal components of the algebraic metric  $g_{ij}$  which give constraint in the  $\beta$ -function equations can be taken zero. Hence, as a choice one can set [7]

$$g_{ij}(t) = a_i^2(t) \delta_{ij}, \quad (18)$$

where  $a_i^2$  are the string frame scale factors. Hence, according to the (9), the Einstein frame scale factors will be  $\bar{a}_i^2 = e^\phi a_i^2$ . If  $\{\sigma^i, i = 1, 2, 3\}$  indicate the left invariant basis of 1-forms, the 3-form  $H$  may be chosen as [7]

$$H = A \sigma^1 \wedge \sigma^2 \wedge \sigma^3. \quad (19)$$

The requirement of  $dH = 0$  implies  $A$  to be a constant.

Substituting the (17) (along with (18)) and (19) in (6) and using the relations (198)–(200), the  $(i, i)$  and time–time components of  $\beta$ -function of metric (6) cast the following general forms, respectively<sup>2</sup>

$$\frac{1}{\alpha'} \bar{\beta}^G i_i = \dot{H}_i + H_i \sum H_k + H_i \dot{\phi} - H_i (\ln \dot{g}_{00}) + V_i^{(1)} - \frac{1}{2} A^2 \bar{a}^{-6} e^{3\phi} + \alpha' (K_i + V_i^{(2)}), \quad (20)$$

<sup>2</sup> Because the results may be complicated, we present the following formalism with more details for a particular isotropic case on Bianchi-type  $V$  in Appendix B.

$$\frac{1}{\alpha'} \tilde{\beta}^G 0 = \sum (\dot{H}_i + H_i^2) + \ddot{\phi} - (\ln \dot{g}_{00})(\dot{\phi} + \sum H_i) - \alpha'(K_0 + V_0^{(2)}), \tag{21}$$

where  $\bar{a}^3 = \bar{a}_1 \bar{a}_2 \bar{a}_3$  and the dot stands for  $d/dt$ . The Hubble coefficients of string frame are defined with  $H_i = (\ln \dot{a}_i(t))$ . The above equations have been written in terms of auxiliary functions of  $V$  and  $K$  in order to get a general and shorthanded form of  $\beta$ -equations in all Bianchi-type models. All the terms with dependence on structure constants of lie-group of models are collected in the  $V$  terms and the others with no dependence on structure constants are gathered in  $K$  part. So, throughout this paper  $V$  and  $K$  denote Bianchi-type dependent and independent terms, respectively. The  $V^{(1)}$  and  $V^{(2)}$  are originated from one-loop and two-loop orders of  $\beta$ -equations, receptively. This terms will be declared for each Bianchi model in the next section. The  $K$  terms of (20) and (21) that are common for all types are given as following

$$K_i = \dot{H}_i^2 + 2H_i^2 \dot{H}_i + H_i^2 \sum H_k^2 + \frac{3}{8} A^4 \bar{a}^{-12} e^{6\phi} - \frac{1}{4} A^2 \bar{a}^{-6} e^{3\phi} (2(f+4)H_i^2 + \sum_{k,j \neq i} (4(2f+1)H_k^2 + 2(4f+11)H_i H_k + (5f+3)H_j H_k)), \tag{22}$$

$$K_0 = - \sum (\dot{H}_i^2 + 2\dot{H}_i H_i^2 + H_i^4) - \frac{1}{2} A^2 \bar{a}^{-6} e^{3\phi} (\sum (\dot{H}_i + 2f H_i^2 - 2 \sum_{k < i} H_k H_i)). \tag{23}$$

Obviously, there is a RS dependence in the  $K_i$  terms expressed by  $f$  parameter which equals to 1 for  $R^2$  scheme and  $-1$  for Gauss–Bonnet scheme. Furthermore, with the considered metric (17), in the Bianchi-type of class  $\mathcal{B}^3$  the  $(0, i)$  components of  $\beta$ -function equations of metric (6) give constraint equations that will be provided in the following section.

Furthermore, in the same way the dilaton  $\beta$ -function (12) casts the following general form

$$\frac{1}{\alpha'} \tilde{\beta}^\phi = -2\ddot{\phi} - \dot{\phi}^2 - \sum (2\dot{\phi} H_i + V_i^{(1)} + H_i^2 + 2\dot{H}_i) - (\sum H_i^2) + 2(\phi' + \sum H_i)(\ln \dot{g}_{00}) + \frac{1}{2} A^2 \bar{a}^{-6} e^{3\phi} + \alpha'[K_\phi + V_\phi^{(2)}], \tag{24}$$

where its  $K$  term is given by

$$K_\phi = - \sum (\dot{H}_i^2 + 2H_i^2 \dot{H}_i + H_i^4) - \sum_{k < i} H_i^2 H_k^2 + \frac{1}{4} A^2 \bar{a}^{-6} e^{-3\phi} (\sum_{k < i} H_k H_i - 5A^2 \bar{a}^{-6} e^{-3\phi}). \tag{25}$$

The Bianchi-type dependent term  $V_\phi^{(2)}$  term which is produced by the two-loop order of dilaton  $\beta$ -function will be given in the next section.

The solutions of the one-loop  $\beta$ -function equations on anisotropic homogeneous space–time with the field strength of (19) have been investigated in [7]. Here, our purpose is to solve the two-loop  $\beta$ -functions on both RS. For performing this, we implement a perturbative series expansion in  $\alpha'$  up to first order on the background fields as following

<sup>3</sup> The classification of Bianchi-type models and their structure constants are given in the [Appendix A](#).

$$\phi = \phi_0 + \alpha' \phi_1, \tag{26}$$

$$\bar{a}_i^2 = a_i^2 e^{\phi_0} (1 + 2\alpha' X_i), \tag{27}$$

$$g_{00} = 1 + 2\alpha' \xi_4. \tag{28}$$

In (27) a series expansion for the Einstein frame scale factors  $a_i^2 e^{\phi}$  has been introduced. In fact, all of the  $\beta$ -function equations are in the string frame. However, we will briefly write them in terms of Einstein frame scale factors.

Now, it is worth to introduce a new time coordinate  $\tau$  [7]

$$d\tau = a^{-3} e^{-\phi} dt, \tag{29}$$

where  $a^3 = a_1 a_2 a_3$ . In the new coordinate  $\tau$ , putting the  $\beta$ -functions (20), (21) and (24) zero, we come to solve the equations. The  $\hat{\beta}^\phi$  equation (24) with using (20) and (21) leads to the following two equations in the zeroth and first order of  $\alpha'$ , respectively

$$\phi_0'' + A^2 e^{2\phi_0} = 0, \tag{30}$$

$$\phi_1'' + 2A^2 e^{2\phi_0} \phi_1 + 2A^2 e^{2\phi_0} \xi_4 - \phi_0' \xi_4' + \rho = 0, \tag{31}$$

where the prime stands for the derivative with respect to the  $\tau$ . The  $\rho$  term denotes the inhomogeneous part of the nonlinear equation and is given as following

$$\rho = -\hat{K}_\phi - \hat{V}_\phi^{(2)} - \sum (\hat{K}_i + \hat{V}_i^{(2)}) + \hat{K}_0 + \hat{V}_0^{(2)}. \tag{32}$$

The hatted  $K$  and  $V$  stand for the corresponding terms in the  $\tau$  coordinate multiplied with a  $a^6 e^{2\phi_0}$  factor. The  $\hat{V}$  terms of every Bianchi model will be given in the next section. The  $\hat{K}$  terms based on (22), (23) and (25) with using (29) are given as follows

$$\begin{aligned} \hat{K}_i = & [((\ln a_i)'' - (\ln a_i)'(\phi_0' + \sum (\ln a_j)'))^2 \\ & + 2(\ln a_i)'^2 ((\ln a_i)'' - (\ln a_i)'(\phi_0' + \sum (\ln a_j)')) \\ & + (\ln a_i)'^2 \sum (\ln a_k)'^2] a^{-6} e^{-2\phi_0} + \frac{3}{8} A^4 a^{-6} e^{2\phi} \\ & - \frac{1}{4} A^2 [2(f+4)(\ln a_i)'^2 + \sum_{k,j \neq i} (4(2f+1)(\ln a_k)')^2 \\ & + 2(4f+11)(\ln a_i)'(\ln a_k)' + (5f+3)(\ln a_j)'(\ln a_k)'], \end{aligned} \tag{33}$$

$$\begin{aligned} \hat{K}_0 = & - \sum [((\ln a_i)'' - (\ln a_i)'(\phi_0' + \sum (\ln a_j)'))^2 \\ & + 2((\ln a_i)'' - (\ln a_i)'(\phi_0' + \sum (\ln a_j)'))(\ln a_i)'] \\ & + (\ln a_i)'^4] a^{-6} e^{-2\phi_0} - \frac{1}{2} A^2 a^{-6} [\sum ((\ln a_i)'' - (\ln a_i)'(\phi_0' + \sum (\ln a_j)')) \\ & + 2f(\ln a_i)'^2 - 2 \sum (\ln a_k)'(\ln a_i)'], \end{aligned} \tag{34}$$

$$\begin{aligned} \hat{K}_\phi = & - \sum_{k < i} [((\ln a_i)'' - (\ln a_i)'(\phi_0' + \sum (\ln a_j)'))^2 \\ & + 2H_i^2 ((\ln a_i)'' - (\ln a_i)'(\phi_0' + \sum (\ln a_j)')) + (\ln a_i)'^4] a^{-6} e^{-2\phi_0} \\ & - \sum_{k < i} (\ln a_k)'^2 (\ln a_i)'^2 a^{-6} e^{-2\phi_0} - \frac{1}{4} [A^2 a^{-6} \sum_{k < i} (\ln a_k)'(\ln a_i)' - 5A^4 a^{-6} e^{2\phi}]. \end{aligned} \tag{35}$$

Now, the solution of (30) gives the zeroth order of dilaton as following

$$\phi_0 = -\ln\left(\frac{A}{n} \cosh(n\tau)\right), \tag{36}$$

and the (31) equation has the following general solutions for the first order of dilaton<sup>4</sup>

$$\begin{aligned} \phi_1 = & Q_1 \tanh(n\tau) + Q_2 (n\tau \tanh(n\tau) - 1) - n \tanh(n\tau) \int \xi_4 d\tau \\ & + \frac{1}{n} [\tanh(n\tau) \int (n\tau \tanh(n\tau) - 1) \rho d\tau - (n\tau \tanh(n\tau) - 1) \int \tanh(n\tau) \rho d\tau], \end{aligned} \tag{37}$$

where the  $n$ ,  $Q_1$  and  $Q_2$  are arbitrary constants. The  $\rho$  varies in Bianchi models and hence the  $\phi_1$  will be different for any type after integrating. The  $\rho$  term of any Bianchi-type will be expressed in the next section.

Furthermore, in the new time coordinate  $\tau$ , the  $(i, i)$  components of  $\beta$ -function equations of metric (20) can be rewritten with using the (30) and (31) to get the following general form

$$\frac{1}{2} (\ln a_i^2 e^{\phi_0})'' + \hat{V}_{i,0}^{(1)} + \alpha' (X_i'' + \hat{V}_{i,1}^{(1)} - \frac{1}{2} (\ln a_i^2 e^{\phi_0})' \xi_4' + g_i) = 0, \tag{38}$$

where the  $g_i$  terms are defined as follows

$$g_i = \frac{\rho}{2} + \hat{K}_i + V_i^{(2)}, \quad i = 1, 2, 3, \tag{39}$$

where the  $\rho$  and  $\hat{K}_i$  are given by (32) and (33). The  $\hat{V}_{i,0}^{(1)}$  and  $\hat{V}_{i,1}^{(1)}$  are taken to indicate the zeroth and first order of the term  $\hat{V}_i^{(1)}$ , i.e.  $\hat{V}_i^{(1)} = \hat{V}_{i,0}^{(1)} + \alpha' \hat{V}_{i,1}^{(1)}$ , and will be obtained expansion of  $\hat{V}_i^{(1)}$  by using the (26) and (27). The  $\hat{V}_i^{(1)}$  terms depend on the three radii  $a_i$  and briefly are called one-loop potentials. The  $\hat{V}_i^{(2)}$  terms depend on  $a_i$ ,  $\phi_0$  and their derivatives. However, the zeroth order of scale factors and dilaton are totally characterized by the solutions of the one-loop order of  $\beta$ -function equations and from the two-loop equations point of view  $V_i^{(2)}$  can be regarded as two-loop potentials. As it will be seen in the next section, in the first order of  $\alpha'$  the  $\hat{V}_{i,1}^{(1)}$  terms will determine the nonlinear part of the equations while the  $g_i$  will be the inhomogeneous part of differential equations. Depending on the  $\hat{V}$  terms, the solutions of (38) differ in each Bianchi-type and will be discussed in the next section.

Moreover, by rewriting the time–time component of metric  $\beta$ -function equation (21) in the new coordinate and using the (30), (31) and (38), the following initial value equation is obtained

$$\begin{aligned} & \frac{1}{2} \left[ \sum_{i < j} (\ln a_i^2 e^{\phi_0})' (\ln a_j^2 e^{\phi_0})' + 2 \sum \hat{V}_{i,0}^{(1)} - \phi_0'^2 - A^2 e^{2\phi_0} \right] (1 + 2\alpha' \xi_4) \\ & + \alpha' \left[ \left( \sum (\ln a_i^2 e^{2\phi_0})' - \phi_0' \right) \sum X_j' - \phi_0' \phi_1' + A^2 e^{2\phi_0} \phi_1 \right. \\ & \left. + \sum (\hat{V}_{i,1}^{(1)} + g_i) + \frac{\rho}{2} + \hat{K}_0 + \hat{V}_0^{(2)} \right] = 0. \end{aligned} \tag{40}$$

<sup>4</sup> An another solution for (31) can be of the following form

$$\begin{aligned} \phi_1 = & Q_1 \tau \tanh(n\tau) - \frac{Q_1}{n} + Q_2 \tanh(n\tau) + Q_3 \ln(\sinh(n\tau)) \tanh(n\tau) \\ & - \tanh(n\tau) \int \coth(n\tau) \int \rho \tanh(n\tau) d\tau d\tau - n \tanh(n\tau) \int \xi_4 d\tau. \end{aligned}$$

The first brackets in this equation (the coefficient of the  $(1 + \alpha'\xi_4)$ ) contain terms which are originated from the leading order of  $\beta$ -function equations. It has been shown in [7] that this is just a constraint between some integrating constants which appears in the solutions of dilaton and scale factors. Here, we have a generalization of this equation which its common feature between all Bianchi-type is that the first brackets and the last brackets of it could not vanish simultaneously. Thus, we regard this equation as an initial value equation which determines the correction of the lapse function,  $\xi_4$  in (28). In this equation the  $\alpha'(-\phi'_0\phi'_1 + A^2e^{2\phi_0}\phi_1)$  term in last brackets brings about a  $\alpha'Q_2$  term. For consistency in the order of  $\alpha'$ , we may choose the  $Q_2$  as a constant in order  $\alpha'^{-1}$  with  $Q_2 = \alpha'^{-1}q_2$  and then, solve the equation (40) to get

$$q_2 = -\frac{1}{2} \left( \sum_{i < j} (\ln a_i^2 e^{\phi_0})' (\ln a_j^2 e^{\phi_0})' + 2 \sum V_{i,0}^{(1)} - \phi_0'^2 - A^2 e^{2\phi_0} \right) \tag{41}$$

and

$$\begin{aligned} \xi_4 = \frac{1}{2q_2} & \left[ (\sum (\ln a_i^2 e^{2\phi_0})' - \phi_0') \sum X_j' + \sum (V_{i,1}^{(1)} + g_i + \frac{\rho}{2} + \hat{K}_0 + \hat{V}_0^{(2)}) \right. \\ & \left. - \frac{\phi_0'}{n} ((\tanh(n\tau))' \int (n\tau \tanh(n\tau) - 1) \rho d\tau - (n\tau \tanh(n\tau))' \int \tanh(n\tau) \rho d\tau) \right] \end{aligned} \tag{42}$$

with demanding  $q_2 \neq 0$ . Here, the general form of  $\xi_4$  is found in terms of  $\rho$  and  $\hat{V}$  and the explicit answers for any case of Bianchi-types will be calculated in the next section.

The solution of two-loop  $\beta$ -function equations (6)–(8) and (12) provides a conformal invariance only to the first order in  $\alpha'$ . In this order, the  $\alpha'$ -corrections of quadratic curvature corrections type  $\alpha'R^2$  are introduced by the  $\beta$ -functions to the effective action which are significant when curvature grows. Principally, recommended by the conformal invariance condition, when  $R\alpha' \gtrsim 1$ ,  $\alpha'H^2 \gtrsim 1$  and  $\alpha'(\nabla\phi)^2 \gtrsim 1$ , all the orders of  $\alpha'$ -corrections series should be considered [50]. However, in this paper, we consider only the first order  $\alpha'$ -corrections. It is worth mentioning, even to this level the higher-derivative corrections are significantly applicable to remove curvature singularities [27,49].

According to (19) and dilaton solution (36), the kinetic terms of the  $B$ -field and dilaton in (10) and the string coupling  $g_s = e^{-\phi}$  are given by

$$H^2 = A^2 \bar{a}^6 e^{-3\phi} = \frac{\cosh(n\tau)^3}{n^3 A} \bar{a}^6, \tag{43}$$

$$(\nabla\phi)^2 = \frac{n^2 (\tanh(n\tau))^2}{A (\cosh(n\tau))^2} \bar{a}^{-6}, \tag{44}$$

$$g_s = \frac{A \cosh(n\tau)}{n}. \tag{45}$$

It will be seen in the next section that the zeroth order scale factors  $\bar{a}_i$  have no dependence on the  $A$  constant. Also, string frame curvatures of the Bianchi model with the considered metric here, are usually proportional to  $A^{-1}$ . Hence, if the field strength magnitude  $A$  be small enough, then the curvature and kinetics of models will be in high limit so the  $\alpha'$ -corrections become important. For instance, if the  $A$  is supposed to be of order  $\alpha'$ , the models will be in  $R\alpha' \gtrsim 1$ ,  $\alpha'H^2 \gtrsim 1$  and  $\alpha'(\nabla\phi)^2 \gtrsim 1$  limit. Furthermore, with a small  $A$  constant the string coupling  $g_s$  start in the weak limit where the loop-corrections in the effective action are negligible. However, the  $g_s$  is an increasing function of time and it may creep into the strong coupling limits at late times.

#### 4. The solutions

In the previous section the general form of  $\beta$ -function equations in terms of dilaton and Hubble coefficients has been presented in (30), (31), (38) and (40). The general solutions of dilaton in zeroth and first order of  $\alpha'$  and the correction of lapse function have been found in (36), (37) and (42). In this section, by finding the Bianchi-type dependent  $\hat{V}$  terms of any Bianchi-type group examples, the solution of the (38) equations is investigated.

##### 4.1. Bianchi-type I

The structure constants of  $I$  type are zero and all  $\hat{V}$  terms this model vanishes. So, with  $\hat{V}_i = \hat{V}_0^{(2)} = \hat{V}_\phi^{(2)} = 0$  the equations of (38) give the following equations respectively in the zeroth and first order of  $\alpha'$

$$(\ln a_i^2 e^{\phi_0})'' = 0, \tag{46}$$

$$X_i'' - \frac{1}{2} (\ln a_i^2 e^{\phi_0})' \xi_4' + g_i = 0. \tag{47}$$

The  $g_i$ , which have been defined in (39), are reduced to  $\frac{\rho}{2} + \hat{K}_i$  in this Bianchi-type. The solution of (46) gives the zeroth order of the Einstein frame scale factors of (27) as following [7]

$$\bar{a}_i^2 = a_i^2 e^{\phi_0} = L_i e^{p_i \tau}, \tag{48}$$

where the  $p_i$  and  $L_i$  are arbitrary constants. Then, with using (48), the solution of (47) determines general form of the first order of the Einstein frame scale factors  $X_i$ , as following

$$X_i = - \iint \left( \frac{\rho}{2} + \hat{K}_i \right) d\tau d\tau + \frac{p_i}{2} \int \xi_4 d\tau. \tag{49}$$

Now, substituting the dilaton (36) and scale factors (48) in the  $\rho$  (32),  $\hat{K}_i$  (33) and  $\hat{K}_0$  (34) we obtain

$$\begin{aligned} \hat{K}_i = & - \frac{3 n e^{-\sum p_i \tau}}{2 A L^3 (\cosh(n\tau))^5} \left( - \frac{a}{12} (\cosh(n\tau))^4 - \frac{b}{6} \sinh(n\tau) (\cosh(n\tau))^3 \right. \\ & \left. + c (\cosh(n\tau))^2 + n^3 \left( (f + \frac{5}{3}) p_i + \left( \frac{17}{24} f + \frac{4}{6} \right) \sum_{j \neq i} p_j \right) \sinh(n\tau) \cosh(n\tau) \right) \\ & - \frac{n^4}{12} (23 f + 28), \end{aligned} \tag{50}$$

$$\begin{aligned} \hat{K}_0 = & \frac{n e^{-\sum p_i \tau}}{2 A L^3 (\cosh(n\tau))^5} \left( d (\cosh(n\tau))^4 \right. \\ & \left. - \left( \frac{n}{4} \left( \sum_{i \neq j} p_i^2 p_j + 6 p_1 p_2 p_3 \right) \right) \sinh(n\tau) (\cosh(n\tau))^3 \right) \\ & + e (\cosh(n\tau))^2 + n^3 (f - 1) \sum p_i \sinh(n\tau) \cosh(n\tau) - n^4 (3/2 f - 1), \end{aligned} \tag{51}$$

$$\begin{aligned} \rho = & \frac{25}{4} \frac{ne^{-\sum p_i \tau}}{A L^3 (\cosh(n\tau))^5} (h (\cosh(n\tau))^4 \\ & - \frac{3n}{25} \sum_{j < k} (p_j p_k + n^2) \sum p_j \sinh(n\tau) (\cosh(n\tau))^3 \\ & + m \sinh(n\tau) (\cosh(n\tau))^2 + n^3 \sum p_i (f + 11/10) \sinh(n\tau) \cosh(n\tau) \\ & - \frac{3}{2} (47/25 + 3f/2) n^4), \end{aligned} \tag{52}$$

where  $L^3 = L_1 L_2 L_3$  and

$$a = n^4 + n^2 (\sum p_j)^2 + \sum_{k, j \neq i} (p_j^2 + \frac{1}{2} (p_i^2 - n^2) p_j p_k), \tag{53}$$

$$b = 2n(n^2 \sum p_j + \frac{1}{2} (n^2 + \sum_{j, k \neq i} (p_j p_i + 2p_j^2 + 2p_j p_k)) p_i), \tag{54}$$

$$\begin{aligned} c = & \frac{1}{12} n^2 ((23f + 24)n^2 + (2f + 4)p_i^2 \\ & + \sum_{j, k \neq i} ((4f + 1)p_j p_i + (4f + 3)p_j^2 + (5f + 4)p_j p_k / 2)), \end{aligned} \tag{55}$$

$$d = (\frac{1}{2} ((\sum p_i)^2 + \sum p_i^2)) n^2 + \sum_{j < i} p_i^2 p_j^2 - p_2 p_3 p_1 \sum p_i, \tag{56}$$

$$e = ((6f - 9)n^2 + 3 \sum p_i^2 f + \sum_{i < j} p_i p_j) n^2 / 4, \tag{57}$$

$$h = \frac{-9}{10} (N^4 + \frac{2n^2}{3} (\frac{1}{2} (3(\sum p_i)^2 + \sum p_i^2))) + \frac{2}{3} (p_2 p_3 p_1^2 + p_2 p_3 p_1 (p_2 + p_3)), \tag{58}$$

$$m = (((f + \frac{9}{10})n^2 + (\frac{4}{25}(f + 1)) \sum p_i^2 + \frac{13}{75}(f + \frac{5}{2}) \sum p_i p_j)) 3n^2 / 2. \tag{59}$$

The  $q_2$  constant in (41) in this Bianchi-type is given by

$$q_2 = (n^2 - \sum_{j < k} p_j p_k) / 2. \tag{60}$$

Now, the appeared integrals in the  $\phi_1$  (37),  $\xi_4$  (42) and  $X_i$  (49) can be calculated for finding the metric and dilaton corrections. For choosing any value for  $p_i$ s, the  $q_2 \neq 0$  condition should be considered.

4.1.1. Some physical properties of an isotropic example in the Bianchi-type I

In Appendix C, a set of answers for metric and dilaton corrections is given with an isotropic parametrization  $p_1 = p_2 = p_3 = 1/3$  and  $n = 1$  through (220)–(222). Regarding the kinetic of  $B$ -field and dilaton in (43) and (44) and the string Ricci scalar of this example which is given by

$$R = \frac{e^{-\tau} (8 (\cosh(\tau))^2 + 9)}{6A L^3 (\cosh(\tau))^3}, \tag{61}$$

it is clear that if the  $A$  supposed to be very small (of order  $\alpha'$ ), the model will be in the high curvature and high kinetics limits and the inclusion of the  $\alpha'$ -correction becomes necessary. The curvature of this model is decreasing so the corrections become significant in the early times near  $\tau = 0$ . Here, we come to investigate some properties of this specific example and consider some physical quantities such as Einstein frame mean Hubble parameter and deceleration parameter which are given by

$$\bar{H} = \frac{\dot{\bar{a}}}{\bar{a}}, \quad q = -\frac{\ddot{\bar{a}}}{\dot{\bar{a}}^2}. \tag{62}$$

Here the dot stands for derivation with respect to Einstein frame time, which is called cosmic time, such that  $\dot{\bar{a}} = d\bar{a}/d\bar{t}$  and  $\ddot{\bar{a}} = d^2\bar{a}/d\bar{t}^2$ . The answers have been given in terms of  $\tau$ . Now, there are two equivalent ways for investigating the behavior of  $\dot{\bar{a}}$  and  $\ddot{\bar{a}}$ ; first, by rewriting the solutions in  $\bar{t}$  terms by finding the relation between  $\bar{t}$  and  $\tau$  using (29) and (9); second, just rewriting the derivatives of  $\dot{\bar{a}}$  and  $\ddot{\bar{a}}$  in terms of  $\tau$  derivatives. The former way is applicable here but it is not generally suitable for other models, where the latter one can be used in every model. Here, following the first procedure, we consider the Einstein frame metric which according to (9), (17) and (18) takes the following form in this isotropic example

$$ds^2 = -N(\bar{t}) d\bar{t}^2 + \bar{a}_1^2(1 + 2 X_i)\delta_{\alpha\beta} dx^\alpha dx^\beta \tag{63}$$

where  $\bar{a}_1^2 = Le^{\tau/3}$ . Accordingly, with  $N = (1 + \alpha'(\sum X_i + \xi_4))$ , the cosmic time  $\bar{t}$  and the  $\tau$ , which has been introduced by (29), would have the following relation

$$\bar{t} + \bar{t}_0 = \int \bar{a}_1^3 d\tau, \tag{64}$$

where the  $\bar{t}_0$  is used for coinciding the beginning of the two time coordinates. Fortunately, in the Bianchi-type  $I$  model with the scale factors given by (48) it is easy to find  $\tau$  in terms of  $\bar{t}$  as following

$$\tau = \ln\left((1 + L^{-3/2}\bar{t})\right), \tag{65}$$

where  $\bar{t}_0$  is fixed as  $\frac{2L^{3/2}}{\sum p_i}$ . Now, substituting (65) (48), (36), (220), (221) and (222), the  $\alpha'$ -corrected dilaton and Einstein frame scale factors are found in terms of cosmic time and their behavior could be investigated.

In this example, if the zeroth order is considered only, the solution (48) will have  $\dot{\bar{a}} > 0$  and  $\ddot{\bar{a}} > 0$  and consequently describe a decelerated expanding universe with no inflation. Furthermore, in this order the strong energy condition is satisfied and there is an initial singularity [7].

The behavior of  $\dot{\bar{a}}$  and  $\ddot{\bar{a}}$  of the corrected scale factors are being investigated, with choosing  $A = \alpha'$ ,  $L = 1.1$ ,  $l_1 = 1/11$ ,  $c_1 = -1$ ,  $\varphi_0 = 1$  and  $Q_1 = 11$  in the Gauss–Bonnet scheme and  $A = \alpha'$ ,  $L = 1.1$ ,  $l_1 = 0.5$ ,  $c_1 = 1.5$ ,  $\varphi_0 = 2.2$  and  $Q_1 = 11$  for the  $R^2$  scheme.<sup>5</sup> In the high curvature limit (which is until about  $\bar{t} = 0.5$  in these parameterizations) we have  $\dot{\bar{a}} > 0$ , but the sign of  $\ddot{\bar{a}}$  changes such that it is negative near  $\bar{t} = 0$  and then, before leaving the high curvature limit, becomes positive. So, until the corrections are significant, the  $\alpha'$ -corrected answer describes an

<sup>5</sup> Regarding the signature of metric to be adopted  $(-1, 1, 1, 1)$  after adding corrections, the constants have been chosen and these set of constants are not unique.

expanding universe which has a decelerated phase in the early times and then becomes accelerated. After leaving the high curvature limit, the behaviors of  $\ddot{a}$  and  $\dot{a}$  turn into the leading order behavior and become negative.

Also, obeying the strong energy condition in the case that the higher order correction terms contribute to the right hand of Einstein frame field equations, may be considered. For the Bianchi-type models the strong energy condition requires  $\dot{k} + \frac{1}{3}k^2 < 0$ , where  $k$  is the extrinsic curvature [51]. With the considered diagonal metric in this paper, we have  $k = \sum \dot{a}_i / \bar{a}_i$  and the strong energy condition leads to  $\ddot{a} < 0$ . In the accelerated phase of our considered example, with  $\ddot{a} > 0$ , there is a violation of the strong energy condition which is a necessary (but not sufficient) condition of avoiding singularity [52]. So, the singularity of the leading order has been cured in some regions in the early time by introducing the higher order correction in the action. This period in the Bianchi-type I model may be regarded as a cosmological inflationary phase [53].

Furthermore, the expansion is rapid at first but then slow down without any singularity in  $\bar{H}$ . Hence, if it is considered as an inflationary phase followed by a standard phase, it will be free of graceful exit problem. This kind of nonsingular behavior (no graceful exit problem) of the Bianchi-type I model with superstring-motivated Gauss–Bonnet term with dilaton and modulus fields have already been investigated in [37].

#### 4.2. Bianchi-type II

In Bianchi-type II the  $\hat{V}$  terms are given by

$$\hat{V}_1^{(1)} = -\hat{V}_2^{(1)} = -\hat{V}_3^{(1)} = \frac{a_1^4 e^{2\phi_0}}{2} (1 + 2\alpha' (2X_1 + \xi_4)), \tag{66}$$

$$\begin{aligned} \hat{V}_1^{(2)} = & \frac{a_1^2}{8a_2^2 a_3^2} (a_1^4 e^{2\phi_0} - 4(\sum (\ln a_i)' ^2 - 2(\ln a_1)' (\ln a_2^2 a_3^2 a_1^{-1})' + (\ln a_2 a_3)') \\ & + 3(f - 1) A^2 e^{2\phi_0}), \end{aligned} \tag{67}$$

$$\begin{aligned} \hat{V}_2^{(2)} = & \frac{a_1^2}{8a_2^2 a_3^2} (5a_1^4 e^{2\phi_0} - 4(\sum (\ln a_i)' ^2 + 4(\ln a_2)' (\ln \frac{a_1}{a_3})' + (\ln a_1)' (\ln \frac{a_1^2}{a_3})') \\ & - (f - 5) A^2 e^{2\phi_0}), \end{aligned} \tag{68}$$

$$\begin{aligned} \hat{V}_3^{(2)} = & \frac{a_1^2}{8a_2^2 a_3^2} (5a_1^4 e^{2\phi_0} - 4(\sum (\ln a_i)' ^2 + 4(\ln a_3)' (\ln \frac{a_1}{a_2})' + (\ln a_1)' (\ln \frac{a_1^2}{a_2})') \\ & - (f - 5) A^2 e^{2\phi_0}), \end{aligned} \tag{69}$$

$$\hat{V}_0^{(2)} = \frac{a_1^2}{2a_2^2 a_3^2} (\sum (\ln a_i)' ^2 + (\ln a_i)' (\ln \frac{a_1^2}{a_2^3 a_3^3})' + (\ln a_2)' (\ln a_3)'), \tag{70}$$

$$\begin{aligned} \hat{V}_\phi = & \frac{a_1^2}{2a_2^2 a_3^2} (-2 \sum (\ln a_i)' - 4(\ln a_1)' + 7(\ln a_1)' (\ln a_2 a_3)' - 5(\ln a_2)' (\ln a_3)') \\ & + \frac{(11a_1^4 + A^2) e^{2\phi_0}}{16}. \end{aligned} \tag{71}$$

With these potentials, the  $\beta$ -function equations (38) give the following nonlinear equations in the zeroth order of  $\alpha'$  [7]

$$(\ln a_1^2 e^{\phi_0})'' + a_1^4 e^{2\phi_0} = 0, \tag{72}$$

$$(\ln a_2^2 e^{\phi_0})'' - a_1^4 e^{2\phi_0} = 0, \tag{73}$$

$$(\ln a_3^2 e^{\phi_0})'' - a_1^4 e^{2\phi_0} = 0, \tag{74}$$

and in the first order of  $\alpha'$  give

$$X_1'' + a_1^4 e^{2\phi_0} (2X_1 + \xi_4) - \frac{1}{2} (\ln a_1^2 e^{\phi_0})' \xi_4' + g_1 = 0, \tag{75}$$

$$X_2'' - a_1^4 e^{2\phi_0} (2X_1 + \xi_4) - \frac{1}{2} (\ln a_2^2 e^{\phi_0})' \xi_4' + g_2 = 0, \tag{76}$$

$$X_3'' - a_1^4 e^{2\phi_0} (2X_1 + \xi_4) - \frac{1}{2} (\ln a_3^2 e^{\phi_0})' \xi_4' + g_3 = 0. \tag{77}$$

The general solutions of (72)–(74) are given by [7]

$$a_1^2 e^{\phi_0} = \frac{p_1}{\cosh(p_1 \tau)}, \tag{78}$$

$$a_2^2 e^{\phi_0} = \cosh(p_1 \tau) e^{p_2 \tau}, \tag{79}$$

$$a_3^2 e^{\phi_0} = \cosh(p_1 \tau) e^{p_3 \tau}. \tag{80}$$

The general solutions of (75)–(77) equations are found as follows

$$\begin{aligned} X_1 = & Q_3 \tanh(p_1 \tau) + Q_4 (p_1 \tau \tanh(p_1 \tau) - 1) - \frac{p_1}{2} \tanh(p_1 \tau) \int \xi_4 d\tau \\ & + \frac{1}{p_1} (\tanh(p_1 \tau) \int (p_1 \tau \tanh(p_1 \tau) - 1) g_1 d\tau \\ & - (p_1 \tau \tanh(p_1 \tau) - 1) \int \tanh(p_1 \tau) g_1 d\tau), \end{aligned} \tag{81}$$

$$X_2 = -X_1 - \iint (g_1 + g_2) d\tau d\tau + \frac{p_2}{2} \int \xi_4 d\tau, \tag{82}$$

$$X_3 = -X_1 - \iint (g_1 + g_3) d\tau d\tau + \frac{p_3}{2} \int \xi_4 d\tau, \tag{83}$$

where the  $Q_3$  and  $Q_4$  are integrating constants and the  $g_i$  has been introduced in (39). With the dilaton (36) and scale factors given by (78)–(80), the explicit forms of  $\rho$  introduced in (32) and  $g_i$  can be derived and then the integrals of  $X_i$  and  $\phi_1$  (37) can be computed. In this Bianchi-type the  $q_2$  constant of (41) is given by

$$q_2 = (n^2 + p_1^2 - p_2 p_3) / 2. \tag{84}$$

Keeping in mind the condition  $q_2 \neq 0$ , one could choose any value for the  $p_i$  and  $n$ . For example, with the special choice of parameterization as  $p_1 = p_2 = p_3 = n = 1$  the  $g_i$  and  $\rho$  are given as following<sup>6</sup>

<sup>6</sup> Here and hereafter, avoiding dense mathematical formulae, we will choose a set of values for the constants of leading order scale factors and present a form of  $g_i$  and  $\rho$  as an example. Final forms of metric and dilaton corrections, obtained after performing their integrals, of these example can be found in the Appendix C.

$$g_1 = \frac{e^{-2\tau}}{16A (\cosh(\tau))^6} \left( -153 (\cosh(\tau))^4 - 144 (\cosh(\tau))^3 \sinh(\tau) + (110f + 561) (\cosh(\tau))^2 + (88f + 444) \sinh(\tau) \cosh(\tau) - 78f - 470 \right), \tag{85}$$

$$g_2 = g_3 = \frac{e^{-2\tau}}{32A (\cosh(\tau))^6} \left( -61 (\cosh(\tau))^4 - 56 (\cosh(\tau))^3 \sinh(\tau) + (170f + 285) (\cosh(\tau))^2 + (136f + 236) \sinh(\tau) \cosh(\tau) - 142f - 246 \right), \tag{86}$$

$$\rho = \frac{e^{-2\tau}}{16A (\cosh(\tau))^6} \left( -153 (\cosh(\tau))^4 - 144 (\cosh(\tau))^3 \sinh(\tau) + (370f + 761) (\cosh(\tau))^2 + (296f + 604) \sinh(\tau) \cosh(\tau) - 298f - 634 \right). \tag{87}$$

Now, using the (37), (42) and (81)–(83), the  $\alpha'$ -corrections of metric and dilaton in both scheme can be found in this special choice of parameters. The detailed answers are given in Appendix C through (223)–(226).

### 4.3. Bianchi-type III

Here, as a general feature of the Bianchi-type of class  $\mathcal{B}$ , the  $\beta$ -function equations are subject to a constraint equation where in this case the (0, 3) component of  $\beta$ -function equations of metric imposes it as following

$$\left( \ln \frac{a_1}{a_3} \right)' \left[ 1 + \alpha' (a_0^{-6} e^{-2\phi} ((\ln a_1)'' - (\ln a_1)' (\ln a_2)' - \phi_0' (\ln a_1)') - a_3^{-2}) - \frac{3A^2 a_0^{-6}}{4} \right] + \alpha' (X_1' - X_3') = 0, \tag{88}$$

which essentially imposes the conditions of  $a_1 = a_3$  and  $X_1 = X_3$ . The  $\beta$ -function equations of  $B$ -field (7) and (8) vanish with the choice (19) in the zeroth order of  $\alpha'$ . But at first order of  $\alpha'$ , despite of the  $\mathcal{A}$  class, in the  $\mathcal{B}$  class models they do not vanish. However, they will be vanished automatically by imposing the conditions that resulted form the constraints equations.

The Bianchi-type dependent terms  $\hat{V}$  of III model are given as following

$$\hat{V}_1^{(1)} = \hat{V}_3^{(1)} = a_1^2 a_2^2 e^{2\phi_0} (1 + 2\alpha' (X_1 + X_2 + \xi_4)), \quad \hat{V}_2^{(1)} = 0, \tag{89}$$

$$\hat{V}_1^{(2)} = \hat{V}_3^{(2)} = a_3^{-2} (-2((\ln a_1)')^2 + (\ln a_3)')^2 - (\ln a_1)' (\ln a_3)') + a_1^2 a_2^2 e^{2\phi} + 3A^2 e^{2\phi} / 2), \tag{90}$$

$$\hat{V}_2^{(2)} = (f + 1) A^2 e^{2\phi} / 2, \tag{91}$$

$$\hat{V}_0^{(2)} = a_3^{-2} ((\ln a_1)' - (\ln a_3)')^2, \tag{92}$$

$$\hat{V}_\phi = a_3^{-2} (-2((\ln a_1)')^2 + (\ln a_3)')^2 - (\ln a_1)' (\ln a_3)') + a_1^2 a_2^2 e^{2\phi_0} + A^2 e^{2\phi_0} / 4). \tag{93}$$

With this form of potentials, the  $\beta$ -function equations (38) lead to the following equations at the zeroth order [7]

$$(\ln a_1^2 e^{\phi_0})'' - 2a_1^2 a_2^2 e^{2\phi_0} = 0, \tag{94}$$

$$(\ln a_2^2 e^{\phi_0})'' = 0, \tag{95}$$

$$(\ln a_3^2 e^{\phi_0})'' - 2a_1^2 a_2^2 e^{2\phi_0} = 0, \tag{96}$$

and at the first order of  $\alpha'$  give

$$X_1'' - 2a_1^2 a_2^2 e^{2\phi_0} (X_1 + X_2 + \xi_4) - \frac{1}{2} (\ln a_1^2 e^{\phi_0})' \xi_4' + g_1 = 0, \tag{97}$$

$$X_2'' - \frac{1}{2} (\ln a_2^2 e^{\phi_0})' \xi_4' + g_2 = 0, \tag{98}$$

$$X_3'' - X_1'' = 0. \tag{99}$$

Solutions of (94)–(96) are given by [7]

$$a_1^2 e^{\phi_0} = a_3^2 e^{\phi_0} = \frac{p_1 e^{p_2 \tau}}{\sinh(p_1 \tau)}, \tag{100}$$

$$a_2^2 e^{\phi_0} = e^{-p_2 \tau}. \tag{101}$$

The equations of (97)–(99) admit the following solutions

$$\begin{aligned} X_1 = X_3 = & -X_2 + Q_3 \coth(p_1 \tau) + Q_4 (p_1 \tau \coth(p_1 \tau) - 1) \\ & + \left( \frac{p_2}{2} - p_1 \coth(p_1 \tau) \right) \int \xi_4 d\tau \\ & + \frac{1}{p_1} \left( \coth(p_1 \tau) \int (p_1 \tau \coth(p_1 \tau) - 1)(g_1 + g_2) d\tau \right. \\ & \left. - (p_1 \tau \coth(p_1 \tau) - 1) \int \coth(p_1 \tau)(g_1 + g_2) d\tau \right), \end{aligned} \tag{102}$$

$$X_2 = -\frac{p_2}{2} \int \xi_4 d\tau - \iint g_2 d\tau d\tau, \tag{103}$$

where the  $Q_3$  and  $Q_4$  are constant. Here the  $q_2$  constant (41) is given by

$$q_2 = (p_2^2 + n^2 - 4 p_1^2)/2. \tag{104}$$

As the other Bianchi-types, the  $g_i$  terms in the answer of metric corrections (102), (103) and  $\rho$  in dilaton (37) can be derived by substituting the zeroth order solution of scale factors and dilaton. For example, with  $p_1 = p_2 = n = 1$  they are given by

$$\begin{aligned} g_2 = & \frac{(\sinh(\tau))^2 e^{-\tau}}{16A (\cosh(\tau))^5} \left( -144 (\cosh(\tau))^6 + (26f + 73) (\cosh(\tau))^4 \right. \\ & + (9f - 7) (\cosh(\tau))^2 + \sinh(\tau) (-26(f + 15/26) \cosh(\tau) \\ & \left. + 144 (\cosh(\tau))^5 - 26(f + 1/26) (\cosh(\tau))^3 + 29f + 38 \right), \end{aligned} \tag{105}$$

$$\begin{aligned} g_2 = & \frac{(\sinh(\tau))^2 e^{-\tau}}{8 (\cosh(\tau))^5} \left( -40 (\cosh(\tau))^6 + (9f + 73/2) (\cosh(\tau))^4 \right. \\ & + 1/18 (f - 55) (\cosh(\tau))^2 + \sinh(\tau) (-6f + 31) \cosh(\tau) \\ & \left. + 40 (\cosh(\tau))^5 - 3(3f + 11/2) (\cosh(\tau))^3 + 29f/2 + 19 \right), \end{aligned} \tag{106}$$

$$\begin{aligned} \rho = & \frac{e^{-\tau}}{8A (\cosh(\tau))^5} \left( (\sinh(\tau))^3 \left( -46 \left( f + \frac{17}{46} \right) (\cosh(\tau))^3 \right. \right. \\ & + 168 (\cosh(\tau))^5 - 50 \left( f + \frac{11}{10} \right) \cosh(\tau) \\ & + (\sinh(\tau))^2 \left( 75 f + 94 + (23 f - 27) (\cosh(\tau))^2 \right. \\ & \left. \left. + (46 f_1 + 101) (\cosh(\tau))^4 - 168 (\cosh(\tau))^6 \right) \right). \end{aligned} \tag{107}$$

The explicit answers for  $X_i$ ,  $\xi_i$  and  $\phi_1$  with this parameterization can be found in [Appendix C](#) through (227)–(230).

#### 4.4. Bianchi-type IV

For this type the constraint equation is

$$\frac{a_3^2}{a_1^2} + \alpha' (2(X_3 - X_1) + \frac{4(\ln a_1)^2 - (\ln \frac{a_2}{a_3})' (\ln \frac{a_2 a_3}{a_1^2})' - A^2}{a_1^4 a_2^2} - \frac{2a_3^2}{a_1^4} - \frac{a_3^4}{2a_1^4 a_2^2}) = 0, \tag{108}$$

where in the zeroth order of  $\alpha'$  leads to the following condition

$$\frac{a_3^2}{a_1^2} = 0. \tag{109}$$

So, all solution in zeroth order of  $\alpha'$  is singular and because of this inconsistency in one-loop order, we abandon its two-loop calculations.

#### 4.5. Bianchi-type V

Bianchi-type V belongs to the  $\mathcal{B}$  class. Here the (0, 1) component of  $\beta$ -function of the metric (6) is a constraint equation and gives

$$\begin{aligned} (\ln \frac{a_1^2}{a_2 a_3})' + \alpha' [2X_1' - X_1' - X_3' + (\ln \frac{a_1^2}{a_2 a_3})' (-\frac{3A^2 a_0^{-6}}{4} - \frac{1}{a_1^2} + a_0^{-6} e^{-2\phi} (\ln a_2 a_3)') \\ + a_0^{-6} e^{-2\phi} ((\ln a_2)'' (\ln \frac{a_3}{a_1})' + (\ln a_3)'' (\ln \frac{a_2}{a_1})' + \phi' ((\ln a_2)' (\ln \frac{a_3}{a_1})' \\ + (\ln a_3)' (\ln \frac{a_2}{a_1})')] = 0. \end{aligned} \tag{110}$$

This equation in zeroth order of  $\alpha'$  implies

$$a_1^2 = a_2 a_3, \tag{111}$$

and its first order of  $\alpha'$  will be discusses afterward.

The potential terms in this Bianchi-type are given by

$$\hat{V}_1^{(1)} = \hat{V}_2^{(1)} = \hat{V}_3^{(1)} = -2a_2^2 a_3^2 e^{2\phi_0} (1 + 2\alpha' (X_2 + X_3 + \xi_4)), \tag{112}$$

$$\hat{V}_1^{(2)} = a_1^{-2}(2(a_2a_3e^{\phi_0})^2 - 2(\ln a_1)'^2 - (\ln a_2)'^2 - (\ln a_3)'^2 + (f + 7)A^2e^{2\phi_0}/2), \tag{113}$$

$$\hat{V}_2^{(2)} = a_1^{-2}(2(a_2a_3e^{\phi_0})^2 - 2(\ln a_1)'^2 - (\ln a_2)'^2 + (\ln a_2)'(\ln \frac{a_1}{a_3})' + (f + 7)A^2e^{2\phi_0}/2), \tag{114}$$

$$\hat{V}_3^{(2)} = a_1^{-2}(2(a_2a_3e^{\phi_0})^2 - 2(\ln a_1)'^2 - (\ln a_3)'^2 + (\ln a_3)'(\ln \frac{a_1}{a_2})' + (f + 7)A^2e^{2\phi_0}/2), \tag{115}$$

$$\hat{V}_0^{(2)} = -a_1^{-2}(2(\ln a_1)'^2 + (\ln a_0)'^2 + (\ln a_3)'^2 - (\ln a_1)'(\ln a_2a_3)'), \tag{116}$$

$$\hat{V}_\phi = a_1^{-2}(-2(\ln a_1')^2 - (\ln a_2')^2 - (\ln a_3')^2 + (\ln a_1)'(\ln a_2a_3)' + (\ln a_2)'(\ln a_3)') \tag{117}$$

$$+ 3a_3^2a_2^2e^{2\phi_0} + 3A^2e^{2\phi_0}/4). \tag{118}$$

With this form of potentials, the  $\beta$ -function equations (38) give the following equations in the zeroth order of  $\alpha'$  [7]

$$(\ln a_1^2e^{\phi_0})'' - 4a_2^2a_3^2e^{2\phi_0} = 0, \tag{119}$$

$$(\ln a_2^2e^{\phi_0})'' - 4a_2^2a_3^2e^{2\phi_0} = 0, \tag{120}$$

and in the first order of  $\alpha^5$  give  $(\ln a_3^2e^{\phi_0})'' - 4a_2^2a_3^2e^{2\phi_0} = 0,$  (121)

$$X_1'' - 4a_2^2a_3^2e^{2\phi_0}(X_2 + X_3 + \xi_4) - \frac{1}{2}(\ln a_1^2e^{\phi_0})'\xi_4' + g_1 = 0, \tag{122}$$

$$X_2'' - 4a_2^2a_3^2e^{2\phi_0}(X_2 + X_3 + \xi_4) - \frac{1}{2}(\ln a_2^2e^{\phi_0})'\xi_4' + g_2 = 0, \tag{123}$$

$$X_3'' - 4a_2^2a_3^2e^{2\phi_0}(X_2 + X_3 + \xi_4) - \frac{1}{2}(\ln a_3^2e^{\phi_0})'\xi_4' + g_3 = 0. \tag{124}$$

The general solutions of (119)–(121) are as following [7]

$$a_1^2e^{\phi_0} = \frac{P}{2 \sinh(p\tau)}, \tag{125}$$

$$a_2^2e^{\phi_0} = \frac{P}{2 \sinh(p\tau)} e^{q\tau}, \tag{126}$$

$$a_3^2e^{\phi_0} = \frac{P}{2 \sinh(p\tau)} e^{-q\tau}. \tag{127}$$

Now, the  $g_i$  terms (39) can be derived. The general solutions admitted by (122)–(124) are as following

$$X_1 = X_2 - \iint (g_1 - g_2)d\tau d\tau - \frac{q}{2} \int \xi_4 d\tau, \tag{128}$$

$$X_2 = Q_3 \coth(p\tau) + Q_4 (p\tau \coth(p\tau) - 1) - \frac{1}{2}(p \coth(p\tau) + q) \int \xi_4 d\tau + \frac{1}{2p} \left( \coth(p\tau) \int (p\tau \coth(p\tau) - 1)(g_2 + g_3) d\tau \right. \tag{129}$$

$$\left. - (p\tau \coth(p\tau) - 1) \int \coth(p\tau)(g_2 + g_3) d\tau \right) + \frac{1}{2} \iint (g_3 - g_2)d\tau d\tau, X_3 = X_2 - \iint (g_3 - g_2)d\tau d\tau - q \int \xi_4 d\tau. \tag{130}$$

Here, the  $q_2$  (41) is given by

$$q_2 = (n^2 - 3p^2 + q^2)/2 \tag{131}$$

With the above solutions, the first order of  $\alpha'$  of the constraint equation (110) imposes  $q = 0$ . This leads to  $g_1 = g_2 = g_3$  and therefore, with ignoring integrating constants,  $X_1 = X_2 = X_3$ . Hence, the Bianchi-type  $V$  model which was allowed to be anisotropic from the leading order  $\beta$ -function solution point of view, is constrained to be isotropic in the solution of two-loop  $\beta$ -function equations. However, an anisotropy may appear because of the integrating constants of (128)–(130), but it can be removed by a redefinition or a shift of time. Generally, with  $q = 0$  the  $g_i$  (39) and  $\rho$  (32) are given by

$$\begin{aligned} g_1 = g_2 = g_3 &= \frac{\sinh(p\tau)n}{4A(\cosh(n\tau))^5 p^3} \left( 4 \sinh(p\tau) \sinh(n\tau) n \cosh(p\tau) (4n^2 + 19p^2) p (\cosh(n\tau))^3 \right. \\ &+ \left( (-5n^4 - 58n^2 p^2 - 33p^4) (\cosh(p\tau))^2 + 5n^4 + 10n^2 p^2 + 33p^4 \right) (\cosh(n\tau))^4 \\ &- 2 \sinh(p\tau) \sinh(n\tau) n^3 \cosh(p\tau) p (58f + 61) \cosh(n\tau) \\ &- n^4 (58f + 76) (\sinh(p\tau))^2 \\ &+ \left( (58f + 39)n^2 + p^2 (58f + 131) \right) (\cosh(p\tau))^2 \\ &\left. - (58f + 39)n^2 + 2p^2 (f + 40) \right) n^2 (\cosh(n\tau))^2, \end{aligned} \tag{132}$$

$$\begin{aligned} \rho &= \frac{3 \sinh(p\tau)n}{2Ap^3 (\cosh(n\tau))^5} \left( \left( 3 (-n^4 - 10n^2 p^2 - 5p^4) (\cosh(p\tau))^2 \right. \right. \\ &+ \left. \left. n^4 + 2n^2 p^2 + 5p^4 \right) (\cosh(n\tau))^4 \right. \\ &+ 12n \sinh(p\tau) \sinh(n\tau) \cosh(p\tau) p (n^2 + 3p^2) (\cosh(n\tau))^3 \\ &+ \left( (50f + 45)n^2 + p^2 (50f + 89) \right) (\cosh(p\tau))^2 \\ &+ (-50f - 45)n^2 - 2p^2 (f + 22) \Big) n^2 (\cosh(n\tau))^2 \\ &- 10 \sinh(p\tau) \sinh(n\tau) n^3 \cosh(p\tau) (10f + 11) p \cosh(n\tau) \\ &\left. - (10f + 188/15)n^4 (\sinh(p\tau))^2 \right) \end{aligned} \tag{133}$$

A detailed solution with  $p = n = 1$  is given in Appendix C in (231)–(233).

#### 4.6. Bianchi-type VI<sub>-1</sub>

In this Bianchi-type case, the (0, 1) component of  $\beta$ -function equations of (6) imposes the following constraint

$$\begin{aligned}
 & (\ln \frac{a_3}{a_2})' + \alpha' [X_3' - X_2' - (\ln \frac{a_3}{a_2})' (\frac{3A^2}{4a^6} + \frac{1}{a_1^2}) \\
 & + \frac{4}{3} a^{-6} e^{-2\phi_0} ((\ln \frac{a_1}{a_2})' (\ln a_2)'' + (\ln \frac{a_3}{a_1})' (\ln a_3)'' \\
 & - (\ln \frac{a_3}{a_2})' ((\ln \frac{a_1}{a_2 a_3})' \phi_0' - (\ln a_2)' (\ln a_3)')] = 0.
 \end{aligned} \tag{134}$$

Its zeroth order requires the condition  $a_3 = a_1$  and imposing it leads to vanishing of the first order constraint equation, automatically. Moreover, the Bianchi-type dependent  $\hat{V}$  terms are given by

$$\hat{V}_1^{(1)} = -2a_2^4 e^{2\phi_0} (1 + 2\alpha' (2X_1 + \xi_4)), \quad \hat{V}_2^{(1)} = \hat{V}_3^{(1)} = 0, \tag{135}$$

$$\hat{V}_1^{(2)} = a_1^{-2} (-2(a_2 a_3 e^{\phi_0})^2 - 2(\ln a_1)'^2 - (\ln a_2)'^2 - (\ln a_3)'^2 + (5 - f)A^2 e^{2\phi_0} / 2), \tag{136}$$

$$\begin{aligned}
 \hat{V}_2^{(2)} = & a_1^{-2} (2(a_2 a_3 e^{\phi_0})^2 - 2(\ln a_1)'^2 - (\ln a_2)'^2 + (\ln a_2)' (\ln a_1 a_3)' \\
 & + (1 + f)A^2 e^{2\phi_0} / 2),
 \end{aligned} \tag{137}$$

$$\begin{aligned}
 \hat{V}_3^{(2)} = & a_1^{-2} (2(a_2 a_3 e^{\phi_0})^2 - 2(\ln a_1)'^2 - (\ln a_3)'^2 + (\ln a_3)' (\ln a_1 a_2)' \\
 & + (1 + f)A^2 e^{2\phi_0} / 2),
 \end{aligned} \tag{138}$$

$$\hat{V}_0^{(2)} = -a_1^{-2} (2(\ln a_1)'^2 + (\ln a_2)'^2 + (\ln a_3)'^2 - (\ln a_1)' (\ln a_2 a_3)'), \tag{139}$$

$$\begin{aligned}
 \hat{V}_\phi = & a_1^{-2} (A^2 e^{2\phi_0} / 4 - 4(\ln a_1)'^2 - 2(\ln a_2)'^2 - 2(\ln a_3)'^2 + 2(\ln a_3)' (\ln a_1 a_2)' \\
 & + 2(\ln a_1)' (\ln a_2)').
 \end{aligned} \tag{140}$$

With these form of potentials and  $a_3 = a_1$  condition, the  $\beta$ -function equations (38) in zeroth order give [7]

$$(\ln a_1^2 e^{\phi_0})'' - 4a_2^4 e^{2\phi_0} = 0, \tag{141}$$

$$(\ln a_2^2 e^{\phi_0})'' = 0, \tag{142}$$

and in the first order of  $\alpha'$  give

$$X_1'' - 4a_2^4 e^{2\phi_0} (2X_1 + \xi_4) - \frac{1}{2} (\ln a_1^2 e^{\phi_0})' \xi_4' + g_1 = 0, \tag{143}$$

$$X_2'' - \frac{1}{2} (\ln a_2^2 e^{\phi_0})' \xi_4' + g_2 = 0, \tag{144}$$

$$X_3'' - X_2'' = 0. \tag{145}$$

The solutions of (141) and (142) are given by [7]

$$a_1^2 e^{\phi_0} = q_1^2 p_1 e^{p_1 \tau} e^{q_2^4 e^{-p_2 \tau}}, \tag{146}$$

$$a_2^2 e^{\phi_0} = a_3^2 e^{\phi_0} = q_2^2 p_2 e^{p_2 \tau}, \tag{147}$$

and (143)–(145) admit the following solutions

$$\begin{aligned}
 X_1 = & -8 \iint (p_2^2 q_2^2 e^{2 p_2 \tau} \iint g_2 d\tau d\tau) d\tau d\tau \\
 & - \iint g_1 d\tau d\tau + (p_2 q_2^2 e^{2 p_2 \tau} + \frac{p_1}{2}) \int \xi_4 d\tau,
 \end{aligned} \tag{148}$$

$$X_2 = X_3 = - \iint g_2 d\tau d\tau + \frac{p_2}{2} \int \xi_4 d\tau. \tag{149}$$

The  $q_2$  constant in this Bianchi-type is given by

$$q_2 = (n^2 - p_1 p_2 - p_2^2)/2. \tag{150}$$

For example with  $p_1 = p_2 = n = 1$  we have

$$\begin{aligned} g_1 = & \frac{12 e^{-e^{2\tau}-3\tau}}{A (\cosh(\tau))^3} \left( \left( -\frac{1}{2} \cosh(\tau) e^\tau + f + 1 \right) e^{2\tau} + (-3 (\cosh(\tau))^2 \right. \\ & + \frac{1}{2} (f + 1) e^{4\tau} + \frac{3}{4} (\cosh(\tau))^2 e^{6\tau} + \frac{25 f}{16} \\ & - \frac{3}{4} ((\cosh(\tau))^2 - \cosh(\tau) \sinh(\tau)) + \frac{67}{32} + \frac{25}{16} \left( f + \frac{11}{10} \right) \tanh(\tau) (1 + e^{2\tau}) \\ & \left. - (\cosh(\tau))^{-2} \left( \frac{25 f}{32} + \frac{47}{48} \right) \right). \end{aligned} \tag{151}$$

$$\begin{aligned} g_2 = & \frac{e^{-e^{2\tau}-3\tau}}{4A (\cosh(\tau))^3} \left( (-4 \cosh(\tau) e^\tau + 7 f + 4 + 8 \tanh(\tau) \left( f + \frac{23}{16} \right)) e^{2\tau} \right. \\ & + 4(-7/2 (\cosh(\tau))^2 + \cosh(\tau) \sinh(\tau) + f + 3/2) e^{4\tau} \\ & + 18 (\cosh(\tau))^2 e^{6\tau} - 12 (\cosh(\tau))^2 - 12 \cosh(\tau) \sinh(\tau) + \frac{29 f}{2} \\ & \left. + \frac{85}{4} + \frac{29}{2} \left( f + \frac{61}{58} \right) \tanh(\tau) \cosh(\tau) - \frac{1}{2 (\cosh(\tau))^2} \left( \frac{29 f}{2} + 19 \right) \right), \end{aligned} \tag{152}$$

$$\begin{aligned} \rho = & \frac{12 e^{-e^{2\tau}-3\tau}}{A (\cosh(\tau))^3} \left( \left( -\frac{1}{2} \cosh(\tau) e^\tau + f + 1 \right) e^{2\tau} + \frac{1}{2} \left( -\frac{3}{2} (\cosh(\tau))^2 + f + 1 \right) e^{4\tau} \right. \\ & + \frac{3}{4} (\cosh(\tau))^2 e^{6\tau} + \frac{25 f}{16} - \frac{3}{4} ((\cosh(\tau))^2 + \cosh(\tau) \sinh(\tau)) \\ & \left. + \frac{67}{32} + \frac{25 \tanh(\tau) (1 + e^{2\tau})}{16} \left( f + \frac{11}{10} \right) - (\cosh(\tau))^{-2} \left( \frac{25 f}{32} + \frac{47}{48} \right) \right). \end{aligned} \tag{153}$$

Unfortunately, there are not a clear results for the first integral of (148).

#### 4.7. Bianchi-type VII<sub>0</sub>

Here, there is no constraint equation and the Bianchi-type dependent  $\hat{V}$  terms are given by

$$\hat{V}_1^{(1)} = -\hat{V}_2^{(1)} = (a_1^4 - a_2^4 + 2\alpha'(2a_1^4 X_1 - 2a_2^4 X_2 + (a_1^4 - a_2^4)\xi_4))e^{2\phi_0}, \tag{154}$$

$$\begin{aligned} \hat{V}_3^{(1)} = & -(a_1^4 + a_2^4 - 2a_1^2 a_2^2 + 2\alpha'(2a_1^4 X_1 + 2a_2^4 X_2 - 2a_1^2 a_2^2 (X_1 + X_2) \\ & + (a_1^4 + a_2^4 - 2a_1^2 a_2^2)\xi_4))e^{2\phi_0}, \end{aligned} \tag{155}$$

$$\hat{V}_1^{(2)} = \frac{1}{2a_3^2} \left( -\frac{a_1^2}{a_2^2} \left( 2(\ln a_1)' \left( \ln \frac{a_1}{a_2^2 a_3^2} \right)' + (\ln a_3)' (\ln a_2)' + \sum (\ln a_i)'^2 \right) \right. \\ \left. - (\ln a_3)' \left( \ln \frac{a_2}{a_3} \right)' - \frac{a_2^2}{a_1^2} \left( 2(\ln a_2)' \left( \ln \frac{a_2}{a_1^2 a_3^2} \right)' + (\ln a_3)' (\ln a_1)' \right) \right) \quad (156)$$

$$+ \sum ((\ln a_i)')^2 + e^{2\phi_0} \left( -2a_2^4 + \frac{a_1^6}{4a_2^2} + \frac{5a_2^6}{4a_1^2} + \frac{a_2^2 a_1^2}{2} \right) \\ + A^2 e^{2\phi_0} \left( 3(f_2 - 1)a_1^2 a_2^{-2} / 4 - (f + 1)/2 - (f - 5)a_2^2 a_1^{-2} / 4 \right), \\ \hat{V}_2^{(2)} = \frac{1}{2a_3^2} \left( -\frac{a_1^2}{a_2^2} \left( (\ln a_1)' \left( \ln \frac{a_1}{a_3^3 a_2^4} \right)' + 4 \ln(a_3)' (\ln a_2)' \right) \right. \\ \left. + \sum (\ln a_i)'^2 \right) - (\ln a_3)' \left( \ln \frac{a_1}{a_3} \right)' - \frac{a_2^2}{a_1^2} \left( (\ln a_2)' \left( \ln \frac{a_2^2}{a_1^4 a_3^3} \right)' \right) \quad (157)$$

$$+ 4(\ln a_3)' (\ln a_1)' + \sum (\ln a_i)'^2 + 2e^{2\phi_0} \left( \frac{a_1^2 a_2^2}{4} - a_2^4 + \frac{5a_1^6}{8a_2^2} + \frac{a_2^6}{8a_1^2} \right) \\ + A^2 e^{2\phi_0} \left( 3(f - 1)a_2^2 a_1^{-2} / 4 - (f + 1)/2 - (f - 5)a_1^2 a_2^{-2} / 4 \right), \\ \hat{V}_3^{(2)} = \frac{1}{2a_3^2} \left( -\frac{a_1^2}{a_2^2} \left( (\ln a_1)' \left( \ln \frac{a_1^2}{a_3^4 a_2^3} \right)' + 4(\ln a_3)' (\ln a_2)' + \sum (\ln a_i)'^2 \right) \right. \\ \left. + (\ln a_3)'^2 + (\ln a_1)' (\ln a_2)' - \frac{a_2^2}{a_1^2} \left( (\ln a_2)' \left( \ln \frac{a_2^2}{a_3^4 a_1^3} \right)' + 4(\ln a_3)' (\ln a_1)' \right) \right) \quad (158)$$

$$+ \sum (\ln a_i)'^2 + e^{2\phi_0} \left( -\frac{a_2^2 a_1^2}{2} + \frac{5a_2^6}{4a_1^2} - \frac{a_1^4}{a_3^2} \right. \\ \left. - 4a_2^4 + \frac{5a_1^6}{4a_2^2 a_3^2} + \frac{(5 - f)A^2}{4} \left( \frac{a_1^2}{a_2^2} - 2 + \frac{a_2^2}{a_1^2} \right) \right), \\ \hat{V}_\phi = \frac{1}{16} a_3^{-2} e^{2\phi_0} \left( -12a_1^4 + 2a_1^2 a_2^2 - 12a_2^4 + \frac{11a_1^6}{a_2^2} + \frac{11a_2^6}{a_1^2} \right. \\ \left. + A^2 \left( \frac{a_1^2}{a_2^2} + \frac{a_2^2}{a_1^2} - 2 \right) \right) + \frac{a_1^2}{a_2^2 a_3^2} \left( -\sum (\ln a_i)'^2 + \frac{1}{2} (7(\ln a_1)')^2 \right. \\ \left. - 5(\ln a_2)' (\ln a_3)' - 2(\ln a_1)'^2 + \frac{7}{2} (\ln a_1)' (\ln a_2)' \right) \quad (159)$$

$$+ \frac{a_2^2}{a_1^2 a_3^2} \left( -\sum (\ln a_i)'^2 + \frac{1}{2} (7(\ln a_2)' - 5(\ln a_1)') (\ln a_3)' - 2(\ln a_2)'^2 \right. \\ \left. + \frac{7}{2} (\ln a_2)' (\ln a_1)' \right) + a_3^{-2} \left( 2(\ln a_3)'^2 - (\ln a_1 a_2)' (\ln a_3)' + (\ln a_1)' (\ln a_2)' \right),$$

With these potentials, at leading order of (38), no general solution has been found unless by setting  $a_1 = a_2$  [7]. In this case, all the  $V$  terms will vanish and the equations will be reduced to the Bianchi-type  $I$  set. Consequently, the solutions can be retrieved from there.

#### 4.8. Bianchi-type VIII

For space–time of VIII type the  $\hat{V}$  terms are given by

$$\hat{V}_1^{(1)} = \frac{1}{2} (a_1^4 - (a_2^2 + a_3^2)^2 + 2\alpha' (2a_1^4 X_1 - 2a_2^4 X_2 - 2a_3^4 X_3 - a_2^2 a_3^2 (X_2 + X_3) \\ + (a_1^4 - (a_2^2 + a_3^2)^2) \xi_4)) e^{2\phi_0}, \quad (160)$$

$$\hat{V}_2^{(1)} = \frac{1}{2}(a_2^4 - (a_3^2 + a_1^2)^2 + 2\alpha'(2a_2^4 X_2 - 2a_1^4 X_1 - 2a_3^4 X_3 - a_2^2 a_3^2(X_1 + X_3) + (a_2^4 - (a_1^2 + a_3^2)^2)\xi_4))e^{2\phi_0}, \tag{161}$$

$$\hat{V}_3^{(1)} = \frac{1}{2}(a_3^4 - (a_1^2 - a_2^2)^2 + 2\alpha'(2a_3^4 X_3 - 2a_3^4 X_2 - 2a_2^4 X_2 + a_1^2 a_2^2(X_1 + X_2) + (a_3^4 - (a_1^2 + a_2^2)^2)\xi_4))e^{2\phi_0}, \tag{162}$$

$$\begin{aligned} \hat{V}_1^{(2)} = & \left( -\sum (\ln a_i)' - 2(\ln a_1)'(\ln \frac{a_1}{a_2^2 a_3^2})' - (\ln a_2)'(\ln a_3)' \right) \frac{a_1^2}{2a_2^2 a_3^2} \\ & - \frac{(\ln a_1'^2 + (\ln a_2)'(\ln a_3)')}{a_1^2} \\ & + \left( -\sum (\ln a_i)' + 4(\ln a_1)'(\ln \frac{a_2}{a_3})' - (\ln a_2)'(\ln \frac{a_2^2}{a_3^2})' \right) \frac{a_2^2}{2a_1^2 a_3^2} \\ & - \frac{(\ln a_2)'((\ln a_2)' - (\ln a_3)')}{a_2^2} \\ & + \left( -\sum (\ln a_i)' - 4(\ln a_1)'(\ln \frac{a_2}{a_3})' - (\ln a_3)'(\ln \frac{a_2^2}{a_3^2})' \right) \frac{a_3^2}{2a_1^2 a_2^2} \\ & - \frac{(\ln a_3)'((\ln a_2)' - (\ln a_3)')}{a_2^2} \end{aligned} \tag{163}$$

$$\begin{aligned} & + e^{2\phi} \left( -\frac{a_2^4}{a_3^2} + \frac{a_3^4}{a_2^2} + \frac{a_2^4}{2a_1^2} + \frac{a_3^4}{2a_1^2} + \frac{a_1^6}{8a_3^2 a_2^2} + \frac{5a_2^6}{8a_3^2 a_1^2} + \frac{5a_3^6}{8a_2^2 a_1^2} \right. \\ & + \frac{1}{2}a_1^2 + a_3^2 - a_2^2 - \frac{a_3^2 a_2^2}{4a_1^2} + \frac{a_3^2 a_1^2}{4a_2^2} + \frac{a_2^2 a_1^2}{4a_3^2} + \frac{A^2}{8} \left( \frac{3(f-1)a_1^2}{a_2^2 a_3^2} \right. \\ & \left. \left. + 2(f+1)\left(\frac{1}{a_2^2} - \frac{1}{a_3^2}\right) + (5-f)\left(\frac{2}{a_1^2} + \frac{a_3^2}{a_2^2 a_1^2} + \frac{a_2^2}{a_3^2 a_1^2}\right) \right) \right), \end{aligned}$$

$$\hat{V}_2^{(2)} = \hat{V}_1^{(2)}(1 \leftrightarrow 2), \tag{164}$$

$$\begin{aligned} \hat{V}_3^{(2)} = & \left( -\sum (\ln a_i)' + 4(\ln a_2)'(\ln \frac{a_3}{a_1})' - 2(\ln a_3)'^2 - 3(\ln a_1)'(\ln a_3)' \right) \frac{a_3^2}{2a_1^2 a_2^2} \\ & + \frac{(\ln a_3')^2 + (\ln a_1)'(\ln a_2)'}{a_3^2} \\ & + \sum_{\substack{k \\ (k \neq i \neq j)}} \left( \left( -\sum (\ln a_i)' - (\ln a_k)'(\ln \frac{a_k^2}{a_j^4 a_3^2})' \right. \right. \\ & \left. \left. - 4(\ln a_j)'(\ln a_3)' \right) \frac{a_k^2}{2a_j^2 a_3^2} - \frac{(\ln a_k)'((\ln a_k)' - (\ln a_j)')}{a_k^2} \right. \\ & \left. + e^{2\phi} \left( \frac{a_3^2 a_k^2}{4a_j^2} - \frac{a_2^2 a_1^2}{4a_3^2} - \frac{a_k^4}{2a_3^2} + \frac{a_j^4}{2a_k^2} + \frac{5a_k^6}{8a_3^2 a_j^2} + \frac{a_3^6}{8a_2^2 a_1^2} - \frac{1}{2}a_3^2 - a_k^2 \right. \right. \\ & \left. \left. + \frac{A^2}{8} \left( \frac{3(f-1)a_3^2}{a_2^2 a_1^2} + 2(f+1)\frac{1}{a_k^2} + (5-f)\left(\frac{a_k^2}{a_j^2 a_3^2} - \frac{2}{a_3^2}\right) \right) \right) \right), \end{aligned} \tag{165}$$

$$\hat{V}_0^{(2)} = \sum_{i \neq j \neq k} \left( (\sum (\ln a_l)' + (\ln a_i)' (\ln \frac{a_i^2}{a_k^4 a_j^3})' + (\ln a_j)' (\ln a_k)') \frac{a_i^2}{2a_j^2 a_k^2} - ((\ln a_i)' - (\ln a_j)' ) ((\ln a_i)' - (\ln a_k)') a_i^{-2} \right), \tag{166}$$

$$\begin{aligned} \hat{V}_\phi = & \sum_{\substack{i \\ j \neq k \neq i}} \left( \frac{a_i^2}{2a_j^2 a_k^2} (-2 \sum (\ln a_l)'^2 - 4(\ln a_i)'^2 + 7(\ln a_i)' ((\ln a_j)' + (\ln a_k)') \right. \\ & - 5(\ln a_j)' (\ln a_k)') + \frac{e^{2\phi_0}}{16} (11a_i^6 a_j^{-2} a_k^{-2} + (2 + A^2) a_i^2 a_j^2 a_k^{-2} \\ & - 12 \sum a_i^4 a_3^{-2} + A^2 (2a_1^{-2} + 2a_2^{-2} - 2a_3^{-2})) \\ & \left. - ((\ln a_2)' (\ln a_1)' - 3(\ln a_3)'^2 - \sum (\ln a_3)' (\ln a_l)') a_3^{-2} \right) \\ & - \sum_{\substack{i \neq 3 \\ j \neq k \neq i}} \left( ((\ln a_j)' (\ln a_3)' + 3(\ln a_i)'^2 - \sum (\ln a_i)' (\ln a_l)') a_i^{-2} + 12a_j^4 a_i^{-2} e^{2\phi_0} \right). \end{aligned} \tag{167}$$

With this form of potentials, the  $\beta$ -function equations (38) recast in the following forms in the zeroth order [7]

$$(\ln a_1^2 e^{\phi_0})'' + (a_1^4 - (a_2^2 + a_3^2)^2) e^{2\phi_0} = 0, \tag{168}$$

$$(\ln a_2^2 e^{\phi_0})'' + (a_2^4 - (a_3^2 + a_1^2)^2) e^{2\phi_0} = 0, \tag{169}$$

$$(\ln a_3^2 e^{\phi_0})'' + (a_3^4 - (a_1^2 - a_2^2)^2) e^{2\phi_0} = 0, \tag{170}$$

and in the first order of  $\alpha'$

$$\begin{aligned} X_1'' + 2(a_1^4 X_1 - a_2^4 X_2 - a_3^4 X_3 - a_2^2 a_3^2 (X_2 + X_3)) e^{2\phi_0} \\ + (a_1^4 - (a_2^2 + a_3^2)^2) e^{2\phi_0} \xi_4 - \frac{1}{2} (\ln a_1^2 e^{\phi_0})' \xi_4' + g_1 = 0, \end{aligned} \tag{171}$$

$$\begin{aligned} X_2'' + 2(a_2^4 X_2 - a_3^4 X_3 - a_1^4 X_1 - a_1^2 a_3^2 (X_1 + X_3)) e^{2\phi_0} \\ + (a_2^4 - (a_3^2 + a_1^2)^2) e^{2\phi_0} \xi_4 - \frac{1}{2} (\ln a_1^2 e^{\phi_0})' \xi_4' + g_2 = 0, \end{aligned} \tag{172}$$

$$\begin{aligned} X_3'' + 2(a_3^4 X_3 - a_1^4 X_1 - a_2^4 X_2 + a_1^2 a_2^2 (X_1 + X_2)) e^{2\phi_0} \\ + (a_3^4 - (a_1^2 - a_2^2)^2) e^{2\phi_0} \xi_4 - \frac{1}{2} (\ln a_1^2 e^{\phi_0})' \xi_4' + g_3 = 0. \end{aligned} \tag{173}$$

General solutions of (168)–(170) are given by [7]

$$\begin{aligned} a_1^2 e^{\phi_0} = a_2^2 e^{\phi_0} = \frac{p_1^2 \cosh(p_2 \tau)}{p_2 \sinh^2(p_1 \tau)}, \\ a_3^2 e^{\phi_0} = \frac{p_2}{\cosh(p_2 \tau)} \end{aligned} \tag{174}$$

and the (171)–(173) admit the following general solutions

$$\begin{aligned}
X_1 = X_2 = -X_3 + Q_3 \coth(p_1 \tau) + Q_4 (p_1 \tau \coth(p_1 \tau) - 1) - p_1 \coth(p_1 \tau) \int \xi_4 d\tau \\
+ \frac{1}{2p_1} \left( \coth(p_1 \tau) \int (p_1 \tau \coth(p_1 \tau) - 1)(g_1 + g_2) d\tau \right. \\
\left. - (p_1 \tau \coth(p_1 \tau) - 1) \int \coth(p_1 \tau)(g_1 + g_2) d\tau \right), \quad (175)
\end{aligned}$$

$$\begin{aligned}
X_3 = Q_5 \tanh(p_2 \tau) + Q_6 (p_2 \tau \tanh(p_2 \tau) - 1) - p_2 \coth(p_2 \tau) \int \xi_4 d\tau \\
+ \frac{1}{p_1} \left( \tanh(p_1 \tau) \int (p_1 \tau \tanh(p_1 \tau) - 1)(g_1 + g_2) d\tau \right. \\
\left. - (p_1 \tau \tanh(p_2 \tau - 1)) \int \tanh(p_2 \tau)(g_1 + g_2) d\tau \right). \quad (176)
\end{aligned}$$

Again, the  $g_i$  (39) and  $\rho$  (32) of this Bianchi-type are obtained by using the (36) and (174). Then, the appeared integral in  $\xi_4$  (42),  $\phi_1$  (37) and the above  $X_i$  could be calculated. In this Bianchi model the  $q_2$  constant (41) is given by

$$q_2 = (n^2 4 - p_1^2 + p_2^2)/2 \quad (177)$$

and for consistency in the solution of  $\xi_4$  (42), any set of parameterization for  $p_i$  and  $n$  should satisfy  $q_2 \neq 0$ . Because of the dense form of  $g_i$  and  $\rho$  terms, here we present an example with a choice of parameters  $p_1 = p_2 = n = 1$  as following

$$\begin{aligned}
g_1 = \frac{1}{16A (\sinh(\tau))^2 (\cosh(\tau))^6} \left( (-7f - 73) (\cosh(\tau))^6 \right. \\
\left. + (85f + 251) (\cosh(\tau))^4 + (-149f - 313) (\cosh(\tau))^2 + 71f + 123 \right), \quad (178)
\end{aligned}$$

$$\begin{aligned}
g_3 = \frac{1}{16A (\sinh(\tau))^2 (\cosh(\tau))^6} \left( (7f - 159) (\cosh(\tau))^6 \right. \\
\left. + (25f + 535) (\cosh(\tau))^4 + (-71f - 623) (\cosh(\tau))^2 + 39f + 235 \right), \quad (179)
\end{aligned}$$

$$\begin{aligned}
\rho = \frac{1}{8A (\sinh(\tau))^2 (\cosh(\tau))^6} \left( (-7f - 169) (\cosh(\tau))^6 \right. \\
\left. + (163f + 637) (\cosh(\tau))^4 + (-305f - 797) (\cosh(\tau))^2 + 149f + 317 \right). \quad (180)
\end{aligned}$$

A detailed form of metric and dilaton corrections for this choice of parameters is given in Appendix C in (234)–(237).

#### 4.9. Bianchi-type IX

The  $\hat{V}$  terms of in Bianchi-type IX case are given as following

$$\begin{aligned}
\hat{V}_i^{(1)} = \frac{1}{2} (a_i^4 - (a_j^2 - a_k^2)^2 + 2\alpha' (a_i^4 X_i - a_j^2 X_j - a_k^2 X_k + a_k^2 a_j^2 (X_k + X_j)) \\
+ (a_i^4 - (a_j^2 - a_k^2)^2) \xi_4) e^{2\phi_0}, \quad (181)
\end{aligned}$$

$$\hat{V}_i^{(2)} = \sum_{\substack{k \neq i \\ (j \neq k)}} \left( -\frac{a_k^2}{2a_j^2 a_i^2} \left( \sum_l (\ln a_l)' + (\ln a_k)' \left( \ln \frac{a_k^2}{a_i^4 a_j^3} \right)' + 4(\ln a_j)' (\ln a_i)' \right) \right. \\ \left. - \frac{a_i^2}{2a_k^2 a_j^2} \left( \sum_l (\ln a_l)' - 4(\ln a_i)' \left( \ln \frac{a_j}{a_k} \right)' + 2(\ln a_j)'^2 + 3(\ln a_k)' (\ln a_j)' \right) \right. \\ \left. + (\ln a_k)' ((\ln a_k)' - (\ln a_j)') a_k^{-2} + (\ln a_j)' ((\ln a_j)' - (\ln a_k)') a_j^{-2} \right. \tag{182} \\ \left. + (\ln a_i)'^2 + (\ln a_k)' (\ln a_i)' a_i^{-2} \right.$$

$$+ e^{2\phi} \left( \frac{a_j^2 a_i^2}{4a_k^2} - \frac{a_j^2 a_k^2}{4a_i^2} - \frac{a_k^4}{a_j^2} - \frac{a_k^4}{2a_i^2} + \frac{a_i^6}{8a_j^2 a_k^2} + \frac{5a_k^6}{8a_j^2 a_i^2} + a_k^2 - \frac{1}{2} a_i^2 \right. \\ \left. + \frac{A^2}{8} \left( (5-f) \left( \frac{a_k^2}{a_i^2 a_j^2} - \frac{2}{a_i^2} \right) + \frac{3(f-1)a_i^2}{a_k^2 a_j^2} - 2(f+1) \frac{1}{a_k^2} \right) \right),$$

$$\hat{V}_0^{(2)} = \sum_{i \neq j \neq k} \left( \left( \sum_l (\ln a_l)' + (\ln a_i)' \left( \ln \frac{a_i^2}{a_k^4 a_j^3} \right)' + (\ln a_j)' (\ln a_k)' \right) \frac{a_i^2}{2a_j^2 a_k^2} \right. \\ \left. - ((\ln a_i)' - (\ln a_j)') ((\ln a_i)' - (\ln a_k)') a_i^{-2} \right), \tag{183}$$

$$\hat{V}_\phi = \sum_i \sum_{j, k \neq i} \left( \frac{a_i^2}{2a_j^2 a_k^2} \left( -2 \sum_l (\ln a_l)'^2 - 4(\ln a_i)'^2 + 12 (\ln a_i)' \sum_{l \neq i} (\ln a_l)' \right) \right. \\ \left. - 5 \sum_{l < j} (\ln a_i)' (\ln a_l)' + (2 (\ln a_i)'^2 - 2(\ln a_i)' \sum_{l \neq i} (\ln a_l)' + (\ln a_k)' (\ln a_j)') a_i^{-2} \right. \tag{184} \\ \left. + \frac{e^{2\phi_0}}{16} \left( (11a_i^6 a_j^{-2} a_k^{-2}) + 2a_i^2 a_j^2 a_k^{-2} + A^2 (a_i^2 a_j^{-2} a_k^{-2} - 2a_i^{-2} - 12a_j^2 a_i^{-4} - 12a_i^2) \right) \right)$$

With this form of potentials, the  $\beta$ -function equation of metric (38) will be, respectively in the zeroth and first order of  $\alpha'$ , as following

$$(\ln a_1^2 e^{\phi_0})'' + (a_i^4 - (a_j^2 - a_k^2)^2) e^{2\phi} = 0, \tag{185}$$

$$X_i'' + 2(a_i^4 X_i - a_j^2 X_j - a_k^2 X_k + a_k^2 a_j^2 (X_k + X_j)) e^{2\phi_0} \\ + (a_i^4 - (a_j^2 - a_k^2)^2) e^{2\phi_0} \xi_4 - \frac{1}{2} (\ln a_i^2 e^{\phi_0})' \xi_4' + g_i = 0. \tag{186}$$

General solutions admitted by the zeroth order are given by [7]

$$a_1^2 e^{\phi_0} = a_3^2 e^{\phi_0} = \frac{p_1^2 \cosh(p_2 \tau)}{p_2 \cosh^2(p_1 \tau)} \\ a_2^2 e^{\phi_0} = \frac{p_2}{\cosh(p_2 \tau)} \tag{187}$$

and solutions of first order are given as following

$$X_1 = X_3 = -X_2 + Q_3 \tanh(p_2 \tau) + Q_4 (p_2 \tau \tanh(p_1 \tau - 1) - p_1 \tanh(p_1 \tau) \int \xi_4 d\tau \\ + \frac{1}{p_1} (\tanh(p_1 \tau) \int (p_1 \tau \tanh(p_1 \tau) - 1)(g_1 + g_2) d\tau \\ - (p_1 \tau \tanh(p_1 \tau) - 1) \int \tanh(p_1 \tau)(g_1 + g_2) d\tau) + c_1, \tag{188}$$

$$\begin{aligned}
X_2 = & Q_5 \tanh(p_2 \tau) + Q_6 (p_2 \tau \tanh(p_2 \tau - 1) - p_2 \tanh(p_2 \tau) \int \xi_4 d\tau \\
& + \frac{1}{p_2} (\tanh(p_2 \tau) \int (p_2 \tau \tanh(p_2 \tau) - 1) g_2 d\tau \\
& - (p_2 \tau \tanh(p_2 \tau) - 1) \int \tanh(p_2 \tau) g_2 d\tau) + c_2.
\end{aligned} \tag{189}$$

In this Bianchi-type the  $q_2$  in (41) is given by

$$q_2 = (n^2 - 4p_1^2 + p_2^2)/2. \tag{190}$$

Now, using the (187) and (36) the  $g_i$  (39) and  $\rho$  (32) can be derived and then by performing the integrals of (37), (42), (188) and (189) the corrections of dilaton and metric in this Bianchi-type are found. Here, as an example, we present a  $g_i$  and  $\rho$  with special choice of parameters as  $n = 1$  and  $p_1 = p_2 = p$ , where the  $q_2 \neq 0$  condition requires  $p \neq \sqrt{3}/3$ .

$$\begin{aligned}
g_1 = & \frac{1}{32A p^3 \cosh(p\tau) (\cosh(\tau))^5} \left( (-33 p^4 - 58 p^2 - 5) (\cosh(\tau))^4 \right. \\
& + \left( (58 f + 131) p^2 + 58 f + 39 \right) (\cosh(\tau))^2 - 58 f - 76 \left. (\cosh(p\tau))^4 \right) \\
& + 76 \left( (p^2 + 5/19) (\cosh(\tau))^2 - 29 f/19 - 61/38 \right) \\
& \times \sinh(p\tau) \sinh(\tau) \cosh(\tau) p (\cosh(p\tau))^3 \\
& + 16 (\cosh(\tau))^2 \left( (p^2 + 4) (\cosh(\tau))^2 - 7/2 f - 19/4 \right) p^2 (\cosh(p\tau))^2 \\
& - 32 \sinh(\tau) \cosh(p\tau) (\cosh(\tau))^3 \sinh(p\tau) p^3 - 64 (\cosh(\tau))^4 p^4,
\end{aligned} \tag{191}$$

$$\begin{aligned}
g_2 = & g_1 + \frac{1}{\cosh(p\tau) (\cosh(\tau))^3} \left( (1 - (p^2 + 1) (\cosh(\tau))^2) (\cosh(p\tau))^2 \right. \\
& + 2 \sinh(\tau) \cosh(p\tau) \cosh(\tau) \sinh(p\tau) p \\
& \left. + 4 (\cosh(\tau))^2 p^2 \right),
\end{aligned} \tag{192}$$

$$\begin{aligned}
\rho = & \frac{1}{16A p^3 \cosh(p\tau) (\cosh(\tau))^5} \left( (-45 p^4 - 90 p^2 - 9) (\cosh(\tau))^4 \right. \\
& + \left( (150 f + 267) p^2 + 150 f_1 + 135 \right) (\cosh(\tau))^2 - 150 f_1 - 188 \left. (\cosh(p\tau))^4 \right) \\
& + 108 \left( (p^2 + 1/3) (\cosh(\tau))^2 - 25 f/9 - 55/18 \right) \\
& \times \sinh(\tau) \sinh(p\tau) \cosh(\tau) p (\cosh(p\tau))^3 \\
& + 16 (\cosh(\tau))^2 \left( (p^2 + 11/2) (\cosh(\tau))^2 - 9 f - 10 \right) p^2 (\cosh(p\tau))^2 \\
& - 32 \cosh(p\tau) (\cosh(\tau))^3 \sinh(p\tau) \sinh(\tau) p^3 - 64 (\cosh(\tau))^4 p^4.
\end{aligned} \tag{193}$$

The string frame Ricci scalar of this  $IX$  model with scale factors of (187) is given by  $R = \frac{3}{2A}$ . Therefore, if the  $A$  is assumed to be a very small constant, for example of order  $\alpha'$ , the model will be in the high curvature limit forever. Also, according to the given discussion in the last paragraph of section 3, the kinetics of dilaton and  $B$ -field will be high as well and the  $\alpha'$ -correction had to be taken into account.

4.9.1. Some physical properties of an example in the Bianchi-type IX

Avoiding of dense mathematical results, as an example we set  $n = 1$  and  $p_1 = p_2 = 1$  and present the resulted forms of  $X_i$ ,  $\phi_1$  and the lapse function  $\xi_4$  in Appendix C in (238)–(241). In this subsection, we are going to investigate the physical behavior of this example.

In this Bianchi model with the given scale factors (187), transforming from  $\tau$  to cosmic time  $\bar{t}$  by (64) and finding the  $\tau$  in terms of  $\bar{t}$ , as we have done for the Bianchi-type I in (65), is not straightforward. Hence, we rewrite the time derivatives in the physical quantities in terms of  $\tau$  derivatives. With the following relation between the cosmic time and  $\tau$

$$d\bar{t} = e^{\frac{\phi}{2}} \sqrt{g_{00}} dt = \bar{a}^{-3} (1 + \alpha' (\sum X_i + \xi_4)) d\tau, \tag{194}$$

up to first order of  $\alpha'$  we have

$$\dot{\bar{a}}_i = \frac{d\bar{a}_i}{d\bar{t}} = \bar{a}^{-3} (\bar{a}'_i (1 - \alpha' (\sum X_j + \xi_4)) + \alpha' (\bar{a}_i X_i)'), \tag{195}$$

$$\ddot{\bar{a}}_i = \frac{d^2\bar{a}_i}{d\bar{t}^2} = \bar{a}^{-6} (\bar{a}''_i - \bar{a}'_i \sum (\ln \bar{a}_j)') (1 - 2\alpha' (\sum X_j + \xi_4)) + \alpha' ((\bar{a}_i X_i)'' - (\bar{a}_i X_i)' \sum (\ln \bar{a}_j)' - (\sum X'_j + \xi'_4) \bar{a}'_i). \tag{196}$$

In this chosen parametrization, the leading order solution of IX (187) is an isotropic model with  $\dot{\bar{a}} < 0$  and  $\ddot{\bar{a}} < 0$ . Hence, it describes a decelerated contracting universe with an initial singularity and no inflation [7].

Considering the two-loop  $\beta$ -function equations solution, the  $\alpha'$ -corrected scale factors (27) with their correction given by (238) and (239), with setting  $A = 1.5\alpha'$ ,  $Q_1 = 2$ ,  $Q_3 = 10$ ,  $Q_4 = -1.8$ ,  $Q_5 = 5$ ,  $Q_6 = -1.2$ ,  $c_1 = -0.1$ ,  $c_2 = 1$  and  $\varphi_0 = 230$ , give rise to  $\dot{\bar{a}}_i > 0$  and  $\ddot{\bar{a}}_1 > 0$ ,  $\ddot{\bar{a}}_3 > 0$  but the  $\ddot{\bar{a}}_2$  changes sign such that  $\ddot{\bar{a}}_2 < 0$  near  $\tau = 0$  and then turn to  $\ddot{\bar{a}}_2 > 0$ , such a way that  $\ddot{\bar{a}} > 0$ .<sup>7</sup> Consequently, the solution of two-loop  $\beta$ -functions in this case of IX model describes an expanding universe which is decelerated at first but then becomes accelerated and there is no collapse to a singularity in the future.

The possibility of recollapse in the Bianchi-type IX can be well understood by the weak energy condition. It is worth considering whether this condition in our solution is satisfied or not. We need to analysis the equation [51]<sup>8</sup>

$$k^2 = -\frac{3}{2} R^{(3)} + 24 T_{\mu\nu} n^\mu n^\nu \tag{197}$$

where  $R^{(3)}$  is the scalar curvature of 3-dimensional homogeneous hypersurface,  $n^\mu$  is a time-like vector, the intrinsic curvature  $k = \sum \dot{\bar{a}}_i / \bar{a}_i$  and the weak energy condition requires  $T_{\mu\nu} n^\mu n^\nu > 0$ . Unlike the other Bianchi models, the  $R^{(3)}$  may become positive in Bianchi-type IX and hence the  $k$  can pass zero and it is possible for an expanding universe of this type to recollapse. But with a negative  $R^{(3)}$ , a change in the sign of  $k$  is not allowed and an expanding model will expand forever. Here, in our example, the  $R^{(3)}$  vanishes in zeroth order and is negative in the first order of  $\alpha'$ . Therefore, being no change in the sign of  $\dot{\bar{a}}$  satisfies the weak energy condition. Furthermore, as we have discussed in the Bianchi-type I, the strong energy condition requires

<sup>7</sup> For choosing the constants, the signature of metric is considered to be adopted  $(-1, 1, 1, 1)$  after adding corrections. Also, a good cosmological behavior of the solutions has been considered and this set of constants is not unique.

<sup>8</sup> Here, we consider the equation given in [51] with  $\Lambda = 0$  and assume that the higher curvature terms contribute to the right hand of Einstein frame field equations in the energy momentum tensor.

the  $\dot{\bar{k}} + \bar{k}^2 < 0$ . Here, the violation of this energy conditions, which is necessary for avoiding the initial singularity [52], occurs soon after  $\tau = 0$  by  $\ddot{\bar{a}} > 0$  in both  $f = \pm 1$  schemes.

In this example, an isotropic parameterization for the zeroth order metric has been chosen. However, the  $\alpha'$ -corrected metric is anisotropic even if one ignores the constants of integration. However, the anisotropy measured by the mean anisotropy parameter  $A_m = \frac{(\bar{H}_I - \bar{H})^2}{\bar{H}^2}$  increases at early times but then decreases as time goes on, and the decrease rate is faster in  $f = 1$  scheme.

## 5. Conclusion

The anisotropic homogeneous space-times have been emerged in the context of string cosmology during the search of backgrounds for describing the early universe evolution. In this work the two-loop (order  $\alpha'$ )  $\beta$ -function equations with dilaton and  $B$ -field on the anisotropic homogeneous space-times, taking into account the two particular cases of RS which are called  $R^2$  and Gauss–Bonnet schemes have been investigated. General forms of two-loop  $\beta$ -function equations in terms of the Hubble coefficients, dilaton and  $B$ -field are derived in both RS. Then, the solutions in the all Bianchi-type space-times with zero  $\Lambda$  are computed and then the metric, dilaton and axion field are found with perturbative  $\alpha'$ -corrections.

The basic forms of the first  $\alpha'$  order solutions have been given in certain integral terms in the all models. Because of mathematical complexity and dense results of the integrals, for obtaining an explicit answer for the first  $\alpha'$  order corrections, fixed value for the arbitrary constants of zeroth order solutions should be chosen. Nevertheless, the integral forms are strongly applicable in every chosen parameterizations of zeroth order as needed for any physical purposes. A sample example of any Bianchi-type with a specific parameter is presented.

Especially, the behavior of the  $\alpha'$ -corrected answers in the Bianchi  $I$  and  $IX$  models have been investigated. In the Bianchi  $I$  model in an isotropic example, the decelerated expanding behavior of leading order solutions are modified in the two-loop solutions and the  $\alpha'$ -corrected answers describe a universe which starts decelerated expanding and then turns to an accelerated expanding phase and then with leaving the high curvature limit, reaches to the leading order solution behavior. The accelerated phase violates the strong energy condition and may describe an inflationary phase. In the Bianchi  $IX$  model, as a special example, an isotropic case of leading order solution is selected which describes a decelerated contraction. The  $\alpha'$ -corrected model of  $IX$  (with choosing a set for arbitrary constants) describes an expanding universe which is decelerated in early times and then becomes accelerated violating the strong energy condition. Although an isotropic parameterization for the leading order has been chosen, the first order correction brings about an anisotropy that decreases as time passes. Also, the condition of the recollapse possibility in the Bianchi-type  $IX$  model is investigated and showed that being no recollapse in the selected solution satisfies the weak energy condition. Beside it, considering this solution in an inflationary framework, no premature recollapse can occur. In both selected Bianchi-type  $I$  and  $IX$ , the  $\alpha'$ -corrected solutions of two-loop  $\beta$ -functions in the both Gauss–Bonnet and  $R^2$  schemes with  $B$ -field contribution are capable to well describe the accelerated expansion of the universe.

The leading order solutions have had an initial singularity with no inflation [7]. We found that the  $\alpha'$ -corrected solutions of two-loop  $\beta$ -function equations in the selected Bianchi-type  $I$  and  $IX$  model in the Bianchi-type  $I$  and  $IX$  model avoid singularity in the violation of strong energy condition context and satisfies the weak energy condition. It has been shown in [55] that including only the  $\alpha'$ -correction of Gauss–Bonnet multiplied by a dilaton field to the leading order effective action does not violate the energy conditions and hence it can not lead to a singularity-

free solution. Therefore, the violation of strong energy condition in our selected solutions in Bianchi-type *I* and *IX* models may be interpreted as a result of contribution of the *B*-field.

Usually, in the recent studies the solutions of the corrected action field equations with dilaton and modulus fields have been investigated where the field strength of the *B*-field has been taken to be zero for simplicity [56]. In this work, we have considered the two-loop  $\beta$ -functions with the contributions of dilaton and *B*-field (with nonzero but constant field strength tensor magnitude), where in the presence of *B*-field a RS dependence appears. Noting the equivalence of the  $\beta$ -function and the equations of motion of the  $\alpha'$ -corrected effective action, the solutions of  $\beta$ -function equations, beside securing the conformal invariance up to two-loop, are higher derivative quadratic gravitational action solutions in the Einstein frame. The dilaton and a *B*-field will appear as an effective matter in the field equations. Hence, from the Einstein frame point of view, our solutions are solutions of quadratic curvature gravity with effective matter associated with the dilaton and *B*-field, where the *B*-field part is capable of violating the strong energy condition.

In [7] the magnitude of the field strength tensor of *B*-field, *A*, have relation with the anisotropy through the initial value equation. But here, a different solution for the zeroth order of dilaton (36) has been chosen which result in the elimination of *A* from the initial value equation constraint (41). Then, taking advantage of the arbitrariness, being in the high curvature and high kinetics limits as the necessary condition for including the  $\alpha'$ -corrections in the action is prescribed by the magnitude of *A* such that it is required to be of order  $\alpha'$ .

Moreover, in the Bianchi-type *V* model, it is observed that although the leading order solution allows an isotropic metric, the two-loop solutions restrict the  $\alpha'$ -corrected metric to be isotropic in its zeroth and first order. The detailed behavior and cosmological implications of these Bianchi-type models would be investigated in another work.

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### Appendix A

In this appendix we present some definitions and the classifications of real 3-dimensional Bianchi groups and their associated vielbeins  $e_\alpha^i(x)$  [7,9].

In Table 1, the  $\{\sigma_i, i = 1, 2, 3\}$  are non-coordinate basis and their dual are  $\{\sigma^i, i = 1, 2, 3\}$ . The relation between coordinate and non-coordinate basis is given by  $\sigma_i = e^\alpha_i(x) dx_\alpha$ . Hence, the non-zero components of the Riemann and Ricci tensors of the metric (17) are expressed as [54]

$$\begin{aligned}
 R^i_{jkl} &= \Gamma_{lj}^m \Gamma_{km}^i - \Gamma_{kj}^m \Gamma_{lm}^i - f^{klm} \Gamma_{mj}^i, \\
 R^0_{j0l} &= \dot{\Gamma}_{lj}^0 + \Gamma_{lj}^m \Gamma_{0m}^0 - \Gamma_{0j}^m \Gamma_{lm}^0,
 \end{aligned} \tag{198}$$

$$\begin{aligned}
 R^0_{jkl} &= \Gamma_{lj}^m \Gamma_{km}^0 - \Gamma_{kj}^m \Gamma_{lm}^0 - f^{klm} \Gamma_{mj}^0, \\
 R_{ij} &= \dot{\Gamma}_{ji}^0 + \Gamma_{ji}^e \Gamma_{de}^d + \Gamma_{ji}^0 \Gamma_{d0}^d + \Gamma_{ji}^0 \Gamma_{00}^0 - \Gamma_{di}^e \Gamma_{je}^d - \Gamma_{di}^0 \Gamma_{j0}^d - \Gamma_{0i}^e \Gamma_{je}^0 - f_{dj}^e \Gamma_{ei}^d, \\
 R_{0j} &= \Gamma_{j0}^e \Gamma_{de}^d - \Gamma_{d0}^e \Gamma_{je}^d,
 \end{aligned} \tag{199}$$

$$R_{00} = \dot{\Gamma}_{d0}^d + \Gamma_{00}^e \Gamma_{de}^d + \Gamma_{00}^0 \Gamma_{d0}^d - \Gamma_{d0}^e \Gamma_{0e}^d,$$

Table 1  
Bianchi-types, their Lie algebras and left invariant one forms.

Algebra	Non-zero commutation relations $[\sigma_i, \sigma_j] = f_{ij}^k \sigma_k$	Class	$g^{-1}dg$
<i>I</i>	$[\sigma_i, \sigma_j] = 0$	$\mathcal{A}$	$dx^i \sigma_i$
<i>II</i>	$[\sigma_2, \sigma_3] = \sigma_1$	$\mathcal{A}$	$dx^1 \sigma_1 + dx^2(\sigma_2 + x_3 \sigma_1) + dx^3 \sigma_3$
<i>III</i>	$[\sigma_1, \sigma_3] = \sigma_3$	$\mathcal{B}$	$dx_1 \sigma_1 + dx^2(\sigma_2 + x^3 \sigma_1) + e^{x^1} dx^3(\sigma_3 - x^2 \sigma_1)$
<i>IV</i>	$[\sigma_1, \sigma_2] = -\sigma_2 + \sigma_3, [\sigma_1, \sigma_3] = -\sigma_3$	$\mathcal{B}$	$dx^1[\sigma_1 - x^2 \sigma_2 + (x^2 - x^3) \sigma_3] + dx^2 \sigma_2 + dx^3 \sigma_3$
<i>V</i>	$[\sigma_1, \sigma_2] = -\sigma_2, [\sigma_1, \sigma_3] = -\sigma_3$	$\mathcal{B}$	$dx^1(\sigma_1 - x^2 \sigma_2 - x^3 \sigma_3) + dx^2 \sigma_2 + dx^3 \sigma_3$
<i>VI<sub>0</sub></i>	$[\sigma_1, \sigma_3] = \sigma_2, [\sigma_2, \sigma_3] = \sigma_1$	$\mathcal{B}$	$dx^1(\sigma_1 \cosh x^3 + \sigma_2 \sinh x^3) + dx^2(\sigma_2 \cosh x^3 + \sigma_1 \sinh x^3) + dx^3 \sigma_3$
<i>VI<sub>-1</sub></i>	$[\sigma_1, \sigma_2] = -\sigma_2, [\sigma_1, \sigma_3] = \sigma_3$	$\mathcal{B}$	$dx^1(\sigma_1 \cosh x^3 + \sigma_2 \sinh x^3) + dx_2(\sigma_2 \cosh x^3 + \sigma_1 \sinh x^3) + dx^3(\sigma_3 - \cosh x^3(x^1 \sigma_2 + x^2 \sigma_1) + \sinh x^3(x^1 \sigma_1 + x^2 \sigma_2))$
<i>VII<sub>0</sub></i>	$[\sigma_1, \sigma_3] = -\sigma_2, [\sigma_2, \sigma_3] = \sigma_1$	$\mathcal{A}$	$dx^1(\sigma_1 \cos x^3 - \sigma_2 \sin x^3) + dx^2(\sigma_2 \cos x^3 + \sigma_1 \sin x^3) + dx^3 \sigma_3$
<i>VII<sub>a</sub></i>	$[\sigma_1, \sigma_2] = -a\sigma_2 + \sigma_3, [\sigma_3, \sigma_1] = \sigma_2 + a\sigma_3$	$\mathcal{B}$	$dx^1[\sigma_1 - (ax^2 + x^3)\sigma_2 - (-ax^3 + x^2)\sigma_3] + dx^2 \sigma_2 + dx^3 \sigma_3$
<i>VIII</i>	$[\sigma_1, \sigma_3] = -\sigma_2, [\sigma_2, \sigma_3] = \sigma_1, [\sigma_1, \sigma_2] = -\sigma_3$	$\mathcal{A}$	$dx^1(\sigma_1 \cosh x^2 \cos x^3 - \sigma_2 \cosh x^2 \sin x^3 - \sigma_3 \sinh x^2) + dx^2(\sigma_2 \cos x^3 + \sigma_1 \sin x^3) + dx^3 \sigma_3$
<i>IX</i>	$[\sigma_1, \sigma_3] = -\sigma_2, [\sigma_2, \sigma_3] = \sigma_1, [\sigma_1, \sigma_2] = \sigma_3$	$\mathcal{A}$	$dx^1(\sigma_1 \cos x^2 \cos x^3 - \sigma_2 \cos x^2 \sin x^3 + \sigma_3 \sin x^2) + dx^2(\sigma_2 \cos x^3 + \sigma_1 \sin x^3) + dx^3 \sigma_3$

where the dot symbol stands for derivative with respect to  $t$  and the connection coefficients are given by

$$\Gamma_{ij}^k = -g^{dk}(f_{id}^e g_{ej} + f_{jd}^e g_{ei}) + f_{ij}^k, \tag{200}$$

$$\Gamma_{ij}^0 = -\frac{1}{2}g^{00} \dot{g}_{ij}, \quad \Gamma_{00}^0 = \frac{1}{2}g^{00} \dot{g}_{00}, \quad \Gamma_{i0}^i = \frac{1}{2}g^{ii} \dot{g}_{ii}.$$

The Riemann and Ricci tensors in coordinate basis are obtained by multiplying the vielbeins  $e_\alpha^i$ , for example  $R_{\alpha\beta} = e_\alpha^i e_\beta^j R_{ij}$  and  $R_{\alpha 0} = e_\alpha^i R_{i0}$ .

### Appendix B

In this appendix, for instructive purposes, we are going to write the given calculation in the sections 3 and 4 with more details for an isotropic example, i.e.  $a_1 = a_2 = a_3$ , in the Bianchi-type  $V$ . In this case, using (198)–(199) and the metric (17) (along with (18)) we have

$$\begin{aligned}
 R^1_{212} &= R^1_{313} = R^2_{323} = a_1^2 H_1^2 - 1, \\
 R^1_{010} &= R^2_{020} = R^3_{030} = -H_1^2 - \dot{H}_1, \\
 R_{11} &= R_{22} = R_{33} = -2 - g_{00}^{-1} a_1^2 (\dot{H}_1 + 3 H_1^2 - H_1 (\ln \dot{g}_{00})), \\
 R_{00} &= 3(-\dot{H}_1 - H_1^2 + H_1 (\ln \dot{g}_{00})), \\
 \Gamma_{ij}^k &= -1, \quad \Gamma_{ij}^0 = -g^{00} a_1^2 H_1, \quad \Gamma_{00}^0 = \frac{1}{2} \ln \dot{g}_{00}, \quad \Gamma_{i0}^i = H_1,
 \end{aligned}
 \tag{201}$$

The constant terms in the Riemann and Ricci tensors and connection coefficients are related to the structure constants  $f_{ij}^k$ , which differ in the Bianchi models and contribute in the Bianchi dependent terms,  $V$ , that have been introduced in the section 3. The other terms in (201) are independent of the structure constants and appear in all Bianchi-type, taking part in Bianchi independent terms  $K$ . In this isotropic case the  $\beta$ -functions of (6) and (12) cast the following forms

$$\frac{1}{\alpha'} \bar{\beta}^G{}_1 = \dot{H}_1 + 3H_1^2 + H_1 \dot{\phi} - H_1 (\ln \dot{g}_{00}) + V_1^{(1)} - \frac{1}{2} A^2 a_1^{-6} + \alpha' (V_1^{(2)} + K_1), \tag{202}$$

$$\frac{1}{\alpha'} \bar{\beta}^G{}_0 = 3(\dot{H}_1 + H_1^2) + \ddot{\phi} - (\ln \dot{g}_{00})(\dot{\phi} + 3\dot{H}_1) - \alpha' (V_0^{(2)} + K_0), \tag{203}$$

$$\begin{aligned}
 \frac{1}{\alpha'} \bar{\beta}^\phi &= -2\ddot{\phi} - \dot{\phi}^2 - 3(2\dot{\phi}H_1 + V_1^{(1)} + H_1^2 - 2\dot{H}_1 + 3H_1^2) \\
 &\quad + 2(\phi' + H_1)(\ln \dot{g}_{00}) + \frac{1}{2} A^2 a_1^{-6} + \alpha' [K_\phi + V_\phi^{(2)}],
 \end{aligned}
 \tag{204}$$

where

$$\begin{aligned}
 V_1^{(1)} &= -2a_1^{-2} g_{00} \\
 V_1^{(2)} + K_1 &= \dot{H}_1^2 + 2H_1 \dot{H}_1 + 3H_1^4 - 4H_1^2 a_1^{-2} + 2a_1^{-4} + \frac{3}{8} A^2 a_1^{-12} \\
 &\quad + \frac{1}{8} A^2 (-H_1^2 (92f + 88) a_1^{-6} + 4(f + 7) a_1^{-8}) \\
 V_0^{(2)} + K_0 &= -3(H_1^2 + \dot{H}_1)^2 - 3A^2 a_1^{-6} (\frac{1}{2} \dot{H}_1 + (f - 1) H_1^2) \\
 K_\phi &= -3(\dot{H}_1^2 + 2H_1^2 \dot{H}_1 + H_1^4) - \frac{1}{4} A^2 a_1^{-6} (3H_1^2 - 5A^2 a_1^{-6}) \\
 V_\phi &= \frac{1}{4} (-3a_1^{-6} H_1^2 + 5A^2 a_1^{-12} + 3a_1^{-8}) A^2.
 \end{aligned}
 \tag{205}$$

Now, the equation of (204) with using (202) and (203) can be rewritten in the following form

$$-\ddot{\phi} - \dot{\phi}^2 - 3\dot{\phi}H_1 - A^2 a_1^{-6} + \alpha' [K_\phi + V_\phi^{(2)} + 3V_1^{(2)} + 3K_1 - V_0^{(2)} - K_0] = 0. \tag{206}$$

By a time redefinition (29) and using the following relations for any function  $B$

$$B' = a_1^3 e^\phi \dot{B}, \quad B'' = a_1^6 e^{2\phi} (\ddot{B} + \dot{B}(3H_1 + \dot{B})), \tag{207}$$

the (206) recast the following form

$$-\phi'' - A^2 e^{2\phi} + \alpha' [\hat{K}_\phi + \hat{V}_\phi^{(2)} + 3\hat{V}_1^{(2)} + 3\hat{K}_1 - \hat{V}_0^{(2)} - \hat{K}_0], \tag{208}$$

where we have used the hatted symbols for the terms in  $\tau$  coordinate with a  $a_1^6 e^{2\phi}$  factor which are obtained from (205) by using (207). Now, using the  $\alpha'$  expansions of (26)–(28) and the series expansion of  $e^\phi$  up to first order of  $\alpha'$  as following

$$e^\phi = e^{\phi_0} (1 + \alpha' \phi_1), \tag{209}$$

the (208) equation leads to the equations (30) and (31) in the zeroth and first order of  $\alpha'$ . Their solutions have been given in (36) and (37). For obtaining the  $\xi_4$ -containing term in the (37) the following relation, which is obtained by integration by part, has been used

$$\begin{aligned} & \frac{1}{n} \left[ \tanh(n\tau) \int (n\tau \tanh(n\tau) - 1) \left( \frac{2n^2 \xi_4}{\cosh^2(n\tau)} + n \tanh(n\tau) \xi_4' \right) d\tau \right. \\ & \quad \left. - (n\tau \tanh(n\tau) - 1) \int \tanh(n\tau) \left( \frac{2n^2 \xi_4}{\cosh^2(n\tau)} + n \tanh(n\tau) \xi_4' \right) d\tau \right] \\ & = -n \tanh(n\tau) \int \xi_4 d\tau. \end{aligned} \tag{210}$$

Then, adding the (208) (with a  $-\frac{1}{2}$  coefficient) to the (202), results in an equation of type (38) which in the zeroth and first order of  $\alpha'$  reads the following equations, respectively

$$(\ln a_1^2 e^{\phi_0})'' - 2a_1^4 e^{2\phi_0} = 0, \tag{211}$$

$$X_1'' - 2a_1^4 e^{2\phi_0} (2X_1 + \xi_4) - \frac{1}{2} (\ln a_1^2 e^{\phi_0})' \xi_4' + g_1 = 0, \tag{212}$$

with

$$g_1 = -\frac{1}{2} (\hat{K}_\phi + \hat{V}_\phi^{(2)} + \hat{V}_1^{(2)} + \hat{K}_1 - \hat{V}_0^{(2)} - \hat{K}_0), \tag{213}$$

where we have used the following relation for rewriting the  $H_i$  in the  $\tau$  coordinate up to the first order of  $\alpha'$

$$H_i = (\ln a_i) = a^{-3} e^{-\phi} [(\ln a_{i0})' + \frac{\alpha' X_i'}{1 + \alpha' X_i}] \simeq a^{-3} e^{-\phi} [(\ln a_{i0})' + \alpha' X_i']. \tag{214}$$

The solutions of equations (211)–(212) are given by

$$a_1^2 e^{\phi_0} = \frac{p}{\sinh(p\tau)}, \tag{215}$$

$$\begin{aligned} X_1 &= Q_3 \coth(p\tau) + Q_4 (p\tau \coth(p\tau) - 1) - \frac{1}{2} p \coth(p\tau) \int \xi_4 d\tau \\ &+ \frac{1}{p} \left( \coth(p\tau) \int (p\tau \coth(p\tau) - 1) (g_1) d\tau \right. \\ &\quad \left. - (p\tau \coth(p\tau) - 1) \int \coth(p\tau) (g_1) d\tau \right). \end{aligned} \tag{216}$$

In finding the above answer the following relation has been used

$$\begin{aligned} & \frac{1}{p} \left[ \coth(p\tau) \int (p\tau \coth(p\tau) - 1) \left( \frac{2p^2 \xi_4}{\sinh^2(p\tau)} - p \coth(p\tau) \xi_4' \right) d\tau \right. \\ & \quad \left. - (p\tau \coth(p\tau) - 1) \int \coth(p\tau) \left( \frac{2p^2 \xi_4}{\sinh^2(p\tau)} - p \coth(p\tau) \xi_4' \right) d\tau \right] \\ & = p \coth(p\tau) \int \xi_4 d\tau. \end{aligned} \tag{217}$$

Now, the time–time component of  $\beta$ -function equation (203) needs to be solved. In the new time coordinate  $\tau$ , using the (26)–(28), (209) and (214) it reads

$$\begin{aligned} &\phi_0'' - \phi_0'(3(\ln a_1)' + \phi_0') + 3(\ln a_1)'' - 3(\ln a_1)'(3(\ln a_1)' + \phi_0') \\ &+ \alpha'[3X_1'' - \frac{1}{2}\phi_1'' + 6(3(\ln a_1)' + \phi_0')X_1' + \phi_0'\phi_1' + 6(\ln a_1)'X_1' + \hat{K}_0 + \hat{V}_0] = 0 \end{aligned} \tag{218}$$

Using the (30), (31), (211), (212) it can be rewritten as following

$$\begin{aligned} &[\frac{3}{2}(\ln a_1^2 e^{\phi_0})'^2 - 6a_1^4 e^{2\phi_0} - \phi_0'^2 - A^2 e^{2\phi_0}](1 + 2\alpha'\xi_4) \\ &+ \alpha'[3(3(\ln a_1^2 e^{2\phi_0})' - \phi_0')X_1' - \phi_0'\phi_1' + A^2 e^{2\phi_0}\phi_1 - 12a_1^4 e^{2\phi_0} \\ &+ 3g_1 + \frac{\rho}{2} + \hat{K}_0 + \hat{V}_0^{(2)}] = 0. \end{aligned} \tag{219}$$

Then, following the given discussion in the first paraphrase of page 7, solution of the above equation gives the  $\xi_4$  and  $q_2 = (n^2 - 3p^2)/2$ . The explicit answers with  $p = n = 1$  which are obtained after performing the integrals of  $X_1$  and  $\phi_1$  can be found in (231)–(233).

### Appendix C

In this appendix some sample examples for discussed models in the section 4 are presented.

#### Bianchi-type I

In Bianchi-type I with choosing  $3p_1 = 3p_2 = 3p_3 = n = 1$  the first  $\alpha'$ -corrections of metric and dilaton are found as following<sup>9</sup>

$$\begin{aligned} X_i = &\frac{1}{AL^3} \left[ \frac{1}{27} Li_2(-e^{2\tau}) - \frac{\ln(e^{2\tau} + 1)(87f - 136)}{216} - \frac{203f + 236}{144(e^{2\tau} + 1)^2} \right. \\ &+ \frac{1}{24} \frac{29f + 38}{(e^{2\tau} + 1)^3} + \frac{29f - 24}{72(e^{2\tau} + 1)} + \frac{\tau(-1766 + 609f + 128\ln(2))}{1728} + \left. \frac{2\tau^2}{27} \right] \tag{220} \\ &+ \frac{P_i}{2} \int \xi_4 d\tau + r_i \tau + c_i, \end{aligned}$$

$$\begin{aligned} \phi_1 = &Q_1 \tanh(\tau) + Q_2(\tau \tanh(\tau) - 1) - \tanh(\tau) \int \xi_4 d\tau \\ &+ \frac{1}{2L^3 A} \left[ \frac{1}{9} \tanh(\tau) \left( -\frac{1}{20(e^{2\tau} + 1)^5} ((5500f + 5100)\tau + 1175f + 6371)e^{2\tau} \right. \right. \\ &+ ((-4000f_1 - 11680)\tau - 7275f - 3243)e^{4\tau} \\ &+ ((12000f + 9560)\tau - 8775f - 8523)e^{6\tau} \\ &+ (50f - 880\tau + 206)e^{8\tau} + (-100f + 148)\tau - 375f + 885) + 4\tau^2 \tag{221} \\ &+ \left. \frac{1}{10} \ln(e^{2\tau} + 1) (-40\tau - 77 + 25f_1) - 2Li_2(-e^{2\tau}) + 77\tau/5 - 5f\tau \right. \\ &\left. - 2(\tau \tanh(\tau) - 1) (-2/9 \ln(\cosh(\tau)) + 2/9\tau + (125f - 1)(\tanh(\tau))^2/96 \right) \end{aligned}$$

<sup>9</sup> The  $Li_2(\tau)$  is Polylogarithm function.

$$+ \frac{\sinh(\tau)}{(\cosh(\tau))^3} (145 f/72 + 173/90) + 1/36 (37/5 - 5 f) \tanh(\tau) - \frac{(39 + 25 f) (\sinh(\tau))^2}{32 (\cosh(\tau))^4} - \frac{\sinh(\tau) (75 f + 94)}{40 (\cosh(\tau))^5} \Big] + \varphi_0,$$

where the  $r_i$ ,  $c_i$  and  $\varphi_0$  are constants of integrating and The initial value equation (42) gives

$$\begin{aligned} \xi_4 = & \frac{1}{5760 L^3 A} [960 (1 - 7/3 \tanh(\tau)) \ln(\cosh(\tau)) - 28335 f \\ & - 960 \tau + 9260 + 2240 \tanh(\tau) (\tau - 1959 f/448 - 5133/1120) \\ & + (\cosh(\tau))^{-4} ((32940 f + 1960) (\cosh(\tau))^2 \\ & + (-28860 f - 20448) \tanh(\tau) (\cosh(\tau))^2 - 4605 f - 11220 \\ & + (34155 f + 43254) \tanh(\tau))] + \sum r_i. \end{aligned} \tag{222}$$

**Bianchi-type II**

For simplicity, with the special choice of parameterization as  $p_1 = p_2 = p_3 = n = 1$ , the  $\alpha'$ -corrections of metric and dilaton are found as following

$$\begin{aligned} X_1 = & Q_3 \tanh(\tau) + Q_4 (\tau \tanh(\tau) - 1) - \frac{1}{2} \tanh(\tau) \int \xi_4 d\tau \\ & + \frac{\tanh(\tau)}{2A} \left( \frac{-e^{-6\tau}}{46080 (\cosh(\tau))^6} ((4896 f + 28422) \tau - 2088 f + 32859) e^{2\tau} \right. \\ & + ((-8280 f - 107100) \tau - 18864 f - 57258) e^{4\tau} \\ & + ((23760 f + 74700) \tau - 16692 f - 85004) e^{6\tau} \\ & + (-810 \tau + 252 f - 2031) e^{10\tau} + (-17820 \tau + 1386 f - 1248) e^{8\tau} \\ & + (-504 f + 1632) \tau - 1050 f + 4330) + \frac{9\tau^2}{8} \\ & + \left. \frac{(-135 \tau + 2 f - 257)}{120} \ln(1 + e^{2\tau}) - \frac{9 Li_2(-e^{2\tau})}{16} - \frac{11 \tau (-24 + f)}{30} \right) \tag{223} \\ & - \frac{\tau \tanh(\tau) - 1}{1920 A} (-1080 \ln(\cosh(\tau)) + 750 f + 1080 \tau - 1525 \\ & - 176 (\frac{21 f}{11} - \frac{68}{11}) \tanh(\tau) + \frac{1}{(\cosh(\tau))^6} ((-1320 f - 2970) (\cosh(\tau))^4 \\ & + 1232 \sinh(\tau) (\frac{72 f}{77} + \frac{43}{14}) (\cosh(\tau))^3 + (1350 f + 9195) (\cosh(\tau))^2 \\ & - 1056 \sinh(\tau) (\frac{17 f}{22} + \frac{62}{11}) \cosh(\tau) - 780 f - 4700)), \end{aligned}$$

$$\begin{aligned} X_2 = & -X_1 + \frac{1}{2} \int \xi_4 d\tau - \frac{1}{30A} \left( -\frac{45 f - 92}{1 + e^{2\tau}} - \frac{90 f + 191}{4(1 + e^{2\tau})^2} \right. \\ & - \frac{3(55 f + 179)}{(1 + e^{2\tau})^4} + \frac{195 f + 604}{(1 + e^{2\tau})^3} + \frac{1}{8} (-105 Li_2(-e^{2\tau}) - 2105 \tau^2 \\ & \left. + \ln(1 + e^{2\tau}) (360 f - 631) - \tau (-1546 + 210 \ln(2) + 375 f)) \right), \end{aligned} \tag{224}$$

$$\begin{aligned}
 \phi_1 = & Q_1 \tanh(\tau) + Q_2 (\tau \tanh(\tau) - 1) \\
 & + \frac{\tanh(\tau)}{2A} \left( \frac{e^{-6\tau}}{23040 (\cosh(\tau))^6} ((21600\tau + 3240)f + 40518\tau \right. \\
 & + 35451)e^{2\tau} + ((-40680\tau - 58320)f - 130140\tau - 88362)e^{4\tau} \\
 & + ((79920\tau - 61420)f + 117900\tau - 118636)e^{6\tau} \\
 & + (-810\tau + 420f - 1839)e^{10\tau} + (-17820\tau + 2310f - 192)e^{8\tau} \\
 & + (-840\tau - 1750)f + 1248\tau + 3530) + \frac{(70f - 135\tau - 239)}{60} \ln(1 + e^{2\tau}) \\
 & + \frac{9}{4}\tau^2 - \frac{9Li_2(-e^{2\tau})}{8} - \frac{1}{30}\tau(70f - 239) \tag{225} \\
 & - \frac{\tau \tanh(\tau) - 1}{960A} (-1080 \ln(\cosh(\tau)) + 1080\tau + 1810f - 605 \\
 & + (-560f + 832) \tanh(\tau) + (\cosh(\tau))^{-6} ((-4440f - 5370) (\cosh(\tau))^4 \\
 & + (4160f + 6056) \sinh(\tau) (\cosh(\tau))^3 + (5610f + 12315) (\cosh(\tau))^2 \\
 & + (-3600f - 7968) \sinh(\tau) \cosh(\tau) - 2980f - 6340)) - \tanh(\tau) \int \xi_4 d\tau.
 \end{aligned}$$

The initial value equation (42) gives

$$\begin{aligned}
 \xi_4 = & Q_4 \left( 2 + \frac{\sinh(\tau)\tau}{(\cosh(\tau))^3} - (\cosh(\tau))^{-2} \right) + \frac{Q_3 \sinh(\tau)}{(\cosh(\tau))^3} \\
 & - \frac{e^{-2\tau}}{16A (\cosh(\tau))^6} (4 \sinh(\tau) \cosh(\tau) (16 (\cosh(\tau))^2 - 16f - 37) \\
 & + 71 (\cosh(\tau))^4 + (-80f - 165) (\cosh(\tau))^2 + 64f + 148) \tag{226} \\
 & + \frac{\ln(\cosh(\tau))}{2A} + \frac{1}{480A} (240 \ln(\cosh(\tau)) - 240\tau - 20f - 2295 \\
 & + (-32f + 1948) \tanh(\tau) + (\cosh(\tau))^{-6} (2070 (\cosh(\tau))^4 \\
 & + (-16f - 1156) \sinh(\tau) (\cosh(\tau))^3 + (60f_1 - 4815) (\cosh(\tau))^2 \\
 & + (-72f + 2568) \sinh(\tau) \cosh(\tau) - 40f + 5040)).
 \end{aligned}$$

**Bianchi-type III**

With choosing a parameterization of  $p_1 = p_2 = p_3 = n = 1$  and calculating the integrals of (37), (102) and (103), we obtain

$$\begin{aligned}
 X_1 = X_3 = & -X_2 + Q_3 \coth(\tau) + Q_4 (\tau \coth(\tau) - 1) + \left( \frac{1}{2} - \coth(\tau) \right) \int \xi_4 d\tau \\
 & + \frac{\coth(\tau)}{A} \left( \frac{11e^{-2\tau}}{16} (3f + \frac{247}{22} + 2f\tau + \frac{45\tau}{11}) + \frac{1}{8} (-\frac{35}{4} - 7\tau) e^{-4\tau} \right. \\
 & - \frac{1}{24(e^{2\tau} + 1)^4} (((398\tau + 63)f - 130\tau - 145)e^{2\tau} \\
 & + ((360\tau - 347)f - 192\tau - 461)e^{4\tau} + ((30\tau - 55)f - 186\tau + 5)e^{6\tau} \\
 & \left. + (68\tau + 7)f - 124\tau - 135) + \frac{f + 25}{16} (-Li_2(-e^{2\tau}) + 2\tau^2) \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{12} \tau (103 f_1 + 232) - \frac{1}{24} \ln(e^{2\tau} + 1) (3 f \tau + 103 f + 75 \tau + 232) \quad (227) \\
& - \frac{1}{48} \frac{1}{A (\cosh(\tau))^3} (\tau \coth(\tau) - 1) (-6 (\cosh(\tau))^3 (f_1 + 25) \ln(\cosh(\tau)) \\
& - 336 (\cosh(\tau))^7 + 336 \sinh(\tau) (\cosh(\tau))^6 + (132 f + 606) (\cosh(\tau))^5 \\
& + (-132 f - 438) \sinh(\tau) (\cosh(\tau))^4 + ((6\tau + 39) f + 150\tau + 45) (\cosh(\tau))^3 \\
& + (68 f - 124) \sinh(\tau) (\cosh(\tau))^2 + (-39 f - 45) \cosh(\tau) \\
& + (58 f + 76) \sinh(\tau)),
\end{aligned}$$

$$\begin{aligned}
X_2 = & -\frac{1}{2} \int \xi_4 d\tau + \frac{1}{24 A} (-\ln(e^{2\tau} + 1) (28 f + 89) \\
& - 1/8 (-348 f - 1560) \tau^2 + 1/8 \tau (348 f \ln(2) + 1560 \ln(2) + 161 f + 952) \\
& - \frac{27 e^{-2\tau} f_1}{4} + 3/4 Li_2(-e^{2\tau}) (29 f + 130) - \frac{41 f + 100}{2 (e^{2\tau} + 1)^2} \quad (228) \\
& + \frac{29 f + 38}{(e^{2\tau} + 1)^3} + \frac{15 e^{-4\tau}}{8} + \frac{2(5 f + 43)}{e^{2\tau} + 1} - \frac{279 e^{-2\tau}}{8}),
\end{aligned}$$

$$\begin{aligned}
\phi_1 = & Q_1 \tanh(\tau) + Q_2 (\tau \tanh(\tau) - 1) - \tanh(\tau) \int \xi_4 d\tau \\
& + \frac{\tanh(\tau)}{A} \left( \frac{(-105 - 84\tau) e^{-4\tau}}{68} + 1/32 (92 f \tau + 138 f + 454\tau + 513) e^{-2\tau} \right. \\
& + \frac{1}{120 (e^{2\tau} + 1)^5} (((880\tau - 1885) f + 9400\tau - 2377) e^{2\tau} \\
& + ((6680\tau + 1125) f + 20540\tau - 459) e^{4\tau} \\
& + ((-420\tau + 2685) f + 8940\tau + 1041) e^{6\tau} \\
& + ((690\tau + 80) f + 3840\tau - 712) e^{8\tau} \\
& + (290\tau - 405) f + 2984\tau - 165) - \frac{5(13 f + 55)}{16} Li_2(-e^{2\tau}) \\
& - ((975\tau + 140) f + 4125\tau - 142) \ln(e^{2\tau} + 1) / 120 \quad (229) \\
& + 5/8 \tau^2 (13 f + 55) + 1/30 \tau (70 f - 71) \\
& - \frac{\tau \tanh(\tau) - 1}{480 A} \left( -3900 (f + \frac{55}{13}) \ln(\cosh(\tau)) - 5040 (\cosh(\tau))^4 \right. \\
& + 5040 \sinh(\tau) (\cosh(\tau))^3 + (2760 f + 18660) (\cosh(\tau))^2 \\
& + (-2760 f - 16140) \sinh(\tau) \cosh(\tau) + (3900\tau - 3195) f \\
& + 16500\tau - 8325 + (-580 f_1 - 5968) \tanh(\tau) \\
& + \frac{1}{(\cosh(\tau))^5} ((60 f - 840) (\cosh(\tau))^3 + (375 f + 585) \cosh(\tau) \\
& \left. + ((340 f + 1696) (\cosh(\tau))^2 - 900 f - 1128) \sinh(\tau) \right),
\end{aligned}$$

$$\begin{aligned}
 \xi_4 = & -2 Q_4 - \frac{1}{960 A} \left( 1920 (f + \frac{45}{32}) \ln (\cosh (\tau)) + (3600 f + 13560) (\cosh (\tau))^2 \right. \\
 & - 10800 (\cosh (\tau))^4 + \frac{1}{\sinh (\tau) (\cosh (\tau))^5} ((8640 (\cosh (\tau))^9 \\
 & - 8640 \sinh (\tau) (\cosh (\tau))^8 + (-1440 f - 19260) (\cosh (\tau))^7 \\
 & + (1440 \sinh (\tau) f + 14940 \sinh (\tau)) (\cosh (\tau))^6 + (2400 f + 10740) (\cosh (\tau))^5 \\
 & + (-1920 f - 9060) \sinh (\tau) (\cosh (\tau))^4 + (-480 f - 180) (\cosh (\tau))^3 \\
 & + (1200 f + 3780) \sinh (\tau) (\cosh (\tau))^2 + (-480 f + 60) \cosh (\tau) \\
 & + (-720 f - 60) \sinh (\tau)) e^{-\tau} + 10800 (\cosh (\tau))^{10} \\
 & + (-3600 f - 18960) (\cosh (\tau))^8 + (5780 f + 5608) (\cosh (\tau))^6 \\
 & - 1920 \sinh (\tau) ((-\frac{65}{32} + \tau) f + \frac{45 \tau}{32} - \frac{549}{128}) (\cosh (\tau))^5 \\
 & + (260 f + 4036) (\cosh (\tau))^4 + (-2280 f - 2820) \sinh (\tau) (\cosh (\tau))^3 \\
 & \left. + (-1540 f - 356) (\cosh (\tau))^2 + (60 f - 15) \sinh (\tau) \cosh (\tau) - 900 f - 1128 \right). \tag{230}
 \end{aligned}$$

**Bianchi-type V**

As an example In this Bianchi-type we set  $p = 1, n = 1, q = 0$  and find the following results for  $X_i$  (128), dilaton (37) and  $\xi_4$  (42)

$$\begin{aligned}
 X_i = & Q_3 \coth(\tau) + Q_4 (\tau \coth(\tau) - 1) - \frac{1}{2} \coth(\tau) \int \xi_4 d\tau \\
 & + \frac{1}{3A} \left( \coth(\tau) [220 \tau f - 134 \tau - \ln(e^{2\tau} + 1)] (110 f - 67) \right. \\
 & + \frac{2}{(e^{2\tau} + 1)^4} (((-446 f + 49) \tau + 52 f - 143) e^{2\tau} \\
 & + ((-330 f + 201) \tau + 452 f_1 + 170) e^{4\tau} \\
 & + ((6 f + 219) \tau + 52 f - 143) e^{6\tau} + (-110 f + 67) \tau) ] \\
 & \left. - \frac{\sinh(\tau)}{(\cosh(\tau))^3} (\tau \coth(\tau) - 1) \left( 110 (\cosh(\tau))^2 f - 67 (\cosh(\tau))^2 + 58 f + 76 \right) \right), \tag{231}
 \end{aligned}$$

$$\begin{aligned}
 \phi_1 = & Q_1 \tanh(\tau) + Q_2 (\tau \tanh(\tau) - 1) - \tanh(\tau) \int \xi_4 d\tau + \\
 & + \frac{1}{10A} \left( - \frac{(\sinh(\tau))^3}{(\cosh(\tau))^5} (\tau \tanh(\tau) - 1) \left( (\cosh(\tau))^2 (-83 + 90 f) + 150 f + 188 \right) \right. \\
 & + \tanh(\tau) (180 \tau f - 166 \tau + \frac{2}{(e^{2\tau} + 1)^5} (((-480 f - 210) \tau + 30 f - 354) e^{2\tau} \\
 & \left. + ((540 f_1 + 1460) \tau + 1290 f + 442) e^{4\tau} \right) \tag{232}
 \end{aligned}$$

$$\begin{aligned}
 &+ ((-1440 f - 630) \tau + 1290 f + 442) e^{6 \tau} \\
 &+ ((30 f + 625) \tau + 30 f_1 - 354) e^{8 \tau} \\
 &+ (-90 f + 83) \tau - \ln \left( e^{2 \tau} + 1 \right) (-83 + 90 f) \Big).
 \end{aligned}$$

The initial value equation (42) gives

$$\begin{aligned}
 \xi_4 = &6 Q_4 - \frac{113 \ln(e^{2 \tau} + 1)}{4 A \cosh(\tau)(\sinh(\tau))^3} \\
 &- \frac{1}{320 A (\cosh(\tau))^5 (\sinh(\tau))^3} \Big( (460 f - 252) (\cosh(\tau))^8 \\
 &+ (-170 f - 197) (\cosh(\tau))^6 - (550 f - 335) \sinh(\tau) \tau (\cosh(\tau))^3 \\
 &+ (-630 f - 1172) (\cosh(\tau))^2 - (290 f + 380) \tau \sinh(\tau) \cosh(\tau) \tag{233} \\
 &+ 270 f + 10436 + ((1100 \tau + 70) f - 670 \tau + 1423) e^{4 \tau} \\
 &+ ((4460 \tau + 800) f - 490 \tau + 4262) e^{2 \tau} + \\
 &+ ((-60 \tau + 800) f - 2190 \tau + 4262) e^{-2 \tau} \\
 &+ (70 f + 1423) e^{-4 \tau} (3300 \tau + 4940) f - 2010 \tau \Big).
 \end{aligned}$$

**VIII**

A sample of answers with choosing a simple parameterization of  $p_1 = p_2 = A = 1$  is found as following

$$\begin{aligned}
 X_1 = X_2 = &Q_3 \coth(\tau) + Q_4 (\tau \coth(\tau) - 1) - \coth(\tau) \int \xi_4 d\tau \\
 &+ \frac{1}{A} \Big( \coth(\tau) \left( \frac{1}{7680 (\sinh(\tau))^2 (\cosh(\tau))^5} ((210 f + 2370) \tau \cosh(5 \tau) \right. \\
 &+ (-4050 f - 3570) \tau \cosh(3 \tau) + (3840 f + 6960) \tau \cosh(\tau) \\
 &+ \frac{e^{-7 \tau}}{2} ((3405 f + 7185) e^{10 \tau} + (-105 f - 1005) e^{12 \tau} \\
 &+ (-3 f + 2721) e^{2 \tau} + (-3441 f - 6613) e^{4 \tau} \tag{234} \\
 &+ (10230 f + 13710) e^{6 \tau} + (-10050 f - 16570) e^{8 \tau} \\
 &- 36 f_1 + 572) - 3/8 (Li_2(-e^{2 \tau}) - Li_2(e^{2 \tau})) \\
 &- (\tau \coth(\tau) - 1) (3/4 \ln(\tanh(\tau)) \\
 &+ \left. \frac{(14 f + 146) (\cosh(\tau))^6 + 12 (\cosh(\tau))^4 + (-85 f - 257) (\cosh(\tau))^2 + 71 f + 123}{32 (\sinh(\tau))^2 (\cosh(\tau))^4} \right) \Big),
 \end{aligned}$$

$$\begin{aligned}
 \phi_1 = & Q_1 \tanh(\tau) + Q_2 (\tau \tanh(\tau) - 1) - \tanh(\tau) \int \xi_4 d\tau \\
 & + \frac{1}{2A} \left( \tanh(\tau) \left( \frac{1}{2304 (\cosh(\tau))^6 \sinh(\tau)} ((126 f + 2826) \tau \sinh(5 \tau) \right. \right. \\
 & + (-5238 f - 8802) \tau \sinh(3 \tau) - (-8940 f - 18804) \tau \sinh(\tau)) \\
 & + \frac{e^{-7\tau}}{4608 (\cosh(\tau))^6 \sinh(\tau)} ((3960 f + 6012) e^{10\tau} + (-63 f - 1629) e^{12\tau} \\
 & + (-808 f - 4564) e^{2\tau} + (2619 f + 729) e^{4\tau} + (-3152 f - 8360) e^{6\tau} \\
 & + (-2407 f - 5425) e^{8\tau} - 149 f - 587)) \\
 & - (\tau \tanh(\tau) - 1) \left( -3 \ln(\tanh(\tau)) + \frac{1}{24 (\cosh(\tau))^6} ((21 f + 471) (\cosh(\tau))^4 \right. \\
 & \left. \left. + (-234 f - 720) (\cosh(\tau))^2 + 149 f + 317) \right) \right), \tag{235}
 \end{aligned}$$

$$\begin{aligned}
 X_3 = & Q_5 \tanh(\tau) + Q_6 (\tau \tanh(\tau) - 1) - \coth(\tau) \int \xi_4 d\tau \\
 & + \frac{1}{2A} \left( \tanh(\tau) \left( \frac{1}{9216 (\cosh(\tau))^6 \sinh(\tau)} (2 \tau ((126 f + 1098) \sinh(5 \tau) \right. \right. \\
 & + (-2430 f - 3546) \sinh(3 \tau) + (4260 f \tau + 7164) \sinh(\tau)) \\
 & + e^{-7\tau} ((1854 f + 1638) e^{10\tau} + (-63 f - 765) e^{12\tau} + (-418 f - 2730) e^{2\tau} \\
 & + (1215 f - 1899) e^{4\tau} + (-1436 f - 5820) e^{6\tau} + (-1081 f - 3855) e^{8\tau} \\
 & \left. \left. - 71 f - 393) \right) + 3/4 (Li_2(-e^{2\tau}) - Li_2(e^{2\tau})) \right) \\
 & - (\tau \tanh(\tau) - 1) \left( -3/2 \ln(\tanh(\tau)) + \frac{1}{48 (\cosh(\tau))^6} ((21 f + 183) (\cosh(\tau))^4 \right. \\
 & \left. \left. + (-117 f - 285) (\cosh(\tau))^2 + 71 f + 123) \right) \right), \tag{236}
 \end{aligned}$$

$$\begin{aligned}
 \xi_4 = & Q_4 - Q_6 - \frac{3 \ln(\tanh(\tau))}{A} \\
 & - \frac{1}{96 (\sinh(\tau))^2 (\cosh(\tau))^6 A} \left( (84 f + 876) (\cosh(\tau))^8 + 418 f_1 \right. \\
 & + (-84 f - 1494) (\cosh(\tau))^6 + (165 f + 2841) (\cosh(\tau))^4 \\
 & \left. + (-583 f - 3503) (\cosh(\tau))^2 + 1352 \right). \tag{237}
 \end{aligned}$$

**IX**

By setting  $n = 1$  and  $p_1 = p_2 = 1$ , the forms of  $X_i$  (188) and (189),  $\phi_1$  (37) and the lapse function  $\xi_4$  (42) are found as following

$$\begin{aligned}
 X_1 = & X_3 = (Q_3 + Q_5) \tanh(n\tau) + (Q_4 + Q_6) (\tau \tanh(\tau) - 1) - \tanh(\tau) \int \xi_4 d\tau \\
 & - \frac{\tanh(\tau) (f - 31)}{31A} \left( \frac{2\tau - e^\tau - 1}{(\cosh(\tau))^2} + (\tau \tanh(\tau) - 1) \tanh(\tau) \right) \\
 & + c_1 - c_2, \tag{238}
 \end{aligned}$$

$$X_2 = Q_5 \tanh(\tau) + Q_6 (\tau \tanh(\tau) - 1) - \tanh(\tau) \int \xi_4 d\tau - \frac{\tanh(\tau)(f+17)}{31A} \left( \frac{2\tau - e^\tau - 1}{(\cosh(\tau))^2} + (\tau \tanh(\tau) - 1) \tanh(\tau) \right) + c_2, \quad (239)$$

$$\phi_1 = Q_1 \tanh(\tau) + Q_2 (\tau \tanh(\tau) - 1) - \tanh(\tau) \int \xi_4 d\tau - \frac{\tanh(\tau)(3f-25)}{31A} \left( \frac{2\tau - e^\tau - 1}{(\cosh(\tau))^2} + (\tau \tanh(\tau) - 1) \tanh(\tau) \right), \quad (240)$$

$$\xi_4 = 4Q_4 - 2Q_6 + \frac{1}{A(\cosh(\tau))^2} (1 - 3/16 f + (\tanh(\tau))^2 ((f - 5/3) (\cosh(\tau))^2 + (-f + 25/3) (1 + \tau \tanh(\tau))))). \quad (241)$$

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