Multipartite entanglement in neutrino oscillations

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Abstract. Particle mixing is related to multi-mode entanglement of single-particle states. The occupation number of both flavor eigenstates and mass eigenstates can be used to define a multiqubit space. In such a framework, flavor neutrino states can be interpreted as multipartite mode-entangled states. By using two different entanglement measures, we analyze the behavior of multipartite entanglement in the phenomenon of neutrino oscillations.

1. Introduction

Quantum entanglement is a crucial physical resource in quantum information and communication science [1]. It has been mainly investigated in systems of condensed matter, atomic physics, and quantum optics. In the domain of particle physics, entanglement has been considered mainly in relation to two-body decay, annihilation, and creation processes, see for instance Refs. [2]. Very recently, multipartite entanglement has been studied in connection with the phenomenon of particle mixing [3, 4]. Such a phenomenon, associated with a mismatch between flavor and mass of the particle, appears in several instances: quarks, neutrinos, and the neutral K-meson system [5, 6]. Particle mixing is at the basis of important effects as neutrino oscillations and CP violation [7]. Flavor mixing for the case of three generations is described by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) in the lepton instance [8, 9]. The matrix elements represent the transition probabilities from one lepton to another. For example, flavor mixing of neutrinos for three generations is described by the 3×3 unitary mixing matrix $\mathbf{U}(\tilde{\theta}, \delta)$ [5],

$$\mathbf{U}(\tilde{\theta}, \delta) = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (1)$$

where $(\tilde{\theta}, \delta) \equiv (\theta_{12}, \theta_{13}, \theta_{23}; \delta)$ and $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$. The parameters θ_{ij} are the mixing angles, and δ is the phase responsible for CP violation. The three-flavor neutrino states are defined as

$$|\underline{\nu}^{(f)}\rangle = \mathbf{U}(\tilde{\theta}, \delta)\, |\underline{\nu}^{(m)}\rangle \quad (2)$$

where $|\underline{\nu}^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$ are the states with definite flavor and $|\underline{\nu}^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$ those with definite mass. From Eq. (2), we see that each flavor state is given by a superposition of mass eigenstates, i.e. $|\nu_\alpha\rangle = U_{\alpha 1}|\nu_1\rangle + U_{\alpha 2}|\nu_2\rangle + U_{\alpha 3}|\nu_3\rangle$. Let us recall that both $\{|\nu_\alpha\rangle\}$ and $\{|\nu_i\rangle\}$ are orthonormal, i.e. $\langle\nu_\alpha|\nu_\beta\rangle = \delta_{\alpha,\beta}$ and $\langle\nu_i|\nu_j\rangle = \delta_{i,j}$. Therefore, one can interpret the label i as denoting a quantum mode, and can establish the following correspondence with three-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle, \quad |\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle, \tag{3}$$

where $|\rangle_i$ denotes states in the Hilbert space for neutrinos with mass m_i. Then, the occupation number allows to interpret the flavor states as constituted by entangled superpositions of the mass eigenstates. Quantum entanglement emerges as a direct consequence of the superposition principle. It is important to remark that the Fock space associated with the neutrino mass eigenstates is physically well defined. In fact, at least in principle, the mass eigenstates can be produced or detected in experiments performing extremely precise kinematical measurements [10]. In this framework, as discussed in Ref. [3], the quantum mechanical state (2) is entangled in the field modes, although being a single-particle state.

Mode entanglement defined for single-photon states of the radiation field or associated with systems of identical particles has been discussed in Ref. [11]. In the particular instance of Eq. (2), the multipartite flavor states can be seen as a generalized class of W states. These are multipartite entangled states that occur in several physical systems and can be engineered with pure quantum optical elements [12]. The concept of mode entanglement in single-particle states has been widely discussed and is by now well established [11, 13]. Successful experimental realizations using single-photon states have been reported as well [14]. Moreover, remarkably, the nonlocality of single-photon states has been experimentally demonstrated [15], verifying a theoretical prediction [16]. Furthermore, the existing schemes to probe nonlocality in single-particle states have been generalized to include massive particles of arbitrary type [17].

In the dynamical regime, flavor mixing (and neutrino mass differences) generates the phenomenon of neutrino oscillations. The mass eigenstates $|\nu_j\rangle$ have definite masses m_j and definite energies E_j. Their propagation can be described by plane wave solutions of the form $|\nu_j(t)\rangle = e^{-iE_j t}|\nu_j\rangle$. The time evolution of the flavor neutrino states Eq.(2) is given by:

$$|\underline{\nu}^{(f)}(t)\rangle = \widetilde{\mathbf{U}}(t)|\underline{\nu}^{(f)}\rangle\,, \qquad \widetilde{\mathbf{U}}(t) \equiv \mathbf{U}(\tilde{\theta},\delta)\,\mathbf{U}_0(t)\,\mathbf{U}(\tilde{\theta},\delta)^{-1}\,, \tag{4}$$

where $|\underline{\nu}^{(f)}\rangle$ are the flavor states at $t = 0$, $\mathbf{U}_0(t) = diag(e^{-iE_1 t}, e^{-iE_2 t}, e^{-iE_3 t})$, and $\widetilde{\mathbf{U}}(t = 0) = \mathrm{I}$.

Flavor neutrino states are well defined in the context of Quantum Field Theory (QFT), where they are obtained as eigenstates of the flavor neutrino charges [19]. In the relativistic limit, the exact QFT flavor states reduce to the usual Pontecorvo flavor states Eq.(2): flavor modes are thus legitimate and physically well-defined individual entities and mode entanglement can be defined and studied in analogy with the static case of Ref.[3]. We can thus establish the following correspondence with three-qubit states:

$$|\nu_e\rangle \equiv |1\rangle_{\nu_e}|0\rangle_{\nu_\mu}|0\rangle_{\nu_\tau}, \quad |\nu_\mu\rangle \equiv |0\rangle_{\nu_e}|1\rangle_{\nu_\mu}|0\rangle_{\nu_\tau}, \quad |\nu_\tau\rangle \equiv |0\rangle_{\nu_e}|0\rangle_{\nu_\mu}|1\rangle_{\nu_\tau}. \tag{5}$$

States $|0\rangle_{\nu_\alpha}$ and $|1\rangle_{\nu_\alpha}$ correspond, respectively, to the absence and the presence of a neutrino in mode α. Entanglement is thus established among flavor modes, in a single-particle setting. Eq. (4) can then be recast as

$$|\nu_\alpha(t)\rangle = \widetilde{\mathbf{U}}_{\alpha e}(t)|1\rangle_{\nu_e}|0\rangle_{\nu_\mu}|0\rangle_{\nu_\tau} + \widetilde{\mathbf{U}}_{\alpha\mu}(t)|0\rangle_{\nu_e}|1\rangle_{\nu_\mu}|0\rangle_{\nu_\tau} + \widetilde{\mathbf{U}}_{\alpha\tau}(t)|0\rangle_{\nu_e}|0\rangle_{\nu_\mu}|1\rangle_{\nu_\tau}\,, \tag{6}$$

with the normalization condition $\sum_\beta |\widetilde{\mathbf{U}}_{\alpha\beta}(t)|^2 = 1$ $(\alpha, \beta = e, \mu, \tau)$. The time-evolved states $|\underline{\nu}^{(f)}(t)\rangle$ are entangled superpositions of the three flavor eigenstates with time-dependent coefficients. Thus, flavor oscillations can be related to multi-mode (flavor) entanglement of single-particle states [4].

It is important to remark difference between Refs. [3] and [4]: In Ref. [3], the multipartite entanglement, associated with the multiqubit space of *mass modes*, has been analyzed in connection with the "decoherence" effects induced by free propagation of oscillating neutrinos. In Ref. [4], on the other hand, the entanglement is quantified with respect to the multiqubit space associated with *flavor modes*, and it arises as a consequence of the non-trivial time evolution of the flavor states.

In this paper, we extend the results of Ref. [4]. We compute the multipartite entanglement associated to three flavor neutrino oscillations, by using two different global measures of entanglement: the von Neumann entropy, based on the entropies related to all possible bipartitions of the system, and the K-component geometric measure, introduced in Ref. [20]. Finally, we compare the results corresponding to the two entanglement measures.

2. Neutrino oscillations

In this section, we briefly review the phenomenon of neutrino oscillations. At time t the probability associated with the transition $\nu_\alpha \to \nu_\beta$ is

$$P_{\nu_\alpha \to \nu_\beta}(t) \,=\, |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 \,=\, |\tilde{\mathbf{U}}_{\alpha\beta}(t)|^2 \,, \tag{7}$$

where $\alpha, \beta = e, \mu, \tau$. The transition probability $P_{\nu_\alpha \to \nu_\beta}(t)$ is a function of the energy differences $\Delta E_{jk} = E_j - E_k$ $(j, k = 1, 2, 3)$ and of the mixing angles. Since detectable neutrinos are ultra-relativistic, the standard adopted approximation is $\Delta E_{jk} \simeq \frac{\Delta m_{jk}^2}{2E}$, where $\Delta m_{jk}^2 = m_j^2 - m_k^2$ and $E = |\overrightarrow{p}|$ is the energy of a massless neutrino (all massive neutrinos are assumed to have the same momentum $\overrightarrow{p}$). The mixing angles θ_{ij}^{PMNS}, and the squared mass differences are fixed at the experimental values reported in Ref. [21]:

$$\sin^2\theta_{12}^{PMNS} \,=\, 0.314\,, \quad \sin^2\theta_{13}^{PMNS} \,=\, 0.8 \times 10^{-2}\,, \quad \sin^2\theta_{23}^{PMNS} \,=\, 0.45\,, \tag{8}$$

$$\begin{aligned} &\Delta m_{21}^2 \,=\, \delta m^2\,, \qquad \Delta m_{31}^2 \,=\, \Delta m^2 + \frac{\delta m^2}{2}\,, \qquad \Delta m_{32}^2 \,=\, \Delta m^2 - \frac{\delta m^2}{2}\,, \\ &\delta m^2 \,=\, 7.92 \times 10^{-5}\, eV^2\,, \qquad \Delta m^2 \,=\, 2.6 \times 10^{-3}\, eV^2\,. \end{aligned} \tag{9}$$

In the forthcoming analysis, we will put to zero the CP-violating phase δ. In Fig. 2 we show the behavior of the transition probabilities $P_{\nu_e \to \nu_\alpha}$ $(\alpha = e, \mu, \tau)$ as a function of the scaled, dimensionless time $T = \frac{2Et}{\Delta m_{12}^2}$.

3. Multipartite entanglement in neutrino oscillations

In this section we characterize the multipartite flavor entanglement possessed by the state (6).

Let us first introduce the entanglement measures which will be exploited to this aim. Bipartite entanglement of pure states is unambiguously quantified by the von Neumann entropy or by any other monotonic function of the former [22]. Moving from the two- to the three-flavor scenario, multipartite entanglement measures have been introduced in terms of functions of bipartite measures [23]. Adopting an approach similar to that of Refs. [23], we define as proper measures of multipartite entanglement the von Neumann entropies associated to every possible bipartition of the system, see Ref. [3]. Let $\rho^{(\alpha)} = |\nu_\alpha(t)\rangle\langle\nu_\alpha(t)|$ be the density operator corresponding to the pure state $|\nu_\alpha(t)\rangle$, Eq. (6). We denote by $\rho_{i,j}^{(\alpha)} \,=\, Tr_k[\rho^{(\alpha)}]$ the density matrix reduced with

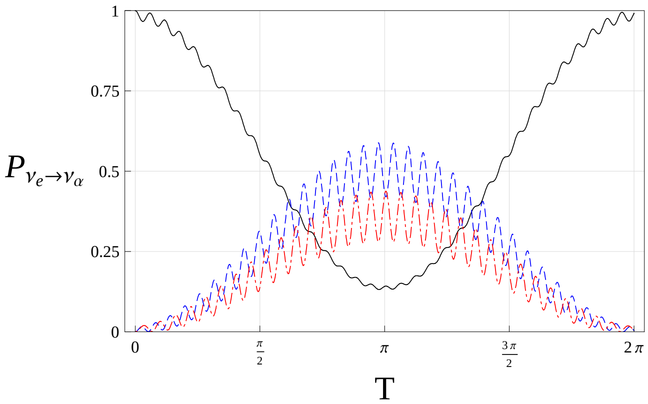

Figure 1. Transition probabilities $P_{\nu_e \to \nu_\alpha}$ as functions of the scaled time T. Parameters θ_{ij} and Δm_{ij}^2 are fixed at the experimental values [21]. Curves correspond to $P_{\nu_e \to \nu_e}$ (full), $P_{\nu_e \to \nu_\mu}$ (dashed), and $P_{\nu_e \to \nu_\tau}$ (dot-dashed). The phase δ is fixed to zero.

respect to flavor k, with $i, j, k = e, \mu, \tau$ and $i \neq j \neq k$. The von Neumann entropy associated with such a bipartition will be given by

$$E_{vN\,\alpha}^{(i,j;k)} = -Tr_{i,j}[\rho_{i,j}^{(\alpha)} \log_2 \rho_{i,j}^{(\alpha)}] \,. \tag{10}$$

At last, we define the average von Neumann entropy

$$\langle E_{vN}^{(2:1)} \rangle = \frac{1}{3} \sum_k E_{vN}^{(i,j;k)} \,, \tag{11}$$

where the sum is intended over all the possible bipartitions of the system.
An alternative characterization of multipartite entanglement is given in Refs. [20, 24]. Given the pure state $|\nu_\alpha(t)\rangle$ belonging to a 3-dimensional Hilbert space, the geometric measure of entanglement is defined as:

$$E_G^{(K)}(|\nu_\alpha(t)\rangle) \equiv E_{G\,\alpha}^{(K)} = -\frac{1}{3} \log_2 \max_{|\Phi^{(K)}\rangle} \left| \langle \Phi^{(K)} | \nu_\alpha(t) \rangle \right|^2 \,, \tag{12}$$

where the maximum is taken with respect to all K-separable pure states $|\Phi^{(K)}\rangle$, with $K = 1, 2$. This measure is intrinsically geometric because it coincides with the distance (in the Hilbert-Schmidt norm) between a given pure state and a set of K-separable reference pure states. For $K = 3$, the reference state $|\Phi^{(3)}\rangle$ is the fully product state:

$$|\Phi^{(3)}\rangle = \prod_k (\cos \xi_k |0\rangle_{\nu_k} + \sin \xi_k |1\rangle_{\nu_k}) \,, \tag{13}$$

where ξ_k $(k = e, \mu, \tau)$ are real parameters. For $K = 2$, the reference state $|\Phi^{(2)}\rangle$ can be written in the normalized form:

$$\begin{aligned} |\Phi^{(2)}\rangle = \; & (\cos \delta_1 |0\rangle_{\nu_i} |0\rangle_{\nu_j} + \sin \delta_1 \cos \delta_2 |0\rangle_{\nu_i} |1\rangle_{\nu_j} + \sin \delta_1 \sin \delta_2 \cos \delta_3 |1\rangle_{\nu_i} |0\rangle_{\nu_j} + \\ & \sin \delta_1 \sin \delta_2 \sin \delta_3 |1\rangle_{\nu_i} |1\rangle_{\nu_j}) \otimes (\cos \delta_4 |0\rangle_{\nu_k} + \sin \delta_5 |1\rangle_{\nu_k}) \,, \end{aligned} \tag{14}$$

where $i, j, k = e, \mu, \tau$ and $i \neq j \neq k$, and δ_k are real parameters. In Fig. 3, panels (a) and (b), we plot the von Neumann entropy $E_{vN\,e}^{(i,j;k)}$ and the geometric component $E_{G\,e}^{(2)}(i, j; k)$ as functions of the scaled time T. Both the entanglement measures exhibit similar behavior. In fact, the curves have the same ordering with respect to each other, and share the same maxima and minima. Next we compare the behavior of the entanglement measures of Fig. 3 with that of the transition probabilities of Fig. 2. We observe that there is a strong correlations between the degree of mixing and the entanglement content of a given bipartition. In order to quantify the

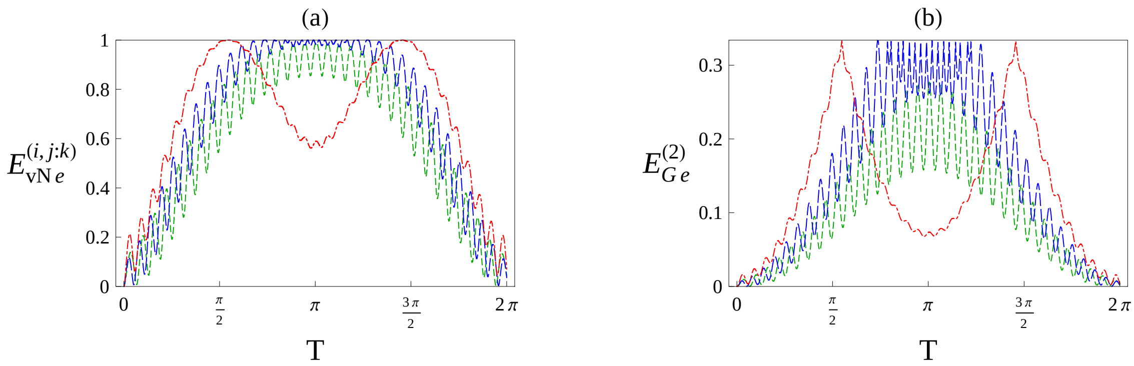

Figure 2. The von Neumann entropy $E^{(i,j;k)}_{vN\,e}$ (a) and the geometric component $E^{(2)}_{G\,e}(i,j;k)$ (b) as functions of the scaled time T. The plot style associated to each bipartition $(i,j;k)$ is: $i=1$, $j=2$, and $k=3$ (dashed line); $i=1$, $j=3$, and $k=2$ (long dashed line); $i=2$, $j=3$, and $k=1$ (dot-dashed line). Parameters θ_{ij} and Δm^2_{ij} are fixed at the experimental values [21]. The phase δ is fixed to zero.

global entanglement of the state $|\nu_e(t)\rangle$, we compute the average von Neumann entropy $E^{(2:1)}_{vN\,e}$ and the geometric component $E^{(3)}_{G\,e}$. These quantities are plotted in Fig. 3, and exhibit a very similar behavior.

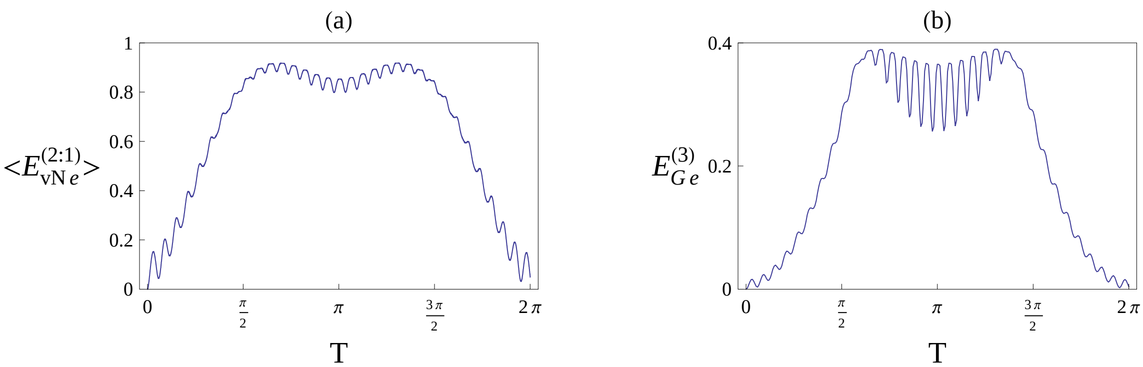

Figure 3. The average von Neumann entropy $\langle E^{(2:1)}_{vN\,e}\rangle$ (a) and the geometric component $E^{(3)}_{G\,e}$ (b) as functions of the scaled time T. Parameters θ_{ij} and Δm^2_{ij} are fixed at the experimental values [21]. The phase δ is fixed to zero.

4. Conclusions

In conclusion, by generalizing the results of a previous paper [4], we have analyzed the amount and distribution of entanglement in the phenomenon of neutrino oscillations. The phenomenon of flavor oscillations can be interpreted in terms of the dynamical behavior of quantum correlations associated with time-evolved flavor states. In Ref.[4], we suggest the possibility of using entangled states of oscillating neutrinos as a resource for quantum information protocols, and propose experimental schemes for transferring it to spatially separated modes of stable leptonic particles.

Acknowledgments
We acknowledge partial financial support from MIUR, INFN, INFM and CNISM.

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