Homotopy algebras, gauge theory, and gravity

Tommaso Macrelli

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Department of Mathematics
University of Surrey
Guildford GU2 7XH, United Kingdom

E-mail: t.macrelli@surrey.ac.uk

[. . .] es el asombro ante el milagro
de que a despecho de infinitos azares,
de que a despecho de que somos
las gotas del río de Heráclito,
perdure algo en nosotros:
inmóvil.

Jorge Luis Borges, *Final de año*, from *Fervor de Buenos Aires*, 1923

Scientific abstract

In this Thesis we discuss applications of homotopy algebras to several aspect of quantum field theories. In an effort to be self-contained, we start introducing L_∞-, A_∞-, and C_∞-algebras, and contextualising them in the framework of Batalin–Vilkovisky formalism, that associates every perturbative Lagrangian field theory to an L_∞-algebra encoding the complete classical theory. Several instances of field theories are reviewed, and their underlying homotopy algebras are discussed in detail. The connection between homotopy algebras and scattering amplitudes are explored, and explicit recursion relations (at tree- and loop-level) are provided and applied to concrete examples. We then apply the homotopy algebra framework to the study of BCJ colour–kinematic duality and double copy prescription for Yang–Mills theory. Following a Lagrangian approach and with the help of an appropriate notion of tensor product for homotopy algebras, we introduce a colour–kinematic factorisation at the level of the L_∞-algebra associated to the theory. We construct a double copied Yang–Mills theory, and we show that it is perturbatively quantum equivalent to $\mathcal{N} = 0$ supergravity, proving the validity of the double copy prescription for Yang–Mills theory at loop-level.

This Thesis is based on the papers [1–6] that I wrote in collaboration with Leron Borsten, Branislav Jurčo, Hyungrok Kim, Lorenzo Raspollini, Christian Saemann, and Martin Wolf.

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Lay summary

The formulation of quantum field theory (QFT) was one of the greatest scientific achievements of the last century, realising a paradigm that conciliate quantum mechanics and special relativity. Nowadays, QFT is our best tool to quantitatively describe Nature, and the Standard Model gives us an incredible precise picture of the fundamental forces in terms of gauge theories. In spite of that huge success, our understanding of fundamental physics is far from being complete: we still miss a consistent quantum description of gravity. Many efforts of present days fundamental physics research are devoted to the aim of grasping a better understanding of gravity, and many unifying descriptions have been proposed, although no one succeeded in obtaining universal consensus in the scientific community. One of these proposals is string theory. Even without debating its ultimate validity as a theory of everything, the sheer amount of advancements in physics and mathematics prompted by string theory is immense. It is precisely in string theory that important mathematical structures, known as *homotopy algebras*, found a natural realisation. It was then discovered that homotopy algebras were almost ubiquitous in theoretical physics: indeed, homotopy structures underpin every classical and quantum field theory, and they encode all the details of their perturbative properties.

This Thesis is devoted to the study of homotopy algebras applications in QFT, and its aim is threefold. First, we want to give a comprehensive description of the Batalin–Vilkovisky formalism, that is the bridge between homotopy algebras and quantum field theories. Second, we want to show that this homotopy algebra framework can be successfully applied to the study of scattering amplitudes, crucial objects in QFT, that link the mathematical description of the theory to the experimental results. In particular, we provide recursion relations for scattering amplitudes, that generalise previous results and interpret them into the homotopy algebra language. Finally, we want to inquire into an intriguing duality between gauge theories and gravity, namely the colour–kinematic duality and the double copy prescription. Inspired by our homotopy algebra technology, we prove a conjecture that links gauge theory scattering amplitudes with gravity scattering amplitudes, potentially opening the way for further conceptual and practical advancements.

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Declaration

This thesis is a result of my own efforts. The work to which it refers is based on my PhD research projects, done in collaboration with Leron Borsten, Branislav Jurčo, Hyungrok Kim, Lorenzo Raspollini, Christian Saemann, and Martin Wolf, which are [1–6]

1. B. Jurčo, T. Macrelli, L. Raspollini, C. Saemann, and M. Wolf, *L_∞-algebras, the BV formalism, and classical fields,* in: "Higher Structures in M-Theory," proceedings of the LMS/EPSRC Durham Symposium, 12-18 August 2018 [1903.02887 [hep-th]]

2. T. Macrelli, C. Saemann, and M. Wolf, *Scattering amplitude recursion relations in BV quantisable theories,* Phys. Rev. D **100** (2019) 045017 [1903.05713 [hep-th]].

3. B. Jurčo, T. Macrelli, C. Saemann, and M. Wolf, *Loop amplitudes and quantum homotopy algebras,* JHEP **2007** (2020) 003 [1912.06695 [hep-th]].

4. B. Jurčo, H. Kim, T. Macrelli, C. Saemann, and M. Wolf, *Perturbative quantum field theory and homotopy algebras,* PoS C **ORFU2019** (2020) 199 [2002.11168 [hep-th]].

5. L. Borsten, B. Jurčo, H. Kim, T. Macrelli, C. Saemann, and M. Wolf, *BRST-Lagrangian Double Copy of Yang-Mills Theory,* Phys. Rev. Lett. **126** (2021) 191601 [2007.13803 [hep-th]].

6. L. Borsten, H. Kim, B. Jurčo, T. Macrelli, C. Saemann, and M. Wolf, *Double Copy from Homotopy Algebras,* 2102.11390 [hep-th].

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Tommaso Macrelli, 4th August 2021

Contents

1

Introduction

In this first Chapter, homotopy algebras are heuristically introduced and motivated in the framework of Batalin–Vilkovisky formalism. The main results of this Thesis are informally presented and contextualised in the landscape of high energy theoretical physics.

1.1. Gauge theory, BV formalism, and homotopy algebras

The conciliation of Special Relativity and Quantum Mechanics into the framework of quantum field theory was one of the major conceptual achievements of the last century. To the present day, quantum field theory is still our best quantitative description of Nature, and it is difficult to overemphasise the role of symmetries in our understanding of it. Symmetry seems indeed inescapably tied with the explanation of the most fundamental bricks of Nature: the identification of elementary particles as the irreducible representations of the symmetries of space–time provided by Wigner's classification is a rigorous answer to the question of what can exist in the universe, a conundrum as old as human speculative thinking.

Every action-based, covariant description of the known fundamental interaction contains an intrinsic redundancy, as it is formulated introducing non-dynamical degrees of freedom. Gauge invariance is the symptom of such redundancy. The most familiar example is provided by the electrodynamics, where the four components of the covariant four-potential A_μ do not correspond to the two helicity states of the photon: the gauge invariance of the theory rules out the non-dynamical components. The same is true for gravity, where the ten degree of freedom of the symmetric metric tensor $h_{\mu\nu}$ are reduced to the two helicity states of the graviton by the diffeomorphism invariance of the theory.

The advantages of a covariant formulation are not priceless, and the quantisation of a gauge theory is the prime example of such difficulties. Heuristically speaking, to quantise a gauge theory means to make sense of its path integral. For the sake of concreteness, let us consider Yang–Mills theory, a fundamental ingredient of the Standard Model:

$$S[A] \;=\; -\frac{1}{4}\int \mathrm{d}^d x\, F^a_{\mu\nu} F^{a\mu\nu} \;. \tag{1.1}$$

Perturbatively, the obvious problem is that the kinematic operator of Yang–Mills theory is not invertible, and we need to introduce a gauge-fixing in order to define a propagator. We have more serious problems at the non-perturbative level: given an observable $O(A)$, a naive expression for its expectation value would be

$$\int \mu(A)\, O(A) \mathrm{e}^{\frac{i}{\hbar}S[A]} \;. \tag{1.2}$$

Unfortunately, this path integral is ill-defined, as we are integrating over gauge-equivalent field configurations with the same weight $O(A)\mathrm{e}^{\frac{i}{\hbar}S[A]}$. We remark that both problems can be seen as direct consequences of the local gauge symmetry.

The Faddev–Popov method and the standard Becchi–Rouet–Stora–Tyutin (BRST) formalism allow us to deal with the aforementioned gauge-fixing and quantisation issues in a covariant way. The inconvenience of a covariant quantisation is the introduction of unphysical states to parametrise gauge freedom, namely *ghosts*: in the Faddev–Popov method they emerge through the Jacobian factor that arises when the (infinite) volume of the local gauge transformation is factored out. The inner product associated to ghosts states (and to unphysical gluon states) is not positively-defined: to obtain a physical Hilbert space we then need additional conditions. This problem is present also in the Abelian case (where ghosts decouple and are not needed for quantisation), where the Gupta–Bleuler condition is imposed on the physical states. In Yang–Mills case, the action constructed with the Faddev–Popov method is invariant under BRST symmetry, a global symmetry associated with a nilpotent, anticommuting conserved charge Q_{BRST}. The original gauge symmetry of the theory is recovered by BRST symmetry, and the ghost field plays the role of the gauge parameter. The physical space of physical states is then constructed completing the pre-Hilbert space given by the cohomology of the differential complex associated to Q_{BRST}.

Despite the great success of this formalism in the quantisation of Yang–Mills gauge theory and in the proof of their renormalisability, there are instances of theories where the

Faddev–Popov method and the BRST quantisation fail. Open algebras gauge theories are theories where the commutator of two gauge transformations is a gauge transformation up to equations of motion: this is equivalent to say that the BRST differential complex is a differential complex only up to equations of motion. Examples of these theories are encountered in the context of (super)gravity. Generally speaking, higher gauge theories, theories where the gauge parameters enjoy themselves gauge freedom (mediated by higher ghosts), feature open symmetry algebras. To gauge-fix and quantise these theories, BRST formalism is not enough. Even for standard gauge theories, exotic gauge-fixing choices cannot be implemented with standard BRST formalism.

Batalin–Vilkovisky (BV) formalism [7–11] (also known as antibracket formalism) can be seen as a generalization of BRST formalism, and was originally introduced to gauge-fix and quantise theories that cannot be handled with the standard BRST approach. Analogously to the BRST approach, the starting point of BV formalism is the introduction of ghosts (and, eventually, higher ghosts) to parametrise gauge freedom. Then, the field content of the theory is doubled: for every field, ghost, higher ghost, we introduce an antifield, a ghost antifield, a higher ghost antifield. In this way we obtain a cotangent bundle, where the original BRST fields Φ^A are the local coordinates on the BRST fields manifold, and the antifields Φ^+_A the fibre coordinates. This comes with a natural symplectic structure, that allows us to define Poisson brackets $\{-,-\}$. The BRST action S_{BRST} is extended (in an essentially unique way in the context of the minimal extension) to a BV action S_{BV}, that satisfies the classical BV master equation

$$\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} \;=\; 0. \tag{1.3}$$

The BRST operator Q_{BRST} is extended to a vector field

$$Q_{\mathrm{BV}} \;=\; \{S_{\mathrm{BV}}, -\} \tag{1.4}$$

that squares to 0. In this formalism, gauge-fixing is imposed evaluating the path integral on a Lagrangian submanifold L of the BV field manifold. This is implemented eliminating the antifields with the introduction of a gauge-fixing fermion Ψ. The gauge independence of the expectation value of an observable O,

$$\langle O\rangle_\Psi \;=\; \int_{\mathfrak{F}_{\mathrm{BV}}} \mu_{\mathrm{BV}}(\Phi,\Phi^+)\,\delta\left(\Phi^+_A - \frac{\delta\Psi}{\delta\Phi^A}\right) O[\Phi,\Phi^+]\mathrm{e}^{\frac{i}{\hbar}S^\hbar_{\mathrm{BV}}[\Phi,\Phi^+]}, \tag{1.5}$$

is expressed by the following statement, proved by Batalin and Vilkovisky: if L_0 and L_1 are Lagrangian submanifolds connected by a continuous family L_t of Lagrangian submanifolds, and the integrand H satisfies $\Delta_{\mathrm{BV}} H = 0$, where

$$\Delta_{\mathrm{BV}} \sim \frac{\delta^2}{\delta\Phi^A \delta\Phi^+_A}, \tag{1.6}$$

then

$$\int_{L_1} \mathrm{d}\lambda_1\, H = \int_{L_0} \mathrm{d}\lambda_0\, H. \tag{1.7}$$

The condition

$$\Delta_{\mathrm{BV}} \left(\mathrm{e}^{\frac{i}{\hbar} S^{\hbar}_{\mathrm{BV}}[\Phi,\Phi^+]} \right) = 0 \tag{1.8}$$

translates to a condition on $S^{\hbar}_{\mathrm{BV}}$,

$$\{S^{\hbar}_{\mathrm{BV}}, S^{\hbar}_{\mathrm{BV}}\} - 2i\hbar \Delta_{\mathrm{BV}} S^{\hbar}_{\mathrm{BV}} = 0, \tag{1.9}$$

that generalise the classical master equation. Equation (1.9) is called quantum BV master equation.

Let us make a step back to the classical BV formalism: the differential algebra associated to Q_{BV} is dual to a codifferential coalgebra, equivalently described as an L_∞-algebra, a homotopy algebra that generalise the notion of a Lie algebra. In more precise terms, the BV differential algebra is the Chevalley–Eilenberg algebra associated to an L_∞-algebra. This L_∞-algebra encodes the complete classical structure of the field theory (symmetries, fields, equations of motion, Noether identities...). At the quantum level, this picture will be extended with a quantum generalisation of the notion of L_∞-algebra: in the same way the classical BV master equation gives rise to an L_∞-algebra, the quantum BV master equations yields an underlying algebraic structure called quantum L_∞-algebra.

Strong homotopy algebras are generalisations of ordinary algebras, such as associative, Leibniz, and Lie algebras, where the structural identities (respectively, associativity, Leibniz identity, and Jacobi identity) hold only up to a coherent homotopy. In general, we can consider homotopy algebras as graded vector spaces, equipped with a differential and multibrackets, called higher products, that obey a homotopy generalisation of the structural identity of the correspondent classical algebra. Prominent examples of homotopy algebras are the already mentioned L_∞-algebras and A_∞-algebras, which generalise the notion of associativity. Starting from the seminal contribution of Masahiro Sugawara [12, 13] in 1957 and the fundamental work of Jim Stasheff [14, 15] in 1963, A_∞-structures were introduced in Mathematics. A historical breakdown of the (intricate) story of the discovery and the development of homotopy algebras in mathematical literature is beyond the purposes of this Thesis: the interested reader can find a detailed account in Stasheff's recent review [16].

Homotopy algebras are ubiquitous in theoretical physics: in the early 80's, their dual, Chevalley–Eilenberg counterpart appeared in supergravity in the work of D'Auria–Fré [17], with the slightly misleading name of free differential algebras or FDAs (in rigorous terms, their FDAs where indeed semifree differential graded algebras). Around the same years, the BV approach to gauge-fixing and quantisation was proposed. Stasheff successively interpreted the BV complex in term of Chevalley–Eilenberg algebras associated to L_∞-algebras [18, 19], and various authors addressed the algebraic structures yielded by BV formalism in gauge theories [20–31]. The identification of L_∞-algebras as the algebraic structures behind Zwiebach's closed string field theory is attributed to Stasheff's comment on Zwiebach's contribution to the 10th and Final Workshop on Grand Unification [32,33] in 1989, and Gaberdiel and Zwiebach [34] recognized A_∞-algebras as the algebraic structures of classical open string field theory. Kajiura and Stasheff proposed an homotopy algebra for classical open–closed string field theory [35], and recently Kunitomo and Sugitomo realised an L_∞ structure associated to heterotic string field theory [36]. Further discussions on homotopy algebras and string field theory can be found in [37–40, 35, 41–43].

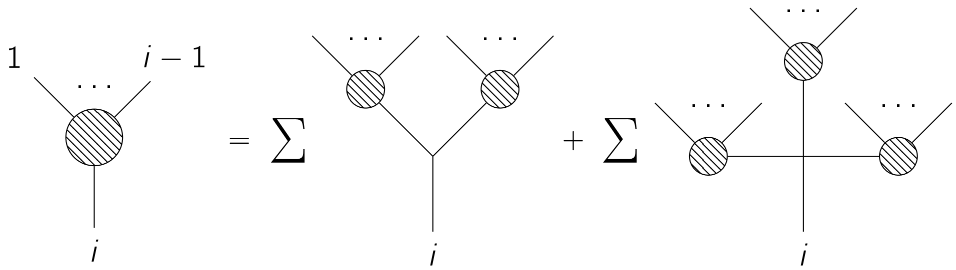
The paper [44] renewed the attention on the homotopy algebra structures underlying every Lagrangian field theory. In the last years, this higher homotopy framework was applied to various aspects of quantum field theory: scattering amplitudes, gravity, double field theory constitute a non exhaustive list of topics where homotopy algebras found natural incarnations [45–72].

1.2. Homotopy algebras and scattering amplitudes

We opened this Introduction remarking how gauge invariance was a common trait of every covariant formulation of the fundamental interactions of Nature. In the last decades, it became evident that the point of view of scattering amplitude (usually opposed to a covariant, action-based formulation) could often give clearer insight into the structure of quantum field theories. The standard textbook approach prescribes: a) to write an action, b) to gauge-fix, and finally c) to compute scattering amplitudes using Feynman rules. But then, in spite of the lengthy diagrammatic calculation a priori required, the astonishing simplicity of MHV formulas for Yang–Mills theory is a clear evidence of how scattering amplitudes could grasp certain aspects of quantum field theory in a more immediate way. In almost every sector of modern theoretical physics, technologies borrowed from the scattering amplitude world (e.g. on-shell methods and generalised unitarity) are common and essential tools.

The homotopy algebra approach to quantum field theory could eventually encompass both the action off-shell perspective and the scattering amplitudes on-shell perspectives: in homotopy algebra terms, the bridge between these two formulation is provided by the notion of *minimal model*. A minimal L_∞-algebra is an L_∞-algebra with trivial differential. Our homotopy framework provides a clear notion of classical equivalence between field theories, namely quasi-isomorphisms. Inside an equivalence class of quasi-isomorphic theories there is a special representative, called minimal model (not to be confused with the homonymous conformal field theory concept). This minimal model can be explicitly constructed starting from the cohomology of the L_∞-algebra that the BV formalism associates to the field theory: through homotopy algebra techniques, the cohomology inherits a minimal L_∞ structure. The minimal model grasps the on-shell, physical data of the theory: indeed, the elements of the cohomology are fields that obey the free equations of motion of the theory, identified up to gauge transformations.

The history of the mentioned Yang–Mills MHV amplitude are indirectly connected to homotopy algebras. In 1987, in a very famous paper Berends and Giele [73] proposed a method to compute in a recursive way gluon scattering amplitudes, proving a number of open conjectures related to amplitudes with most of the gluon with the same helicity. The objects recursively computed in Berends–Giele recursion relations are tree-level off-shell currents, scattering processes involving $i-1$ on-shell fields and an ith off-shell field.



(1.10)

In Yang–Mills theory, the recursive nature of these diagrammatic objects is a simple combinatorial evidence. Complete on-shell scattering amplitude are then computed contracting the off-shell leg of the relevant current with the appropriate polarisation, and imposing momentum conservation. This diagrammatic construction is naturally interpreted in the context of homotopy algebras: the key to the dictionary between these two formulations is to realise that tree-level off-shell currents codify the quasi-isomorphism between the minimal model and the original L_∞-algebra. The minimal model construction hence yield a homotopy algebra generalisation of Berend–Giele recursion relations, valid for every Lagrangian field theory [2], see Section 5.2.. Using the BV approach to quantisation, this recursive homotopy algebra construction can be further generalised to loop-level. This approach to off-shell recursion relations in quantum field theory can be useful to prove in a convenient way properties of tree- and loop-level amplitudes [3], see Section 5.3.. Our homotopy algebra perspective was also followed by Lopez–Arcos and Quintero Vélez to link the perturbiner expansion to the L_∞-algebra formalism [63].

1.3. Homotopy algebras and gauge–gravity dualities

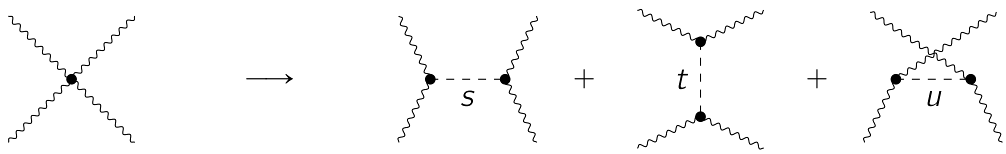
The study of the dualities between gauge theory and gravity are among the most fruitful, recent research lines in the context of quantum field theory. A paradigm that turned out to be a very powerful insight is the possibility to realise a gravity theory as a squared gauge theory. Heuristically speaking, an intuition that motivates this idea is that we can identify the tensor product of two colour-stripped gauge potential $A, \bar{A}$ with the field content of $\mathcal{N} = 0$ supergravity, namely the NS–NS sector of the $\alpha' \to 0$ limit of closed string theory:

$$\text{`}A_\mu \otimes \bar{A}_\nu \;=\; g_{\mu\nu} \oplus B_{\mu\nu} \oplus \varphi\text{'}\;, \tag{1.11}$$

where $g_{\mu\nu}$ is the metric, $B_{\mu\nu}$ the antisymmetric Kalb–Ramond Abelian gauge potential and φ the dilaton. It is not difficult to realise this identification at the level of on-shell states. However, extending this construction to the full theory is far from being immediate.

The first concrete incarnation of this principle came from string theory, in the guise of KLT relations [74]. Yang–Mills theory comes from the low energy limit of open string theory, while gravity arises in the low energy limit of closed string theory: closed string spectra are given by the tensor product of left- and right-moving open string spectra. KLT relations express tree-level closed string amplitudes as sum of products of open string amplitudes, giving a quantitative formulation to the heuristic duality (1.11), albeit intrinsically tied to the tree-level.

Advancement in scattering amplitudes made possible a more recent, purely field theoretic approach to the 'gravity = gauge × gauge' paradigm, namely <u>*BCJ colour–kinematics duality*</u> and <u>*double copy prescription*</u>, that suggested the possibility to extend this gauge–gravity duality to the loop-level. For a pedagogical review of these topics and further perspectives, see [75–80]. We start with a simple observation: we can blow-up Yang–Mills four-gluon interaction vertex into trivalent components, that can be absorbed in the three interaction channels s, t, u.



(1.12)

This means that we can organise a L-loop Yang–Mills amplitude as a sum of trivalent contributions:

$$\mathscr{A}_{n,L} \;=\; (-\mathrm{i})^{n-3+3L} g^{n-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{\mathrm{d}^d p_l}{(2\pi)^d S_i} \frac{\mathsf{c}_i \mathsf{n}_i}{d_i} \;. \tag{1.13}$$

Here i runs over all L-loops trivalent graphs, g is the coupling constant, S_i is the symmetry factor, and d_i are the denominators that come from propagators. The numerators can be split into two factor: a colour factor c_i, composed of gauge group structure constants, and a kinematic factor n_i, obtained from Lorentz-invariant contractions of polarisations and momenta. Importantly, kinematic factors are not univocally determined, and this is at the heart of the BCJ colour–kinematic statement

Conjecture 1.1. *(Bern–Carrasco–Johansson, [81, 82]) There exists a choice of kinematic numerators of the trivalent diagrams entering the scattering amplitude $\mathscr{A}_{n,L}$ such that*

- *if a triple of trivalent diagrams (i, j, k) has colour numerators obeying the Jacobi identity*

$$\mathsf{c}_i + \mathsf{c}_j + \mathsf{c}_k \;=\; 0 \;, \tag{1.14a}$$

  then the corresponding kinematic numerators obey the same identity

$$\mathsf{n}_i + \mathsf{n}_j + \mathsf{n}_k \;=\; 0 \;; \tag{1.14b}$$

- *in any individual diagram, if the colour numerator is mapped from c_i to $-\mathsf{c}_i$ under the permutation of two legs, then the corresponding kinematic numerator is mapped from n_i to $-\mathsf{n}_i$.*

We will call (1.14b) kinematic Jacobi identities. If this statement holds true, then the double copy prescription allow us to compute gravity amplitude from Yang–Mills ones: if we replace the colour factors of Equation (1.13) with kinematic numerators $\tilde{\mathsf{n}}_i$ (having that $\tilde{\mathsf{n}}_i$ or n_i are BCJ-compliant) and Yang–Mills coupling constant g with $\left(\frac{\kappa}{2}\right)$ (where κ is the gravitational coupling constant), we obtain a legitimate $\mathcal{N} = 0$ supergravity amplitude [81–83]

$$\mathscr{M}_{n,L} = (-\mathrm{i})^{n-3+3L}\left(\frac{\kappa}{2}\right)^{n-2+2L}\sum_i\int\prod_{l=1}^{L}\frac{\mathrm{d}^d p_l}{(2\pi)^d S_i}\frac{\mathsf{n}_i\tilde{\mathsf{n}}_i}{d_i} . \tag{1.15}$$

This is an all-loop statement, the problem is that it relies on the validity of colour–kinematic duality. While proven at tree-level [84, 85], at loop-level colour–kinematic duality remains a conjecture, despite being supported by many evidences [82, 86–103].

The range of the applications of colour–kinematic duality and double copy is not limited to scattering amplitudes computation: we can mention for example the study of (non-perturbative) classical solutions in gravity and bi-adjoint scalar theory [104–129], classical black hole scattering [130–143], connections with string theory [84, 85, 144–148], ambitwistor strings and scattering equations [149–159].

It is natural to suspect that colour–kinematic duality could be made manifest at the level of the action. Indeed, the explicit formulation of a non-local reformulation of Yang–Mills theory action that produces tree-level, BCJ-compliant numerators for on-shell gluons scattering amplitudes, was presented by Tolotti and Weinzierl in [160]. Following the earlier step in this direction presented in [83], where an effective Lagrangian producing BCJ-compliant numerators for tree level scattering amplitude up to six point was introduced, Tolotti and Weinzierl proposed the Lagrangian

$$\mathscr{L}^{\mathrm{YM}} = \sum_n \mathscr{L}_n^{\mathrm{YM}} , \tag{1.16}$$

in which the n-th order term is

$$\mathscr{L}_n^{\mathrm{YM}} = \sum_{\Gamma\in\mathrm{Tree}_{3,n}} O_{n,\Gamma}^{\mu_1\cdots\mu_n}\frac{\mathrm{tr}\left\{[A_{\mu_{\sigma(1)}}, A_{\mu_{\sigma(2)}}]\,[\ldots[A_{\mu_{\sigma(3)}}, A_{\mu_{\sigma(4)}}]\ldots, A_{\mu_{\sigma(n)}}]\right\}}{\square_{j_{n,\Gamma,1}}\cdots\square_{j_{n,\Gamma,n-3}}} , \tag{1.17}$$

where $\mathrm{Tree}_{3,n}$ is the set of trivalent tree diagrams with n external vertices. The permutation σ is determined by the diagram Γ and $O_{n,\Gamma}^{\mu_1\cdots\mu_n}$ is a sum of polynomials in the inverse Minkowski metric $\eta^{\mu\nu}$ and $n-2$ partial differential operators ∂_μ acting on one of the n occurrences of the field A in the numerator. The operators $\square_{j_{n,\Gamma,k}}$ in the denominator act on the kth internal edge of Γ. This expression is algebraically equal to ordinary Yang–Mills Lagrangian: Jacobi identity vanishes the higher-order vertices. Tolotti–Weinzierl action expresses how these vertices are distributed into trivalent trees. This action is one of the starting points of our work: at the price of introducing an infinite tower of auxiliary fields, we can make this action local and at most cubic in the interactions.

Our claim is that homotopy algebras can help us to solve the all-loop conundrum, and the route we chose to validate loop-level double copy does not involve a direct proof of loop-level colour–kinematic duality. On-shell methods were fundamental in revealing this structure, hidden in the standard action-based formulation of the theory. We propose an off-shell, Lagrangian approach to the colour–kinematic duality and double copy paradigm, with homotopy algebras being instrumental in manifesting this structure at the level of the associate L_∞-algebras, and eventually at the level of the actions. Following this approach, the remarkable result is that we can directly prove the double copy prescription at arbitrary high loop level, without relying on the validity of colour–kinematic duality for loops. The key technical construction for this homotopy algebra interpretation is the introduction of an adequate notion of tensor product, such that we can factorise Yang–Mills theory L_∞-algebra into three components:

$$\mathfrak{L}^{\mathsf{YM}} = \mathfrak{g} \otimes \mathfrak{Kin} \otimes_\tau \mathfrak{Scal} , \tag{1.18}$$

where $\mathfrak{g}$ is the gauge Lie algebra, $\mathfrak{Kin}$ a graded vector space whose basis corresponds to the Poincaré representation of the field content of the theory, and $\mathfrak{Scal}$ the A_∞-algebra of a scalar theory. The tensor product we introduce in this construction is suitably twisted with the introduction of a twist datum τ, that codifies how $\mathfrak{Kin}$ acts on $\mathfrak{Scal}$ as a kinematic operator algebra. The double copied theory is then realised replacing the $\mathfrak{g}$ factor with a copy of $\mathfrak{Kin}$:

$$\mathfrak{L}^{\mathsf{DC}} = \mathfrak{Kin} \otimes_\tau \mathfrak{Kin} \otimes_\tau \mathfrak{Scal} . \tag{1.19}$$

The theory associated to this L_∞-algebra is perturbatively quantum equivalent to $\mathcal{N} = 0$ supergravity [5, 6], see Section 9.5., and this implies the validity of double copy at loop level. Alternatively, one can replace $\mathfrak{Kin}$ with a copy of the gauge Lie algebra $\mathfrak{g}$ (or a different one): in this case, we obtain the L_∞-algebra of a biadjoint scalar theory.

$$
\begin{array}{ccccc}
\text{Biadjoint scalar field theory} & \longleftarrow & \text{Yang–Mills theory} & \longrightarrow & \mathcal{N}=0 \text{ supergravity} \\
\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{Scal} & & \mathfrak{g}\otimes\mathfrak{Kin}\otimes_\tau\mathfrak{Scal} & & \mathfrak{Kin}\otimes_\tau\mathfrak{Kin}\otimes_\tau\mathfrak{Scal}
\end{array}
\tag{1.20}
$$

1.4. Outlook

The recent progress of homotopy algebras applications to high energy theoretical physics shows how these sophisticated mathematical techniques could be helpful to provide new insight into the structure of field theories and to suggest solutions to relevant open problems. Restricting our attentions to the themes of the present Thesis, we can identify some interesting research lines, where our homotopy algebra-based approach could provide new results. Some of these research directions are natural generalisations of the results discussed in this Thesis.

Scattering amplitudes recursion relations. Homotopy algebra minimal model construction encodes and generalise off-shell Berends–Giele recursion relations. Since tree-level on-shell scattering amplitudes are completely grasped by the minimal model structure associated to the field theory, it is reasonable to expect that also on-shell recursion relations (e.g., BCFW recursion relations) could be interpreted and eventually generalised in terms of homotopy algebras. In this context, a recursive construction based on Hartogs extension theorem was proposed in [161].

Colour–kinematic duality and double copy. The double copy paradigm opened new perspectives on quantum gravity, providing both deep conceptual advancements and new, crucial computational developments [162, 90, 91, 93, 99, 96, 100, 101, 163, 164]. A growing zoology of gravity theories could be constructed from double copy [162,82,83,165,91,166–169, 166, 170–179, 103, 180–186], and under some assumptions our Lagrangian realisation of this paradigm could be extended to them. This would imply the validity of double copy prescription to all loop order for many relevant theories. A simpler example, the non-linear sigma model (whose double copy is the special galileon), is discussed in [6]. A natural follow-up of pure Yang–Mills theory double copy case would be the inclusion of supersymmetry: we are free to extract the kinematic factor $\mathfrak{Kin}$ and the twist datum τ from theories different from pure Yang–Mills. For example, from the factors of pure Yang–Mills theory and $\mathcal{N} = 1$ Yang–Mills theory it should be possible to realise $\mathcal{N} = 1$ supergravity minimally coupled to a single chiral multiplet, see also [126]. Almost all $\mathcal{N} \geqslant 2$ ungauged supergravity theories [182], (super) Einstein–Yang–Mills–scalar theories [173], and gauged supergravity (with Poincaré background) [184] could be realised from double copy. Other candidates of double copy-constructible theories are Abelian Dirac–Born–Infeld theory [153, 187, 159], massive gravity [188], and conformal gravity [180, 189]. Ambitiously, homotopy algebra techniques could be used to directly prove loop-level and even off-shell colour–kinematic duality [190].

String theory. From the perspective of string theory, this relation between gauge theories and gravity is a reflection of a more fundamental 'open $\otimes$ open = closed' duality, as suggested by KLT relations. Inquiring into the stringy origin of colour–kinematic duality and double copy could give us a better understanding of the structures involved in our formulation, like the homotopy algebra factorisation that we introduce in Chapter 6. Moreover, homotopy algebras are the natural language of string field theory: the homotopy algebra interpretation of double copy could be a valid framework to investigate and generalise open/closed string dualities.

1.5. Plan of the Thesis

In this Section we present a short summary of the content of the following Chapters and Appendices.

In Chapter 2 we give an overview of the homotopy algebras relevant for our physical applications. Chapter 2 is based on [6].

In Chapter 3 we review BV formalism, and we show how homotopy algebras describe every perturbative field theory. Chapter 3 is based on [6].

In Chapter 4 we show several concrete examples of applications of the homotopy algebra framework to the formulation of field theories. We introduce here the field theories relevant to Yang–Mills theory double copy. Chapter 4 is based on [2, 6].

In Chapter 5 we focus on minimal models and homotopy algebra applications to scattering amplitudes. Chapter 5 is based on [2, 3, 6].

In Chapter 6 we introduce a notion of factorisation for (strict) homotopy algebras, that we will adopt to give an homotopy algebra description of colour–kinematic duality. Chapter 6 is based on [6].

In Chapter 7 we expose the homotopy algebra factorisation underlying Yang–Mills theory double copy at linear level. Chapter 7 is based on [6].

In Chapter 8 we collect several field theoretic observations, that will prepare the ground for extending the linear result to the full, interacting picture. Chapter 8 is based on [5, 6].

In Chapter 9 we finally show the perturbative quantum equivalence between Yang–Mills theory double copy and $\mathcal{N} = 0$ supergravity. Chapter 9 is based on [5, 6].

In Appendix A we present a proof of minimal model recursive construction for L_∞-algebras. Appendix A is based on [2].

In Appendix B we discuss a further generalisation of Berends–Giele recursive relations. Appendix B is based on [2].

2 Homotopy algebras

The homotopy algebras that appear naturally in the context of field theories, namely A_∞-, C_∞-, and L_∞-algebras are homotopy versions of associative, commutative and Lie algebras. In particular, associativity and the Jacobi identity only hold up to coherent homotopies.[1] In this first Chapter, we list the main definitions and several technical results that will be relevant for our field theoretic applications, as well as the conventions that we adopted in this Thesis. For more details on L_∞-algebras and some of the calculations detailed in this Chapter, see e.g. [52, 1]; our conventions match the ones in these references. Other helpful references with original results listed in this Chapter are [191, 40, 192]. A unifying description of all the homotopy algebras and their cyclic structures listed below is given by operads, but we refrain from introducing this additional layer of abstraction.

The material in this Chapter is borrowed from [6].

2.1. A_∞-algebras

A_∞-algebras. An *A_∞-algebra* or *strong homotopy associative algebra* is a graded vector space $\mathfrak{A} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{A}_i$ together with *higher products* which are i-linear maps $\mathsf{m}_i : \mathfrak{A}\times\cdots\times\mathfrak{A} \to \mathfrak{A}$ of degree $2-i$ that satisfy the homotopy associativity relation

$$\sum_{i_1+i_2+i_3=i} (-1)^{i_1 i_2+i_3}\mathsf{m}_{i_1+i_3+1}(\mathrm{id}^{\otimes i_1}\otimes \mathsf{m}_{i_2}\otimes \mathrm{id}^{\otimes i_3}) \;=\; 0 \tag{2.1}$$

[1]But graded commutativity (in the case of C_∞-algebras) and graded anti-symmetry (in the case of L_∞-algebras) are not relaxed.

for all $i \in \mathbb{N}^+$. The lowest identities read as

$$\begin{aligned} \mathsf{m}_1(\mathsf{m}_1(\ell_1)) &= 0\,, \\ \mathsf{m}_1(\mathsf{m}_2(\ell_1,\ell_2)) &= \mathsf{m}_2(\mathsf{m}_1(\ell_1),\ell_2) + (-1)^{|\ell_1|_{\mathfrak{A}}}\mathsf{m}_2(\ell_1,\mathsf{m}_1(\ell_2))\,, \\ \mathsf{m}_1(\mathsf{m}_3(\ell_1,\ell_2,\ell_3)) + \mathsf{m}_3(\mathsf{m}_1(\ell_1),\ell_2,\ell_3) &+ (-1)^{|\ell_1|_{\mathfrak{A}}}\mathsf{m}_3(\ell_1,\mathsf{m}_1(\ell_2),\ell_3) + \\ + (-1)^{|\ell_1|_{\mathfrak{A}}+|\ell_2|_{\mathfrak{A}}}\mathsf{m}_3(\ell_1,\ell_2,\mathsf{m}_1(\ell_3)) &= \mathsf{m}_2(\mathsf{m}_2(\ell_1,\ell_2),\ell_3) - \mathsf{m}_2(\ell_1,\mathsf{m}_2(\ell_2,\ell_3))\,, \\ &\vdots \end{aligned} \tag{2.2}$$

for $\ell_1,\ldots,\ell_i \in \mathfrak{A}$ elements of homogenous degree $|\ell_1|_{\mathfrak{A}},\ldots,|\ell_i|_{\mathfrak{A}}$. We thus see that the unary product m_1 is a differential and a derivation for the binary product m_2. Importantly, the ternary product m_3 captures the failure of the binary product m_2 to be associative.

Cyclic A_∞-algebras. A *cyclic A_∞-algebra* $(\mathfrak{A},\langle -,-\rangle_{\mathfrak{A}})$ is an A_∞-algebra $\mathfrak{A}$ equipped with a non-degenerate graded-symmetric bilinear form $\langle -,-\rangle_{\mathfrak{A}} : \mathfrak{A}\times\mathfrak{A}\to\mathbb{R}$ such that

$$\langle \ell_1, \mathsf{m}_i(\ell_2,\ldots,\ell_{i+1})\rangle_{\mathfrak{A}} = (-1)^{i+i(|\ell_1|_{\mathfrak{A}}+|\ell_{i+1}|_{\mathfrak{A}})+|\ell_{i+1}|_{\mathfrak{A}}\sum_{j=1}^{i}|\ell_j|_{\mathfrak{A}}}\langle \ell_{i+1}, \mathsf{m}_i(\ell_1,\ldots,\ell_i)\rangle_{\mathfrak{A}} \tag{2.3}$$

for all $\ell_i \in \mathfrak{A}$. When it is clear from the context, we shall suppress the subscript $\mathfrak{A}$ on the inner products.

Homotopy Maurer–Cartan theory. Each A_∞-algebra comes with a homotopy Maurer–Cartan theory, where the *gauge potential* is an element $a \in \mathfrak{A}_1$ whose *curvature* $f \in \mathfrak{A}_2$ is defined as

$$f := \mathsf{m}_1(a) + \mathsf{m}_2(a,a) + \cdots = \sum_{i\geqslant 1}\mathsf{m}_i(a,\ldots,a) \tag{2.4}$$

and satisfies the Bianchi identity

$$\sum_{i\geqslant 0}\sum_{j=0}^{i}(-1)^{i+j}\mathsf{m}_{i+1}(\underbrace{a,\ldots,a}_{j},f,\underbrace{a,\ldots,a}_{i-j}) = 0\,. \tag{2.5}$$

If the *homotopy Maurer–Cartan equation*

$$f = 0 \tag{2.6}$$

holds, we say that a is a *homotopy Maurer–Cartan element*. Provided $\mathfrak{A}$ is cyclic with pairing of degree -3, homotopy Maurer–Cartan elements are the stationary points of the

<u>*homotopy Maurer–Cartan action*</u>

$$S^{\rm hMC}[a] \;:=\; \sum_{i\geqslant 1}\frac{1}{i+1}\langle a,\mathsf{m}_i(a,\ldots,a)\rangle_{\mathfrak{A}}\;. \tag{2.7}$$

Infinitesimal gauge transformations are mediated by elements $c_0\in\mathfrak{A}_0$ and are given by

$$\delta_{c_0}a \;:=\; \sum_{i\geqslant 0}\sum_{j=0}^{i}(-1)^{i+j}\mathsf{m}_{i+1}(\underbrace{a,\ldots,a}_{j},c_0,\underbrace{a,\ldots,a}_{i-j})\;. \tag{2.8}$$

One may check that the action (2.7) is invariant under the transformations (2.8), and the curvature (2.4) transforms as

$$\delta_{c_0}f \;=\; \sum_{i\geqslant 0}\sum_{j=0}^{i}\sum_{k=0}^{i-j}(-1)^k\mathsf{m}_{i+2}(\underbrace{a,\ldots,a}_{j},f,\underbrace{a,\ldots,a}_{i-j},c_0,\underbrace{a,\ldots,a}_{i-j-k})\;. \tag{2.9}$$

To verify these statements, one makes use of (2.1).

2.2. C_∞-algebras

Permutations, shuffles, and unshuffles. Let S_n be the permutation group of degree $n\in\mathbb{N}^+$. We shall write for a permutation $\sigma\in S_n$

$$\sigma \;:=\; \begin{pmatrix}1 & 2 & \cdots & n\\ \sigma(1) & \sigma(2) & \cdots & \sigma(n)\end{pmatrix}. \tag{2.10}$$

A <u>(p,q)*-shuffle*</u> for $p,q\in\mathbb{N}^+$ is a permutation $\sigma\in S_{p+q}$ which satisfies the condition that if $1\leqslant\sigma(i)<\sigma(j)\leqslant p$ or $p+1\leqslant\sigma(i)<\sigma(j)\leqslant p+q$ then $i<j$. We denote the set of all (p,q)-shuffles in S_{p+q} by $\mathrm{Sh}(p;p+q)$. Consider, for instance, S_3. We have the permutations

$$S_3 \;=\; \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix},\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\right\}. \tag{2.11}$$

Then, the sets of $(1,2)$- and $(2,1)$-shuffles are given by

$$\begin{aligned}\mathrm{Sh}(1;3) \;&=\; \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\right\},\\ \mathrm{Sh}(2;3) \;&=\; \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\right\}.\end{aligned} \tag{2.12}$$

Likewise, a *(p,q)-unshuffle* for $p,q \in \mathbb{N}^+$ is a permutation $\sigma \in S_{p+q}$ which satisfies the condition that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. We denote the set of all (p,q)-unshuffles in S_{p+q} by $\overline{\mathsf{Sh}}(p;p+q)$. For instance, the sets of $(1,2)$- and $(2,1)$-unshuffles in S_3 are given by

$$\begin{aligned} \overline{\mathsf{Sh}}(1;3) &= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}, \\ \overline{\mathsf{Sh}}(2;3) &= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}. \end{aligned} \tag{2.13}$$

It follows from the above definitions, and it is evident from the explicit examples (2.12) and (2.13), that a permutation is a (p,q)-shuffle if and only if its inverse is a (p,q)-unshuffle, and vice versa.

C_∞-algebras. A *C_∞-algebra* or *strong homotopy commutative algebra* is an A_∞-algebra $\mathfrak{C} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{C}_i$ where the higher products m_i, in addition to (2.1), also satisfy the *homotopy commutativity relations*

$$\sum_{\sigma \in \mathsf{Sh}(i_1;i)} \chi(\sigma;\ell_1,\ldots\ell_i)\, \mathsf{m}_i(\ell_{\sigma(1)},\ldots,\ell_{\sigma(i_1)},\ell_{\sigma(i_1+1)},\ldots,\ell_{\sigma(i)}) = 0 \tag{2.14}$$

for all $0 < i_1 < i$ and for all $\ell_1,\ldots,\ell_i \in \mathfrak{C}$. Here, $\chi(\sigma;\ell_1,\ldots,\ell_i)$ is the *Koszul sign* for total graded anti-symmetrisation defined by

$$\ell_1 \wedge \ldots \wedge \ell_i = \chi(\sigma;\ell_1,\ldots,\ell_i)\, \ell_{\sigma(1)} \wedge \ldots \wedge \ell_{\sigma(i)} . \tag{2.15}$$

The lowest four homotopy commutativity relations are

$$\begin{aligned} \mathsf{m}_2(\ell_1,\ell_2) - (-1)^{|\ell_1|_\mathfrak{C}|\ell_2|_\mathfrak{C}} \mathsf{m}_2(\ell_2,\ell_1) &= 0 , \\ \mathsf{m}_3(\ell_1,\ell_2,\ell_3) - (-1)^{|\ell_2|_\mathfrak{C}|\ell_3|_\mathfrak{C}} \mathsf{m}_3(\ell_1,\ell_3,\ell_2) + (-1)^{(|\ell_1|_\mathfrak{C}+|\ell_2|_\mathfrak{C})|\ell_3|_\mathfrak{C}} \mathsf{m}_3(\ell_3,\ell_1,\ell_2) &= 0 , \\ \mathsf{m}_4(\ell_1,\ell_2,\ell_3,\ell_4) - (-1)^{|\ell_1|_\mathfrak{C}|\ell_2|_\mathfrak{C}} \mathsf{m}_4(\ell_2,\ell_1,\ell_3,\ell_4) + & \\ + (-1)^{|\ell_1|_\mathfrak{C}(|\ell_2|_\mathfrak{C}+|\ell_3|_\mathfrak{C})} \mathsf{m}_4(\ell_2,\ell_3,\ell_1,\ell_4) - (-1)^{|\ell_1|_\mathfrak{C}(|\ell_2|_\mathfrak{C}+|\ell_3|_\mathfrak{C}+|\ell_4|_\mathfrak{C})} \mathsf{m}_4(\ell_2,\ell_3,\ell_4,\ell_1) &= 0 , \\ \mathsf{m}_4(\ell_1,\ell_2,\ell_3,\ell_4) - (-1)^{|\ell_2|_\mathfrak{C}|\ell_3|_\mathfrak{C}} \mathsf{m}_4(\ell_1,\ell_3,\ell_2,\ell_4) + & \\ + (-1)^{|\ell_2|_\mathfrak{C}(|\ell_3|_\mathfrak{C}+|\ell_4|_\mathfrak{C})} \mathsf{m}_4(\ell_1,\ell_3,\ell_4,\ell_2) + (-1)^{(|\ell_1|_\mathfrak{C}+|\ell_2|_\mathfrak{C})|\ell_3|_\mathfrak{C}} \mathsf{m}_4(\ell_3,\ell_1,\ell_2,\ell_4) - & \\ -(-1)^{(|\ell_1|_\mathfrak{C}+|\ell_2|_\mathfrak{C})|\ell_3|_\mathfrak{C}+|\ell_2|_\mathfrak{C}|\ell_4|_\mathfrak{C}} \mathsf{m}_4(\ell_3,\ell_1,\ell_4,\ell_2) + & \\ + (-1)^{(|\ell_1|_\mathfrak{C}+|\ell_2|_\mathfrak{C})|\ell_3|_\mathfrak{C}+(|\ell_1|_\mathfrak{C}+|\ell_2|_\mathfrak{C})|\ell_4|_\mathfrak{C}} \mathsf{m}_4(\ell_3,\ell_4,\ell_1,\ell_2) &= 0 , \end{aligned} \tag{2.16}$$

and we see that the product m_2 is indeed graded commutative. Note that, a priori, there are two relations for m_3 given by the $(2,1)$- and $(1,2)$-shuffles. However, the $(1,2)$-shuffles for (ℓ_1,ℓ_2,ℓ_3) are the same as the $(2,1)$-shuffles for (ℓ_3,ℓ_2,ℓ_1). Since ℓ_1, ℓ_2, and ℓ_3 are arbitrary elements of $\mathfrak{C}$, the two relations thus reduce to one relation. Generally, the number of independent relations for m_i is $\lfloor \frac{i}{2} \rfloor$.

Cyclic C_∞-algebras. A cyclic C_∞-algebra is a cyclic A_∞-algebra satisfying the homotopy commutativity relations (2.14).

2.3. L_∞-algebras

L_∞-algebras. An *L_∞-algebra* or *strong homotopy Lie algebra* is a graded vector space $\mathfrak{L} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{L}_i$ together with *higher products* which are graded anti-symmetric i-linear maps $\mu_i : \mathfrak{L}\times\cdots\times\mathfrak{L}\to\mathfrak{L}$ of degree $2-i$ that satisfy the *homotopy Jacobi identities*

$$\sum_{i_1+i_2=i}\ \sum_{\sigma\in\overline{\mathrm{Sh}}(i_1;i)} (-1)^{i_2}\chi(\sigma;\ell_1,\ldots,\ell_i)\mu_{i_2+1}(\mu_{i_1}(\ell_{\sigma(1)},\ldots,\ell_{\sigma(i_1)}),\ell_{\sigma(i_1+1)},\ldots,\ell_{\sigma(i)}) \;=\; 0\;. \tag{2.17}$$

for all $\ell_1,\ldots,\ell_i\in\mathfrak{L}$ and $i\in\mathbb{N}^+$; see Section 2.2. and Equation (2.15) for the definitions of the unshuffles $\overline{\mathrm{Sh}}(i_1;i)$ and of the Koszul sign $\chi(\sigma;\ell_1,\ldots,\ell_i)$. The lowest homotopy Jacobi identities, slightly rewritten, read as

$$\begin{aligned}
\mu_1(\mu_1(\ell_1)) \;&=\; 0\;,\\
\mu_1(\mu_2(\ell_1,\ell_2)) \;&=\; \mu_2(\mu_1(\ell_1),\ell_2)+(-1)^{|\ell_1|_{\mathfrak{L}}}\mu_2(\ell_1,\mu_1(\ell_2))\;,\\
\mu_2(\mu_2(\ell_1,\ell_2),\ell_3)+(-1)^{|\ell_1|_{\mathfrak{L}}\,|\ell_2|_{\mathfrak{L}}}\mu_2(\ell_2,\mu_2(\ell_1,\ell_3))-\mu_2(\ell_1,\mu_2(\ell_2,\ell_3)) \;&=\\
=\; \mu_1(\mu_3(\ell_1,\ell_2,\ell_3))+\mu_3(\mu_1(\ell_1),\ell_2,\ell_3)&+(-1)^{|\ell_1|_{\mathfrak{L}}}\mu_3(\ell_1,\mu_1(\ell_2),\ell_3)+\\
&+(-1)^{|\ell_1|_{\mathfrak{L}}+|\ell_2|_{\mathfrak{L}}}\mu_3(\ell_1,\ell_2,\mu_1(\ell_3))\;,\\
&\;\;\vdots
\end{aligned} \tag{2.18}$$

and we can interpret them as follows. The unary product μ_1 is a differential and a derivation with respect to the binary product μ_2. In addition, the ternary product μ_3 captures the failure of the binary product μ_2 to satisfy the standard Jacobi identity. Roughly speaking, the ternary product μ_3 correspond to a homotopy that control the violation of standard

Jacobi identity[1].

We note that any A_∞-algebra yields an L_∞-algebra with higher products obtained from total anti-symmetrisation,

$$\mu_i(\ell_1,\ldots,\ell_i) \;=\; \sum_{\sigma\in S_i}\chi(\sigma;\ell_1,\ldots,\ell_i)\,\mathsf{m}_i(\ell_{\sigma(1)},\ldots,\ell_{\sigma(i)})\ . \tag{2.19}$$

In particular, the Lie algebra arising from the commutator on any matrix algebra is an L_∞-algebra. Likewise, the anti-symmetrisation of a C_∞-algebra is an L_∞-algebra with $\mu_i = 0$ for $i \geqslant 2$ due to the homotopy commutativity relations (2.14).

We call an L_∞-algebra *nilpotent*, if all nested higher products vanish, i.e.

$$\mu_i(\mu_j(-,\ldots,-),\ldots,-) \;=\; 0 \quad \text{for all } i,j \geqslant 1\ . \tag{2.20}$$

Cyclic L_∞-algebras. A *cyclic L_∞-algebra* $(\mathfrak{L},\langle-,-\rangle_{\mathfrak{L}})$ is an L_∞-algebra $\mathfrak{L}$ equipped with a non-degenerate graded-symmetric bilinear form $\langle-,-\rangle_{\mathfrak{L}}:\mathfrak{L}\times\mathfrak{L}\to\mathbb{R}$ such that

$$\langle\ell_1,\mu_i(\ell_2,\ldots,\ell_{i+1})\rangle_{\mathfrak{L}} \;=\; (-1)^{i+i(|\ell_1|_{\mathfrak{L}}+|\ell_{i+1}|_{\mathfrak{L}})+|\ell_{i+1}|_{\mathfrak{L}}\sum_{j=1}^{i}|\ell_j|_{\mathfrak{L}}}\langle\ell_{i+1},\mu_i(\ell_1,\ldots,\ell_i)\rangle_{\mathfrak{L}} \tag{2.21}$$

for all $\ell_i\in\mathfrak{L}$. As before, when it is clear from the context, we shall suppress the subscript $\mathfrak{L}$ on the inner products.

Homotopy Maurer–Cartan theory. Similar to A_∞-algebras, any L_∞-algebra $(\mathfrak{L},\mu_i)$ comes with its homotopy Maurer–Cartan theory. In particular, a *gauge potential* is an element $a\in\mathfrak{L}_1$, and its *curvature* is

$$f \;:=\; \mu_1(a)+\tfrac{1}{2}\mu_2(a,a)+\cdots \;=\; \sum_{i\geqslant 1}\frac{1}{i!}\mu_i(a,\ldots,a) \;\in\; \mathfrak{L}_2\ . \tag{2.22}$$

The Bianchi identity reads here as

$$\sum_{i\geqslant 0}\frac{1}{i!}\mu_{i+1}(a,\ldots,a,f) \;=\; 0\ . \tag{2.23}$$

[1] To be more precise, a *cochain homotopy* between two morphisms of cochain complexes $\phi,\psi:(C,\mathsf{d})\to(C',\mathsf{d}')$ is a family of morphisms of degree -1, $h_k: C^{k+1}\to C'^k$, such that $\phi_k-\psi_k = h_k\circ\mathsf{d}_k+\mathsf{d}'_{k-1}\circ h_k$. The operator appearing on the right-hand-side of this expression can be interpreted as a coboundary operator, and, in turn, if we compare this to the third identity of Equation (2.18), we see that the right-hand-side of this identity can be written in terms of this coboundary operator.

Homotopy Maurer–Cartan elements, i.e. gauge potentials with vanishing curvature $f = 0$, are the stationary points of the homotopy Maurer–Cartan action

$$S^{\mathrm{hMC}}[a] \;:=\; \sum_{i\geqslant 1} \frac{1}{(i+1)!}\langle a, \mu_i(a,\ldots,a)\rangle_{\mathfrak{L}} \tag{2.24}$$

provided $\mathfrak{L}$ comes with a cyclic pairing $\langle -,-\rangle_{\mathfrak{L}}$ of degree -3[1]. Similarly to (2.8), infinitesimal gauge transformations are of the form

$$\delta_{c_0} a \;:=\; \sum_{i\geqslant 0} \frac{1}{i!}\mu_{i+1}(a,\ldots,a,c_0) \tag{2.25}$$

and are parametrised by elements $c_0 \in \mathfrak{L}_0$. The action is invariant under such transformations, and the curvature behaves as

$$\delta_{c_0} f \;=\; \sum_{i\geqslant 0} \frac{1}{i!}\mu_{i+2}(a,\ldots,a,f,c_0)\ . \tag{2.26}$$

To verify these statements, one makes use of (2.17). Using Equation (2.17), one may show that

$$[\delta_{c_0},\delta_{c_0'}]a \;=\; \delta_{c_0''}a + \sum_{i\geqslant 0} \frac{1}{i!}\mu_{i+3}(a,\ldots,a,f,c_0,c_0')\ , \tag{2.27a}$$

where

$$c_0'' \;:=\; \sum_{i\geqslant 0} \frac{1}{i!}\mu_{i+2}(a,\ldots,a,c_0,c_0')\ . \tag{2.27b}$$

In general gauge transformations are not closed: a sufficient condition to ensure closure is $f = 0$.

Covariant derivative. Given an L_∞-algebra $(\mathfrak{L},\mu_i)$, consider $\varphi \in \mathfrak{L}_k$ for some $k \in \mathbb{Z}$ and require that under infinitesimal gauge transformations, φ transforms adjointly, that is,

$$\delta_{c_0}\varphi \;:=\; \sum_{i\geqslant 0} \frac{1}{i!}\mu_{i+2}(a,\ldots,a,\varphi,c_0) \tag{2.28}$$

for $c_0 \in \mathfrak{L}_0$. We then define the *covariant derivative* $\nabla : \mathfrak{L}_k \to \mathfrak{L}_{k+1}$ by

$$\nabla\varphi \;:=\; \mu_1(\varphi) + \mu_2(a,\varphi) + \cdots \;=\; \sum_{i\geqslant 0} \frac{1}{i!}\mu_{i+1}(a,\ldots,a,\varphi) \tag{2.29}$$

[1]A cyclic structure of degree -3 is needed in order to have an action of degree 0.

for $a \in \mathfrak{L}_1$. Using (2.17), one can show that under infinitesimal gauge transformations (2.25) and (2.28), $\nabla\varphi$ transforms as

$$\delta_{c_0}(\nabla\varphi) = \sum_{i\geqslant 0}\frac{1}{i!}\mu_{i+2}(a,\ldots,a,\nabla\varphi,c_0)+\sum_{i\geqslant 0}\frac{1}{i!}\mu_{i+3}(a,\ldots,a,f,\varphi,c_0)\ , \tag{2.30}$$

where f is the curvature (2.22) of a. Thus, for homotopy Maurer–Cartan elements a, the covariant derivative transforms adjointly as well.[1] Using (2.17) again, we obtain in addition

$$\nabla^2\varphi = \sum_{i\geqslant 0}\frac{1}{i!}\mu_{i+2}(a,\ldots,a,f,\varphi)\ . \tag{2.31}$$

Curved morphisms of L_∞-algebras. Morphisms between Lie algebras are maps preserving the Lie bracket. In the context of L_∞-algebras, this notion generalises and one obtains what is known as a *curved morphism (of L_∞-algebras)*. Specifically, a curved morphism $\phi : (\mathfrak{L},\mu_i)\to(\tilde{\mathfrak{L}},\tilde\mu_i)$ between two L_∞-algebras $(\mathfrak{L},\mu_i)$ and $(\tilde{\mathfrak{L}},\tilde\mu_i)$ is a collection of i-linear graded anti-symmetric maps $\phi_i : \mathfrak{L}\times\cdots\times\mathfrak{L}\to\tilde{\mathfrak{L}}$ of degree $1-i$ such that

$$\begin{aligned}\sum_{i_1+i_2=i}\ \sum_{\sigma\in\overline{\mathrm{Sh}}(i_1;i)}&(-1)^{i_2}\chi(\sigma;\ell_1,\ldots,\ell_i)\phi_{i_2+1}(\mu_{i_1}(\ell_{\sigma(1)},\ldots,\ell_{\sigma(i_1)}),\ell_{\sigma(i_1+1)},\ldots,\ell_{\sigma(i)}) =\\ =\sum_{j\geqslant 1}\frac{1}{j!}\sum_{k_1+\cdots+k_j=i}\ &\sum_{\sigma\in\overline{\mathrm{Sh}}(k_1,\ldots,k_{j-1};i)}\chi(\sigma;\ell_1,\ldots,\ell_i)\zeta(\sigma;\ell_1,\ldots,\ell_i)\times\\ &\times\tilde\mu_j\Big(\phi_{k_1}\big(\ell_{\sigma(1)},\ldots,\ell_{\sigma(k_1)}\big),\ldots,\phi_{k_j}\big(\ell_{\sigma(k_1+\cdots+k_{j-1}+1)},\ldots,\ell_{\sigma(i)}\big)\Big)\end{aligned} \tag{2.32a}$$

for $i\in\mathbb{N}^+\cup\{0\}$ with $\chi(\sigma;\ell_1,\ldots,\ell_i)$ the Koszul sign (2.15) and $\zeta(\sigma;\ell_1,\ldots,\ell_i)$ given by

$$\zeta(\sigma;\ell_1,\ldots,\ell_i) := (-1)^{\sum_{1\leqslant m<n\leqslant j}k_mk_n+\sum_{m=1}^{j-1}k_m(j-m)+\sum_{m=2}^{j}(1-k_m)\sum_{k=1}^{k_1+\cdots+k_{m-1}}|\ell_{\sigma(k)}|_{\mathfrak{L}}}\ . \tag{2.32b}$$

[1] It will always transform adjointly when $\mu_i = 0$ for all $i > 2$, that is, for differential graded Lie algebras also known as strict L_∞-algebras, cf. Section 2.4..

Note that $\phi_0 : \mathbb{R} \to \tilde{\mathfrak{L}}_1$ is the constant map, and we identify $\phi_0 = \phi_0(1)$. Explicitly, the lowest expressions of (2.32) read as

$$\begin{gathered}
0 \;=\; \sum_{i\geqslant 1} \frac{1}{i!}\tilde{\mu}_i(\phi_0,\ldots,\phi_0)\;,\\
\phi_1(\mu_1(\ell_1)) \;=\; \tilde{\mu}_1(\phi_1(\ell_1)) + \sum_{i\geqslant 1}\frac{1}{i!}\tilde{\mu}_{i+1}(\phi_0,\ldots,\phi_0,\phi_1(\ell_1))\;,\\
\phi_1(\mu_2(\ell_1,\ell_2)) - \phi_2(\mu_1(\ell_1),\ell_2) + (-1)^{|\ell_1|_{\mathfrak{L}}|\ell_2|_{\mathfrak{L}}}\phi_2(\mu_1(\ell_2),\ell_1) \;=\\
=\; \tilde{\mu}_1(\phi_2(\ell_1,\ell_2)) + \tilde{\mu}_2(\phi_1(\ell_1),\phi_1(\ell_2)) +\\
+\sum_{i\geqslant 1}\frac{1}{i!}\tilde{\mu}_{i+1}(\phi_0,\ldots,\phi_0,\phi_2(\ell_1,\ell_2)) + \sum_{i\geqslant 1}\frac{1}{i!}\tilde{\mu}_{i+2}(\phi_0,\ldots,\phi_0,\phi_1(\ell_1),\phi_1(\ell_2))\;,\\
\vdots
\end{gathered} \tag{2.33}$$

It is easily seen that this definition reduces to the standard definition of a Lie algebra morphism in the context of Lie algebras. Note that a curved morphism is simply called an <u>*(uncurved) morphism (of L_∞-algebras)*</u> whenever $\phi_0 = 0$, and this notion of morphisms is usually used in the literature when discussing L_∞-algebras. As we will see below, we shall need the more general notion of curved morphisms to reinterpret gauge transformations as morphisms of L_∞-algebras.

Evidently, the first equation of (2.33) implies that ϕ_0 is necessarily a homotopy Maurer–Cartan element of $\tilde{\mathfrak{L}}$. For such ϕ_0, we now set

$$\tilde{\mu}_i^{\phi_0}(\tilde{\ell}_1,\ldots,\tilde{\ell}_i) \;:=\; \sum_{j\geqslant 0}\frac{1}{j!}\tilde{\mu}_{i+j}(\phi_0,\ldots,\phi_0,\tilde{\ell}_1,\ldots,\tilde{\ell}_i) \tag{2.34}$$

for all $\tilde{\ell}_1,\ldots,\tilde{\ell}_i \in \tilde{\mathfrak{L}}$ and $i \in \mathbb{N}^+$. By virtue of (2.31), we immediately have that $\tilde{\mu}_1^{\phi_0}$ is a differential. In fact, one can show that $(\tilde{\mathfrak{L}}, \tilde{\mu}_i^{\phi_0})$ forms an L_∞-algebra, that is, the $\tilde{\mu}_i^{\phi_0}$ satisfy the homotopy Jacobi identities (2.17) thus defining another L_∞-structure on $\tilde{\mathfrak{L}}$. From (2.32) we may then conclude that any curved morphism between two L_∞-algebras $(\mathfrak{L},\mu_i)$ and $(\tilde{\mathfrak{L}},\tilde{\mu}_i)$ can be viewed as an uncurved morphism between $(\mathfrak{L},\mu_i)$ and $(\tilde{\mathfrak{L}},\tilde{\mu}_i^{\phi_0})$.

Maurer–Cartan elements and curved morphisms. Consider $a \in \mathfrak{L}_1$ and let $f \in \mathfrak{L}_2$ be its curvature (2.22). We define the image of a gauge potential under a curved morphism $\phi : (\mathfrak{L},\mu_i) \to (\tilde{\mathfrak{L}},\tilde{\mu}_i)$ as

$$\tilde{a} \;:=\; \phi_0 + \phi_1(a) + \tfrac{1}{2}\phi_2(a,a) + \cdots \;=\; \sum_{i\geqslant 0}\frac{1}{i!}\phi_i(a,\ldots,a) \;\in\; \tilde{\mathfrak{L}}_1\;. \tag{2.35}$$

The curvature of $\tilde{a}$ is then

$$\tilde{f} \;=\; \sum_{i\geqslant 1}\frac{1}{i!}\tilde{\mu}_i(\tilde{a},\ldots,\tilde{a}) \;=\; \sum_{i\geqslant 0}\frac{1}{i!}\phi_{i+1}(a,\ldots,a,f) \;\in\; \tilde{\mathfrak{L}}_2\;, \tag{2.36}$$

which one can verify using (2.17) and (2.32). Hence, homotopy Maurer–Cartan elements in $\mathfrak{L}$ are mapped to homotopy Maurer–Cartan elements in $\tilde{\mathfrak{L}}$.

Let us extend the above observation to gauge orbits. Consider gauge transformations (2.25) $a \mapsto a+\delta_{c_0}a$ and $\tilde{a}\mapsto\tilde{a}+\delta_{\tilde{c}_0}\tilde{a}$ with the image of the gauge parameter $c_0\in\mathfrak{L}_0$ given by

$$\tilde{c}_0 \;:=\; \phi_1(c_0)+\phi_2(a,c_0)+\cdots \;=\; \sum_{i\geqslant 0}\frac{1}{i!}\phi_{i+1}(a,\ldots,a,c_0) \;\in\; \tilde{\mathfrak{L}}_0\;. \tag{2.37}$$

A short calculation involving (2.17) reveals that

$$\delta_{\tilde{c}_0}\tilde{a} \;=\; -\sum_{i\geqslant 0}\frac{1}{i!}\phi_{i+2}(a,\ldots,a,f,c_0)+\sum_{i\geqslant 0}\frac{1}{i!}\phi_{i+1}(\delta_{c_0}a,a,\ldots,a)\;. \tag{2.38}$$

This immediately yields

$$\begin{aligned}\sum_{i\geqslant 0}\frac{1}{i!}\phi_i(a+\delta_{c_0}a,\ldots,a+\delta_{c_0}a) \;&=\; \sum_{i\geqslant 0}\frac{1}{i!}\phi_i(a,\ldots,a)+\sum_{i\geqslant 0}\frac{1}{i!}\phi_{i+1}(\delta_{c_0}a,a,\ldots,a)\\ &=\; \tilde{a}+\delta_{\tilde{c}_0}\tilde{a}+\sum_{i\geqslant 0}\frac{1}{i!}\phi_{i+2}(a,\ldots,a,f,c_0)\end{aligned} \tag{2.39}$$

at linear order. Consequently, gauge equivalence classes of homotopy Maurer–Cartan elements in $\mathfrak{L}$ are mapped to gauge equivalence classes of homotopy Maurer–Cartan elements in $\tilde{\mathfrak{L}}$ under (curved) morphisms.

Morphisms of cyclic L_∞-algebras. Consider an uncurved morphism between two L_∞-algebras $(\mathfrak{L},\mu_i)$ and $(\tilde{\mathfrak{L}},\tilde{\mu}_i)$, that is, a curved morphism with $\phi_0=0$. If, in addition, we have inner products $\langle -,-\rangle_{\mathfrak{L}}$ on $\mathfrak{L}$ and $\langle -,-\rangle_{\tilde{\mathfrak{L}}}$ on $\tilde{\mathfrak{L}}$, then a morphism of cyclic L_∞-algebras has to satisfy

$$\langle \ell_1,\ell_2\rangle_{\mathfrak{L}} \;=\; \langle\phi_1(\ell_1),\phi_1(\ell_2)\rangle_{\tilde{\mathfrak{L}}} \tag{2.40a}$$

for all $\ell_{1,2}\in\mathfrak{L}$ and for all $i\geqslant 3$ and $\ell_1,\ldots,\ell_i\in\mathfrak{L}$

$$\sum_{\substack{i_1+i_2=i\\ i_1,i_2\geqslant 1}}\langle\phi_{i_1}(\ell_1,\ldots,\ell_{i_1}),\phi_{i_2}(\ell_{i_1+1},\ldots,\ell_i)\rangle_{\tilde{\mathfrak{L}}} \;=\; 0\;. \tag{2.40b}$$

We note that the morphisms of cyclic L_∞-algebras defined here require ϕ_1 to be injective. More general notions of such morphisms can be defined using Lagrangian correspondences, cf. [193].

Suppose now that the inner product $\langle -,-\rangle_{\mathfrak{L}}$ on $\mathfrak{L}$ and $\langle -,-\rangle_{\tilde{\mathfrak{L}}}$ on $\tilde{\mathfrak{L}}$ of degree -3 so that the homotopy Maurer–Cartan equations, $f=0$ and $\tilde{f}=0$, are variational. Then, under a morphism $\phi:(\mathfrak{L},\mu_i)\to(\tilde{\mathfrak{L}},\tilde{\mu}_i)$, we obtain

$$\begin{aligned}\sum_{i\geqslant 1}\frac{1}{(i+1)!}\langle a,\mu_i(a,\ldots,a)\rangle_{\mathfrak{L}} &= S^{\mathrm{hMC}}[a]\\ &= \tilde{S}^{\mathrm{hMC}}[\tilde{a}] \;=\; \sum_{i\geqslant 1}\frac{1}{(i+1)!}\langle \tilde{a},\tilde{\mu}_i(\tilde{a},\ldots,\tilde{a})\rangle_{\tilde{\mathfrak{L}}}\end{aligned} \tag{2.41}$$

by virtue of (2.40) and (2.35).

Curved quasi-isomorphisms of L_∞-algebras. Recall that the homotopy Jacobi identities (2.17) (see also (2.18)) imply that $\mu_1^2=0$. Hence, we may consider the cohomology

$$H^\bullet_{\mu_1}(\mathfrak{L}) = \bigoplus_{k\in\mathbb{Z}} H^k_{\mu_1}(\mathfrak{L}) \quad\text{with}\quad H^k_{\mu_1}(\mathfrak{L}) := \ker(\mu_1|_{\mathfrak{L}_k})/\mathrm{im}(\mu_1|_{\mathfrak{L}_{k-1}})\ . \tag{2.42}$$

A curved morphism of L_∞-algebras $\phi:(\mathfrak{L},\mu_i)\to(\tilde{\mathfrak{L}},\tilde{\mu}_i)$ is called a *curved quasi-isomorphism (of L_∞-algebras)* whenever ϕ_1 induces an isomorphism $H^\bullet_{\mu_1}(\mathfrak{L})\cong H^\bullet_{\tilde{\mu}_1^{\phi_0}}(\tilde{\mathfrak{L}})$; the products $\tilde{\mu}_i^{\phi_0}$ were defined in Equation (2.34). There is a bijection between the moduli spaces of gauge equivalence classes of homotopy Maurer–Cartan elements of $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$: indeed, every quasi-isomorphism admits an inverse, and by means of this one can show that the moduli spaces are equivalent, see [191, 40, 194]. A curved quasi-isomorphism is called an *(uncurved) quasi-isomorphism* whenever $\phi_0=0$. A (uncurved) quasi-isomorphism is called an *(uncurved) isomorphism* if ϕ_1 is invertible.

Gauge transformations as curved morphisms. Let us revisit the infinitesimal gauge transformations (2.25) and first explain how they arise from partially flat homotopies. In particular, set $I:=[0,1]\subseteq\mathbb{R}$ and consider the tensor product

$$\mathfrak{L}_\Omega := \Omega^\bullet(I)\otimes\mathfrak{L} = \bigoplus_{k\in\mathbb{Z}}(\mathfrak{L}_\Omega)_k \quad\text{with}\quad (\mathfrak{L}_\Omega)_k = \mathscr{C}^\infty(I)\otimes\mathfrak{L}_k\oplus\Omega^1(I)\otimes\mathfrak{L}_{k-1} \tag{2.43}$$

between the de Rham complex $(\Omega^\bullet(I), \mathrm{d})$ on the interval I and an L_∞-algebra $(\mathfrak{L}, \mu_i)$. $\mathfrak{L}_\Omega$ carries an L_∞-structure, given by

$$\hat{\mu}_1(\alpha_1 \otimes \ell_1) \;:=\; \mathrm{d}\alpha_1 \otimes \ell_1 + (-1)^{|\alpha_1|_{\Omega^\bullet(I)}} \alpha_1 \otimes \mu_1(\ell_1) \tag{2.44a}$$

and

$$\begin{aligned}\hat{\mu}_i(\alpha_1 \otimes \ell_1, \ldots, \alpha_i \otimes \ell_i) \;:=\;& (-1)^{i\sum_{j=1}^{i}|\alpha_j|_{\Omega^\bullet(I)} + \sum_{j=0}^{i-2}|\alpha_{i-j}|_{\Omega^\bullet(I)}\sum_{k=1}^{i-j-1}|\ell_k|_{\mathfrak{L}}} \times \\ &\times\; (\alpha_1 \wedge \cdots \wedge \alpha_i) \otimes \mu_i(\ell_1, \ldots, \ell_i)\;,\end{aligned} \tag{2.44b}$$

where $\alpha_1, \ldots, \alpha_i \in \Omega^\bullet(I)$ of degree $|\alpha_1|_{\Omega^\bullet(I)}, \ldots, |\alpha_i|_{\Omega^\bullet(I)}$, and $\ell_1, \ldots, \ell_i \in \mathfrak{L}$. A general element $\mathsf{a} \in (\mathfrak{L}_\Omega)_1$ is of the form $\mathsf{a}(t) = a(t) + \mathrm{d}t \otimes c_0(t)$ with $a(t) \in \mathscr{C}^\infty(I) \otimes \mathfrak{L}_1$ and $c_0(t) \in \mathscr{C}^\infty(I) \otimes \mathfrak{L}_0$. Its curvature $\mathsf{f} \in (\mathfrak{L}_\Omega)_2$ is then

$$\mathsf{f}(t) \;=\; f(t) + \mathrm{d}t \otimes \left\{ \frac{\partial a(t)}{\partial t} - \sum_{i\geqslant 0} \frac{1}{i!}\mu_{i+1}(a(t), \ldots, a(t), c_0(t)) \right\}, \tag{2.45}$$

where $f(t) \in \mathscr{C}^\infty(I) \otimes \mathfrak{L}_2$ is the curvature of $a(t)$. The requirement of partial flatness of $\mathsf{f}(t)$ amounts to saying that $\mathsf{f}(t)$ has no components along $\mathrm{d}t$. Thus,

$$\frac{\partial a(t)}{\partial t} \;=\; \sum_{i\geqslant 0} \frac{1}{i!}\mu_{i+1}(a(t), \ldots, a(t), c_0(t)) \tag{2.46}$$

and we recover the gauge transformations (2.25) from

$$\delta_{c_0} a \;=\; \left.\frac{\partial a(t)}{\partial t}\right|_{t=0} \tag{2.47}$$

with $a = a(0)$ and $c_0 = c_0(0)$. Furthermore, upon solving the ordinary differential equation (2.46), we will obtain finite gauge transformations. Let us now explain how one can understand this as a curved morphism that preserves the products μ_i.

Concretely, we consider (2.35) and (2.37) and make the ansatz

$$a(t) \;:=\; \sum_{i\geqslant 0} \frac{1}{i!}\phi_i(t)(a, \ldots, a) \quad \text{and} \quad c_0(t) \;:=\; \sum_{i\geqslant 0} \frac{1}{i!}\phi_{i+1}(t)(a, \ldots, a, c_0)\;. \tag{2.48}$$

Here, we again set $a = a(0)$ and $c_0 = c_0(0)$ which, in turn, translates to the conditions $\phi_i(0) = 0$ for all $i \neq 1$ and $\phi_1(0) = 1$. Upon substituting the ansatz (2.48) into (2.46) and remembering (2.38), we obtain

$$\begin{aligned}\frac{\partial a(t)}{\partial t} \;&=\; \sum_{i\geqslant 0} \frac{1}{i!}\frac{\partial \phi_i(t)}{\partial t}(a, \ldots, a) \\ &=\; -\sum_{i\geqslant 0}\frac{1}{i!}\phi_{i+2}(t)(a, \ldots, a, f, c_0) + \sum_{i\geqslant 0}\frac{1}{i!}\phi_{i+1}(t)(\delta_{c_0} a, a, \ldots, a)\;,\end{aligned} \tag{2.49}$$

where f is the curvature of a. Thus, solving the ordinary differential equation (2.46) for gauge transformations is equivalent to solving the ordinary differential equation (2.49) for a curved morphism ϕ_i on the L_∞-algebra that preserves the L_∞-algebra structure. Put differently, finite gauge transformations are given by curved morphisms that arise as solutions to (2.49).

Let us exemplify these discussions by considering a standard Lie algebra valued one-form gauge potential on Minkowski space $\mathbb{M}^d$. Here, $a = A \in \Omega^1(\mathbb{M}^d) \otimes \mathfrak{g}$ and $c_0 = c \in \mathscr{C}^\infty(\mathbb{M}^d) \otimes \mathfrak{g}$ for a Lie algebra $\mathfrak{g}$. Moreover, in this case it is enough to consider $\phi_0(t)$ and $\phi_1(t)$ and set $\phi_i(t) = 0$ for all $i > 1$. Consequently, the ordinary differential equation (2.49) reduces to

$$\frac{\partial A(t)}{\partial t} \;=\; \frac{\partial \phi_0(t)}{\partial t} + \frac{\partial \phi_1(t)}{\partial t}(A) \;=\; \phi_1(t)(\mathrm{d}c + [A, c]) \tag{2.50}$$

and is solved by $A(t) = \phi_0(t) + \phi_1(t)(A)$ and $c(t) = \phi_1(t)(c)$ with[1]

$$\begin{aligned}
\phi_0(t) &= t\mathrm{d}c + \tfrac{t^2}{2!}[\mathrm{d}c, c] + \tfrac{t^3}{3!}[[\mathrm{d}c, c], c] + \cdots = \mathrm{e}^{-tc}\,\mathrm{d}\mathrm{e}^{tc} \,, \\
\phi_1(t)(A) &= A + t[A, c] + \tfrac{t^2}{2!}[[A, c], c] + \tfrac{t^3}{3!}[[[A, c], c], c] + \cdots = \mathrm{e}^{-tc}\,A\,\mathrm{e}^{tc} \,, \\
\phi_1(t)(c) &= c
\end{aligned} \tag{2.51}$$

as a short calculation reveals; recall from Equation (2.33) that $\phi_0(t)$ must be a homotopy Maurer–Cartan element.

2.4. Structure theorems

In the following, the term 'homotopy algebra' refers to either an A_∞-, C_∞-, or L_∞-algebra[2]. Note that the unary higher product is a differential for any homotopy algebra. We call a homotopy algebra *minimal* provided the unary product vanishes. A homotopy algebra is called *strict* if only the unary and binary products are non-vanishing. Moreover, a homotopy algebra is called *linearly contractible* if only the unary product is nonvanishing and it has trivial cohomology.

[1] We can also consider the more general case $\phi_0(t) = g^{-1}(t)\,\mathrm{d}g(t)$, $\phi_1(t)(A) = g^{-1}(t)\,A\,g(t)$, and $\phi_1(t)(c) = g^{-1}(t)\,\partial_t g(t)$ for $g \in \mathscr{C}^\infty(I, \mathsf{G})$ with $g(0) = 1$, that is, g solves the ordinary differential equation $\partial_t g(t) = g(t)\,c(t)$; note that $\partial_t g(t)|_{t=0} = c$.

[2] The notions of morphism, quasi-isomorphism and isomorphism for A_∞-algebras are analogous to their L_∞-algebras counterparts.

Structure theorems. We now have the following structure theorems:

1. The *decomposition theorem*: any homotopy algebra is isomorphic to the direct sum of a minimal and a linearly contractible one; see e.g. [192] for the case of A_∞-algebras.
2. The *minimal model theorem*: any homotopy algebra is quasi-isomorphic to a minimal one. This follows directly from the decomposition theorem, see also [195, 192] for the case of L_∞-algebras.
3. The *strictification theorem*: any homotopy algebra is quasi-isomorphic to a strict one [196, 197].

We note that strict A_∞-, C_∞-, and L_∞-algebras are simply differential graded associative, differential graded commutative, and differential graded Lie algebras, respectively. We also note that mathematicians would probably use the term ‘rectify’ over ‘strictify’; we found the latter term more descriptive.

Remark 2.1. *We also would like to make a few remarks on the relations between the homotopy algebras:*

1. *As we saw above in Equation (2.19), any A_∞-algebra carries an L_∞-structure by (graded) anti-symmetrisation the higher products.*
2. *All higher products of a C_∞-algebra (which is also in particular an A_∞-algebra) except for the differential vanish after anti-symmetrisation.*

3

Batalin–Vilkovisky formalism

In the following, we summarise how perturbative quantum field theory is naturally formulated in the language of homotopy algebras. The bridge between field theories and homotopy algebras is provided by the Batalin–Vilkovisky (BV) formalism [8, 198]. Our discussion follows the treatment in [52, 1]; see also [4] for a pedagogical summary and [199] for a detailed discussion of Feynman diagrams. We start with the Becchi–Rouet–Stora–Tyutin (BRST) formalism for the archetypal example of Yang–Mills theory. This will also prepare our discussion in Chapter 4.

The material in this Chapter is borrowed from [6].

3.1. Motivation

Yang–Mills action. We consider d-dimensional Minkowski space $\mathbb{M}^d := \mathbb{R}^{1,d-1}$ with metric $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, \ldots, 1)$ with $\mu, \nu, \ldots = 0, 1, \ldots, d-1$ and local coordinates x^μ. Let $\mathfrak{g}$ be a semi-simple compact matrix Lie algebra with basis e_a with $a, b, \ldots = 1, 2, \ldots, \dim(\mathfrak{g})$, $[\mathrm{e}_a, \mathrm{e}_b] = f_{ab}{}^c \mathrm{e}_c$ with $[-,-]$ the Lie bracket on $\mathfrak{g}$, and $\langle \mathrm{e}_a, \mathrm{e}_b \rangle := -\mathrm{tr}(\mathrm{e}_a \mathrm{e}_b) = \delta_{ab}$ with 'tr' the matrix trace.

The action for Yang–Mills theory in R_ξ-gauge for some real constant ξ in the BRST formalism reads as

$$S^{\mathrm{YM}}_{\mathrm{BRST}} := \int \mathrm{d}^d x \left\{ -\tfrac{1}{4} F_{a\mu\nu} F^{a\mu\nu} - \bar{c}_a \partial^\mu (\nabla_\mu c)^a + \tfrac{\xi}{2} b_a b^a + b_a \partial^\mu A^a_\mu \right\} \tag{3.1a}$$

with

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^a g A^b_\mu A^c_\nu \quad \text{and} \quad (\nabla_\mu c)^a := \partial_\mu c^a + g f_{bc}{}^a A^b_\mu c^c \ , \tag{3.1b}$$

where g is the Yang–Mills coupling constant, A^a_μ are the components of the $\mathfrak{g}$-valued one-form gauge potential on $\mathbb{M}^d$, and c^a, b^a, and $\bar{c}^a$ are the components of $\mathfrak{g}$-valued functions corresponding to the ghost, the Nakanishi–Lautrup field, and the anti-ghost field, respectively.

$\mathbb{Z}$-graded vector spaces. We note that the fields in the action (3.1a) are graded by their *ghost number* as detailed in Table 3.1. Therefore, we should view them as coordinate functions on a *$\mathbb{Z}$-graded vector space* $\mathfrak{V} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{V}_k$. Elements of $\mathfrak{V}_k$ are said to be *homogeneous* of degree k, and we shall use the notation $|\ell|_{\mathfrak{V}}$ to denote the degree of a homogeneous element $\ell \in \mathfrak{V}$.

field Φ^I	c^a	A^a_μ	b^a	$\bar{c}^a$
ghost number $\lvert\Phi^I\rvert_{\text{gh}}$	1	0	0	-1

Table 3.1: Ghost numbers of the fields in Yang–Mills theory.

The *tensor product* of two $\mathbb{Z}$-graded vector spaces $\mathfrak{V}$ and $\mathfrak{W}$ is defined as

$$\mathfrak{V} \otimes \mathfrak{W} = \bigoplus_{k\in\mathbb{Z}} (\mathfrak{V} \otimes \mathfrak{W})_k \quad \text{with} \quad (\mathfrak{V} \otimes \mathfrak{W})_k := \bigoplus_{i+j=k} \mathfrak{V}_i \otimes \mathfrak{W}_j \ , \tag{3.2}$$

and the degree in $\mathfrak{V} \otimes \mathfrak{W}$ is thus the sum of the degrees in $\mathfrak{V}$ and $\mathfrak{W}$.

We shall denote the *dual* of a $\mathbb{Z}$-graded vector space $\mathfrak{V}$ by $\mathfrak{V}^*$,[1] and we have

$$\mathfrak{V}^* = \bigoplus_{k\in\mathbb{Z}} (\mathfrak{V}^*)_k \quad \text{with} \quad (\mathfrak{V}^*)_k := (\mathfrak{V}_{-k})^* \ . \tag{3.3}$$

In particular, elements in $\mathfrak{V}_k$ have the opposite degree of elements in $(\mathfrak{V}_k)^*$.

Given a $\mathbb{Z}$-graded vector space $\mathfrak{V}$, we can introduce the *degree-shifted $\mathbb{Z}$-graded vector space* $\mathfrak{V}[l]$ for $l \in \mathbb{Z}$ by

$$\mathfrak{V}[l] = \bigoplus_{k\in\mathbb{Z}} (\mathfrak{V}[l])_k \quad \text{with} \quad (\mathfrak{V}[l])_k := \mathfrak{V}_{k+l} \ . \tag{3.4}$$

For an ordinary vector space $\mathfrak{V} \equiv \mathfrak{V}_0$, for instance, $\mathfrak{V}[1]$ consists of elements of degree -1 since only $(\mathfrak{V}[1])_{-1} = \mathfrak{V}_0$ is non-trivial. Note that $(\mathfrak{V} \otimes \mathfrak{W})[l] = \mathfrak{V}[l] \otimes \mathfrak{W} = \mathfrak{V} \otimes \mathfrak{W}[l]$

[1] We will not discuss the analytical subtleties of this construction in the infinite-dimensional case, except to note that the dual spaces will be degree-wise topological duals.

and $(\mathfrak{V}[l])^* = \mathfrak{V}^*[-l]$ for all $l \in \mathbb{Z}$. For convenience, we introduce the notion of a *shift isomorphism*

$$\sigma : \mathfrak{V} \rightarrow \mathfrak{V}[1] \tag{3.5}$$

which lowers the degree of every element of $\mathfrak{V}$, that is, $\sigma : \mathfrak{V}_k \rightarrow (\mathfrak{V}[1])_{k-1}$.

We note that the action (3.1a) is built of polynomial functions and their derivatives. By the algebra of *polynomial functions* on a $\mathbb{Z}$-graded vector space $\mathfrak{V}$, we mean the $\mathbb{Z}$-graded symmetric tensor algebra $\mathscr{C}^\infty(\mathfrak{V}) := \bigodot^\bullet \mathfrak{V}^*$. Basis elements of $\mathfrak{V}^*$ can be regarded as the coordinate functions on $\mathfrak{V}$. Explicitly, such a function looks like

$$\mathsf{f} = f + \xi^\alpha f_\alpha + \tfrac{1}{2}\xi^\alpha \xi^\beta f_{\alpha\beta} + \cdots \in \mathscr{C}^\infty(\mathfrak{V}) , \tag{3.6}$$

where ξ^α are basis elements of $\mathfrak{V}^*$ and $f, f_\alpha, f_{\alpha\beta}, \ldots$ are constants. We have $\xi^\alpha\xi^\beta = (-1)^{|\xi^\alpha|_{\mathfrak{V}^*}|\xi^\beta|_{\mathfrak{V}^*}}\xi^\beta\xi^\alpha$. Note that if $\mathfrak{V}$ is a vector space of some suitably smooth functions or, more generally, sections of some vector bundle, then the dual $\mathfrak{V}^*$, being the space of distributions, contains not only the ordinary dual coordinate functions but also all of their derivatives.

BRST operator in Yang–Mills theory. The reason for introducing ghosts in the first place is the gauge symmetry of Yang–Mills theory, which in the BRST and BV formalisms is captured in a dual formulation as a differential on a differential graded commutative algebra that is called the *Chevalley–Eilenberg algebra*. More specifically, this is the algebra of polynomial functions, and the differential is a nilquadratic vector field $Q : \mathscr{C}^\infty(\mathfrak{V}) \rightarrow \mathscr{C}^\infty(\mathfrak{V})$ of degree one, $Q^2 = 0$, known as the *homological vector field*. A $\mathbb{Z}$-graded vector space with such a homological vector field is called a *Q-vector space*.

The prime example of a Q-vector space is that of an ordinary vector space $\mathfrak{g}$ with basis e_a for $a, b, \ldots = 1, \ldots, \dim(\mathfrak{g})$, regarded as the $\mathbb{Z}$-graded vector space $\mathfrak{g}[1]$. On $\mathfrak{g}[1]$, we have coordinates ξ^a only in degree one and thus, the most general vector field $Q : \mathscr{C}^\infty(\mathfrak{g}[1]) \rightarrow \mathscr{C}^\infty(\mathfrak{g}[1])$ of degree one is of the form

$$Q := \tfrac{1}{2}\xi^b\xi^c f_{cb}{}^a \frac{\partial}{\partial \xi^a} \quad \Rightarrow \quad Q\xi^a = \tfrac{1}{2}\xi^b\xi^c f_{cb}{}^a \tag{3.7}$$

for some constants $f_{ab}{}^c = -f_{ba}{}^c$. The condition $Q^2 = 0$ is equivalent to the Jacobi identity for the $f_{ab}{}^c$ so that Q induces a Lie bracket $[\mathsf{e}_a, \mathsf{e}_b] = f_{ab}{}^c \mathsf{e}_c$ on $\mathfrak{g}$. The differential graded algebra $(\mathscr{C}^\infty(\mathfrak{g}[1]), Q)$ is the Chevalley–Eilenberg algebra of the Lie algebra $(\mathfrak{g}, [-, -])$ to which we alluded above. In order to translate between Q and $[-,-]$, it is useful to define the <u>*contracted coordinate functions*</u>[1]

$$\mathsf{a} \;:=\; \xi^a \otimes \mathsf{e}_a \;\in\; (\mathfrak{g}[1])^* \otimes \mathfrak{g} \tag{3.8}$$

of degree one in $(\mathfrak{g}[1])^* \otimes \mathfrak{g}$. Consequently,

$$\begin{aligned} Q\mathsf{a} \;&:=\; (Q\xi^a) \otimes \mathsf{e}_a \\ &=\; \tfrac{1}{2}\xi^b\xi^c f_{cb}{}^a \otimes \mathsf{e}_a \\ &=\; -\tfrac{1}{2}\xi^b\xi^c \otimes f_{bc}{}^a \mathsf{e}_a \\ &=\; -\tfrac{1}{2}\xi^b\xi^c \otimes [\mathsf{e}_b, \mathsf{e}_c] \\ &=: \; -\tfrac{1}{2}[\xi^b \otimes \mathsf{e}_b, \xi^c \otimes \mathsf{e}_c] \\ &=\; -\tfrac{1}{2}[\mathsf{a}, \mathsf{a}] \;. \end{aligned} \tag{3.9}$$

More general vector fields arise in the Chevalley–Eilenberg algebras of L_∞-algebras and L_∞-algebroids, cf. e.g. [52] for further details. In the case of Yang–Mills theory, the homological vector field $Q^{\mathrm{YM}}_{\mathrm{BRST}}$ describing the gauge symmetry acts according to

$$\begin{aligned} c^a \;&\xmapsto{Q^{\mathrm{YM}}_{\mathrm{BRST}}}\; -\tfrac{1}{2} g f_{bc}{}^a c^b c^c \;, \qquad & \bar{c}^a \;&\xmapsto{Q^{\mathrm{YM}}_{\mathrm{BRST}}}\; b^a \;, \\ A^a_\mu \;&\xmapsto{Q^{\mathrm{YM}}_{\mathrm{BRST}}}\; (\nabla_\mu c)^a \;, & b^a \;&\xmapsto{Q^{\mathrm{YM}}_{\mathrm{BRST}}}\; 0 \;. \end{aligned} \tag{3.10}$$

These transformations are known as the <u>*BRST transformations*</u> and $Q^{\mathrm{YM}}_{\mathrm{BRST}}$ as the <u>*BRST operator*</u>. One readily verifies that $(Q^{\mathrm{YM}}_{\mathrm{BRST}})^2 = 0$, that is, Q_{BRST} is a differential. In addition, the action (3.1a) is $Q^{\mathrm{YM}}_{\mathrm{BRST}}$-closed, that is, $Q^{\mathrm{YM}}_{\mathrm{BRST}} S^{\mathrm{YM}}_{\mathrm{BRST}} = 0$, which ensures gauge choice independence.

We shall denote the minimal field space[2] for gauge-fixed Yang–Mills theory by $\mathfrak{L}^{\mathrm{YM}}_{\mathrm{BRST}}$, but the ghost number is the degree of coordinate functions on $\mathfrak{L}^{\mathrm{YM}}_{\mathrm{BRST}}[1]$. Explicitly,

$$\begin{gathered} \mathfrak{L}^{\mathrm{YM}}_{\mathrm{BRST}} \;=\; \mathfrak{L}^{\mathrm{YM}}_{\mathrm{BRST},0} \oplus \mathfrak{L}^{\mathrm{YM}}_{\mathrm{BRST},1} \oplus \mathfrak{L}^{\mathrm{YM}}_{\mathrm{BRST},2} \;, \\ \mathfrak{L}^{\mathrm{YM}}_{\mathrm{BRST},0} \;:=\; \mathscr{C}^\infty(\mathbb{M}^d) \otimes \mathfrak{g} \;, \quad \mathfrak{L}^{\mathrm{YM}}_{\mathrm{BRST},1} \;:=\; (\Omega^1(\mathbb{M}^d) \oplus \mathscr{C}^\infty(\mathbb{M}^d)) \otimes \mathfrak{g} \;, \\ \mathfrak{L}^{\mathrm{YM}}_{\mathrm{BRST},2} \;:=\; \mathscr{C}^\infty(\mathbb{M}^d) \otimes \mathfrak{g} \end{gathered} \tag{3.11}$$

[1]These are often used in the string field theory literature, albeit shifted such that a is of degree zero.

[2]This graded vector space is, in fact, the space of sections of a graded vector bundle, and fields and their derivatives are sections of the corresponding jet bundle; but these details would not enlighten our discussion any further so we suppress them.

and c, A, b, and $\bar{c}$ are coordinate functions on $(\mathfrak{L}^{\rm YM}_{\rm BRST}[1])_{-1}$, $(\mathfrak{L}^{\rm YM}_{\rm BRST}[1])_0$, $(\mathfrak{L}^{\rm YM}_{\rm BRST}[1])_0$, and $(\mathfrak{L}^{\rm YM}_{\rm BRST}[1])_1$ and thus of degrees 1, 0, 0, and -1, respectively. Moreover, the action (3.1a) is a polynomial function $S^{\rm YM}_{\rm BRST} \in \mathscr{C}^\infty(\mathfrak{L}^{\rm YM}_{\rm BRST}[1])$ on $\mathfrak{L}^{\rm YM}_{\rm BRST}[1]$ of total ghost number zero, $|S^{\rm YM}_{\rm BRST}|_{\mathscr{C}^\infty(\mathfrak{L}^{\rm YM}_{\rm BRST}[1])} = 0$. In the following, we shall write $|-|_{\rm gh}$ as a shorthand for both $|-|_{(\mathfrak{L}^{\rm YM}_{\rm BRST}[1])^*}$ and $|-|_{\mathscr{C}^\infty(\mathfrak{L}^{\rm YM}_{\rm BRST}[1])}$.

The Q-vector space $(\mathfrak{L}^{\rm YM}_{\rm BRST}[1], Q^{\rm YM}_{\rm BRST})$ describes the Lie algebra of gauge transformations as well as its action on the various fields, which together form an action Lie algebroid. This becomes clear when comparing (3.10) to (3.9); the latter is the evident generalisation, e.g. to the corresponding formulas for a differential graded Lie algebra.

We note that gauge-invariant objects are $Q^{\rm YM}_{\rm BRST}$-closed and that gauge-trivial objects are $Q^{\rm YM}_{\rm BRST}$-exact. Therefore, physical observables are in the cohomology of $Q_{\rm BRST}$. The pair of fields $(b, \bar{c})$ is known as a *trivial pair*, as $Q^{\rm YM}_{\rm BRST}$ links the two fields by an identity map. They vanish in the $Q^{\rm YM}_{\rm BRST}$-cohomology and thus are not observable.

As in Equation (3.8), it will turn out useful to define the contracted coordinates

$$\begin{aligned} \mathsf{a} \;:=\; \int \mathrm{d}^d x \Big\{ & c^a(x) \otimes (\mathsf{e}_a \otimes \mathsf{s}_x) + A^a_\mu(x) \otimes (\mathsf{e}_a \otimes \mathsf{v}^\mu \otimes \mathsf{s}_x) + \\ & + b^a(x) \otimes (\mathsf{e}_a \otimes \mathsf{s}_x) + \bar{c}^a(x) \otimes (\mathsf{e}_a \otimes \mathsf{s}_x) \Big\}\,, \end{aligned} \tag{3.12a}$$

where e_a, v^μ, and s_x are basis vectors on $\mathfrak{g}$, $T^*_x\mathbb{M}^d$, and $\mathscr{C}^\infty(\mathbb{M}^d)$, respectively (and thus, we have an identification $\mathsf{v}^\mu \;\hat{=}\; \mathrm{d}x^\mu$). It should be noted that a is an element of $(\mathfrak{L}^{\rm YM}_{\rm BRST}[1])^* \otimes \mathfrak{L}^{\rm YM}_{\rm BRST}$ of degree one, and it can be regarded as a superfield which contains all the fields of different ghost numbers. The component fields can be recovered by projecting onto the respective ghost numbers. In the following, we will write symbolically

$$\mathsf{a} \;=\; \Phi^I \otimes \mathsf{e}_I \tag{3.12b}$$

for DeWitt indices $I, J, \ldots$, which contain Lorentz and gauge indices as well as space-time position. A contraction of DeWitt indices involves sums over all discrete indices and evident integrals over the continuous ones.

3.2. Batalin–Vilkovisky formalism and L_∞-algebras

The above example of Yang–Mills theory has demonstrated how $\mathbb{Z}$-graded vector spaces and homological vector fields enter into the description of a gauge field theory in the BRST formalism. In particular, gauge-invariant observables were contained in the cohomology of Q_{BRST}. To fully characterise classical observables, however, we also need to impose the equations of motion. This is the purpose of the more general Batalin–Vilkovisky (BV) formalism. As a byproduct, the BV formalism can cater for *open* gauge symmetries which are gauge symmetries for which Q_{BRST} is a differential only on-shell. The BV operator Q_{BV}, which generalises the BRST operator Q_{BRST}, encodes the Chevalley–Eilenberg description of a cyclic L_∞-algebra (i.e. an L_∞-algebra with a notion of inner product). The gauge-fixed form of this cyclic L_∞-algebra will be crucial for our formulation of the double copy of amplitudes.

BV operator. Let $\mathfrak{L}_{\mathsf{BRST}}[1]$ be a $\mathbb{Z}$-graded vector space of fields of a general field theory. Then we have also a correspondence between the fields and the coordinate functions on this space. In order to encode the field equations for all the fields in the action of an operator Q_{BV}, we 'double' this vector space such that we have for each field Φ^I of ghost number $|\Phi^I|_{\mathsf{gh}}$ an *anti-field* Φ^+_I of ghost number $|\Phi^+_I|_{\mathsf{gh}} := -1 - |\Phi^I|_{\mathsf{gh}}$ so that

$$Q_{\mathsf{BV}}\Phi^+_I \;:=\; (-1)^{|\Phi^I|}\frac{\delta S_{\mathsf{BRST}}}{\delta \Phi^I} + \cdots \,. \tag{3.13}$$

Here, the ellipsis denotes terms at least linear in the anti-fields. Formally, this doubling amounts to considering the cotangent space

$$\mathfrak{L}_{\mathsf{BV}}[1] \;:=\; T^*[-1](\mathfrak{L}_{\mathsf{BRST}}[1]) \quad\Leftrightarrow\quad \mathfrak{L}_{\mathsf{BV}} \;:=\; T^*[-3]\mathfrak{L}_{\mathsf{BRST}}\,, \tag{3.14}$$

which yields a canonical symplectic form

$$\omega \;:=\; \delta\Phi^+_I \wedge \delta\Phi^I \tag{3.15}$$

of ghost number -1. This symplectic form ω, in turn, induces a Poisson bracket, also known as the *anti-bracket*. It reads explicitly as[1]

$$\{F,G\} \;=\; (-1)^{|\Phi^I|_{\mathsf{gh}}(|F|_{\mathsf{gh}}+1)}\frac{\delta F}{\delta\Phi^I}\frac{\delta G}{\delta\Phi^+_I} - (-1)^{(|\Phi^I|_{\mathsf{gh}}+1)(|F|_{\mathsf{gh}}+1)}\frac{\delta F}{\delta\Phi^+_I}\frac{\delta G}{\delta\Phi^I}\,, \tag{3.16}$$

and it is of ghost number one so that $\{F,G\} = -(-1)^{(|F|_{\mathsf{gh}}+1)(|G|_{\mathsf{gh}}+1)}\{G,F\}$.

[1]The signs arise as follows. Hamiltonian vector fields V_F are given by $V_F \lrcorner\, \omega = \delta F$ for some function F. The Poisson bracket is then given by $\{F,G\} := V_F \lrcorner\, V_G \lrcorner\, \omega = V_F(G)$ from which the signs follow using the explicit form (3.15) of ω. The signs are often absorbed using left- and right-derivatives; however, for clarity we shall keep them explicitly.

The *classical Batalin–Vilkovisky action* is now a function $S_{\rm BV} \in \mathscr{C}^\infty(\mathfrak{L}_{\rm BV}[1])$ of ghost number zero, which obeys the *classical master equation*

$$\{S_{\rm BV}, S_{\rm BV}\} \;=\; 0\;, \tag{3.17a}$$

which extends the original action S_0 of the field theory (without ghosts or trivial pairs)[1]

$$S_{\rm BV}|_{\Phi^+_I=0} \;=\; S_0\;, \tag{3.17b}$$

and whose Hamiltonian vector field extends the BRST differential,

$$(Q_{\rm BV}\Phi^I)|_{\Phi^+_J=0} \;=\; Q_{\rm BRST}\Phi^I \tag{3.17c}$$

with

$$Q_{\rm BV} \;:=\; \{S_{\rm BV}, -\}\;. \tag{3.18}$$

We note that $Q^2_{\rm BV} = 0$ and (3.17a) are equivalent.

The last two conditions fix the terms of $S_{\rm BV}$ which are constant and linear in the anti-fields to read as

$$S_{\rm BV} \;=\; S_0 + (-1)^{|\Phi^I|_{\rm gh}}\Phi^+_I Q_{\rm BRST}\Phi^I + \cdots\;. \tag{3.19}$$

General theorems now state that for each action and compatible BRST operator, there is a corresponding BV action and a BV operator, cf. [200].

In a general theory, we will usually have a physical field a of ghost number zero as well as ghosts c_0 together with higher ghosts c_{-k} of each ghost number $-k+1$ as coordinate functions on $\mathfrak{L}_{\rm BV}[1]$. Higher ghosts are non-trivial only in theories with higher gauge invariance. All fields come with the corresponding anti-fields a^+, c^+_0, and c^+_{-k}. To accommodate gauge fixing, we will have to expand the field space further by trivial pairs and corresponding anti-fields, as already encountered in the previous section.

The equations of motion generate an ideal $\mathscr{I}$ in $\mathscr{C}^\infty(\mathfrak{L}_{\rm BRST}[1])$, and the functions on the solutions space are the quotient $\mathscr{C}^\infty(\mathfrak{L}_{\rm BRST}[1])/\mathscr{I}$. Because of (3.18),

$$Q_{\rm BV}\Phi^+_I \;=\; (-1)^{|\Phi^I|}\frac{\delta S_{\rm BV}}{\delta\Phi^I}\;, \tag{3.20}$$

and the gauge-invariant functions on the solutions space are described by the $Q_{\rm BV}$-cohomology.

[1] Here, $|_{\Phi^+_I=0}$ is the restriction to the subspace of BV field space where all anti-fields are zero.

L_∞-algebras. Following (3.12), we define again a superfield

$$\mathsf{a} \ := \ \mathsf{a}^{\boldsymbol{I}} \otimes \mathsf{e}_{\boldsymbol{I}} \ = \ \Phi^I \otimes \mathsf{e}_I + \Phi_I^+ \otimes \mathsf{e}^I \tag{3.21}$$

of degree one in $(\mathfrak{L}_{\mathrm{BV}}[1])^* \otimes \mathfrak{L}_{\mathrm{BV}}$, where $\boldsymbol{I}$ runs over all fields, ghosts, ghosts for ghosts and the corresponding anti-fields, as well as space-time and Lie algebra indices. As in (3.9), we may extend the action of Q_{BV} to elements in $(\mathfrak{L}_{\mathrm{BV}}[1])^* \otimes \mathfrak{L}_{\mathrm{BV}}$ by left action and write

$$Q_{\mathrm{BV}}\mathsf{a} \ = \ \{S_{\mathrm{BV}}, \mathsf{a}\} \ = \ -\mathsf{f}(\mathsf{a}) \quad \text{with} \quad \mathsf{f}(\mathsf{a}) \ =: \ \sum_{i\geqslant 1} \frac{1}{i!}\mu_i'(\mathsf{a}, \ldots, \mathsf{a}) \ . \tag{3.22a}$$

The μ_i' now encode i-ary graded anti-symmetric linear maps $\mu_i : \mathfrak{L}_{\mathrm{BV}} \times \cdots \times \mathfrak{L}_{\mathrm{BV}} \to \mathfrak{L}_{\mathrm{BV}}$, which can be extracted by the formulas

$$\begin{aligned} \mu_1'(\mathsf{a}) \ &= \ (-1)^{|\mathsf{a}^{\boldsymbol{I}}|_{\mathrm{gh}}} \mathsf{a}^{\boldsymbol{I}} \otimes \mu_1(\mathsf{e}_{\boldsymbol{I}}) \ , \\ \mu_i'(\mathsf{a}, \ldots, \mathsf{a}) \ &= \ (-1)^{i\sum_{j=1}^i |\mathsf{a}^{\boldsymbol{I}_j}|_{\mathrm{gh}} + \sum_{j=2}^i |\mathsf{a}^{\boldsymbol{I}_j}|_{\mathrm{gh}} \sum_{k=1}^{j-1} |\mathsf{e}_{\boldsymbol{I}_k}|_{\mathfrak{L}_{\mathrm{BV}}}} \mathsf{a}^{\boldsymbol{I}_1} \cdots \mathsf{a}^{\boldsymbol{I}_i} \otimes \mu_i(\mathsf{e}_{\boldsymbol{I}_1}, \ldots, \mathsf{e}_{\boldsymbol{I}_i}) \ , \end{aligned} \tag{3.22b}$$

see [52] for a much more detailed exposition.[1] The condition $Q_{\mathrm{BV}}^2 = 0$ then amounts to the *homotopy Jacobi identities* (2.17), and the pair $(\mathfrak{L}_{\mathrm{BV}}, \mu_i)$ with products μ_i subject to (2.17) is called an *L_∞-algebra*, cf. Section 2.3.. In our present setting, $\mathfrak{L}_{\mathrm{BV}}$ is, in fact, a *cyclic L_∞-algebra* because of the presence of the symplectic form ω. Specifically, if we consider the shift isomorphism (3.5), then ω induces the (indefinite) inner product[2]

$$\langle \ell_1, \ell_2 \rangle \ := \ (-1)^{|\ell_1|_{\mathfrak{L}_{\mathrm{BV}}}} \omega(\sigma(\ell_1), \sigma(\ell_2)) \tag{3.23a}$$

of degree -3 in $\mathfrak{L}_{\mathrm{BV}}$ and of ghost number zero. It is cyclic in the sense that

$$\langle \ell_1, \mu_i(\ell_2, \ldots, \ell_{i+1}) \rangle \ = \ (-1)^{i+i(|\ell_1|_{\mathfrak{L}_{\mathrm{BV}}} + |\ell_{i+1}|_{\mathfrak{L}_{\mathrm{BV}}}) + |\ell_{i+1}|_{\mathfrak{L}_{\mathrm{BV}}} \sum_{j=1}^i |\ell_j|_{\mathfrak{L}_{\mathrm{BV}}}} \langle \ell_{i+1}, \mu_i(\ell_1, \ldots, \ell_i) \rangle \ , \tag{3.23b}$$

which is a consequence of the vanishing of the Lie derivative of ω along Q_{BV}. This is equivalent to saying that the higher products μ_i, with the first $i-1$ arguments fixed, act as graded derivations on $\langle -, - \rangle$.

Correspondence between actions and L_∞-algebras. Every cyclic L_∞-algebra $(\mathfrak{L}_{\mathrm{BV}}, \mu_i)$ comes with a *homotopy Maurer–Cartan action*, cf. Chapter 2. In particular, the functional

$$S^{\mathrm{hMC}} \ := \ \sum_{i\geqslant 1} \frac{1}{(i+1)!} \langle a, \mu_i(a, \ldots, a) \rangle \tag{3.24}$$

[1] Note that the μ_i' define, in fact, an L_∞-structure on $\mathscr{C}^\infty(\mathfrak{L}_{\mathrm{BV}}[1]) \otimes \mathfrak{L}_{\mathrm{BV}}$.

[2] We will, in the bulk of the Thesis, deviate from this sign convention in order to simplify the signs arising in our double copy formalism.

for $a \in \mathfrak{L}_{\mathrm{BV},1}$ reproduces the action for the physical fields. Using the superfield a defined in (3.21), we can write down a more general homotopy Maurer–Cartan action

$$S^{\mathrm{shMC}} := \sum_{i\geqslant 1} \frac{1}{(i+1)!}\langle \mathsf{a}, \mu'_i(\mathsf{a},\ldots,\mathsf{a})\rangle' , \tag{3.25a}$$

where we define

$$\langle f_1^{\boldsymbol{I}} \otimes \mathsf{e}_{\boldsymbol{I}}, f_2^{\boldsymbol{J}} \otimes \mathsf{e}_{\boldsymbol{J}}\rangle' := (-1)^{|f_1^{\boldsymbol{I}}|_{\mathrm{gh}}+|f_2^{\boldsymbol{J}}|_{\mathrm{gh}}+|\mathsf{e}_{\boldsymbol{I}}|_{\mathfrak{L}_{\mathrm{BV}}}|f_2^{\boldsymbol{J}}|_{\mathrm{gh}}}\, f_1^{\boldsymbol{I}} f_2^{\boldsymbol{J}}\, \langle \mathsf{e}_{\boldsymbol{I}}, \mathsf{e}_{\boldsymbol{J}}\rangle \tag{3.25b}$$

for $f_{1,2}^{\boldsymbol{I}} \in \mathscr{C}^\infty(\mathfrak{L}_{\mathrm{BV}}[1])$. This superfield version of the homotopy Maurer–Cartan action is, in fact, the full BV action S_{BV}. Put differently, (3.25a) satisfies the quantum master equation (3.34) if and only if the μ_i in μ'_i via (3.22b) satisfy the homotopy Jacobi identities (2.17). We shall refer to the action (3.25a) as the *superfield homotopy Maurer–Cartan action* of the L$_\infty$-algebra $(\mathfrak{L}_{\mathrm{BV}}, \mu_i)$.

In summary, the BV formalism provides an equivalence between classical field theories and cyclic L$_\infty$-algebras, where the BV operator plays the role of the Chevalley–Eilenberg differential of the L$_\infty$-algebra. Clearly, the BV action corresponding to an L$_\infty$-algebra $\mathfrak{L}_{\mathrm{BV}}$ is physically only interesting if its degree-one part is non-trivial. To read off the L$_\infty$-algebra from a particular action functional, we note that using (3.25b) we have

$$\begin{aligned} \langle \mathsf{a}, \mu'_i(\mathsf{a},\ldots,\mathsf{a})\rangle' &= \langle \mathsf{a}^{\boldsymbol{I}_{i+1}} \otimes \mathsf{e}_{\boldsymbol{I}_{i+1}}, \mu'_i(\mathsf{a}^{\boldsymbol{I}_1}\otimes \mathsf{e}_{\boldsymbol{I}_1},\ldots,\mathsf{a}^{\boldsymbol{I}_i}\otimes \mathsf{e}_{\boldsymbol{I}_i})\rangle' \\ &= \zeta(\boldsymbol{I}_1,\ldots,\boldsymbol{I}_i)\, \mathsf{a}^{\boldsymbol{I}_{i+1}}\mathsf{a}^{\boldsymbol{I}_1}\cdots \mathsf{a}^{\boldsymbol{I}_i}\langle \mathsf{e}_{\boldsymbol{I}_{i+1}}, \mu_i(\mathsf{e}_{\boldsymbol{I}_1},\ldots,\mathsf{e}_{\boldsymbol{I}_i})\rangle \end{aligned} \tag{3.26a}$$

with the sign $\zeta(\boldsymbol{I}_1,\ldots,\boldsymbol{I}_i)$ given by

$$\zeta(\boldsymbol{I}_1,\ldots,\boldsymbol{I}_i) := (-1)^{\sum_{k=1}^i |\mathsf{a}^{\boldsymbol{I}_k}|_{\mathrm{gh}}(i+k+\sum_{j=k}^i |\mathsf{a}^{\boldsymbol{I}_j}|_{\mathrm{gh}})} . \tag{3.26b}$$

More explicitly,

$$\begin{aligned} \langle \mathsf{a}, \mu'_1(\mathsf{a})\rangle' &= (-1)^{|\mathsf{a}^{\boldsymbol{I}_1}|_{\mathrm{gh}}}\, \mathsf{a}^{\boldsymbol{I}_2}\mathsf{a}^{\boldsymbol{I}_1}\langle \mathsf{e}_{\boldsymbol{I}_2}, \mu_1(\mathsf{e}_{\boldsymbol{I}_1})\rangle , \\ \langle \mathsf{a}, \mu'_2(\mathsf{a},\mathsf{a})\rangle' &= (-1)^{(|\mathsf{a}^{\boldsymbol{I}_1}|_{\mathrm{gh}}+1)|\mathsf{a}^{\boldsymbol{I}_2}|_{\mathrm{gh}}}\, \mathsf{a}^{\boldsymbol{I}_3}\mathsf{a}^{\boldsymbol{I}_1}\mathsf{a}^{\boldsymbol{I}_2}\langle \mathsf{e}_{\boldsymbol{I}_3}, \mu_2(\mathsf{e}_{\boldsymbol{I}_1},\mathsf{e}_{\boldsymbol{I}_2})\rangle , \end{aligned} \tag{3.27}$$

and we shall make use of these formulas later.

Remark 3.1. *The exchange of the coordinate functions on field space with the actual fields can easily lead to confusion. Let us therefore summarise the situation once more. Actual fields (usually sections of a bundle or connections and their generalisations) are* *elements of a graded vector space $\mathfrak{L}_{\mathsf{BV}}$. The L_∞-algebra structure is defined on the vector space $\mathfrak{L}_{\mathsf{BV}}$. The symbols appearing in an action S are, technically speaking, not fields but coordinate functions on the grade-shifted field space $\mathfrak{L}_{\mathsf{BV}}[1]$, the same way that in differential geometry one writes the metric in terms of the symbols x^μ, which are not points in space-time but rather real-valued coordinate functions defined on space-time. Once we evaluate the action for particular fields, the coordinate functions are replaced by their values. Similarly, the BV operator, the anti-bracket etc. all act on or take as arguments polynomial functions on $\mathfrak{L}_{\mathsf{BV}}[1]$, which are given by polynomial expressions in the coordinate functions as well as their derivatives, which are also contained in $(\mathfrak{L}_{\mathsf{BV}}[1])^*$. To simplify notation, the coordinate function for a field (e.g. in an action) will be denoted by the same symbol as the field (element of the L_∞-algebra), as commonly done in quantum field theory.*

Remark 3.2. *The integral defining the action S of a classical field theory is mathematically usually not well defined. At a classical level, this does not matter because we are never interested in the value of S itself, and we can treat all integrals as formal expressions. For definiteness, mathematicians often drop the action and work with the Lagrangian instead. This can easily be done in the L_∞-algebra picture, working with graded modules over the ring of functions instead of graded vector spaces.*

At quantum level, however, the value of S for particular field configurations does play a role, and one needs to carefully restrict the field space such that all integrals are indeed well-defined, cf. [2]. One suitable restriction offers itself for the perturbative treatment. We split the field space into interacting fields, $\mathfrak{F}_{\mathsf{int}}$, which can simply be identified with Schwartz functions on Minkowski space $\mathscr{S}(\mathbb{M}^d)$, and free fields $\mathfrak{F}_{\mathsf{free}}$, which can be identified with solutions to the free equations of motion (i.e. fields in the kernel of μ_1), which are Schwartz type for any fixed time-slice of Minkowski space,

$$\mathfrak{F} \;:=\; \mathfrak{F}_{\mathsf{int}} \oplus \mathfrak{F}_{\mathsf{free}} \quad \text{with} \quad \mathfrak{F}_{\mathsf{int}} \;:=\; \mathscr{S}(\mathbb{M}^d) \quad \text{and} \quad \mathfrak{F}_{\mathsf{free}} \;:=\; \ker_{\mathscr{S}}(\mu_1)\;. \tag{3.28}$$

The elements of $\ker_{\mathscr{S}}(\mu_1)$ are, of course, the states that label the asymptotic on-shell states in perturbation theory. On the other hand, the fields in $\mathscr{S}(\mathbb{M}^d)$ are the propagating degrees of freedom found on internal lines in Feynman diagrams. The decomposition (3.28) is very much in the spirit of the homological perturbation lemma, which can be used to construct the scattering amplitudes, as we shall discuss below.

We note that the wave operator is invertible on $\mathscr{S}(\mathbb{M}^d)$ *and the inverse is indeed the propagator* h*, as we shall discuss in more detail below. This allows us to define the operators* $\sqrt{\Box}$ *and* $\frac{1}{\sqrt{\Box}}$ *on* $\mathscr{S}(\mathbb{M}^d)$*, which we continue to all of* $\mathfrak{F}$ *by mapping elements of* $\ker_{\mathscr{S}}(\mu_1)$ *to zero. This fact will play an important role later.*

Gauge fixing. The next step in the BV formalism is the implementation of gauge fixing. This is achieved by a canonical transformation

$$S^{\rm gf}_{\rm BV}\big[\Phi^I,\tilde{\Phi}^+_I\big] \ := \ S_{\rm BV}\left[\Phi^I,\Phi^+_I+\frac{\delta\Psi}{\delta\Phi^I}\right] \tag{3.29}$$

which is mediated by a choice of *gauge-fixing fermion*, the generating functional for the canonical transformation, which is a function $\Psi\in\mathscr{C}^\infty(\mathfrak{L}_{\rm BV}[1])$ of ghost number -1. The action (3.29) is then gauge-fixed if its Hessian is invertible. This requires a careful choice of Ψ: the trivial choice $\Psi=0$ leads back to the original action. When the classical BV action is only linear in the anti-fields, as is e.g. the case for Yang–Mills theory and all the field theories we are dealing with, we may set the anti-fields in $S^{\rm gf}_{\rm BV}$ to zero after gauge-fixing, without loss of generality since the BV operator reduces to a BRST operator.

Note that to construct the gauge-fixing fermion Ψ of ghost number -1, we will have to introduce additional fields of negative ghost number together with their anti-fields, arranged as trivial pairs, such as e.g. the anti-ghost $\bar{c}$ and the Nakanishi–Lautrup field b in the case of Yang–Mills theory. If we do not change the $Q_{\rm BV}$-cohomology, these new fields do not affect the observables. This can trivially be achieved if $Q_{\rm BV}$ maps one field to another,

$$\bar{c}\ \xrightarrow{Q_{\rm BV}}\ b\ ,\qquad b\ \xrightarrow{Q_{\rm BV}}\ 0\ ,\qquad \bar{c}^+\ \xrightarrow{Q_{\rm BV}}\ 0\ ,\qquad b^+\ \xrightarrow{Q_{\rm BV}}\ -\bar{c}^+\ , \tag{3.30}$$

cf. Equation (3.10). We shall encounter a number of more involved examples in Chapter 4.

Quantum master equation and quantum L_∞-algebras. Besides the canonical symplectic form (3.15), we also have a canonical second-order differential operator on $\mathscr{C}^\infty(\mathfrak{L}_{\rm BV}[1])$, called the *Batalin–Vilkovisky Laplacian*, and defined as

$$\Delta F \ := \ (-1)^{|\Phi^I|_{\rm gh}+|F|_{\rm gh}}\,\frac{\delta^2F}{\delta\Phi^+_I\,\delta\Phi^I} \tag{3.31}$$

for $F\in\mathscr{C}^\infty(\mathfrak{L}_{\rm BV}[1])$.

The BV Laplacian plays a key role in the path integral quantisation of a theory. In particular, the gauge fixing (3.29) is implemented at the path-integral level as

$$Z_\Psi \;:=\; \int_{\mathfrak{L}_{\rm BV}} \mu(\Phi^I,\Phi^+_I)\,\delta\left(\Phi^+_I-\frac{\delta\Psi}{\delta\Phi^I}\right)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S^\hbar_{\rm qBV}[\Phi^I,\Phi^+_I]}\;, \tag{3.32}$$

where μ is a measure that is compatible with the symplectic form ω, δ is a functional delta distribution, $\hbar$ is a formal parameter, and $S^\hbar_{\rm qBV}\in\mathscr{C}^\infty(\mathfrak{L}_{\rm BV}[1])$ is a functional of ghost number zero with

$$S^\hbar_{\rm qBV}|_{\hbar=0} \;=\; S_{\rm BV}\;. \tag{3.33}$$

For Z_Ψ to be independent of the choice of gauge-fixing fermion Ψ, $S^\hbar_{\rm qBV}$ must satisfy the <u>*quantum master equation*</u> [8][1]

$$\Delta\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S^\hbar_{\rm qBV}} \;=\; 0 \qquad\Longleftrightarrow\qquad \{S^\hbar_{\rm qBV},S^\hbar_{\rm qBV}\}-2\mathrm{i}\hbar\Delta S^\hbar_{\rm qBV} \;=\; 0\;. \tag{3.34}$$

Consequently, we obtain as generalisation of (3.18) the quantum BRST-BV operator

$$Q_{\rm qBV} \;:=\; \{S^\hbar_{\rm qBV},-\}-2\mathrm{i}\hbar\Delta\;, \tag{3.35}$$

and the quantum master equation (3.34) is equivalent to $Q^2_{\rm qBV}=0$. Note that contrary to the classical version, the quantum version (3.35) is no longer a derivation. Solutions $S^\hbar_{\rm qBV}$ to (3.34) are called <u>*quantum Batalin–Vilkovisky actions*</u>. We may now solve (3.34) order by order in $\hbar$ generalising the products μ'_i in (3.25a) to products $\mu'_{i,L}$ for $L=0,1,2,\ldots$ to reflect the $\hbar$-dependence with $\mu'_{i,L=0}=\mu'_i$ and $\mu'_{i,L=-1}:=0$. Consequently, we consider the ansatz

$$S^{\rm qshMC} \;:=\; \sum_{\substack{i\geqslant 1\\ L\geqslant 0}}\frac{\hbar^L}{(i+1)!}\langle \mathsf{a},\mu'_{i,L}(\mathsf{a},\ldots,\mathsf{a})\rangle' \tag{3.36}$$

for the superfield (3.21). The action (3.36) satisfies the quantum master equation (3.34) if and only if the $\mu_{i,L}$ satisfy the <u>*quantum homotopy Jacobi identities*</u> [33, 38, 49]

$$\begin{gathered}\sum_{\substack{i_1+i_2=i\\ L_1+L_2=L}}\;\sum_{\sigma\in\overline{\rm Sh}(i_1;i)}(-1)^{i_2}\chi(\sigma;\ell_1,\ldots,\ell_i)\mu_{i_2+1,L_2}(\mu_{i_1,L_1}(\ell_{\sigma(1)},\ldots,\ell_{\sigma(i_1)}),\ell_{\sigma(i_1+1)},\ldots,\ell_{\sigma(i)})-\\ -\,\mathrm{i}\mu_{i+2,L-1}(\mathsf{e}^I,\mathsf{e}_I,\ell_1,\ldots,\ell_i) \;=\; 0\end{gathered} \tag{3.37}$$

[1]Specifically, one requires $Z_{\Psi+\delta\Psi}=Z_\Psi$ for an infinitesimal deformation $\delta\Psi$ of Ψ; the space of gauge-fixing fermions Ψ (whose Hessians may not be invertible) is contractible, so Z_Ψ is globally independent of Ψ.

for $\ell_1, \ldots, \ell_i \in \mathfrak{L}_{\mathrm{BV}}$, where the $\mu_{i,L}$ are as in (3.22b) via the $\mu'_{i,L}$. Here $\mathsf{e}^I := \mathsf{e}_J\omega^{JI}$, where ω^{IJ} is the inverse of the symplectic form (3.15) when written as $\omega = \frac{1}{2}\delta\mathsf{a}^I \wedge \omega_{IJ}\delta\mathsf{a}^J$. Furthermore, (3.22a) generalises to

$$Q_{\mathrm{qBV}}\mathsf{a} \;=\; -\sum_{\substack{i\geqslant 1\\ L\geqslant 0}} \frac{1}{i!}\mu'_{i,L}(\mathsf{a},\ldots,\mathsf{a}) \;. \tag{3.38}$$

The tuple $(\mathfrak{L}_{\mathrm{BV}}, \mu_{i,L}, \omega)$ with the products $\mu_{i,L}$ subject to (3.37) is called a *quantum* or *loop L_∞-algebra*. In the classical limit $\hbar \to 0$, the higher products $\mu_{i,L}$ for $L > 0$ become trivial, and we recover a cyclic L_∞-algebra. Note that for scalar field theory, Yang–Mills theory, and also Chern–Simons theory, the classical BV action also satisfies the quantum master equation and hence, in those cases, we may set $S^\hbar_{\mathrm{qBV}} = S_{\mathrm{BV}}$, in which case $\mu_{i,L} = 0$ for $L > 0$. Even though the classical BV action satisties the quantum master equation, one still requires knowledge of the quantum deformation of L_∞-algebras in order to undertake the computation in Section 5.3. of the recursion relations for loop-level amplitudes.

4

Field theories, BV complexes, and homotopy algebras

In this Chapter we will discuss in detail how the mathematical framework that we have introduced so far (homotopy algebras, BV formalism) applies to concrete examples of field theories. In the following, we review the actions, the BV complexes and the dual L_∞-algebra structures of different field theories, in particular the ones relevant to our homotopy algebraic treatment of the double copy. We note that many of the theories we discuss in this Chapter does not require the BV formalism for quantisation. As explained before, however, it does make the link to homotopy algebras evident and clarifies the freedom we have in choosing gauges, an important aspect in our later discussion.

The material in this Chapter is borrowed from [2, 6].

4.1. Scalar field theory

As an introductory example illustrating the construction of an L_∞-algebra for a classical field theory, we consider scalar field theory on d-dimensional Minkowski space $\mathbb{M}^d := (\mathbb{R}^{1,d-1}, \eta)$ with η the Minkowski metric. In the following, $\mu, \nu, \ldots = 0, \ldots, d-1$, and we shall write $x \cdot y := \eta_{\mu\nu} x^\mu y^\nu = x_\mu y^\mu$ and $\Box := \partial^\mu \partial_\mu$.

Instead of plain φ^4-theory, we start from the action

$$S^{\text{scal}} := \int \mathrm{d}^d x \, \left\{ \tfrac{1}{2}\varphi(\Box - m^2)\varphi - \tfrac{\kappa}{3!}\varphi^3 - \tfrac{\lambda}{4!}\varphi^4 \right\} . \tag{4.1}$$

Scalar L_∞-algebra. The associated L_∞-algebra of this field theory is obtained as usual from the BV formalism.[1] Here, we merely note that in a field theory without (gauge) sym-

[1] See also [29] for pure φ^4-theory and [52] for a discussion closer to ours.

metry to be factored out, the BV action agrees with the classical action. The homological vector field Q_{BV} therefore acts only non-trivially on the anti-field φ^+, and we have

$$Q_{\mathrm{BV}}\varphi^+ \;=\; \{S_{\mathrm{BV}},\varphi^+\} \;=\; \frac{\delta S}{\delta\varphi} \;=\; \sum_{i\geqslant 1}\frac{1}{i!}\mu_i(\varphi,\dots,\varphi)\ , \tag{4.2}$$

where $*$ denotes the trivial vector space 0. The resulting L_∞-algebra is therefore

$$\underbrace{*}_{=:\,\mathfrak{L}_0} \longrightarrow \underbrace{\mathscr{C}^\infty(\mathbb{M}^d)}_{=:\,\mathfrak{L}_1} \xrightarrow{\Box-m^2} \underbrace{\mathscr{C}^\infty(\mathbb{M}^d)}_{=:\,\mathfrak{L}_2} \longrightarrow \underbrace{*}_{=:\,\mathfrak{L}_3} \tag{4.3a}$$

with products

$$\begin{gathered}\mu_1(\varphi_1) \;:=\; (\Box-m^2)\varphi_1\ ,\quad \mu_2(\varphi_1,\varphi_2) \;:=\; -\kappa\varphi_1\varphi_2\ ,\\ \mu_3(\varphi_1,\varphi_2,\varphi_3) \;:=\; -\lambda\varphi_1\varphi_2\varphi_3\end{gathered} \tag{4.3b}$$

for $\varphi_{1,2,3}\in\mathscr{C}^\infty(\mathbb{M}^d)$. The homotopy Maurer–Cartan action for this L_∞-algebra becomes S.

4.2. Biadjoint scalar field theory

The simplest field theory relevant for the double copy discussion is that of a biadjoint scalar field theory with cubic interaction. This theory appeared in the scattering amplitudes and double copy literature in various incarnations [201, 202, 149, 203, 153, 104, 173, 204, 105, 205, 174, 107, 108, 206, 183, 207].

In particular, let $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ be two semi-simple compact matrix Lie algebras. For $(\mathfrak{g}\otimes\bar{\mathfrak{g}})$-valued functions on Minkowski space $\mathbb{M}^d$, we define a symmetric bracket and an inner product by linearly extending

$$\begin{aligned}[\mathsf{e}_1\otimes\bar{\mathsf{e}}_1,\mathsf{e}_2\otimes\bar{\mathsf{e}}_2]_{\mathfrak{g}\otimes\bar{\mathfrak{g}}} \;&:=\; [\mathsf{e}_1,\mathsf{e}_2]_{\mathfrak{g}}\otimes[\bar{\mathsf{e}}_1,\bar{\mathsf{e}}_2]_{\bar{\mathfrak{g}}}\ ,\\ \langle\mathsf{e}_1\otimes\bar{\mathsf{e}}_1,\mathsf{e}_2\otimes\bar{\mathsf{e}}_2\rangle_{\mathfrak{g}\otimes\bar{\mathfrak{g}}} \;&:=\; \mathrm{tr}_{\mathfrak{g}}(\mathsf{e}_1\mathsf{e}_2)\,\mathrm{tr}_{\bar{\mathfrak{g}}}(\bar{\mathsf{e}}_1\bar{\mathsf{e}}_2)\end{aligned} \tag{4.4}$$

for all $\mathsf{e}_{1,2}\in\mathfrak{g}$ and $\bar{\mathsf{e}}_{1,2}\in\bar{\mathfrak{g}}$.

BV action and BV operator. The BV action for biadjoint scalar field theory then reads as

$$S^{\mathrm{biadj}} \;:=\; \int\mathrm{d}^dx\,\Big\{\tfrac{1}{2}\langle\varphi,\Box\varphi\rangle_{\mathfrak{g}\otimes\bar{\mathfrak{g}}}-\tfrac{\lambda}{3!}\langle\varphi,[\varphi,\varphi]_{\mathfrak{g}\otimes\bar{\mathfrak{g}}}\rangle_{\mathfrak{g}\otimes\bar{\mathfrak{g}}}\Big\}\ , \tag{4.5}$$

where λ is a coupling constant, $\square := \eta^{\mu\nu}\partial_\mu\partial_\nu$, and φ is a scalar field taking values in $\mathfrak{g}\otimes\bar{\mathfrak{g}}$. We write $\varphi \in (\mathfrak{g}\otimes\bar{\mathfrak{g}})\otimes\mathfrak{F}$ where $\mathfrak{F}$ is a suitable function space discussed shortly. Introducing basis vectors e_a and $\bar{\mathsf{e}}_{\bar{a}}$ on $\mathfrak{g}$ and $\bar{\mathfrak{g}}$, respectively, we can rewrite this action in component form

$$S^{\text{biadj}} = \int \mathrm{d}^d x \left\{ \tfrac{1}{2}\varphi_{a\bar{a}}\square\varphi^{a\bar{a}} - \tfrac{\lambda}{3!} f_{abc} f_{\bar{a}\bar{b}\bar{c}} \varphi^{a\bar{a}}\varphi^{b\bar{b}}\varphi^{c\bar{c}} \right\} , \tag{4.6a}$$

where

$$\begin{aligned} \mathrm{tr}_{\mathfrak{g}}(\mathsf{e}_a\mathsf{e}_b) &= -\delta_{ab} , \quad & \mathrm{tr}_{\bar{\mathfrak{g}}}(\bar{\mathsf{e}}_{\bar{a}}\bar{\mathsf{e}}_{\bar{b}}) &= -\delta_{\bar{a}\bar{b}} , \\ f_{abc} &:= -\mathrm{tr}_{\mathfrak{g}}(\mathsf{e}_a[\mathsf{e}_b,\mathsf{e}_c]_{\mathfrak{g}}) , \quad & f_{\bar{a}\bar{b}\bar{c}} &:= -\mathrm{tr}_{\bar{\mathfrak{g}}}(\bar{\mathsf{e}}_{\bar{a}}[\bar{\mathsf{e}}_{\bar{b}},\bar{\mathsf{e}}_{\bar{c}}]_{\bar{\mathfrak{g}}}) . \end{aligned} \tag{4.6b}$$

Besides the field φ, we also have the anti-field φ^+ and the BV operator (3.18) acts according to

$$\varphi^{a\bar{a}} \xmapsto{Q_{\text{BV}}} 0 \quad \text{and} \quad \varphi^{+a\bar{a}} \xmapsto{Q_{\text{BV}}} \square\varphi^{a\bar{a}} - \tfrac{\lambda}{2} f_{bc}{}^{a} f_{\bar{b}\bar{c}}{}^{\bar{a}} \varphi^{b\bar{b}}\varphi^{c\bar{c}} . \tag{4.7}$$

L_∞-algebra. The BV operator (4.7) is the Chevalley–Eilenberg differential of an L_∞-algebra $\mathfrak{L}^{\text{biadj}}_{\text{BV}}$ which has the underlying cochain complex

$$* \longrightarrow \underbrace{\overset{\varphi^{a\bar{a}}}{(\mathfrak{g}\otimes\bar{\mathfrak{g}})\otimes\mathfrak{F}}}_{\mathfrak{L}^{\text{biadj}}_{\text{BV},1}} \xrightarrow{\square} \underbrace{\overset{\varphi^{+a\bar{a}}}{(\mathfrak{g}\otimes\bar{\mathfrak{g}})\otimes\mathfrak{F}}}_{\mathfrak{L}^{\text{biadj}}_{\text{BV},2}} \longrightarrow * \tag{4.8}$$

with cyclic inner product

$$\langle \varphi, \varphi^+ \rangle := \int \mathrm{d}^d x \, \varphi^{a\bar{a}}\varphi^+_{a\bar{a}} , \tag{4.9}$$

and the only non-trivial higher product is

$$(\varphi^{a\bar{a}}, \varphi^{b\bar{b}}) \xmapsto{\mu_2} -\lambda f_{bc}{}^{a} f_{\bar{b}\bar{c}}{}^{\bar{a}} \varphi^{b\bar{b}}\varphi^{c\bar{c}} . \tag{4.10}$$

At this point it is important to recall Remark 3.1 and that we always use the same symbol for a coordinate function on field space and the corresponding elements of field space.

The field space $\mathfrak{F}$ can roughly be thought of as the smooth functions of Minkowski space $\mathscr{C}^\infty(\mathbb{M}^d)$. More precisely, however, the field space is the direct sum of interacting fields and solutions to the (colour-stripped) equations of motion, cf. Remark 3.2.

4.3. Yang–Mills theory

A key player in the double copy is Yang–Mills theory on d-dimensional Minkowski space $\mathbb{M}^d$ with a semi-simple compact matrix Lie algebra $\mathfrak{g}$ as gauge algebra. The gauge potential

A^a_μ is a one-form on $\mathbb{M}^d$ taking values in $\mathfrak{g}$. Let ∇ be the connection with respect to A. Infinitesimal gauge transformations act according to

$$A^a_\mu \;\mapsto\; \tilde{A}^a_\mu \;:=\; A^a_\mu + (\nabla_\mu c)^a \quad \text{for all} \quad c \;\in\; \mathscr{C}^\infty(\mathbb{M}^d) \otimes \mathfrak{g} \;. \tag{4.11}$$

BV action and BV operator. The list of all the fields required in the BV formulation of Yang–Mills theory together with their properties is found in Table 4.1, and the BV action is [8]

$$S^{\text{YM}}_{\text{BV}} \;:=\; \int \mathrm{d}^d x \left\{ -\tfrac{1}{4} F_{a\mu\nu} F^{a\mu\nu} + A^+_{a\mu} (\nabla^\mu c)^a + \tfrac{g}{2} f_{bc}{}^a c^+_a c^b c^c - b^a \bar{c}^+_a \right\} . \tag{4.12}$$

As in Section 3.1., all the fields are rescaled such that the Yang–Mills coupling constant g appears in all interaction vertices. Consequently, the BV operator (3.18) acts as

$$\begin{gathered}
c^a \xmapsto{Q_{\text{BV}}} -\tfrac{g}{2} f_{bc}{}^a c^b c^c \;, \quad c^{+a} \xmapsto{Q_{\text{BV}}} -(\nabla^\mu A^+_\mu)^a - g f_{bc}{}^a c^b c^{+c} \;, \\
A^a_\mu \xmapsto{Q_{\text{BV}}} (\nabla_\mu c)^a \;, \quad A^{+a}_\mu \xmapsto{Q_{\text{BV}}} (\nabla^\nu F_{\nu\mu})^a - g f_{bc}{}^a A^{+b}_\mu c^c \;, \\
b^a \xmapsto{Q_{\text{BV}}} 0 \;, \quad b^{+a} \xmapsto{Q_{\text{BV}}} -\bar{c}^{+a} \;, \\
\bar{c}^a \xmapsto{Q_{\text{BV}}} b^a \;, \quad \bar{c}^{+a} \xmapsto{Q_{\text{BV}}} 0 \;.
\end{gathered} \tag{4.13}$$

fields					anti-fields			
	role	$\lvert - \rvert_{\text{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim		$\lvert - \rvert_{\text{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim
c^a	ghost field	1	0	$\frac{d}{2} - 2$	c^{+a}	-2	3	$\frac{d}{2} + 2$
A^a_μ	physical field	0	1	$\frac{d}{2} - 1$	A^{+a}_μ	-1	2	$\frac{d}{2} + 1$
b^a	Nakanishi–Lautrup field	0	1	$\frac{d}{2}$	b^{+a}	-1	2	$\frac{d}{2}$
$\bar{c}^a$	anti-ghost field	-1	2	$\frac{d}{2}$	$\bar{c}^{+a}$	0	1	$\frac{d}{2}$

Table 4.1: The full set of BV fields for Yang–Mills theory on $\mathbb{M}^d$ with gauge Lie algebra $\mathfrak{g}$, including their ghost numbers, their L_∞-degrees, and their mass dimensions. The mass dimension of the coupling constant g is $2 - \frac{d}{2}$.

L_∞-algebra. The BV operator (4.13) is the Chevalley–Eilenberg differential of an L_∞-algebra which we shall denote by $\mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV}}$. This L_∞-algebra has the underlying complex[1]

$$
\begin{array}{ccccccc}
 & & \overset{A^a_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} & \xrightarrow{-(\partial_\nu\partial^\mu-\delta^\mu_\nu\Box)} & \overset{A^{+a}_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} & & \\
 & \overset{-\partial_\mu}{\nearrow} & & & & \overset{-\partial^\mu}{\searrow} & \\
 & & \overset{b^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}} & \overset{\mathsf{id}}{\searrow}\;\;\nearrow & \overset{b^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}} & & \\
 & & & \nearrow\;\;\underset{-\mathsf{id}}{\searrow} & & & \\
\underbrace{\overset{c^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},0}} & & \underbrace{\overset{\bar c^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},1}} & & \underbrace{\overset{\bar c^{a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},2}} & & \underbrace{\overset{c^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},3}}
\end{array}
\tag{4.14a}
$$

We shall label the subspaces $\mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},i}$ to which the various fields belong by the corresponding subscripts, that is,

$$
\begin{aligned}
\mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},0} &= \mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},0,c}~, & \mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},1} &= \bigoplus_{\phi\in(A,\,b,\,\bar c^+)} \mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},1,\phi}~,\\
\mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},2} &= \bigoplus_{\phi\in(A^+,\,b^+,\,\bar c)} \mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},2,\phi}~, & \mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},3} &= \mathfrak{L}^{\mathsf{YM}}_{3,c^+}~,
\end{aligned}
\tag{4.14b}
$$

and the non-trivial actions of the differential μ_1 in $\mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},i}$ are

$$
\begin{aligned}
c^a &\xmapsto{\mu_1} -\partial_\mu c^a \in \mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},1,A}~,\\
\begin{pmatrix} A^a_\mu \\ b^a \\ \bar c^{+a}\end{pmatrix} &\xmapsto{\mu_1} \begin{pmatrix} -(\partial_\mu\partial^\nu-\delta^\nu_\mu\Box)A^a_\nu \\ -\bar c^{+a} \\ b^a\end{pmatrix} \in \bigoplus_{\phi\in(A^+,\,b^+,\,\bar c)} \mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},2,\phi}~,\\
A^{+a}_\mu &\xmapsto{\mu_1} -\partial^\mu A^{+a}_\mu \in \mathfrak{L}^{\mathsf{YM}}_{\mathsf{BV},3,c^+}~.
\end{aligned}
\tag{4.14c}
$$

[1]This complex has been rediscovered several times in the literature. For early references, see [26, 28]; more detailed historical references are found in [52].

The non-vanishing higher products are

$$
\begin{aligned}
(c^a, c^b) &\xmapsto{\mu_2} g f_{bc}{}^a c^b c^c \in \mathfrak{L}^{\rm YM}_{{\rm BV},\,0,\,c}\ ,\\
(A^a_\mu, c^b) &\xmapsto{\mu_2} -g f_{bc}{}^a A^b_\mu c^c \in \mathfrak{L}^{\rm YM}_{{\rm BV},\,1,\,A}\ ,\\
(A^{+a}_\mu, c^b) &\xmapsto{\mu_2} -g f_{bc}{}^a A^{+b}_\mu c^c \in \mathfrak{L}^{\rm YM}_{{\rm BV},\,2,\,A^+}\ ,\\
(A^a_\mu, A^b_\nu) &\xmapsto{\mu_2} 2g f_{bc}{}^a \Big(\partial^\nu(A^b_\nu A^c_\mu) + 2A^{b\nu}\partial_{[\nu}A^c_{\mu]}\Big) \in \mathfrak{L}^{\rm YM}_{{\rm BV},\,2,\,A^+}\ ,\\
(c^a, c^{+b}) &\xmapsto{\mu_2} g f_{bc}{}^a c^b c^{+c} \in \mathfrak{L}^{\rm YM}_{{\rm BV},\,3,\,c^+}\ ,\\
(A^a_\mu, A^{+b}_\nu) &\xmapsto{\mu_2} -g f_{bc}{}^a A^b_\mu A^{+c\mu} \in \mathfrak{L}^{\rm YM}_{{\rm BV},\,3,\,c^+}\ ,\\
(A^a_\mu, A^b_\nu, A^c_\kappa) &\xmapsto{\mu_3} 3!g^2 A^{\nu c} A^d_\nu A^e_\mu f_{ed}{}^b f_{bc}{}^a \in \mathfrak{L}^{\rm YM}_{{\rm BV},\,2,\,A^+}\ ,
\end{aligned}
\tag{4.14d}
$$

and the general expressions follow from graded antisymmetry of higher products and Equation (2.17). We have that $(\mathfrak{L}^{\rm YM}_{\rm BV}, \mu_i)$ forms an L_∞-algebra, and with the inner products

$$
\begin{aligned}
\langle A, A^+\rangle &:= \int {\rm d}^d x\, A^a_\mu A^{+\mu}_a\ , & \langle b, b^+\rangle &:= \int {\rm d}^d x\, b^a b^+_a\ ,\\
\langle c, c^+\rangle &:= \int {\rm d}^d x\, c^a c^+_a\ , & \langle \bar c, \bar c^+\rangle &:= -\int {\rm d}^d x\, \bar c^a \bar c^+_a\ ,
\end{aligned}
\tag{4.15}
$$

it becomes a cyclic L_∞-algebra. Note that the superfield homotopy Maurer–Cartan action (3.25a) reduces to the BV action (4.12) when using these higher products and inner products together with (3.26).

Gauge fixing. We have discussed the general gauge-fixing procedure in the BV formalism in Section 3.1.. Here, to implement R_ξ-gauge for some real parameter ξ, we choose the gauge-fixing fermion

$$
\Psi := -\int {\rm d}^d x\, \bar c_a\big(\partial^\mu A^a_\mu + \tfrac{\xi}{2} b^a\big)\ . \tag{4.16}
$$

Following (3.29) and (3.32), the Lagrangian of the resulting gauge-fixed BV action is

$$
\begin{aligned}
S^{\rm YM,\,gf}_{\rm BV} = \int {\rm d}^d x\, \Big\{ &-\tfrac14 F_{a\mu\nu}F^{a\mu\nu} - \bar c_a \partial^\mu(\nabla_\mu c)^a + \tfrac{\xi}{2} b_a b^a + b_a \partial^\mu A^a_\mu +\\
&+ A^+_{a\mu}(\nabla_\mu c)^a + \tfrac{g}{2} f_{bc}{}^a c^+_a c^b c^c - b^a \bar c^+_a \Big\}\ ,
\end{aligned}
\tag{4.17}
$$

and after putting to zero the anti-fields, we obtain

$$
S^{\rm YM}_{\rm BRST} = \int {\rm d}^d x\, \Big\{ -\tfrac14 F_{a\mu\nu}F^{a\mu\nu} - \bar c_a \partial^\mu(\nabla_\mu c)^a + \tfrac{\xi}{2} b_a b^a + b_a \partial^\mu A^a_\mu \Big\}\ . \tag{4.18}
$$

This is precisely the action appearing in (3.1a).

4.4. Free Kalb–Ramond 2-form

The next theory which we would like to discuss is that of a free two-form gauge potential $B \in \Omega^2(\mathbb{M}^d)$. It has a three-form curvature given by

$$H_{\mu\nu\kappa} \ := \ \partial_\mu B_{\nu\kappa} + \partial_\nu B_{\kappa\mu} + \partial_\kappa B_{\mu\nu} \quad \in \Omega^3(\mathbb{M}^d) \tag{4.19}$$

and transforms under the infinitesimal gauge transformations as

$$B_{\mu\nu} \ \mapsto \ \tilde{B}_{\mu\nu} \ := \ B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \ , \tag{4.20}$$

where $\Lambda \in \Omega^1(\mathbb{M}^d)$ is the one-form gauge parameter. Note that the gauge parameters themselves transform under a *higher gauge symmetry*,

$$\Lambda_\mu \ \mapsto \ \tilde{\Lambda}_\mu \ := \ \Lambda_\mu + \partial_\mu \lambda \ , \tag{4.21}$$

where $\lambda \in \mathscr{C}^\infty(\mathbb{M}^d)$ is the (scalar) higher gauge parameter.

fields					anti-fields			
	role	$\lvert - \rvert_{\text{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim		$\lvert - \rvert_{\text{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim
λ	ghost–for–ghost field	2	-1	$\frac{d}{2}-3$	λ^+	-3	4	$\frac{d}{2}+3$
Λ_μ	ghost field	1	0	$\frac{d}{2}-2$	Λ^+_μ	-2	3	$\frac{d}{2}+2$
γ	trivial pair partner of ε	1	0	$\frac{d}{2}-1$	γ^+	-2	3	$\frac{d}{2}+1$
$B_{\mu\nu}$	physical field	0	1	$\frac{d}{2}-1$	$B^+_{\mu\nu}$	-1	2	$\frac{d}{2}+1$
α_μ	Nakanishi–Lautrup field	0	1	$\frac{d}{2}$	α^+_μ	-1	2	$\frac{d}{2}$
ε	trivial pair partner of γ	0	1	$\frac{d}{2}-1$	ε^+	-1	2	$\frac{d}{2}+1$
$\bar{\Lambda}_\mu$	anti-ghost field	-1	2	$\frac{d}{2}$	$\bar{\Lambda}^+_\mu$	0	1	$\frac{d}{2}$
$\bar{\gamma}$	trivial pair partner of $\bar{\lambda}$	-1	2	$\frac{d}{2}+1$	$\bar{\gamma}^+$	0	1	$\frac{d}{2}-1$
$\bar{\lambda}$	trivial pair partner of $\bar{\gamma}$	-2	3	$\frac{d}{2}+1$	$\bar{\lambda}^+$	1	0	$\frac{d}{2}-1$

Table 4.2: The full set of BV fields for the free Kalb–Ramond field, including their ghost numbers, their L_∞-degrees, and their mass dimension. Besides the physical field, the ghost field, and ghost–for–ghost field, we also introduced trivial pairs $(\alpha, \bar{\Lambda})$, (γ, ε), and $(\bar{\gamma}, \bar{\lambda})$ together with their anti-fields.

BV action and BV operator. The full set of fields required for gauge fixing in the BV formalism is given by what is known as the *Batalin–Vilkovisky triangle* [9], see also [52] for a recent review in the notation used here. In Batalin–Vilkovisky triangle, the lowest level trivial pair is used to gauge-fix gauge potentials and ghosts, the next-to-lowest level trivial pairs are needed to gauge-fix the lowest higher ghost, and so on. In Kalb–Ramond theory we have a higher antighost, and for this reason we precisely need three trivial pairs and their associated antifields. The complete list of BV fields is given in Table 4.2. Following the discussion of [9], the BV action reads as

$$S^{\mathrm{KR}}_{\mathrm{BV}} := \int \mathrm{d}^d x \left\{ -\tfrac{1}{12} H_{\mu\nu\kappa} H^{\mu\nu\kappa} + 2 B^+_{\mu\nu} \partial^\mu \Lambda^\nu - \Lambda^+_\mu \partial^\mu \lambda - \bar{\Lambda}^+_\mu \alpha^\mu + \bar{\lambda}^+ \bar{\gamma} + \varepsilon^+ \gamma \right\}, \quad (4.22)$$

where the factor of two has been introduced for later convenience. Consequently, the BV operator acts (3.18) as

$$\begin{aligned}
\lambda &\xmapsto{Q_{\mathrm{BV}}} 0\,, & \lambda^+ &\xmapsto{Q_{\mathrm{BV}}} \partial^\mu \Lambda^+_\mu\,,\\
\Lambda_\mu &\xmapsto{Q_{\mathrm{BV}}} \partial_\mu \lambda\,, & \Lambda^+_\mu &\xmapsto{Q_{\mathrm{BV}}} -2\partial^\nu B^+_{\nu\mu}\,,\\
\gamma &\xmapsto{Q_{\mathrm{BV}}} 0\,, & \gamma^+ &\xmapsto{Q_{\mathrm{BV}}} \varepsilon^+\,,\\
B_{\mu\nu} &\xmapsto{Q_{\mathrm{BV}}} \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu\,, & B^+_{\mu\nu} &\xmapsto{Q_{\mathrm{BV}}} \tfrac{1}{2}\partial^\kappa H_{\kappa\mu\nu}\,,\\
\alpha_\mu &\xmapsto{Q_{\mathrm{BV}}} 0\,, & \alpha^+_\mu &\xmapsto{Q_{\mathrm{BV}}} \bar{\Lambda}^+_\mu\,,\\
\varepsilon &\xmapsto{Q_{\mathrm{BV}}} \gamma\,, & \varepsilon^+ &\xmapsto{Q_{\mathrm{BV}}} 0\,,\\
\bar{\Lambda}_\mu &\xmapsto{Q_{\mathrm{BV}}} \alpha_\mu\,, & \bar{\Lambda}^+_\mu &\xmapsto{Q_{\mathrm{BV}}} 0\,,\\
\bar{\gamma} &\xmapsto{Q_{\mathrm{BV}}} 0\,, & \bar{\gamma}^+ &\xmapsto{Q_{\mathrm{BV}}} \bar{\lambda}^+\,,\\
\bar{\lambda} &\xmapsto{Q_{\mathrm{BV}}} \bar{\gamma}\,, & \bar{\lambda}^+ &\xmapsto{Q_{\mathrm{BV}}} 0\,.
\end{aligned} \quad (4.23)$$

L_∞-algebra. The BV operator (4.23) is the Chevalley–Eilenberg differential of an L_∞-algebra $\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV}}$, which has the underlying complex

$$
\begin{array}{ccccccccccc}
\overset{\lambda}{\mathscr{C}^\infty(\mathbb{M}^d)} & \xrightarrow{-\partial_\mu} & \overset{\Lambda_\mu}{\Omega^1(\mathbb{M}^d)} & \xrightarrow{2\partial_{[\nu}} & \overset{B_{\mu\nu}}{\Omega^2(\mathbb{M}^d)} & \xrightarrow{\mu_1} & \overset{B^+_{\mu\nu}}{\Omega^2(\mathbb{M}^d)} & \xrightarrow{2\partial^\nu} & \overset{\Lambda^+_\mu}{\Omega^1(\mathbb{M}^d)} & \xrightarrow{-\partial^\mu} & \overset{\lambda^+}{\mathscr{C}^\infty(\mathbb{M}^d)}
\end{array}
$$

$$
\begin{array}{ccc}
\overset{\bar{\Lambda}^+_\mu}{\Omega^1(\mathbb{M}^d)} & & \overset{\bar{\Lambda}_\mu}{\Omega^1(\mathbb{M}^d)} \\
 & \overset{\mathsf{id}}{\searrow} \; \underset{-\mathsf{id}}{\nearrow} & \\
\overset{\alpha_\mu}{\Omega^1(\mathbb{M}^d)} & & \overset{\alpha^+_\mu}{\Omega^1(\mathbb{M}^d)}
\end{array}
$$

$$
\begin{array}{ccccccc}
\overset{\gamma}{\mathscr{C}^\infty(\mathbb{M}^d)} & \xrightarrow{\mathsf{id}} & \overset{\varepsilon}{\mathscr{C}^\infty(\mathbb{M}^d)} & \qquad & \overset{\varepsilon^+}{\mathscr{C}^\infty(\mathbb{M}^d)} & \xrightarrow{-\mathsf{id}} & \overset{\gamma^+}{\mathscr{C}^\infty(\mathbb{M}^d)} \\[2ex]
\overset{\bar{\lambda}^+}{\mathscr{C}^\infty(\mathbb{M}^d)} & \xrightarrow{\mathsf{id}} & \overset{\bar{\gamma}^+}{\mathscr{C}^\infty(\mathbb{M}^d)} & \qquad & \overset{\bar{\gamma}}{\mathscr{C}^\infty(\mathbb{M}^d)} & \xrightarrow{-\mathsf{id}} & \overset{\bar{\lambda}}{\mathscr{C}^\infty(\mathbb{M}^d)}
\end{array}
$$

$$
\underbrace{\qquad}_{=:\,\mathfrak{L}^{\mathsf{KR}}_{-1}} \quad \underbrace{\qquad}_{=:\,\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},0}} \quad \underbrace{\qquad}_{=:\,\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},1}} \quad \underbrace{\qquad}_{=:\,\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},2}} \quad \underbrace{\qquad}_{=:\,\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},3}} \quad \underbrace{\qquad}_{=:\,\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},4}}
\tag{4.24a}
$$

with

$$
\begin{aligned}
\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},-1} &= \mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},-1,\lambda}\,,\\
\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},0} &= \bigoplus_{\phi\in(\Lambda,\gamma,\bar{\lambda}^+)} \mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},0,\phi}\,,\\
\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},1} &= \bigoplus_{\phi\in(B,\bar{\Lambda}^+,\alpha,\varepsilon,\bar{\gamma}^+)} \mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},1,\phi}\,,\\
\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},2} &= \bigoplus_{\phi\in(B^+,\bar{\Lambda},\alpha^+,\varepsilon^+,\bar{\gamma})} \mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},2,\phi}\,,\\
\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},3} &= \bigoplus_{\phi\in(\Lambda^+,\gamma^+,\bar{\lambda})} \mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},3,\phi}\,,\\
\mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},4} &= \mathfrak{L}^{\mathsf{KR}}_{\mathsf{BV},4,\lambda^+}\,,
\end{aligned}
\tag{4.24b}
$$

and the non-vanishing action of the differential μ_1 given by

$$
\begin{aligned}
\lambda &\xmapsto{\mu_1} -\partial_\mu \lambda \in \mathfrak{L}^{\mathrm{KR}}_{\mathrm{BV},0,\Lambda} ,\\
\begin{pmatrix} \Lambda_\mu \\ \gamma \\ \bar{\lambda}^+ \end{pmatrix} &\xmapsto{\mu_1} \begin{pmatrix} -2\partial_{[\mu}\Lambda_{\nu]} \\ \gamma \\ \bar{\lambda}^+ \end{pmatrix} \in \bigoplus_{\phi \in (B,\varepsilon,\bar{\gamma}^+)} \mathfrak{L}^{\mathrm{KR}}_{\mathrm{BV},1,\phi} ,\\
\begin{pmatrix} B_{\mu\nu} \\ \bar{\Lambda}^+_\mu \\ \alpha_\mu \end{pmatrix} &\xmapsto{\mu_1} \begin{pmatrix} \frac{1}{2}\partial^\kappa H_{\kappa\mu\nu} \\ \alpha_\mu \\ -\bar{\Lambda}^+_\mu \end{pmatrix} \in \bigoplus_{\phi \in (B^+,\bar{\Lambda},\alpha^+)} \mathfrak{L}^{\mathrm{KR}}_{\mathrm{BV},2,\phi} ,\\
\begin{pmatrix} B^+_{\mu\nu} \\ \varepsilon^+ \\ \bar{\gamma} \end{pmatrix} &\xmapsto{\mu_1} \begin{pmatrix} 2\partial^\nu B^+_{\mu\nu} \\ -\varepsilon^+ \\ -\bar{\gamma} \end{pmatrix} \in \bigoplus_{\phi \in (\Lambda^+,\gamma^+,\bar{\lambda})} \mathfrak{L}^{\mathrm{KR}}_{\mathrm{BV},3,\phi} ,\\
\Lambda^+_\mu &\xmapsto{\mu_1} -\partial^\mu \Lambda^+_\mu \in \mathfrak{L}^{\mathrm{KR}}_{\mathrm{BV},4,\lambda^+} ,
\end{aligned}
\tag{4.24c}
$$

There are no higher products because the theory is free. The L_∞-algebra $\mathfrak{L}^{\mathrm{KR}}_{\mathrm{BV}}$ becomes cyclic upon introducing

$$
\begin{aligned}
\langle \lambda, \lambda^+ \rangle &:= -\int \mathrm{d}^d x\, \lambda\lambda^+ , & \langle \bar{\lambda}, \bar{\lambda}^+ \rangle &:= -\int \mathrm{d}^d x\, \bar{\lambda}\bar{\lambda}^+ ,\\
\langle \Lambda, \Lambda^+ \rangle &:= \int \mathrm{d}^d x\, \Lambda^\mu \Lambda^+_\mu , & \langle \bar{\Lambda}, \bar{\Lambda}^+ \rangle &:= -\int \mathrm{d}^d x\, \bar{\Lambda}^\mu \bar{\Lambda}^+_\mu ,\\
\langle B, B^+ \rangle &:= \int \mathrm{d}^d x\, B^{\mu\nu} B^+_{\mu\nu} , & & \\
\langle \alpha, \alpha^+ \rangle &:= \int \mathrm{d}^d x\, \alpha^\mu \alpha^+_\mu , & \langle \varepsilon, \varepsilon^+ \rangle &:= \int \mathrm{d}^d x\, \varepsilon\varepsilon^+ ,\\
\langle \gamma, \gamma^+ \rangle &:= \int \mathrm{d}^d x\, \gamma\gamma^+ , & \langle \bar{\gamma}, \bar{\gamma}^+ \rangle &:= -\int \mathrm{d}^d x\, \bar{\gamma}\bar{\gamma}^+ .
\end{aligned}
\tag{4.25}
$$

Again, the superfield homotopy Maurer–Cartan action (3.25a) of $\mathfrak{L}^{\mathrm{KR}}_{\mathrm{BV}}$ with higher products (3.26) is the BV action (4.22).

Gauge fixing. Recall the general gauge-fixing procedure in the BV formalism from Section 3.1.. The most general Lorentz covariant linear gauge choices are implemented by the gauge-fixing fermion

$$
\Psi := -\int \mathrm{d}^d x \left\{ \bar{\Lambda}^\nu \left(\partial^\mu B_{\mu\nu} + \tfrac{\zeta_1}{2} \alpha_\nu \right) - \bar{\lambda} \left(\partial^\mu \Lambda_\mu + \zeta_2 \gamma \right) + \varepsilon \left(\partial^\mu \bar{\Lambda}_\mu + \zeta_3 \bar{\gamma} \right) \right\}
\tag{4.26}
$$

for some real parameters $\zeta_{1,2,3}$. The resulting gauge-fixed action (after putting to zero the anti-fields) is

$$\begin{aligned} S^{\rm KR}_{\rm BRST} \;=\; \int {\rm d}^d x\, \Big\{ & \tfrac{1}{4} B_{\mu\nu} \Box B^{\mu\nu} + \tfrac{1}{2} (\partial^\mu B_{\mu\nu})(\partial_\kappa B^{\kappa\nu}) - \bar\Lambda_\mu \Box \Lambda^\mu - \\ & - (\partial^\mu \bar\Lambda_\mu)(\partial_\nu \Lambda^\nu) - \bar\lambda \Box \lambda + \tfrac{\zeta_1}{2} \alpha_\mu \alpha^\mu + \alpha^\nu \partial^\mu B_{\mu\nu} + \\ & + \varepsilon \partial_\mu \alpha^\mu - (\zeta_2 + \zeta_3)\, \bar\gamma \gamma + \gamma \partial_\mu \bar\Lambda^\mu - \bar\gamma \partial_\mu \Lambda^\mu \Big\} \,. \end{aligned} \tag{4.27}$$

4.5. Einstein–Hilbert gravity

The fourth relevant theory is Einstein–Hilbert gravity on a d-dimensional Lorentzian manifold M^d with metric $g \in \Gamma(M^d, \odot^2 T^* M^d)$. Let ∇ be the Levi–Civita connection for g. Recall that infinitesimal gauge transformations of the metric are parametrised by a vector field χ and act as

$$g_{\mu\nu} \;\mapsto\; \tilde g_{\mu\nu} \;:=\; g_{\mu\nu} + (\mathcal{L}_\chi g)_{\mu\nu} \,, \tag{4.28}$$

where $\mathcal{L}_\chi$ denotes the Lie derivative along χ.

BV action and BV operator. The list of all the fields required in the BV formulation of Einstein–Hilbert gravity together with their properties is found in Table 4.3 and the BV action (cf. e.g. [208] or [209] for the gauge-fixed version) is

$$S^{\rm EH}_{\rm BV} \;:=\; \int {\rm d}^d x \Big\{ -\tfrac{1}{\kappa^2} \sqrt{-g}\, R + g^{+\mu\nu} (\mathcal{L}_\chi g)_{\mu\nu} + \tfrac{1}{2} \chi^+_\mu (\mathcal{L}_\chi \chi)^\mu - \varrho^\mu \bar\chi^+_\mu \Big\} \,, \tag{4.29}$$

where R denotes the Ricci scalar and $2\kappa^2 = 16\pi G_{\rm N}^{(d)}$ Einstein's gravitational constant. Consequently, the BV operator (3.18) acts as

$$\begin{array}{llll} \chi^\mu \xmapsto{Q_{\rm BV}} -\tfrac{1}{2} (\mathcal{L}_\chi \chi)^\mu \,, & \chi^+_\mu \xmapsto{Q_{\rm BV}} -2\nabla^\nu g^+_{\nu\mu} + (\mathcal{L}_\chi \chi^+)_\mu \,, \\ g_{\mu\nu} \xmapsto{Q_{\rm BV}} (\mathcal{L}_\chi g)_{\mu\nu} \,, & g^{+\mu\nu} \xmapsto{Q_{\rm BV}} -\tfrac{1}{\kappa^2} \sqrt{-g} \big(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R \big) + (\mathcal{L}_\chi g^+)^{\mu\nu} \,, \\ \varrho^\mu \xmapsto{Q_{\rm BV}} 0 \,, & \varrho^+_\mu \xmapsto{Q_{\rm BV}} -\bar\chi_\mu \,, \\ \bar\chi^\mu \xmapsto{Q_{\rm BV}} \varrho^\mu \,, & \bar\chi^+_\mu \xmapsto{Q_{\rm BV}} 0 \,, \end{array} \tag{4.30}$$

where $R_{\mu\nu}$ is the Ricci tensor.

fields					anti-fields			
	role	$\|-\|_{\mathrm{gh}}$	$\|-\|_{\mathfrak{L}}$	dim		$\|-\|_{\mathrm{gh}}$	$\|-\|_{\mathfrak{L}}$	dim
χ^μ	ghost field	1	0	-1	χ^+_μ	-2	3	$d+1$
$g_{\mu\nu}$	physical field	0	1	0	$g^{+\mu\nu}$	-1	2	d
ϱ^μ	Nakanishi–Lautrup field	0	1	$\frac{d}{2}$	ϱ^+_μ	-1	2	$\frac{d}{2}$
$\bar\chi^\mu$	anti-ghost field	-1	2	$\frac{d}{2}$	$\bar\chi^+_\mu$	0	1	$\frac{d}{2}$

Table 4.3: The full set of BV fields for Einstein–Hilbert gravity, including their ghost numbers, their L_∞-degrees, and their mass dimensions. The mass dimension of the coupling constant κ is $1-\frac{d}{2}$. Note that all fields are tensors and all anti-fields are tensor densities.

Perturbation theory. Let us now restrict to a Lorentzian manifold M^d for which the metric can be seen as a fluctuation $h_{\mu\nu}$ about the Minkowski metric $\eta_{\mu\nu}$ on $\mathbb{M}^d$, that is,

$$g_{\mu\nu} \;=:\; \eta_{\mu\nu}+\kappa h_{\mu\nu}\;. \tag{4.31a}$$

For future reference, we note that

$$g^{\mu\nu} \;=\; \eta^{\mu\nu}-\kappa h^{\mu\nu}+\kappa^2 h^{\mu\rho}h_\rho{}^\nu-\kappa^3 h^{\mu\rho}h_\rho{}^\sigma h_\sigma{}^\nu+\mathcal{O}(\kappa^4)\;, \tag{4.31b}$$

where $h_\mu{}^\nu:=\eta^{\nu\lambda}h_{\mu\lambda}$ and $h^{\mu\nu}:=\eta^{\mu\kappa}\eta^{\nu\lambda}h_{\kappa\lambda}$. Likewise,

$$\begin{aligned}\sqrt{-g} \;=\; &1+\tfrac{1}{2}\kappa\mathring{h}+\kappa^2\big(\tfrac{1}{8}\mathring{h}^2-\tfrac{1}{4}h_\mu{}^\nu h_\nu{}^\mu\big)+\\ &+\kappa^3\big(\tfrac{1}{48}\mathring{h}^3-\tfrac{1}{8}\mathring{h}h_\mu{}^\nu h_\nu{}^\mu+\tfrac{1}{6}h_\mu{}^\nu h_\nu{}^\rho h_\rho{}^\mu\big)+\mathcal{O}(\kappa^4)\;,\end{aligned} \tag{4.31c}$$

where $\mathring{h}:=\eta^{\mu\nu}h_{\mu\nu}$.

We also introduce the following rescaled anti-fields and unphysical fields:

$$\begin{gathered}h^{+\mu\nu} \;:=\; \tfrac{\kappa}{\sqrt{-g}}g^{+\mu\nu}\;,\\ X^\mu \;:=\; \tfrac{1}{\kappa}\chi^\mu\;,\quad X^+_\mu \;:=\; \tfrac{\kappa}{\sqrt{-g}}\chi^+_\mu\;,\quad \bar X^\mu \;:=\; \bar\chi^\mu\;,\quad \bar X^+_\mu \;:=\; \tfrac{1}{\sqrt{-g}}\bar\chi^+_\mu\;,\\ \varpi^\mu \;:=\; \varrho^\mu\;,\quad \varpi^+_\mu \;:=\; \tfrac{1}{\sqrt{-g}}\varrho^+_\mu\;.\end{gathered} \tag{4.32}$$

In addition to these, we introduce two trivial pairs (β,δ) and $(\pi,\bar\beta)$, together with the corresponding anti-fields. These do not modify the physical observables; as we shall see later, however, they do arise rather naturally in the double copy and are crucial once the dilaton enters. We also note that precisely these trivial pairs were also introduced in [210]

fields					anti-fields			
	role	$\vert - \vert_{\mathrm{gh}}$	$\vert - \vert_{\mathfrak{L}}$	dim		$\vert - \vert_{\mathrm{gh}}$	$\vert - \vert_{\mathfrak{L}}$	dim
X^μ	ghost field	1	0	$\frac{d}{2}-2$	X^+_μ	-2	3	$\frac{d}{2}+2$
β	trivial pair partner of δ	1	0	$\frac{d}{2}-1$	β^+	-2	3	$\frac{d}{2}+1$
$h_{\mu\nu}$	physical field	0	1	$\frac{d}{2}-1$	$h^{+\mu\nu}$	-1	2	$\frac{d}{2}+1$
ϖ^μ	Nakanishi–Lautrup field	0	1	$\frac{d}{2}$	ϖ^+_μ	-1	2	$\frac{d}{2}$
π	trivial pair partner of $\bar\beta$	0	1	$\frac{d}{2}+1$	π^+	-1	2	$\frac{d}{2}-1$
δ	trivial pair partner of β	0	1	$\frac{d}{2}-1$	δ^+	-1	2	$\frac{d}{2}+1$
$\bar X^\mu$	anti-ghost field	-1	2	$\frac{d}{2}$	$\bar X^+_\mu$	0	1	$\frac{d}{2}$
$\bar\beta$	trivial pair partner of π	-1	2	$\frac{d}{2}+1$	$\bar\beta^+$	0	1	$\frac{d}{2}-1$

Table 4.4: The full set of BV fields for perturbative Einstein–Hilbert gravity, including their ghost numbers, their L_∞-degrees, and their mass dimension. All the fields are regarded as tensors on Minkowski space.

in order to explain a unimodular gauge fixing in the BV formalism. The full list of fields and their properties is given in Table 4.4.

The action itself can now be expanded in orders of κ,

$$\begin{aligned} S^{\mathrm{eEH}}_{\mathrm{BV}} &= \int \mathrm{d}^d x\,\sqrt{-g}\,\Big\{-\tfrac{1}{\kappa^2}R + \tfrac{2}{\sqrt{-g}}g^{+\mu\nu}\nabla_\mu\chi_\nu + \\ &\qquad + \tfrac{1}{2\sqrt{-g}}\chi^+_\mu(\mathcal{L}_\chi\chi)^\mu - \tfrac{1}{\sqrt{-g}}\varpi^\mu\bar\chi^+_\mu + \beta\delta^+ + \pi\bar\beta^+\Big\} \\ &= \int \mathrm{d}^d x\,\sqrt{-g}\,\Big\{-\tfrac{1}{\kappa^2}R + 2h^{+\mu\nu}\nabla_\mu X_\nu + \tfrac{\kappa}{2}X^+_\mu(\mathcal{L}_X X)^\mu - \varpi^\mu\bar\chi^+_\mu + \beta\delta^+ + \pi\bar\beta^+\Big\} \\ &=: \int \mathrm{d}^d x\,\sum_{n=0}^\infty \kappa^n \mathscr{L}^{\mathrm{eEH}}_n \end{aligned} \tag{4.33}$$

with indices now raised and lowered with the Minkowski metric. The lowest-order Lagrangian $\mathscr{L}_0$ is given by the <u>*Fierz–Pauli Lagrangian*</u> plus the terms containing ghosts and other unphysical fields,

$$\begin{aligned} \mathscr{L}^{\mathrm{eEH}}_0 &= -\tfrac{1}{4}\partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} + \tfrac{1}{2}\partial^\mu h^{\nu\rho}\partial_\nu h_{\mu\rho} - \tfrac{1}{2}\partial^\mu \mathring{h}\partial^\nu h_{\mu\nu} + \tfrac{1}{4}\partial^\mu\mathring{h}\partial_\mu\mathring{h} + \\ &\qquad + 2h^{+\mu\nu}\partial_\mu X_\nu - \varpi^\mu\bar X^+_\mu + \beta\delta^+ + \pi\bar\beta^+ \ , \end{aligned} \tag{4.34}$$

cf. e.g. [211]. To first order in κ, we have

$$
\begin{aligned}
\mathscr{L}_1^{\mathsf{eEH}} \;=\; & -h^{\mu\nu}\Big\{\tfrac{1}{2}\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} - \tfrac{1}{4}\eta_{\mu\nu}\partial_\sigma h_{\tau\rho}\partial^\sigma h^{\tau\rho} + \partial_\nu \mathring{h}\left(\partial_\rho h_\mu{}^\rho - \tfrac{1}{2}\partial_\mu \mathring{h}\right) + \\
& + \partial_\nu h_\mu{}^\rho \partial_\rho \mathring{h} - \partial_\rho \mathring{h}\partial^\rho h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\partial^\rho \mathring{h}\left(\partial_\sigma h_\rho{}^\sigma - \tfrac{1}{2}\partial_\rho \mathring{h}\right) + \partial^\rho h_{\mu\nu}\partial_\sigma h_\rho{}^\sigma - \\
& - 2\partial_\nu h_{\rho\sigma}\partial^\sigma h_\mu{}^\rho - \partial_\rho h_{\nu\sigma}\partial^\sigma h_\mu{}^\rho + \partial_\sigma h_{\nu\rho}\partial^\sigma h_\mu{}^\rho + \tfrac{1}{2}\eta_{\mu\nu}\partial_\rho h_{\tau\sigma}\partial^\sigma h^{\tau\rho}\Big\} + \\
& + 2h^{+\mu\nu}\Big\{h_{\nu\lambda}\partial_\mu X^\lambda + \tfrac{1}{2}(\partial_\mu h_{\lambda\nu} + \partial_\lambda h_{\mu\nu} - \partial_\nu h_{\mu\lambda})X^\lambda\Big\} + \\
& + \tfrac{1}{2}X^+_\mu(\mathcal{L}_X X)^\mu + \tfrac{1}{2}\mathring{h}(-\varpi^\mu \bar{X}^+_\mu + \beta\delta^+ + \pi\bar{\beta}^+)\;.
\end{aligned}
\tag{4.35}
$$

L_∞-algebra. The full L_∞-algebra $\mathfrak{L}^{\mathsf{eEH}}_{\mathsf{BV}}$ of Einstein–Hilbert gravity has the underlying complex

$$
\begin{array}{ccccccc}
\overset{X_\mu}{\Omega^1(\mathbb{M}^d)} & \xrightarrow{2\partial_{(\nu}} & \overset{h_{\mu\nu}}{\Omega^2(\mathbb{M}^d)} & \xrightarrow{\mu_1} & \overset{h^+_{\mu\nu}}{\Omega^2(\mathbb{M}^d)} & \xrightarrow{2\partial^\nu} & \overset{X^+_\mu}{\Omega^1(\mathbb{M}^d)} \\
 & & \overset{\bar{X}^+_\mu}{\Omega^1(\mathbb{M}^d)} & \overset{\mathsf{id}}{\nearrow}\ \underset{-\mathsf{id}}{\searrow} & \overset{\bar{X}_\mu}{\Omega^1(\mathbb{M}^d)} & & \\
 & & \overset{\varpi_\mu}{\Omega^1(\mathbb{M}^d)} & & \overset{\varpi^+_\mu}{\Omega^1(\mathbb{M}^d)} & & \\
 & & \overset{\bar{\beta}^+}{\mathscr{C}^\infty(\mathbb{M}^d)} & \overset{-\mathsf{id}}{\nearrow}\ \underset{\mathsf{id}}{\searrow} & \overset{\bar{\beta}}{\mathscr{C}^\infty(\mathbb{M}^d)} & & \\
 & & \overset{\pi}{\mathscr{C}^\infty(\mathbb{M}^d)} & & \overset{\pi^+}{\mathscr{C}^\infty(\mathbb{M}^d)} & & \\
\overset{\beta}{\mathscr{C}^\infty(\mathbb{M}^d)} & \xrightarrow{\mathsf{id}} & \overset{\delta}{\mathscr{C}^\infty(\mathbb{M}^d)} & & \overset{\delta^+}{\mathscr{C}^\infty(\mathbb{M}^d)} & \xrightarrow{-\mathsf{id}} & \overset{\beta^+}{\mathscr{C}^\infty(\mathbb{M}^d)} \\
\underbrace{\qquad}_{=:\,\mathfrak{L}^{\mathsf{eEH}}_{\mathsf{BV},0}} & & \underbrace{\qquad}_{=:\,\mathfrak{L}^{\mathsf{eEH}}_{\mathsf{BV},1}} & & \underbrace{\qquad}_{=:\,\mathfrak{L}^{\mathsf{eEH}}_{\mathsf{BV},2}} & & \underbrace{\qquad}_{=:\,\mathfrak{L}^{\mathsf{eEH}}_{\mathsf{BV},3}}
\end{array}
\tag{4.36a}
$$

with

$$
\begin{aligned}
\mathfrak{L}^{\text{eEH}}_{\text{BV},0} &= \mathfrak{L}^{\text{eEH}}_{\text{BV},0,X} \oplus \mathfrak{L}^{\text{eEH}}_{\text{BV},0,\beta} ,\\
\mathfrak{L}^{\text{eEH}}_{\text{BV},1} &= \bigoplus_{\phi\in(h,\bar{X}^+,\varpi,\bar{\beta}^+,\pi,\delta)} \mathfrak{L}^{\text{eEH}}_{\text{BV},1,\phi} ,\\
\mathfrak{L}^{\text{eEH}}_{\text{BV},2} &= \bigoplus_{\phi\in(h^+,\bar{X},\varpi^+,\bar{\beta},\pi^+,\delta^+)} \mathfrak{L}^{\text{eEH}}_{\text{BV},2,\phi} ,\\
\mathfrak{L}^{\text{eEH}}_{\text{BV},3} &= \mathfrak{L}^{\text{eEH}}_{\text{BV},0,X^+} \oplus \mathfrak{L}^{\text{eEH}}_{\text{BV},0,\beta^+} .
\end{aligned}
\tag{4.36b}
$$

The L_∞-algebra $\mathfrak{L}^{\text{eEH}}_{\text{BV}}$ comes with non-trivial higher products of arbitrarily high order, with μ_i encoding the Lagrangian $\mathscr{L}^{\text{eEH}}_{\text{BV},i-1}$. Below, we merely list μ_1 and μ_2 to prepare our discussion of the double copy later on. The differentials are

$$
\begin{aligned}
&\begin{pmatrix} X_\mu \\ \beta \end{pmatrix} \xmapsto{\mu_1} \begin{pmatrix} -\partial_{(\mu}X_{\nu)} \\ \beta \end{pmatrix} \in \bigoplus_{\phi\in(h,\delta)} \mathfrak{L}^{\text{eEH}}_{\text{BV},1,\phi} ,\\
&\begin{pmatrix} h_{\mu\nu} \\ \bar{X}^+_\mu \\ \varpi^\mu \\ \bar{\beta}^+ \\ \pi \end{pmatrix} \xmapsto{\mu_1} \begin{pmatrix} \left[\frac{1}{2}(\delta^\rho_\mu\delta^\sigma_\nu - \eta_{\mu\nu}\eta^{\rho\sigma})\Box - (\delta^\sigma_\nu\eta_{\mu\kappa} - \delta^\sigma_\kappa\eta_{\mu\nu})\partial^\rho\partial^\kappa\right] h_{\rho\sigma} \\ -\varpi^\mu \\ \bar{X}^+_\mu \\ \pi \\ -\bar{\beta}^+ \end{pmatrix} \\
&\qquad\qquad \in \bigoplus_{\phi\in(h^+,\bar{X},\varpi^+,\bar{\beta},\pi^+)} \mathfrak{L}^{\text{eEH}}_{\text{BV},2,\phi} ,\\
&\begin{pmatrix} h^+_{\mu\nu} \\ \delta^+ \end{pmatrix} \xmapsto{\mu_1} \begin{pmatrix} -\partial^\nu h_{\nu\mu} \\ -\delta^+ \end{pmatrix} \in \bigoplus_{\phi\in(X^+,\beta^+)} \mathfrak{L}^{\text{eEH}}_{\text{BV},3,\phi} ,
\end{aligned}
\tag{4.36c}
$$

and the cubic interactions are encoded in the binary products

$$
\begin{gathered}
(X_{1\mu}, X_{2\nu}) \xmapsto{\mu_2} (\mathcal{L}_{X_1}X_2)_\mu \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,0,\,X} ~, \\
(X_\mu, X^+_\nu) \xmapsto{\mu_2} (\partial_\mu X^\nu)X^+_\nu + \partial_\nu(X^\nu X^+_\mu) \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,3,\,X^+} ~, \\
(\varpi, \bar{X}^+_\mu) \xmapsto{\mu_2} \tfrac{1}{2}\varpi^\rho \bar{X}^+_\rho \eta_{\mu\nu} \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,2,\,h^+} ~, \\
(h_{\mu\nu}, \varpi) \xmapsto{\mu_2} -\tfrac{1}{2}\mathring{h}\varpi_\mu \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,2,\,\bar{X}} ~, \\
(h_{\mu\nu}, \bar{X}^+_\rho) \xmapsto{\mu_2} \tfrac{1}{2}\mathring{h}\bar{X}^+_\mu \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,2,\,\varpi^+} ~, \\
(\beta, \delta^+) \xmapsto{\mu_2} \tfrac{1}{2}\beta\delta^+\eta_{\mu\nu} \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,2,\,h^+} ~, \\
(h_{\mu\nu}, \beta) \xmapsto{\mu_2} \tfrac{1}{2}\mathring{h}\beta \in \mathfrak{L}^{\rm eEH}_{1,\delta} ~, \\
(h_{\mu\nu}, \delta^+) \xmapsto{\mu_2} -\tfrac{1}{2}\mathring{h}\delta^+ \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,3,\,\beta^+} ~, \\
(\pi, \bar{\beta}^+) \xmapsto{\mu_2} \tfrac{1}{2}\pi\bar{\beta}^+\eta_{\mu\nu} \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,2,\,h^+} ~, \quad (h_{\mu\nu}, \pi) \xmapsto{\mu_2} -\tfrac{1}{2}\mathring{h}\pi \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,2,\,\bar{\beta}} ~, \\
(h_{\mu\nu}, \bar{\beta}^+) \xmapsto{\mu_2} \tfrac{1}{2}\mathring{h}\bar{\beta}^+ \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,2,\,\pi^+} ~, \\
(X_\mu, h_{\nu\rho}) \xmapsto{\mu_2} -2h_{\nu\kappa}\partial_\mu X^\kappa - (\partial_\mu h_{\kappa\nu} + \partial_\kappa h_{\mu\nu} - \partial_\nu h_{\mu\kappa})X^\kappa \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,1,\,h} ~,
\end{gathered}
$$

$$
\begin{aligned}
(h^+_{\mu\nu}, h_{\rho\sigma}) &\xmapsto{\mu_2} -2\partial_\kappa(h^{+\kappa\nu}h_{\nu\mu}) + h^{+\kappa\nu}(\partial_\kappa h_{\mu\nu} + \partial_\mu h_{\kappa\nu} - \partial_\nu h_{\kappa\mu}) \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,3,\,X^+} ~, \\
(h^+_{\mu\nu}, X_\rho) &\xmapsto{\mu_2} -2h^+_{\kappa\mu}\partial^\kappa X_\nu + \partial^\kappa(h^+_{\kappa\nu}X_\mu) + \partial_\kappa(h^{+\mu\nu}X^\kappa) - \partial^\kappa(h^+_{\mu\kappa}X_\nu) \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,2,\,h^+} ~, \\
(h_{1\mu\nu}, h_{2\rho\sigma}) &\xmapsto{\mu_2} 3\Big\{\tfrac{1}{2}\partial_\mu h_1^{\rho\sigma}\partial_\nu h_{2\rho\sigma} - \tfrac{1}{4}\eta_{\mu\nu}\partial_\sigma h_{1\tau\rho}\partial^\sigma h_2^{\tau\rho} + \partial_\nu \mathring{h}_1\left(\partial_\rho h_{2\mu}{}^\rho - \tfrac{1}{2}\partial_\mu\mathring{h}_2\right) + \\
&\qquad + \partial_\nu h_{1\mu}{}^\rho\partial_\rho\mathring{h}_2 - \partial_\rho\mathring{h}_1\partial^\rho h_{2\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\partial^\rho\mathring{h}_1\left(\partial_\sigma h_{2\rho}{}^\sigma - \tfrac{1}{2}\partial_\rho\mathring{h}_2\right) + \\
&\qquad + \partial^\rho h_{1\mu\nu}\partial_\sigma h_{2\rho}{}^\sigma - 2\partial_\nu h_{1\rho\sigma}\partial^\sigma h_{2\mu}{}^\rho - \partial_\rho h_{1\nu\sigma}\partial^\sigma h_{2\mu}{}^\rho + \\
&\qquad + \partial_\sigma h_{1\nu\rho}\partial^\sigma h_{2\mu}{}^\rho + \tfrac{1}{2}\eta_{\mu\nu}\partial_\rho h_{1\tau\sigma}\partial^\sigma h_2^{\tau\rho}\Big\} + \\
&\quad +(1 \leftrightarrow 2) \in \mathfrak{L}^{\rm eEH}_{{\rm BV},\,2,\,h^+} ~,
\end{aligned}
\tag{4.36d}
$$

The cyclic structure is given by the following integrals:

$$
\begin{aligned}
\langle X, X^+\rangle &:= \int {\rm d}^d x\, X^\mu X^+_\mu ~, & \langle \bar{X}, \bar{X}^+\rangle &:= -\int {\rm d}^d x\, \bar{X}^\mu \bar{X}^+_\mu ~, \\
\langle \beta, \beta^+\rangle &:= \int {\rm d}^d x\, \beta\beta^+ ~, & \langle \bar{\beta}, \bar{\beta}^+\rangle &:= -\int {\rm d}^d x\, \bar{\beta}\bar{\beta}^+ ~, \\
\langle h, h^+\rangle &:= \int {\rm d}^d x\, h^{\mu\nu} h^+_{\mu\nu} ~, & \langle \varpi, \varpi^+\rangle &:= \int {\rm d}^d x\, \varpi^\mu \varpi^+_\mu ~, \\
\langle \pi, \pi^+\rangle &:= \int {\rm d}^d x\, \pi\pi^+ ~, & \langle \delta, \delta^+\rangle &:= \int {\rm d}^d x\, \delta\delta^+ ~.
\end{aligned}
\tag{4.37}
$$

Gauge fixing. Gauge fixing proceeds as usual in the BV formalism, but due to our two additional trivial pairs, we can now write down a much more general gauge fixing fermion. We restrict ourselves to

$$\begin{aligned}\Psi_0 \;&:=\; -\int \mathrm{d}^d x \left\{\bar{X}^\nu\big(\zeta_4\partial^\mu h_{\mu\nu} - \tfrac{\zeta_5}{2}\varpi_\nu + \zeta_6\partial_\nu \mathring{h} - \zeta_7\partial_\nu\delta + \zeta_8\frac{\partial_\nu\pi}{\Box}\big) + \right.\\ &\qquad\left. + \bar{\beta}\big(\zeta_9\mathring{h} - \zeta_{10}\delta + \zeta_{11}\frac{\partial^\mu\partial^\nu h_{\mu\nu}}{\Box}\big)\right\}\,,\end{aligned} \tag{4.38}$$

and this is the freedom required for the discussion of the double copy. The resulting Lagrangian, to lowest order in κ, reads as

$$\begin{aligned}\mathscr{L}_0^{\mathrm{eEH,\,gf}} \;&=\; \tfrac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} + \tfrac{1}{2}(\partial^\mu h_{\mu\nu})^2 + \tfrac{1}{2}\mathring{h}\partial^\mu\partial^\nu h_{\mu\nu} - \tfrac{1}{4}\mathring{h}\Box\mathring{h} + \\ &\quad + \zeta_4\varpi^\nu\partial^\mu h_{\mu\nu} - \tfrac{\zeta_5}{2}\varpi^\mu\varpi_\mu + \zeta_6\varpi^\mu\partial_\mu\mathring{h} - \zeta_7\varpi^\mu\partial_\mu\delta + \zeta_8\varpi^\mu\frac{\partial_\mu\pi}{\Box} - \\ &\quad - \pi\zeta_9\mathring{h} + \zeta_{10}\pi\delta - \zeta_{11}\pi\frac{\partial^\mu\partial^\nu h_{\mu\nu}}{\Box} + \\ &\quad + \zeta_4(\partial^\mu\bar{X}^\nu + \partial^\nu\bar{X}^\mu)\partial_\mu X_\nu + \zeta_6(\partial_\kappa\bar{X}^\kappa)^2 - \zeta_9\bar{\beta}\partial^\mu X_\mu - \zeta_{11}\frac{\partial^\mu\partial^\nu\bar{\beta}}{\Box}\partial_\mu X_\nu + \\ &\quad - \zeta_7\beta\partial_\nu\bar{X}^\nu - \zeta_{10}\bar{\beta}\beta\;,\end{aligned} \tag{4.39}$$

after putting to zero the anti-fields.

4.6. $\mathcal{N} = 0$ supergravity

The actions for the free Kalb–Ramond field and Einstein–Hilbert gravity are combined and coupled to an additional scalar field φ, the dilaton, in $\mathcal{N} = 0$ supergravity. This is the common, or universal, Neveu–Schwarz-Neveu–Schwarz sector of the $\alpha' \to 0$ limit of closed string theories, and the action reads as

$$S^{\mathcal{N}=0} \;:=\; \int \mathrm{d}^d x\,\sqrt{-g}\left\{-\tfrac{1}{\kappa^2}R - \tfrac{1}{d-2}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{12}\mathrm{e}^{-\frac{4\kappa}{d-2}\varphi}H_{\mu\nu\kappa}H^{\mu\nu\kappa}\right\}. \tag{4.40}$$

The solutions of the associated equations of motions give backgrounds (with vanishing cosmological constant) around which strings can be quantised to lowest order in α' and string coupling. They also ensure conformal invariance of the string is non-anomalous in critical dimensions.

We note that the free part of $\mathcal{N} = 0$ supergravity is a sum of the free Kalb–Ramond two form, Einstein–Hilbert gravity and a free scalar field. Therefore, the free parts of the

BV formalism as well as the L_∞-algebra description just add in a straightforward manner. The interaction terms then consist of the interaction terms of Einstein–Hilbert gravity as presented in the previous Section, as well as additional terms of arbitrary order involving the dilaton and the Kalb–Ramond two-form. These are readily read off (4.40), but their explicit form will not be of relevance to us.

5 Minimal model and scattering amplitudes

In the previous Chapters, we saw that actions of field theories are encoded in cyclic L_∞-algebras. The same holds for tree-level scattering amplitudes, and loop-level scattering amplitudes are encoded in quantum L_∞-algebras, as we shall explain in this Chapter.

The material in this Chapter is borrowed from [2, 3, 6].

5.1. Equivalence of field theories

Classically, two physical theories are equivalent, if they have an isomorphic space of observables.[1] Translated to the BV formalism, this implies that classically equivalent physical theories have isomorphic Q_{BV}-cohomology. Dually, this implies that two physical theories are classically equivalent, if they have *quasi-isomorphic* L_∞-algebras, which is also mathematically the natural notion of equivalence for L_∞-algebras; see Chapter 2 for more details.

Given two L_∞-algebras $(\mathfrak{L}_{\text{BV}}, \mu_i)$ and $(\tilde{\mathfrak{L}}_{\text{BV}}, \tilde{\mu}_i)$ constructed from a BV action, a *morphism* $\phi : \mathfrak{L}_{\text{BV}} \to \tilde{\mathfrak{L}}_{\text{BV}}$ of L_∞-algebras is a collection of i-linear totally graded anti-symmetric maps $\phi_i : \mathfrak{L}_{\text{BV}} \times \cdots \times \mathfrak{L}_{\text{BV}} \to \tilde{\mathfrak{L}}_{\text{BV}}$ of degree $1 - i$ subject to the conditions (2.32). We note that the homotopy Jacobi identities (2.17) imply that μ_1 and $\tilde{\mu}_1$ are differentials. Therefore, we may consider their cohomologies $H^\bullet_{\mu_1}(\mathfrak{L}_{\text{BV}}) := \bigoplus_{k\in\mathbb{Z}} H^k_{\mu_1}(\mathfrak{L}_{\text{BV}})$ and $H^\bullet_{\tilde{\mu}_1}(\tilde{\mathfrak{L}}_{\text{BV}}) := \bigoplus_{k\in\mathbb{Z}} H^k_{\tilde{\mu}_1}(\tilde{\mathfrak{L}}_{\text{BV}})$. We also note that the identity (2.32) implies that ϕ_1 is a cochain map, that is, $\tilde{\mu}_1 \circ \phi_1 = \phi_1 \circ \mu_1$ and thus descends to a map $H^\bullet_{\mu_1}(\mathfrak{L}_{\text{BV}}) \to H^\bullet_{\tilde{\mu}_1}(\tilde{\mathfrak{L}}_{\text{BV}})$

[1] This is weaker than the statement that tree-level scattering amplitudes coincide. To define asymptotic in- and out-states in the same Hilbert space, one needs the additional data of the symplectic form ω. Two classical theories have the same tree-level scattering amplitudes if they are related by a quasi-isomorphisms compatible with the cyclic structures. Again, see [52] for some more details.

on the cohomologies. Quasi-isomorphisms are those morphisms for which ϕ_1 induces an isomorphism on cohomology.

Under quasi-isomorphisms, the physical theory remains unchanged as is explained in Chapter 2, see also [40, 192, 52, 1, 4]. In particular, the BV actions S_{BV} and $\tilde{S}_{\mathrm{BV}}$ for $\mathfrak{L}_{\mathrm{BV}}$ and $\tilde{\mathfrak{L}}_{\mathrm{BV}}$ are related as $\tilde{S}_{\mathrm{BV}} = \phi^* S_{\mathrm{BV}}$, where we used the pullback $\phi^* : \mathscr{C}^\infty(\tilde{\mathfrak{L}}_{\mathrm{BV}}[1]) \to \mathscr{C}^\infty(\mathfrak{L}_{\mathrm{BV}}[1])$ dual to the morphism ϕ. Consequently, quasi-isomorphisms constitute the correct notion of equivalence[1].

Because the Q_{BV}-cohomologies in ghost numbers different from zero (i.e. dual to L_∞-degree one) are not measurable, one may wonder if the notion of a full quasi-isomorphism is not too restrictive. For perturbation theory, agreement in $H^1_{\mu_1}(\mathfrak{L}_{\mathrm{BV}})$ is certainly sufficient, and this can often be extended to an agreement in further cohomologies, cf. e.g. [54, Appendix C]. Moreover, some fields in L_∞-degree zero, such as e.g. anti-fields of anti-ghosts and Nakanishi–Lautrup fields, are often unphysical, and arise only as internal fields in loop diagrams. Therefore their contributions to $H^1_{\mu_1}(\mathfrak{L}_{\mathrm{BV}})$ can also be disregarded when identifying physical observables. At a technical level, one can restrict these fields such that the kernel of the differential operator describing their linearised equations of motion vanishes, cf. Remark 3.2.

5.2. Tree-level scattering amplitudes

There is an L_∞-structure μ_i° with vanishing differential μ_1° on the cohomology $\mathfrak{L}^\circ_{\mathrm{BV}} := H^\bullet_{\mu_1}(\mathfrak{L}_{\mathrm{BV}})$ of an L_∞-algebra $(\mathfrak{L}_{\mathrm{BV}}, \mu_i)$ such that $\mathfrak{L}^\circ_{\mathrm{BV}}$ and $\mathfrak{L}_{\mathrm{BV}}$ are quasi-isomorphic. This L_∞-algebra $\mathfrak{L}^\circ_{\mathrm{BV}}$ is called the *minimal model* of $\mathfrak{L}_{\mathrm{BV}}$, cf. Chapter 2. The minimal model corresponds to a field theory equivalent to the original field theory, but without any propagating degrees of freedom. Its higher products therefore have to be the tree-level scattering amplitudes [33, 52, 161, 2].

The relation between $\mathfrak{L}_{\mathrm{BV}}$ and $\mathfrak{L}^\circ_{\mathrm{BV}}$ can be understood as follows. We start from the underlying cochain complexes and the following diagram:

$$\mathsf{h} \circlearrowright (\mathfrak{L}_{\mathrm{BV}}, \mu_1) \underset{\mathsf{e}}{\overset{\mathsf{p}}{\rightleftarrows}} (\mathfrak{L}^\circ_{\mathrm{BV}}, 0) \ . \tag{5.1a}$$

[1]Here, we are a bit cavalier about the inclusion of the cyclic structure; again, see [52] for some more details.

Here, p is the obvious projection onto the cohomology, and e is a choice of embedding (involving choices, e.g. a choice of gauge for gauge theories). The quasi-isomorphism also gives rise to a contracting homotopy h, which is a linear map of degree -1. The maps e and h can be chosen such that

$$\begin{aligned} \mathrm{id} &= \mu_1 \circ \mathsf{h} + \mathsf{h} \circ \mu_1 + \mathsf{e} \circ \mathsf{p} \,, \\ \mathsf{p} \circ \mathsf{e} &= \mathrm{id} \,, \\ \mathsf{p} \circ \mathsf{h} &= \mathsf{h} \circ \mathsf{e} = \mathsf{h} \circ \mathsf{h} = 0 \,, \\ \mathsf{p} \circ \mu_1 &= \mu_1 \circ \mathsf{e} = 0 \,. \end{aligned} \tag{5.1b}$$

Moreover, we now have a decomposition[1]

$$\begin{gathered} \mathfrak{L}_{\mathrm{BV}} \cong \mathfrak{L}_{\mathrm{BV}}^{\mathrm{harm}} \oplus \mathfrak{L}_{\mathrm{BV}}^{\mathrm{ex}} \oplus \mathfrak{L}_{\mathrm{BV}}^{\mathrm{coex}} \,, \\ \mathfrak{L}_{\mathrm{BV}}^{\mathrm{harm}} := \mathrm{im}(\mathsf{e} \circ \mathsf{p}) \,, \quad \mathfrak{L}_{\mathrm{BV}}^{\mathrm{ex}} := \mathrm{im}(\mu_1 \circ \mathsf{h}) \,, \quad \mathfrak{L}_{\mathrm{BV}}^{\mathrm{coex}} := \mathrm{im}(\mathsf{h} \circ \mu_1) \end{gathered} \tag{5.2}$$

with $\mathfrak{L}_{\mathrm{BV}}^{\mathrm{harm}} \cong \mathfrak{L}_{\mathrm{BV}}^{\circ}$. It is rather straightforward to verify that

$$\begin{gathered} \mathrm{im}(\mathsf{e}) \cong \mathfrak{L}_{\mathrm{BV}}^{\mathrm{harm}} \,, \quad \mathrm{im}(\mu_1) \cong \mathfrak{L}_{\mathrm{BV}}^{\mathrm{ex}} \,, \quad \mathrm{im}(\mathsf{h}) \cong \mathfrak{L}_{\mathrm{BV}}^{\mathrm{coex}} \,, \\ \ker(\mathsf{p}) \cong \mathfrak{L}_{\mathrm{BV}}^{\mathrm{ex}} \oplus \mathfrak{L}_{\mathrm{BV}}^{\mathrm{coex}} \,, \quad \ker(\mu_1) \cong \mathfrak{L}_{\mathrm{BV}}^{\mathrm{harm}} \oplus \mathfrak{L}_{\mathrm{BV}}^{\mathrm{ex}} \,, \quad \ker(\mathsf{h}) \cong \mathfrak{L}_{\mathrm{BV}}^{\mathrm{harm}} \oplus \mathfrak{L}_{\mathrm{BV}}^{\mathrm{coex}} \,. \end{gathered} \tag{5.3}$$

Mathematically, this is an abstract *Hodge–Kodaira decomposition*. The map h in L_∞-degree two turns out to be the (Feynman–'t Hooft) propagator of the physical theory in question [212–214], see also [215] and references therein.

We directly extend the diagram (5.1a) to the Chevalley–Eilenberg picture, where we have

$$\mathsf{H}_0 \circlearrowright \left(\mathscr{C}^\infty(\mathfrak{L}_{\mathrm{BV}}[1]), Q_{\mathrm{BV},0}\right) \underset{\mathsf{P}_0}{\overset{\mathsf{E}_0}{\rightleftarrows}} \left(\mathscr{C}^\infty(\mathfrak{L}_{\mathrm{BV}}^{\circ}[1]), 0\right)$$

$$\begin{aligned} \mathrm{id} &= \mathsf{P}_0 \circ \mathsf{E}_0 + Q_{\mathrm{BV},0} \circ \mathsf{H}_0 + \mathsf{H}_0 \circ Q_{\mathrm{BV},0} \,, \\ \mathsf{E}_0 \circ \mathsf{P}_0 &= \mathrm{id} \,, \\ \mathsf{E}_0 \circ \mathsf{H}_0 &= \mathsf{H}_0 \circ \mathsf{P}_0 = \mathsf{H}_0 \circ \mathsf{H}_0 = 0 \,, \\ \mathsf{E}_0 \circ Q_{\mathrm{BV},0} &= Q_{\mathrm{BV},0} \circ \mathsf{P}_0 = 0 \,, \end{aligned} \tag{5.4a}$$

where $Q_{\mathrm{BV},0}$ is the linear part of Q_{BV}, which encodes the differential μ_1. The maps E_0, P_0, and H_0 are defined by the 'tensor trick' [216] as

$$\mathsf{F}_0 = \sum_{i \geqslant 1} \frac{1}{i!} (\mathsf{F}_0)^i \quad \text{for} \quad \mathsf{F}_0 \in \{\mathsf{E}_0, \mathsf{P}_0, \mathsf{H}_0\} \tag{5.4b}$$

[1] The superscripts are borrowed from the Hodge decomposition of a differential form into harmonic, exact, and co-exact parts, see [52, Section 5.2] for the corresponding formulas.

with

$$(\mathsf{E}_0)^i := (\mathsf{e}^*)^{\odot i}~,\quad (\mathsf{P}_0)^i := (\mathsf{p}^*)^{\odot i}~,\quad (\mathsf{H}_0)^i := \sum_{k+l=i-1} 1^{\odot k}\odot \mathsf{h}^*\odot(\mathsf{p}^*\circ\mathsf{e}^*)^{\odot l}~. \tag{5.4c}$$

We can now regard the non-linear part

$$\delta := Q_{\mathrm{BV}} - Q_{\mathrm{BV},0} \tag{5.5}$$

of Q_{BV} as a perturbation and use the homological perturbation lemma [216, 217], which asserts that there is a contracting homotopy

$$\mathsf{H} \circlearrowright (\mathscr{C}^\infty(\mathfrak{L}_{\mathrm{BV}}[1]), Q_{\mathrm{BV}}) \underset{\mathsf{P}}{\overset{\mathsf{E}}{\rightleftarrows}} (\mathscr{C}^\infty(\mathfrak{L}^\circ_{\mathrm{BV}}[1]), Q^\circ_{\mathrm{BV}})$$

$$\begin{gathered}
\mathrm{id} = \mathsf{P}\circ\mathsf{E} + Q_{\mathrm{BV}}\circ\mathsf{H} + \mathsf{H}\circ Q_{\mathrm{BV}}~,\\
\mathsf{E}\circ\mathsf{P} = \mathrm{id}~,\\
\mathsf{E}\circ\mathsf{H} = \mathsf{H}\circ\mathsf{P} = \mathsf{H}\circ\mathsf{H} = 0,\\
\mathsf{E}\circ Q_{\mathrm{BV}} = Q^\circ_{\mathrm{BV}}\circ\mathsf{E}~,\quad Q_{\mathrm{BV}}\circ\mathsf{P} = \mathsf{P}\circ Q^\circ_{\mathrm{BV}}
\end{gathered} \tag{5.6a}$$

in the deformed setting. In particular,

$$\begin{aligned}
\mathsf{E} &= \mathsf{E}_0\circ(\mathrm{id}+\delta\circ\mathsf{H}_0)^{-1}~,\quad \mathsf{H} = \mathsf{H}_0\circ(\mathrm{id}+\delta\circ\mathsf{H}_0)^{-1}~,\\
\mathsf{P} &= \mathsf{P}_0 - \mathsf{H}\circ\delta\circ\mathsf{P}_0~,\quad Q^\circ_{\mathrm{BV}} = \mathsf{E}\circ\delta\circ\mathsf{P}_0~,
\end{aligned} \tag{5.6b}$$

and Q°_{BV} is the Chevalley–Eilenberg differential encoding the higher products of the minimal model and thus its tree-level scattering amplitudes. Note that here, the inverse operators are to be seen as geometric series.[1] We regard δ as a small parameter, and this is consistent with the standard perturbative approach in Quantum Field Theory, since δ is at least linear in the coupling constants. The formula for Q°_{BV} is then recursive, which has interesting consequences [2, 3].

Translated to the dual picture, the homological perturbation lemma yields the following

[1]Because we are interested in perturbation theory, we do not have to concern ourselves with convergence issues.

formulas for the quasi-isomorphism $\phi : (\mathfrak{L}_{\mathsf{BV}}, \mu_i) \to (\mathfrak{L}^\circ_{\mathsf{BV}}, \mu_i^\circ)$ [192]:

$$
\begin{aligned}
\phi_1(\ell_1^\circ) &:= \mathsf{e}(\ell_1^\circ) \,, \\
\phi_2(\ell_1^\circ, \ell_2^\circ) &:= -(\mathsf{h} \circ \mu_2)(\phi_1(\ell_1^\circ), \phi_1(\ell_2^\circ)) \,, \\
&\;\vdots \\
\phi_i(\ell_1^\circ, \dots, \ell_i^\circ) &:= -\sum_{j=2}^{i} \frac{1}{j!} \sum_{k_1+\cdots+k_j=i} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1,\dots,k_{j-1};i)} \chi(\sigma; \ell_1^\circ, \dots, \ell_i^\circ) \zeta(\sigma; \ell_1^\circ, \dots, \ell_i^\circ) \times \\
&\quad \times (\mathsf{h} \circ \mu_j)\Big(\phi_{k_1}\big(\ell^\circ_{\sigma(1)}, \dots, \ell^\circ_{\sigma(k_1)}\big), \dots, \phi_{k_j}\big(\ell^\circ_{\sigma(k_1+\cdots+k_{j-1}+1)}, \dots, \ell^\circ_{\sigma(i)}\big)\Big) \,,
\end{aligned}
\tag{5.7a}
$$

and the products $\mu_i^\circ : \mathfrak{L}^\circ_{\mathsf{BV}} \times \cdots \times \mathfrak{L}^\circ_{\mathsf{BV}} \to \mathfrak{L}^\circ_{\mathsf{BV}}$ are constructed recursively as

$$
\begin{aligned}
\mu_1^\circ(\ell_1^\circ) &:= 0 \,, \\
\mu_2^\circ(\ell_1^\circ, \ell_2^\circ) &:= (\mathsf{p} \circ \mu_2)(\phi_1(\ell_1^\circ), \phi_1(\ell_2^\circ)) \,, \\
&\;\vdots \\
\mu_i^\circ(\ell_1^\circ, \dots, \ell_i^\circ) &:= \sum_{j=2}^{i} \frac{1}{j!} \sum_{k_1+\cdots+k_j=i} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1,\dots,k_{j-1};i)} \chi(\sigma; \ell_1^\circ, \dots, \ell_i^\circ) \zeta(\sigma; \ell_1^\circ, \dots, \ell_i^\circ) \times \\
&\quad \times (\mathsf{p} \circ \mu_j)\Big(\phi_{k_1}\big(\ell^\circ_{\sigma(1)}, \dots, \ell^\circ_{\sigma(k_1)}\big), \dots, \phi_{k_j}\big(\ell^\circ_{\sigma(k_1+\cdots+k_{j-1}+1)}, \dots, \ell^\circ_{\sigma(i)}\big)\Big) \,,
\end{aligned}
\tag{5.7b}
$$

where $\ell_1^\circ, \dots, \ell_i^\circ \in \mathfrak{L}^\circ_{\mathsf{BV}}$. Here, χ and ζ are again the Koszul sign (**??**) and the sign factor (2.32b), respectively.

Using the higher products of the minimal model, n-point tree-level scattering amplitudes of the free fields $a_1^\circ, \dots, a_n^\circ \in H^1_{\mu_1}(\mathfrak{L}_{\mathsf{BV}})$ are then computed using formula [2] (see also [40, 192] for the case of string field theory)

$$
\mathscr{A}_{n,0}(a_1^\circ, \dots, a_n^\circ) = \mathrm{i}\langle a_1^\circ, \mu^\circ_{n-1}(a_2^\circ, \dots, a_n^\circ)\rangle \,. \tag{5.8}
$$

Furthermore, in [2] it was shown that the recursion relations (5.7a) encode the famous Berends–Giele recursion relations [73] for gluon scattering in Yang–Mills theory. For a related discussion of the S-matrix in the language of L_∞-algebras, see also [60] as well as [161, 218] for an interpretation of tree-level on-shell recursion relations in terms of L_∞-algebras.

5.3. Loop-level scattering amplitudes

In order to extend the above discussion to recursion relations for loop-level amplitudes, we follow [49, 3, 4]. Recall that in the transition from the classical to the quantum master equation, the classical BV operator is deformed in powers of $\hbar$ according to

$$Q_{\mathsf{BV}} \; := \; \{S_{\mathsf{BV}}, -\} \quad \rightarrow \quad Q_{\mathsf{qBV}} \; := \; \{S^{\hbar}_{\mathsf{qBV}}, -\} - 2\mathrm{i}\hbar\Delta \quad \text{with} \quad S^{\hbar}_{\mathsf{qBV}} \; = \; S_{\mathsf{BV}} + \mathcal{O}(\hbar) \; . \tag{5.9}$$

Consequently, the perturbation

$$\delta \; := \; Q_{\mathsf{qBV}} - Q_{\mathsf{qBV},0} \; = \; Q_{\mathsf{qBV}} - Q_{\mathsf{BV},0} \tag{5.10}$$

between the full and linearised part of Q_{qBV} is now also deformed in powers of $\hbar$. Starting again from the diagram (5.20b), we use the homological perturbation lemma to obtain a contracting homotopy

$$\mathsf{H} \circlearrowright (\mathscr{C}^\infty(\mathfrak{L}_{\mathsf{BV}}[1]), Q_{\mathsf{qBV}}) \underset{\mathsf{P}}{\overset{\mathsf{E}}{\rightleftarrows}} (\mathscr{C}^\infty(\mathfrak{L}^{\circ}_{\mathsf{BV}}[1]), Q^{\circ}_{\mathsf{qBV}})$$

$$\begin{gathered} \mathsf{id} \; = \; \mathsf{P} \circ \mathsf{E} + Q_{\mathsf{qBV}} \circ \mathsf{H} + \mathsf{H} \circ Q_{\mathsf{qBV}} \; , \\ \mathsf{E} \circ \mathsf{P} \; = \; \mathsf{id} \; , \\ \mathsf{E} \circ \mathsf{H} \; = \; \mathsf{H} \circ \mathsf{P} \; = \; \mathsf{H} \circ \mathsf{H} \; = \; 0 \; , \\ \mathsf{E} \circ Q_{\mathsf{qBV}} \; = \; Q^{\circ}_{\mathsf{qBV}} \circ \mathsf{E} \; , \quad Q_{\mathsf{qBV}} \circ \mathsf{P} \; = \; \mathsf{P} \circ Q^{\circ}_{\mathsf{qBV}} \; , \end{gathered} \tag{5.11a}$$

where

$$\begin{gathered} \mathsf{E} \; = \; \mathsf{E}_0 \circ (\mathsf{id} + \delta \circ \mathsf{H}_0)^{-1} \; , \quad \mathsf{H} \; = \; \mathsf{H}_0 \circ (\mathsf{id} + \delta \circ \mathsf{H}_0)^{-1} \; , \\ \mathsf{P} \; = \; \mathsf{P}_0 - \mathsf{H} \circ \delta \circ \mathsf{P}_0 \; , \quad Q^{\circ}_{\mathsf{qBV}} \; = \; \mathsf{E} \circ \delta \circ \mathsf{P}_0 \; . \end{gathered} \tag{5.11b}$$

Note that because δ contains the second order differential operator Δ, none of the maps will be algebra morphisms in general; this is just a consequence of the fact that Q°_{qBV} defines a loop homotopy algebra.

Importantly, the differential Q° can be written as [219, 49]

$$Q^{\circ}_{\mathsf{qBV}} \; = \; \{W^{\hbar}_{\mathsf{qBV}}, -\}^{\circ} - 2\mathrm{i}\hbar\Delta^{\circ} \; , \tag{5.12}$$

where $\{-,-\}^{\circ}$ and Δ° are the anti-bracket and the BV Laplacian on $\mathscr{C}^\infty(\mathfrak{L}^{\circ}_{\mathsf{BV}}[1])$, respectively, and $W^{\hbar}_{\mathsf{qBV}}$ is of the form (3.36) but with $\mu^{\circ}_{1,L=0} = 0$. Altogether, we obtain $(\mathfrak{L}^{\circ}_{\mathsf{BV}}[1], Q^{\circ}_{\mathsf{qBV}})$ which corresponds to a quantum L_∞-structure on $H^{\bullet}_{\mu_{1,L=0}}(\mathfrak{L}_{\mathsf{BV}})$ with a differential that vanishes to zeroth order in $\hbar$.

The quantum BV action $W^{\hbar}_{\rm qBV}$ is the action that encodes all scattering amplitudes to arbitrary loop order in perturbation theory.[1] In particular, for theories for which the classical BV action also satisfies the quantum master equation, which includes scalar field theory, Yang–Mills theory, and Chern–Simons theory, L coincides with the loop expansion order and hence, the products $\mu^{\circ}_{n-1,L}$ are the L-loop integrands for the n-point scattering amplitude. Consequently, (5.8) generalises to

$$\mathscr{A}_{n,L}(a^{\circ}_1,\ldots,a^{\circ}_n) \;=\; \mathrm{i}\langle a^{\circ}_1, \mu^{\circ}_{n-1,L}(a^{\circ}_2,\ldots,a^{\circ}_n)\rangle\;. \tag{5.13}$$

To construct the $\mu_{i,L}$, we note that (5.11) immediately implies

$$\mathsf{E} \;=\; \mathsf{E}_0 - \mathsf{E}\circ\delta\circ\mathsf{H}_0 \tag{5.14}$$

which is a recursion relation for E. Hence, we can iterate this equation to obtain E recursively and substitute the result into $Q^{\circ}_{\rm qBV} = \mathsf{E}\circ\delta\circ\mathsf{P}_0$ from (5.11) with P_0 given in (5.4c). We conclude, in analogy with (3.38), that

$$Q^{\circ}_{\rm qBV}\mathsf{a}^{\circ} \;=\; -\sum_{\substack{i\geqslant 1\\ L\geqslant 0}}\frac{\hbar^L}{i!}\mu'^{\circ}_{i,L}(\mathsf{a}^{\circ},\ldots,\mathsf{a}^{\circ})\;, \tag{5.15}$$

from which the $\mu'^{\circ}_{i,L}$ and thus the $\mu^{\circ}_{i,L}$ can be read off. We refer to [3, 4] for full details. It is not difficult to see that for $\hbar\to 0$, the recursion relation (5.14) coincides with the recursion relation (5.7a) and (5.15) with that for the products (5.7b) for the minimal model at the tree level.

The homological perturbation lemma correctly takes into account the symmetry factor of each Feynman diagram contributing to the scattering amplitude, see [199] for a detailed discussion of the scalar field theory case.

5.4. Coalgebra picture

Let us discuss in some detail the dual, coalgebra picture, mostly useful when discussing scattering amplitudes applications of homotopy algebra. For the sake of convenience, we will consider the (quantum) minimal model associated to a (quantum) A_∞-algebra: these can be directly related to BV formalism, as they give rise to L_∞-algebras from total

[1] One should not confuse the quantum BV action with the one-particle-irreducible effective action or the Wilsonian effective action, even though it has the form of $\hbar$-corrections to the classical action.

antisymmetrisation, just as the commutator on a matrix algebra induces a Lie algebra structure, see Equation (2.19). In particular, we can interpret every Lagrangian field theory as the homotopy Maurer-Cartan theory associated to a cyclic A_∞-algebra $(\mathfrak{A}, \langle -,-\rangle)$ with action (2.7). We consider the tensor algebra

$$\mathsf{T}^\bullet(\mathfrak{A}) \;:=\; \bigoplus_{k=0}^{\infty} \mathsf{T}^k(\mathfrak{A}) \;=\; \mathbb{R} \oplus \mathfrak{A} \oplus (\mathfrak{A}\otimes\mathfrak{A}) \oplus \cdots \;, \tag{5.16}$$

and extend the higher products m_i as coderivations M_i from $\mathfrak{A}$ to $\mathsf{T}^\bullet(\mathfrak{A})$. For instance, for $\varphi_{1,\dots,4} \in \mathfrak{A}_1$ we set

$$\mathsf{M}_3(\varphi_1 \otimes \cdots \otimes \varphi_4) \;:=\; \mathsf{m}_3(\varphi_1, \varphi_2, \varphi_3) \otimes \varphi_4 + \varphi_1 \otimes \mathsf{m}_3(\varphi_2, \varphi_3, \varphi_4) \tag{5.17}$$

and $\mathsf{M}_1(\mathbb{R}) = 0$, $\mathsf{M}_2(\varphi_1) = 0$, etc. These coderivations combine into a linear map $\mathsf{D} : \mathsf{T}^\bullet(\mathfrak{A}) \to \mathsf{T}^\bullet(\mathfrak{A})$,

$$\mathsf{D} \;:=\; \sum_i \mathsf{M}_i \;, \tag{5.18}$$

which is a codifferential. An A_∞-algebra can indeed be defined to be a $\mathbb{Z}$-graded vector space with a codifferential on its tensor algebra.

Tree-level. The minimal model construction is analogous to the case of L_∞ algebras. To induce an A_∞-structure on the cohomology $\mathfrak{A}^\circ := H^\bullet_{\mu_1}(\mathfrak{A})$, we start with an abstract Hodge–Kodaira decomposition

$$\mathsf{h} \circlearrowright (\mathfrak{A}, \mathsf{m}_1) \underset{\mathsf{e}}{\overset{\mathsf{p}}{\rightleftarrows}} (\mathfrak{A}^\circ, 0) \;. \tag{5.19a}$$

where p is the obvious projection, e is a choice of embedding, and h is the contracting homotopy, such that

$$\begin{aligned} 1 \;&=\; \mathsf{m}_1 \circ \mathsf{h} + \mathsf{h} \circ \mathsf{m}_1 + \mathsf{e} \circ \mathsf{p} \;, \\ \mathsf{p} \circ \mathsf{e} \;&=\; 1 \;, \\ \mathsf{p} \circ \mathsf{h} \;=\; \mathsf{h} \circ \mathsf{e} \;&=\; \mathsf{h} \circ \mathsf{h} \;=\; 0 \;, \\ \mathsf{p} \circ \mathsf{m}_1 \;&=\; \mathsf{m}_1 \circ \mathsf{e} \;=\; 0 \;. \end{aligned} \tag{5.19b}$$

We can extend both p and e trivially to corresponding maps P_0 and E_0 between $\mathsf{T}^\bullet(\mathfrak{A})$ and $\mathsf{T}^\bullet(\mathfrak{A}^\circ)$,

$$\mathsf{P}_0|_{\mathsf{T}^k(\mathfrak{A})} \;:=\; \mathsf{p}^{\otimes k} \quad \text{and} \quad \mathsf{E}_0|_{\mathsf{T}^k(\mathfrak{A}^\circ)} \;:=\; \mathsf{e}^{\otimes k} \;. \tag{5.20a}$$

The contracting homotopy h is extended to a map $\mathsf{H}_0 : \mathsf{T}^\bullet(\mathfrak{A}) \to \mathsf{T}^\bullet(\mathfrak{A})$ via the tensor trick,

$$\mathsf{H}_0|_{\mathsf{T}^k(\mathfrak{A})} \;:=\; \sum_{i+j=k-1} 1^{\otimes^i} \otimes \mathsf{h} \otimes (\mathsf{e} \circ \mathsf{p})^{\otimes^j} \;. \tag{5.20b}$$

Splitting D into the 'free' part $\mathsf{D}_0 := \mathsf{M}_1$ and the 'interaction' part $\mathsf{D}_{\text{int}} := \sum_{i>1} \mathsf{M}_i$, we recover (5.19) with the maps m_1, p, e, and h replaced by M_1, P_0, E_0, and H_0.

The homological perturbation lemma allows us to deform M_1 to the codifferential D, regarding D_{int} as a perturbation, which induces a codifferential D° on $\mathsf{T}^\bullet(\mathfrak{A}^\circ)$,

$$\begin{aligned} \mathsf{P} \;&=\; \mathsf{P}_0 \circ (1 + \mathsf{D}_{\text{int}} \circ \mathsf{H}_0)^{-1}, \quad \mathsf{H} \;=\; \mathsf{H}_0 \circ (1 + \mathsf{D}_{\text{int}} \circ \mathsf{H}_0)^{-1} \;, \\ \mathsf{E} \;&=\; (1 + \mathsf{H}_0 \circ \mathsf{D}_{\text{int}})^{-1} \circ \mathsf{E}_0, \quad \mathsf{D}^\circ \;=\; \mathsf{P} \circ \mathsf{D}_{\text{int}} \circ \mathsf{E}_0 \;. \end{aligned} \tag{5.21}$$

We have a picture analogous to (5.19), with the maps m_1, p, e, and h replaced by D, P, E, and H. Moreover, E and P satisfy the evident relations

$$\mathsf{P} \circ \mathsf{D} \;=\; \mathsf{D}^\circ \circ \mathsf{P} \quad \text{and} \quad \mathsf{D} \circ \mathsf{E} \;=\; \mathsf{E} \circ \mathsf{D}^\circ \;. \tag{5.22}$$

The equations for E and H in (5.21) imply

$$\mathsf{D}^\circ \;=\; \mathsf{P}_0 \circ \mathsf{D}_{\text{int}} \circ \mathsf{E} \;, \tag{5.23a}$$

$$\mathsf{E} \;=\; \mathsf{E}_0 - \mathsf{H}_0 \circ \mathsf{D}_{\text{int}} \circ \mathsf{E} \;. \tag{5.23b}$$

Substituting (5.23b) back into itself yields a recursion relation in the powers of the coupling constants since D_{int} adds one power of either κ or λ. Equation (5.23a) then allows us to construct $\mathsf{D}^\circ = \sum_{i=2}^{\infty} \mathsf{M}_i^\circ$ and hence, the products m_i° entering the amplitude (5.24). By construction, $\mathsf{M}_1^\circ = 0$ and so $\mathsf{m}_1^\circ = 0$. If we restrict the action of E to $\mathsf{T}^n(\mathfrak{A}^\circ)$ and project the result onto $\mathfrak{A} = \mathsf{T}^1(\mathfrak{A}) \subseteq \mathsf{T}^\bullet(\mathfrak{A})$, we recover the aforementioned generalisation of tree-level n-point Berends–Giele currents. The tree-level scattering amplitude reads as

$$\begin{aligned} \mathscr{A}_{n,0}(a_1^\circ, \ldots, a_n^\circ) \;&=\; \mathrm{i} \sum_{\sigma \in S_{n-1}} \langle a_n^\circ, \mathsf{m}_{n-1}^\circ(a_{\sigma(1)}^\circ, \ldots, a_{\sigma(n-1)}^\circ) \rangle \\ &=\; \mathrm{i} \sum_{\sigma \in S_n/\mathbb{Z}_n} \langle a_{\sigma(1)}^\circ, \mathsf{m}_{n-1}^\circ(a_{\sigma(2)}^\circ, \ldots, a_{\sigma(n)}^\circ) \rangle \;, \end{aligned} \tag{5.24}$$

Loop-level. The BV formalism gives a clear indication as how to generalise the above to the quantum case: the codifferential D is the dual of the classical BV differential. In the quantum case, the term $-\mathrm{i}\hbar\Delta$ is added to this differential, where Δ is the usual BV

Laplacian featuring in the quantum master equation. In the coalgebra picture, this amounts to adding $-\mathrm{i}\hbar\Delta^*$ which inserts a complete set of fields and antifields in any possible way into the tensor product, preserving the order of the original factors. Considering the example of a scalar theory in Section 4.1., for $\varphi_{1,2} \in \mathfrak{A}$, for instance,

$$\begin{aligned}\Delta^*(\varphi_1 \otimes \varphi_2) \;=\; \int \frac{\mathrm{d}^d k}{(2\pi)^d}\Big\{&\psi(k) \otimes \psi^+(k) \otimes \varphi_1 \otimes \varphi_2 + \psi(k) \otimes \varphi_1 \otimes \psi^+(k) \otimes \varphi_2 + \cdots + \\ &+ \psi^+(k) \otimes \psi(k) \otimes \varphi_1 \otimes \varphi_2 + \psi^+(k) \otimes \varphi_1 \otimes \psi(k) \otimes \varphi_2 + \cdots \Big\} ,\end{aligned} \tag{5.25}$$

where $\psi(k)$ is a (momentum space) basis of the field space $\mathfrak{A}_1$ and $\psi^+(k)$ of the antifield space $\mathfrak{A}_2$.

To compute the loop-level scattering amplitudes, we replace the perturbation,

$$\mathsf{D}_{\text{int}} \;\to\; \mathsf{D}_{\text{int}} - \mathrm{i}\hbar\Delta^* , \tag{5.26}$$

in the homological perturbation lemma (see also [219, 49]). This generalises (5.23) to

$$\mathsf{D}^\circ \;=\; \mathsf{P}_0 \circ (\mathsf{D}_{\text{int}} - \mathrm{i}\hbar\Delta^*) \circ \mathsf{E} , \tag{5.27a}$$

$$\mathsf{E} \;=\; \mathsf{E}_0 - \mathsf{H}_0 \circ (\mathsf{D}_{\text{int}} - \mathrm{i}\hbar\Delta^*) \circ \mathsf{E} . \tag{5.27b}$$

Contrary to the tree-level case, P and E are no longer coalgebra morphisms but only morphisms of graded vector spaces. Importantly, the substitution (5.26) is justified for any theory whose classical BV action also satisfies the quantum master equation. This includes scalar field theory, Chern–Simons theory, and also Yang–Mills theory.

As before, (5.27) yields a recursion relation, now in the powers of both the coupling constants and $\hbar$. The former counts the number of interaction vertices while the latter counts the number of loops.[1] The map E encodes all currents, and we introduce the restrictions to j factors in the input and i factors in the output tensor product,

$$\mathsf{E}^{i,j} \;:=\; \big(\mathrm{pr}_{\mathsf{T}^i(\mathfrak{A})} \circ \mathsf{E}\big)\big|_{\mathsf{T}^j(\mathfrak{A}^\circ)} \quad \text{and} \quad \mathsf{D}^{i,j}_{\text{int}} \;:=\; \big(\mathrm{pr}_{\mathsf{T}^i(\mathfrak{A})} \circ \mathsf{D}_{\text{int}}\big)\big|_{\mathsf{T}^j(\mathfrak{A})} . \tag{5.28}$$

If we further restrict to currents with ℓ loops and v vertices, (5.27) becomes the recursion relation

$$\mathsf{E}^{i,j}_{\ell,v} \;=\; \delta^0_\ell \delta^0_v \delta^{ij} \mathsf{E}_0|_{\mathsf{T}^i(\mathfrak{A}^\circ)} - \mathsf{H}_0|_{\mathsf{T}^i(\mathfrak{A})} \circ \sum_{k=2}^{i+2} \mathsf{D}^{i,k}_{\text{int}} \circ \mathsf{E}^{k,j}_{\ell,v-1} + \mathrm{i}\hbar\, \mathsf{H}_0|_{\mathsf{T}^i(\mathfrak{A})} \circ \Delta^*|_{\mathsf{T}^{i-2}(\mathfrak{A})} \circ \mathsf{E}^{i-2,j}_{\ell-1,v} \tag{5.29}$$

[1] When a classical BV action does not satisfy the quantum master equation, one first has to construct the quantum BV action which is given as a series expansion in powers of $\hbar$. In this case, the parameter ℓ in (5.29) is no longer the loop expansion parameter.

for the scalar field theory in Section 4.1.. Here, we put $\mathsf{E}^{i,j}_{\ell,v} = 0$ for $\ell < 0$ or $v < 0$ and this implies that the recursion relation terminates for each finite number of ℓ and v.

Just as the currents E, we can also decompose the higher products according to their loop order, $\mathsf{m}^\circ_i = \sum_{\ell=0}^{\infty} \hbar^\ell \mathsf{m}^\circ_{i,\ell}$ with $\mathsf{m}^\circ_{1,0} = 0$. The ℓ-loop scattering amplitude reads as

$$\begin{aligned}\mathscr{A}_{n,\ell}(a^\circ_1, \dots, a^\circ_n) &= \mathrm{i} \sum_{\sigma \in S_{n-1}} \langle a^\circ_n, \mathsf{m}^\circ_{n-1,\ell}(a^\circ_{\sigma(1)}, \dots, a^\circ_{\sigma(n-1)}) \rangle \\ &= \mathrm{i} \sum_{\sigma \in S_n/\mathbb{Z}_n} \langle a^\circ_{\sigma(1)}, \mathsf{m}^\circ_{n-1,\ell}(a^\circ_{\sigma(2)}, \dots, a^\circ_{\sigma(n)}) \rangle \,. \end{aligned} \tag{5.30}$$

$\mathfrak{A}^\circ := (H^\bullet_{\mathsf{m}_1}(\mathfrak{A}), \mathsf{m}^\circ_i)$ constitutes (the minimal model of) a quantum A_∞-algebra.

5.5. Berends–Giele recursion relations

In this Section we interpret the original Berends–Giele recursion relations for Yang–Mills theory with gauge group $\mathfrak{su}(N)$ in the context of homotopy algebra minimal models. For convenience, we will adopt the differential form language over space–time indices conventions. The cohomology of the cochain complex (4.14a) reads as $\mathfrak{L}^\circ_{\mathrm{YM}} = \mathfrak{su}(N) \otimes \mathfrak{L}^\circ_{\mathrm{Maxwell}}$ with

$$\mathfrak{L}^\circ_{\mathrm{Maxwell}} := \left(\mathbb{R} \longrightarrow \ker(\mathrm{d}^\dagger \mathrm{d})/\mathrm{im}(\mathrm{d}) \longrightarrow \ker(\mathrm{d}^\dagger \mathrm{d})/\mathrm{im}(\mathrm{d}) \longrightarrow \mathbb{R} \right) . \tag{5.31}$$

We choose the projectors p_k to be the evident L^2-projectors onto the subspaces $\mathfrak{L}^\circ_{\mathrm{YM},k} \subseteq \mathfrak{L}_{\mathrm{YM},k}$ and we have the trivial embeddings e_k. To compute the L_∞-structure on $\mathfrak{L}^\circ_{\mathrm{YM}}$, we need also a contracting homotopy $\mathsf{h} = (\mathsf{h}_k)$ with $\mathsf{h}_k : \mathfrak{L}_k \to \mathfrak{L}_{k-1}$ which satisfies (5.1b). Some algebra shows that[1]

$$\mathsf{h}_1 := G^F \mathrm{d}^\dagger \,, \quad \mathsf{h}_2 := G^F P_{\mathrm{ex}} \,, \quad \text{and} \quad \mathsf{h}_3 := G^F \mathrm{d} \tag{5.32a}$$

is a possible choice. Here, G^F is the Green operator, that formally obeys

$$G^F \circ \mu_1|_{\mathscr{S}(\mathbb{M}^d)} = \mu_1 \circ G^F = \mathrm{id}_{\mathscr{S}(\mathbb{M}^d)} \,. \tag{5.32b}$$

see e.g. [220, Chapter 14] for more details. P_{ex} is the projector onto the exact part under the abstract Hodge–Kodaira decomposition as discussed in Section 5.2. i.e. onto the

[1]See [52] for details on the compact case.

image of $\mathsf{d}^\dagger\mathsf{d}$. Explicitly, in momentum space and suppressing the gauge algebra for the moment, we have

$$\hat{\mathsf{h}}_2^{\mu\nu}(k) = \frac{1}{k^2+\mathrm{i}\varepsilon}\hat{P}_{\mathrm{ex}}^{\mu\nu}(k)\ , \quad \text{with} \quad \hat{P}_{\mathrm{ex}}^{\mu\nu}(k) = \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\ . \tag{5.32c}$$

Recall that our formulas (5.7a) were derived under the assumption that $\mathsf{h}_1(A) = 0$, cf. (A.4). Here, this implies that we work in Lorenz gauge $\mathsf{d}^\dagger A = 0$, and the propagator $G^F P_{\mathrm{ex}}$ is indeed the corresponding gluon propagator.

It remains to insert the projectors and contracting homotopies into (5.7a) to write down the quasi-isomorphism as well as the higher products for the minimal model.

Yang–Mills Berends–Giele gluon recursion relation. Let us denote the generators in the fundamental representation of $\mathfrak{su}(N)$ by τ_a and set the conventions (for this Section only):

$$[\tau_a,\tau_b] = f_{ab}{}^c\tau_c \quad \text{and} \quad g_{ab} := \mathrm{tr}(\tau_a^\dagger\tau_b) = -\mathrm{tr}(\tau_a\tau_b) = \tfrac{1}{2}\delta_{ab}\ . \tag{5.33}$$

Using g_{ab}, we may rewrite the structure constants $f_{abc} := f_{ab}{}^d g_{dc}$ as $f_{abc} = -\mathrm{tr}([\tau_a,\tau_b]\tau_c)$. Furthermore, with the help of the completeness relation

$$g^{ab}(\tau_a)_m{}^n(\tau_b)_k{}^l = -\delta_m^l\delta_k^n + \tfrac{1}{N}\delta_m^n\delta_k^l \tag{5.34}$$

we immediately obtain

$$\begin{aligned} g^{ab}\mathrm{tr}(X\tau_a)\mathrm{tr}(\tau_b Y) &= -\mathrm{tr}(XY) + \tfrac{1}{N}\mathrm{tr}(X)\mathrm{tr}(Y)\ , \\ g^{a_1a_2}g^{b_1b_2}\mathrm{tr}(X\tau_{a_1})\mathrm{tr}(Y\tau_{b_1})f_{a_2b_2c} &= -\mathrm{tr}([X,Y]\tau_c) \end{aligned} \tag{5.35}$$

for any two matrices X and Y. Consequently, all commutators appearing below can be expressed in terms of such traces.

Consider now a plane wave $A = A_\mu\,\mathrm{d}x^\mu$ with $A_\mu = \varepsilon_\mu(k)\,\mathrm{e}^{\mathrm{i}k\cdot x}X$, where k_μ is the four-momentum and ε_μ the polarisation vector with $k^2 = 0$ and $k\cdot\varepsilon = 0$, and $X \in \mathfrak{su}(N)$. We shall also write

$$A_i := A_{i\mu}\,\mathrm{d}x^\mu \quad \text{with} \quad A_{i\mu} := \underbrace{\varepsilon_\mu(k_i)}_{=:\,J_\mu(i)}\,\mathrm{e}^{\mathrm{i}k_i\cdot x}X_i\ , \tag{5.36}$$

to denote the 'i-th gluon'.

Then, the action of ϕ_1 in (5.7a) on A_1 is simply given by

$$\phi_1(A_1) = \mathsf{e}(A_1) = J_\mu(1)\,\mathrm{e}^{\mathrm{i}k_1\cdot x}X_1\,\mathrm{d}x^\mu\ , \tag{5.37}$$

with e acting as the identity map. Moreover, the action of ϕ_2 is

$$\phi_2(A_1, A_2) \;=\; -(\mathsf{h}_2 \circ \mu_2)(\phi_1(A_1), \phi_1(A_2)) \tag{5.38a}$$

and with (5.37) and (4.14), we find

$$\begin{aligned}
\mu_2(A_1, A_2) &= \mathrm{d}^\dagger[A_1, A_2] + \star[A_1, \star\mathrm{d}A_2] + \star[A_2, \star\mathrm{d}A_1] \\
&= \big\{2(J(1)\cdot k_2)J_\mu(2) - 2(J(2)\cdot k_1)J_\mu(1) + \\
&\qquad + (J(1)\cdot J(2))(k_1 - k_2)_\mu\big\}\mathrm{e}^{\mathrm{i}(k_1+k_2)\cdot x}\,[X_1, X_2]\,\mathrm{d}x^\mu \\
&= [\![J(1), J(2)]\!]_\mu\,\mathrm{e}^{\mathrm{i}(k_1+k_2)\cdot x}\,[X_1, X_2]\,\mathrm{d}x^\mu\ ,
\end{aligned} \tag{5.38b}$$

where

$$[\![J(1), J(2)]\!]_\mu \;:=\; 2(J(1)\cdot k_2)J_\mu(2) - 2(J(2)\cdot k_1)J_\mu(1) + (J(1)\cdot J(2))(k_1 - k_2)_\mu\ . \tag{5.38c}$$

Consequently, using the contracting homotopy (5.32), we obtain

$$\begin{aligned}
\phi_2(A_1, A_2) &= -P_{\mathrm{ex}}\left(\frac{[\![J(1), J(2)]\!]_\mu}{(k_1+k_2)^2}\,\mathrm{e}^{\mathrm{i}(k_1+k_2)\cdot x}\,[X_1, X_2]\,\mathrm{d}x^\mu\right) \\
&= -\underbrace{\frac{[\![J(1), J(2)]\!]_\mu}{(k_1+k_2)^2}}_{=:\,J_\mu(1,2)}\,\mathrm{e}^{\mathrm{i}(k_1+k_2)\cdot x}\,[X_1, X_2]\,\mathrm{d}x^\mu \\
&= -\frac{1}{2}\sum_{\sigma\in S_2} J_\mu(\sigma(1), \sigma(2))\,\mathrm{e}^{\mathrm{i}(k_{\sigma(1)}+k_{\sigma(2)})\cdot x}\,[X_{\sigma(1)}, X_{\sigma(2)}]\,\mathrm{d}x^\mu\ ,
\end{aligned} \tag{5.38d}$$

where in the second step, we used that P_{ex} acts trivially and the sum is over all permutations. Equation (5.38d) yields indeed the 2-gluon current that can be found in Berends–Giele [73]. It is also instructive to give the next level expression before turning to the general case. In particular, the action of ϕ_3 is

$$\begin{aligned}
\phi_3(A_1, A_2, A_3) &= -(\mathsf{h}_2 \circ \mu_2)(\phi_1(A_1), \phi_2(A_2, A_3)) - \\
&\quad - (\mathsf{h}_2 \circ \mu_2)(\phi_1(A_2), \phi_2(A_1, A_3)) - \\
&\quad - (\mathsf{h}_2 \circ \mu_2)(\phi_1(A_3), \phi_2(A_1, A_2)) - \\
&\quad - (\mathsf{h}_2 \circ \mu_3)(\phi_1(A_1), \phi_1(A_2), \phi_1(A_3))\ .
\end{aligned} \tag{5.39a}$$

From (4.14), we have

$$\begin{aligned}
&\mu_3(A_1, A_2, A_3) = \\
&\quad = \sum_{\sigma\in\mathbb{Z}_3} \star[A_{\sigma(1)}, \star[A_{\sigma(2)}, A_{\sigma(3)}]] \\
&\quad = -\sum_{\sigma\in\mathbb{Z}_3} [\![J(\sigma(1)), J(\sigma(2)), J(\sigma(3))]\!]_\mu\,\mathrm{e}^{\mathrm{i}(k_{\sigma(1)}+k_{\sigma(2)}+k_{\sigma(3)})\cdot x}\,[X_{\sigma(1)}, [X_{\sigma(2)}, X_{\sigma(3)}]]\,\mathrm{d}x^\mu\ ,
\end{aligned} \tag{5.39b}$$

where the sum is over cyclic permutations only and

$$[\![J(1),J(2),J(3)]\!]_\mu \;:=\; (J(1)\cdot J(3))J_\mu(2)-(J(1)\cdot J(2))J_\mu(3)\;. \tag{5.39c}$$

Combining this with the expression (5.38d) and using the contracting homotopy (5.32), we immediately find that ϕ_3 is given by

$$\begin{aligned}\phi_3(A_1,A_2,A_3) \;&=\\ &= P_{\rm ex}\sum_{\sigma\in\mathbb{Z}_3}\tilde{J}_\mu(\sigma(1),\sigma(2),\sigma(3))\,\mathrm{e}^{\mathrm{i}(k_{\sigma(1)}+k_{\sigma(2)}+k_{\sigma(3)})\cdot x}\,[X_{\sigma(1)},[X_{\sigma(2)},X_{\sigma(3)}]]\,\mathrm{d}x^\mu\;,\end{aligned} \tag{5.39d}$$

where

$$\tilde{J}_\mu(1,2,3)\;:=\;\frac{[\![J(1),J(2,3)]\!]_\mu+[\![J(1),J(2),J(3)]\!]_\mu}{(k_1+k_2+k_3)^2}\;. \tag{5.39e}$$

The expression for the 3-gluon current as given by Berends–Giele [73] is simply

$$J_\mu(1,2,3)\;:=\;\tilde{J}_\mu(1,2,3)-\tilde{J}_\mu(3,1,2)\;, \tag{5.39f}$$

and, upon using the antisymmetry and the Jacobi identity for the Lie bracket $[-,-]$, a short calculation reveals that (5.39d) becomes

$$\begin{aligned}\phi_3(A_1,A_2,A_3)\;&=\\ &=\frac{1}{3}\sum_{\sigma\in S_3}J_\mu(\sigma(1),\sigma(2),\sigma(3))\,\mathrm{e}^{\mathrm{i}(k_{\sigma(1)}+k_{\sigma(2)}+k_{\sigma(3)})\cdot x}\,[X_{\sigma(1)},[X_{\sigma(2)},X_{\sigma(3)}]]\,\mathrm{d}x^\mu\;,\end{aligned} \tag{5.39g}$$

where the sum here is over all permutations and $P_{\rm ex}$ acts again trivially.

Let us now turn to the general case. The above discussion for 2- and 3-points motivates us to define

$$J_a(1,\ldots,i)\;=\;g_{ab}J^b(1,\ldots,i)\;:=\;-\mathrm{tr}(\phi_i(A_1,\ldots,A_i)\tau_a) \tag{5.40}$$

with g_{ab} as given in (5.33). Hence,

$$\phi_i(A_1,\ldots,A_i)\;=\;J^a(1,\ldots,i)\tau_a\;. \tag{5.41}$$

Furthermore, we also define

$$\begin{aligned}J^a(1,\ldots,i)\;&=:\;g^{ab}\sum_{\sigma\in S_i}\mathrm{tr}(X_{\sigma(1)}\cdots X_{\sigma(i)}\tau_b)J_\mu(\sigma(1),\ldots,\sigma(i))\,\mathrm{e}^{\mathrm{i}(k_\sigma(1)+\cdots+k_{\sigma(i)})\cdot x}\mathrm{d}x^\mu\\ J(1,\ldots,i)\;&:=\;J_\mu(1,\ldots,i)\,\mathrm{d}x^\mu\end{aligned} \tag{5.42}$$

similar to Berends–Giele [73]. Then, the first term in the quasi-isomorphism

$$\begin{aligned}
&\phi_i(A_1,\ldots,A_i) \;= \\
&= -\frac{1}{2!}\sum_{k_1+k_2=i}\;\sum_{\sigma\in\mathrm{Sh}(k_1;i)}\times \\
&\qquad \times\,(\mathsf{h}_2\circ\mu_2)\big(\phi_{k_1}\big(A_{\sigma(1)},\ldots,A_{\sigma(k_1)}\big),\phi_{k_2}\big(A_{\sigma(k_1+1)},\ldots,A_{\sigma(i)}\big)\big)- \\
&\quad -\frac{1}{3!}\sum_{k_1+k_2+k_3=i}\;\sum_{\sigma\in\mathrm{Sh}(k_1,k_2;i)}\times \\
&\qquad \times\,(\mathsf{h}_2\circ\mu_3)\big(\phi_{k_1}\big(A_{\sigma(1)},\ldots,A_{\sigma(k_1)}\big),\ldots,\phi_{k_3}\big(A_{\sigma(k_1+k_2+1)},\ldots,A_{\sigma(i)}\big)\big)
\end{aligned} \tag{5.43}$$

is given by

$$\begin{aligned}
(\mathrm{I}) \;&:=\; -\frac{1}{2!}\sum_{k_1+k_2=i}\;\sum_{\sigma\in\mathrm{Sh}(k_1;i)}\times \\
&\qquad \times\,\mu_2\big(\phi_{k_1}\big(A_{\sigma(1)},\ldots,A_{\sigma(k_1)}\big),\phi_{k_2}\big(A_{\sigma(k_1+1)},\ldots,A_{\sigma(i)}\big)\big) \\
&=\; -\frac{1}{2!}\sum_{\sigma\in S_i}\sum_{j=1}^{i-1}\frac{1}{j!(i-j)!}\times \\
&\qquad \times\,\mu_2\big(\phi_j\big(A_{\sigma(1)},\ldots,A_{\sigma(j)}\big),\phi_{i-j}\big(A_{\sigma(j+1)},\ldots,A_{\sigma(i)}\big)\big) \\
&=\; -\frac{1}{2!}\sum_{\sigma\in S_i}\sum_{j=1}^{i-1}\frac{1}{j!(i-j)!}[\![J^a(\sigma(1),\ldots,\sigma(j)),J^b(\sigma(j+1),\ldots,\sigma(i))]\!]f_{abc}g^{cd}\tau_d \\
&=\; \sum_{\sigma\in S_i}\sum_{j=1}^{i-1}[\![J(\sigma(1),\ldots,\sigma(j)),J(\sigma(j+1),\ldots,\sigma(i))]\!]\times \\
&\qquad \times\,\mathrm{e}^{\mathrm{i}(k_{\sigma}(1)+\cdots+k_{\sigma(i)})\cdot x}\,g^{ab}\mathrm{tr}(X_{\sigma(1)}\cdots X_{\sigma(i)}\tau_b)\tau_a\;,
\end{aligned} \tag{5.44}$$

where we have substituted (5.42) and used (5.35). In addition, $[\![-,-]\!]$ is the bracket defined in (5.38c).

Likewise, the second term in (5.43) is given by

$$\begin{aligned}
(\mathrm{II}) \;&:=\; -\frac{1}{3!}\sum_{k_1+k_2+k_3=i}\;\sum_{\sigma\in\mathrm{Sh}(k_1,k_2;i)}\times \\
&\qquad \times\,\mu_3\big(\phi_{k_1}\big(A_{\sigma(1)},\ldots,A_{\sigma(k_1)}\big),\ldots,\phi_{k_3}\big(A_{\sigma(k_1+k_2+1)},\ldots,A_{\sigma(i)}\big)\big) \\
&=\; -\frac{1}{3!}\sum_{\sigma\in S_i}\sum_{j=1}^{i-2}\sum_{k=j+1}^{i-1}\frac{1}{j!(k-j)!(i-k)!}\mu_3\big(\phi_j\big(A_{\sigma(1)},\ldots,A_{\sigma(j)}\big), \\
&\qquad\qquad \phi_{k-j}\big(A_{\sigma(j+1)},\ldots,A_{\sigma(k)}\big),\phi_{i-k}\big(A_{\sigma(k+1)},\ldots,A_{\sigma(i)}\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2!}\sum_{\sigma\in S_i}\sum_{j=1}^{i-2}\sum_{k=j+1}^{i-1}\frac{1}{j!(k-j)!(i-k)!}\times\\
&\qquad\times [\![J^a(\sigma(1),\ldots,\sigma(j)),J^b(\sigma(j+1),\ldots,\sigma(k)),J^c(\sigma(k+1),\ldots,\sigma(i))]\!]\times\\
&\qquad\times f_{bcd}f_{aef}g^{de}g^{fg}\tau_g\\
&= \sum_{\sigma\in S_i}\sum_{j=1}^{i-2}\sum_{k=j+1}^{i-1}[\![J(\sigma(1),\ldots,\sigma(j)),J(\sigma(j+1),\ldots,\sigma(k)),J(\sigma(k+1),\ldots,\sigma(i))]\!]'\times\\
&\qquad\times \mathrm{e}^{\mathrm{i}(k_{\sigma(1)}+\cdots+k_{\sigma(i)})\cdot x}\,g^{ab}\mathrm{tr}(X_{\sigma(1)}\cdots X_{\sigma(i)}\tau_b)\tau_a\ ,
\end{aligned}
\tag{5.45}
$$

where we have again substituted (5.42), used twice the relations (5.35), and defined

$$
[\![J(1),J(2),J(3)]\!]' \ :=\ [\![J(1),J(2),J(3)]\!]-[\![J(3),J(1),J(2)]\!] \tag{5.46}
$$

with $[\![-,-,-]\!]$ the bracket introduced in (5.39c). Hence, upon adding (I) and (II) and applying the contracting homotopy h_2 from (5.32), we find

$$
\begin{aligned}
J(1,\ldots,i)\ &=\\
&= \frac{1}{(k_1+\cdots+k_i)^2}\times\\
&\qquad\times\hat{P}_{\text{ex}}\left\{\sum_{j=1}^{i-1}[\![J(1,\ldots,j),J(j+1,\ldots,i)]\!]+\right.\\
&\qquad\qquad\left.+\sum_{j=1}^{i-2}\sum_{k=j+1}^{i-1}[\![J(1,\ldots,j),J(j+1,\ldots,k),J(k+1,\ldots,i)]\!]'\right\}.
\end{aligned}
\tag{5.47}
$$

This is precisely the Berends–Giele recursion [73] modulo the appearance of the projector $\hat{P}_{\text{ex}}$. As before, it acts trivially, as follows from the current conservation property of the expression inside the curly bracket, that is, $(k_1+\cdots+k_i)\cdot\{\cdots\}=0$.

Altogether, we conclude that the quasi-isomorphism between the L_∞-algebra governing Yang–Mills theory in the second-order formulation and its minimal model encodes the Berends–Giele gluon current recursion relations. The actual scattering amplitudes $\mathscr{A}(1,\ldots,i)$ now follow directly from the homotopy Maurer–Cartan action for the minimal model brackets (5.7a) for this quasi-isomorphism. For $i\geqslant 2$, we have

$$
\mathscr{A}(A_1,\ldots,A_{i+1})\ =\ \mathrm{i}\langle A_1,\mu_i^\circ(A_2,\ldots,A_{i+1})\rangle_{\mathfrak{L}_{\text{YM}}} \tag{5.48a}
$$

with

$$\begin{aligned}\mu_i^\circ(A_1,\ldots,A_i) &= \\ &= -\sum_{\sigma\in S_i}(k_1+\cdots+k_i)^2\,J_\mu(\sigma(1),\ldots,\sigma(i))\,\mathrm{e}^{\mathrm{i}(k_{\sigma(1)}+\cdots+k_{\sigma(i)})\cdot x}\times \\ &\qquad\times g^{ab}\mathrm{tr}(X_{\sigma(1)}\cdots X_{\sigma(i)}\tau_b)\tau_a\,\mathrm{d}x^\mu\Big|_{(k_1+\cdots+k_i)^2=0}\,,\end{aligned} \tag{5.48b}$$

where $J_\mu(1,\ldots,i)$ as given in (5.47). Note that the expression $\mu_i^\circ(A_1,\ldots,A_i)$ is already co-closed and hence, the projection p in (5.7a) acts by requiring that $(k_1+\cdots+k_i)^2=0$ in the case at hand. Note also that the symmetry of the amplitude (5.48a) under the exchange of any two gluons is due to the cyclic property (2.21).

5.6. Colour structure of scattering amplitudes

To further demonstrate the power of our formalism, we examine the colour structure of scattering amplitudes in YM theory. This is facilitated by our generalisation from the L_∞-algebras from the BV formalism to A_∞-algebras.

Consider plane waves $A_i = a_i X_i = a_{i\mu}\,\mathrm{d}x^\mu X_i \in H^1_{\mathsf{m}_1}(\mathfrak{a})$ with $a_{i\mu} := \varepsilon_\mu(k_i)\,\mathrm{e}^{\mathrm{i}k_i\cdot x}$, where k_i is the on-shell momentum, $\varepsilon(k_i)$ is the polarisation in Lorenz gauge $k_i\cdot\varepsilon(k_i)=0$, and $X_i\in\mathfrak{u}(N)$ is the colour part. The scattering amplitude then is

$$\begin{aligned}\mathscr{A}_n(A_1,A_2,\ldots,A_n) &= \mathrm{i}\sum_{\sigma\in S_{n-1}}\langle A_n,\mathsf{m}^\circ_{n-1}(A_{\sigma(1)},\ldots,A_{\sigma(n-1)})\rangle \\ &= \mathrm{i}\sum_{\sigma\in S_n/\mathbb{Z}_n}\langle A_{\sigma(1)},\mathsf{m}^\circ_{n-1}(A_{\sigma(2)},\ldots,A_{\sigma(n)}\rangle\,,\end{aligned} \tag{5.49a}$$

where

$$\mathsf{m}_i^\circ = \left(\mathrm{pr}_{\mathsf{T}^1(\mathfrak{A}^\circ)}\circ\mathsf{P}_0\circ\mathsf{D}_{\rm int}\circ\mathsf{E}\right)\Big|_{\mathsf{T}^i(\mathfrak{A}^\circ)} = \sum_{\ell=0}^\infty\hbar^\ell\mathsf{m}^\circ_{i,\ell} \tag{5.49b}$$

as follows from Equation (5.23), and with E satisfying again the recursion relation (5.27b). The interaction vertices m_i in $\mathsf{D}_{\rm int}$, as given by (4.14), lead to products of the colour parts and kinematic functions. Given (composite) fields $\Phi_i=\phi_iX_i\in\mathfrak{A}_1$, we can define colour-stripped interactions m_i by

$$\mathsf{m}_i(\Phi_1,\ldots,\Phi_i) =: m_i(\phi_1,\ldots,\phi_i)X_1\cdots X_i \tag{5.50}$$

and D_{int} acts on tensor products as in (5.17), e.g.

$$\begin{aligned}\mathsf{D}_{\text{int}}(\Phi_1 \otimes \Phi_2 \otimes \Phi_3) &= m_2(\phi_1, \phi_2)X_1X_2 \otimes \phi_3 X_3 + \phi_1 X_1 \otimes m_2(\phi_2, \phi_3)X_2X_3 + \\ &\quad + m_3(\phi_1, \phi_2, \phi_3)X_1X_2X_3 \ .\end{aligned} \tag{5.51}$$

Moreover, Δ^* acts similarly as in (5.25) on the components ϕ_i of Φ_i by inserting in all possible places of the tensor product of the Φ_is a complete pair of field and antifield components,

$$\Psi^+_\ominus = \psi^+_\theta(k, \varepsilon)|a)(b| \quad \text{and} \quad \Psi^\ominus = \psi^\theta(k, \varepsilon)|b)(a| \ , \tag{5.52}$$

where $|a)(b|$ is the $(N \times N)$-matrix with the only non-vanishing entry 1 at position (a, b) and $\ominus$ are multi-indices including particle species (labelled by θ), momenta (labelled by k), polarisations (labelled by ε), and colours (labelled by a and b). Contractions of $\ominus$ thus imply sums and integrals.

If Δ^* is applied once in the recursion, the colour factor of the amplitude contains terms of the form

$$\sum_{a,b=1}^{N} X_1 \otimes \cdots \otimes X_j \otimes |a)(b| \otimes |b)(a| \otimes X_{j+1} \otimes \cdots \otimes X_i \tag{5.53a}$$

and

$$\sum_{a,b=1}^{N} X_1 \otimes \cdots \otimes X_j \otimes |a)(b| \otimes X_{j+1} \otimes \cdots \otimes X_k \otimes |b)(a| \otimes X_{k+1} \otimes \cdots \otimes X_i \ . \tag{5.53b}$$

Contributing to the amplitude (5.49a) are exactly those expressions in which all the tensor products in the colour factors have been turned into matrix products by the D_{int}. The terms (5.53a), with neighbouring insertion points, enter into planar Feynman diagrams and they come with an additional factor of N. The terms (5.53b) enter into non-planar Feynman diagrams.

More generally, it is clear that the ℓ-loop n-point amplitude has maximally $t = \max\{\ell, n\}$ traces in its colour factor and that contributions with t traces come with a factor $N^{\ell - t + 1}$. Thus, as well-known, planar Feynman diagrams dominate in the large-N limit.

5.7. One-loop structure

Let us look at the structure of one-loop scattering amplitudes in more detail. Upon iterating (5.27b), we find

$$\begin{aligned}
\mathsf{m}^{\circ}_{i,1} &= \left(\mathrm{pr}_{\mathsf{T}^1(\mathfrak{A}^\circ)} \circ \mathsf{P}|_{\mathcal{O}(\hbar^0)} \circ (-\mathrm{i}\Delta^*) \circ \mathsf{E}|_{\mathcal{O}(\hbar^0)}\right)\Big|_{\mathsf{T}^i(\mathfrak{A}^\circ)} , \\
\mathsf{P}|_{\mathcal{O}(\hbar^0)} &= \mathsf{P}_0 \circ (1 + \mathsf{D}_{\mathrm{int}} \circ \mathsf{H}_0)^{-1} , \\
\mathsf{E}|_{\mathcal{O}(\hbar^0)} &= (1 + \mathsf{H}_0 \circ \mathsf{D}_{\mathrm{int}})^{-1} \circ \mathsf{E}_0 ;
\end{aligned} \tag{5.54}$$

see also (5.21). The form of the interaction vertices and our above considerations directly yield

$$\begin{aligned}
\mathsf{m}^{\circ}_{i,1}(A_1, \ldots, A_i) &= \kappa^{i-1}\Big[N J_{i,1}(1, \ldots, i)\, \mathrm{e}^{\mathrm{i}k_{1i}\cdot x} X_1 \cdots X_i \, + \\
&\quad + \sum_{j=1}^{i-1} K^j_{i,1}(1, \ldots, i)\, \mathrm{e}^{\mathrm{i}k_{1i}\cdot x} X_1 \cdots X_j \,\mathrm{tr}(X_{j+1} \cdots X_i)\Big]\Big|_{k^2_{1i}=0}
\end{aligned} \tag{5.55}$$

with $k_{ij} := k_i + \cdots + k_j$ for $i \leqslant j$. The currents $J_{i,1}, K^j_{i,1} \in \Omega^1$ contain all the kinematical information and eventually form the one-loop generalisation of the tree-level Berends–Giele current [73] after symmetrisation.

The general form of the one-loop amplitude thus is

$$\begin{aligned}
\mathscr{A}_{n,1}(A_1, \ldots, A_n) &= N \sum_{\sigma \in S_n/\mathbb{Z}_n} \alpha^0_{n,1}(\sigma(1), \ldots, \sigma(n))\, \mathrm{tr}(X_{\sigma(1)} \cdots X_{\sigma(n)}) \, + \\
&\quad + \sum_{m=1}^{n-1} \sum_{\sigma \in S_n/(\mathbb{Z}_m \times \mathbb{Z}_{n-m})} \alpha^m_{n,1}(\sigma(1), \ldots, \sigma(n)) \, \times \\
&\qquad \times \mathrm{tr}(X_{\sigma(1)} \cdots X_{\sigma(m)}) \mathrm{tr}(X_{\sigma(m+1)} \cdots X_{\sigma(n)}) ,
\end{aligned} \tag{5.56}$$

where $\alpha^0_{n,1}$ is a linear combination of (the components of) $J_{n-1,1}$ and the $\alpha^m_{n,1}$ of $K^{m-1}_{n-1,1}$. The result (5.56) was first derived in [221] using different methods.

In [222] it was shown that the $\alpha^m_{n,1}$ are linear combinations of the $\alpha^0_{n,1}$ so that the full scattering amplitude can be constructed from its planar part. Explicitly,

$$\alpha^m_{n,1}(1, \ldots, n) = (-1)^m \sum_{\sigma \in \mathrm{COP}_{m,n}} \alpha^0_{n,1}(\sigma(1), \ldots, \sigma(n)) , \tag{5.57}$$

where $\mathrm{COP}_{m,n}$ are all permutations of $(1, \ldots, n)$ which preserve the position of n as well as the cyclic orders of $(1, \ldots, m)$ and $(m + 1, \ldots, n)$.

The relation (5.57) can be derived from our recursion relation, but the derivation simplifies significantly if we use the strictification theorem for homotopy algebras (see e.g. [197]): any A_∞-algebra is quasi-isomorphic (read: equivalent for all physical purposes, cf. [2,52]) to a strict A_∞-algebra, which is an A_∞-algebra with $m_i = 0$ for $i \geqslant 3$. YM theory admits a first-order formulation which constitutes a strictification, see [223–225,31,2,52] (see also [226, 227]) for the L_∞-algebra description and the quasi-isomorphism, and we readily apply our formalism. Specifically, we compute again scattering amplitudes using formulas (5.49), but now $m_3 = 0$, which simplifies the discussion, and the plane waves have to be replaced by their pre-image under the (strict!) isomorphism that links the minimal models of the original A_∞-algebra and of its minimal model.

As in the ordinary case, m_2 is anti-symmetric also in the strict case. Moreover, m_2° cannot change the order of the colour parts X_i, and so, $\alpha^m_{n,1}$ arises from the terms

$$\begin{aligned}\sum_{k=m}^{n-1}\sum_{\sigma\in C_m}\Big\langle &\mathrm{e}(A_n), \mathcal{M}\big(\mathsf{D}_{\text{tree}}(A_{m+1}\otimes\cdots\otimes A_k\otimes \mathsf{h}(\Psi^+_\ominus)\otimes \\ &\otimes A_{\sigma(1)}\otimes\cdots\otimes A_{\sigma(m)}\otimes\Psi^\ominus\otimes A_{k+1}\otimes\cdots\otimes A_{n-1})\big) + \\ &+\mathcal{M}\big(\mathsf{D}_{\text{tree}}(A_{m+1}\otimes\cdots\otimes A_k\otimes\Psi^\ominus\otimes A_{\sigma(1)}\otimes \\ &\otimes\cdots\otimes A_{\sigma(m)}\otimes \mathsf{h}(\Psi^+_\ominus)\otimes A_{k+1}\otimes\cdots\otimes A_{n-1})\big)\Big\rangle\,,\end{aligned} \tag{5.58}$$

where $\mathsf{D}_{\text{tree}} := \mathsf{D}_{\text{int}}\circ(\mathsf{H}\circ\mathsf{D}_{\text{int}})^{n-1}$ produces a formal sum of full binary trees with $n+1$ leaves corresponding to the $n+1$ arguments and nodes corresponding to the map m_2 applied to their children. We call these trees non-planar trees and the leaves corresponding to the $A_1,\ldots,A_m$ inner leaves, while all other leaves are outer leaves. For any tree, the sequence of arguments corresponding to the leaves of the tree will be called its leaf sequence.

Similarly, the planar trees relevant in the planar contributions arise from expressions

$$\begin{aligned}\sum_{k=0}^{n-1}\sum_{\sigma\in \mathrm{COP}_{m,n}}\Big\langle &\mathrm{e}(A_n), \mathcal{M}(\mathsf{D}_{\text{tree}}(A_{\sigma(1)}\otimes\cdots\otimes A_{\sigma(k)}\otimes \\ &\otimes(\mathsf{h}(\Psi^+_\ominus)\otimes\Psi^\ominus+\Psi^\ominus\otimes\mathsf{h}(\Psi^+_\ominus))\otimes A_{\sigma(k+1)}\otimes\cdots\otimes A_{\sigma(n-1)}))\Big\rangle\,.\end{aligned} \tag{5.59}$$

For both the non-planar and planar trees, the linear function $\mathcal{M}$ assigns a combinatorial factor to each tree, arising from the various sequences of the operations $\mathsf{H}\circ\mathsf{D}_{\text{int}}$ and $\mathsf{H}\circ\Delta^*$ in the recursion relation (5.27b).

Upon stripping off the colour factor in each tree, $\mathrm{tr}(X_1\cdots X_m)\mathrm{tr}(X_{m+1}\cdots X_n)$, we obtain two formal sums of binary trees with nodes corresponding to m_2 and leaf sequences consisting of a_i, $\psi^\theta(k,\varepsilon)$ and $\mathsf{h}(\psi^+_\theta(k,\varepsilon))$.

There is now a one-to-one correspondence between the two sets of full binary trees with leaf sequence $A_1,\dots,A_k$ and with leaf sequence $A_k,\dots,A_1$, by inverting the order of children in each of the $k-1$ nodes ('flipping the nodes'), which gives rise to a factor of $(-1)^{k-1}$.

In each non-planar binary tree with inner leaves, we can now flip common ancestor of a ψ, turning inner leaves into outer leaves. We start from common ancestors closest to the leaves. In each flip, k inner leaves are turned into outer leaves, and together with the initial flip, fully reversing their ordering leads to a relative factor of $(-1)^k$. We stop this process when all m inner leaves have become outer leaves, with a relative factor of $(-1)^m$.

This map from non-planar to planar trees is clearly injective. It is, however, not surjective since its image does not contain planar trees which have vertices who have a ψ and a root of a subtree containing both inner and outer leaves as descendants. These, however, cancel pairwise: pick any outer leaf, and flip the first common ancestor with an inner leaf. This leads to a negative contribution from another tree, which is included in (5.59) due to the sum over the COP permutations.

It remains to compare the multiplicities $\mathcal{M}$ for non-planar and planar trees. Flipping a node does not change the combinatorial factor for applying $\mathsf{H}\circ\mathsf{D}_{\mathrm{int}}$ in different ways. It can, however, affect the multiplicity arising from applying $\mathsf{H}\circ\Delta^*$ at different positions since in the planar trees, inner and outer leaves can be joined to subtrees before applying $\mathsf{H}\circ\Delta^*$, which was not possible in the non-planar case. These subtrees are of the type discussed in the previous paragraph and they cancel again pairwise.

6

Factorisation of homotopy algebras and colour ordering

The tensor product between arbitrary homotopy structures is not defined in general. An adequate notion of factorisation of homotopy algebras is instrumental to our interpretation of the colour–kinematic duality and double copy: in this Chapter we consider a notion of tensor product between (strict) homotopy algebras, and we generalise it with the introduction of a twist. In Chapter 7 and Chapter 9 this construction will be applied to the factorisation of the L_∞-algebras of biadjoint scalar theory, Yang–Mills theory and $\mathcal{N} = 0$ supergravity, providing a Lagrangian, homotopy algebra realisation of double copy.

The material in this Chapter is borrowed from [6].

6.1. Tensor products of homotopy algebras

Tensor products of strict homotopy algebras. Let Ass, Com, and Lie denote (the categories of) associative, commutative, and Lie algebras, respectively. Schematically, we have *tensor products* of the form

$$\begin{aligned} \otimes : \mathsf{Ass} \times \mathsf{Ass} &\rightarrow \mathsf{Ass} , \quad \otimes : \mathsf{Com} \times \mathsf{Ass} \rightarrow \mathsf{Ass} , \quad \otimes : \mathsf{Ass} \times \mathsf{Com} \rightarrow \mathsf{Ass} , \\ \otimes : \mathsf{Com} \times \mathsf{Com} &\rightarrow \mathsf{Com} , \quad \otimes : \mathsf{Com} \times \mathsf{Lie} \rightarrow \mathsf{Lie} , \quad \otimes : \mathsf{Lie} \times \mathsf{Com} \rightarrow \mathsf{Lie} . \end{aligned} \tag{6.1}$$

In particular, let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras from this list for which there is a tensor product. The vector space underlying the tensor product algebra $\mathfrak{A} \otimes \mathfrak{B}$ is simply the ordinary tensor product of vector spaces and the product $\mathsf{m}_2^{\mathfrak{A}\otimes\mathfrak{B}}$ is given by

$$\mathsf{m}_2^{\mathfrak{A}\otimes\mathfrak{B}}(a_1 \otimes b_1, a_2 \otimes b_2) \;:=\; \mathsf{m}_2^{\mathfrak{A}}(a_1, a_2) \otimes \mathsf{m}_2^{\mathfrak{B}}(b_1, b_2) \tag{6.2}$$

for $a_1, a_2 \in \mathfrak{A}$ and $b_1, b_2 \in \mathfrak{B}$.

On the other hand, the tensor product of two cochain complexes $(\mathfrak{A}, \mathsf{m}_1^{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{m}_1^{\mathfrak{B}})$ is defined as the tensor product of the underlying (graded) vector spaces $\mathfrak{A}$ and $\mathfrak{B}$,

$$\mathfrak{A} \otimes \mathfrak{B} = \bigoplus_{k \in \mathbb{Z}} (\mathfrak{A} \otimes \mathfrak{B})_k \quad \text{with} \quad (\mathfrak{A} \otimes \mathfrak{B})_k := \bigoplus_{i+j=k} \mathfrak{A}_i \otimes \mathfrak{B}_j \ , \tag{6.3a}$$

cf. (3.2). The differential $\mathsf{m}_1^{\mathfrak{A} \otimes \mathfrak{B}}$ is defined as

$$\mathsf{m}_1^{\mathfrak{A} \otimes \mathfrak{B}}(a \otimes b) := \mathsf{m}_1^{\mathfrak{A}}(a) \otimes b + (-1)^{|a|_{\mathfrak{A}}} a \otimes \mathsf{m}_1^{\mathfrak{B}}(b) \tag{6.3b}$$

for $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$.

Strict A_∞-, C_∞-, and L_∞-algebras are nothing but differential graded associative, commutative, and Lie algebras, respectively. For such algebras $\mathfrak{A}$ and $\mathfrak{B}$, the above formulas combine to

$$\begin{aligned} \mathsf{m}_1^{\mathfrak{A} \otimes \mathfrak{B}}(a_1 \otimes b_1) &:= \mathsf{m}_1^{\mathfrak{A}}(a_1) \otimes b_1 + (-1)^{|a_1|_{\mathfrak{A}}} a_1 \otimes \mathsf{m}_1^{\mathfrak{B}}(b_1) \ , \\ \mathsf{m}_2^{\mathfrak{A} \otimes \mathfrak{B}}(a_1 \otimes b_1, a_2 \otimes b_2) &:= (-1)^{|b_1|_{\mathfrak{B}} |a_2|_{\mathfrak{A}}} \mathsf{m}_2^{\mathfrak{A}}(a_1, a_2) \otimes \mathsf{m}_2^{\mathfrak{B}}(b_1, b_2) \end{aligned} \tag{6.4}$$

for $a_1, a_2 \in \mathfrak{A}$ and $b_1, b_2 \in \mathfrak{B}$. If, in addition, the two differential graded algebras carry cyclic inner products $\langle -, - \rangle_{\mathfrak{A}}$ and $\langle -, - \rangle_{\mathfrak{B}}$, then the tensor product carries the cyclic inner product

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{\mathfrak{A} \otimes \mathfrak{B}} := (-1)^{|b_1|_{\mathfrak{B}} |a_2|_{\mathfrak{A}} + s(|a_1|_{\mathfrak{A}} + |a_2|_{\mathfrak{A}})} \langle a_1, a_2 \rangle_{\mathfrak{A}} \langle b_1, b_2 \rangle_{\mathfrak{B}} \tag{6.5}$$

for $a_1, a_2 \in \mathfrak{A}$ and $b_1, b_2 \in \mathfrak{B}$. Here, $s := |\langle -, - \rangle_{\mathfrak{B}}|_{\mathfrak{B}}$ is the degree of the inner product on $\mathfrak{B}$.

Tensor products of general homotopy algebras. There is a simple argument that extends the above tensor product of strict homotopy algebras to general homotopy algebras, using not much more than the homological perturbation lemma. Let us therefore also briefly consider this case, even though we will only make use of it in passing when discussing colour-stripping of Yang–Mills amplitudes.

An extension from the strict case to the general case can be performed as follows. Recall that the strictification theorem asserts that every homotopy algebra is quasi-isomorphic to a strict homotopy algebra, see Section 2.4. for details. Using this theorem, we first strictify each of the factors $\mathfrak{A}$ and $\mathfrak{B}$ in the tensor product $\mathfrak{A} \otimes \mathfrak{B}$ we wish to define. We then compute the tensor product $\mathfrak{A}^{\mathrm{st}} \otimes \mathfrak{B}^{\mathrm{st}}$ of the strictified factors. This is a homotopy algebra whose underlying cochain complex $\mathsf{Ch}(\mathfrak{A}^{\rm st}\otimes\mathfrak{B}^{\rm st})$ is quasi-isomorphic to the tensor product $\mathsf{Ch}(\mathfrak{A})\otimes\mathsf{Ch}(\mathfrak{B})$ of the two differential complexes underlying the factors $\mathfrak{A}$ and $\mathfrak{B}$. We can then use the homological perturbation lemma, most readily in the form used e.g. in [3] for the coalgebra formulation of homotopy algebras, to transfer the full homotopy structure from $\mathsf{Ch}(\mathfrak{A}^{\rm st}\otimes\mathfrak{B}^{\rm st})$ to $\mathsf{Ch}(\mathfrak{A})\otimes\mathsf{Ch}(\mathfrak{B})$ along the quasi-isomorphism between the cochain complexes. This yields a homotopy algebra structure on $\mathsf{Ch}(\mathfrak{A})\otimes\mathsf{Ch}(\mathfrak{B})$ together with a quasi-isomorphism to the tensor product of the strictified factors. We stress that this transfer is not unique and depends on the choice of contracting homotopy (essentially, a choice of gauge).

We stress that the fact that the tensor products (6.1) lift to corresponding tensor products of homotopy algebras is found in the literature for special cases, see e.g. [228,229] for the case of A_∞-algebras, as well as [230, Appendix B] for the case of tensor products of C_∞-algebras with Lie algebras.

Tensor products of matrix and Lie algebras with homotopy algebras. To capture the colour decomposition of amplitudes in Yang–Mills theory, it suffices to consider the tensor product between homotopy algebras and matrix (Lie) algebras. In particular, given a matrix algebra (or, more generally, an associative algebra) $\mathfrak{a}$ and an A_∞-algebra $(\mathfrak{A},\mathsf{m}_i)$, then the tensor product $\mathfrak{a}\otimes\mathfrak{A}$ is equipped with the higher products

$$\mathsf{m}_i^{\mathfrak{a}\otimes\mathfrak{A}}(\mathsf{e}_1\otimes a_1,\ldots,\mathsf{e}_i\otimes a_i) \;:=\; \mathsf{e}_1\cdots\mathsf{e}_i\otimes\mathsf{m}_i(a_1,\ldots,a_i) \tag{6.6}$$

for all $\mathsf{e}_1,\ldots,\mathsf{e}_i\in\mathfrak{a}$ and $a_1,\ldots,a_i\in\mathfrak{A}$ and $i\in\mathbb{N}^+$. Evidently, these formulas can also be applied to the tensor product between a matrix algebra and a C_∞-algebra, however, the result will, in general, be an A_∞-algebra rather than a C_∞-algebra as, for instance, the binary product on the tensor product will not necessarily be graded commutative.

Next, we may consider the tensor product $\mathfrak{g}\otimes\mathfrak{C}$ between a Lie algebra $(\mathfrak{g},[-,-])$ and a C_∞-algebra $(\mathfrak{C},\mathsf{m}_i)$. We obtain an L_∞-algebra $(\mathfrak{L},\mu_i)$ with $\mathfrak{L}:=\mathfrak{g}\otimes\mathfrak{C}$, however, the higher products μ_i are less straightforward than the ones in (6.6) for A_∞-algebras. Nevertheless, they can be computed iteratively, and we obtain for the lowest products[1]

$$\begin{aligned}\mu_1(\mathsf{e}_1\otimes c_1) \;&:=\; \mathsf{e}_1\otimes\mathsf{m}_1(c_1)~,\\ \mu_2(\mathsf{e}_1\otimes c_1,\mathsf{e}_2\otimes c_2) \;&:=\; [\mathsf{e}_1,\mathsf{e}_2]\otimes\mathsf{m}_2(c_1,c_2)~,\end{aligned} \tag{6.7a}$$

[1]As detailed in (2.19), the graded anti-symmetrisation of any A_∞-algebra yields an L_∞-algebra, and so the form of the higher products can be gleaned from the graded anti-symmetrisation of (6.6).

and

$$
\begin{aligned}
\mu_3(e_1 \otimes c_1, e_2 \otimes c_2, e_3 \otimes c_3) \;&:=\; [e_1,[e_2,e_3]] \otimes m_3(c_1,c_2,c_3) - \\
&\quad - (-1)^{|c_1|_{\mathfrak{C}}|c_2|_{\mathfrak{C}}}[e_1,[e_2,e_3]] \otimes m_3(c_2,c_1,c_3) + \\
&\quad + (-1)^{|c_1|_{\mathfrak{C}}|c_2|_{\mathfrak{C}}}[[e_1,e_2],e_3] \otimes m_3(c_2,c_1,c_3)\;, \\
\mu_4(e_1 \otimes c_1, e_2 \otimes c_2, e_3 \otimes c_3, e_4 \otimes c_4) \;&:=\; [e_1,[e_2,[e_3,e_4]]] \otimes m_4(c_1,c_2,c_3,c_4) - \\
&\quad - (-1)^{|c_2|_{\mathfrak{C}}|c_3|_{\mathfrak{C}}}[e_1,[e_3,[e_2,e_4]]] \otimes m_4(c_1,c_3,c_2,c_4) - \\
&\quad - (-1)^{|c_1|_{\mathfrak{C}}|c_2|_{\mathfrak{C}}}[e_2,[e_1,[e_3,e_4]]] \otimes m_4(c_2,c_1,c_3,c_4) + \\
&\quad + (-1)^{|c_1|_{\mathfrak{C}}(|c_2|_{\mathfrak{C}}+|c_3|_{\mathfrak{C}})}[[[e_1,e_4],e_3],e_2] \otimes m_4(c_2,c_3,c_1,c_4) - \\
&\quad - (-1)^{(|c_1|_{\mathfrak{C}}+|c_2|_{\mathfrak{C}})|c_3|_{\mathfrak{C}}}[[e_1,[e_2,e_4]],e_3] \otimes m_4(c_3,c_1,c_2,c_4) - \\
&\quad - (-1)^{|c_1|_{\mathfrak{C}}(|c_2|_{\mathfrak{C}}+|c_3|_{\mathfrak{C}})+|c_2|_{\mathfrak{C}}|c_3|_{\mathfrak{C}}}[[[e_1,e_4],e_2],e_3] \otimes m_4(c_3,c_2,c_1,c_4) \\
&\;\;\vdots
\end{aligned}
\tag{6.7b}
$$

for all $e_1, \ldots, e_4 \in \mathfrak{g}$ and $c_1, \ldots, c_4 \in \mathfrak{C}$. We list these formulas here as they are useful in colour-stripping in Yang–Mills theory and we have not been able to find them in the literature.

6.2. Colour-stripping in Yang–Mills theory

As an example of the above factorisations, let us discuss colour-stripping in Yang–Mills theory and show that this is nothing but a factorisation of homotopy algebras. For concreteness, let us consider the gauge-fixed action (4.17) and the corresponding L_∞-algebra $\mathfrak{L}^{\mathrm{YM,\,gf}}_{\mathrm{BV}}$.

If the gauge Lie algebra $\mathfrak{g}$ is a matrix Lie algebra, then the L_∞-algebra $\mathfrak{L}^{\mathrm{YM,\,gf}}_{\mathrm{BV}}$ is the total anti-symmetrisation via (2.19) of a family of A_∞-algebras. One of these is special in that it is totally graded anti-symmetric [3] and thus is also a C_∞-algebra.

More generally, we can factorise $\mathfrak{L}^{\mathrm{YM,\,gf}}_{\mathrm{BV}}$ into a gauge Lie algebra $\mathfrak{g}$ and a colour C_∞-algebras $\mathfrak{C}^{\mathrm{YM,\,gf}}_{\mathrm{BV}}$ using formula (6.6),

$$
\mathfrak{L}^{\mathrm{YM,\,gf}}_{\mathrm{BV}} \;=\; \mathfrak{g} \otimes \mathfrak{C}^{\mathrm{YM,\,gf}}_{\mathrm{BV}}
\tag{6.8}
$$

Explicitly, the C_∞-algebra $\mathfrak{C}_{\mathrm{BV}}^{\mathrm{YM,\,gf}}$ has the underlying cochain complex

$$
\begin{array}{ccccccc}
 & & \overset{A_\mu}{\Omega^1(\mathbb{M}^d)} & \xrightarrow{-(\partial_\nu\partial^\mu-\delta_\nu^\mu\Box)} & \overset{A_\mu^+}{\Omega^1(\mathbb{M}^d)} & & \\
 & \overset{-\partial_\mu}{\nearrow} & & \overset{\partial_\mu}{\nearrow}\ \overset{-\partial^\mu}{\searrow} & & \overset{-\partial^\mu}{\searrow} & \\
 & & \overset{b}{\mathscr{C}^\infty(\mathbb{M}^d)} & \xrightarrow{\ \xi\ } & \overset{b^+}{\mathscr{C}^\infty(\mathbb{M}^d)} & & \\
\underbrace{\overset{c}{\mathscr{C}^\infty(\mathbb{M}^d)}}_{=:\,\mathfrak{C}_{\mathrm{BV},0}^{\mathrm{YM,\,gf}}} & \xrightarrow{-\Box} & \underbrace{\overset{\bar c^+}{\mathscr{C}^\infty(\mathbb{M}^d)}}_{=:\,\mathfrak{C}_{\mathrm{BV},1}^{\mathrm{YM,\,gf}}} & & \underbrace{\overset{\bar c}{\mathscr{C}^\infty(\mathbb{M}^d)}}_{=:\,\mathfrak{C}_{\mathrm{BV},2}^{\mathrm{YM,\,gf}}} & \xrightarrow{-\Box} & \underbrace{\overset{c^+}{\mathscr{C}^\infty(\mathbb{M}^d)}}_{=:\,\mathfrak{C}_{\mathrm{BV},3}^{\mathrm{YM,\,gf}}}
\end{array}
\tag{6.9a}
$$

where we label subspaces again by the fields parametrising them

$$
\begin{gathered}
\mathfrak{C}_{\mathrm{BV},0}^{\mathrm{YM,\,gf}} = \mathfrak{C}_{\mathrm{BV},0,c}^{\mathrm{YM,\,gf}}~,\quad \mathfrak{C}_{\mathrm{BV},1}^{\mathrm{YM,\,gf}} = \bigoplus_{\phi\in(A,b,\bar c^+)} \mathfrak{C}_{\mathrm{BV},1,\phi}^{\mathrm{YM,\,gf}}~,\\
\mathfrak{C}_{\mathrm{BV},2}^{\mathrm{YM,\,gf}} = \bigoplus_{\phi\in(A^+,b^+,\bar c)} \mathfrak{C}_{\mathrm{BV},2,\phi}^{\mathrm{YM,\,gf}}~,\quad \mathfrak{C}_{\mathrm{BV},3}^{\mathrm{YM,\,gf}} = \mathfrak{C}_{\mathrm{BV},3,c^+}^{\mathrm{YM,\,gf}}~.
\end{gathered}
\tag{6.9b}
$$

The non-trivial actions of the differential m_1 are

$$
\begin{aligned}
c &\xmapsto{\mathsf{m}_1} \begin{pmatrix} -\partial_\mu c \\ 0 \\ -\Box c \end{pmatrix} \in \bigoplus_{\phi\in(A,b,\bar c^+)} \mathfrak{C}_{\mathrm{BV},1,\phi}^{\mathrm{YM,\,gf}}~,\\
\begin{pmatrix} A_\mu \\ b \\ \bar c^+ \end{pmatrix} &\xmapsto{\mathsf{m}_1} \begin{pmatrix} -(\partial_\mu\partial^\nu-\delta_\mu^\nu\Box)A_\nu - \partial_\mu b \\ \partial^\mu A_\mu + \xi b \\ 0 \end{pmatrix} \in \bigoplus_{\phi\in(A^+,b^+,\bar c)} \mathfrak{C}_{\mathrm{BV},2,\phi}^{\mathrm{YM,\,gf}}~,\\
\begin{pmatrix} A_\mu^+ \\ b^+ \\ \bar c \end{pmatrix} &\xmapsto{\mathsf{m}_1} -\partial^\mu(A_\mu^+ + \partial_\mu \bar c) \in \mathfrak{C}_{\mathrm{BV},3,c^+}^{\mathrm{YM,\,gf}}~,
\end{aligned}
\tag{6.9c}
$$

the binary product m_2 acts as

$$
\begin{aligned}
(c_1, c_2) &\xmapsto{\mathsf{m}_2} g c_1 c_2 \in \mathfrak{C}^{\mathrm{YM,\,gf}}_{\mathrm{BV},\,0,\,c}\ ,\\
\left(c, \begin{pmatrix} A_\mu \\ b \\ \bar{c}^+ \end{pmatrix}\right) &\xmapsto{\mathsf{m}_2} g \begin{pmatrix} -cA_\mu \\ 0 \\ -\partial^\mu(cA_\mu) \end{pmatrix} \in \bigoplus_{\phi\in(A,\,b,\,\bar{c}^+)} \mathfrak{C}^{\mathrm{YM,\,gf}}_{\mathrm{BV},\,1,\,\phi}\ ,\\
\left(c, \begin{pmatrix} A^+_\mu \\ \bar{c} \\ b^+ \end{pmatrix}\right) &\xmapsto{\mathsf{m}_2} g \begin{pmatrix} c(A^+_\mu + \partial_\mu \bar{c}) \\ 0 \\ 0 \end{pmatrix} \in \bigoplus_{\phi\in(A^+,\,b^+,\,\bar{c})} \mathfrak{C}^{\mathrm{YM,\,gf}}_{\mathrm{BV},\,2,\,\phi}\ ,\\
(c, c^+) &\xmapsto{\mathsf{m}_2} g c c^+ \in \mathfrak{C}^{\mathrm{YM,\,gf}}_{\mathrm{BV},\,3,\,c^+}\ ,\\
\left(\begin{pmatrix} A_\mu \\ b \\ \bar{c}^+ \end{pmatrix}, \begin{pmatrix} A^+_\nu \\ \bar{c} \\ b^+ \end{pmatrix}\right) &\xmapsto{\mathsf{m}_2} -gA^\mu(A^+_\mu + \partial_\mu \bar{c}) \in \mathfrak{C}^{\mathrm{YM,\,gf}}_{\mathrm{BV},\,3,\,c^+}\ ,\\
\left(\begin{pmatrix} A_{1\mu} \\ b_1 \\ \bar{c}^+_1 \end{pmatrix}, \begin{pmatrix} A_{2\nu} \\ b_2 \\ \bar{c}^+_2 \end{pmatrix}\right) &\xmapsto{\mathsf{m}_2} 2g \begin{pmatrix} \partial^\nu(A_{1[\nu}A_{2\mu]}) + A^\nu_1 \partial_{[\nu}A_{2\mu]} - \partial_{[\nu}A_{1\mu]}A^\nu_2 \\ 0 \\ 0 \end{pmatrix}\\
&\qquad \in \bigoplus_{\phi\in(A^+,\,b^+,\,\bar{c})} \mathfrak{C}^{\mathrm{YM,\,gf}}_{\mathrm{BV},\,2,\,\phi}\ ,
\end{aligned}
\tag{6.9d}
$$

and the ternary product m_3 acts as

$$
\begin{aligned}
\left(\begin{pmatrix} A_{1\mu} \\ b_1 \\ \bar{c}^+_1 \end{pmatrix}, \begin{pmatrix} A_{2\nu} \\ b_2 \\ \bar{c}^+_2 \end{pmatrix}, \begin{pmatrix} A_{3\kappa} \\ b_3 \\ \bar{c}^+_4 \end{pmatrix}\right) &\xmapsto{\mathsf{m}_3} -2g^2 \begin{pmatrix} A^\nu_1 A_{2[\mu}A_{3\nu]} - A_{1[\mu}A_{2\nu]}A^\nu_3 \\ 0 \\ 0 \end{pmatrix}\\
&\qquad \in \bigoplus_{\phi\in(A^+,b^+,\bar{c})} \mathfrak{C}^{\mathrm{YM,\,gf}}_{2,\,\phi}\ .
\end{aligned}
\tag{6.9e}
$$

It is a straightforward exercise to check that these higher products do indeed satisfy the C_∞-algebra relations (2.1) and (2.14).

The factorisation (6.8) descends to the minimal model $\mathfrak{L}^{\mathrm{YM,\,gf}\,\circ}_{\mathrm{BV}}$,

$$
\mathfrak{L}^{\mathrm{YM,\,gf}\,\circ}_{\mathrm{BV}} = \mathfrak{g} \otimes \mathfrak{C}^{\mathrm{YM,\,gf}\,\circ}_{\mathrm{BV}}\ , \tag{6.10}
$$

and the higher products in the C_∞-algebra $\mathfrak{C}^{\mathrm{YM,\,gf}\,\circ}$ describes the colour-ordered tree-level scattering amplitudes. We set

$$
\mathscr{A}_{n,0}(1,\ldots,n) =: \mathrm{i} \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{tr}(\mathsf{e}_{a_{\sigma(1)}}\cdots\mathsf{e}_{a_{\sigma(n)}}) A_{n,0}(\sigma(1),\ldots,\sigma(n))\ , \tag{6.11}
$$

and we have the formula

$$A_{n,0}(1,\dots,n) \;=\; \langle n, \mathsf{m}^\circ_{n-1}(1,\dots,n-1)\rangle\;, \tag{6.12}$$

where the numbers $1,\dots,n$ represent the external free fields. The symmetry of the colour-stripped amplitude is reflected in the graded anti-symmetry of the higher products m°_i in the C_∞-algebra $\mathfrak{C}^{\mathrm{YM,\,gf}\,\circ}$, because all fields are of degree one.

6.3. Twisted tensor products of strict homotopy algebras

The factorisation of the L_∞-algebras corresponding to the field theories involved in the double copy is a twisted factorisation and we define our notion of twisted tensor products in the following.

Cochain complexes. A graded vector space is a particular example of a cochain complex with trivial differential. In our situation, we would like the vector space to act as an Abelian Lie algebra on the cochain complex. We therefore generalise the usual tensor product as follows. Given a graded vector space $\mathfrak{V}$ together with a cochain complex $(\mathfrak{A},\mathsf{m})$, we define a *twist datum* τ_1 to be a linear map

$$\begin{aligned}\tau_1 \,:\, \mathfrak{V} \;&\rightarrow\; \mathfrak{V}\otimes \mathrm{End}(\mathfrak{A})\;,\\ v \;&\mapsto\; \tau_1(v) \;:=\; \sum_\pi \tau_1^{\pi,1}(v)\otimes\tau_1^{\pi,2}(v)\;,\end{aligned} \tag{6.13}$$

where the index π labels the summands in the twist element $\tau_1(v)$.[1] Given a twist datum τ_1, the *twisted differential* is defined by

$$\mathsf{m}_1^{\tau_1}(v\otimes a) \;:=\; \sum_\pi (-1)^{|\tau_1^{\pi,1}(v)|_{\mathfrak{V}}}\;\tau_1^{\pi,1}(v)\otimes \mathsf{m}_1(\tau_1^{\pi,2}(v)(a)) \tag{6.15}$$

for $v\otimes a\in\mathfrak{V}\otimes\mathfrak{A}$. This rather cumbersome formula describes a rather simple procedure and it will become fully transparent in concrete examples. Evidently, there are constraints on admissible twist data. Firstly, $\mathsf{m}_1^{\tau_1}$ has to be differential and satisfy

$$\mathsf{m}_1^{\tau_1}\circ\mathsf{m}_1^{\tau_1} \;=\; 0\;, \tag{6.16}$$

[1] In Sweedler notation, popular e.g. in the context of Hopf algebras, we would simply write

$$\tau_1(v) := \tau_1^{(1)}(v)\otimes\tau_1^{(2)}(v)\;. \tag{6.14}$$

and secondly, $\mathsf{m}_1^{\tau_1}$ has to be cyclic with respect to the inner product (6.5) on the tensor product $\mathfrak{V}\otimes\mathfrak{A}$. We note that as it stands, the twisted tensor product is not necessarily compatible with quasi-isomorphisms as its cohomology is, in general, independent of those of the underlying factors. This is not an issue for our constructions, but explains why the above twist is not readily found in the mathematical literature.

As we shall see momentarily, one of the key roles of the twist is the construction of a complex of differential forms from a complex of functions. The following toy example exemplifies what we have in mind.

Example 6.1. *Consider the graded vector space* $\mathfrak{V}$ *and the cochain complex* $(\mathfrak{A},\mathsf{m}_1)$ *defined by*

$$\mathfrak{V} := \underbrace{\mathbb{M}^d\oplus\mathbb{R}}_{=:\,\mathfrak{V}_0} \quad \textit{and} \quad \mathfrak{A} := \Big(\underbrace{\mathscr{C}^\infty(\mathbb{M}^d)}_{=:\,\mathfrak{A}_1}\xrightarrow{\ \mathrm{id}\ }\underbrace{\mathscr{C}^\infty(\mathbb{M}^d)}_{=:\,\mathfrak{A}_2}\Big)~. \tag{6.17}$$

For a basis $(\mathrm{v}^\mu,\mathrm{n})$ *of* $\mathbb{M}^d\oplus\mathbb{R}$*, a choice of twist datum is given by*

$$\tau_1(\mathrm{v}^\mu) := 0\otimes 0 \quad \textit{and} \quad \tau_1(\mathrm{n}) := \mathrm{v}^\mu\otimes\frac{\partial}{\partial x^\mu}~. \tag{6.18}$$

The complex $\mathfrak{V}\otimes_\tau\mathfrak{A}$ *with the twisted differential* m_1^τ *is then*

$$\mathfrak{V}\otimes_\tau\mathfrak{A} = \left(\begin{array}{ccc} \Omega^1(\mathbb{M}^d)\cong\mathbb{M}^d\otimes\mathscr{C}^\infty(\mathbb{M}^d) & & \Omega^1(\mathbb{M}^d) = \mathbb{M}^d\otimes\mathscr{C}^\infty(\mathbb{M}^d)\\ \oplus & \overset{\mathrm{d}}{\nearrow} & \oplus \\ \mathscr{C}^\infty(\mathbb{M}^d)\cong\mathbb{R}\otimes\mathscr{C}^\infty(\mathbb{M}^d) & & \mathscr{C}^\infty(\mathbb{M}^d) = \mathbb{R}\otimes\mathscr{C}^\infty(\mathbb{M}^d)\end{array}\right) \tag{6.19}$$

Hence, we obtain a description of the cochain complex $(\mathscr{C}^\infty(\mathbb{M}^d)\oplus\Omega^1(\mathbb{M}^d),\mathrm{d})$*, albeit with some amount of redundancy.*

Differential graded algebras. Twisted tensor products for unital algebras were discussed in various places in the literature, e.g., in [231]. We would like to twist the ordinary tensor product of differential graded algebras introduced in Section 6.1., by extending the notion of twist datum from cochain complexes as follows. Given a graded vector space $\mathfrak{V}$ and a differential graded algebra $(\mathfrak{A},\mathsf{m}_1,\mathsf{m}_2)$, a *twist datum* is a pair of maps, one linear and the other one bilinear,

$$\begin{aligned}\tau_1 : \mathfrak{V} &\to \mathfrak{V}\otimes\mathrm{End}(\mathfrak{A})~,\\ v &\mapsto \tau_1(v) := \sum_\pi \tau_1^{\pi,1}(v)\otimes\tau_1^{\pi,2}(v)~,\end{aligned} \tag{6.20a}$$

and

$$\begin{aligned} \tau_2 \,:\, \mathfrak{V} \otimes \mathfrak{V} \;&\rightarrow\; \mathfrak{V} \otimes \mathsf{End}(\mathfrak{A}) \otimes \mathsf{End}(\mathfrak{A}) \;, \\ v_1 \otimes v_2 \;&\mapsto\; \tau_2(v_1, v_2) \;:=\; \sum_\pi \tau_2^{\pi,1}(v_1, v_2) \otimes \tau_2^{\pi,2}(v_1, v_2) \otimes \tau_2^{\pi,3}(v_1, v_2) \;, \end{aligned} \tag{6.20b}$$

where we again label summands in the tensor product by π. The twisted tensor product has then higher maps

$$\begin{aligned} \mathsf{m}_1^{\tau_1}(v \otimes a) \;&:=\; \sum_\pi (-1)^{|\tau_1^{\pi,1}(v)|_{\mathfrak{V}}} \tau_1^{\pi,1}(v) \otimes \mathsf{m}_1(\tau_1^{\pi,2}(v)(a)) \;, \\ \mathsf{m}_2^{\tau_2}(v_1 \otimes a_1, v_2 \otimes a_2) \;&:= \\ &:=\; (-1)^{|v_2|_{\mathfrak{V}}\, |a_1|_{\mathfrak{A}}} \sum_\pi \tau_2^{\pi,1}(v_1, v_2) \otimes \mathsf{m}_2(\tau_2^{\pi,2}(v_1, v_2)(a_1), \tau_2^{\pi,3}(v_1, v_2)(a_2)) \;. \end{aligned} \tag{6.21}$$

Note that in general, one may want to insert an additional sign $(-1)^{|\tau_2^{\pi,3}(v_1,v_2)|_{\mathfrak{V}}\, |a_1|_{\mathfrak{A}}}$ into this equation; all our twist, however, satisfy $|\tau_2^{\pi,3}(v_1, v_2)|_{\mathfrak{V}} = 0$.

Clearly, not every twist datum leads to a valid homotopy algebra, and just as in the case of cochain complexes, one has to check that this works for a given twist by hand. We also note that the twist datum relevant for the double copy will be able to mix types of homotopy algebras, that is, for $\mathfrak{A}$ an L_∞-algebra, we obtain a C_∞-algebra and for $\mathfrak{A}$ a C_∞-algebra, we obtain again an L_∞-algebra.

Altogether, our twisted tensor products are a way of factorising strict homotopy algebras in a unique fashion as necessary for the double copy. However, it remains to be seen if our construction in its present form is mathematically interesting in a wider context.

7

Factorisation of free field theories and free double copy

The first step toward the realisation of $\mathcal{N} = 0$ supergravity as the double copy of Yang-Mills theory is at the level of the free theories. In this Chapter, we expose the factorisation of the cochain complexes associated to the L_∞-algebras of the theories relevant to our interpretation of the double copy prescription, namely biadjoint scalar field theory, Yang–Mills theory, and $\mathcal{N} = 0$ supergravity. We obtain explicit field redefinitions that link Yang–Mills theory double copy and $\mathcal{N} = 0$ supergravity at linear level.

The double copy of supersymmetric gauge theories will be discussed in the upcoming paper [190]. The material in this Chapter is borrowed from [6].

Summary. Recall that the unary product μ_1 in any L_∞-algebra is a differential. Consequently, any L_∞-algebra $(\mathfrak{L}, \mu_i)$ naturally comes with an underlying cochain complex

$$\mathsf{Ch}(\mathfrak{L}) := \left(\cdots \xrightarrow{\mu_1} \mathfrak{L}_0 \xrightarrow{\mu_1} \mathfrak{L}_1 \xrightarrow{\mu_1} \mathfrak{L}_2 \xrightarrow{\mu_1} \mathfrak{L}_3 \xrightarrow{\mu_1} \cdots \right) . \tag{7.1}$$

In an L_∞-algebra corresponding to a field theory, the cochain complex $\mathsf{Ch}(\mathfrak{L})$ is the L_∞-algebra of the free theory with all coupling constants put to zero. In each factorisation, we thus expose the field content as well as the free fields that parametrise the theory's scattering amplitudes.

We will obtain the following factorisations of cochain complexes isomorphic to the cochain complexes underlying the L_∞-algebras of biadjoint scalar field theory, Yang–Mills theory in R_ξ-gauge, and gauge-fixed $\mathcal{N}=0$ supergravity:

$$\begin{aligned} \mathsf{Ch}(\mathfrak{L}^{\text{biadj}}_{\text{BRST}}) &= \mathsf{Ch}(\tilde{\mathfrak{L}}^{\text{biadj}}_{\text{BRST}}) = \mathfrak{g}\otimes(\bar{\mathfrak{g}}\otimes\mathsf{Ch}(\mathfrak{Scal}))\,, \\ \mathsf{Ch}(\mathfrak{L}^{\text{YM}}_{\text{BRST}}) &\cong \mathsf{Ch}(\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST}}) = \mathfrak{g}\otimes(\mathfrak{Kin}\otimes_{\tau_1}\mathsf{Ch}(\mathfrak{Scal}))\,, \\ \mathsf{Ch}(\mathfrak{L}^{\mathcal{N}=0}_{\text{BRST}}) &\cong \mathsf{Ch}(\tilde{\mathfrak{L}}^{\mathcal{N}=0}_{\text{BRST}}) = \mathfrak{Kin}\otimes_{\tau_1}(\mathfrak{Kin}\otimes_{\tau_1}\mathsf{Ch}(\mathfrak{Scal}))\,, \end{aligned} \tag{7.2}$$

where $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ are semi-simple compact matrix Lie algebras corresponding to the colour factors, $\mathfrak{Kin}$ is a graded vector space and $\mathfrak{Scal}$ is the L_∞-algebra of a scalar field theory. $\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST}}$, $\tilde{\mathfrak{L}}^{\text{biadj}}_{\text{BRST}}$, and $\tilde{\mathfrak{L}}^{\mathcal{N}=0}_{\text{BRST}}$ are L_∞-algebras associated to field redefinitions of biadjoint scalar theory, Yang–Mills theory, and $\mathcal{N}=0$ supergravity. We see that the cochain complex $\mathsf{Ch}(\tilde{\mathfrak{L}}^{\mathcal{N}=0}_{\text{BRST}})$ is fully determined by the factorisation of $\mathsf{Ch}(\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST}})$, which is nothing but the double copy at the linearised level.

There are two points to note concerning the factorisations of all those field theories but that of biadjoint scalar field theory. Firstly, these factorisations are most conveniently performed in particular field bases. We explain the required changes of basis, which are canonical transformations on the relevant BV field spaces. Secondly, these factorisations are twisted factorisation of cochain complexes of the type introduced in Section 6.3., with common twist datum τ_1, as indicated in (7.2). We remark that the twist we will consider is dictated only by Yang–Mills theory L_∞-algebra $\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST}}$. In general, applying different twists one obtains inequivalent theories.

7.1. Factorisation of the cochain complex of biadjoint scalar field theory

Let us start with the case of biadjoint scalar field theory as introduced in Section 4.2.. This case is particularly simple as its cochain complex $\mathsf{Ch}(\mathfrak{L}^{\text{biadj}}_{\text{BRST}})$ factorises as an ordinary tensor product.

Factorisation of the cochain complex. We can factor out the colour Lie algebras $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ leaving us with the L_∞-algebra $\mathfrak{Scal}$ of a plain scalar theory,

$$\mathsf{Ch}(\mathfrak{L}^{\text{biadj}}_{\text{BRST}}) = \mathfrak{g}\otimes(\bar{\mathfrak{g}}\otimes\mathsf{Ch}(\mathfrak{Scal}))\,, \tag{7.3}$$

where $\mathfrak{Scal}$ is a homotopy algebra of cubic scalar field theory which we will fully identify later in (9.4). The natural cochain complex is[1]

$$\mathsf{Ch}(\mathfrak{Scal}) \;:=\; \left(\underbrace{\overset{\mathsf{s}_x}{\mathfrak{F}[-1]}}_{\mathfrak{Scal}_1} \xrightarrow{\;\Box\;} \underbrace{\overset{\mathsf{s}^+_x}{\mathfrak{F}[-2]}}_{\mathfrak{Scal}_2} \right) , \tag{7.4}$$

concentrated in degrees one and two, cf. [52, 2]. Here, s_x and s^+_x are basis vectors for the function spaces $\mathfrak{F}[-1]$ and $\mathfrak{F}[-2]$ with $\mathfrak{F}$ given in (3.28). Their inner product is given by

$$\langle \mathsf{s}_{x_1}, \mathsf{s}^+_{x_2} \rangle \;:=\; \delta^{(d)}(x_1 - x_2) \;. \tag{7.5}$$

fields				anti-fields			
	$\lvert - \rvert_{\mathrm{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim		$\lvert - \rvert_{\mathrm{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim
s_x	0	1	$\frac{d}{2} - 1$	s^+_x	-1	2	$\frac{d}{2} + 1$

Table 7.1: The basis vectors of $\mathfrak{Scal}$ with their L_∞-degrees, their ghost numbers, and their mass dimensions.

fields				anti-fields			
factorisation	$\lvert - \rvert_{\mathrm{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim	factorisation	$\lvert - \rvert_{\mathrm{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim
$\varphi = \mathsf{e}_a \bar{\mathsf{e}}_{\bar{a}} \mathsf{s}_x \varphi^{a\bar{a}}(x)$	0	1	$\frac{d}{2} - 1$	$\varphi^+ = \mathsf{e}_a \bar{\mathsf{e}}_{\bar{a}} \mathsf{s}^+_x \varphi^{+a\bar{a}}(x)$	-1	2	$\frac{d}{2} + 1$

Table 7.2: Factorisation of the BV fields in the theory of biadjoint scalars. Note that we suppressed the integrals over x and the tensor products for simplicity.

The L_∞-degrees correspond to the evident ghost numbers and the differential induces mass dimensions, and both are summarised in Table 7.1. The factorisation of the BV fields is listed in Table 7.2. The differential $\mu_1 : \mathfrak{L}^{\mathrm{biadj}}_{\mathrm{BRST},1} \to \mathfrak{L}^{\mathrm{biadj}}_{\mathrm{BRST},2}$ is given by (6.3b) for the untwisted tensor product,

$$\begin{aligned} \mu_1(\varphi) \;&=\; \mu_1 \left(\mathsf{e}_a \otimes \bar{\mathsf{e}}_{\bar{a}} \otimes \int \mathrm{d}^d x \, \mathsf{s}_x \varphi^{a\bar{a}}(x) \right) \\ &=\; \mathsf{e}_a \otimes \bar{\mathsf{e}}_{\bar{a}} \otimes \mu_1^{\mathfrak{Scal}} \left(\int \mathrm{d}^d x \, \mathsf{s}_x \varphi^{a\bar{a}}(x) \right) \;=\; \Box \varphi \;, \end{aligned} \tag{7.6}$$

[1]See (3.4) for the notation $\mathfrak{F}[k]$.

where $\mu_1^{\mathfrak{Scal}}$ is the product appearing in (7.4). Furthermore, the inner product is

$$\begin{aligned}\langle\varphi,\varphi^+\rangle &= \mathrm{tr}_{\mathfrak{g}}(\mathrm{e}_a\mathrm{e}_b)\,\mathrm{tr}_{\bar{\mathfrak{g}}}(\bar{\mathrm{e}}_{\bar a}\bar{\mathrm{e}}_{\bar b})\int \mathrm{d}^d x_1\int \mathrm{d}^d x_2\,\langle \mathrm{s}_{x_1},\mathrm{s}^+_{x_2}\rangle\,\varphi^{a\bar a}(x_1)\varphi^{+b\bar b}(x_2)\\ &= \int \mathrm{d}^d x\,\varphi^{a\bar a}(x)\varphi^+_{a\bar a}(x)\ .\end{aligned} \tag{7.7}$$

In conclusion, we have thus verified the factorisation of the cochain complex (7.3).

7.2. Factorisation of the cochain complex of Yang–Mills theory

The case of Yang–Mills theory is more involved than the previous one. We start with the gauge fixed BV action (4.18) and perform a canonical transformation on BV field space, which then allows for a convenient factorisation of the resulting cochain complex $\mathrm{Ch}(\tilde{\mathfrak{L}}^{\mathrm{YM}}_{\mathrm{BRST}})$. For the following discussion, recall the gauge-fixing procedure and the gauge-fixed action from Section 4.3..

Canonical transformation. We note that the term $\partial^\mu A^a_\mu$ will vanish for physical states due to the polarisation condition $p\cdot\varepsilon = 0$ where p_μ is the momentum and ε_μ is the polarisation vector for A^a_μ. Off-shell, and at the level of the action, our gauge fixing terms allow us to absorb quadratic terms in $\partial^\mu A^a_\mu$ in a field redefinition[1] of the Nakanishi–Lautrup field b^a. We further rescale the field b^a in order to homogenise its mass dimension with that of A^a_μ, which will prove useful in our later discussion. Explicitly, we perform the field redefinitions

$$\begin{aligned}\tilde c^a &:= c^a\ , & \tilde c^{+a} &:= c^{+a}\ ,\\ \tilde A^a_\mu &:= A^a_\mu\ , & \tilde A^{+a}_\mu &:= A^{+a}_\mu + \frac{1-\sqrt{1-\xi}}{\xi}\,\partial_\mu b^{+a}\ ,\\ \tilde b^a &:= \sqrt{\frac{\xi}{\Box}}\left(b^a + \frac{1-\sqrt{1-\xi}}{\xi}\,\partial^\mu A^a_\mu\right)\ , & \tilde b^{+a} &:= \sqrt{\frac{\Box}{\xi}}\,b^{+a}\ ,\\ \tilde{\bar c}^a &:= \bar c^a\ , & \tilde{\bar c}^{+a} &:= \bar c^{+a}\ .\end{aligned} \tag{7.8}$$

Under these field redefinitions, the action (4.18)

$$S^{\mathrm{YM}}_{\mathrm{BRST}} = \int \mathrm{d}^d x\left\{\tfrac12 A_{a\mu}\Box A^{a\mu} + \tfrac12(\partial^\mu A^a_\mu)^2 - \bar c_a\Box c^a + \tfrac{\xi}{2}b_a b^a + b_a\partial^\mu A^a_\mu\right\} + S^{\mathrm{YM,\,int}}_{\mathrm{BRST}}\ , \tag{7.9}$$

[1] The redefinition of the anti-fields preserves the cyclic structure of the L_∞-algebra; it is mostly irrelevant for our discussion.

where $S^{\text{YM, int}}_{\text{BRST}}$ represents the interaction terms, turns into

$$\tilde{S}^{\text{YM}}_{\text{BRST}} := \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \tilde{A}_{a\mu} \Box \tilde{A}^{a\mu} - \tilde{\bar{c}}_a \Box \tilde{c}^a + \tfrac{1}{2} \tilde{b}_a \Box \tilde{b}^a + \tilde{\xi}\, \tilde{b}_a \sqrt{\Box}\, \partial^\mu \tilde{A}^a_\mu \right\} + \tilde{S}^{\text{YM, int}}_{\text{BRST}} , \quad (7.10)$$

where we rewrote the gauge-fixing parameter as

$$\tilde{\xi} := \sqrt{\frac{1-\xi}{\xi}} . \quad (7.11)$$

Note that at the level of the BV field space, the redefinitions (7.8) constitute a canonical transformation. For a more detailed discussion, including the precise meaning of the inverses of the $\Box$ operator, see Remark 3.2.

L_∞-algebra. The action (7.10) is now the superfield homotopy Maurer–Cartan action (3.25b) for an L_∞-algebra $\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST}}$. The complex underlying $\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST}}$ is given as

$$\begin{array}{ccc}
\overset{\tilde{A}^a_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} & \xrightarrow{\ \Box\ } & \overset{\tilde{A}^{+a}_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} \\
 & \searrow^{-\tilde{\xi}\sqrt{\Box}\,\partial^\mu} \quad \nearrow_{\tilde{\xi}\sqrt{\Box}\,\partial_\mu} & \\
\overset{\tilde{b}^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}} & \xrightarrow[\ \Box\ ]{} & \overset{\tilde{b}^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}
\end{array}$$

$$\underbrace{\overset{\tilde{c}^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},0}} \xrightarrow{-\Box} \underbrace{\overset{\tilde{\bar{c}}^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},1}} \qquad \underbrace{\overset{\tilde{\bar{c}}^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},2}} \xrightarrow{-\Box} \underbrace{\overset{\tilde{c}^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},3}} \quad (7.12a)$$

with

$$\begin{gathered}
\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},0} = \tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},0,\tilde{c}} , \quad \tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},1} = \bigoplus_{\phi\in(\tilde{A},\tilde{b},\tilde{\bar{c}}^+)} \tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},1,\phi} , \\
\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},2} = \bigoplus_{\phi\in(\tilde{A}^+,\tilde{b}^+,\tilde{\bar{c}})} \tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},1,\phi} , \quad \tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},3} = \tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},3,\tilde{c}^+} .
\end{gathered} \quad (7.12b)$$

The differential μ_1 acts on the various fields as follows

$$\begin{aligned}
(\tilde{c}^a) &\xmapsto{\mu_1} -\Box\, \tilde{c}^a \in \tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},1,\tilde{\bar{c}}^+} , \\
\begin{pmatrix} \tilde{A}^a_\mu \\ \tilde{b}^a \end{pmatrix} &\xmapsto{\mu_1} \begin{pmatrix} \Box\, \tilde{A}^a_\mu - \tilde{\xi}\sqrt{\Box}\,\partial_\mu \tilde{b}^a \\ \Box\, \tilde{b}^a + \tilde{\xi}\sqrt{\Box}\,\partial^\mu \tilde{A}^a_\mu \end{pmatrix} \in \bigoplus_{\phi\in(\tilde{A}^+,\tilde{b}^+)} \tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},2,\phi} , \\
(\tilde{\bar{c}}^a) &\xmapsto{\mu_1} -\Box\, \tilde{\bar{c}}^a \in \tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST},3,\tilde{c}^+}
\end{aligned} \quad (7.12c)$$

with all other actions trivial. The non-vanishing images of the higher products μ_2 and μ_3 are

$$
\begin{aligned}
(\tilde{A}^a_\mu, \tilde{c}^b) &\xmapsto{\mu_2} -g f_{bc}{}^a \partial^\mu(\tilde{A}^b_\mu \tilde{c}^c) \in \tilde{\mathfrak{L}}^{\rm YM}_{\rm BRST,\,1,\,\tilde{\bar{c}}^+} ,\\
(\tilde{c}^a, \tilde{\bar{c}}^b) &\xmapsto{\mu_2} -g f_{bc}{}^a \tilde{c}^b \partial_\mu \tilde{\bar{c}}^c \in \tilde{\mathfrak{L}}^{\rm YM}_{\rm BRST,\,2,\,\tilde{A}^+} ,\\
(\tilde{A}^a_\mu, \tilde{A}^b_\nu) &\xmapsto{\mu_2} 3! g f_{bc}{}^a \partial^\nu(\tilde{A}^b_\nu \tilde{A}^c_\mu) \in \tilde{\mathfrak{L}}^{\rm YM}_{\rm BRST,\,2,\,\tilde{A}^+} ,\\
(\tilde{A}^a_\mu, \tilde{\bar{c}}^b) &\xmapsto{\mu_2} -g f_{bc}{}^a \tilde{A}^b_\mu \partial^\mu \tilde{\bar{c}}^c \in \tilde{\mathfrak{L}}^{\rm YM}_{\rm BRST,\,3,\,\tilde{c}^+} ,\\
(\tilde{A}^a_\mu, \tilde{A}^b_\nu, \tilde{A}^c_\kappa) &\xmapsto{\mu_3} -3! g^2 f_{bc}{}^a f_{de}{}^b \tilde{A}^{\nu c} \tilde{A}^d_\nu \tilde{A}^e_\mu \in \tilde{\mathfrak{L}}^{\rm YM}_{\rm BRST,\,2,\,\tilde{A}^+} ,
\end{aligned}
\tag{7.12d}
$$

and the general expressions follow from anti-symmetrisation and polarisation. We note that the formulas (3.26) are useful in the derivation of the explicit form of these higher products.

By construction, $(\tilde{\mathfrak{L}}^{\rm YM}_{\rm BRST}, \mu_i)$ forms an L_∞-algebra, and with the inner products

$$
\begin{aligned}
\langle \tilde{A}, \tilde{A}^+ \rangle &:= \int {\rm d}^d x\, \tilde{A}^a_\mu \tilde{A}^{+\mu}_a , \qquad \langle \tilde{b}, \tilde{b}^+ \rangle := \int {\rm d}^d x\, \tilde{b}^a \tilde{b}^+_a ,\\
\langle \tilde{c}, \tilde{c}^+ \rangle &:= \int {\rm d}^d x\, \tilde{c}^a \tilde{c}^+_a , \qquad \langle \tilde{\bar{c}}, \tilde{\bar{c}}^+ \rangle := -\int {\rm d}^d x\, \tilde{\bar{c}}^a \tilde{\bar{c}}^+_a ,
\end{aligned}
\tag{7.13}
$$

it is cyclic.

We stress that the Chevalley–Eilenberg differential of the L_∞-algebra $\tilde{\mathfrak{L}}^{\rm YM}_{\rm BRST}$ is *not* the usual gauge-fixed BV operator[1]

$$
\tilde{Q}^{\rm YM,\,gf}_{\rm BV} := \{\tilde{S}^{\rm YM,\,gf}_{\rm BV}, -\}|_{\tilde{\Phi}^+_I=0} ,
\tag{7.14}
$$

where $\tilde{S}^{\rm YM,\,gf}_{\rm BV}$ is the gauge-fixed BV action that is obtained from (3.29) by the canonical transformation determined by the gauge fixing fermion (4.16). Instead, we are merely using the general correspondence between Lagrangians and L_∞-algebras as pointed out in Section 3.2.. This is reflected in the images of all higher products of (7.12a) lying in spaces parametrised by anti-fields.

Factorisation of the cochain complex. As explained in Section 6.2., we may factor out the gauge Lie algebra $\mathfrak{g}$, and we are left with a C_∞-algebra. This C_∞-algebra can be further factorised into a twisted tensor product, extending Example 6.1, and we obtain

$$
{\rm Ch}(\tilde{\mathfrak{L}}^{\rm YM}_{\rm BRST}) = \mathfrak{g} \otimes (\mathfrak{Kin} \otimes_{\tau_1} {\rm Ch}(\mathfrak{Scal})) .
\tag{7.15}
$$

[1]Here, $|_{\tilde{\Phi}^+_I=0}$ is again the restriction to the subspace of the BV field space where all anti-fields are zero.

Here, $\mathfrak{g}$ is the colour Lie algebra, $\mathsf{Ch}(\mathfrak{Scal})$ is the cochain complex (7.4), and $\mathfrak{Kin}$ is the graded vector space[1]

$$\mathfrak{Kin} \;:=\; \underbrace{\overset{\mathtt{g}}{\mathbb{R}[1]}}_{=:\,\mathfrak{Kin}_{-1}} \;\oplus\; \underbrace{\big(\overset{\mathtt{v}^\mu}{\mathbb{M}^d} \oplus \overset{\mathtt{n}}{\mathbb{R}}\big)}_{=:\,\mathfrak{Kin}_0} \;\oplus\; \underbrace{\overset{\mathtt{a}}{\mathbb{R}[-1]}}_{=:\,\mathfrak{Kin}_1}\;, \tag{7.16}$$

where the typewriter letters label basis elements of the corresponding vector spaces. The natural degree-zero inner product on $\mathfrak{Kin}$ is given by

$$\langle \mathtt{g}, \mathtt{a}\rangle \;:=\; -1\;,\quad \langle \mathtt{v}^\mu, \mathtt{v}^\nu\rangle \;:=\; \eta^{\mu\nu}\;,\quad \langle \mathtt{n}, \mathtt{n}\rangle \;:=\; 1\;. \tag{7.17}$$

The elements of $\mathfrak{Kin}$ also carry mass dimensions, which are listed in Table 7.3.

We summarise the factorisation of individual Yang–Mills fields in Table 7.4. A few remarks about the structure of the factorisation are in order. Whilst fields always have a factor of s_x, anti-fields always have a factor of s_x^+. This guarantees that the inner product is indeed that of the factorisation: (7.13) is reproduced correctly using the factorisations given in Table 7.4 and Equation (7.17) complemented by the inner product $\langle \mathsf{e}_a, \mathsf{e}_b\rangle = -\mathrm{tr}(\mathsf{e}_a\mathsf{e}_b) = \delta_{ab}$ on $\mathfrak{g}$:

$$\begin{aligned}
\langle \tilde c, \tilde c^+\rangle \;&=\; \left\langle \mathsf{e}_a \otimes \mathtt{g} \otimes \int \mathrm{d}^d x_1\, \mathsf{s}_{x_1} \tilde c^a(x_1), \mathsf{e}_b \otimes \mathtt{a} \otimes \int \mathrm{d}^d x_2\, \mathsf{s}^+_{x_2} \tilde c^{+b}(x_2)\right\rangle\\
&=\; -\langle \mathsf{e}_a, \mathsf{e}_b\rangle \langle \mathtt{g}, \mathtt{a}\rangle \int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2\, \delta^{(d)}(x_1 - x_2) \tilde c^a(x_1)\, \tilde c^{+b}(x_2)\\
&=\; \int \mathrm{d}^d x\, \tilde c^a(x)\, \tilde c^+_a(x)\;,\\
\langle \tilde A, \tilde A^+\rangle \;&=\; \left\langle \mathsf{e}_a \otimes \mathtt{v}^\mu \otimes \int \mathrm{d}^d x_1\, \mathsf{s}_{x_1} \tilde A^a_\mu(x_1), \mathsf{e}_b \otimes \mathtt{v}^\nu \otimes \int \mathrm{d}^d x_2\, \mathsf{s}^+_{x_2} \tilde A^{+b}_\nu(x_2)\right\rangle\\
&=\; \langle \mathsf{e}_a, \mathsf{e}_b\rangle \langle \mathtt{v}^\mu, \mathtt{v}^\nu\rangle \int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2\, \delta^{(d)}(x_1 - x_2) \tilde A^a_\mu(x_1)\, \tilde A^{+b}_\nu(x_2)\\
&=\; \int \mathrm{d}^d x\, \tilde A^a_\mu(x)\, \tilde A^{+\mu}_a(x)\;,
\end{aligned} \tag{7.18a}$$

[1]See (3.4) for the notation $\mathbb{R}[k]$, etc.

$$
\begin{aligned}
\langle \tilde{b}, \tilde{b}^+ \rangle &= \left\langle \mathrm{e}_a \otimes \mathtt{n} \otimes \int \mathrm{d}^d x_1\, \mathrm{s}_{x_1} \tilde{b}^a(x_1), \mathrm{e}_b \otimes \mathtt{n} \otimes \int \mathrm{d}^d x_2\, \mathrm{s}^+_{x_2} \tilde{b}^{+b}(x_2) \right\rangle \\
&= \langle \mathrm{e}_a, \mathrm{e}_b \rangle \langle \mathtt{n}, \mathtt{n} \rangle \int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2\, \delta^{(d)}(x_1 - x_2) \tilde{c}^a(x_1)\, \tilde{c}^{+b}(x_2) \\
&= \int \mathrm{d}^d x\, \tilde{b}^a(x)\, \tilde{b}^+_a(x) \ , \\
\langle \tilde{\bar{c}}, \tilde{\bar{c}}^+ \rangle &= \left\langle \mathrm{e}_a \otimes \mathtt{a} \otimes \int \mathrm{d}^d x_1\, \mathrm{s}_{x_1} \tilde{\bar{c}}^a(x_1), \mathrm{e}_b \otimes \mathtt{g} \otimes \int \mathrm{d}^d x_2\, \mathrm{s}^+_{x_2} \tilde{\bar{c}}^{+b}(x_2) \right\rangle \\
&= -\langle \mathrm{e}_a, \mathrm{e}_b \rangle \langle \mathtt{a}, \mathtt{g} \rangle \int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2\, \delta^{(d)}(x_1 - x_2) \tilde{\bar{c}}^a(x_1) \tilde{\bar{c}}^{+b}(x_2) \\
&= -\int \mathrm{d}^d x\, \tilde{\bar{c}}^a(x)\, \tilde{\bar{c}}^+_a(x) \ .
\end{aligned}
\tag{7.18b}
$$

Note that the kinematic factor $\mathfrak{Kin}$ essentially arranges the fields in a quartet: the physical field has a ghost, a Nakanishi–Lautrup field, and an anti-ghost. These patterns reoccur in the double copy.

	$\lvert - \rvert_{\mathrm{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim
$\mathtt{g}$	1	−1	−1
$\mathtt{v}^\mu$	0	0	0
$\mathtt{n}$	0	0	0
$\mathtt{a}$	−1	1	1

Table 7.3: The elements of $\mathfrak{Kin}$ with their L_∞-degrees, their ghost numbers, and their mass dimensions.

To extend this factorisation of graded vector spaces to a factorisation of cochain complexes, we introduce the twist datum τ_1 given by

$$
\tau_1(\mathtt{g}) := \mathtt{g} \otimes \mathsf{id} \ , \quad
\begin{aligned}
\tau_1(\mathtt{v}^\mu) &:= \mathtt{v}^\mu \otimes \mathsf{id} + \tilde{\xi}\mathtt{n} \otimes \frac{1}{\sqrt{\Box}} \partial^\mu \ , \\
\tau_1(\mathtt{n}) &:= \mathtt{n} \otimes \mathsf{id} - \tilde{\xi}\mathtt{v}^\mu \otimes \frac{1}{\sqrt{\Box}} \partial_\mu \ ,
\end{aligned}
\quad \tau_1(\mathtt{a}) := \mathtt{a} \otimes \mathsf{id} \ ,
\tag{7.19}
$$

and we shall use the convenient shorthand notation

$$
\tau_1(\mathtt{v}^\mu, \mathtt{n}) \begin{pmatrix} \int \mathrm{d}^d x\, \mathrm{s}_x \tilde{A}^a_\mu(x) \\ \int \mathrm{d}^d x\, \mathrm{s}_x \tilde{b}^a(x) \end{pmatrix} = (\mathtt{v}^\mu, \mathtt{n}) \otimes \begin{pmatrix} \mathsf{id} & -\frac{\xi}{\sqrt{\Box}} \partial_\mu \\ \frac{\xi}{\sqrt{\Box}} \partial^\mu & \mathsf{id} \end{pmatrix} \begin{pmatrix} \int \mathrm{d}^d x\, \mathrm{s}_x \tilde{A}^a_\mu(x) \\ \int \mathrm{d}^d x\, \mathrm{s}_x \tilde{b}^a(x) \end{pmatrix} . \tag{7.20}
$$

The twisted differentials on $\mathfrak{g} \otimes (\mathfrak{Kin} \otimes_{\tau_1} \mathfrak{Scal})$ are now indeed those of (7.12c):

$$
\mu_1(\tilde{c}) = \mu_1 \left(\mathrm{e}_a \otimes \mathtt{g} \otimes \int \mathrm{d}^d x\, \mathrm{s}_x \tilde{c}^a(x) \right)
$$

fields				anti-fields			
factorisation	$\lvert - \rvert_{\mathrm{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim	factorisation	$\lvert - \rvert_{\mathrm{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim
$\tilde{c} = \mathsf{e}_a \mathtt{g} \mathsf{s}_x \tilde{c}^a(x)$	1	0	$\frac{d}{2}-2$	$\tilde{c}^+ = \mathsf{e}_a \mathtt{a} \mathsf{s}_x^+ \tilde{c}^{+a}(x)$	-2	3	$\frac{d}{2}+2$
$\tilde{A} = \mathsf{e}_a \mathtt{v}^\mu \mathsf{s}_x \tilde{A}^a_\mu(x)$	0	1	$\frac{d}{2}-1$	$\tilde{A}^+ = \mathsf{e}_a \mathtt{v}^\mu \mathsf{s}_x^+ \tilde{A}^{+a}_\mu(x)$	-1	2	$\frac{d}{2}+1$
$\tilde{b} = \mathsf{e}_a \mathtt{n} \mathsf{s}_x \tilde{b}^a(x)$	0	1	$\frac{d}{2}-1$	$\tilde{b}^+ = \mathsf{e}_a \mathtt{n} \mathsf{s}_x^+ \tilde{b}^{+a}(x)$	-1	2	$\frac{d}{2}+1$
$\tilde{\bar{c}} = \mathsf{e}_a \mathtt{a} \mathsf{s}_x \tilde{\bar{c}}^a(x)$	-1	2	$\frac{d}{2}$	$\tilde{\bar{c}}^+ = \mathsf{e}_a \mathtt{g} \mathsf{s}_x^+ \tilde{\bar{c}}^{+a}(x)$	0	1	$\frac{d}{2}$

Table 7.4: Factorisation of the redefined BV fields for Yang–Mills theory from Table 4.1 after the field redefinitions (7.8). Here, e_a denote the basis vectors of $\mathfrak{g}$. Likewise, $\mathtt{g}$, $\mathtt{n}$, $\mathtt{v}^\mu$, and $\mathtt{a}$ denote the basis vectors of $\mathfrak{Kin}$ defined in (7.16). Furthermore, s_x and s_x^+ are the basis vectors of $\mathfrak{Scal}$ from Table 7.1. Note that we suppressed the integrals over x and the tensor products for simplicity.

$$
\begin{aligned}
&= -\mathsf{e}_a \otimes \mathtt{g} \otimes \mu_1^{\mathfrak{Scal}} \left(\int \mathrm{d}^d x\, \mathsf{s}_x \tilde{c}^a(x) \right) \\
&= \mathsf{e}_a \otimes \mathtt{g} \otimes \int \mathrm{d}^d x\, \mathsf{s}_x^+ \big\{ -\Box\, \tilde{c}^a(x) \big\} \ , \qquad (7.21a)
\end{aligned}
$$

$$
\begin{aligned}
\mu_1 \begin{pmatrix} \tilde{A} \\ \tilde{b} \end{pmatrix} &= \mu_1 \left(\mathsf{e}_a \otimes (\mathtt{v}^\mu, \mathtt{n}) \otimes \begin{pmatrix} \int \mathrm{d}^d x\, \mathsf{s}_x \tilde{A}^a_\mu(x) \\ \int \mathrm{d}^d x\, \mathsf{s}_x \tilde{b}^a(x) \end{pmatrix} \right) \\
&= \mathsf{e}_a \otimes (\mathtt{v}^\mu, \mathtt{n}) \otimes \mu_1^{\mathfrak{Scal}} \left(\begin{pmatrix} \mathrm{id} & -\frac{\tilde{\xi}}{\sqrt{\Box}}\partial_\mu \\ \frac{\tilde{\xi}}{\sqrt{\Box}}\partial^\mu & \mathrm{id} \end{pmatrix} \begin{pmatrix} \int \mathrm{d}^d x\, \mathsf{s}_x \tilde{A}^a_\mu(x) \\ \int \mathrm{d}^d x\, \mathsf{s}_x \tilde{b}^a(x) \end{pmatrix} \right) \\
&= \mathsf{e}_a \otimes (\mathtt{v}^\mu, \mathtt{n}) \otimes \begin{pmatrix} \int \mathrm{d}^d x\, \mathsf{s}_x^+ \big\{ \Box\, \tilde{A}^a_\mu(x) - \tilde{\xi}\sqrt{\Box}\,\partial_\mu \tilde{b}^a(x) \big\} \\ \int \mathrm{d}^d x\, \mathsf{s}_x^+ \big\{ \Box\, \tilde{b}^a(x) + \tilde{\xi}\sqrt{\Box}\,\partial^\mu \tilde{A}^a_\mu(x) \big\} \end{pmatrix} \\
&= \mathsf{e}_a \otimes \begin{pmatrix} \mathtt{v}^\mu \otimes \int \mathrm{d}^d x\, \mathsf{s}_x^+ \big\{ \Box\, \tilde{A}^a_\mu(x) - \tilde{\xi}\sqrt{\Box}\,\partial_\mu \tilde{b}^a(x) \big\} \\ \mathtt{n} \otimes \int \mathrm{d}^d x\, \mathsf{s}_x^+ \big\{ \Box\, \tilde{b}^a(x) + \tilde{\xi}\sqrt{\Box}\,\partial^\mu \tilde{A}^a_\mu(x) \big\} \end{pmatrix} , \qquad (7.21b)
\end{aligned}
$$

$$
\begin{aligned}
\mu_1(\tilde{\bar{c}}) &= \mu_1 \left(\mathsf{e}_a \otimes \mathtt{a} \otimes \int \mathrm{d}^d x\, \mathsf{s}_x \tilde{\bar{c}}^a(x) \right) \\
&= -\mathsf{e}_a \otimes \mathtt{a} \otimes \mu_1^{\mathfrak{Scal}} \left(\int \mathrm{d}^d x\, \mathsf{s}_x \tilde{\bar{c}}^a(x) \right) \\
&= \mathsf{e}_a \otimes \mathtt{a} \otimes \int \mathrm{d}^d x\, \mathsf{s}_x^+ \big\{ -\Box\, \tilde{\bar{c}}^a(x) \big\} \ . \qquad (7.21c)
\end{aligned}
$$

Altogether, we saw that the factorisation (7.15) is valid for twist datum τ_1.

7.3. Canonical transformation for the free Kalb–Ramond two-form

To keep our discussion manageable, we shall discuss the canonical transformations for the free Kalb–Ramond two-form and Einstein–Hilbert gravity separately. For the following discussion, recall the gauge-fixing procedure and the gauge-fixed action from Section 4.4..

Canonical transformation. Analogously to the case of Yang–Mills theory, we can now perform a field redefinition in order to eliminate the quadratic terms that would vanish on-shell in Lorenz gauge due to contractions between momenta and polarisation tensors. We also insert inverses of the wave operator to match the mass dimensions of fields of L_∞-degree one. The field redefinitions are

$$
\begin{aligned}
\tilde{\lambda} &:= \lambda\,, & \tilde{\lambda}^+ &:= \lambda^+\,,\\
\tilde{\Lambda}_\mu &:= \Lambda_\mu\,, & \tilde{\Lambda}^+_\mu &:= \Lambda^+_\mu + \frac{1-\sqrt{1-\xi}}{\xi}\partial_\mu\gamma^+\,,\\
\tilde{\gamma} &:= \sqrt{\frac{\xi}{\Box}}\left(\gamma + \frac{1-\sqrt{1-\xi}}{\xi}\partial^\mu\Lambda_\mu\right), & \tilde{\gamma}^+ &:= \sqrt{\frac{\Box}{\xi}}\gamma^+\,,\\
\tilde{B}_{\mu\nu} &:= B_{\mu\nu}\,, & \tilde{B}^+_{\mu\nu} &:= B^+_{\mu\nu} + \frac{1-\sqrt{1-\xi}}{\xi}\partial_{[\mu}\alpha^+_{\nu]}\,,\\
\tilde{\alpha}_\mu &:= \sqrt{\frac{\xi}{\Box}}\Big(\alpha_\mu - \partial_\mu\varepsilon\, - & \tilde{\alpha}^+_\mu &:= \sqrt{\frac{\Box}{\xi}}\left(\alpha^+_\mu + \frac{1-\xi}{2\Box}\partial_\mu\varepsilon^+\right),\\
&\qquad - \frac{1-\xi}{2\Box}\partial_\mu\partial^\nu\alpha_\nu\, + \\
&\qquad + \frac{1-\sqrt{1-\xi}}{\xi}\partial^\nu B_{\nu\mu}\Big)\,,\\
\tilde{\varepsilon} &:= \varepsilon + \frac{1-\xi}{2\Box}\partial^\mu\alpha_\mu\,, & \tilde{\varepsilon}^+ &:= \frac{1+\xi}{2}\varepsilon^+ - \partial^\mu\alpha^+_\mu\,,\\
\tilde{\bar{\Lambda}}_\mu &:= \bar{\Lambda}_\mu\,, & \tilde{\bar{\Lambda}}^+_\mu &:= \bar{\Lambda}^+_\mu + \frac{1-\sqrt{1-\xi}}{\xi}\partial_\mu\bar{\gamma}^+\,,\\
\tilde{\bar{\gamma}} &:= \sqrt{\frac{\xi}{\Box}}\left(\bar{\gamma} + \frac{1-\sqrt{1-\xi}}{\xi}\partial^\mu\bar{\Lambda}_\mu\right), & \tilde{\bar{\gamma}}^+ &:= \sqrt{\frac{\Box}{\xi}}\bar{\gamma}^+\,,\\
\tilde{\bar{\lambda}} &:= \bar{\lambda}\,, & \tilde{\bar{\lambda}}^+ &:= \bar{\lambda}^+\,,
\end{aligned}
\tag{7.22a}
$$

with

$$
\xi := \xi_1 = \xi_3 - \xi_2\,. \tag{7.22b}
$$

These redefinitions constitute canonical transformations on the BV field space. Upon applying these transformations to the action (4.27), we obtain

$$
\begin{aligned}
\tilde{S}^{\mathsf{KR}}_{\mathsf{BRST}} \;:=\; \int \mathrm{d}^d x \Big\{ & \tfrac{1}{4}\tilde{B}_{\mu\nu}\Box\tilde{B}^{\mu\nu} - \tilde{\bar{\Lambda}}_\mu\Box\tilde{\Lambda}^\mu + \tfrac{1}{2}\tilde{\alpha}_\mu\Box\tilde{\alpha}^\mu - \tfrac{\tilde{\xi}^2}{2}(\partial^\mu\tilde{\alpha}_\mu)^2 + \tfrac{1}{2}\tilde{\varepsilon}\Box\tilde{\varepsilon} - \tilde{\bar{\lambda}}\Box\tilde{\lambda} - \\
& - \tilde{\bar{\gamma}}\Box\tilde{\gamma} + \tilde{\xi}\tilde{\alpha}^\nu\sqrt{\Box}\,\partial^\mu\tilde{B}_{\mu\nu} + \tilde{\xi}\tilde{\gamma}\sqrt{\Box}\,\partial_\mu\tilde{\bar{\Lambda}}^\mu - \tilde{\xi}\tilde{\bar{\gamma}}\sqrt{\Box}\,\partial_\mu\tilde{\Lambda}^\mu \Big\}\,,
\end{aligned}
\tag{7.23}
$$

where we have again used the shorthand $\tilde{\xi} := \sqrt{\frac{1-\xi}{\xi}}$, cf. (7.11).

L_∞-algebra. The action (7.23) is the superfield homotopy Maurer–Cartan action (3.25b) of an L_∞-algebra, denoted by $\tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST}}$, that is given by

$$
\overset{\tilde{\varepsilon}}{\mathscr{C}^\infty(\mathbb{M}^d)} \xrightarrow{\;\Box\;} \overset{\tilde{\varepsilon}^+}{\mathscr{C}^\infty(\mathbb{M}^d)}
$$

$$
\begin{array}{ccc}
\overset{\tilde{\Lambda}_\mu}{\Omega^1(\mathbb{M}^d)} & \xrightarrow{\;-\Box\;} & \overset{\tilde{\bar{\Lambda}}^+_\mu}{\Omega^1(\mathbb{M}^d)} \\
 & \searrow^{-\tilde{\xi}\sqrt{\Box}\,\partial^\mu} \quad \nearrow_{\tilde{\xi}\sqrt{\Box}\,\partial_\mu} & \\
\underset{\tilde{\gamma}}{\mathscr{C}^\infty(\mathbb{M}^d)} & \xrightarrow{\;-\Box\;} & \underset{\tilde{\bar{\gamma}}^+}{\mathscr{C}^\infty(\mathbb{M}^d)}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\overset{\tilde{\bar{\Lambda}}_\mu}{\Omega^1(\mathbb{M}^d)} & \xrightarrow{\;-\Box\;} & \overset{\tilde{\Lambda}^+_\mu}{\Omega^1(\mathbb{M}^d)} \\
 & \searrow^{-\tilde{\xi}\sqrt{\Box}\,\partial^\mu} \quad \nearrow_{\tilde{\xi}\sqrt{\Box}\,\partial^\mu} & \\
\underset{\tilde{\bar{\gamma}}}{\mathscr{C}^\infty(\mathbb{M}^d)} & \xrightarrow{\;-\Box\;} & \underset{\tilde{\gamma}^+}{\mathscr{C}^\infty(\mathbb{M}^d)}
\end{array}
$$

$$
\underbrace{\overset{\tilde{\lambda}}{\mathscr{C}^\infty(\mathbb{M}^d)}}_{=:\,\tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,-1}} \xrightarrow{\;\Box\;} \underbrace{\overset{\tilde{\bar{\lambda}}^+}{\mathscr{C}^\infty(\mathbb{M}^d)}}_{=:\,\tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,0}}
\qquad
\underbrace{\begin{array}{c} \overset{\tilde{B}_{\mu\nu}}{\Omega^2(\mathbb{M}^d)} \\ \\ \overset{\tilde{\alpha}_\mu}{\Omega^1(\mathbb{M}^d)} \end{array}}_{=:\,\tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,1}}
\begin{array}{c} \xrightarrow{\;\Box\;} \\ \searrow^{-\tilde{\xi}\sqrt{\Box}\,\partial^\nu} \quad \nearrow_{\tilde{\xi}\sqrt{\Box}\,\partial_{[\nu}} \\ \xrightarrow[\tilde{\xi}^2\partial_\nu\partial^\mu]{\;\Box\;} \end{array}
\underbrace{\begin{array}{c} \overset{\tilde{B}^+_{\mu\nu}}{\Omega^2(\mathbb{M}^d)} \\ \\ \overset{\tilde{\alpha}^+_\mu}{\Omega^1(\mathbb{M}^d)} \end{array}}_{=:\,\tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,2}}
\qquad
\underbrace{\overset{\tilde{\bar{\lambda}}}{\mathscr{C}^\infty(\mathbb{M}^d)}}_{=:\,\tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,3}} \xrightarrow{\;\Box\;} \underbrace{\overset{\tilde{\lambda}^+}{\mathscr{C}^\infty(\mathbb{M}^d)}}_{=:\,\tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,4}}
\tag{7.24a}
$$

with

$$
\begin{gathered}
\tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,-1} = \tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,-1,\,\tilde{\lambda}}\,, \quad \tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,0} = \bigoplus_{\phi\in(\tilde{\Lambda},\,\tilde{\gamma},\,\tilde{\bar{\lambda}}^+)} \tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,0,\,\phi}\,, \\
\tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,1} = \bigoplus_{\phi\in(\tilde{\varepsilon},\tilde{\bar{\Lambda}}^+,\,\tilde{\bar{\gamma}}^+,\,\tilde{B},\,\tilde{\alpha})} \tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,1,\,\phi}\,, \quad \tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,2} = \bigoplus_{\phi\in(\tilde{\varepsilon}^+,\tilde{\bar{\Lambda}},\tilde{\bar{\gamma}},\,\tilde{B}^+,\,\tilde{\alpha}^+)} \tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,2,\,\phi}\,, \\
\tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,3} = \bigoplus_{\phi\in(\tilde{\Lambda}^+,\,\tilde{\gamma}^+,\,\tilde{\bar{\lambda}})} \tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,3,\,\phi}\,, \quad \tilde{\mathfrak{L}}^{\mathsf{KR}}_{\mathsf{BRST},\,4} = \tilde{\mathfrak{L}}^{\mathsf{YM}}_{\mathsf{BRST},\,4,\,\tilde{\lambda}^+}\,,
\end{gathered}
\tag{7.24b}
$$

and the non-vanishing differential

$$
\begin{aligned}
(\tilde{\lambda}) &\xmapsto{\mu_1} \Box\tilde{\lambda} \in \tilde{\mathfrak{L}}^{\mathrm{KR}}_{\mathrm{BRST},0,\tilde{\tilde{\lambda}}^+} ,\\
\begin{pmatrix}\tilde{\Lambda}_\mu\\ \tilde{\gamma}\end{pmatrix} &\xmapsto{\mu_1} -\begin{pmatrix}\Box\tilde{\Lambda}_\mu - \tilde{\xi}\sqrt{\Box}\,\partial_\mu\tilde{\gamma}\\ \Box\tilde{\gamma}+\tilde{\xi}\sqrt{\Box}\,\partial^\mu\tilde{\Lambda}_\mu\end{pmatrix} \in \bigoplus_{\phi\in(\tilde{\tilde{\Lambda}}^+,\tilde{\tilde{\gamma}}^+)} \tilde{\mathfrak{L}}^{\mathrm{KR}}_{\mathrm{BRST},1,\phi} ,\\
\begin{pmatrix}\tilde{B}_{\mu\nu}\\ \tilde{\alpha}_\mu\end{pmatrix} &\xmapsto{\mu_1} \begin{pmatrix}\Box\tilde{B}_{\mu\nu} - 2\tilde{\xi}\sqrt{\Box}\,\partial_{[\mu}\tilde{\alpha}_{\nu]}\\ \Box\tilde{\alpha}_\mu+\tilde{\xi}\sqrt{\Box}\,\partial^\nu\tilde{B}_{\nu\mu}+\tilde{\xi}^2\partial_\mu\partial^\nu\tilde{\alpha}_\nu\end{pmatrix} \in \bigoplus_{\phi\in(\tilde{B}^+,\tilde{\alpha}^+)} \tilde{\mathfrak{L}}^{\mathrm{KR}}_{\mathrm{BRST},2,\phi} ,\\
\begin{pmatrix}\tilde{\tilde{\Lambda}}_\mu\\ \tilde{\tilde{\gamma}}\end{pmatrix} &\xmapsto{\mu_1} -\begin{pmatrix}\Box\tilde{\tilde{\Lambda}}_\mu - \tilde{\xi}\sqrt{\Box}\,\partial_\mu\tilde{\tilde{\gamma}}\\ \Box\tilde{\tilde{\gamma}}+\tilde{\xi}\sqrt{\Box}\,\partial^\mu\tilde{\tilde{\Lambda}}_\mu\end{pmatrix} \in \bigoplus_{\phi\in(\tilde{\Lambda}^+,\tilde{\gamma}^+)} \tilde{\mathfrak{L}}^{\mathrm{KR}}_{\mathrm{BRST},3,\phi} ,\\
(\tilde{\tilde{\lambda}}) &\xmapsto{\mu_1} \Box\tilde{\tilde{\lambda}} \in \tilde{\mathfrak{L}}^{\mathrm{KR}}_{\mathrm{BRST},4,\tilde{\lambda}^+} .
\end{aligned}
\tag{7.24c}
$$

There are no additional higher products because the theory is free. The expressions

$$
\begin{aligned}
\langle\tilde{\lambda},\tilde{\lambda}^+\rangle &:= -\int \mathrm{d}^d x\,\tilde{\lambda}\tilde{\lambda}^+ , & \langle\tilde{\tilde{\lambda}},\tilde{\tilde{\lambda}}^+\rangle &:= -\int \mathrm{d}^d x\,\tilde{\tilde{\lambda}}\tilde{\tilde{\lambda}}^+ ,\\
\langle\tilde{\Lambda},\tilde{\Lambda}^+\rangle &:= \int \mathrm{d}^d x\,\tilde{\Lambda}^\mu\tilde{\Lambda}^+_\mu , & \langle\tilde{\tilde{\Lambda}},\tilde{\tilde{\Lambda}}^+\rangle &:= -\int \mathrm{d}^d x\,\tilde{\tilde{\Lambda}}^\mu\tilde{\tilde{\Lambda}}^+_\mu ,\\
\langle\tilde{B},\tilde{B}^+\rangle &:= \tfrac{1}{2}\int \mathrm{d}^d x\,\tilde{B}^{\mu\nu}\tilde{B}^+_{\mu\nu} , & &\\
\langle\tilde{\alpha},\tilde{\alpha}^+\rangle &:= \int \mathrm{d}^d x\,\tilde{\alpha}^\mu\tilde{\alpha}^+_\mu , & \langle\tilde{\varepsilon},\tilde{\varepsilon}^+\rangle &:= \int \mathrm{d}^d x\,\tilde{\varepsilon}\tilde{\varepsilon}^+ ,\\
\langle\tilde{\gamma},\tilde{\gamma}^+\rangle &:= \int \mathrm{d}^d x\,\tilde{\gamma}\tilde{\gamma}^+ , & \langle\tilde{\tilde{\gamma}},\tilde{\tilde{\gamma}}^+\rangle &:= -\int \mathrm{d}^d x\,\tilde{\tilde{\gamma}}\tilde{\tilde{\gamma}}^+
\end{aligned}
\tag{7.25}
$$

define a cyclic inner product on $(\tilde{\mathfrak{L}}^{\mathrm{YM}}_{\mathrm{BRST}},\mu_1)$.

7.4. Canonical transformation for Einstein–Hilbert gravity with dilaton

The case of Einstein–Hilbert gravity with dilaton is now more involved that of the free Kalb–Ramond field. For the following discussion, recall the gauge-fixing procedure and the gauge-fixed action from Section 4.5..

Canonical transformations. We start from the Lagrangian (4.39) but add a scalar kinetic term for the dilaton φ,

$$
\mathscr{L}_0^{\mathrm{eEHD,gf}} := \mathscr{L}_0^{\mathrm{eEH,gf}} + \tfrac{1}{2}\varphi\Box\varphi . \tag{7.26}
$$

We perform a field redefinition analogous to the case of Yang–Mills theory and the Kalb–Ramond field, absorbing various terms that vanish on-shell, as well as the trace of $h_{\mu\nu}$ in δ and ensuring that all fields come with the right propagators. For the fields of non-vanishing ghost number, the transformation read as

$$\begin{aligned}
\tilde{X}_\mu &:= X_\mu \, , & \tilde{X}^+_\mu &:= X^+_\mu \, , \\
\tilde{\beta} &:= \frac{1}{\sqrt{\Box}}\beta \, , & \tilde{\beta}^+ &:= \sqrt{\Box}\beta^+ \\
\tilde{\bar{X}}_\mu &:= \bar{X}_\mu \, , & \tilde{\bar{X}}^+_\mu &:= \bar{X}^+_\mu - \frac{1-\sqrt{1-\xi}}{\sqrt{\xi}}\partial_\mu\bar{\beta}^+ \, , \\
\tilde{\bar{\beta}} &:= \frac{1}{\sqrt{\Box}}\left(\bar{\beta} - \frac{1-\sqrt{1-\xi}}{\sqrt{\xi}}\partial^\mu\bar{X}_\mu\right) \, , & \tilde{\bar{\beta}}^+ &:= \sqrt{\Box}\bar{\beta}^+ \, ,
\end{aligned} \tag{7.27a}$$

where we worked in the special gauge

$$\begin{aligned}
\zeta_4 &= 1 \, , \quad \zeta_5 = \frac{1-\sqrt{1-\xi}}{\sqrt{\xi}} \, , \quad \zeta_6 = -\tfrac{1}{2} \, , \quad \zeta_7 = -\frac{4(\xi + 2\sqrt{1-\xi}\xi - \sqrt{1-\xi} - 1)}{\sqrt{\xi}(4\xi-3)} \, , \\
\zeta_8 &= \frac{1}{4(3-4\xi)^2\sqrt{\xi}}\Big(50(1+\sqrt{1-\xi}) - \xi\big(5(34+29\sqrt{1-\xi}) + \\
&\qquad + 8\xi(-23 - 15\sqrt{1-\xi} + 2(4+\sqrt{1-\xi})\xi)\big)\Big) \, , \\
\zeta_9 &= 0 \, , \quad \zeta_{10} = \frac{1}{2\xi + \sqrt{1-\xi} - 1} \, , \quad \zeta_{11} = 0 \, .
\end{aligned} \tag{7.27b}$$

From the expressions for ζ_7 and ζ_8, it is already apparent that the field redefinitions we would like to perform here are much more involved than in the case of the Kalb–Ramond field.[1] Because the resulting expressions for the fields of ghost number zero are too involved and not very illuminating, we restrict ourselves to the case $\xi = 1$ corresponding to Feynman gauge in Yang–Mills theory. Here, we have the inverse field transformations

$$\begin{aligned}
h_{\mu\nu} &= \tilde{h}_{\mu\nu} + \frac{\partial_\mu\partial_\nu\mathring{\tilde{h}}}{\Box} - 2\frac{\partial_\mu\partial^\kappa\tilde{h}_{\kappa\nu}}{\Box} - \frac{\partial_\mu\tilde{\varpi}_\nu + \partial_\nu\tilde{\varpi}_\mu}{\sqrt{\Box}} \, , \\
\varpi_\mu &= -\partial_\mu\tilde{\delta} - \partial^\kappa\tilde{h}_{\mu\kappa} - \sqrt{\Box}\tilde{\varpi}_\mu \, , \\
\pi &= -2\Box\tilde{\delta} + \Box\tilde{\pi} - \partial^\mu\partial^\nu\tilde{h}_{\mu\nu} \, , \\
\delta &= \frac{\tilde{\delta}}{2} + \frac{\tilde{\pi}}{4} + \frac{\partial^\mu\partial^\nu\tilde{h}_{\mu\nu}}{4\Box} \, , \\
\varphi &= \frac{1}{\sqrt{2}}\mathring{\tilde{h}} - \frac{1}{\sqrt{2}\Box}\partial^\mu\partial^\nu\tilde{h}_{\mu\nu}
\end{aligned} \tag{7.27c}$$

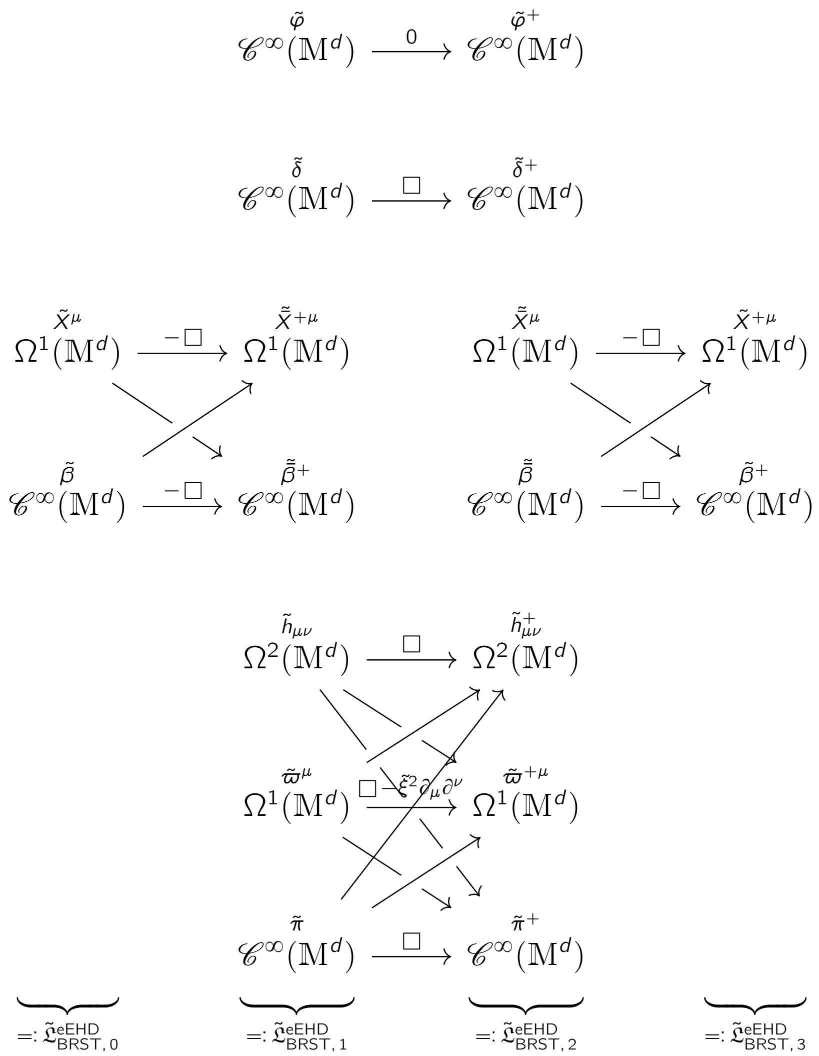
[1]We suspect that there is a simpler field redefinition in a simpler gauge which we have not been able to identify yet.

with readily computed antifield transformations. We note that the field redefinition for φ agrees precisely with the expectation of how the dilaton should be extracted from the double copied metric perturbation $\tilde{h}$.

For general ξ, the total Lagrangian to lowest order in κ, reads as

$$\begin{aligned}\mathscr{L}^{\text{eEHD}}_{\text{BRST},0} = {} & \tfrac{1}{4}\tilde{h}_{\mu\nu}\Box\tilde{h}^{\mu\nu} + \tfrac{1}{2}\tilde{\varpi}_\mu\Box\tilde{\varpi}^\mu + \tfrac{1}{2}\tilde{\xi}^2(\partial^\mu\tilde{\varpi}_\mu)^2 + \tilde{\xi}\tilde{\varpi}^\nu\sqrt{\Box}\partial^\mu\tilde{h}_{\mu\nu} - \\ & - \tfrac{1}{2}\tilde{\delta}\Box\tilde{\delta} + \tfrac{1}{4}\tilde{\pi}\Box\tilde{\pi} + \tilde{\xi}\tilde{\pi}\sqrt{\Box}\partial_\mu\tilde{\varpi}^\mu + \tfrac{1}{2}\tilde{\xi}^2\tilde{\pi}\partial_\mu\partial_\nu\tilde{h}^{\mu\nu} - \\ & - \tilde{\bar{X}}_\mu\Box\tilde{X}^\mu - \tilde{\bar{\beta}}\Box\tilde{\beta} + \tilde{\xi}\tilde{\beta}\sqrt{\Box}\partial_\mu\tilde{\bar{X}}^\mu - \tilde{\xi}\tilde{\bar{\beta}}\sqrt{\Box}\partial_\mu\tilde{X}^\mu\,.\end{aligned} \tag{7.28}$$

This is the quadratic part of the Lagrangian of the superfield homotopy Maurer–Cartan action (3.25b) for an L_∞-algebra $\tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST}}$. The latter has underlying complex



(7.29a)

with

$$\begin{aligned}
\tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},0} &= \bigoplus_{\phi\in(\tilde{\beta},\tilde{X})} \tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},0,\phi}~, & \tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},1} &= \bigoplus_{\phi\in(\tilde{\delta},\tilde{\tilde{X}}^+,\tilde{\tilde{\beta}}^+,\tilde{h},\tilde{\varpi},\tilde{\pi})} \tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},1,\phi}~,\\
\tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},3} &= \bigoplus_{\phi\in(\tilde{\beta}^+,\tilde{X}^+)} \tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},3,\phi}~, & \tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},2} &= \bigoplus_{\phi\in(\tilde{\delta}^+,\tilde{\tilde{X}},\tilde{\tilde{\beta}},\tilde{h}^+,\tilde{\varpi}^+,\tilde{\pi}^+)} \tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},2,\phi}~,
\end{aligned} \tag{7.29b}$$

and the lowest non-vanishing products

$$\begin{aligned}
&\begin{pmatrix}\tilde{X}_\mu\\ \tilde{\beta}\end{pmatrix} \xmapsto{\mu_1} -\begin{pmatrix}\Box\tilde{X}_\mu - \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\beta}\\ \Box\tilde{\beta} + \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{X}^\mu\end{pmatrix} \in \bigoplus_{\phi\in(\tilde{\tilde{X}}^+,\tilde{\tilde{\beta}}^+)} \tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},1,\phi}~,\\
&\begin{pmatrix}\tilde{h}_{\mu\nu}\\ \tilde{\varpi}_\mu\\ \tilde{\pi}\end{pmatrix} \xmapsto{\mu_1} \begin{pmatrix}\Box\tilde{h}_{\mu\nu} - 2\tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\varpi}_\nu + \tilde{\xi}^2\partial_\mu\partial_\nu\tilde{\pi}\\ \Box\tilde{\varpi}_\mu + \tilde{\xi}\sqrt{\Box}\partial^\mu\tilde{h}_{\mu\nu} - \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\pi} - \tilde{\xi}^2\partial_\mu\partial^\nu\tilde{\varpi}_\nu\\ \Box\tilde{\pi}_\mu(x) + 2\tilde{\xi}\sqrt{\Box}\partial^\mu\tilde{\varpi}_\mu(x) + \tilde{\xi}^2\partial^\mu\partial^\nu\tilde{h}_{\mu\nu}\end{pmatrix} \in \bigoplus_{\phi\in(\tilde{h}^+,\tilde{\varpi}^+,\tilde{\pi}^+)} \tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},2,\phi}~,\\
&\begin{pmatrix}\tilde{\tilde{X}}_\mu\\ \tilde{\tilde{\beta}}\end{pmatrix} \xmapsto{\mu_1} -\begin{pmatrix}\Box\tilde{\tilde{X}}_\mu - \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\tilde{\beta}}\\ \Box\tilde{\tilde{\beta}} + \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\tilde{X}}^\mu\end{pmatrix} \in \bigoplus_{\phi\in(\tilde{X}^+,\tilde{\beta}^+)} \tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST},3,\phi}~.
\end{aligned} \tag{7.29c}$$

The $\tilde{\mathfrak{L}}^{\text{eEHD}}_{\text{BRST}}$ algebra is endowed with the following cyclic structure:

$$\begin{aligned}
\langle\tilde{X},\tilde{X}^+\rangle &:= \int \mathrm{d}^d x\, \tilde{X}^\mu\tilde{X}^+_\mu~, & \langle\tilde{\tilde{X}},\tilde{\tilde{X}}^+\rangle &:= -\int \mathrm{d}^d x\, \tilde{\tilde{X}}^\mu\tilde{\tilde{X}}^+_\mu~,\\
\langle\tilde{\beta},\tilde{\beta}^+\rangle &:= \int \mathrm{d}^d x\, \tilde{\beta}\tilde{\beta}^+~, & \langle\tilde{\tilde{\beta}},\tilde{\tilde{\beta}}^+\rangle &:= -\int \mathrm{d}^d x\, \tilde{\tilde{\beta}}\tilde{\tilde{\beta}}^+~,\\
\langle\tilde{h},\tilde{h}^+\rangle &:= \tfrac{1}{2}\int \mathrm{d}^d x\, \tilde{h}^{\mu\nu}\tilde{h}^+_{\mu\nu}~,\\
\langle\tilde{\varpi},\tilde{\varpi}^+\rangle &:= \int \mathrm{d}^d x\, \tilde{\varpi}^\mu\tilde{\varpi}^+_\mu~,\\
\langle\tilde{\pi},\tilde{\pi}^+\rangle &:= \tfrac{1}{2}\int \mathrm{d}^d x\, \tilde{\pi}\tilde{\pi}^+~, & \langle\tilde{\delta},\tilde{\delta}^+\rangle &:= -\int \mathrm{d}^d x\, \tilde{\delta}\tilde{\delta}^+~.
\end{aligned} \tag{7.30}$$

7.5. Factorisation of the cochain complex of $\mathcal{N} = 0$ supergravity

The factorisation of the cochain complex of the L_∞-algebra for Yang–Mills theory now fixes completely the factorisation of the cochain complex of the L_∞-algebra of $\mathcal{N} = 0$ supergravity. In view of (7.15), it thus merely remains to verify that

$$\mathsf{Ch}(\tilde{\mathfrak{L}}^{\mathcal{N}=0}_{\text{BRST}}) = \mathfrak{Kin} \otimes_{\tau_1} (\mathfrak{Kin} \otimes_{\tau_1} \mathsf{Ch}(\mathfrak{Scal})) \tag{7.31}$$

at the level of cochain complexes, where $\mathfrak{Kin}$ is given in (7.16) and $\mathsf{Ch}(\mathfrak{Scal})$ in (7.4). Furthermore, the twist in the outer tensor product of (7.31) will only affect $\mathsf{Ch}(\mathfrak{Scal})$ and commute with the other factor of $\mathfrak{Kin}$. Let us stress that we could have allowed for two different twist parameters for each of the tensor products. This, however, would make our discussion unnecessarily involved.

Factorisation of fields. It is not surprising that the identification works at the level of graded vector spaces for the physical fields. This is merely the statement that a rank-two (covariant) tensor decomposes into its symmetric part and its anti-symmetric part. The symmetric part splits further into the trace, which can be identified with the dilaton, and the remaining components, which describe gravitational modes. More interesting is the sector of unphysical fields, and the complete factorisation of all fields is given in Table 7.5.

The elements of $\mathfrak{Kin}$ form a quartet, which is reflected in the well-known quartet of fields in the gauge-fixed Yang–Mills action:

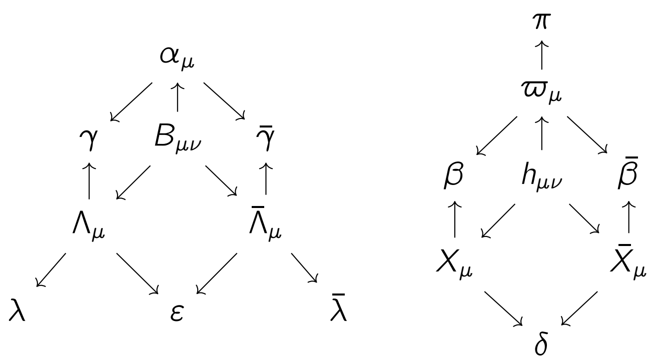
$$\begin{array}{ccc} & \mathtt{n} & \\ & \uparrow & \\ & \mathtt{v}^{\mu} & \\ \swarrow & & \searrow \\ \mathtt{g} & & \mathtt{a} \end{array} \quad \longrightarrow \quad \begin{array}{ccc} & b^a & \\ & \uparrow & \\ & A^a_{\mu} & \\ \swarrow & & \searrow \\ c^a & & \bar{c}^a \end{array} \tag{7.32}$$

In the last diagram, and in the following ones, a field is connected to the associated Nakanishi–Lautrup field, ghost and BRST antighost by an upward arrow, a left downward arrow and a right downward arrow, respectively. The relationships between the terms in the diagram correspond to the entries in Table 7.4. Each field in $\mathsf{Ch}(\tilde{\mathfrak{L}}^{\mathcal{N}=0}_{\mathrm{BRST}})$ thus lives in the tensor product of two such quartets. This tensor product further splits into (graded) symmetric, anti-symmetric, and trace parts, which belong to the two-form $B_{\mu\nu}$, the graviton modes $h_{\mu\nu}$, and the dilaton φ. Because the product of two ghosts $\mathtt{g}\tilde{\mathtt{g}}$ is automatically anti-symmetric, only the B-field has a ghost for ghost λ. On the graviton/dilaton side, we do not have the higher gauge transformations, but contrary to Yang–Mills theory, the ghost is a vector. We can summarise the relations between the fields in the following two

fields				anti-fields		
factorisation	$\lvert - \rvert_{\mathrm{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim	factorisation	$\lvert - \rvert_{\mathfrak{L}}$	dim
$\tilde{\lambda} = -[\mathrm{g},\mathrm{g}]\mathrm{s}_x\frac{1}{2}\tilde{\lambda}(x)$	2	-1	$\frac{d}{2}-3$	$\tilde{\lambda}^+ = -[\mathrm{a},\mathrm{a}]\mathrm{s}_x^+\frac{1}{2}\tilde{\lambda}^+(x)$	4	$\frac{d}{2}+3$
$\tilde{\Lambda} = [\mathrm{g},\mathrm{v}^\mu]\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\Lambda}_\mu(x)$	1	0	$\frac{d}{2}-2$	$\tilde{\Lambda}^+ = [\mathrm{a},\mathrm{v}^\mu]\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\Lambda}^+_\mu$	3	$\frac{d}{2}+2$
$\tilde{\gamma} = [\mathrm{g},\mathrm{n}]\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\gamma}(x)$	1	0	$\frac{d}{2}-2$	$\tilde{\gamma}^+ = [\mathrm{a},\mathrm{n}]\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\gamma}^+(x)$	3	$\frac{d}{2}+2$
$\tilde{B} = [\mathrm{v}^\mu,\mathrm{v}^\nu]\mathrm{s}_x\frac{1}{2\sqrt{2}}\tilde{B}_{\mu\nu}(x)$	0	1	$\frac{d}{2}-1$	$\tilde{B}^+ = [\mathrm{v}^\mu,\mathrm{v}^\nu]\mathrm{s}_x^+\frac{1}{2\sqrt{2}}\tilde{B}^+_{\mu\nu}(x)$	2	$\frac{d}{2}+1$
$\tilde{\alpha} = [\mathrm{n},\mathrm{v}^\mu]\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\alpha}_\mu(x)$	0	1	$\frac{d}{2}-1$	$\tilde{\alpha}^+ = [\mathrm{n},\mathrm{v}^\mu]\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\alpha}^+_\mu(x)$	2	$\frac{d}{2}+1$
$\tilde{\varepsilon} = -[\mathrm{g},\mathrm{a}]\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\varepsilon}(x)$	0	1	$\frac{d}{2}-1$	$\tilde{\varepsilon}^+ = -[\mathrm{g},\mathrm{a}]\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\varepsilon}^+(x)$	2	$\frac{d}{2}+1$
$\tilde{\tilde{\Lambda}} = [\mathrm{a},\mathrm{v}^\mu]\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\tilde{\Lambda}}_\mu(x)$	-1	2	$\frac{d}{2}$	$\tilde{\tilde{\Lambda}}^+ = [\mathrm{g},\mathrm{v}^\mu]\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\tilde{\Lambda}}^+_\mu(x)$	1	$\frac{d}{2}$
$\tilde{\tilde{\gamma}} = [\mathrm{a},\mathrm{n}]\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\tilde{\gamma}}(x)$	-1	2	$\frac{d}{2}$	$\tilde{\tilde{\gamma}}^+ = [\mathrm{g},\mathrm{n}]\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\tilde{\gamma}}^+(x)$	1	$\frac{d}{2}$
$\tilde{\tilde{\lambda}} = -[\mathrm{a},\mathrm{a}]\mathrm{s}_x\frac{1}{2}\tilde{\tilde{\lambda}}(x)$	-2	3	$\frac{d}{2}+1$	$\tilde{\tilde{\lambda}}^+ = -[\mathrm{g},\mathrm{g}]\mathrm{s}_x^+\frac{1}{2}\tilde{\tilde{\lambda}}^+(x)$	0	$\frac{d}{2}-1$
$\tilde{X} = (\mathrm{g},\mathrm{v}^\mu)\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{X}_\mu(x)$	1	0	$\frac{d}{2}-2$	$\tilde{X}^+ = (\mathrm{a},\mathrm{v}^\mu)\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{X}^+_\mu(x)$	3	$\frac{d}{2}+2$
$\tilde{\beta} = (\mathrm{g},\mathrm{n})\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\beta}(x)$	1	0	$\frac{d}{2}-2$	$\tilde{\beta}^+ = (\mathrm{a},\mathrm{n})\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\beta}^+(x)$	3	$\frac{d}{2}+2$
$\tilde{h} = (\mathrm{v}^\mu,\mathrm{v}^\nu)\mathrm{s}_x\frac{1}{2\sqrt{2}}\tilde{h}_{\mu\nu}(x)$	0	1	$\frac{d}{2}-1$	$\tilde{h}^+ = (\mathrm{v}^\mu,\mathrm{v}^\nu)\mathrm{s}_x^+\frac{1}{2\sqrt{2}}\tilde{h}^+_{\mu\nu}(x)$	2	$\frac{d}{2}+1$
$\tilde{\varpi} = -(\mathrm{n},\mathrm{v}^\mu)\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\varpi}_\mu(x)$	0	1	$\frac{d}{2}-1$	$\tilde{\varpi}^+ = -(\mathrm{n},\mathrm{v}^\mu)\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\varpi}^+_\mu(x)$	2	$\frac{d}{2}+1$
$\tilde{\pi} = (\mathrm{n},\mathrm{n})\mathrm{s}_x\frac{1}{2\sqrt{2}}\tilde{\pi}(x)$	0	1	$\frac{d}{2}-1$	$\tilde{\pi}^+ = (\mathrm{n},\mathrm{n})\mathrm{s}_x^+\frac{1}{2\sqrt{2}}\tilde{\pi}^+(x)$	2	$\frac{d}{2}+1$
$\tilde{\delta} = -(\mathrm{g},\mathrm{a})\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\delta}(x)$	0	1	$\frac{d}{2}-1$	$\tilde{\delta}^+ = -(\mathrm{g},\mathrm{a})\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\delta}^+(x)$	2	$\frac{d}{2}+1$
$\tilde{\tilde{X}} = (\mathrm{a},\mathrm{v}^\mu)\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\tilde{X}}_\mu(x)$	-1	2	$\frac{d}{2}$	$\tilde{\tilde{X}}^+ = (\mathrm{g},\mathrm{v}^\mu)\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\tilde{X}}_\mu(x)$	1	$\frac{d}{2}$
$\tilde{\tilde{\beta}} = (\mathrm{a},\mathrm{n})\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\tilde{\beta}}(x)$	-1	2	$\frac{d}{2}$	$\tilde{\tilde{\beta}}^+ = (\mathrm{g},\mathrm{n})\mathrm{s}_x^+\frac{1}{\sqrt{2}}\tilde{\tilde{\beta}}^+(x)$	1	$\frac{d}{2}$

Table 7.5: Factorisation of the redefined BV fields for $\mathcal{N} = 0$ supergravity. Just as in the case of Yang–Mills theory, all fields have a factor of s_x, while all anti-fields have a factor of s_x^+. Here, we again suppressed the integrals over x and we used the notation $[\mathrm{x},\mathrm{y}] := \mathrm{x}\otimes\mathrm{y} - (-1)^{|\mathrm{x}|\,|\mathrm{y}|}\mathrm{y}\otimes\mathrm{x}$ and $(\mathrm{x},\mathrm{y}) := \mathrm{x}\otimes\mathrm{y} + (-1)^{|\mathrm{x}|\,|\mathrm{y}|}\mathrm{y}\otimes\mathrm{x}$ for $\mathrm{x},\mathrm{y}\in\mathfrak{Kin}$.

diagrams:



(7.33)

where upper, lower left, and lower right arrows point to fields where a vector factor $\mathtt{v}^\mu$ has been replaced by a factor $\mathtt{n}$, $\mathtt{g}$, and $\mathtt{a}$, respectively. The L_∞-degrees of the fields are the same in each column, increasing from left to right by one.

Factorisation as cyclic complex. From Table 7.5, it is clear that the tensor product (7.31) is indeed correct at the level of graded vector spaces. The inner product structure on the anti-symmetric part is given by

$$\begin{aligned} \langle\tilde{\lambda},\tilde{\lambda}^+\rangle &= \left\langle -\mathtt{g}\otimes\mathtt{g}\otimes\int \mathrm{d}^d x_1\,\mathtt{s}_{x_1}\tilde{\lambda}(x_1), -\mathtt{a}\otimes\mathtt{a}\otimes\int \mathrm{d}^d x_2\,\mathtt{s}^+_{x_2}\tilde{\lambda}^+(x_2)\right\rangle \\ &= -\langle\mathtt{g},\mathtt{a}\rangle\langle\mathtt{g},\mathtt{a}\rangle\int\mathrm{d}^d x_1\int\mathrm{d}^d x_2\,\delta^{(d)}(x_1-x_2)\tilde{\lambda}(x_1)\tilde{\lambda}^+(x_2) \\ &= -\int\mathrm{d}^d x\;\tilde{\lambda}(x)\tilde{\lambda}^+(x)\ , \end{aligned} \tag{7.34a}$$

Similarly,

$$\begin{aligned} \langle\tilde{\Lambda},\tilde{\Lambda}^+\rangle &= \int\mathrm{d}^d x\,\tilde{\Lambda}^\mu(x)\tilde{\Lambda}^+_\mu(x)\ , & \langle\tilde{\bar{\Lambda}},\tilde{\bar{\Lambda}}^+\rangle &= -\int\mathrm{d}^d x\,\tilde{\bar{\Lambda}}^\mu(x)\tilde{\bar{\Lambda}}^+_\mu(x)\ , \\ \langle\tilde{\gamma},\tilde{\gamma}^+\rangle &= \int\mathrm{d}^d x\,\tilde{\gamma}(x)\tilde{\gamma}^+(x)\ , & \langle\tilde{\bar{\gamma}},\tilde{\bar{\gamma}}^+\rangle &= -\int\mathrm{d}^d x\,\tilde{\bar{\gamma}}(x)\tilde{\bar{\gamma}}^+(x)\ , \\ \langle\tilde{B},\tilde{B}^+\rangle &= \tfrac{1}{2}\int\mathrm{d}^d x\,\tilde{B}^{\mu\nu}(x)\tilde{B}^+_{\mu\nu}(x)\ , & \langle\tilde{\varepsilon},\tilde{\varepsilon}^+\rangle &= \int\mathrm{d}^d x\,\tilde{\varepsilon}(x)\tilde{\varepsilon}^+(x)\ , \\ \langle\tilde{\alpha},\tilde{\alpha}^+\rangle &= \int\mathrm{d}^d x\,\tilde{\alpha}^\mu(x)\tilde{\alpha}^+_\mu(x)\ , & \langle\tilde{\bar{\lambda}},\tilde{\bar{\lambda}}^+\rangle &= -\int\mathrm{d}^d x\,\tilde{\bar{\lambda}}(x)\tilde{\bar{\lambda}}^+(x)\ . \end{aligned} \tag{7.34b}$$

On the symmetric part, we have analogously

$$\begin{aligned}
\langle\tilde{X},\tilde{X}^+\rangle &= \int \mathrm{d}^d x\,\tilde{X}^\mu(x)\tilde{X}^+_\mu(x)\ , & \langle\tilde{\pi},\tilde{\pi}^+\rangle &= \tfrac{1}{2}\int \mathrm{d}^d x\,\tilde{\pi}(x)\tilde{\pi}^+(x)\ ,\\
\langle\tilde{\beta},\tilde{\beta}^+\rangle &= \int \mathrm{d}^d x\,\tilde{\beta}(x)\tilde{\beta}^+(x)\ , & \langle\tilde{\delta},\tilde{\delta}^+\rangle &= -\int \mathrm{d}^d x\,\tilde{\delta}(x)\tilde{\delta}^+(x)\ ,\\
\langle\tilde{h},\tilde{h}^+\rangle &= \tfrac{1}{2}\int \mathrm{d}^d x\,\tilde{h}^{\mu\nu}(x)\tilde{h}^+_{\mu\nu}(x)\ , & \langle\tilde{\tilde{X}},\tilde{\tilde{X}}^+\rangle &= -\int \mathrm{d}^d x\,\tilde{\tilde{X}}^\mu(x)\tilde{\tilde{X}}^+_\mu(x)\ ,\\
\langle\tilde{\varpi},\tilde{\varpi}^+\rangle &= \int \mathrm{d}^d x\,\tilde{\varpi}^\mu(x)\tilde{\varpi}^+_\mu(x)\ , & \langle\tilde{\tilde{\beta}},\tilde{\tilde{\beta}}^+\rangle &= -\int \mathrm{d}^d x\,\tilde{\tilde{\beta}}(x)\tilde{\tilde{\beta}}^+(x)\ .
\end{aligned} \tag{7.34c}$$

Next, we compute the action of the differential μ_1, which is completely fixed by the tensor product $\mathfrak{Kin}\otimes_{\tau_1}(\mathfrak{Kin}\otimes_{\tau_1}\mathfrak{Scal})$, cf. definition (6.15). Following the notation described in Table 7.5, we have, for example,

$$\begin{aligned}
\mu_1(\tilde{\lambda}) &= \mu_1\left(-[\mathrm{g},\mathrm{g}]\otimes\tfrac{1}{2}\int\mathrm{d}^d x\,\mathrm{s}_x\tilde{\lambda}(x)\right) = -[\mathrm{g},\mathrm{g}]\otimes\tfrac{1}{2}\mu_1\left(\int\mathrm{d}^d x\,\mathrm{s}_x\tilde{\lambda}(x)\right) = \Box\tilde{\lambda}\ ,\\
\mu_1\begin{pmatrix}\tilde{\Lambda}\\ \tilde{\gamma}\end{pmatrix} &= \mu_1\left(([\mathrm{g},\mathrm{v}^\mu],[\mathrm{g},\mathrm{n}])\otimes\begin{pmatrix}\int\mathrm{d}^d x\,\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\Lambda}_\mu(x)\\ \int\mathrm{d}^d x\,\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\gamma}(x)\end{pmatrix}\right)\\
&= -([\mathrm{g},\mathrm{v}^\mu],[\mathrm{g},\mathrm{n}])\otimes\mu_1\left(\begin{pmatrix}\mathrm{id} & -\tilde{\xi}\Box^{-\frac{1}{2}}\partial_\mu\\ \tilde{\xi}\Box^{-\frac{1}{2}}\partial^\mu & \mathrm{id}\end{pmatrix}\begin{pmatrix}\int\mathrm{d}^d x\,\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\Lambda}_\mu(x)\\ \int\mathrm{d}^d x\,\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\gamma}(x)\end{pmatrix}\right)\\
&= -([\mathrm{g},\mathrm{v}^\mu],[\mathrm{g},\mathrm{n}])\otimes\begin{pmatrix}\int\mathrm{d}^d x\,\mathrm{s}^+_x\frac{1}{\sqrt{2}}\{\Box\tilde{\Lambda}_\mu(x)-\tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\gamma}(x)\}\\ \int\mathrm{d}^d x\,\mathrm{s}^+_x\frac{1}{\sqrt{2}}\{\Box\tilde{\gamma}(x)+\tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\Lambda}^\mu(x)\}\end{pmatrix},\\
\mu_1\begin{pmatrix}\tilde{B}\\ \tilde{\alpha}\end{pmatrix} &= \mu_1\left(([\mathrm{v}^\mu,\mathrm{v}^\nu],[\mathrm{n},\mathrm{v}^\mu])\otimes\begin{pmatrix}\int\mathrm{d}^d x\,\mathrm{s}_x\frac{1}{2\sqrt{2}}\tilde{B}_{\mu\nu}(x)\\ \int\mathrm{d}^d x\,\mathrm{s}_x\frac{1}{\sqrt{2}}\tilde{\alpha}_\mu(x)\end{pmatrix}\right)\\
&= ([\mathrm{v}^\mu,\mathrm{v}^\nu],[\mathrm{n},\mathrm{v}^\mu])\otimes\begin{pmatrix}\int\mathrm{d}^d x\,\mathrm{s}^+_x\frac{1}{\sqrt{2}}\{\frac{1}{2}\Box\tilde{B}_{\mu\nu}(x)-\tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\alpha}_\nu(x)\}\\ \int\mathrm{d}^d x\,\mathrm{s}^+_x\frac{1}{\sqrt{2}}\{\Box\tilde{\alpha}_\mu(x)+\tilde{\xi}\sqrt{\Box}\partial^\nu\tilde{B}_{\nu\mu}(x)+\tilde{\xi}^2\partial_\mu\partial^\nu\tilde{\alpha}_\nu(x)\}\end{pmatrix},\\
\mu_1\begin{pmatrix}\tilde{h}\\ \tilde{\varpi}\\ \tilde{\pi}\end{pmatrix} &= \mu_1\left(((\mathrm{v}^\mu,\mathrm{v}^\nu),(\mathrm{n},\mathrm{v}^\mu),(\mathrm{n},\mathrm{n}))\otimes\begin{pmatrix}\int\mathrm{d}^d x\,\mathrm{s}_x\frac{1}{2\sqrt{2}}\tilde{h}_{\mu\nu}(x)\\ \int\mathrm{d}^d x\,\mathrm{s}_x\left(-\frac{1}{\sqrt{2}}\tilde{\varpi}_\mu(x)\right)\\ \int\mathrm{d}^d x\,\mathrm{s}_x\frac{1}{2\sqrt{2}}\tilde{\pi}_\mu(x)\end{pmatrix}\right)\\
&= ((\mathrm{v}^\mu,\mathrm{v}^\nu),(\mathrm{n},\mathrm{v}^\mu),(\mathrm{n},\mathrm{n}))\otimes M
\end{aligned} \tag{7.35a}$$

with

$$M := \begin{pmatrix} \int \mathrm{d}^d x\, \mathrm{s}_x^+\{\frac{1}{2\sqrt{2}}\Box\,\tilde{h}_{\mu\nu}(x) - \frac{1}{\sqrt{2}}\tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\varpi}_\nu(x) + \frac{1}{2\sqrt{2}}\tilde{\xi}^2\partial_\mu\partial_\nu\tilde{\pi}(x)\} \\ \int \mathrm{d}^d x\, \mathrm{s}_x^+\{-\frac{1}{\sqrt{2}}\Box\,\tilde{\varpi}_\mu(x) - \frac{1}{\sqrt{2}}\tilde{\xi}\sqrt{\Box}\partial^\mu\tilde{h}_{\mu\nu}(x) + \frac{1}{\sqrt{2}}\tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\pi}(x) + \frac{1}{\sqrt{2}}\tilde{\xi}^2\partial_\mu\partial^\nu\tilde{\varpi}_\nu(x)\} \\ \int \mathrm{d}^d x\, \mathrm{s}_x^+\{\frac{1}{2\sqrt{2}}\Box\,\tilde{\pi}_\mu(x) + \frac{1}{\sqrt{2}}\tilde{\xi}\sqrt{\Box}\partial^\mu\tilde{\varpi}_\mu(x) + \frac{1}{2\sqrt{2}}\tilde{\xi}^2\partial^\mu\partial^\nu\tilde{h}_{\mu\nu}\} \end{pmatrix}. \tag{7.35b}$$

Furthermore, we have

$$\begin{aligned}
\mu_1\begin{pmatrix}\tilde{\bar{\Lambda}}\\ \tilde{\bar{\gamma}}\end{pmatrix} &= -([\mathrm{a},\mathrm{v}^\mu],[\mathrm{a},\mathrm{n}])\otimes\begin{pmatrix}\int \mathrm{d}^d x\,\mathrm{s}_x^+\frac{1}{\sqrt{2}}\{\Box\,\tilde{\bar{\Lambda}}_\mu(x) - \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\bar{\gamma}}(x)\}\\ \int \mathrm{d}^d x\,\mathrm{s}_x^+\frac{1}{\sqrt{2}}\{\Box\,\tilde{\bar{\gamma}}(x) + \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\bar{\Lambda}}^\mu(x)\}\end{pmatrix}, \\
\mu_1(\tilde{\varepsilon}) &= \Box\,\tilde{\varepsilon}, \\
\mu_1(\tilde{\bar{\lambda}}) &= \Box\,\tilde{\bar{\lambda}}, \\
\mu_1\begin{pmatrix}\tilde{X}\\ \tilde{\beta}\end{pmatrix} &= -((\mathrm{g},\mathrm{v}^\mu),(\mathrm{g},\mathrm{n}))\otimes\begin{pmatrix}\int \mathrm{d}^d x\,\mathrm{s}_x^+\frac{1}{\sqrt{2}}\{\Box\,\tilde{X}_\mu(x) - \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\beta}(x)\}\\ \int \mathrm{d}^d x\,\mathrm{s}_x^+\frac{1}{\sqrt{2}}\{\Box\tilde{\beta}(x) + \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{X}^\mu(x)\}\end{pmatrix}, \\
\mu_1\begin{pmatrix}\tilde{\bar{X}}\\ \tilde{\bar{\beta}}\end{pmatrix} &= -([\mathrm{a},\mathrm{v}^\mu],[\mathrm{a},\mathrm{n}])\otimes\begin{pmatrix}\int \mathrm{d}^d x\,\mathrm{s}_x^+\frac{1}{\sqrt{2}}\{\Box\,\tilde{\bar{X}}_\mu(x) - \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\bar{\beta}}(x)\}\\ \int \mathrm{d}^d x\,\mathrm{s}_x^+\frac{1}{\sqrt{2}}\{\Box\tilde{\bar{\beta}}(x) + \tilde{\xi}\sqrt{\Box}\partial_\mu\tilde{\bar{X}}^\mu(x)\}\end{pmatrix}, \\
\mu_1(\tilde{\delta}) &= \Box\,\tilde{\delta}.
\end{aligned} \tag{7.35c}$$

The resulting superfield homotopy Maurer–Cartan action (3.25a) for the superfield $\mathsf{a} = \tilde{\lambda}+\tilde{\Lambda}+\cdots+\tilde{B}+\tilde{h}$ is

$$\begin{aligned}
\tilde{S}_0^{\mathrm{DC}} := \int \mathrm{d}^d x\,\Big\{ &\tfrac{1}{4}\tilde{B}_{\mu\nu}\Box\,\tilde{B}^{\mu\nu} - \tilde{\bar{\Lambda}}_\mu\Box\,\tilde{\Lambda}^\mu + \tfrac{1}{2}\tilde{\alpha}_\mu\Box\,\tilde{\alpha}^\mu - \tfrac{\tilde{\xi}^2}{2}(\partial^\mu\tilde{\alpha}_\mu)^2 + \tfrac{1}{2}\tilde{\varepsilon}\Box\,\tilde{\varepsilon} - \tilde{\bar{\lambda}}\Box\,\tilde{\lambda} - \\
&- \tilde{\bar{\gamma}}\Box\,\tilde{\gamma} + \tilde{\xi}\tilde{\alpha}^\nu\sqrt{\Box}\partial^\mu\tilde{B}_{\mu\nu} + \tilde{\xi}\tilde{\gamma}\sqrt{\Box}\partial_\mu\tilde{\bar{\Lambda}}^\mu - \tilde{\xi}\tilde{\bar{\gamma}}\sqrt{\Box}\partial_\mu\tilde{\Lambda}^\mu + \\
&+ \tfrac{1}{4}\tilde{h}_{\mu\nu}\Box\,\tilde{h}^{\mu\nu} - \tilde{\bar{X}}_\mu\Box\,\tilde{X}^\mu + \tfrac{1}{2}\tilde{\varpi}_\mu\Box\,\tilde{\varpi}^\mu + \tfrac{\tilde{\xi}^2}{2}(\partial^\mu\tilde{\varpi}_\mu)^2 - \\
&- \tfrac{1}{2}\tilde{\delta}\Box\,\tilde{\delta} + \tfrac{1}{4}\tilde{\pi}\Box\,\tilde{\pi} - \tilde{\bar{\beta}}\Box\tilde{\beta} + \tilde{\xi}\tilde{\varpi}^\nu\sqrt{\Box}\partial^\mu\tilde{h}_{\mu\nu} + \tilde{\xi}\tilde{\pi}\sqrt{\Box}\partial_\mu\tilde{\varpi}^\mu + \\
&+ \tfrac{1}{2}\tilde{\xi}^2\tilde{\pi}\partial_\mu\partial_\nu\tilde{h}^{\mu\nu} + \tilde{\xi}\tilde{\beta}\sqrt{\Box}\partial_\mu\tilde{\bar{X}}^\mu - \tilde{\xi}\tilde{\bar{\beta}}\sqrt{\Box}\partial_\mu\tilde{X}^\mu\Big\}.
\end{aligned} \tag{7.36}$$

This action is precisely the sum of the transformed Kalb–Ramond action (7.23) and of the transformed zeroth-order gravity action augmented by a dilaton kinetic term (7.28). Consequently, we see that our double copy prescription, arising from the factorisation of the L_∞-algebras of Yang–Mills theory and $\mathcal{N}=0$ supergravity into three factors, works at the level of cochain complexes.

8 Quantum field theoretic preliminaries

After completing the discussion of double copy for the free theories, the objective is to extend our picture to the full, interacting level.

As discussed in Section 1.3., to double copy Yang–Mills amplitudes we need to reformulate them in terms of diagrams with trivalent vertices only. Interpreted in the context of homotopy algebras, this corresponds to a strictification of the original theory, associated to a physically equivalent action with only cubic interaction terms. We will give explicit formulas for the lowest orders in coupling constants.

The strictification of the underlying L_∞-algebra of the theory allows us to factorise it accordingly with our twisted tensor product notion, introduced in Chapter 6, and to construct the Lagrangian double copy theory associated with Yang–Mills theory. We will refrain from giving fully explicit expressions for this action.

In this Chapter we introduce a set of quantum field theoretic observations, that prepare the ground for the proof of the quantum equivalence between Yang–Mills theory double copy theory and $\mathcal{N} = 0$ supergravity given in the final Chapter.

In the following, we shall always clearly distinguish between scattering amplitudes $\mathscr{A}(\cdots)$ and correlation functions $\langle\cdots\rangle$. Correlation functions, contain operators that create and annihilate arbitrary fields without any constraints. Scattering amplitudes, on the other hand, are labelled by *external fields*, which usually are physical fields with on-shell momenta and physical polarisations. For our arguments, it is convenient to lift the restriction to physical polarisations and work with the BRST-extended Hilbert space of external fields which, in the case of Yang–Mills theory, includes gluons of arbitrary polarisations as well as the ghosts and anti-ghosts, as we will explain in the following.

The material in this Chapter is borrowed from [5, 6].

8.1. BRST-extended Hilbert space and Ward identities

The tree-level scattering amplitudes of Yang–Mills theory are parametrised by degree one elements of the minimal model of the L_∞-algebra (7.12). These are the physical, on-shell states. A convenient set of coordinates for these are the gluon's momentum p_μ as well as a discrete label indicating the gluon's helicity. More conveniently, we can replace the discrete labels by a linearly independent set of polarisation vectors ε_μ that satisfy

$$(\varepsilon_\mu) = \begin{pmatrix} 0 \\ \vec{\varepsilon} \end{pmatrix}, \quad \vec{p}\cdot\vec{\varepsilon} = 0 , \quad \text{and} \quad |\vec{\varepsilon}| = 1 . \tag{8.1}$$

BRST-extended Hilbert space. We can extend this conventional Hilbert space of external fields to the full BRST field space $\mathfrak{H}^{\rm YM}_{\rm BRST}$ as done, e.g., in [232]. We thus have two additional, unphysical polarisations of the gluon, called *forward* and *backward* and denoted by $A^{\uparrow a}_\mu$ and $A^{\downarrow a}_\mu$, respectively. We can be a bit more explicit for general gluons with light-like momenta. Here, the polarisation vector $\varepsilon^\uparrow_\mu$ is proportional to the momentum p_μ and the backwards polarisation vector $\varepsilon^\downarrow_\mu$ is obtained by reversing the spatial part,

$$(\varepsilon^\uparrow_\mu) = \frac{1}{\sqrt{2}|\vec{p}\,|}\begin{pmatrix} p_0 \\ \vec{p} \end{pmatrix} \quad \text{and} \quad (\varepsilon^\downarrow_\mu) = \frac{1}{\sqrt{2}|\vec{p}\,|}\begin{pmatrix} p_0 \\ -\vec{p} \end{pmatrix}, \tag{8.2a}$$

so that

$$\varepsilon^\uparrow\cdot\varepsilon^\uparrow = 0 , \quad \varepsilon^\downarrow\cdot\varepsilon^\downarrow = 0 , \quad \text{and} \quad \varepsilon^\uparrow\cdot\varepsilon^\downarrow = -1 . \tag{8.2b}$$

We also have ghost and anti-ghost states. All scattering amplitudes we shall consider will be built from the Hilbert space $\mathfrak{H}^{\rm YM}_{\rm BRST}$. We note that the S-matrix of the physical Hilbert space $\mathfrak{H}^{\rm YM}_{\rm phys}$ is then the restriction of the S-matrix for the BRST extended Hilbert space $\mathfrak{H}^{\rm YM}_{\rm BRST}$. Although there are scattering amplitudes producing unphysical particles in $\mathfrak{H}^{\rm YM}_{\rm BRST}$ from physical gluons in $\mathfrak{H}^{\rm YM}_{\rm phys}$, this is consistent, because the restricted S-matrix is unitary. This is a consequence of the full S-matrix on $\mathfrak{H}^{\rm YM}_{\rm BRST}$ being unitary and BRST symmetry, cf. [233, Section 16.4].

Evidently, $\mathfrak{H}^{\rm YM}_{\rm BRST}$ carries an action of the linearisation of the BRST operator, denoted by $Q^{\rm lin}_{\rm BRST}$, cf. again [232] or the discussion in [233, Section 16.4]. Note that after gauge-fixing, the full BRST transformations are given by the restriction of the BV transformations (4.13) since the gauge-fixing fermion is assumed to be independent of the anti-fields. We have

$$\begin{aligned} c^a &\xmapsto{Q^{\text{YM}}_{\text{BRST}}} -\tfrac{1}{2} g f_{bc}{}^a c^b c^c\ , & \bar{c}^a &\xmapsto{Q^{\text{YM}}_{\text{BRST}}} b^a\ , \\ A^a_\mu &\xmapsto{Q^{\text{YM}}_{\text{BRST}}} (\nabla_\mu c)^a\ , & b^a &\xmapsto{Q^{\text{YM}}_{\text{BRST}}} 0\ , \end{aligned} \tag{8.3}$$

and $(Q^{\text{YM}}_{\text{BRST}})^2 = 0$ off-shell. In momentum space, it is then easy to see that the *transversely-polarised* or *physical* gluon states $A^{\perp\, a}_\mu$ are singlets under the action of the linearised BRST operator, $Q^{\text{YM, lin}}_{\text{BRST}} A^{\perp\, a}_\mu = 0$, since $\partial_\mu c$ is parallel to k_μ. The remaining four states arrange into two doublets,

$$A^{\uparrow\, a}_\mu \xmapsto{Q^{\text{YM, lin}}_{\text{BRST}}} \partial_\mu c^a \quad \text{and} \quad \bar{c}^a \xmapsto{Q^{\text{YM, lin}}_{\text{BRST}}} b^a = \frac{1}{\xi}\partial^\mu A^{\downarrow\, a}_\mu + \cdots\ , \tag{8.4}$$

where the ellipsis indicates terms that would arise from the shift of the gauge-fixing fermion in (8.20).

Connected correlation functions. In our later analysis of the double copy, we shall compare correlation functions at the tree level. Recall that the partition function Z and the free energy $W := \log(Z)$ are the generating functionals for the correlation functions and the connected correlation functions, respectively. Evidently, this implies that the connected correlation functions can be written as linear combinations of products of correlation functions. This simplifies our analysis as we can restrict ourselves to the contributions of connected Feynman diagrams to correlation functions.

Observation 8.1. *The set of connected correlation functions is BRST-invariant because the connected correlation functions can be written as linear combinations of products of correlation functions.*

Ward identities for scattering amplitudes. In order to translate colour–kinematics duality for scattering amplitudes from gluons to ghosts, we shall use supersymmetric on-shell Ward identities, cf. [76, 77], and we focus on the supersymmetry generated by the linearised BRST operator $Q^{\text{YM, lin}}_{\text{BRST}}$ acting on the BRST-extended Hilbert space $\mathfrak{H}^{\text{YM}}_{\text{BRST}}$, whose elements label our scattering amplitudes.

The free vacuum is certainly invariant under the action of $Q^{\mathrm{YM,\,lin}}_{\mathrm{BRST}}$, cf. again [232] or [233, Section 16.4]. We therefore have the on-shell Ward identity

$$0 \;=\; \langle 0|[Q^{\mathrm{YM,\,lin}}_{\mathrm{BRST}}, \mathscr{O}_1 \cdots \mathscr{O}_n]|0\rangle \;. \tag{8.5}$$

In order to use this Ward identity to link scattering amplitudes with k ghost–anti-ghost pairs to amplitudes with $k+1$ such pairs, we consider the special case

$$\mathscr{O}_1 \cdots \mathscr{O}_n \;=\; A^{\uparrow}\bar{c}(c\bar{c})^k A^{\perp}_1 \cdots A^{\perp}_{n-2k-2} \;, \tag{8.6}$$

where the gluon $A^{\uparrow\,a}_{\mu}$ is forward polarised while all other gluons have physical polarisation. In this special case, the on-shell Ward identity (8.5) directly implies

$$\begin{aligned} p_{A^{\uparrow}}\langle 0|(c\bar{c})^{k+1}A^{\perp}_1 \cdots A^{\perp}_{n-2k-2}|0\rangle + \langle 0|A^{\uparrow} b(c\bar{c})^k A^{\perp}_1 \cdots A^{\perp}_{n-2k-2}|0\rangle +& \\ + \sum_{j=0}^{k-1}\langle 0|A^{\uparrow}\bar{c}(c\bar{c})^j cb(c\bar{c})^{k-j-1}A^{\perp}_1 \cdots A^{\perp}_{n-2k-2}|0\rangle \;=\; 0 \;.& \end{aligned} \tag{8.7}$$

Observation 8.2. *Any amplitude with $k+1$ ghost–anti-ghost pairs and all gluons transversely polarised is given by a sum of amplitudes with k ghost pairs.*

The simplest non-trivial concrete example to illustrate Observation 8.2 is the case $n=4$, $k=0$ in Yang–Mills theory (the three-point scattering amplitudes vanish). We may then identify

$$\begin{aligned} &\langle 0|\hat{A}^{\uparrow\,a}(p_1)\hat{b}^b(p_2)\hat{A}^{\perp\,c}_1(p_3)\hat{A}^{\perp\,d}_2(p_2)|0\rangle \;= \\ &\qquad = \; p^0_2\mathscr{A}_{AAAA}\big(\varepsilon^{\uparrow}(p_1),p_1,a;\varepsilon^{\downarrow}(p_2),p_2,b;\varepsilon^{\perp}_1(p_3),p_3,c;\varepsilon^{\perp}_2(p_4),p_4,d\big) \end{aligned} \tag{8.8a}$$

and

$$\begin{aligned} &\langle 0|\hat{c}^a(p_1)\hat{\bar{c}}^b(p_2)\hat{A}^{\perp\,c}_1(p_3)\hat{A}^{\perp\,d}_2(p_4)|0\rangle \;= \\ &\qquad = \; p^0_1\mathscr{A}_{c\bar{c}AA}\big(p_1,a;p_2,b;\varepsilon^{\perp}_1(p_3),p_3,c;\varepsilon^{\perp}_2(p_4),p_4,d\big) \;, \end{aligned} \tag{8.8b}$$

where $\mathscr{A}_{AAAA}$ and $\mathscr{A}_{c\bar{c}AA}$ denote the four-gluon and two-ghost–two-gluon scattering amplitudes, respectively, with external particles labelled by polarisation vectors, momenta, and colour indices. The hat indicates the Fourier transform. A standard Feynman diagram computation then shows that

$$\begin{aligned}
p_2^0 \mathscr{A}_{AAAA} \;=\; & \frac{f^{ade} f_e{}^{bc}}{\sqrt{2}} \Big\{ (\varepsilon_2 \cdot \varepsilon_4)\big[(p_1 \cdot \varepsilon_3) + 2(p_2 \cdot \varepsilon_3)\big] - (\varepsilon_3 \cdot \varepsilon_4)\big[(p_1 \cdot \varepsilon_2) + 2(p_3 \cdot \varepsilon_2)\big] - \\
& \qquad - \frac{p_2^0 (p_2 \cdot \varepsilon_3)(p_1 \cdot \varepsilon_4)}{\sqrt{2}\big((p_1 \cdot p_2) + (p_1 \cdot p_3)\big)} - (\varepsilon_2 \cdot \varepsilon_3)(p_1 \cdot \varepsilon_4) \\
& \qquad - 2(\varepsilon_2 \cdot \varepsilon_3)(p_2 \cdot \varepsilon_4) - \sqrt{2} p_2^0 (\varepsilon_3 \cdot \varepsilon_4) \Big\} + \\
& + \frac{f^{abe} f_e{}^{cd}}{\sqrt{2}} \Big\{ - \frac{p_2^0}{\sqrt{2}(p_1 \cdot p_2)} \big[2(p_1 \cdot \varepsilon_4)(p_2 \cdot \varepsilon_3) - 2(p_1 \cdot \varepsilon_3)(p_2 \cdot \varepsilon_4)\big] - \\
& \qquad - \frac{p_2^0}{\sqrt{2}(p_1 \cdot p_2)} \big[(p_1 \cdot p_2) - 2(p_1 \cdot p_3)\big](\varepsilon_3 \cdot \varepsilon_4) - \\
& \qquad - (\varepsilon_2 \cdot \varepsilon_3)\big[(p_1 \cdot \varepsilon_4) + 2(p_2 \cdot \varepsilon_4)\big] + \\
& \qquad + (\varepsilon_2 \cdot \varepsilon_4)\big[(p_1 \cdot \varepsilon_3) + 2(p_2 \cdot \varepsilon_3)\big] - \\
& \qquad - (\varepsilon_3 \cdot \varepsilon_4)\big[(p_1 \cdot \varepsilon_2) + 2(p_3 \cdot \varepsilon_2)\big] \Big\} + \\
& + \frac{f^{ace} f_e{}^{bd}}{\sqrt{2}} \Big\{ \frac{p_2^0 (p_1 \cdot \varepsilon_3)(p_2 \cdot \varepsilon_4)}{\sqrt{2}(p_1 \cdot p_3)} + (\varepsilon_2 \cdot \varepsilon_3)\big[(p_1 \cdot \varepsilon_4) + 2(p_2 \cdot \varepsilon_4)\big] - \\
& \qquad - (\varepsilon_2 \cdot \varepsilon_4)\big[(p_1 \cdot \varepsilon_3) + 2(p_2 \cdot \varepsilon_3)\big] + \\
& \qquad + (\varepsilon_3 \cdot \varepsilon_4)\big((p_1 \cdot \varepsilon_2) + 2(p_3 \cdot \varepsilon_2)\big] \\
& \qquad + \sqrt{2} p_2^0 (\varepsilon_3 \cdot \varepsilon_4) \Big\}
\end{aligned} \tag{8.9a}$$

and

$$\begin{aligned}
p_1^0 \mathscr{A}_{c\bar{c}AA} \;=\; & f^{ace} f_e{}^{bd} \, \frac{p_2^0 (p_1 \cdot \varepsilon_3)(p_2 \cdot \varepsilon_4)}{2(p_1 \cdot p_3)} \\
& + f^{abe} f_e{}^{cd} \, \frac{p_2^0}{(p_1 \cdot p_2)} \Big\{ (p_1 \cdot \varepsilon_3)(p_2 \cdot \varepsilon_4) - (p_1 \cdot \varepsilon_4)(p_2 \cdot \varepsilon_3) + \\
& \qquad + \big[\tfrac{1}{2}(p_1 \cdot p_2) + (p_1 \cdot p_3)\big](\varepsilon_3 \cdot \varepsilon_4) \Big\} - \\
& - f^{ade} f_e{}^{bc} \, \frac{p_2^0 (p_1 \cdot \varepsilon_4)(p_2 \cdot \varepsilon_3)}{2\big[(p_1 \cdot p_2) + (p_1 \cdot p_3)\big]} \, .
\end{aligned} \tag{8.9b}$$

The sum of both terms vanishes,

$$p_2^0 \mathscr{A}_{AAAA} + p_1^0 \mathscr{A}_{c\bar{c}AA} \;=\; 0 \, , \tag{8.10}$$

upon using momentum conservation, transversality of the physically polarised gluons, the explicit form of the on-shell polarisation vectors (8.2), and the Jacobi identity. That is, the s-, t-, and u-channels are not related separately. This is not very surprising: the four-point gluon vertex can be distributed in different ways to the various channels and each distribution would imply a different relation between the channels of the two amplitudes. If we ensured colour–kinematics duality for the four-point vertex, however, then the relation between the two amplitudes would hold for each individual channel.

When we come to discussing the double copy theory, we will be able to ensure BRST invariance of the action only on-shell. However, from the construction of correlators from Feynman diagrams it is clear that the action of $Q^{\mathrm{YM,\,lin}}_{\mathrm{BRST}}$ on the on-shell BRST-extended Hilbert space will still be preserved, and we again have (8.5) with the corresponding link between scattering amplitudes with different number of ghost–anti-ghost pairs:

Observation 8.3. *Suppose that $Q^{\mathrm{YM}}_{\mathrm{BRST}}S^{\mathrm{YM}}_{\mathrm{BRST}} = 0$ and $(Q^{\mathrm{YM}}_{\mathrm{BRST}})^2 = 0$ only on-shell. Then, we still have an identification of scattering amplitudes with $k+1$ ghost–anti-ghost pairs and all gluons transversely polarised and a sum of amplitudes with k ghost–anti-ghost pairs.*

Off-shell Ward identities. BRST invariance of the action, being a global symmetry, induces an off-shell Ward identity for correlation functions,

$$\langle(\partial^\mu j_\mu(x))\mathscr{O}_1(x_1)\cdots\mathscr{O}_n(x_n)\rangle \;=\; \sum_{i=1}^{n} \mp\delta^{(d)}(x-x_i)\left\langle (Q_{\mathrm{BRST}}\mathscr{O}_i(x_i))\prod_{j\neq i}\mathscr{O}_j(x_j)\right\rangle, \qquad (8.11)$$

where j_μ is the BRST current and the sign is the Koszul sign arising from permuting operators of non-vanishing ghost number. Note that in general, $Q^{\mathrm{YM}}_{\mathrm{BRST}}$ is the renormalised BRST operator of the full quantum theory, cf. [234, Chapter 17.2]. As we will always discuss tree-level correlators, however, we can restrict ourselves to the classical BRST operator with action (8.3). We note that the left-hand side of (8.11) vanishes after integration over x and the Ward identity simplifies to

$$0 \;=\; \sum_{i=1}^{n} \pm\left\langle (Q^{\mathrm{YM}}_{\mathrm{BRST}}\mathscr{O}_i(x_i))\prod_{j\neq i}\mathscr{O}_j(x_j)\right\rangle. \qquad (8.12)$$

When applying Ward identities to correlation functions, we can use Observation 8.1 to restrict the correlation functions to purely connected correlators, i.e. the contribution arising from connected Feynman diagrams. Moreover, we can restrict the correlation functions to a particular order in the coupling constant g. This implies that for operators linear in the fields we can truncate the action of the BRST operator $Q^{\mathrm{YM}}_{\mathrm{BRST}}$ to the Abelian action.

As a short explicit example, let us consider (8.12) for the special case $n = 3$ with

$$\hat{\mathscr{O}}_1 = \hat{A}^{a\uparrow}_{\mu}(p_1) \ , \quad \hat{\mathscr{O}}_2 = \hat{\bar{c}}^{b}(p_2) \ , \quad \hat{\mathscr{O}}_3 = \hat{A}^{c\uparrow}_{\mu}(p_3) \ , \tag{8.13}$$

and we switched to momentum space for simplicity. We obtain the identity

$$\begin{aligned} \mathrm{P}^{\uparrow\,\mu'}_{\mu}(p_1)\mathrm{P}^{\uparrow\,\nu'}_{\nu}(p_3)\,\big(\langle\hat{A}^{a\uparrow}_{\mu'}(p_1)\hat{b}^{b}(p_2)\hat{A}^{c\uparrow}_{\nu'}(p_3)\rangle + \langle p_{1\,\mu'}\hat{c}^{a}(p_1)\hat{\bar{c}}^{b}(p_2)\hat{A}^{c\uparrow}_{\nu'}(p_3)\rangle - \\ - \langle\hat{A}^{a\uparrow}_{\mu'}(p_1)\hat{\bar{c}}^{b}(p_2)p_{3\,\nu'}\hat{c}^{c}(p_3)\rangle\big) \ = \ 0 \ , \end{aligned} \tag{8.14}$$

where $\mathrm{P}^{\uparrow\,\mu'}_{\mu}(p)$ is the projector onto (off-shell) forward polarised gluons. Explicitly,

$$\begin{aligned} \mathrm{P}^{\uparrow\,\nu}_{\mu}(p) &:= p_\mu\frac{(p\cdot\tilde{p})}{(p\cdot\tilde{p})^2-(p\cdot p)^2}\left[\tilde{p}^{\nu}-\frac{(p\cdot p)}{(p\cdot\tilde{p})}p^{\nu}\right] , \\ \mathrm{P}^{\downarrow\,\nu}_{\mu}(p) &:= \tilde{p}_\mu\frac{(p\cdot\tilde{p})}{(p\cdot\tilde{p})^2-(p\cdot p)^2}\left[p^{\nu}-\frac{(p\cdot p)}{(p\cdot\tilde{p})}\tilde{p}^{\nu}\right] , \end{aligned} \tag{8.15}$$

where $\tilde{p}_\mu$ is p_μ with spatial components reverted.

The relevant vertices are clearly the cubic gluon vertex to which $\hat{b}^b(p_2)$ is linked by a propagator, as well as the ghost–anti-ghost–gluon vertex. At tree-level, we thus obtain

$$\begin{aligned} &\mathrm{P}^{\uparrow\,\mu'}_{\mu}(p_1)\mathrm{P}^{\uparrow\,\nu'}_{\nu}(p_3)\langle\hat{A}^{a\uparrow}_{\mu'}(p_1)\hat{b}^{b}(p_2)\hat{A}^{c\uparrow}_{\nu'}(p_3)\rangle \ = \\ &\quad = \ f^{abc}\mathrm{P}^{\uparrow\,\mu'}_{\mu}(p_1)\mathrm{P}^{\uparrow\,\nu'}_{\nu}(p_3)\big[p_{2\mu'}p_{1\nu'}-p_{3\mu'}p_{2\nu'}+\eta_{\mu'\nu'}(p_3-p_1)\cdot(\mathrm{P}^{\downarrow}(p_2)\cdot p_2)\big] \ , \end{aligned} \tag{8.16a}$$

$$\mathrm{P}^{\uparrow\,\mu'}_{\mu}(p_1)\mathrm{P}^{\uparrow\,\nu'}_{\nu}(p_3)\langle p_{1\,\mu'}\hat{c}^{a}(p_1)\hat{\bar{c}}^{b}(p_2)\hat{A}^{c\uparrow}_{\nu'}(p_3)\rangle \ = \ f^{abc}\mathrm{P}^{\uparrow\,\mu'}_{\mu}(p_1)\mathrm{P}^{\uparrow\,\nu'}_{\nu}(p_3)p_{1\mu'}p_{2\nu'} \ , \tag{8.16b}$$

and

$$\mathrm{P}^{\uparrow\,\mu'}_{\mu}(p_1)\mathrm{P}^{\uparrow\,\nu'}_{\nu}(p_3)\langle\hat{A}^{a\uparrow}_{\mu'}(p_1)\hat{\bar{c}}^{b}(p_2)p_{3\,\nu'}\hat{c}^{c}(p_3)\rangle \ = \ f^{cba}\mathrm{P}^{\uparrow\,\mu'}_{\mu}(p_1)\mathrm{P}^{\uparrow\,\nu'}_{\nu}(p_3)p_{3\nu'}p_{2\mu'} \ . \tag{8.16c}$$

Summing these three terms according to the signs set in (8.14) we obtain

$$f^{abc}\mathrm{P}^{\uparrow\,\mu'}_{\mu}(p_1)\mathrm{P}^{\uparrow\,\nu'}_{\nu}(p_3)\eta_{\mu'\nu'}\big[(p_3-p_1)\cdot(\mathrm{P}^{\downarrow}(p_2)\cdot p_2)\big] \ , \tag{8.17}$$

which vanishes after inserting the explicit expressions (8.15).

We conclude with the following observation.

Observation 8.4. *We have Ward identities between tree-level correlation functions for the linearised BRST operator.*

8.2. Quantum equivalence, correlation functions, and field redefinitions

Let us now leave the special case of Yang–Mills theory for a moment and reconsider notions of equivalence between field theories in general. As discussed in Chapter 5, two field theories are classically equivalent if they are quasi-isomorphic and thus have a common minimal model. In the same Chapter, it was explained how the minimal model of a field theory is constructed using the homological perturbation lemma.

Perturbative quantum equivalence. For the full quantum equivalence at the perturbative level, we have the following evident statement.

Observation 8.5. *Two field theories are perturbatively quantum equivalent if all correlators built from polynomials of fields and their derivatives agree to any finite order in coupling constant and loop level. Since correlators can be glued together from tree-level correlators (up to regularisation issues), it suffices if the tree level correlators agree.*

We stress that we are only interested in the integrands of scattering amplitudes, which allows us to ignore all issues related to regularisation.

To provide a link between the double-copied action and the action of $\mathcal{N} = 0$ supergravity, we will need to perform a sequence of field redefinitions. The field content of the theories will be the same from the outset, and we choose to work with the same path integral measure in both cases. We are therefore interested in field redefinitions that leave the path integral measure invariant.

There are large classes of such field redefinitions. The most evident such class of field redefinitions is

$$\phi \;\mapsto\; \tilde{\phi} \;:=\; \phi + f(\phi'_1, \ldots, \phi'_n)\;, \tag{8.18}$$

where f is a polynomial function of a set of fields $\{\phi'_1, \ldots, \phi'_n\}$ and their derivatives with $\phi \notin \{\phi'_1, \ldots, \phi'_n\}$. Under such a field redefinition, the path integral measure remains unchanged; this becomes evident when imagining the finite-dimensional analogue of volume forms and a coordinate shifted by a function of different coordinates.

More subtle is the fact that field redefinitions of the form

$$\phi \;\mapsto\; \tilde{\phi} \;:=\; \phi + \mathcal{O}(\phi^2)\;, \tag{8.19}$$

where $\mathcal{O}(\phi^2)$ denotes local polynomial functions in arbitrary fields and their derivatives which are at least of quadratic order in ϕ can also be considered as leaving the path integral measure invariant.

Invariance of the S-matrix under (8.19) without derivatives is captured by the Chisholm–Kamefuchi–O'Raifeartaigh–Salam equivalence theorem [235, 236]. A proof using the BV formalism of perturbative quantum equivalence for local field redefinitions of the form (8.19) allowing for derivatives was given in [237]. This is sufficient for our purposes as we are only concerned with the integrands of scattering amplitudes. Note, however, the well-known need to choose the counter-terms consistently, as emphasised in [237]. With this in mind, the simplest approach is to use dimensional regularisation, since (8.19) produces a Jacobian which is then regulated to unity, see [238, 239] as well as [240, Sections 18.2.3–4].

We sum up the above discussion as follows.

Observation 8.6. *A shift of a field by products of fields and their derivatives which do not involve the field itself does not change the path integral measure. Local field redefinitions that are trivial at linear order are quantum mechanically safe as they produce a Jacobian that can be regulated to unity in dimensional regularisation.*

Nakanishi–Lautrup field shifts and changes of gauge. Besides field redefinitions, we also adjust our choice of gauge to link equivalent field theories. In particular, we can shift the usual choice (4.16) for R_ξ-gauge to

$$\Psi \;\mapsto\; \Psi + \Xi \quad \text{with} \quad \Xi \;:=\; \int \mathrm{d}^d x\, \bar{c}^a Y_a\ . \tag{8.20}$$

Here, Y^a is of ghost number zero, and we limit ourselves to terms Y^a that are independent of the Nakanishi–Lautrup field. The shift (8.20) leads to a shift of the gauge-fixed Lagrangian (4.18) given by

$$\mathscr{L}^{\text{YM}}_{\text{BRST}} \;\mapsto\; \mathscr{L}^{\text{YM}}_{\text{BRST}} + \frac{\delta\Xi}{\delta A^a_\mu}(\nabla_\mu c)^a + \frac{g}{2} f_{bc}{}^a \frac{\delta\Xi}{\delta c^a} c^b c^c - b^a \frac{\delta\Xi}{\delta \bar{c}^a}\ . \tag{8.21}$$

Evidently, this new Lagrangian is quantum-equivalent to the one with $Y^a = 0$, as we merely chose to work in a different gauge.

Subsequently, we may perform the shift

$$b^a \;\mapsto\; b^a + Z^a \tag{8.22}$$

in the Nakanishi–Lautrup field with Z^a polynomials in the fields and their derivatives. The combination of this shift and (8.20) results in

$$\begin{aligned}\mathscr{L}^{\mathrm{YM}}_{\mathrm{BRST}} \;\mapsto\; \mathscr{L}^{\mathrm{YM}}_{\mathrm{BRST}} &+ \frac{\delta\Xi}{\delta A^a_\mu}(\nabla_\mu c)^a + \frac{g}{2}f_{bc}{}^a\frac{\delta\Xi}{\delta c^a}c^b c^c + \\ &+ \frac{\xi}{2}Z_a Z^a + Z_a(\xi b^a + \partial^\mu A^a_\mu) - (b^a + Z^a)\frac{\delta\Xi}{\delta\bar{c}^a}\,.\end{aligned} \tag{8.23}$$

We shall assume that Z^a is independent of the Nakanishi–Lautrup field as this will yield a quantum-equivalent Lagrangian by Observation 8.6. We shall also assume that Z^a depends at least quadratically on the other fields and their derivatives to preserve the linearised BRST action on the BRST-extended Hilbert space introduced in Section 8.1..

Interaction terms linear in the Nakanishi–Lautrup fields. Let us now consider the following special case: suppose that we are in R_ξ-gauge and that our Lagrangian contains a term $Z_a\partial^\mu A^a_\mu$ with Z^a independent of the Nakanishi–Lautrup field and at least quadratic in the fields and their derivatives. On the physical Hilbert space with transversely polarised gluons, such expressions vanish. Off-shell, we can still remove such terms by the shifts (8.22). Given the need to shift by Z^a, we can then iteratively construct a Y^a which cancels any new terms linear in b^a, as is clear from (8.23). Explicitly, we solve the equation

$$0 \;=\; \xi Z_a - \frac{\delta\Xi}{\delta\bar{c}^a} \;=\; \xi Z_a - Y_a + \bar{c}^b\frac{\partial Y_b}{\partial\bar{c}^a} + \cdots\;, \tag{8.24}$$

where the ellipsis denotes terms containing partial derivatives with respect to derivatives of the anti-ghost field $\bar{c}^b$. Clearly, for consistency, Y^a needs to be at least quadratic in the fields and their derivatives because Z^a is. We are left with the terms

$$-\frac{\xi}{2}Z_a Z^a + \frac{\delta\Xi}{\delta A^a_\mu}(\nabla_\mu c)^a + \frac{g}{2}f_{bc}{}^a\frac{\delta\Xi}{\delta c^a}c^b c^c\;, \tag{8.25}$$

which are either at least quartic in the fields or at least cubic in the fields, containing ghost fields. The ability to remove any terms of the form $Z_a(\partial^\mu A_\mu)^a$ through local shifts of the Nakanishi–Lautrup field, absorbing them into b^a, and a compensating gauge choice is the 'off-shell' Lagrangian analogue of being able to impose that the on-shell external gluons in an amplitude are transverse. We summarise as follows.

Observation 8.7. *Interaction terms in the Lagrangian of degree $n \geqslant 3$ of the form $Z_a(\partial^\mu A_\mu)^a$ with Z^a independent of the Nakanishi–Lautrup field can be removed in R_ξ-gauge* *by shifting the Nakanishi–Lautrup field according to* (8.22)*. This creates the additional terms* (8.25) *which do no modify the scattering amplitudes by Observation 8.6 and, in addition, contribute only to interaction vertices of degree n with more ghost–anti-ghost pairs or to interaction vertices of degree greater than n.*

We also note that a shift of the gauge-fixing fermion by itself (8.20) allows us to absorb physical terms proportional to the Nakanishi–Lautrup field without further affecting the physical sector.

Observation 8.8. *Terms in the action that are proportional to the Nakanishi–Lautrup field can be absorbed by choosing a suitable term Y^a. This leaves the physical sector invariant but it may modify the ghost sector. Because Nakanishi–Lautrup fields appear via trivial pairs in the BV action, this extends to general gauge theories, e.g. with several Nakanishi–Lautrup fields and ghosts–for–ghosts.*

Actions related by field redefinitions. Let us return to a general setting. Suppose that we are given two classical field theories which are specified by local actions S and $\tilde{S}$, as power series in the fields and their derivatives, whose corresponding L_∞-algebras have the same minimal model, the same field content and the same kinetic parts.

Consider the cubic interaction terms $\mathscr{L}_3$ and $\tilde{\mathscr{L}}_3$ in S and $\tilde{S}$. Since the three-point amplitudes agree, these interaction terms can differ at most in terms that vanish on external fields. Therefore, these terms have to be proportional to either the on-shell equation for an external field or to a field with unphysical polarisation which is not contained in the external fields. Both types of terms can be cancelled by a local field redefinition which shifts the discrepancy into the quartic and higher interaction terms. Such field redefinitions constitute a quasi-isomorphism of L_∞-algebras and leaves the tree-level scattering amplitudes unmodified. We are left with two theories with the same tree-level scattering amplitudes and which agree to cubic order in the interaction terms.

The discrepancy between the total quartic terms of both field theories after the above field redefinition is again invisible at the level of external fields, because the tree-level scattering amplitudes still agree. We then compensate again by further field redefinitions, shifting the discrepancy into quintic and higher interaction terms. In this way, we can remove the differences between the Lagrangians order by order in the interaction vertices, field-redefining the difference away to higher order interaction vertices. Since we are merely interested in perturbation theory, agreements to arbitrary finite orders are completely sufficient.

Altogether, we can conclude that for the purpose of perturbative quantum field theory, we can regard the actions S and $\tilde{S}$ to be related by local field redefinitions. In certain cases it is even possible to give closed all order expression for (part of) the field redefinitions, providing a formal non-perturbative equivalence.

Observation 8.9. *If two field theories have the same tree-level scattering amplitudes, then the minimal models of the corresponding* L_∞*-algebras coincide, cf. [52, 2]. If also the associated actions are local and given by power series of the fields and their derivatives, and have the same field content and kinetic parts, then they are related by local (invertible) field redefinitions.*

The explicit example of Yang–Mills theory may be instructive. Consider the action (7.10) of Yang–Mills theory in R_ξ-gauge with the field redefinitions (7.8) implemented as in Section 7.2. and consider an action $\tilde{S}$ with the same fields, the same kinematic parts and identical tree-level scattering amplitudes. The discrepancies in the interaction vertices at each order are proportional to (at least) one of the terms

$$\tilde{A}^{\uparrow a}_{\mu}~, \quad \sqrt{\Box}\,\tilde{b}^a + \tilde{\xi}\partial^\mu \tilde{A}^a_\mu~, \quad \Box\tilde{A}^a_\mu~, \quad \Box\tilde{c}^a~, \quad \Box\tilde{\bar{c}}^a~, \quad \text{and} \quad \Box\tilde{b}^a~. \tag{8.26}$$

Given the BRST invariance, we can always exclude terms proportional to $\tilde{A}^{\uparrow a}_{\mu}$, as these can be absorbed by residual gauge transformations. Terms proportional to $\sqrt{\Box}\tilde{b}^a + \tilde{\xi}\partial^\mu \tilde{A}^a_\mu$ can be absorbed by a field redefinition of the Nakanishi–Lautrup field due to Observation 8.7. All remaining differences are sums of terms proportional to $\Box\tilde{A}^a_\mu$, $\Box\tilde{c}^a$, $\Box\tilde{\bar{c}}^a$, or $\Box\tilde{b}^a$, and they can be absorbed by iterative field redefinitions, starting with the three-point amplitudes. There is an evident field redefinition of the relevant field, quadratic in the fields and their derivatives, such that the kinetic term of redefined Yang–Mills theory produces the difference in kinetic terms. Since such a field redefinition is a quasi-isomorphism of the corresponding L_∞-algebras, it preserves the minimal model and thus the tree-level amplitudes. Moreover, such a field redefinition is clearly local.

8.3. Strictification of Yang–Mills theory

Generalities. An important structure theorem for homotopy algebras is the strictification theorem, cf. Section 2.4.. In particular, it implies that any L_∞-algebra is quasi-isomorphic to a strict L_∞-algebra, i.e. an L_∞-algebra with $\mu_i = 0$ for $i \geqslant 3$, better known as a differential graded Lie algebra.

From a field theory perspective, this implies that any classical field theory is equivalent to a classical field theory with interaction terms which are all cubic in the fields. Generically, a strictifying quasi-isomorphism may produce non-local terms, but there is always a systematic choice of strictification that is entirely local. This is quite evident for the interactions of scalar fields, since we can 'blow up' n-ary vertices to cubic graphs with edges corresponding to propagating auxiliary fields, cf. e.g. the discussions in [52, 2].

As a simple example of a strictification, consider the first-order formalism of Yang–Mills theory on four-dimensional Euclidean space $\mathbb{R}^4$ [226], in which an additional self-dual two form $B_+ \in \Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g}$ in the adjoint representation of the gauge Lie algebra is added to the field content,

$$S^{\mathrm{YM}_1} := \int \mathrm{d}^4 x \left\{ \tfrac{1}{2}\varepsilon^{\mu\nu\kappa\lambda} F_{a\mu\nu} B^a_{+\kappa\lambda} + \tfrac{1}{4}\varepsilon^{\mu\nu\kappa\lambda} B_{+a\mu\nu} B^a_{+\kappa\lambda} \right\} . \tag{8.27}$$

The L_∞-algebra corresponding to the full BV complex of this theory is indeed strict; see [31, 52] for a quasi-isomorphism between this L_∞-algebra and that of the ordinary, second-order formulation of Yang–Mills theory.

Note, however, that the full strictification of gauge theories including ghosts is a bit more involved: the equations of motion of the introduced auxiliary fields would be at least quadratic in other fields, and if these transform in the adjoint representation or as connections, the gauge transformations of auxiliary fields are at least cubic in fields and ghosts, leading to quartic or higher terms in the BV action. The strictification theorem still guarantees the existence of an equivalent formulation as a field theory with cubic interaction vertices, but we may have to extend our field space not merely by adding fields, but by switching e.g. to its loop space. This is due to the fact that cubic gauge transformations in an L_∞-algebra are encoded in a μ_3, which in turn corresponds to a particular three-cocycle. The latter can be transgressed to a two-cocycle over loop space, which merely corresponds to a Lie algebra extension and thus, is turned into a higher product μ_2. For fully gauge-fixed actions, however, this problem never arises.

We also note that the factorisation in the double copy is most easily performed in a specific strictification[1], which is *not* the first order formulation (8.27). Its precise form is discussed in the following.

[1] It is actually a family of strictifications.

Colour–kinematics-dual form and cubic diagrams. Recall from 1.3. that the tree-level Yang–Mills amplitudes can be rearranged in colour–kinematics-dual form, which is by now a well-established fact [85, 84, 241, 144, 242–246].

Observation 8.10. *The tree amplitudes of Yang–Mills theory can be written in colour–kinematics-dual form.*

Explicitly, one can construct a Lagrangian whose Feynman diagrams generate colour–kinematics-dual tree-level amplitudes of physical (transverse) gluons in Yang–Mills theory, making colour–kinematics duality manifest at the Lagrangian level. This is achieved by adding non-local interaction terms $\mathcal{O}(A^n)$, for all $n > 5$, to the action that vanish identically due to the colour Jacobi identity. The necessary terms were first constructed in [83] up to six points. The algorithm of Tolotti–Weinzierl [160] is a prescription of how to find the necessary terms to arbitrary order.

Since the new terms are identically zero they obviously leave the theory and amplitudes invariant, but nonetheless change the individual kinematic numerators to realise colour–kinematics duality. Moreover, the new terms can be rendered cubic and local through the introduction of auxiliary fields [5], as demonstrated explicitly at five points in [83]. Roughly speaking, one starts from Yang–Mills theory and strictifies the already present quartic interaction vertex by inserting an auxiliary field, redistributing the contributions to ensure colour–kinematics duality for four-point amplitudes. The colour–kinematics duality of the five-point amplitudes then requires a new interaction term $\mathcal{O}(A^5)$ which vanishes due to the Jacobi identity. This vertex is then strictified by inserting further auxiliary fields, etc. The overall action is thus trivially equivalent to Yang–Mills theory. We note that the form of the strictification is encoded in the action produced by the Tolotti–Weinzierl algorithm. We shall be completely explicit below, but let us first summarise the situation.

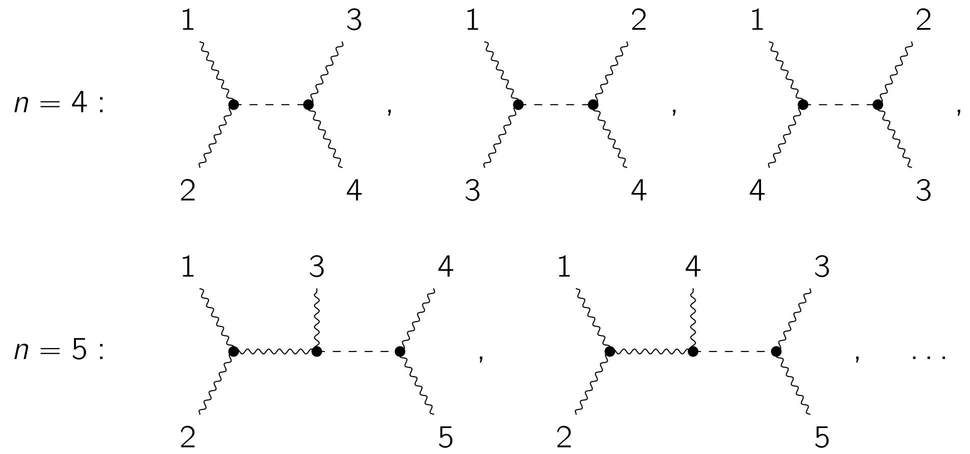
Observation 8.11. *Given tree-level physical gluon amplitudes in colour–kinematics-dual form, there is a corresponding purely cubic Lagrangian whose Feynman diagrams (summed over identical topologies) produce kinematic numerators satisfying the kinematic Jacobi identities* (1.14b)*.*

To illustrate the strictification, let us consider the four- and five-point contributions, which were already computed in [83]:

$$
\begin{aligned}
\mathscr{L}^{(4)} &\sim \operatorname{tr}\big\{[A_\mu, A_\nu][A^\mu, A^\nu]\big\} = -\eta^{\mu\nu}\eta^{\kappa\rho}\eta^{\lambda\sigma}\ \partial^{12}_\mu \partial^{34}_\nu\ \frac{\operatorname{tr}\big\{[A_\kappa, A_\lambda][A_\rho, A_\sigma]\big\}}{\Box_{12}}\ ,\\
\mathscr{L}^{(5)} &\sim \operatorname{tr}\left\{[A^\nu, A^\rho]\frac{1}{\Box}\left(\left[[\partial_\mu A_\nu, A_\rho], \frac{\Box}{\Box}A^\mu\right] + \right.\right. \\
&\qquad \left.\left. + \left[[A_\rho, A^\mu], \frac{\Box}{\Box}\partial_\mu A_\nu\right] + \left[[A^\mu, \partial_\mu A_\nu], \frac{\Box}{\Box}A_\rho\right]\right)\right\}.
\end{aligned}
\tag{8.28}
$$

We immediately note that $\mathscr{L}^{(5)}$ vanishes by the colour Jacobi identity. Its presence, however, is required for the kinematic Jacobi identity to hold after factorisation.

As explained in 1.3., these terms reflect a 'blow up' of n-point interaction vertices into trees with trivalent vertices and all symmetries taken into account:



(8.29)

Here, an internal wavy line comes with a propagator in Feynman gauge $\frac{1}{\Box}$, while a dashed line corresponds to the identity operator $\frac{\Box}{\Box}$.

The general Lagrangian at n-th order is then of the form

$$
\mathscr{L}^{(n)} = f_{M_1\cdots M_k} E_1^{M_1} D_1(E_2^{M_2} D_2(E_3^{M_3} D_3 \cdots))\ , \tag{8.30}
$$

where D_i stands for either $\frac{1}{\Box}$ or $\frac{\Box}{\Box}$ and the M_is are Lorentz multi-indices. Note that all the E_is are polynomials of degree one or two in the fields. In the tree picture, the wave operators in the denominator correspond precisely to the edges in the trees.

Strictification. To strictify the non-local action, we now iteratively insert auxiliary fields $G^M_{n,\Gamma,i}$ and $\bar{G}^{n,\Gamma,i}_M$ for each operator D_i. If we are dealing with an operator of the form $\frac{\Box}{\Box}$, we first use partial integration

$$\frac{E_1^{M_1}\,\Box_1\,E_2^{M_2}}{\Box_1} = -\frac{(\partial_\mu E_1^{M_1})(\partial^\mu E_2^{M_2})}{\Box_1}\ , \tag{8.31}$$

where E_i^M is an arbitrary expression in the fields, derivatives, and auxiliary fields. We then use the fact that the Lagrangians

$$E_1^M \frac{1}{\Box} E_M^2 \tag{8.32a}$$

and

$$-\,G^M_{n,\Gamma,i}\,\Box\,\bar{G}_M^{n,\Gamma,i} + G^M_{n,\Gamma,i}E_M^2 + E_1^M\bar{G}_M^{n,\Gamma,i} \tag{8.32b}$$

are physically equivalent after integrating out the auxiliary fields $G^M_{n,\Gamma,i}$ and $\bar{G}_M^{n,\Gamma,i}$. We iterate this process until all the inverse wave operators have been replaced in this manner.

We note that in each iteration, E_1^M and E_M^2 are both polynomials of degree at least two in the fields. Introducing the auxiliary fields reduces the polynomial degree at least by one, and in the end, the action has indeed only cubic interaction terms and thus is a strictification of the original action. We also note that two auxiliary fields can be combined into one if they have identical equations of motion.

Homotopy algebraic perspective. The strictification $\mathfrak{L}^{\rm YM,\,st}_{\rm BRST}$ of the L_∞-algebra $\mathfrak{L}^{\rm YM}_{\rm BRST}$ or, equivalently, of the colour–kinematics-dual action is nothing but a quasi-isomorphism (see Section 2.3.)

$$\phi\ :\ \mathfrak{L}^{\rm YM}_{\rm BRST} \ \to\ \mathfrak{L}^{\rm YM,\,st}_{\rm BRST}\ , \tag{8.33}$$

and the map ϕ is given by

$$A^{\rm st} + \sum_{n,\Gamma,i} G_{n,\Gamma,i} = \phi_1(A) + \tfrac{1}{2}\phi_2(A,A) + \cdots = \sum_{k\geqslant 1}\tfrac{1}{k!}\phi_k(A,\ldots,A)\ , \tag{8.34}$$

where $A^{\rm st}$ is the gauge potential in $\mathfrak{L}^{\rm YM,\,st}$,

$$A^{\rm st} = \phi_1(A)\ , \tag{8.35}$$

and the higher maps are such that $G_{n,\Gamma,i}$ are given by their equations of motion, fully reduced to expressions in the original gauge potential A.

Let us work out the details for the example of the fourth- and fifth-order terms (8.28). The explicit form of the corresponding strictified Lagrangian is already found in [83],

$$\mathscr{L}^{\rm YM,\,st} := \tfrac{1}{2}{\rm tr}\big\{A_\mu\,\Box\,A^\mu\big\} + \mathscr{L}_4^{\rm YM,\,st} + \mathscr{L}_5^{\rm YM,\,st}\ , \tag{8.36a}$$

with

$$
\begin{aligned}
\mathscr{L}_4^{\mathrm{YM,\,st}} &:= \mathrm{tr}\Big\{ -\tfrac{1}{2} G^{\mu\nu\kappa}_{4,\Gamma,1} \Box\, G^{4,\Gamma,1}_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}}\partial^\kappa G^{4,\Gamma,1}_{\kappa\mu\nu})[A^\mu, A^\nu] \Big\} ~,\\
\mathscr{L}_5^{\mathrm{YM,\,st}} &:= \mathrm{tr}\Big\{ G^{\mu\nu}_{5,\Gamma,1} \Box\, \bar{G}^{5,\Gamma,1}_{\mu\nu} + G^{\mu\nu\kappa}_{5,\Gamma,2} \Box\, \bar{G}^{5,\Gamma,2}_{\mu\nu\kappa} + G^{\mu\nu\kappa\lambda}_{5,\Gamma,3} \Box\, \bar{G}^{5,\Gamma,3}_{\mu\nu\kappa\lambda} + \\
&\qquad + g G^{\mu\nu}_{5,\Gamma,1}[A_\mu, A_\nu] + g\partial_\mu G^{\mu\nu\kappa}_{5,\Gamma,2}[A_\nu, A_\kappa] - \tfrac{g}{2}\partial_\mu G^{\mu\nu\kappa\lambda}_{5,\Gamma,3}[\partial_{[\nu}A_{\kappa]}, A_\lambda] + \\
&\qquad + g\bar{G}^{\mu\nu}_{5,\Gamma,1}\big(\tfrac{1}{2}[\partial^\kappa \bar{G}^{5,\Gamma,2}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa \bar{G}^{5,\Gamma,3}_{\kappa\lambda\nu[\mu}, A^{\lambda]}]\big) \Big\} ~.
\end{aligned}
\tag{8.36b}
$$

Consequently, the resulting quasi-isomorphism reads as

$$
\begin{aligned}
&\phi_1(A) + \tfrac{1}{2}\phi_2(A,A) + \tfrac{1}{3!}\phi_3(A,A,A) ~=~\\
&\qquad = \begin{pmatrix} A^s_\mu \\ G^{4,\Gamma,1}_{\mu\nu\kappa} \\ G^{5,\Gamma,1}_{\mu\nu} \\ \bar{G}^{5,\Gamma,1}_{\mu\nu} \\ G^{\mu\nu\kappa}_{5,\Gamma,2} \\ \bar{G}^{5,\Gamma,2}_{\mu\nu\kappa} \\ G^{\mu\nu\kappa\lambda}_{5,\Gamma,3} \\ \bar{G}^{5,\Gamma,3}_{\mu\nu\kappa\lambda} \end{pmatrix} = \begin{pmatrix} A_\mu \\ \frac{g}{2\Box}\partial_\mu[A_\nu, A_\kappa] \\ -\frac{g^2}{2\Box}\big([[A_\lambda, A_\mu], \partial^\lambda A_\nu] - [[\partial_{[\lambda}A_{\nu]}, A_\mu], A^\lambda]\big) \\ -\frac{g}{\Box}[A_\mu, A_\nu] \\ -\frac{g^2}{2\Box}\partial^\mu\big[\partial^\nu A_\lambda, \frac{1}{\Box}[A^\kappa, A^\lambda]\big] \\ \frac{g}{\Box}\partial_\mu[A_\nu, A_\kappa] \\ -\frac{g^2}{\Box}\partial^\mu\big[A^\nu, \frac{1}{\Box}[A^\lambda, A^\kappa]\big] \\ -\frac{g}{2\Box}\partial_\mu[\partial_{[\nu}A_{\kappa]}, A_\lambda] \end{pmatrix} ~.
\end{aligned}
\tag{8.37}
$$

Note that the decomposition into the images of the maps ϕ_i corresponds to the decomposition of the image into monomials of power i in the fields.

Tree-level double copy. As reviewed in Section 1.3., the double copy of the kinematic numerators in the scattering amplitudes of the strictified Yang–Mills theory produces the tree-level scattering amplitudes of $\mathcal{N} = 0$ supergravity [81–83].

Observation 8.12. *Double copying the Yang–Mills tree-level scattering amplitudes of physical gluons in colour–kinematics-dual form yields the physical tree-level scattering amplitudes of $\mathcal{N} = 0$ supergravity.*

Compatibility with quantisation. It is clear that quantisation does not commute with quasi-isomorphisms: classically equivalent field theories can have very different quantum field theories. A simple example making this evident is the L_∞-algebra of Yang–Mills theory $\mathfrak{L}^{\mathrm{YM}}_{\mathrm{BRST}}$ and one of its quasi-isomorphic minimal models $\mathfrak{L}^{\mathrm{YM}\circ}_{\mathrm{BRST}}$. The vector space of $\mathfrak{L}^{\mathrm{YM}\circ}_{\mathrm{BRST}}$ is simply the free fields labelling external states in Yang–Mills scattering amplitudes, together with some irrelevant cohomological remnants in the ghosts, Nakanishi–Lautrup fields, and anti-ghosts. The tree-level scattering amplitudes of $\mathfrak{L}^{\text{YM}}_{\text{BRST}}$ are given by the higher products of $\mathfrak{L}^{\text{YM}\circ}_{\text{BRST}}$. They are also the tree-level scattering amplitudes of $\mathfrak{L}^{\text{YM}\circ}_{\text{BRST}}$ since there are no propagating degrees of freedom left. Clearly, however, there are loop-level scattering amplitudes in Yang–Mills theory which $\mathfrak{L}^{\text{YM}}_{\text{BRST}}$ can describe but which are absent in $\mathfrak{L}^{\text{YM}\circ}_{\text{BRST}}$. Thus, the quantum theories described by the quasi-isomorphic L_∞-algebras $\mathfrak{L}^{\text{YM}}_{\text{BRST}}$ and $\mathfrak{L}^{\text{YM}\circ}_{\text{BRST}}$ differ.

Certainly, there are quasi-isomorphisms which are compatible with quantisation. In particular, any quasi-isomorphism that corresponds to integrating out fields which appears at most quadratically in the action are of this type: we can simply complete the square in the path integral and perform the Gaußian integral. This amounts to replacing each auxiliary field by the equation of motion.

This is precisely the case in the above strictification of Yang–Mills theory, and the original formulation is quantum equivalent to its strictification. This is also clear at the level of Feynman diagrams: as the kinematic terms are all of the form $-G^M_{n,\Gamma,i}\,\Box\,\bar{G}^{n,\Gamma,i}_M$, each auxiliary field propagates into precisely one other auxiliary field. Moreover, each auxiliary field G appears in precisely one type of vertex and then only as one leg. That is, once a propagator ends in one of the auxiliary fields, the continuation of the diagram at this end is unique until all the remaining open legs are non-auxiliaries. There are no loops consisting of purely auxiliary fields. All loops containing at least one gluon propagator are simply contracted to gluon loops. It is thus clear that the degrees of freedom running around loops in the strictified theory are the same as those running around in ordinary Yang–Mills theory.

8.4. Colour–kinematics duality for unphysical states

The action and factorisation we have presented so far are the complete data to double copy tree-level gauge theory amplitudes to gravity amplitudes. For the full double copy at the loop level, however, we need to work a bit harder, as explained in our previous paper [5].

So far, colour–kinematics duality is only ensured for all on-shell gluon states with physical polarisation. Our goal will be to double copy arbitrary tree-level correlators, which can have unphysical polarisations of gluons as well as ghost states on external legs. We therefore need to ensure that colour–kinematics duality holds more generally. In order to establish the off-shell double copy it is sufficient to guarantee colour–kinematics duality for on-shell states in the BRST-extended Hilbert space from Section 8.1..

Unphysical states. Colour–kinematics duality is not affected by forward-polarised gluons, as these can be absorbed by residual gauge transformations. Furthermore, colour–kinematics duality for backward-polarised gluons can be achieved by adding new terms to the action, which are physically irrelevant since they are introduced only through the gauge-fixing fermion. Colour–kinematics duality for ghosts is then achieved by transferring colour–kinematics duality for longitudinal gluons to the ghost sector by Observation 8.2 via the BRST Ward identities. We now explain the procedure in detail.

We perform the corrections order by order in the degree n of the vertices and for each degree order by order in the number k of ghost–anti-ghost pairs. The first vertex to consider is $n = 4$, and we start at $k = 0$. Colour–kinematics duality for four on-shell gluons in the BRST-extended Hilbert space can only be violated by terms proportional to $\xi b^a + \partial^\mu A^a_\mu$ and we can introduce a vertex compensating these violations in the Lagrangian. We do this directly in a BRST-invariant fashion, and a short calculation shows that the appropriate addition to the Lagrangian is

$$\mathscr{L}^{\text{YM, comp}}_{n=4,\,k=0} \;=\; -\xi\left\{b^b A^{c\mu}\frac{1}{\Box}\big[(\partial^\nu A^d_\mu)A^e_\nu\big] - \bar{c}^b Q_{\text{BRST}}\left(A^{c\mu}\frac{1}{\Box}\big[(\partial^\nu A^d_\mu)A^e_\nu\big]\right)\right\} f_{ed}{}^a f_{acb}\;. \tag{8.38}$$

Here, the first term compensates the colour–kinematics duality violating term for four gluons and the second term renders the compensation BRST-invariant, thus ensuring

$$Q_{\text{BRST}}\mathscr{L}^{\text{YM, corr}}_{n=4,\,k=0} \;=\; 0\;. \tag{8.39}$$

To show that these terms are indeed unphysical and that they do not modify the tree-level correlation functions, we use Observation 8.7 and Observation 8.8: these terms are produced by a shift (8.22) of the form

$$Z^a \;:=\; -A^{c\mu}\frac{1}{\Box}\big[(\partial^\nu A^d_\mu)A^e_\nu)\big]f_{ed}{}^b f_{bc}{}^a \quad\text{and}\quad Y^a \;:=\; \frac{1}{\xi}Z^a\;. \tag{8.40}$$

We note that the terms in $\mathscr{L}^{\text{YM, comp}}_{n=4,\,k=0}$ come with a canonical strictification given by the colour structure. This strictification then yields colour–kinematics-dual four-point gluon amplitudes.

The next case to consider is $n = 4$, $k = 1$. We now use Observation 8.4 to relate the four-gluon correlation function to this correlation function, and, correspondingly, the four-gluon tree-level correlator to the two gluon, one ghost-anti-ghost pair correlator. We obtain colour–kinematics duality for amplitudes consisting of a ghost–anti-ghost pair as well as two physically polarised gluons. Generalising the latter to two arbitrary gluons in the BRST-extended Hilbert space, we expect colour–kinematics duality violating terms proportional to $\xi b^a + \partial^\mu A^a_\mu$. It turns out that these terms happen to vanish and there is nothing left to do. Note that if these terms had not vanished, we would have compensated for them again by inserting physically irrelevant terms to the action in a BRST-invariant fashion.

Observation 8.3 now immediately implies that the amplitudes for $n = 4$, $k = 2$ are colour–kinematics-dual, because those for $n = 4$, $k = 1$ are.

So far, we constructed a strict Lagrangian for Yang–Mills theory with the same tree-level scattering amplitudes for the BRST-extended Hilbert space as ordinary Yang–Mills theory, but with a manifestly colour–kinematics-dual factorisation of the four-point amplitudes.

We now turn to $n = 5$, $k = 0$ and iterate our procedure in the evident fashion:

Step 1) Identify the colour–kinematics duality violating terms. They are necessarily proportional to $\xi b^a + \partial^\mu A^a_\mu$.

Step 2) Compensate by inserting a corresponding non-local vertex. Complete the compensating term to a BRST-invariant one, which may be deduced directly via the gauge-fixing fermion.

Step 3) The colour structure of the vertices induces a canonical strictification, implement this strictification.

Step 4) Use Observation 8.3 to transfer colour–kinematics duality to tree level correlators with one more ghost–anti-ghost pair, but all other gluons physically polarised.

Step 5) Continue with Step 1), if there is room for backward-polarised gluons. Otherwise turn to the next higher n-point scattering amplitudes.

The outcome of this construction is a strictified BRST action for Yang–Mills theory which is perturbatively quantum equivalent to ordinary Yang–Mills theory and whose scattering amplitudes come canonically factorised in colour–kinematics-dual form.

We note that this action comes with a BRST operator which is cubic in the fields of the BRST-extended Hilbert space, but of higher order in its action on the auxiliary fields introduced in strictification.

9 Double copy from factorisation of homotopy algebras

In this final Chapter we use the notion of twisted tensor products of differential graded algebras to factorise the (strictified) L_∞ algebra associated to the full, interacting Yang–Mills theory

$$\tilde{\mathfrak{L}}^{\mathsf{YM,\,st}}_{\mathsf{BRST}} \;=\; \mathfrak{g}\otimes(\mathfrak{Kin}^{\mathsf{st}}\otimes_\tau \mathfrak{Scal}). \tag{9.1}$$

Relying on the results exposed in the previous Chapters, we finally show that the double copied theory

$$\tilde{\mathfrak{L}}^{\mathsf{DC}}_{\mathsf{BRST}} \;=\; \mathfrak{Kin}^{\mathsf{st}}\otimes_\tau(\mathfrak{Kin}^{\mathsf{st}}\otimes_\tau \mathfrak{Scal}) \tag{9.2}$$

is perturbatively equivalent to $\mathcal{N}=0$ supergravity.

The material in this Chapter is borrowed from [5, 6].

9.1. Biadjoint scalar field theory

Before discussing the factorisation of full Yang–Mills theory, let us examine the simpler case of interacting biadjoint scalar field theory, cf. Section 4.2.. The factorisation of the free theory cochain complexes (7.3) does not require any twist, and can be lifted to the full (strict) L_∞-algebra

$$\mathfrak{L}^{\mathsf{biadj}}_{\mathsf{BRST}} \;=\; \mathfrak{g}\otimes(\bar{\mathfrak{g}}\otimes\mathfrak{Scal}). \tag{9.3}$$

A technicality: in Equation (9.3) we have $\bar{\mathfrak{g}}\otimes\mathfrak{Scal}$, the tensor product between $\bar{\mathfrak{g}}$ and $\mathfrak{Scal}$. While the tensor product between a Lie algebra and an L_∞-algebra (in general not defined) does not appear in the list (6.1) of possible tensor product between strict homotopy algebras, in this special case the product is well defined. Indeed, for nilpotent L_∞-algebras, i.e. L_∞-algebras with $\mu_i\circ\mu_j=0$, the product exists and yields a C_∞-algebra.

The full L_∞-algebra is then obtained tensoring the latter by a Lie algebra, as exposed in Section 6.1..

L_∞-algebra $\mathfrak{Scal}$. Explicitly, the L_∞-algebra $\mathfrak{Scal}$ is built from the cochain complex (7.4),

$$\mathfrak{Scal} \;:=\; \left(\overset{\mathsf{s}_x}{\mathfrak{F}[-1]} \xrightarrow{\;\square\;} \overset{\mathsf{s}_x^+}{\mathfrak{F}[-2]} \right) , \tag{9.4a}$$

and the only non-vanishing higher product beyond the differential $\mu_1^{\mathfrak{Scal}}$ is

$$\mu_2^{\mathfrak{Scal}}\left(\int \mathrm{d}^d x_1\, \mathsf{s}_{x_1} \varphi_1(x_1), \int \mathrm{d}^d x_2\, \mathsf{s}_{x_2} \varphi_2(x_2) \right) \;:=\; \lambda \int \mathrm{d}^d x\, \mathsf{s}_x^+ \varphi_1(x) \varphi_2(x) \;. \tag{9.4b}$$

Evidently, $\mathfrak{Scal}$ is nilpotent.

Factorisation. Following the prescription for the untwisted tensor product of strict homotopy algebras from Section 6.1., we obtain the binary product

$$\mu_2(\mathsf{e}_a \otimes \bar{\mathsf{e}}_{\bar{a}} \otimes \mathsf{s}_{x_1}, \mathsf{e}_b \otimes \bar{\mathsf{e}}_{\bar{b}} \otimes \mathsf{s}_{x_2}) \;=\; [\mathsf{e}_a, \mathsf{e}_b] \otimes [\bar{\mathsf{e}}_{\bar{a}}, \bar{\mathsf{e}}_{\bar{b}}] \otimes \lambda \delta^{(d)}(x_1 - x_2) \mathsf{s}_{x_1}^+ \;, \tag{9.5}$$

which, together with the differential

$$\mu_1(\mathsf{e}_a \otimes \bar{\mathsf{e}}_{\bar{a}} \otimes \mathsf{s}_{x_1}) \;=\; \mathsf{e}_a \otimes \bar{\mathsf{e}}_{\bar{a}} \otimes \square\, \mathsf{s}_{x_1}^+ \;, \tag{9.6}$$

and the cyclic structure

$$\langle \varphi, \varphi^+ \rangle \;=\; \int \mathrm{d}^d x\, \varphi^{a\bar{a}}(x) \varphi^+_{a\bar{a}}(x) \;, \tag{9.7}$$

forms the cyclic L_∞-algebra $\mathfrak{L}^{\text{biadj}}_{\text{BRST}}$. The homotopy Maurer–Cartan action of this L_∞-algebra is then the action (4.5) of biadjoint scalar field theory,

$$\begin{aligned} S^{\text{biadj}} \;&=\; \tfrac{1}{2}\langle \varphi, \mu_1(\varphi) \rangle + \tfrac{1}{3!}\langle \varphi, \mu_2(\varphi, \varphi) \rangle \\ &=\; \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \varphi_{a\bar{a}} \square \varphi^{a\bar{a}} - \tfrac{\lambda}{3!} f_{abc} f_{\bar{a}\bar{b}\bar{c}} \varphi^{a\bar{a}} \varphi^{b\bar{b}} \varphi^{c\bar{c}} \right\} \;, \end{aligned} \tag{9.8}$$

which verifies (9.3).

9.2. Strictified Yang–Mills theory

General considerations. The strictification of Yang–Mills theory formulated in Section 8.3. is now readily extended to a BV action, which can then be gauge fixed and converted into a strict L_∞-algebra $\tilde{\mathfrak{L}}^{\text{YM, st}}_{\text{BRST}}$.

The full strictification of Yang–Mills theory involves an infinite number of additional auxiliary fields and corresponding interaction terms in the Lagrangian. Thus, our discussion cannot be fully explicit and has to remain somewhat conceptual, but as before, we shall give explicit lowest order terms to exemplify our discussion. Recall, however, that for computing n-point correlation function at the tree-level, only a finite number of auxiliary fields and interaction terms are necessary. Moreover, for computing n-point scattering amplitudes up to ℓ loops, only a finite number of correlators is necessary. Therefore, we can always truncate the Yang–Mills action to finitely many auxiliary fields to perform our computations.

We note that gauge fixing of Yang–Mills theory is fully equivalent to gauge fixing of the strictified theory. Moreover, the additional interaction vertices that arise from the BV formalism are all cubic, except for the terms involving anti-fields of the auxiliary fields; the latter, however, will not contribute.

The last point implies that the L_∞-algebra $\tilde{\mathfrak{L}}^{\text{YM, st}}_{\text{BRST}}$ for the strictified and gauge-fixed form of Yang–Mills theory contains the cochain complex of the L_∞-algebra $\tilde{\mathfrak{L}}^{\text{YM}}_{\text{BRST}}$ which we have computed in Section 4.3.. This cochain complex is enlarged by the kinematic terms for all the auxiliary fields. We then have additional binary products encoding the cubic interactions.

L_∞-algebra of Yang–Mills theory. We consider the strictification up to quartic terms, as explained in Section 8.3.. By the arguments given there, however, it is clear that our discussion trivially generalises to strictifications up to an arbitrary order. The Lagrangian, including the strictification of the colour–kinematics duality producing terms (8.38), reads as

$$
\begin{aligned}
\mathscr{L}^{\mathrm{YM,\,st}}_{\mathrm{BRST},\,4} \;=\;& \tfrac12 \tilde A_{a\mu}\,\Box\,\tilde A^{\mu a} - \tilde{\bar c}_a\,\Box\,\tilde c^a + \tfrac12 \tilde b_a\,\Box\,\tilde b^a + \tilde\xi\,\tilde b_a\sqrt{\Box}\,\partial_\mu \tilde A^{\mu a} - g f_{abc}\bar c^a\partial^\mu(\tilde A^b_\mu \tilde c^c) - \\
&- \tfrac12 \tilde G_a^{\mu\nu\kappa}\,\Box\,\tilde G^a_{\mu\nu\kappa} + g f_{abc}\Big(\partial_\mu \tilde A^a_\nu + \tfrac{1}{\sqrt2}\partial^\kappa \tilde G^a_{\kappa\mu\nu}\Big)\tilde A^{\mu b}\tilde A^{\nu c} - \\
&- \tilde K^\mu_{1a}\,\Box\,\tilde{\bar K}^{1a}_\mu - \tilde K^\mu_{2a}\,\Box\,\tilde{\bar K}^{2a}_\mu - \\
&- g f_{abc}\Big\{\tilde K_1^{a\mu}(\partial^\nu \tilde A^b_\mu)\tilde A^c_\nu + \Big[\Big(-\sqrt{\tfrac{\Box}{\xi}}\,\tilde b^a + \tfrac{1-\sqrt{1-\xi}}{\sqrt{\xi}}\partial^\kappa \tilde A^a_\kappa\Big)\tilde A^{b\mu} - \tilde{\bar c}^a\partial^\mu \tilde c^b\Big]\tilde{\bar K}^{1c}_\mu\Big\} + \\
&+ g f_{abc}\Big\{\tilde K_2^{a\mu}\Big[(\partial^\nu\partial_\mu \tilde c^b)\tilde A^c_\nu + (\partial^\nu \tilde A^b_\mu)\partial_\nu \tilde c^c\Big] + \tilde{\bar c}^a \tilde A^{b\mu}\tilde{\bar K}^{2c}_\mu\Big\}\ ,
\end{aligned}
\tag{9.9}
$$

where $K_i^{a\mu}$ and $\bar K^{ai}_\mu$ are auxiliary $\mathfrak{g}$-valued one-forms, strictifying $\mathscr{L}^{\mathrm{YM,\,comp}}_{\mathrm{BRST},\,n=4,\,k=0}$, and we used the shorthand $\tilde G^a_{\mu\nu\kappa} := \tilde G^{4,\gamma,1,a}_{\mu\nu\kappa}$. The field content is summarised in Table 9.1. Note that $K_1^{a\mu}$ and $\bar K^{a1}_\mu$ are of ghost number zero, while $K_2^{a\mu}$ and $\bar K^{a2}_\mu$ carry ghost numbers -1 and $+1$, respectively. The L_∞-algebra $\tilde{\mathfrak{L}}^{\mathrm{YM,\,st}}_{\mathrm{BRST}}$ to quartic order has underlying cochain complex

$$
\overset{(\tilde K_1^{a\mu},\,\tilde{\bar K}^{1a}_\mu)}{\mathbb{R}^2\otimes\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} \xrightarrow{\;\Box\;} \overset{(\tilde K_1^{+a\mu},\,\tilde{\bar K}^{1+a}_\mu)}{\mathbb{R}^2\otimes\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}}
$$

$$
\overset{\tilde G^a_{\mu\nu\kappa}}{\otimes^3\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} \xrightarrow{\;\Box\;} \overset{\tilde G^{+a}_{\mu\nu\kappa}}{\otimes^3\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}}
$$

$$
\begin{array}{ccc}
\overset{\tilde A^a_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} & \xrightarrow{\;\Box\;} & \overset{\tilde A^{+a}_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} \\
 & {\scriptstyle \tilde\xi\sqrt{\Box}\,\partial^\mu}\searrow \quad \nearrow{\scriptstyle -\tilde\xi\sqrt{\Box}\,\partial_\mu} & \\
\overset{\tilde b^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}} & \xrightarrow[\;\Box\;]{} & \overset{\tilde b^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}
\end{array}
$$

$$
\begin{array}{ccccccc}
\overset{\tilde{\bar K}^{2a}_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} & \xrightarrow{-\Box} & \overset{\tilde K_2^{+a\mu}}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} & \qquad & \overset{\tilde K_2^{a\mu}}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} & \xrightarrow{-\Box} & \overset{\tilde{\bar K}^{2+a}_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} \\
\underbrace{\overset{\tilde c^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\tilde{\mathfrak{L}}^{\mathrm{YM,\,st}}_{\mathrm{BRST},\,0}} & \xrightarrow{-\Box} & \underbrace{\overset{\tilde{\bar c}^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\tilde{\mathfrak{L}}^{\mathrm{YM,\,st}}_{\mathrm{BRST},\,1}} & & \underbrace{\overset{\tilde{\bar c}^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\tilde{\mathfrak{L}}^{\mathrm{YM,\,st}}_{\mathrm{BRST},\,2}} & \xrightarrow{-\Box} & \underbrace{\overset{\tilde c^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=:\,\tilde{\mathfrak{L}}^{\mathrm{YM,\,st}}_{\mathrm{BRST},\,3}}
\end{array}
\tag{9.10a}
$$

Besides the differentials in (9.10a), we also have the following higher products

$$\left(\begin{pmatrix}\tilde{K}_1^{a\mu}\\ \tilde{\tilde{K}}_\mu^{1a}\\ \tilde{G}^a_{\mu\nu\kappa}\\ \tilde{A}^a_\mu\\ \tilde{b}\end{pmatrix},\begin{pmatrix}\tilde{\tilde{K}}_\mu^{2a}\\ \tilde{c}^a\end{pmatrix}\right)\xmapsto{\mu_2} gf_{bc}{}^a\begin{pmatrix}(\partial^\nu\tilde{A}^b_\mu)\partial_\nu\tilde{c}^c-\tilde{A}^b_\nu\partial^\nu\partial_\mu\tilde{c}^c\\ -\partial^\mu(\tilde{A}^b_\mu\tilde{c}^c)-\tilde{\tilde{K}}^b_{1\mu}(\partial^\mu\tilde{c}^c)+\tilde{A}^{b\mu}\tilde{\tilde{K}}^{2c}_\mu\end{pmatrix}$$

$$\in\bigoplus_{\phi\in(\tilde{K}_2^+,\tilde{\tilde{c}}^+)}\tilde{\mathfrak{L}}^{\mathrm{YM,\,st}}_{\mathrm{BRST},\,1,\,\phi}\,, \tag{9.10b}$$

$$\left(\begin{pmatrix}\tilde{\tilde{K}}_\mu^{2a}\\ \tilde{c}^a\end{pmatrix},\begin{pmatrix}\tilde{K}_2^{a\mu}\\ \tilde{\tilde{c}}^a\end{pmatrix}\right)\xmapsto{\mu_2} gf_{bc}{}^a\begin{pmatrix}-(\partial^\mu\tilde{c}^b)\tilde{\tilde{c}}^c\\ -\tilde{\tilde{K}}_\mu^{2b}\tilde{\tilde{c}}^c+(\partial_\mu\partial_\nu\tilde{c}^b)\tilde{K}_2^{c\nu}+\partial^\nu(\partial_\nu\tilde{c}^b\tilde{K}^c_{2\mu})-\tilde{c}^b\partial_\mu\tilde{\tilde{c}}^c\end{pmatrix}$$

$$\in\bigoplus_{\phi\in(\tilde{\tilde{K}}_1^+,\tilde{A}^+)}\tilde{\mathfrak{L}}^{\mathrm{YM,\,st}}_{\mathrm{BRST},\,1,\,\phi}\,, \tag{9.10c}$$

$$\left(\begin{pmatrix}\tilde{K}_1^{a\mu}\\ \tilde{\tilde{K}}_\mu^{1a}\\ \tilde{G}^a_{\mu\nu\kappa}\\ \tilde{A}^a_\mu\\ \tilde{b}^a\end{pmatrix},\begin{pmatrix}\tilde{K}_1^{a\mu}\\ \tilde{\tilde{K}}_\mu^{1a}\\ \tilde{G}^a_{\mu\nu\kappa}\\ \tilde{A}^a_\mu\\ \tilde{b}^a\end{pmatrix}\right)\xmapsto{\mu_2} gf_{bc}{}^a\begin{pmatrix}2(\partial^\nu\tilde{A}^b_\mu)\tilde{A}^c_\nu\\ 2\frac{1-\sqrt{1-\xi}}{\sqrt{\xi}}(\partial^\kappa\tilde{A}^b_\kappa)\tilde{A}^{c\mu}+2\sqrt{\frac{\Box}{\xi}}(\tilde{A}^b_\mu\tilde{b}^c)\\ \sqrt{2}\partial_\mu(\tilde{A}^b_\nu\tilde{A}^c_\kappa)\\ R^{\tilde{A}^+}_{bc\mu}\\ -2\sqrt{\frac{\Box}{\xi}}(\tilde{\tilde{K}}^{1b}_\mu\tilde{A}^{c\mu})\end{pmatrix}$$

$$\in\bigoplus_{\phi\in(\tilde{K}_1^+,\tilde{\tilde{K}}^{1+},\tilde{G}^+,\tilde{A}^+,\tilde{b}^+)}\tilde{\mathfrak{L}}^{\mathrm{YM,\,st}}_{\mathrm{BRST},\,2,\,\phi}\,, \tag{9.10d}$$

$$\begin{aligned}R^{\tilde{A}^+}_{bc\mu}\;:=\;&-3!\partial^\nu(\tilde{A}^b_\nu\tilde{A}^c_\mu)-\sqrt{8}\tilde{A}^{\nu b}\partial^\kappa\tilde{G}^c_{\kappa\nu\mu}-4\tilde{K}_1^{b\nu}\partial_\mu\tilde{A}^c_\nu-\\&-4\frac{1-\sqrt{1-\xi}}{\sqrt{\xi}}(\partial^\kappa\tilde{A}^b_\kappa)\tilde{\tilde{K}}^{1c}_\mu+2\tilde{\tilde{K}}^{1b}_\mu\sqrt{\frac{\Box}{\xi}}\tilde{b}^c\,,\end{aligned}$$

and

$$\left(\begin{pmatrix}\tilde{K}_1^{a\mu}\\ \tilde{\tilde{K}}_\mu^{1a}\\ \tilde{G}^a_{\mu\nu\kappa}\\ \tilde{A}^a_\mu\\ \tilde{b}^a\end{pmatrix},\begin{pmatrix}\tilde{K}_\mu^{2a}\\ \tilde{\tilde{c}}^a\end{pmatrix}\right)\xmapsto{\mu_2} gf_{bc}{}^a\begin{pmatrix}\tilde{A}^b_\mu\tilde{\tilde{c}}^c\\ -\tilde{A}^b_\mu\partial^\mu\tilde{\tilde{c}}^c-\partial^\mu(\tilde{\tilde{K}}^{1b}_\mu\tilde{\tilde{c}}^c)+\partial^\nu\partial_\mu(\tilde{A}^b_\nu\tilde{K}_2^{c\mu})\end{pmatrix}$$
$$\in\bigoplus_{\phi\in(\tilde{\tilde{K}}^{2+},\,\tilde{c}^+)}\tilde{\mathfrak{L}}^{\mathsf{YM,\,st}}_{\mathsf{BRST},\,1,\,\phi}\,,\tag{9.10e}$$

and the cyclic structure is given by

$$\begin{aligned}
\langle\tilde{A},\tilde{A}^+\rangle &:= \int\mathrm{d}^dx\,\tilde{A}^a_\mu\tilde{A}^{+\mu}_a\,, & \langle\tilde{b},\tilde{b}^+\rangle &:= \int\mathrm{d}^dx\,\tilde{b}^a\tilde{b}^+_a\,,\\
\langle\tilde{c},\tilde{c}^+\rangle &:= \int\mathrm{d}^dx\,\tilde{c}^a\tilde{c}^+_a\,, & \langle\tilde{\tilde{c}},\tilde{\tilde{c}}^+\rangle &:= -\int\mathrm{d}^dx\,\tilde{\tilde{c}}^a\tilde{\tilde{c}}^+_a\,,\\
\langle\tilde{K}_1,\tilde{K}_1^+\rangle &:= -\int\mathrm{d}^dx\,\tilde{K}_1^{a\mu}\tilde{K}^+_{1a\mu}\,, & \langle\tilde{\tilde{K}}^1,\tilde{\tilde{K}}^{1+}\rangle &:= -\int\mathrm{d}^dx\,\tilde{\tilde{K}}^{1a}_\mu\tilde{\tilde{K}}^{1+\mu}_a\,,\\
\langle\tilde{K}_2,\tilde{K}_2^+\rangle &:= -\int\mathrm{d}^dx\,\tilde{K}_2^{a\mu}\tilde{K}^+_{2a\mu}\,, & \langle\tilde{\tilde{K}}^2,\tilde{\tilde{K}}^{2+}\rangle &:= \int\mathrm{d}^dx\,\tilde{\tilde{K}}^{2a}_\mu\tilde{\tilde{K}}^{2+\mu}_a\,,\\
\langle\tilde{G},\tilde{G}^+\rangle &:= -\int\mathrm{d}^dx\,\tilde{G}^a_{\mu\nu\kappa}\tilde{G}^{+\mu\nu\kappa}_a\,.
\end{aligned}\tag{9.10f}$$

Factorisation and twist datum. We factorise this L_∞-algebra as

$$\tilde{\mathfrak{L}}^{\mathsf{YM,\,st}}_{\mathsf{BRST}} = \mathfrak{g}\otimes(\mathfrak{Kin}^{\mathsf{st}}\otimes_\tau\mathfrak{Scal})\,,\tag{9.11}$$

where $\mathfrak{g}$ is the usual colour Lie algebra, $\mathfrak{Kin}^{\mathsf{st}}$ the graded vector space

$$\mathfrak{Kin}^{\mathsf{st}} := \left(\underbrace{\begin{matrix}\overset{\bar{\mathtt{t}}_2^\mu}{\mathbb{M}^d}\\ \oplus\\ \overset{\mathtt{g}}{\mathbb{R}[1]}\end{matrix}}_{=:\,\mathfrak{Kin}^{\mathsf{st}}_{-1}}\oplus\underbrace{\begin{matrix}\overset{\mathtt{t}^1_\mu,\ \bar{\mathtt{t}}^\mu_1}{\mathbb{M}^d\oplus\mathbb{M}^d}\\ \oplus\\ \overset{\mathtt{t}_0^{\mu\nu\kappa}}{\mathbb{M}^d\otimes(\mathbb{M}^d\wedge\mathbb{M}^d)}\\ \oplus\\ \overset{\mathtt{v}^\mu}{\mathbb{M}^d}\\ \oplus\\ \underset{\mathtt{n}}{\mathbb{R}}\end{matrix}}_{=:\,\mathfrak{Kin}^{\mathsf{st}}_0}\oplus\underbrace{\begin{matrix}\overset{\mathtt{t}^2_\mu}{\mathbb{M}^d}\\ \oplus\\ \overset{\mathtt{a}}{\mathbb{R}[-1]}\end{matrix}}_{=:\,\mathfrak{Kin}^{\mathsf{st}}_1}\right),\tag{9.12}$$

and $\mathfrak{Scal}$ the L_∞-algebra defined in (9.4). This L_∞-algebra is cyclic with the inner products given by (7.17) together with

$$\begin{gathered}\langle \mathtt{t}^1_\mu, \bar{\mathtt{t}}^\nu_1 \rangle := -\delta^\nu_\mu\ ,\quad \langle \bar{\mathtt{t}}^\nu_1, \mathtt{t}^1_\mu \rangle := -\delta^\nu_\mu\ ,\quad \langle \mathtt{t}^2_\mu, \bar{\mathtt{t}}^\nu_2 \rangle := \delta^\nu_\mu\ ,\quad \langle \bar{\mathtt{t}}^\nu_2, \mathtt{t}^2_\mu \rangle := \delta^\nu_\mu\ ,\\ \langle \mathtt{t}^{\mu\nu\kappa}_0, \mathtt{t}^{\lambda\rho\sigma}_0 \rangle := -\tfrac{1}{2}\eta^{\mu\lambda}(\eta^{\nu\rho}\eta^{\kappa\sigma} - \eta^{\nu\sigma}\eta^{\kappa\rho})\ .\end{gathered} \tag{9.13}$$

fields				anti-fields			
factorisation	$\lvert - \rvert_{\rm gh}$	$\lvert - \rvert_{\mathfrak{L}}$	dim	factorisation	$\lvert - \rvert_{\rm gh}$	$\lvert - \rvert_{\mathfrak{L}}$	dim
$\tilde{c} = \mathsf{e}_a \mathtt{g} \mathsf{s}_x \tilde{c}^a(x)$	1	0	$\frac{d}{2} - 2$	$\tilde{c}^+ = \mathsf{e}_a \mathtt{a} \mathsf{s}^+_x \tilde{c}^{+a}(x)$	-2	3	$\frac{d}{2} + 2$
$\tilde{A} = \mathsf{e}_a \mathtt{v}^\mu \mathsf{s}_x \tilde{A}^a_\mu(x)$	0	1	$\frac{d}{2} - 1$	$\tilde{A}^+ = \mathsf{e}_a \mathtt{v}^\mu \mathsf{s}^+_x \tilde{A}^{+a}_\mu(x)$	-1	2	$\frac{d}{2} + 1$
$\tilde{b} = \mathsf{e}_a \mathtt{n} \mathsf{s}_x \tilde{b}^a(x)$	0	1	$\frac{d}{2} - 1$	$\tilde{b}^+ = \mathsf{e}_a \mathtt{n} \mathsf{s}^+_x \tilde{b}^{+a}(x)$	-1	2	$\frac{d}{2} + 1$
$\tilde{\bar{c}} = \mathsf{e}_a \mathtt{a} \mathsf{s}_x \tilde{\bar{c}}^a(x)$	-1	2	$\frac{d}{2}$	$\tilde{\bar{c}}^+ = \mathsf{e}_a \mathtt{g} \mathsf{s}^+_x \tilde{\bar{c}}^{+a}(x)$	0	1	$\frac{d}{2}$
$\tilde{K}_1 = \mathsf{e}_a \mathtt{t}^1_\mu \mathsf{s}_x \tilde{K}^\mu_1(x)$	0	1	$\frac{d}{2} - 1$	$\tilde{K}^+_1 = \mathsf{e}_a \mathtt{t}^1_\mu \mathsf{s}^+_x \tilde{K}^{+a\mu}_1(x)$	-1	2	$\frac{d}{2} - 1$
$\tilde{\bar{K}}^1 = \mathsf{e}_a \bar{\mathtt{t}}^\mu_1 \mathsf{s}_x \tilde{\bar{K}}^{1a}_\mu(x)$	0	1	$\frac{d}{2} - 1$	$\tilde{\bar{K}}^{1+} = \mathsf{e}_a \bar{\mathtt{t}}^\mu_1 \mathsf{s}^+_x \tilde{\bar{K}}^{1+a}_\mu(x)$	-1	2	$\frac{d}{2} - 1$
$\tilde{K}_2 = \mathsf{e}_a \mathtt{t}^2_\mu \mathsf{s}_x \tilde{K}^\mu_2(x)$	-1	2	$\frac{d}{2} - 1$	$\tilde{K}^+_2 = \mathsf{e}_a \mathtt{t}^2_\mu \mathsf{s}^+_x \tilde{K}^{+a\mu}_2(x)$	0	1	$\frac{d}{2} - 1$
$\tilde{\bar{K}}^2 = \mathsf{e}_a \bar{\mathtt{t}}^\mu_2 \mathsf{s}_x \tilde{\bar{K}}^{2a}_\mu(x)$	1	0	$\frac{d}{2} - 1$	$\tilde{\bar{K}}^{2+} = \mathsf{e}_a \bar{\mathtt{t}}^\mu_2 \mathsf{s}^+_x \tilde{\bar{K}}^{2+a}_\mu(x)$	-2	3	$\frac{d}{2} - 1$
$\tilde{G} = \mathsf{e}_a \mathtt{t}^{\mu\nu\kappa}_0 \mathsf{s}_x \tilde{G}^a_{\mu\nu\kappa}(x)$	0	1	$\frac{d}{2} - 1$	$\tilde{G}^+ = \mathsf{e}_a \mathtt{t}^{\mu\nu\kappa}_0 \mathsf{s}^+_x \tilde{G}^{+a}_{\mu\nu\kappa}(x)$	-1	2	$\frac{d}{2} - 1$

Table 9.1: Factorisation of the fields in the L_∞-algebra corresponding to the Lagrangian $\mathscr{L}^{\rm YM,\,st}_{\rm BRST,\,4}$. Note that we suppressed the integrals over x and the tensor products for simplicity.

The twist datum τ, see (6.20) for the general definition, in the factorisation (9.11) is then given by the maps

$$\tau_1(\mathtt{g}) := \mathtt{g}\otimes\mathsf{id}\ ,\quad \begin{gathered}\tau_1(\mathtt{t}^i_\mu) := \mathtt{t}^i_\mu\otimes\mathsf{id}\ ,\quad \tau_1(\bar{\mathtt{t}}^\mu_i) := \bar{\mathtt{t}}^\mu_i\otimes\mathsf{id}\ ,\\ \tau_1(\mathtt{t}^{\mu\nu\kappa}_0) := \mathtt{t}^{\mu\nu\kappa}_0\otimes\mathsf{id}\ ,\\ \tau_1(\mathtt{v}^\mu) := \mathtt{v}^\mu\otimes\mathsf{id} + \tilde{\xi}\mathtt{n}\otimes\Box^{-\frac{1}{2}}\partial^\mu\ ,\\ \tau_1(\mathtt{n}) := \mathtt{n}\otimes\mathsf{id} - \tilde{\xi}\mathtt{v}^\mu\otimes\Box^{-\frac{1}{2}}\partial_\mu\ ,\end{gathered}\quad \tau_1(\mathtt{a}) := \mathtt{a}\otimes\mathsf{id} \tag{9.14a}$$

and

$$\begin{aligned}\tau_2(\mathtt{g},\mathtt{v}^\mu) &:= -\mathtt{g}\otimes(\mathsf{id}\otimes\partial^\mu + \partial^\mu\otimes\mathsf{id}) + \mathtt{t}^\mu_2\otimes(\partial^\nu\otimes\partial_\nu - \partial^\mu\partial_\nu\otimes\mathsf{id})\ ,\\ \tau_2(\mathtt{v}^\mu,\mathtt{g}) &:= \mathtt{g}\otimes(\mathsf{id}\otimes\partial^\mu + \partial^\mu\otimes\mathsf{id}) - \mathtt{t}^\mu_2\otimes(\partial^\nu\otimes\partial_\nu - \mathsf{id}\otimes\partial^\mu\partial_\nu)\ ,\\ \tau_2(\mathtt{g},\bar{\mathtt{t}}^1_\mu) &:= -\mathtt{g}\otimes\partial_\mu\otimes\mathsf{id}\ ,\\ \tau_2(\bar{\mathtt{t}}^1_\mu,\mathtt{g}) &:= \mathtt{g}\otimes\mathsf{id}\otimes\partial_\mu\ ,\end{aligned}$$

$$
\begin{aligned}
\tau_2(\bar{\mathtt{t}}_2^\mu, \mathtt{v}^\nu) &:= \eta^{\mu\nu}\mathtt{g}\otimes \mathsf{id}\otimes\mathsf{id}\ ,\\
\tau_2(\mathtt{v}^\mu, \bar{\mathtt{t}}_2^\nu) &:= -\eta^{\mu\nu}\mathtt{g}\otimes \mathsf{id}\otimes\mathsf{id}\ ,\\
\tau_2(\mathtt{g}, \mathtt{a}) &:= \mathtt{v}^\mu\otimes\mathsf{id}\otimes\partial_\mu + \bar{\mathtt{t}}_1^\mu\otimes\partial_\mu\otimes\mathsf{id}\ ,\\
\tau_2(\mathtt{a}, \mathtt{g}) &:= -\mathtt{v}^\mu\otimes\partial_\mu\otimes\mathsf{id} - \bar{\mathtt{t}}_1^\mu\otimes\mathsf{id}\otimes\partial_\mu\ ,\\
\tau_2(\bar{\mathtt{t}}_2^\mu, \mathtt{a}) &:= \mathtt{v}^\mu\otimes\mathsf{id}\otimes\mathsf{id}\ ,\\
\tau_2(\mathtt{a}, \bar{\mathtt{t}}_2^\mu) &:= -\mathtt{v}^\mu\otimes\mathsf{id}\otimes\mathsf{id}\ ,\\
\tau_2(\mathtt{g}, \mathtt{t}_2^\mu) &:= -\mathtt{v}^\nu\otimes\partial_\nu\partial^\mu\otimes\mathsf{id} - \mathtt{v}^\mu\otimes\Box\otimes\mathsf{id} - \mathtt{v}^\mu\otimes\partial_\nu\otimes\partial^\nu\ ,\\
\tau_2(\mathtt{t}_2^\mu, \mathtt{g}) &:= \mathtt{v}^\nu\otimes\mathsf{id}\otimes\partial_\nu\partial^\mu + \mathtt{v}^\mu\otimes\mathsf{id}\otimes\Box + \mathtt{v}^\mu\otimes\partial_\nu\otimes\partial^\nu\ ,\\
\tau_2(\mathtt{v}^\mu, \mathtt{a}) &:= -\bar{\mathtt{t}}_\mu^2\otimes\mathsf{id}\otimes\mathsf{id} + \mathtt{a}\otimes\mathsf{id}\otimes\partial^\mu\ ,\\
\tau_2(\mathtt{a}, \mathtt{v}^\mu) &:= \bar{\mathtt{t}}_\mu^2\otimes\mathsf{id}\otimes\mathsf{id} - \mathtt{a}\otimes\partial^\mu\otimes\mathsf{id}\ ,\\
\tau_2(\bar{\mathtt{t}}_1^\mu, \mathtt{a}) &:= \mathtt{a}\otimes(\partial^\mu\otimes\mathsf{id} + \mathsf{id}\otimes\partial^\mu)\ ,\\
\tau_2(\mathtt{a}, \bar{\mathtt{t}}_1^\mu) &:= -\mathtt{a}\otimes(\partial^\mu\otimes\mathsf{id} + \mathsf{id}\otimes\partial^\mu)\ ,\\
\tau_2(\mathtt{v}^\mu, \mathtt{t}_2^\nu) &:= -\mathtt{a}\otimes(\partial^\mu\partial^\nu\otimes\mathsf{id} + \partial^\mu\otimes\partial^\nu + \partial^\nu\otimes\partial^\mu + \mathsf{id}\otimes\partial^\mu\partial^\nu)\ ,\\
\tau_2(\mathtt{t}_2^\mu, \mathtt{v}^\nu) &:= \mathtt{a}\otimes(\partial^\mu\partial^\nu\otimes\mathsf{id} + \partial^\mu\otimes\partial^\nu + \partial^\nu\otimes\partial^\mu + \mathsf{id}\otimes\partial^\mu\partial^\nu)\ ,\\
\tau_2(\mathtt{v}^\mu, \mathtt{v}^\nu) &:= \mathtt{t}_1^\mu\otimes\partial^\nu\otimes\mathsf{id} - \mathtt{t}_1^\nu\otimes\mathsf{id}\otimes\partial^\mu +\\
&\qquad + \frac{1-\sqrt{1-\xi}}{\sqrt{\xi}}\big(\bar{\mathtt{t}}_1^\nu\otimes\partial^\mu\otimes\mathsf{id} - \bar{\mathtt{t}}_1^\mu\otimes\mathsf{id}\otimes\partial^\nu\big) -\\
&\qquad - 3\Big[\mathtt{v}^\nu\otimes(\partial^\mu\otimes\mathsf{id} + \mathsf{id}\otimes\partial^\mu) - \mathtt{v}^\mu\otimes(\partial^\nu\otimes\mathsf{id} + \mathsf{id}\otimes\partial^\nu)\Big] +\\
&\qquad + \sqrt{2}\big(\mathtt{t}_0^{\kappa\mu\nu}\otimes\partial_\kappa\otimes\mathsf{id} + \mathtt{t}_0^{\kappa\mu\nu}\otimes\mathsf{id}\otimes\partial_\kappa\big)\ ,\\
\tau_2(\mathtt{v}^\mu, \mathtt{n}) &:= \bar{\mathtt{t}}_1^\mu\otimes\sqrt{\frac{\Box}{\xi}}^{(1)}\otimes\sqrt{\frac{\Box}{\xi}}^{(2)}\ ,\\
\tau_2(\mathtt{n}, \mathtt{v}^\mu) &:= -\bar{\mathtt{t}}_1^\mu\otimes\sqrt{\frac{\Box}{\xi}}^{(1)}\otimes\sqrt{\frac{\Box}{\xi}}^{(2)}\ ,\\
\tau_2(\mathtt{v}^\mu, \mathtt{t}_0^{\nu\kappa\lambda}) &:= -\frac{\sqrt{2}}{2}\big(\eta^{\mu\kappa}\mathtt{v}^\lambda\otimes\mathsf{id}\otimes\partial^\nu - \eta^{\mu\lambda}\mathtt{v}^\kappa\otimes\mathsf{id}\otimes\partial^\nu\big)\ ,\\
\tau_2(\mathtt{t}_0^{\nu\kappa\lambda}, \mathtt{v}^\mu) &:= \frac{\sqrt{2}}{2}\big(\eta^{\mu\kappa}\mathtt{v}^\lambda\otimes\partial^\nu\otimes\mathsf{id} - \eta^{\mu\lambda}\mathtt{v}^\kappa\otimes\partial^\nu\otimes\mathsf{id}\big)\ ,\\
\tau_2(\mathtt{t}_1^\mu, \mathtt{v}^\nu) &:= -2\eta^{\mu\nu}\mathtt{v}^\kappa\otimes\mathsf{id}\otimes\partial^\kappa\ ,\\
\tau_2(\mathtt{v}^\nu, \mathtt{t}_1^\mu) &:= 2\eta^{\mu\nu}\mathtt{v}^\kappa\otimes\partial^\kappa\otimes\mathsf{id}\ ,\\
\tau_2(\mathtt{v}^\nu, \bar{\mathtt{t}}_1^\mu) &:= -2\frac{1-\sqrt{1-\xi}}{\sqrt{\xi}}\mathtt{v}^\mu\otimes\partial^\nu\otimes\mathsf{id} + \eta^{\mu\nu}\mathtt{n}\otimes\sqrt{\frac{\Box}{\xi}}^{(1)}\otimes\sqrt{\frac{\Box}{\xi}}^{(2)}\ ,
\end{aligned}
\tag{9.14b}
$$

$$
\begin{aligned}
\mathcal{T}_2(\bar{\mathtt{t}}_1^\mu, \mathtt{v}^\nu) &:= 2\frac{1-\sqrt{1-\xi}}{\sqrt{\xi}}\mathtt{v}^\mu \otimes \mathsf{id} \otimes \partial^\nu - \eta^{\mu\nu}\mathtt{n} \otimes \sqrt{\frac{\Box}{\xi}}^{(1)} \otimes \sqrt{\frac{\Box}{\xi}}^{(2)} ,\\
\mathcal{T}_2(\bar{\mathtt{t}}_1^\mu, \mathtt{n}) &:= \mathtt{v}^\mu \otimes \mathsf{id} \otimes \sqrt{\frac{\Box}{\xi}} ,\\
\mathcal{T}_2(\mathtt{n}, \bar{\mathtt{t}}_1^\mu) &:= -\mathtt{v}^\mu \otimes \sqrt{\frac{\Box}{\xi}} \otimes \mathsf{id} ,
\end{aligned}
$$

where we defined

$$
\left(\sqrt{\frac{\Box}{\xi}}^{(1)} \otimes \sqrt{\frac{\Box}{\xi}}^{(2)}\right)(f \otimes g) := \sqrt{\frac{\Box}{\xi}}(fg) . \tag{9.14c}
$$

We note that the twisted tensor product $\mathfrak{Kin}^{\mathrm{st}} \otimes_\tau \mathfrak{Scal}$ is a (strict) C_∞-algebra, which becomes an L_∞-algebra after the tensor product with the colour Lie algebra $\mathfrak{g}$; see Section 6.1. for details.

9.3. BRST Lagrangian double copy

A key feature of our double copy prescription based on factorisations of the L_∞-algebras of gauge-fixed BRST Lagrangians is that not only the action but also the BRST operator double copies. This fact guarantees that the double copy creates the appropriate gauge-fixing sectors which is crucial in considering the double copy at the loop level. In the following, we give a general discussion of what we called the *BRST Lagrangian double copy* in [5].

Strictification of BRST-invariant actions. As discussed in Section 8.3., any field theory can be strictified to a classically equivalent field theory with purely cubic interaction terms, and this equivalence extends to the quantum level. Consider a general strictified field theory

$$
S = \frac{1}{2}\Phi^I \mathsf{g}_{IJ}\Phi^J + \frac{1}{3!}\Phi^I \mathsf{f}_{IJK}\Phi^J\Phi^K , \tag{9.15}
$$

where g_{IJ} and f_{IJK} are some structure constants. As in Section 3.1., $I, J, \ldots$ are DeWitt indices that include labels for the field species, the gauge and Lorentz representations, as well as the space-time position.

Let us now consider a theory which is invariant under a gauge symmetry. We extend the action of this theory to its BV form by including ghosts, anti-ghosts, and the Nakanishi–Lautrup field, as done in Chapter 4. We then strictify the full BV action to an action with cubic interaction vertices. Restricting to gauge-fixing fermions which are quadratic in the fields[1] guarantees that the action remains cubic after gauge fixing. The resulting BRST operator Q_{BRST}, given by (3.17c), is then automatically at most quadratic in the fields, and we can write

$$\Phi^I \xmapsto{Q_{\text{BRST}}} \mathsf{Q}^I_J \Phi^J + \frac{1}{2}\mathsf{Q}^I_{JK}\Phi^J\Phi^K \tag{9.16}$$

for some structure constants Q^I_J and Q^I_{JK}.

	$\mathfrak{V}$	$\bar{\mathfrak{V}}$
Biadjoint scalar field theory	$\mathfrak{g}$	$\bar{\mathfrak{g}}$
Yang–Mills theory	$\mathfrak{g}$	$\mathfrak{Kin}$
$\mathcal{N} = 0$ supergravity	$\mathfrak{Kin}$	$\mathfrak{Kin}$

Table 9.2: Factors appearing in the field space factorisation (9.17) with $\mathfrak{Kin}$ given in (7.16) and $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ the colour Lie algebras.

Factorisation of structure constants. As indicated previously, the key to the double copy is the factorisation of the field space $\mathfrak{L}$ into

$$\mathfrak{L} \;:=\; \mathfrak{V} \otimes \bar{\mathfrak{V}} \otimes \mathscr{C}^\infty(\mathbb{M}^d) \;, \tag{9.17}$$

where $\mathfrak{V}$ and $\bar{\mathfrak{V}}$ are two (graded) vector spaces. In our preceeding discussion, we have encountered the three examples in Table 9.2. Consequently, in our formulas, we shall split the multi indices into triples, that is, $I = (\alpha, \bar{\alpha}, x)$, and write (see e.g. (3.12b))

$$(\mathfrak{L}[1])^* \otimes \mathfrak{L} \;\ni\; \mathsf{a} \;=\; \Phi^I \otimes \mathsf{e}_I \;=\; \int \mathrm{d}^d x\, \Phi^{\alpha\bar{\alpha}}(x) \otimes (\mathsf{e}_\alpha \otimes \bar{\mathsf{e}}_{\bar{\alpha}} \otimes \mathsf{s}_x) \;. \tag{9.18}$$

We also demand that the structure constants g_{IJ} and f_{IJK} that appear in the action (9.15) as well as the structure constants Q^I_J and Q^I_{JK} that appear in the BRST operator (9.16) are local in the sense that they vanish unless all the space-time points in the multi-indices agree.

We write

$$\mathsf{g}_{IJ} \;=:\; \mathsf{g}_{\alpha\beta}\,\bar{\mathsf{g}}_{\bar{\alpha}\bar{\beta}}\,\square \;, \tag{9.19}$$

[1] This is the case for all explicit gauge-fixing fermions used in this paper.

where $\mathsf{g}_{\alpha\beta}$ and $\bar{\mathsf{g}}_{\bar{\alpha}\bar{\beta}}$ are differential operators, mapping $\mathscr{C}^\infty(\mathbb{M}^d)$ to itself. In more detail, we have

$$\mathsf{g}_{IJ}\Phi^J \equiv \int \mathrm{d}^d y\ \mathsf{g}_{(\alpha,\bar{\alpha},x);(\beta,\bar{\beta},y)}\Phi^{\beta\bar{\beta},y} = \int \mathrm{d}^d y \int \mathrm{d}^d z\ \mathsf{g}_{\alpha\beta}(x,y)\bar{\mathsf{g}}_{\bar{\alpha}\bar{\beta}}(y,z)\Box\,\Phi^{\beta\bar{\beta},z}\ , \tag{9.20a}$$

where the integral kernels are of the fom

$$\mathsf{g}_{\alpha\beta}(x,y) = \delta^{(d)}(x-y)\mathsf{g}_{\alpha\beta}(x) \quad \text{and} \quad \bar{\mathsf{g}}_{\bar{\alpha}\bar{\beta}}(y,z) = \delta^{(d)}(y-z)\bar{\mathsf{g}}_{\bar{\alpha}\bar{\beta}}(y) \tag{9.20b}$$

due to our assumption about locality, and we assume that $\mathsf{g}_{\alpha\beta}(x)$ and $\bar{\mathsf{g}}_{\bar{\alpha}\bar{\beta}}(y)$ are invertible.

Analogously, we write

$$\mathsf{f}_{IJK} = \mathsf{f}_{(\alpha,\bar{\alpha},x);(\beta,\bar{\beta},y);(\gamma,\bar{\gamma},z)} =: \mathsf{p}\,\mathsf{f}_{\alpha\beta\gamma}\,\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\ , \tag{9.20c}$$

where $\mathsf{f}_{\alpha\beta\gamma}$ and $\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}$ are bi-differential operators $\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathscr{C}^\infty(\mathbb{M}^d) \to \mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathscr{C}^\infty(\mathbb{M}^d)$ and

$$\mathsf{p} : \mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathscr{C}^\infty(\mathbb{M}^d) \to \mathscr{C}^\infty(\mathbb{M}^d) \tag{9.20d}$$

is the natural diagonal product of functions. For the integral kernels of $\mathsf{f}_{\alpha\beta\gamma}$ and $\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}$ we have again the locality condition

$$\begin{aligned}\mathsf{f}_{\alpha\beta\gamma}(x_1,x_2;y_1,y_2) &= \delta^{(d)}(x_1-y_1)\delta^{(d)}(x_2-y_2)\mathsf{f}_{\alpha\beta\gamma}(y_1,y_2)\ ,\\ \bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}(x_1,x_2;y_1,y_2) &= \delta^{(d)}(x_1-y_1)\delta^{(d)}(x_2-y_2)\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}(y_1,y_2)\ .\end{aligned} \tag{9.20e}$$

We note that there is some ambiguity in the definition (9.20c) due to the projection onto the diagonal involved in p, but this redundancy never arises in any formula. To give a clearer picture of what the above construction is doing, we can expand the $\mathsf{f}_{\alpha\beta\gamma}$ and the $\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}$ further in a basis of differential operators ∂^M for M a Lorentz multiindex, and we have

$$(\mathsf{p}\,\mathsf{f}_{\alpha\beta\gamma}\,\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}})(\Phi\otimes\Phi) = \mathsf{f}_{\alpha\beta M_1\gamma M_2}\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}N_1\bar{\gamma}N_2}(\partial^{M_1}\partial^{N_1}\Phi^{\beta\bar{\beta}})(\partial^{M_2}\partial^{N_2}\Phi^{\gamma\bar{\gamma}})\ . \tag{9.21}$$

For convenience, we also introduce the operators $\mathsf{f}^{\alpha}_{\beta\gamma}$ and $\bar{\mathsf{f}}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}$ by

$$\mathsf{p}\,\mathsf{f}_{\alpha\beta\gamma} =: \mathsf{g}_{\alpha\delta}\,\mathsf{p}\,\mathsf{f}^{\delta}_{\beta\gamma} \quad \text{and} \quad \mathsf{p}\,\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} =: \bar{\mathsf{g}}_{\bar{\alpha}\bar{\delta}}\,\mathsf{p}\,\bar{\mathsf{f}}^{\bar{\delta}}_{\bar{\beta}\bar{\gamma}}\ , \tag{9.22}$$

which is possible due to the invertibility of $\mathsf{g}_{\alpha\beta}$ and $\bar{\mathsf{g}}_{\bar{\alpha}\bar{\beta}}$ as well as the form of the integral kernels (9.20e) . Evidently, $\mathsf{f}^{\alpha}_{\beta\gamma}$ and $\bar{\mathsf{f}}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}$ are again bi-differential operators, just as $\mathsf{f}_{\alpha\beta\gamma}$ and $\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}$.

With the factorisation restriction, the action (9.15) becomes

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2}\Phi^{\alpha\bar\alpha}\mathsf{g}_{\alpha\beta}\bar{\mathsf{g}}_{\bar\alpha\bar\beta}\Box\Phi^{\beta\bar\beta} + \frac{1}{3!}\Phi^{\alpha\bar\alpha}(\mathsf{p}\,\mathsf{f}_{\alpha\beta\gamma}\,\bar{\mathsf{f}}_{\bar\alpha\bar\beta\bar\gamma})(\Phi^{\beta\bar\beta}\otimes\Phi^{\gamma\bar\gamma}) \right\} . \tag{9.23}$$

For the BRST operator Q_{BRST}, the factorisation of indices and the linearity of Q_{BRST} imply the decomposition

$$Q_{\mathrm{BRST}} =: q_{\mathrm{BRST}} + \bar q_{\mathrm{BRST}} , \tag{9.24}$$

where q_{BRST} and $\bar q_{\mathrm{BRST}}$ are BRST operators acting in a non-trivial way on the factors $\mathfrak{V}\otimes\mathscr{C}^\infty(\mathbb{M}^d)$ and $\bar{\mathfrak{V}}\otimes\mathscr{C}^\infty(\mathbb{M}^d)$ in the factorisation (9.18), respectively. By this, we mean that the structure constants Q^I_J and Q^I_{JK} decompose as $\mathsf{Q}^I_J \to (\mathsf{q}^I_J, \bar{\mathsf{q}}^I_J)$ and $\mathsf{Q}^I_{JK} \to (\mathsf{q}^I_{JK}, \bar{\mathsf{q}}^I_{JK})$. More explicitly,

$$\begin{aligned} \mathsf{q}^{(\alpha,\bar\alpha,x)}_{(\beta,\bar\beta,y)} &= \delta^{(d)}(x-y)\mathsf{q}^\alpha_\beta(x)\delta^{\bar\alpha}_{\bar\beta} , & \mathsf{q}^{(\alpha,\bar\alpha,x)}_{(\beta,\bar\beta,y);(\gamma,\bar\gamma,z)} &= \delta^{(d)}(x-y)\delta^{(d)}(x-z)\mathsf{q}^\alpha_{\beta\gamma}(x)\bar{\mathsf{f}}^{\bar\alpha}_{\bar\beta\bar\gamma}(x) , \\ \bar{\mathsf{q}}^{(\alpha,\bar\alpha,x)}_{(\beta,\bar\beta,y)} &= \delta^{(d)}(x-y)\delta^\alpha_\beta\bar{\mathsf{q}}^{\bar\alpha}_{\bar\beta}(x) , & \bar{\mathsf{q}}^{(\alpha,\bar\alpha,x)}_{(\beta,\bar\beta,y);(\gamma,\bar\gamma,z)} &= \delta^{(d)}(x-y)\delta^{(d)}(x-z)\mathsf{f}^\alpha_{\beta\gamma}(x)\bar{\mathsf{q}}^{\bar\alpha}_{\bar\beta\bar\gamma}(x) , \end{aligned} \tag{9.25}$$

where q^α_β and $\bar{\mathsf{q}}^{\bar\alpha}_{\bar\beta}$ are differential operators and $\mathsf{q}^\alpha_{\beta\gamma}$ and $\bar{\mathsf{q}}^{\bar\alpha}_{\bar\beta\bar\gamma}$ are again bi-differential operators, just as $\mathsf{f}^\alpha_{\beta\gamma}$ and $\bar{\mathsf{f}}^{\bar\alpha}_{\bar\beta\bar\gamma}$, with locality again restricting their integral kernels. Note that in this splitting, the association of terms of the form $\delta^{(d)}(x-y)\delta^\alpha_\beta\delta^{\bar\alpha}_{\bar\beta}$ and $\delta^{(d)}(x-y)\delta^{(d)}(x-z)\mathsf{f}^\alpha_{\beta\gamma}(y,z)\bar{\mathsf{f}}^{\bar\alpha}_{\bar\beta\bar\gamma}(y,z)$ is not unique; we assign half of each of these terms to $(\mathsf{q}^I_J, \mathsf{q}^I_{JK})$ and half to $(\bar{\mathsf{q}}^I_J, \bar{\mathsf{q}}^I_{JK})$.

Example. To make our rather abstract discussion more concrete, let us briefly consider the case of Yang–Mills theory (4.12). We refrain from discussing the details of the strictification of the BV action, but it is clear that $\mathfrak{V} = \mathfrak{g}$ and $\bar{\mathfrak{V}} = \mathfrak{Kin}'$ with $\mathfrak{Kin}'$ some extension of $\mathfrak{Kin}$ allowing for auxiliary fields, similar to $\mathfrak{Kin}^{\mathrm{st}}$ defined in (9.13). It is then also clear that $\mathsf{g}_{\alpha\beta}$ and $\mathsf{f}^\alpha_{\beta\gamma}$ are the Killing form and the structure constants of the gauge Lie algebra $\mathfrak{g}$.

On $\mathfrak{Kin}'$, the integral kernel for the differential operator $\bar{\mathsf{g}}_{\mu\nu}$ is given by

$$\bar{\mathsf{g}}_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{\Box}\partial_\mu\partial_\nu . \tag{9.26}$$

We note that $\mathsf{q}^\alpha_\beta = 0$ and $\bar{\mathsf{q}}^{\bar\alpha}_{\bar\beta}$ is only non-trivial for $\bar\alpha$ labelling ghost and Nakanishi–Lautrup fields, and $\bar\beta$ labelling the gauge potential and the anti-ghost field, all colour-stripped. Working out all other structure constants is a straightforward but tedious process; since no more insights would be obtained from it, we refrain from listing them here. We only note that for Yang–Mills theory, the ambiguity in assigning terms to q and $\bar q$ is absent.

Double copy. We now note that the decomposition of the Lagrangian matches precisely the decomposition of scattering amplitudes in the discussion of colour–kinematics duality, cf. Section 1.3., which is the starting point for the double copy. We merely extended the factorisation of the interaction vertices to a factorisation of the whole BRST structure.

In the usual double copy, we start from the factorisation for Yang–Mills theory and replace the colour factor by a kinematic factor. More generally, however, we can certainly replace any one of the (graded) vector spaces $\mathfrak{V}$ and $\bar{\mathfrak{V}}$ and the corresponding structure constants with (graded) vector spaces and structure constants from other theories. This gives us a new action, which we shall denote by $\tilde{S}^{\rm DC}_{\rm BRST}$. The corresponding BRST operator $\tilde{Q}^{\rm DC}_{\rm BRST}$ is obtained by replacing one set of kinematic structure constants in the decomposition of the BRST operator (9.23) with those from the new factor.

BRST Lagrangian double copy. In order to obtain a consistent and quantisable theory, we demand the new BRST structure to be consistent. Specifically,

$$\tilde{Q}^{\rm DC}_{\rm BRST}\tilde{S}^{\rm DC}_{\rm BRST} = 0 \quad \text{and} \quad (\tilde{Q}^{\rm DC}_{\rm BRST})^2 = 0 \,. \tag{9.27}$$

By construction, we have again a decomposition $\tilde{Q}^{\rm DC}_{\rm BRST} =: \tilde{q}^{\rm DC}_{\rm BRST} + \tilde{\bar{q}}^{\rm DC}_{\rm BRST}$. The condition $Q^2_{\rm BRST} = 0$ implies $q^2_{\rm BRST} = 0$, and we decompose the latter into linear, quadratic, and cubic terms in the fields,

$$q^2_{\rm BRST}\Phi^{\cdots} =: q^{(2,0)}_1 + q^{(2,0)}_2 + q^{(2,0)}_3 \ , \tag{9.28}$$

and analogously for $\bar{q}^2_{\rm BRST}$, $(\tilde{q}^{\rm DC}_{\rm BRST})^2$, and $(\tilde{\bar{q}}^{\rm DC}_{\rm BRST})^2$, respectively. Schematically, the summands read as

$$\begin{aligned} q^{(2,0)}_1 &= \cdots \mathsf{q}^\alpha_\beta \mathsf{q}^\beta_\gamma \cdots \ , \\ q^{(2,0)}_2 &= \cdots (\mathsf{q}^\alpha_\delta \mathsf{q}^\delta_{\beta\gamma} + \mathsf{q}^\delta_\beta \mathsf{q}^\alpha_{\delta\gamma} \pm \mathsf{q}^\delta_\gamma \mathsf{q}^\alpha_{\beta\delta})\bar{\mathsf{f}}^{\bar\alpha}_{\bar\beta\bar\gamma} \cdots \ , \\ q^{(2,0)}_3 &= \cdots (\mathsf{q}^\varepsilon_{\beta\gamma} \mathsf{q}^\alpha_{\varepsilon\delta} \bar{\mathsf{f}}^{\bar\varepsilon}_{\bar\beta\bar\gamma} \bar{\mathsf{f}}^{\bar\alpha}_{\bar\varepsilon\bar\delta} \pm \mathsf{q}^\varepsilon_{\beta\gamma} \mathsf{q}^\alpha_{\delta\varepsilon} \bar{\mathsf{f}}^{\bar\varepsilon}_{\bar\beta\bar\gamma} \bar{\mathsf{f}}^{\bar\alpha}_{\bar\delta\bar\varepsilon}) \cdots \ , \end{aligned} \tag{9.29a}$$

and

$$\begin{aligned} \tilde{q}^{(2,0)}_1 &= \cdots \mathsf{q}^\alpha_\beta \mathsf{q}^\beta_\gamma \cdots \ , \\ \tilde{q}^{(2,0)}_2 &= \cdots (\mathsf{q}^\alpha_\delta \mathsf{q}^\delta_{\beta\gamma} + \mathsf{q}^\delta_\beta \mathsf{q}^\alpha_{\delta\gamma} \pm \mathsf{q}^\delta_\gamma \mathsf{q}^\alpha_{\beta\delta})\tilde{\bar{\mathsf{f}}}^{\bar\alpha}_{\bar\beta\bar\gamma} \cdots \ , \\ \tilde{q}^{(2,0)}_3 &= \cdots (\mathsf{q}^\varepsilon_{\beta\gamma} \mathsf{q}^\alpha_{\varepsilon\delta} \tilde{\bar{\mathsf{f}}}^{\bar\varepsilon}_{\bar\beta\bar\gamma} \tilde{\bar{\mathsf{f}}}^{\bar\alpha}_{\bar\varepsilon\bar\delta} \pm \mathsf{q}^\varepsilon_{\beta\gamma} \mathsf{q}^\alpha_{\delta\varepsilon} \tilde{\bar{\mathsf{f}}}^{\bar\varepsilon}_{\bar\beta\bar\gamma} \tilde{\bar{\mathsf{f}}}^{\bar\alpha}_{\bar\delta\bar\varepsilon}) \cdots \ , \end{aligned} \tag{9.29b}$$

where $\tilde{\mathsf{f}}^\alpha_{\beta\gamma}$ and $\tilde{\bar{\mathsf{f}}}^{\bar\alpha}_{\bar\beta\bar\gamma}$ denote the kinematic constants in $\tilde{S}^{\rm DC}_{\rm BRST}$. It is now clear that $\tilde{q}^{(2,0)}_1$ and $\tilde{q}^{(2,0)}_2$ vanish if $q^2_{\rm BRST} = 0$ and thus, $q^{(2,0)}_1$ and $q^{(2,0)}_2$ vanish on arbitrary fields.

So far, our discussion was fairly general and nothing singled out colour–kinematics-dual theories from other theories. This changes with the condition that $q_3^{(2,0)} = 0$ must imply $\tilde{q}_3^{(2,0)} = 0$. Vanishing of $q_3^{(2,0)}$ relies on a transfer of the symmetry properties of the open indices of $\bar{\mathsf{f}}^{\bar{\varepsilon}}_{\bar{\beta}\bar{\gamma}}\bar{\mathsf{f}}^{\bar{\alpha}}_{\bar{\varepsilon}\bar{\delta}}$ and $\bar{\mathsf{f}}^{\bar{\varepsilon}}_{\bar{\beta}\bar{\gamma}}\bar{\mathsf{f}}^{\bar{\alpha}}_{\bar{\delta}\bar{\varepsilon}}$ via the contracting fields (in which the expression is totally symmetric) to $\mathsf{q}^{\varepsilon}_{\beta\gamma}\mathsf{q}^{\alpha}_{\varepsilon\delta}$ and $\mathsf{q}^{\varepsilon}_{\beta\gamma}\mathsf{q}^{\alpha}_{\delta\varepsilon}$. It follows that if the symmetry properties of the open indices in the terms quadratic in $\bar{\mathsf{f}}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}$ are the same as for the terms quadratic in $\tilde{\bar{\mathsf{f}}}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}$ then $\tilde{q}_3^{(2,0)} = 0$. The colour–kinematics duality provides such a condition.

The same argument shows that $(\tilde{\bar{q}}^{\rm DC}_{\rm BRST})^2 = 0$, and we can directly turn to the cross terms and split them again into linear, quadratic, and cubic pieces,

$$(q_{\rm BRST}\bar{q}_{\rm BRST} + \bar{q}_{\rm BRST}q_{\rm BRST})\Phi^{\cdots} \;=:\; q_1^{(1,1)} + q_2^{(1,1)} + q_3^{(1,1)}\;, \tag{9.30a}$$

and

$$(\tilde{q}^{\rm DC}_{\rm BRST}\tilde{\bar{q}}^{\rm DC}_{\rm BRST} + \tilde{\bar{q}}^{\rm DC}_{\rm BRST}\tilde{q}^{\rm DC}_{\rm BRST})\Phi^{\cdots} \;=:\; \tilde{q}_1^{(1,1)} + \tilde{q}_2^{(1,1)} + \tilde{q}_3^{(1,1)}\;. \tag{9.30b}$$

We note that the conditions $q_1^{(1,1)} = 0$ and $\tilde{q}_1^{(1,1)} = 0$ are implied directly when q_1 and $\bar{q}_1$ and $\tilde{q}_1$ and $\tilde{\bar{q}}_1$ anti-commute, respectively, which is always the case in the theories we study. Moreover, we have, again schematically, the conditions

$$\begin{aligned} q_2^{(1,1)} &= \cdots\mathsf{q}^{\alpha}_{\beta\gamma}(\bar{\mathsf{q}}^{\bar{\alpha}}_{\bar{\delta}}\bar{\mathsf{f}}^{\bar{\delta}}_{\bar{\beta}\bar{\gamma}} \pm \bar{\mathsf{q}}^{\bar{\delta}}_{\bar{\beta}}\bar{\mathsf{f}}^{\bar{\alpha}}_{\bar{\delta}\bar{\gamma}} \pm \bar{\mathsf{q}}^{\bar{\delta}}_{\bar{\gamma}}\bar{\mathsf{f}}^{\bar{\alpha}}_{\bar{\beta}\bar{\delta}})\cdots + \cdots\bar{\mathsf{q}}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}(\mathsf{q}^{\alpha}_{\delta}\mathsf{f}^{\delta}_{\beta\gamma} \pm \mathsf{q}^{\delta}_{\beta}\mathsf{f}^{\alpha}_{\delta\gamma} \pm \mathsf{q}^{\delta}_{\gamma}\mathsf{f}^{\alpha}_{\beta\delta})\cdots\;,\\ q_3^{(1,1)} &= \cdots(\mathsf{q}^{\alpha}_{\varepsilon\delta}\bar{\mathsf{f}}^{\bar{\alpha}}_{\bar{\varepsilon}\bar{\delta}}\mathsf{f}^{\varepsilon}_{\beta\gamma}\bar{\mathsf{q}}^{\bar{\varepsilon}}_{\bar{\beta}\bar{\gamma}} \pm \mathsf{q}^{\alpha}_{\beta\varepsilon}\bar{\mathsf{f}}^{\bar{\alpha}}_{\bar{\beta}\bar{\varepsilon}}\mathsf{f}^{\varepsilon}_{\gamma\delta}\bar{\mathsf{q}}^{\bar{\varepsilon}}_{\bar{\gamma}\bar{\delta}} \pm \mathsf{f}^{\alpha}_{\varepsilon\delta}\bar{\mathsf{q}}^{\bar{\alpha}}_{\bar{\varepsilon}\bar{\delta}}\mathsf{q}^{\varepsilon}_{\beta\gamma}\bar{\mathsf{f}}^{\bar{\varepsilon}}_{\bar{\beta}\bar{\gamma}} \pm \mathsf{f}^{\alpha}_{\beta\varepsilon}\bar{\mathsf{q}}^{\bar{\alpha}}_{\bar{\beta}\bar{\varepsilon}}\mathsf{q}^{\varepsilon}_{\gamma\delta}\bar{\mathsf{f}}^{\bar{\varepsilon}}_{\bar{\gamma}\bar{\delta}})\cdots\;. \end{aligned} \tag{9.31}$$

We see that $q_2^{(1,1)} = 0$ splits into two separate conditions on the indices in $\mathfrak{V}$ and $\bar{\mathfrak{V}}$ and thus it implies $\tilde{q}_2^{(1,1)} = 0$. The condition $\tilde{q}_3^{(1,1)} = 0$ can, in principle, be non-trivial, but again colour–kinematics duality as well as the special form of the BRST operator in the theories in which we are interested renders $\tilde{q}_3^{(1,1)} = 0$ equivalent to $q_3^{(1,1)} = 0$.

Finally, we have to check that $\tilde{Q}^{\rm DC}_{\rm BRST}\tilde{S}^{\rm DC}_{\rm BRST} = 0$, and we consider

$$q_{\rm BRST}S \;=:\; s_2^{(1,0)} + s_3^{(1,0)} + s_4^{(1,0)}\;, \tag{9.32}$$

where $s_2^{(1,0)}$, $s_3^{(1,0)}$, and $s_4^{(1,0)}$ are quadratic, cubic, and quartic in the fields. Analogously, we have $\tilde{q}^{\rm DC}_{\rm BRST}\tilde{S}^{\rm DC}_{\rm BRST} =: \tilde{s}_2^{(1,0)} + \tilde{s}_3^{(1,0)} + \tilde{s}_4^{(1,0)}$, and the discussion for $\bar{q}_{\rm BRST}$ and $\tilde{\bar{q}}^{\rm DC}_{\rm BRST}$ is similar. Schematically, we compute

$$\begin{aligned} s_2^{(1,0)} &= \int \mathrm{d}^d x\,\cdots(\mathsf{q}^{\gamma}_{\alpha}\mathsf{g}_{\gamma\beta}\bar{\mathsf{g}}_{\bar{\alpha}\bar{\beta}}\Box)\cdots\;,\\ s_3^{(1,0)} &= \int \mathrm{d}^d x\,\cdots(\mathsf{g}_{\alpha\delta}\Box\,\mathsf{q}^{\delta}_{\beta\gamma} + \mathsf{f}_{\alpha\delta\gamma}\mathsf{q}^{\delta}_{\beta} + \mathsf{f}_{\alpha\beta\delta}\mathsf{q}^{\delta}_{\beta})\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\cdots\;,\\ s_4^{(1,0)} &= \int \mathrm{d}^d x\,\cdots(\mathsf{f}_{\alpha\varepsilon\delta}\mathsf{q}^{\varepsilon}_{\beta\gamma}\bar{\mathsf{f}}_{\bar{\alpha}\bar{\varepsilon}\bar{\delta}}\bar{\mathsf{f}}^{\bar{\varepsilon}}_{\bar{\beta}\bar{\gamma}} + \mathsf{f}_{\alpha\beta\varepsilon}\mathsf{q}^{\varepsilon}_{\gamma\delta}\bar{\mathsf{f}}_{\bar{\alpha}\bar{\beta}\bar{\varepsilon}}\bar{\mathsf{f}}^{\bar{\varepsilon}}_{\bar{\gamma}\bar{\delta}})\cdots\;, \end{aligned} \tag{9.33}$$

where we have assumed that q_{BRST} commutes with the differential and bi-differential operators in the action, which is the case in all our theories. We see that $s_2^{(1,0)} = 0$ and $s_3^{(1,0)} = 0$ imply $\tilde{s}_2^{(1,0)} = 0$ and $\tilde{s}_3^{(1,0)} = 0$, respectively. The relation $\tilde{s}_4^{(1,0)} = 0$ can, in principle, lead to additional conditions. In a theory with colour–kinematics duality, however, the contraction of the kinematic structure constants $\bar{\mathsf{f}}_{\bar{\beta}\bar{\gamma}}^{\bar{\alpha}}$ appears as in the Jacobi identity, and $s_4^{(1,0)}$ as well as $\tilde{s}_4^{(1,0)}$ vanish automatically.

In general, if we have a theory where $Q_{\text{BRST}}^2 = 0$, $Q_{\text{BRST}} S = 0$ are satisfied only because of the algebraic properties of the structure constants, and if we replace a set of structure constants with a new set of structure constants that obey the same algebraic properties of the old ones, we obtain an action $\tilde{S}$ and a BRST operator $\tilde{Q}_{\text{BRST}}$ such that $\tilde{Q}_{\text{BRST}}^2 = 0$, $\tilde{Q}_{\text{BRST}}\tilde{S} = 0$. Colour–kinematic duality provides precisely this condition.

Partial BRST Lagrangian double copy. There are few theories where we expect the BRST Lagrangian double copy to work perfectly. The reason is that in most formulations, colour–kinematics duality will not hold. In Yang–Mills theory, for example, it is not known if colour–kinematics duality can be made manifest for off-shell fields.[1]

Now if colour–kinematics duality fails to hold up to certain terms, say the ideal of functions of the fields vanishing on-shell as in the case of Yang–Mills theory, then the equation $\tilde{Q}_{\text{BRST}}^{\text{DC}}\tilde{S}_{\text{BRST}}^{\text{DC}} = 0$ will also fail to hold up to the same ideal. Consequently, $\tilde{Q}_{\text{BRST}}^{\text{DC}}\tilde{S}_{\text{BRST}}^{\text{DC}}$ is a product of factors whose vanishing amounts to the equations of motion possibly multiplied by other fields and their derivatives.

9.4. BRST Lagrangian double copy of Yang–Mills theory

After the above general discussion, we now focus our attention on the instance of BRST Lagrangian double copy that constitute the main object of our interest:

$$\tilde{\mathfrak{L}}_{\text{BRST}}^{\text{DC}} := \mathfrak{Kin}^{\text{st}} \otimes_{\tau} (\mathfrak{Kin}^{\text{st}} \otimes_{\tau} \mathfrak{Scal}) , \tag{9.34}$$

where $\mathfrak{Kin}^{\text{st}}$ is given in Equation (9.12) and $\mathfrak{Scal}$ in Equation (9.4), respectively.

[1]Recall that we only extended colour–kinematics to the BRST-extended Hilbert space in Section 8.4., but with all fields still on-shell.

Field content. From the analysis at the level of cochain complexes in Section 7.5., we already know that the field content of double-copied BRST-extended Hilbert space of Yang–Mills theory agrees with the field content of the BRST-extended Hilbert space of $\mathcal{N}=0$ supergravity. We shall continue to label fields as in Table 7.5.

However, when we consider the homotopy algebra associated to the full, interactive picture, we have an additional infinite tower of auxiliary fields, coming from the infinitely many additional auxiliary fields of colour–kinematics duality preserving, strictified Yang–Mills theory. In Chapter 8, we wrote explicitly five of the auxiliary fields in Yang–Mills theory,

$$\tilde{K}_1^{a\mu}\,,\quad \tilde{\tilde{K}}_1^{a\mu}\,,\quad G^a_{\mu\nu\kappa}\,,\quad \tilde{K}^{2a}_\mu\,,\quad \tilde{\tilde{K}}^{2a}_\mu\,, \tag{9.35}$$

which correspond to the additional basis elements

$$\mathsf{t}^1_\mu\,,\quad \bar{\mathsf{t}}^\mu_1\,,\quad \mathsf{t}^{\mu\nu\kappa}_0\,,\quad \mathsf{t}^2_\mu\,,\quad \bar{\mathsf{t}}^\mu_2 \tag{9.36}$$

in $\mathfrak{Kin}^{\mathrm{st}}$. From them, the tensor product (9.34) produces 40 auxiliary fields involving one auxiliary kinetic factor and another 25 auxiliary fields involving two auxiliary kinetic factors. Instead of giving these auxiliary fields individual labels, we collectively denote them by ${}_{\mathtt{k}_1}\Upsilon_{\mathtt{k}_2}$, where $\mathtt{k}_1$ and $\mathtt{k}_2$ denote the first and second kinematic factors, respectively. For example,

$$\begin{aligned}
{}_{\mathrm{g}}\Upsilon_{\mathrm{g}} &:= \mathrm{g}\otimes\mathrm{g}\otimes\left(\int \mathrm{d}^dx\,\mathrm{s}_x\varphi^{\mathrm{gg}}(x)\right) = \tilde{\lambda}\,,\\
{}_{\mathrm{v}}\Upsilon_{\mathrm{v}} &:= \mathrm{e}_a\otimes\mathrm{v}^\mu\otimes\mathrm{v}^\nu\otimes\left(\int \mathrm{d}^dx\,\mathrm{s}_x\varphi^{\mathrm{vv}}{}_{\mu\nu}(x)\right) = \tilde{h}+\tilde{B}\,,\\
{}_{\mathrm{t}^1}\Upsilon_{\mathrm{t}_0} &:= \mathrm{t}^1_\mu\otimes\mathrm{t}_0^{\nu\kappa\lambda}\otimes\left(\int \mathrm{d}^dx\,\mathrm{s}_x\varphi^{\mathrm{t}^1\mathrm{t}_0\mu}{}_{\nu\kappa\lambda}(x)\right)\,.
\end{aligned} \tag{9.37}$$

Higher products. The twist (9.14a) and (9.14b) determines the products μ_1 and μ_2 between the elements of $\tilde{\mathfrak{L}}^{\mathrm{DC}}_{\mathrm{BRST}}$. The formulas from Section 6.3. with all the appropriate signs included read as

$$\begin{aligned}
&\mu_1(\mathrm{x}_1\otimes\mathrm{y}_1\otimes\varphi_1) := (-1)^{|\tau_1^{(1)}(\mathrm{x}_1)|+|\tau_1^{(1)}(\mathrm{y}_1)|}\,\tau_1^{(1)}(\mathrm{x}_1)\otimes\tau_1^{(1)}(\mathrm{y}_1)\otimes\left(\tau_1^{(2)}(\mathrm{x}_1)(\tau_2^{(2)}(\mathrm{y}_1)(\varphi_1))\right)\,,\\
&\mu_2(\mathrm{x}_1\otimes\mathrm{y}_1\otimes\varphi_1\,,\,\mathrm{x}_2\otimes\mathrm{y}_2\otimes\varphi_2) :=\\
&\quad := (-1)^{(|\mathrm{y}_1|+|\varphi_1|)|\mathrm{x}_2|+|\varphi_1|\,|\mathrm{y}_2|}\times\\
&\qquad\times\tau_2^{(1)}(\mathrm{x}_1,\mathrm{x}_2)\otimes\tau_2^{(1)}(\mathrm{y}_1,\mathrm{y}_2)\otimes\left(\tau_2^{(2)}(\mathrm{x}_1,\mathrm{x}_2)\varphi_1(x)\right)\left(\tau_2^{(2)}(\mathrm{y}_1,\mathrm{y}_2)\varphi_2(x)\right)\,.
\end{aligned} \tag{9.38}$$

Note that there are no additional signs because our $\tau_i^{(2)}$ are always even. While the computation is readily performed, listing the higher products for all 81 fields is not particularly enlightening.

Action. The factorisation (9.34) induces the following cyclic structure:

$$\begin{aligned}&\langle \mathrm{x}_1 \otimes \mathrm{y}_1 \otimes \varphi_1, \mathrm{x}_2 \otimes \mathrm{y}_2 \otimes \varphi_2 \rangle := \\ &\quad := (-1)^{|\mathrm{x}_2|_{\mathfrak{Kin}}(|\mathrm{y}_1|_{\mathfrak{Kin}}+|\varphi_1|_{\mathfrak{Scal}})+|\mathrm{x}_2|_{\mathfrak{Kin}}|\varphi_1|_{\mathfrak{Scal}}} \langle \mathrm{x}_1, \mathrm{x}_2 \rangle \langle \mathrm{y}_1, \mathrm{y}_2 \rangle \langle \varphi_1, \varphi_2 \rangle \ .\end{aligned} \tag{9.39}$$

Together with the formulas for the super homotopy Maurer–Cartan action (3.27), we can compute the (gauge-fixed) BRST action corresponding to the L_∞-algebra $\tilde{\mathfrak{L}}^{\mathrm{DC}}_{\mathrm{BRST}}$. Again, listing all the terms would not provide much insight, but we stress that we obtain all the expected terms, in particular the lowest terms of the Fierz–Pauli version of the $\mathcal{N} = 0$ supergravity action as well as the evident terms involving ghosts.

Double copy of the BRST operator. We now consider the double copy of the BRST operator to a BRST operator $\tilde{Q}^{\mathrm{DC}}_{\mathrm{BRST}}$. For our purposes, the double copy of the linearised part without considering the auxiliary fields will be sufficient. We start from Yang–Mills theory with the factors $\mathfrak{V} := \mathfrak{g}$ and $\bar{\mathfrak{V}} := \mathfrak{Kin}$ in (9.17) and the usual BRST relations in terms of coordinate functions on $\tilde{\mathfrak{L}}^{\mathrm{YM}}_{\mathrm{BRST}}$,

$$\begin{aligned}&\tilde{A}^a_\mu \xmapsto{Q^{\mathrm{YM,\,lin}}_{\mathrm{BRST}}} \delta^a_b \partial_\mu \tilde{c}^b \ , \qquad \tilde{b}^a \xmapsto{Q^{\mathrm{YM,\,lin}}_{\mathrm{BRST}}} \delta^a_b \frac{1-\sqrt{1-\xi}}{\sqrt{\xi}} \sqrt{\Box}\, \tilde{c}^b \ , \\ &\tilde{c}^a \xmapsto{Q^{\mathrm{YM,\,lin}}_{\mathrm{BRST}}} 0 \ , \qquad \tilde{\bar{c}}^a \xmapsto{Q^{\mathrm{YM,\,lin}}_{\mathrm{BRST}}} \delta^a_b \left(\sqrt{\frac{\Box}{\xi}} \tilde{b}^b - \frac{1-\sqrt{1-\xi}}{\xi} \partial^\mu \tilde{A}^b_\mu \right) .\end{aligned} \tag{9.40}$$

We thus have $\mathsf{q}^\alpha_\beta = \delta^\alpha_\beta$, and the non-vanishing components of $\bar{\mathsf{q}}^{\bar{\alpha}}_{\bar{\beta}}$ are given by

$$\bar{\mathsf{q}}^{\bar{\alpha}}_{\bar{\beta}} = \begin{cases} \partial_\mu & \text{for} \quad \bar{\alpha} = \mathrm{g}^* \ , \quad \bar{\beta} = \mathrm{v}^*_\mu \\ \frac{1-\sqrt{1-\xi}}{\sqrt{\xi}} \sqrt{\Box} & \text{for} \quad \bar{\alpha} = \mathrm{g}^* \ , \quad \bar{\beta} = \mathrm{n}^* \\ \sqrt{\frac{\Box}{\xi}} & \text{for} \quad \bar{\alpha} = \mathrm{n}^* \ , \quad \bar{\beta} = \mathrm{a}^* \\ -\frac{1-\sqrt{1-\xi}}{\xi} \partial^\mu & \text{for} \quad \bar{\alpha} = \mathrm{v}^*_\mu \ , \quad \bar{\beta} = \mathrm{a}^* \end{cases} . \tag{9.41}$$

After the double copy, we have $\mathfrak{V} := \mathfrak{Kin} =: \bar{\mathfrak{V}}$ and, correspondingly, $\mathsf{q}^\alpha_\beta = \bar{\mathsf{q}}^\alpha_\beta$. The linearisation of the double-copied BRST operator is then non-trivial on a field containing a factor of $\mathtt{v}^\mu$ or $\mathtt{a}$ and we have in the anti-symmetrised sector

$$
\begin{aligned}
\tilde{\lambda} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} 0 \, , \\
\tilde{\Lambda}_\mu &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} \partial_\mu \tilde{\lambda} \, , \\
\tilde{\gamma} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} \frac{1-\sqrt{1-\xi}}{\sqrt{\xi}} \sqrt{\Box}\,\tilde{\lambda} \, , \\
\tilde{B}_{\mu\nu} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} \partial_\mu \tilde{\Lambda}_\nu - \partial_\nu \tilde{\Lambda}_\mu \, , \\
\tilde{\alpha}_\mu &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} \frac{1-\sqrt{1-\xi}}{\sqrt{\xi}} \sqrt{\Box}\,\tilde{\Lambda}_\mu - \partial_\mu \tilde{\gamma} \, , \\
\tilde{\varepsilon} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} \sqrt{\frac{\Box}{\xi}} \tilde{\gamma} - \frac{1-\sqrt{1-\xi}}{\xi} \partial^\mu \tilde{\Lambda}_\mu \, , \\
\tilde{\bar{\Lambda}}_\mu &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} \partial_\mu \tilde{\varepsilon} + \sqrt{\frac{\Box}{\xi}} \tilde{\alpha}_\mu - \frac{1-\sqrt{1-\xi}}{\xi} \partial^\nu \tilde{B}_{\mu\nu} \, , \\
\tilde{\bar{\gamma}} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} \frac{1-\sqrt{1-\xi}}{\sqrt{\xi}} \sqrt{\Box}\,\tilde{\varepsilon} + \frac{1-\sqrt{1-\xi}}{\xi} \partial^\mu \tilde{\alpha}_\mu \, , \\
\tilde{\bar{\lambda}} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} \sqrt{\frac{\Box}{\xi}} \tilde{\bar{\gamma}} - \frac{1-\sqrt{1-\xi}}{\xi} \partial^\mu \tilde{\bar{\Lambda}}_\mu \, ,
\end{aligned}
\tag{9.42a}
$$

and in the symmetrised sector

$$
\begin{aligned}
\tilde{X}^\mu &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} 0 \, , \\
\tilde{\beta} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} 0 \, , \\
\tilde{h}_{\mu\nu} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} \partial_\mu \tilde{X}_\nu + \partial_\nu \tilde{X}_\mu \, , \\
\tilde{\varpi}^\mu &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} -\frac{1-\sqrt{1-\xi}}{\sqrt{\xi}} \sqrt{\Box}\,\tilde{X}^\mu - \partial^\mu \tilde{\beta} \, , \\
\tilde{\pi} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} 2\frac{1-\sqrt{1-\xi}}{\sqrt{\xi}} \sqrt{\Box}\tilde{\beta} \, , \\
\tilde{\delta} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} \sqrt{\frac{\Box}{\xi}} \tilde{\beta} - \frac{1-\sqrt{1-\xi}}{\xi} \partial_\mu \tilde{X}^\mu \, , \\
\tilde{\bar{X}}^\mu &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} -\partial^\mu \tilde{\delta} - \sqrt{\frac{\Box}{\xi}} \tilde{\varpi}^\mu - \frac{1-\sqrt{1-\xi}}{\xi} \partial_\nu \tilde{h}^{\nu\mu} \, , \\
\tilde{\bar{\beta}} &\xmapsto{\tilde{Q}^{\text{DC, lin}}_{\text{BRST}}} -\frac{1-\sqrt{1-\xi}}{\sqrt{\xi}} \sqrt{\Box}\,\tilde{\delta} + \frac{1-\sqrt{1-\xi}}{\xi} \partial_\mu \tilde{\varpi}^\mu + \sqrt{\frac{\Box}{\xi}} \tilde{\pi} \, .
\end{aligned}
\tag{9.42b}
$$

Importantly, this BRST operator is related to the usual linearised BRST operator for $\mathcal{N}=0$ supergravity, (4.23) and (4.30),

$$
\begin{aligned}
\lambda &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} 0\,, & \varphi &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} 0\,,\\
\Lambda_\mu &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} \partial_\mu\lambda\,, & X^\mu &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} 0\,,\\
\gamma &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} 0\,, & \beta &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} 0\,,\\
B_{\mu\nu} &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} \partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu\,, & h_{\mu\nu} &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} \partial_\mu X_\nu+\partial_\nu X_\mu\,,\\
\alpha_\mu &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} 0\,, & \varpi^\mu &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} 0\,,\\
\varepsilon &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} \gamma\,, & \delta &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} \beta\,,\\
\bar\Lambda_\mu &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} \alpha_\mu\,, & \bar X^\mu &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} \varpi^\mu\,,\\
\bar\gamma &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} 0\,, & \bar\beta &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} \pi\,,\\
\bar\lambda &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} \bar\gamma\,, & \pi &\xrightarrow{Q_{\text{BRST}}^{\mathcal{N}=0,\,\text{lin}}} 0\,.
\end{aligned}
\tag{9.42c}
$$

by the field redefinitions (7.22) and (7.27), respectively.

9.5. Equivalence of the double copied action and $\mathcal{N}=0$ supergravity

We now conclude this final Chapter by showing that the double copied action $\tilde{S}^{\text{DC}}_{\text{BRST}}$ we constructed in Section 9.4. is fully perturbatively quantum equivalent to the suitably gauge fixed version of the BV action of $\mathcal{N}=0$ supergravity, $S^{\mathcal{N}=0}_{\text{BRST}}$, defined in Section 4.6.. For this, we have to show that up to a field redefinition, both theories have the same tree-level correlation functions. A crucial point in our discussion will be the BRST Lagrangian double copy formalism developed in the previous section.

Before exposing with our argument, let us introduce some terminology: we shall speak of 'auxiliary fields connected to a field ϕ' by which we mean all auxiliary fields which appear together with ϕ in Feynman diagrams containing only propagators of auxiliary fields. In other terms, an auxiliary field ψ connected to a field ϕ can have an interaction vertex with ϕ or interact with an auxiliary field that propagates to an auxiliary field that non-trivially interacts with ϕ, etc.:

(9.43)

where a dashed line denotes an auxiliary field. We also use the adjectives *physical* and *unphysical* when referring to fields, interaction terms and scattering amplitudes. The unphysical fields are all ghosts, anti-ghosts, and Nakanishi–Lautrup fields as well as auxiliary fields connected to these. Physical fields are the remaining fields, consisting of the metric, the Kalb–Ramond field and the dilaton as well as a number of auxiliary fields. Physical interaction vertices are those consisting exclusively of physical fields. Physical scattering amplitudes are those with physical fields as external labels.

Physical tree-level scattering amplitudes. We first note that the auxiliary fields in the double copied action $\tilde{S}^{\rm DC}_{\rm BRST}$ can be integrated out, after which the field content and the kinematic terms in both actions fully agree, up to the field redefinitions we discussed in Chapter 7. Implementing these field redefinitions on $S^{\mathcal{N}=0}_{\rm BRST}$, we obtain the action $S^{\mathcal{N}=0}_{\rm BRST,\,0}$.

Moreover, the physical tree-level scattering amplitudes computed from the interaction vertices of the action $\tilde{S}^{\rm DC}_{\rm BRST}$ are by design precisely those arising in the usual double copy prescription for the construction of $\mathcal{N} = 0$ supergravity tree amplitudes from a factorisation of Yang–Mills amplitudes. The tree-level double copy has been demonstrated to hold, cf. Observation 8.12, and therefore the physical tree-level scattering amplitudes of $\tilde{S}^{\rm DC}_{\rm BRST}$ and $S^{\mathcal{N}=0}_{\rm BRST,\,0}$ agree.

If we put all unphysical fields to zero, the resulting theories $\tilde{S}^{\rm DC}_{\rm BRST,\,phys}$ and $S^{\mathcal{N}=0}_{\rm BRST,\,phys}$ are classically equivalent by Observation 8.9. In the homotopy algebraic picture, this corresponds to a restriction $\mathfrak{L}^{\mathcal{N}=0}_{\rm BRST,\,phys}$ and $\tilde{\mathfrak{L}}^{\rm DC}_{\rm phys}$ to two quasi-isomorphic L_∞-subalgebras.

In order to improve this restricted classical equivalence to a full perturbative quantum equivalence, we need to adjust and modify the actions or, equivalently, the corresponding L_∞-algebras. We shall do this now in a sequence of steps, expanding the discussion in [5].

Auxiliary fields of ghost number zero. The reformulation of the tree-level scattering amplitudes of $\mathcal{N} = 0$ supergravity used in the double copy defines a local strictification of the physical part of the action $S^{\mathcal{N}=0}_{\mathrm{BRST}}$ to the action $S^{\mathcal{N}=0}_{\mathrm{BRST},1}$ by promoting all cubic interaction vertices to cubic interaction terms. This is fully analogous to the strictification implied by the manifestly colour–kinematics-dual form of the Yang–Mills action explained in Section 8.3..

By construction, the actions $S^{\mathcal{N}=0}_{\mathrm{BRST},1}$ and $\tilde{S}^{\mathrm{DC}}_{\mathrm{BRST,\,phys}}$ have the same field content, the same kinematical terms for the physical and auxiliary fields and identical tree-level scattering amplitudes for the physical fields.

Let us now consider the tree-level scattering amplitudes which have auxiliary fields of ghost number zero on their external legs. Such amplitudes are fully determined by the (iterated) collinear limits of physical tree-level scattering amplitudes. Because, again, the physical tree-level scattering amplitudes of $S^{\mathcal{N}=0}_{\mathrm{BRST},1}$ and $\tilde{S}^{\mathrm{DC}}_{\mathrm{BRST,\,phys}}$ agree, the tree-level scattering amplitudes with physical and auxiliary fields of ghost number zero on external legs agree.

By Observation 8.9, we then have a local field redefinition of $S^{\mathcal{N}=0}_{\mathrm{BRST},1}$ to $S^{\mathcal{N}=0}_{\mathrm{BRST},2}$ such that both actions agree after all fields except for physical ones and auxiliary fields of ghost number zero are put to zero. If we integrated out all auxiliary fields in both actions, the resulting actions would agree in their purely physical parts.

Nakanishi–Lautrup fields. In the next step, we deal with the difference between $\tilde{S}^{\mathrm{DC}}_{\mathrm{BRST}}$ and $S^{\mathcal{N}=0}_{\mathrm{BRST},2}$ proportional to any of the Nakanishi–Lautrup fields $(\tilde{\bar{\beta}}, \tilde{\varpi}_\mu, \tilde{\pi}, \tilde{\gamma}, \tilde{\alpha}_\mu, \tilde{\bar{\gamma}})$; we shall come to the ghost field β later. After integrating out all auxiliary fields, this difference can be compensated by Observation 8.8. That is, we can modify the gauge-fixing fermion and perform a field redefinition of the Nakanishi–Lautrup fields such that this difference is removed. We note that neither of these two processes modifies the physical parts of the action and both preserve quantum equivalence. We can thus replace all terms in $S^{\mathcal{N}=0}_{\mathrm{BRST},2}$ containing Nakanishi–Lautrup fields by the terms in $\tilde{S}^{\mathrm{DC}}_{\mathrm{BRST}}$ containing Nakanishi–Lautrup fields as well as auxiliary fields connected to Nakanishi–Lautrup fields. We call the resulting action $S^{\mathcal{N}=0}_{\mathrm{BRST},3}$.

Recall that there is a ghost number -2 field $\bar{\lambda}$ which is paired with the Nakanishi–Lautrup-type field γ in the gauge fixing fermion (4.26), allowing us to absorb any term proportional to γ in a different gauge choice. This is not the case for the corresponding Nakanishi–Lautrup-type field in the gravity sector, β. Any discrepancy proportional to β

between $S^{\mathcal{N}=0}_{\mathrm{BRST},3}$ and $\tilde{S}^{\mathrm{DC}}_{\mathrm{BRST}}$ (again, after integrating out all the auxiliary fields) should instead be absorbed by shifting the gauge fixing fermion Ψ from (4.38) by a term δP, where βP is the discrepancy. This will generate the desired corrections. This will also lead to new ghost terms, which we will treat separately in the next step.

Ghost sector. Let us now examine the ghost interactions. We know that the action $S^{\mathcal{N}=0}_{\mathrm{BRST},3}$ comes with a BRST operators $Q^{\mathcal{N}=0}_{\mathrm{BRST},3}$ which satisfies

$$(Q^{\mathcal{N}=0}_{\mathrm{BRST},3})^2 \;=\; 0 \quad \text{and} \quad Q^{\mathcal{N}=0}_{\mathrm{BRST},3} S^{\mathcal{N}=0}_{\mathrm{BRST},3} \;=\; 0\ . \tag{9.44}$$

From our discussion in the previous section, we know that the double-copied BRST operator $\tilde{Q}^{\mathrm{DC}}_{\mathrm{BRST}}$ satisfies

$$(\tilde{Q}^{\mathrm{DC}}_{\mathrm{BRST}})^2 \;\in\; \mathscr{I} \quad \text{and} \quad \tilde{Q}^{\mathrm{DC}}_{\mathrm{BRST}} \tilde{S}^{\mathrm{DC}}_{\mathrm{BRST}} \;\in\; \mathscr{I}\ , \tag{9.45}$$

where $\mathscr{I}$ is the ideal of polynomials in the fields and their derivatives which vanishes for on-shell fields. We also know from the discussion around (9.42) that the linearisations of the BRST operators satisfy

$$\tilde{Q}^{\mathrm{DC,\,lin}}_{\mathrm{BRST}} \;=\; Q^{\mathcal{N}=0,\,\mathrm{lin}}_{\mathrm{BRST},3}\ . \tag{9.46}$$

After integrating out all auxiliary fields, these BRST operators link the physical tree-level scattering amplitudes to tree-level scattering amplitudes containing ghosts by the on-shell Ward identities, cf. Observation 8.2.

At the level of the BRST operators $\tilde{Q}^{\mathrm{DC,\,lin}}_{\mathrm{BRST}}$ and $Q^{\mathcal{N}=0,\,\mathrm{lin}}_{\mathrm{BRST},3}$ the situation is more involved, but we still end up with similar on-shell Ward identities. The BRST doublets in the BRST-extended Hilbert space of Yang–Mills theory double copy to BRST doublets of auxiliary and non-auxiliary fields.

Therefore, the tree-level scattering amplitudes for the BRST-extended Hilbert spaces of $S^{\mathcal{N}=0}_{\mathrm{BRST},3}$ and $\tilde{S}^{\mathrm{DC}}_{\mathrm{BRST}}$ are fully determined via on-shell Ward identities by the tree-level scattering amplitudes of the physical and auxiliary fields of ghost number zero. We conclude that all these tree-level scattering amplitudes agree between both theories.

Full quantum equivalence. For full quantum equivalence, it remains to show that there is a local field redefinition that links the action $S^{\mathcal{N}=0}_{\mathrm{BRST},3}$ to $\tilde{S}^{\mathrm{DC}}_{\mathrm{BRST}}$. Both actions fully agree in their kinematic terms and their interaction vertices that contain exclusively fields of ghost number zero. Moreover, they have identical tree-level scattering amplitudes on their BRST-extended (i.e. full) Hilbert spaces. We can therefore invoke Observation 8.9 one final time in order to provide us with a local field redefinition that shifts the discrepancies between both actions to interaction vertices of arbitrarily high degree. This renders the actions fully quantum equivalent from the perspective of perturbative quantum field theory.

Appendices

$\mathcal{A}$

Minimal model recursive construction

To derive the recursion relations (5.7a), we need to construct a quasi-isomorphism $\phi : \mathfrak{L}^\circ \to \mathfrak{L}$ that allows us to pull back the higher products on $\mathfrak{L}$ to $\mathfrak{L}^\circ$ via formula (2.32) (with $\phi_0 = 0$). Our construction of ϕ follows the idea of [247], where essentially the same construction was given in the case of A_∞-algebras. In particular, we assume that we have a Maurer–Cartan element a° in $\mathfrak{L}^\circ$ and map it to an element a in $\mathfrak{L}$. The fact that Maurer–Cartan elements are mapped to Maurer–Cartan elements under quasi-isomorphisms, cf. (2.36), together with the assumption that a° (and therefore a) is small, will give us enough constraints to determine the quasi-isomorphisms and the higher products on $\mathfrak{L}^\circ$.

The material in this Appendix is borrowed from [2].

A.1. Minimal model recursive construction

We start from the contracting homotopy

$$\mathsf{h} \circlearrowright (\mathfrak{L}, \mu_1) \underset{\mathsf{e}}{\overset{\mathsf{p}}{\rightleftarrows}} (\mathfrak{L}^\circ, 0) \ . \tag{A.1}$$

where we can assume that $\mathsf{h}^2 = 0$ and $\mathsf{e} \circ \mathsf{p}$, $\mu_1 \circ \mathsf{h}$ and $\mathsf{h} \circ \mu_1$ are projectors onto $\mathfrak{L}^{\text{harm}}$, $\mathfrak{L}^{\text{ex}}$, and $\mathfrak{L}^{\text{coex}}$, respectively. Moreover, let $a^\circ \in \mathfrak{L}^\circ_1$ be a Maurer–Cartan element. Under a quasi-isomorphism ϕ, a° is mapped to

$$a = \sum_{i \geqslant 1} \frac{1}{i!} \phi_i(a^\circ, \ldots, a^\circ) \ . \tag{A.2}$$

A convenient choice is $\phi_1 = \mathsf{e}$, and it remains to identify ϕ_i for $i > 1$. We will do this by fixing a as a function of a°.

Recall that (5.2) yields the unique decomposition

$$a \;=\; a_{\text{harm}} + a_{\text{ex}} + a_{\text{coex}}\ , \quad \text{with} \quad a_{\text{harm, ex, coex}} \in \mathfrak{L}_{\text{harm, ex, coex}}\ . \tag{A.3}$$

There is some freedom in the choice of ϕ and without of loss of generality, we may impose the *gauge fixing* condition

$$\mathsf{h}(a) \;=\; 0\ . \tag{A.4}$$

This is, in fact, a generalisation of the Lorenz gauge fixing condition from ordinary gauge theory. Consequently, $a_{\text{ex}} = (\mu_1 \circ \mathsf{h})(a) = 0$. Moreover, the fact that μ_1 is a chain map implies that $\mu_1(a_{\text{harm}}) = (\mu_1 \circ \mathsf{e} \circ \mathsf{p})(a) = 0$ so that the homotopy Maurer–Cartan equation for a becomes

$$\mu_1(a_{\text{coex}}) + \sum_{i\geqslant 2} \frac{1}{i!}\mu_i(a_{\text{harm}} + a_{\text{coex}}, \ldots, a_{\text{harm}} + a_{\text{coex}}) \;=\; 0\ . \tag{A.5}$$

Upon acting with h on both sides of this equation, we obtain

$$a_{\text{coex}} \;=\; -\sum_{i\geqslant 2} \frac{1}{i!}(\mathsf{h} \circ \mu_i)(a_{\text{harm}} + a_{\text{coex}}, \ldots, a_{\text{harm}} + a_{\text{coex}})\ . \tag{A.6}$$

If we now assume that a° is small, say of order $\mathcal{O}(g)$ with $g \ll 1$ for g a formal parameter, we may rewrite (A.2) as

$$\begin{aligned} a \;&=\; \sum_{i\geqslant 1} \frac{g^i}{i!}\phi_i(a^\circ, \ldots, a^\circ) \;=\; g\,\underbrace{\mathsf{e}(a^\circ)}_{=:\,a^{(1)}} + \frac{g^2}{2}\underbrace{\phi_2(a^\circ, a^\circ)}_{=:\,a^{(2)}} + \cdots \\ &=\; g\big(a^{(1)}_{\text{harm}} + a^{(1)}_{\text{coex}}\big) + \frac{g^2}{2}\big(a^{(2)}_{\text{harm}} + a^{(2)}_{\text{coex}}\big) + \cdots \end{aligned} \tag{A.7a}$$

We can then compute the solution a of the homotopy Maurer–Cartan equation order by order in g using (A.6). In this process, we can choose to put $a^{(i)}_{\text{harm}} = 0$ for $i > 1$ so that

$$a \;=\; \underbrace{g\,a^{(1)}_{\text{harm}}}_{=\,a_{\text{harm}}} + \underbrace{\sum_{i\geqslant 2} \frac{g^i}{i!}a^{(i)}_{\text{coex}}}_{=\,a_{\text{coex}}} \;=\; a_{\text{harm}} + a_{\text{coex}}\ . \tag{A.7b}$$

Observe that $a^{(1)}_{\text{coex}} = 0$ follows from the condition $\mu_1(a^{(1)}_{\text{coex}}) = 0$ obtained at linear order from Equation (A.5). Substituting the expansion (A.7) into (A.6), we arrive at the recursion relation

$$a^{(i)}_{\text{coex}} \;=\; -\sum_{j=2}^{i} \frac{1}{j!} \sum_{k_1+\cdots+k_j=i} (\mathsf{h} \circ \mu_j)(a^{(k_1)}_{\text{harm}} + a^{(k_1)}_{\text{coex}}, \ldots, a^{(k_j)}_{\text{harm}} + a^{(k_j)}_{\text{coex}}) \tag{A.8}$$

for a_{coex}. Comparison with (A.2) then yields the quasi-isomorphism (5.7a) when evaluated at degree 1 elements.

To recover also the brackets μ_i° on $\mathfrak{L}^\circ$ listed in (5.7a) by pullback, we note that upon applying the projector p to (A.5) and using the fact that p is a chain map, we immediately find that

$$\sum_{i\geqslant 2}\frac{1}{i!}(\mathsf{p}\circ\mu_i)(a_{\text{harm}}+a_{\text{coex}},\ldots,a_{\text{harm}}+a_{\text{coex}}) \;=\; 0\;. \tag{A.9}$$

Hence, after substituting the expansion (A.7), we recover the brackets (5.7a) for degree 1 elements.

Our derivation above is strictly speaking only applicable to Maurer–Cartan elements, which are elements of the L_∞-algebra of degree 1. As noted in [52], however, we may enlarge every L_∞-algebra $\mathfrak{L}$ to the L_∞-algebra $\mathfrak{L}_{\mathscr{C}} := \mathscr{C}^\infty(\mathfrak{L}[1])\otimes\mathfrak{L}$ where $\mathscr{C}^\infty(\mathfrak{L}[1])$ are the smooth functions on the grade-shifted vector space $\mathfrak{L}[1]$. Then, every element in $\mathfrak{L}$ gives rise to a degree 1 element in $\mathfrak{L}_{\mathscr{C}}$, and, applying the above construction to $\mathfrak{L}_{\mathscr{C}}$ yields the full L_∞-quasi-isomorphism and brackets listed in (5.7a).

Cyclic L_∞-algebras. Finally, we note that the above construction also extends to the cyclic case. For this, we need h chosen such that

$$\langle\mathfrak{L}^{\text{coex}},\mathfrak{L}^{\text{coex}}\rangle_{\mathfrak{L}} \;=\; 0\;. \tag{A.10}$$

This is always possible since cyclicity (2.21) for μ_1 implies in general that

$$\langle\mathfrak{L}^{\text{ex}},\mathfrak{L}^{\text{ex}}\rangle_{\mathfrak{L}} \;=\; \langle\mathfrak{L}^{\text{harm}},\mathfrak{L}^{\text{ex}}\rangle_{\mathfrak{L}} \;=\; 0\;. \tag{A.11}$$

The remaining freedom in the choice of h can therefore be used to ensure that the only non-vanishing entries of the underlying metric are

$$\langle\mathfrak{L}^{\text{harm}},\mathfrak{L}^{\text{harm}}\rangle_{\mathfrak{L}}\;,\quad \langle\mathfrak{L}^{\text{ex}},\mathfrak{L}^{\text{coex}}\rangle_{\mathfrak{L}}\;,\quad\text{and}\quad \langle\mathfrak{L}^{\text{coex}},\mathfrak{L}^{\text{ex}}\rangle_{\mathfrak{L}}\;. \tag{A.12}$$

If we now pull-back the cyclic structure from $\mathfrak{L}$ to $\mathfrak{L}^\circ$ and define

$$\langle\ell_1^\circ,\ell_2^\circ\rangle_{\mathfrak{L}^\circ} \;:=\; \langle\phi_1(\ell_1^\circ),\phi_1(\ell_2^\circ)\rangle_{\mathfrak{L}}\;, \tag{A.13}$$

we have satisfied the first condition in (2.40) on a morphism of cyclic L_∞-algebras. The second condition in (2.40) is implied by (A.10) together with $\text{im}(\phi)\subseteq\mathfrak{L}^{\text{coex}}$.

B

A generalisation of Berends–Giele recursion relations

Let us present a derivation of the Berends–Giele recursion from the quasi-isomorphism (5.43) in the case of a general gauge group not necessarily simple and compact, which relies on the Dynkin–Specht–Wever lemma.

The material in this Appendix is borrowed from [2].

B.1. Dynkin–Specht–Wever lemma

Statement. For simplicity, let $\mathfrak{a}$ be a matrix algebra and $\mathfrak{l}$ be the Lie subalgebra generated by the elements that generate $\mathfrak{a}$, that is, the *free Lie algebra* over $\mathfrak{a}$. Consider the *Dynkin map* $D : \mathfrak{a} \to \mathfrak{l}$ defined by

$$\mathfrak{a} \ni \sum_{\sigma\in S_i} \lambda_\sigma X_{\sigma(1)} \cdots X_{\sigma(i)} \mapsto \sum_{\sigma\in S_i} \lambda_\sigma [X_{\sigma(1)}, [X_{\sigma(2)}, \ldots [X_{\sigma(i-1)}, X_{\sigma(i)}] \cdots]] \in \mathfrak{l} , \tag{B.1}$$

where $X_1, \ldots, X_i \in \mathfrak{a}$ and the coefficients λ_σ are some numbers, and by the identity if $i = 1$. The *Dynkin–Specht–Wever lemma* then asserts that if $p(X) := \sum_{\sigma\in S_{i_p}} \lambda^{(p)}_\sigma X_{\sigma(1)} \cdots X_{\sigma(i_p)} \in \mathfrak{l}$ then

$$D(p(X)) = i_p\, p(X) . \tag{B.2}$$

Hence, for any homogeneous polynomial $p(X) \in \mathfrak{a}$ of degree i_p, we obtain $(D\circ D)(p(X)) = i_p D(p(X))$.

Proof. To prove (B.2), we follow [248]. Firstly, we set $\mathrm{ad}(X)(Y) := [X, Y]$. Then, one can show by induction on the degree of the polynomial $p(X)$ that if $p(X) \in \mathfrak{l}$ then

$$\mathrm{ad}(p(X)) = p(\mathrm{ad}(X)) \tag{B.3a}$$

with

$$p(\mathrm{ad}(X)) \;:=\; \sum_{\sigma\in S_{i_p}} \lambda_\sigma^{(p)}\, \mathrm{ad}(X_{\sigma(1)})\circ\cdots\circ \mathrm{ad}(X_{\sigma(i_p)}) \;. \tag{B.3b}$$

Secondly, (B.2) is certainly true for $i_p = 1$ so let us assume it is true for $i_p > 1$ and prove the statement by induction. To this end, let $p(X) \in \mathfrak{l}$ and $q(X) \in \mathfrak{l}$ be homogeneous polynomials of degrees i_p and i_q, respectively. Then,

$$\begin{aligned} D(p(X)q(X)) &= \sum_{\sigma\in S_{i_p}} \lambda_\sigma^{(p)}[X_{\sigma(1)},[X_{\sigma(2)},\ldots[X_{\sigma(i_p-1)},[X_{\sigma(i_p)},D(q(X))]]\cdots]] \\ &= p(\mathrm{ad}(X))(D(q(X)) \\ &= \mathrm{ad}(p(X))(D(q(X)) \\ &= [p(X),D(q(X))] \\ &= i_q[p(X),q(X)] \;, \end{aligned} \tag{B.4}$$

where in the third step we have used (B.3a) since $q(X) \in \mathfrak{l}$ and in the fifth step the induction hypothesis. Thus,

$$D([p(X),q(X)]) \;=\; (i_p+i_q)[p(X),q(X)] \;. \tag{B.5}$$

This concludes the proof of (B.2).

Applications. Consider now

$$\begin{aligned} D(X_1\cdots X_i) &= [X_1,[X_2,\ldots[X_{i-1},X_i]\cdots]] \\ &= \sum_{j=0}^{i-1}\sum_{\sigma\in\mathrm{Sh}(j;i-1)} (-1)^{i+j+1} X_{\sigma(1)}\cdots X_{\sigma(j)}X_iX_{\sigma(i-1)}\cdots X_{\sigma(j+1)} \\ &= \frac{1}{i}\sum_{j=0}^{i-1}\sum_{\sigma\in\mathrm{Sh}(j;i-1)} (-1)^{i+j+1} D(X_{\sigma(1)}\cdots X_{\sigma(j)}X_iX_{\sigma(i-1)}\cdots X_{\sigma(j+1)}) \;, \end{aligned} \tag{B.6}$$

where in the third step we have used (B.2).

Then, again using (B.2), we obtain

$$\begin{aligned} \underbrace{[D(X_1\cdots X_i),D(X_{i+1}\cdots X_{i+j})]}_{=:\,(i+j)\sum_{\sigma\in S_{i+j}}\lambda_\sigma^{(i;i+j)}X_{\sigma(1)}\cdots X_{\sigma(i+j)}} &= \frac{1}{i+j}D([D(X_1\cdots X_i),D(X_{i+1}\cdots X_{i+j})]) \\ &= \sum_{\sigma\in S_{i+j}} \lambda_\sigma^{(i;i+j)} D(X_{\sigma(1)}\cdots X_{\sigma(i+j)}) \;, \end{aligned} \tag{B.7}$$

where the $\lambda_\sigma^{(i;i+j)}$ are given in terms of the coefficients in (B.6).

Likewise, again using (B.2), we have

$$\begin{aligned}
&[D(X_1\cdots X_i),[D(X_{i+1}\cdots X_{i+j}),D(X_{i+j+1}\cdots X_{i+j+k})]] \;=\\
&\quad= \frac{1}{(j+k)(i+j+k)}D([D(X_1\cdots X_i),D([D(X_{i+1}\cdots X_{i+j}),D(X_{i+j+1}\cdots X_{i+j+k})])])\\
&\quad= \frac{1}{(i+j+k)}\sum_{\sigma_2\in S_{j+k}}\lambda_{\sigma_2}^{(j;j+k)}D([D(X_1\cdots X_i),D(X_{i+\sigma_2(1)}\cdots X_{i+\sigma_2(j+k)})])\\
&\quad= \sum_{\substack{\sigma_1\in S_{i+j+k}\\ \sigma_2\in S_{j+k}}}\lambda_{\sigma_1}^{(i;i+j+k)}\lambda_{\sigma_2}^{(j;j+k)}D(X_{\sigma_1(1)}\cdots X_{\sigma_1(i)}X_{\sigma_1(i+\sigma_2(1))}\cdots X_{\sigma_1(i+\sigma_2(j+k))})\\
&\quad=: \sum_{\sigma\in S_{i+j+k}}\lambda_\sigma^{(i,j;i+j+k)}D(X_{\sigma(1)}\cdots X_{\sigma(i+j+k)})\;,
\end{aligned} \tag{B.8a}$$

where the coefficients $\lambda_\sigma^{(i,j)}$ are defined as follows: letting

$$\sigma_3 \;:=\; \sigma_1\circ\tau_{\sigma_2}\;,\quad \text{with}\quad \tau_{\sigma_2}(\ell)\;:=\;\begin{cases}\ell & \text{for}\quad \ell\in\{1,\ldots,i\}\;,\\ i+\sigma_2(\ell-i) & \text{for}\quad \ell\in\{i+1,\ldots,i+j+k\}\;,\end{cases} \tag{B.8b}$$

we obtain

$$\begin{aligned}
&\sum_{\sigma_1\in S_{i+j+k}}\sum_{\sigma_2\in S_{j+k}}\lambda_{\sigma_1}^{(i;i+j+k)}\lambda_{\sigma_2}^{(j;j+k)}D(X_{\sigma_1(1)}\cdots X_{\sigma_1(i)}X_{\sigma_1(i+\sigma_2(1))}\cdots X_{\sigma_1(i+\sigma_2(j+k))}) \;=\\
&\quad= \sum_{\sigma_3\in S_{i+j+k}}\sum_{\sigma_2\in S_{j+k}}\lambda_{\sigma_3\circ\tau_{\sigma_2}^{-1}}^{(i;i+j+k)}\lambda_{\sigma_2}^{(j;j+k)}D(X_{\sigma_3(1)}\cdots X_{\sigma_3(i+j+k)})\;,
\end{aligned} \tag{B.8c}$$

since when σ_1 runs over all of S_{i+j+k} so does σ_3. Consequently, we may set

$$\lambda_\sigma^{(i,j;i+j+k)} \;:=\; \sum_{\sigma'\in S_{j+k}}\lambda_{\sigma\circ\tau_{\sigma'}^{-1}}^{(i;i+j+k)}\lambda_{\sigma'}^{(j;j+k)}\;. \tag{B.8d}$$

B.2. Gluon recursion for general Lie groups

We again consider plane waves of the form (5.36) and make the ansatz

$$\begin{aligned}
\phi_i(A(1),\ldots,A(i)) \;=\; -\frac{(-1)^i}{i}\sum_{\sigma\in S_i}J_\mu(\sigma(1),\ldots,\sigma(i))\,\mathrm{e}^{\mathrm{i}(k_{\sigma(1)}+\cdots+k_{\sigma(i)})\cdot x}\times\\
\times[X_{\sigma(1)},[X_{\sigma(2)},[\ldots,[X_{\sigma(i-2)},[X_{\sigma(i-1)},X_{\sigma(i)}]]\cdots]]\,\mathrm{d}x^\mu\;.
\end{aligned} \tag{B.9}$$

Upon substituting this into (5.43) and using the contracting homotopy (5.32), a straightforward calculation shows that

$$
\begin{aligned}
J_\mu(1,\ldots,i) &= \\
&= \frac{1}{(k_1+\cdots+k_i)^2}\times \\
&\quad\times P_{\rm ex}\left\{\sum_{j=1}^{i-1}[\![J(1,\ldots,j),J(j+1,\ldots,i)]\!]'_\mu+\right. \\
&\qquad\left.+\sum_{j=1}^{i-2}\sum_{k=j+1}^{i-1}[\![J(1,\ldots,j),J(j+1,\ldots,k),J(k+1,\ldots,i)]\!]''_\mu\right\}
\end{aligned}
\tag{B.10a}
$$

with

$$
\begin{aligned}
&[\![J(1,\ldots,j)),J(j+1,\ldots,i)]\!]'_\mu := \\
&\quad := \frac{i}{2j(i-j)}\sum_{\sigma\in S_i}\lambda^{(j;i)}_{\sigma^{-1}}[\![J(\sigma(1),\ldots,\sigma(j)),J(\sigma(j+1),\ldots,\sigma(i))]\!]_\mu\,, \\
&[\![J(1,\ldots,j),J(j+1,\ldots,k),J(k+1,\ldots,i)]\!]'_\mu := \\
&\quad := \frac{i}{3j(k-j)(i-k)}\sum_{\sigma\in S_i}\lambda^{(j,k-j;i)}_{\sigma^{-1}}\times \\
&\qquad\times[\![J(\sigma(1),\ldots,\sigma(j)),J(\sigma(j+1),\ldots,\sigma(k)),J(\sigma(k+1),\ldots,\sigma(i))]\!]_\mu\,, \\
&[\![J(1),J(2),J(3)]\!]''_\mu := [\![J(1),J(2),J(3)]\!]'_\mu-[\![J(3),J(1),J(2)]\!]'_\mu\,,
\end{aligned}
\tag{B.10b}
$$

where $[\![-,-]\!]_\mu$ and $[\![-,-,-]\!]_\mu$ were introduced in (5.38c) and (5.39c) and the λ-coefficients are defined in (B.7) and (B.8), respectively. This is the Berends–Giele recursion for any matrix gauge algebra.

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