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RESEARCH ARTICLE

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A note on abelian arithmetic BF-theory

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Abstract
We compute some arithmetic path integrals for BF-theory over the ring of integers of a totally imaginary field, which evaluate to natural arithmetic invariants associated to $\mathbb{G}_m$ and abelian varieties.

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1 | TOWARDS ARITHMETIC BF THEORY

BF-theory is a rare example of a topological field theory that can be defined in any dimension [5]. Let M be an oriented n-manifold, G a compact Lie group and P a principal G-bundle on M. Assume given also a finite-dimensional representation ρ of the Lie algebra $\mathfrak{g}$ of G. The BF functional depends on two fields, a connection A on P and an $(n-2)$-form $B \in \Omega^{n-2}(M, adP)$ with values in the adjoint bundle of P. Then

$$BF(B, A) := \int_M Tr[\rho(B) \wedge \rho(F_A)],$$

where F_A is the curvature form of A. (There seem to be some other possibilities for the invariant function on $\mathfrak{g}$ that goes into the integral.) A number of properties of BF-theories make them easier to deal with than Chern–Simons theories. Primary among them for our purposes is that the two variables allow us to extend the definition to non-orientable manifolds, simply by letting the field B vary over $\Omega^{n-2}(M, adP \otimes \omega_M)$, where ω_M is the orientation bundle of M.

In some earlier papers, a preliminary attempt to define and compute arithmetic analogues of Chern–Simons functions was made [1, 3, 7, 8, 10]. Also, moduli spaces of ‘arithmetic gauge fields’ have been applied to Diophantine geometry [2, 11]. One of the obstructions to developing a full-fledged arithmetic topological field theory based on Chern–Simons theory is that natural arithmetic dualities involve a sheaf μ_n or $\hat{\mathbb{Z}}(1)$, which are not trivialisable in general. That is, arithmetic schemes are not orientable. There are various ways to circumvent this problem, all of which introduce some difficulties for working out interesting examples. The purpose of this paper is to suggest, by way of two brief computations, that BF-theory provides a simpler way to link number theory to topological field theory.

Let $X = \mathrm{Spec}(\mathcal{O}_F)$, where F is a totally imaginary number field. Then

Proposition 1.1.

$$\sum_{(a,b)\in H^1(X,\mathbb{Z}/n)\times H^1(X,\mu_n)} \exp(2\pi i BF(a,b)) = |n\,\mathrm{Cl}_F[n^2]| \cdot |\mathcal{O}_X^\times/(\mathcal{O}_X^\times)^n| \cdot |\,\mathrm{Cl}_F/n|.$$

Thus, for n divisible by the exponent of Cl_F, the ‘finite path integral’ will capture exactly

$$|\mathcal{O}_X^\times/(\mathcal{O}_X^\times)^n| \cdot |\,\mathrm{Cl}_F\,|,$$

a quantity of the form (regulator × class number). The precise notation and definitions that go into this proposition as well as the next one will be explained in subsequent sections.

Now let A and B be dual abelian varieties over F with semi-stable reduction at all places. Let n be an integer coprime to the order of the group of connected components of both A and B as well as to the places of bad reduction for them. Assume that the Tate–Shafarevich groups Ш(A) and Ш(B) are finite and Ш$(B)[n] =$ Ш$(B)[n^2]$. Then

Proposition 1.2.

$$\sum_{(a,b)\in H^1(X,\mathcal{A}[n])\times H^1(X,\mathcal{B}[n])} \exp(2\pi i BF(a,b)) = |A(F)/n| \cdot |B(F)/n| \cdot |\text{Ш}(A)[n]|$$

Here, $\mathcal{A}$ and $\mathcal{B}$ are the Neron models of A and B, respectively. Note that the expression is symmetric in A and B because $|\text{Ш}(A)[n]| = |\text{Ш}(B)[n]|$ via the Cassel–Tate pairing.

We view these formulae as some (weak) evidence for the suggestion made in [10] that an arithmetic topological functional on moduli spaces of Galois representations will have something to do with L-functions. In any case, it is rather striking to find expressions for the orders of class groups and Tate–Shafarevich groups as exponential sums, which perhaps have not appeared heretofore in the literature. Equally notable is that a path integral for the BF functional for three manifolds leads to the Alexander polynomial of a knot in the physics literature [5], which is well known to be an analogue of the p-adic L-function [15].

In forthcoming work, we will develop BF-theory for arithmetic schemes with boundary, that is, $\mathrm{Spec}(\mathcal{O}_F[1/S])$ for a finite set S of places, as well as a p-adic theory. Also interesting would be to develop arithmetic BF-theory for arithmetic schemes in higher dimension by way of the duality theory of [9]. However, in the present announcement, the primary goal is to illustrate with a minimum of clutter the relationship between a path integral in the sense of physicists and important arithmetic invariants.

2 | SOME FINITE PATH INTEGRALS FOR $\mathbb{G}_m$: PROOF OF PROPOSITION 1.1

As before, let $X = \mathrm{Spec}(\mathcal{O}_F)$ for F a totally imaginary number field and let μ_n and $\mathbb{Z}/n$ be the usual finite flat group schemes viewed as sheaves in the flat topology. We have the Bockstein map

$$\delta : H^1(X, \mu_n) \longrightarrow H^2(X, \mu_n)$$

coming from the exact sequence

$$1 \longrightarrow \mu_n \longrightarrow \mu_{n^2} \longrightarrow \mu_n \longrightarrow 1$$

and the invariant isomorphism [12]

$$\int : H^3(X, \mu_n) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Define the BF-functional on

$$\mathcal{M} := H^1(X, \mathbb{Z}/n) \times H^1(X, \mu_n)$$

by

$$BF(a, b) = \int (a \cup \delta b).$$

(The class δb is the analogue of the curvature F.)

Remark 2.1. One can also define a BF-functional on $\mathcal{M}$ as $BF'(a, b) = \int(\delta a \cup b)$, where $\delta : H^1(X, \mathbb{Z}/n) \to H^2(X, \mathbb{Z}/n^2)$ is the Bockstein map coming from the exact sequence

$$1 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2 \longrightarrow \mathbb{Z}/n \longrightarrow 1.$$

However, $BF' = BF$, since we have an equality $\delta a \cup b = a \cup \delta b$ by Lemma *2.1* and the proof of Lemma *2.2* in [7].

We will now calculate the path integral

$$\sum_{(a,b)\in\mathcal{M}} \exp(2\pi i BF(a, b)).$$

First, let us calculate the groups $H^i(X, \mu_n)$. We define $\mathrm{Div}\, F$ to be the group of fractional ideals of F and for $x \in F^*$, we let $\mathrm{div}(x)$ be the associated principal ideal. We claim that

$$H^i(X, \mu_n) = \begin{cases} \mu_n(F) & \text{for } i = 0 \\ Z_1/B_1 & \text{for } i = 1 \\ \mathrm{Cl}_F/n & \text{for } i = 2 \\ \mathbb{Z}/n & \text{for } i = 3 \\ 0 & \text{for } i > 3, \end{cases} \tag{2.1}$$

where

$$Z_1 = \{(x, I) \in F^* \oplus \mathrm{Div}\, F |\ nI = -\mathrm{div}(x)\}$$

and $B_1 = \{(x^n, -\mathrm{div}(x)) \in F^* \oplus \mathrm{Div}\, F| \quad x \in F^*\}$. To see this, we first claim that $\mu_n = \mathscr{H}om(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,X})$ is quasi-isomorphic to $R\mathscr{H}om(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,X})$, so that $H^i(X, \mu_n) = \mathrm{Ext}^i_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,X})$. By applying $\mathscr{H}om(-, \mathbb{G}_{m,X})$ to the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

we get a long exact sequence

$$0 \to \mu_n \to \mathbb{G}_{m,X} \xrightarrow{\cdot n} \mathbb{G}_{m,X} \to \mathscr{E}xt^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,X}) \to \mathscr{E}xt^1(\mathbb{Z}, \mathbb{G}_{m,X}) \to \cdots .$$

Now, since $\mathbb{Z}_X$ is free, $R\mathscr{H}om(\mathbb{Z}_X, \mathbb{G}_{m,X}) = \mathscr{H}om(\mathbb{Z}, \mathbb{G}_{m,X}) = \mathbb{G}_{m,X}$, so $\mathscr{E}xt^i(\mathbb{Z}, \mathbb{G}_{m,X}) = 0$ for $i > 0$ and $\mathscr{E}xt^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,X})$ naturally identifies with the cokernel of the multiplication by n map on $\mathbb{G}_{m,X}$, which is zero, since we are working with the fppf topology. By the local-to-global spectral sequence, we then get that $H^i(X, \mu_n) = \mathrm{Ext}^i_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,X})$. Now, the Ext-groups $\mathrm{Ext}^i_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,X})$ in the fppf topology coincides with the Ext-groups in the étale topology, since both $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{G}_{m,X}$ are smooth (see [13, III, Theorem 3.9]), and these Ext-groups have been computed in [4, Corollary 4.3] and the values match those in the display 2.1. The Bockstein map

$$\delta : H^1(X, \mu_n) \longrightarrow H^2(X, \mu_n)$$

is the composite of two maps: the first is the surjective map which takes $(x, I) \in H^1(X, \mu_n) \cong Z_1/B_1$ to $I \in \mathrm{Cl}_F[n]$ and the second is the reduction map $\mathrm{Cl}_F[n] \to \mathrm{Cl}_F/n$; we prove this claim in the appendix in Lemma A.1. By noting that the kernel of the first map is $\mathcal{O}_X^\times/(\mathcal{O}_X^\times)^n$ and that the kernel of the second map is $n\,\mathrm{Cl}_F[n^2]$, we see that our sum becomes

$$|\mathcal{O}_X^\times/(\mathcal{O}_X^\times)^n| \cdot |n\,\mathrm{Cl}_F[n^2]| \sum_{(a,b) \in H^1(X, \mathbb{Z}/n\mathbb{Z}) \times \mathrm{Cl}_F[n]/n\,\mathrm{Cl}_F[n^2]} \exp(2\pi i a \cup b),$$

where $b \in \mathrm{Cl}_F/n$ is, by abuse of notation, the reduction of b. But for b non-trivial, it is clear that this sum is zero, giving us

$$\sum_{(a,b) \in \mathcal{M}} \exp(2\pi i BF(a, b)) = |n\,\mathrm{Cl}_F[n^2]| \cdot |\mathcal{O}_X^\times/(\mathcal{O}_X^\times)^n| \cdot |\,\mathrm{Cl}_F/n|.$$

In particular, if $\mathrm{Cl}_F[n] = \mathrm{Cl}_F$, we see that the sum evaluates to $|\mathcal{O}_X^\times/(\mathcal{O}_X^\times)^n| \cdot |\,\mathrm{Cl}_F|$.

3 | SOME FINITE PATH INTEGRALS FOR ABELIAN VARIETIES: PROOF OF PROPOSITION 1.2

Let $\mathcal{A}$ be the Néron model of an abelian variety A over F and let $\mathcal{B}$ be the Néron model of the dual B. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have semi-stable reduction and let n be a sufficiently large positive integer. More precisely, suppose that both A and B have good reduction at all places dividing n. Further assume that n is coprime to $|\Phi_A| \cdot |\Phi_B|$, where Φ_A and Φ_B are the groups of connected components of $\mathcal{A}$ and $\mathcal{B}$. According to [6, Theorem 1.1(a)(ii)] (for further details see Lemma A.2 of the Appendix), we have isomorphisms

$$H^1(X, \mathcal{A}[n]) \cong Sel(F, A[n]),$$

$$H^1(X, \mathcal{B}[n]) \cong Sel(F, B[n]),$$

where the left-hand side is flat cohomology (in the fppf site). We claim that our assumption that $\mathcal{B}$ has semi-stable reduction, together with our assumptions on n, implies that multiplication by n on $\mathcal{B}$ is an epimorphism in the category of fppf sheaves. To see this, note that by [6, Lemma B.4.], $\mathcal{B}^0 \xrightarrow{\cdot n} \mathcal{B}^0$ is faithfully flat, thus an epimorphism of fppf sheaves. Further, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{B}^0 & \longrightarrow & \mathcal{B} & \longrightarrow & \Phi_B & \longrightarrow & 0 \\ & & \downarrow n & & \downarrow n & & \downarrow n & & \\ 0 & \longrightarrow & \mathcal{B}^0 & \longrightarrow & \mathcal{B} & \longrightarrow & \Phi_B & \longrightarrow & 0 \end{array}$$

and since n is prime to $|\Phi_B|$, multiplication by n on Φ_B is actually an isomorphism. Thus, by the snake lemma, multiplication by n on $\mathcal{B}$ is an epimorphism. By [14, III,Corollary 3.4] there is a perfect pairing

$$\cup : H^1(X, \mathcal{A}[n]) \times H^2(X, \mathcal{B}[n]) \to H^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}.$$

Using the exact sequence

$$0 \longrightarrow \mathcal{B}[n] \longrightarrow \mathcal{B}[n^2] \xrightarrow{n} \mathcal{B}[n] \longrightarrow 0,$$

we get the Bockstein map

$$\delta : H^1(X, \mathcal{B}[n]) \longrightarrow H^2(X, \mathcal{B}[n]).$$

We now define the BF-functional

$$BF : H^1(X, \mathcal{A}[n]) \times H^1(X, \mathcal{B}[n]) \longrightarrow H^3(X, \mathbb{G}_m)[n] \cong H^3(X, \mu_n) \cong \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

$$BF(a, b) = \int (a \cup \delta b),$$

where $\int$ is the isomorphism $H^3(X, \mu_n) \cong \frac{1}{n}\mathbb{Z}/\mathbb{Z}$, as in the previous section. Our goal is now to calculate the path integral

$$\sum_{(a,b) \in H^1(X, \mathcal{A}[n]) \times H^1(X, \mathcal{B}[n])} \exp(2\pi i BF(a, b)).$$

To this end, note that, by Lemma A.3 and our assumptions on n, we have

$$H^1(X,\mathcal{A})[n] \cong Ш(A)[n], \;\; H^1(X,\mathcal{B})[n] \cong Ш(B)[n].$$

Assuming the finiteness of the Tate–Shafarevich group, we can take n large enough so that

$$H^1(X,\mathcal{B})[n^2] = H^1(X,\mathcal{B})[n].$$

We now identify the kernel of δ. By using the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{B}/n^2\mathcal{B} & \longrightarrow & H^1(X,\mathcal{B}[n^2]) & \longrightarrow & H^1(X,\mathcal{B})[n^2] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \\ 0 & \longrightarrow & \mathcal{B}/n\mathcal{B} & \longrightarrow & H^1(X,\mathcal{B}[n]) & \longrightarrow & H^1(X,\mathcal{B})[n] & \longrightarrow & 0 \end{array}$$

and the fact that the rightmost map is zero, we see that $Ker(\delta) = Im(\mathcal{B}/n\mathcal{B})$. Thus, we get an induced injection

$$\bar{\delta} : H^1(X,\mathcal{B})[n] \hookrightarrow H^2(X,\mathcal{B}[n]).$$

By an argument identical to the case of μ_n above, since

$$|H^1(X,\mathcal{A}[n])| = |A(F)/n| \cdot |Ш(A)[n]|,$$

we get

$$\begin{aligned} \sum_{(a,b)\in H^1(X,\mathcal{A}[n])\times H^1(X,\mathcal{B}[n])} \exp(2\pi i BF(a,b)) &= |B(F)/n| \sum_{(a,c)\in H^1(X,\mathcal{A}[n])\times Im(\delta)} \exp(2\pi i a\cup c) \\ &= |B(F)/n| \sum_{(a,c)\in H^1(X,\mathcal{A}[n])\times \mathrm{Im}(\delta)} \exp(2\pi i a \cup c) \\ &= |A(F)/n| \cdot |B(F)/n| \cdot |Ш(A)[n]|. \end{aligned}$$

APPENDIX A: A FEW TECHNICAL LEMMAS

The purpose of this appendix is to gather, and prove, the main lemmas that are needed for our computations. The lemmas presented here are fairly straightforward to prove and are more or less already proven in [1] and [6], but some work needs to be done, so to spare the reader unnecessary work, we prove the required lemmas in this appendix. Throughout this section, $X = \mathrm{Spec}(\mathcal{O}_F)$ is the ring of integers of a totally imaginary number field.

Lemma A.1. *The Bockstein map*

$$H^1(X,\mu_n) \longrightarrow H^2(X,\mu_n)$$

is, under the natural identifications $H^1(X,\mu_n) \cong Z_1/B_1$ *and* $H^2(X,\mu_n) \cong \mathrm{Cl}_F/n$*, the map which takes a class* $(x,I) \in Z_1/B_1$ *to* $I \in \mathrm{Cl}_F/n$*.*

Proof. This is shown in the [1, Proof of Lemma 4.1]. Namely, in the proof of Lemma 4.1 in loc. cit., it is proven that the connecting homomorphism $\mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X}) \to \mathrm{Ext}^2(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X})$ coming from the exact sequence

$$1 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2 \longrightarrow \mathbb{Z}/n \longrightarrow 1 \tag{A.1}$$

coincides, under the identifications $\mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X}) \cong Z_1/B_1, \mathrm{Ext}^2(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X}) \cong \mathrm{Cl}_F/n$ with the map as in the statement of our lemma. If we apply RHom to the exact sequence A.1, we get a map $\mathrm{RHom}(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X}) \to \mathrm{RHom}(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X})[1]$ in the derived category and we get $\mathrm{Ext}^i_X(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X}) \to \mathrm{Ext}^{i+1}(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X})$ by applying H^1. Now, naturally, $R\Gamma\circ R\mathscr{H}om(-,\mathbb{G}_m) = \mathrm{RHom}(-,\mathbb{G}_m)$, and by applying $R\mathscr{H}om(-,\mathbb{G}_m)$ to the exact sequence A.1 we get the exact triangle

$$\mu_n \longrightarrow \mu_{n^2} \longrightarrow \mu_n \longrightarrow \mu_n[1],$$

since as we explained in Section 2, $R\mathscr{H}om(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X}) = \mathscr{H}om(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X}) = \mu_n$. Thus the Bockstein

$$H^1(X,\mu_n) \longrightarrow H^2(X,\mu_n)$$

coincides under natural identifications, with the map $\mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X}) \to \mathrm{Ext}^2(\mathbb{Z}/n\mathbb{Z},\mathbb{G}_{m,X})$ which is computed in [1, Lemma 4.1]. □

Recall now that if $\mathcal{A}$ is the Néron model of an abelian variety A over F, then we have the exact sequence (see [6, Appendix B])

$$0 \to \mathcal{A}^0 \to \mathcal{A} \to \Phi_A \to 0,$$

where Φ_A is the group of connected components of $\mathcal{A}$. We have that $\Phi_A = \oplus_s (i_s)_* \Phi_{A_s}$, where s ranges over all the closed points of X, and Φ_{A_s} is the étale group of connected components of the base change of $\mathcal{A}$ to Spec $k(s)$. It thus makes sense to speak of the order of Φ_A.

Lemma A.2. *Let* $\mathcal{A}$ *be the Néron model of an abelian variety over* F *and assume that* $\mathcal{A}$ *has semi-stable reduction. Let* n *be a positive integer such that* A *has good reduction at all places dividing* n *and assume further that* n *is coprime to the order of the group of connected components of* $\mathcal{A}$*. Then we have that* $H^1(X,\mathcal{A}[n])$ *is isomorphic to* $Sel(F,\mathcal{A}[n])$*.*

Proof. This follows directly from [6, Theorem 1.1(a)(ii)]. Indeed, under the assumptions of our Lemma, the degree of the isogeny $A \xrightarrow{n} A$ is prime to the Tamagawa factors, and since F is totally imaginary, [6, Theorem 1.1(a)(ii)] tells us that $H^1(X,\mathcal{A}[n]) = Sel(F,\mathcal{A}[n])$ inside $H^1(K,A[n])$ and our statement follows. □

Lemma A.3. *Let* $\mathcal{A}$ *be the Néron model of an abelian variety over* F *and assume that* $\mathcal{A}$ *has semi-stable reduction. Let* n *be a positive integer such that* A *has good reduction at all places dividing* n

and assume further that n is coprime to $|\Phi_A|$, *where* Φ_A *is the group of connected components of* $\mathcal{A}$. *Then* $H^1(X,\mathcal{A})[n]$ *is isomorphic to* $Ш(A)[n]$.

Proof. This is a direct consequence of [6, Proposition 4.5(c)–(d)]. To see this, we use the connected-étale exact sequence $0 \to \mathcal{A}^0 \to \mathcal{A} \to \Phi_A \to 0$. By applying cohomology, we obtain the long exact sequence

$$0 \to H^0(X,\mathcal{A}^0) \to H^0(X,\mathcal{A}) \to H^0(X,\Phi_A) \to H^1(X,\mathcal{A}^0) \to H^1(X,\mathcal{A}) \to H^1(X,\Phi_A) \to \cdots$$

Since multiplication by n on Φ_A is an isomorphism by our assumptions, we see that

$$H^1(X,\mathcal{A})[n] = \mathrm{Im}(H^1(X,\mathcal{A}^0) \to H^1(X,\mathcal{A}))[n],$$

and by [6, Proposition 4.5 (c)]

$$\mathrm{Im}(H^1(X,\mathcal{A}^0) \to H^1(X,\mathcal{A})) = Ш(\mathcal{A}).$$

Lastly, since we have assumed F to be totally imaginary, [6, Proposition 4.5 (d)] implies that $Ш(\mathcal{A}) = Ш(A)$. □

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