

Kerr black hole shadow: non-parametric description and analytic methods of spin extraction

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Abstract. In this paper we study the asymmetry of the shadow cast by a rotating Kerr black hole. Using Bardeen parametric formulas we plot contours of black hole shadows for different values of black hole's angular momentum and different observer's positions. In the case of the equatorial observer we present an equation that explains the behaviour of the numeric Hioki-Maeda curve, which is used to measure the black hole angular momentum from its shadow. Also in this case we introduce a new observable of the shadow and derive a simple formula that allows to constrain the black hole spin via measuring this observable. Finally we come out with the formula that provides the non-parametric description of the black hole shadow.

1. Introduction

It is well-known, that the space-time geometry created by a rotating uncharged black hole with the mass M and the angular momentum J directed along z -axis is determined within Einstein's general relativity by the Kerr metric, that in Boyer-Lindquist coordinates defines the interval as [1]

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{4aMr \sin^2 \theta}{\rho^2} dt d\varphi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \sin^2 \theta \left(r^2 + a^2 + \frac{2a^2 Mr \sin^2 \theta}{\rho^2} \right) d\varphi^2, \quad (1)$$

where we denote

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2, \quad a = \frac{J}{M}. \quad (2)$$

Parameter a is often referred to as the spin of the black hole. Here and everywhere in this article we use metric signature $(+, -, -, -)$ and geometric unit system $c = G = 1$, so quantities r , t and a are measured in mass units. The event horizon of a black hole, which is a hypersurface that allows particles and light only inside-directed movement, in our case is defined by equation [1]

$$r = r_h = M + \sqrt{M^2 - a^2}. \quad (3)$$



One can see, that the geometry specified by (1) provides a description of a black hole only if $a \leq M$, otherwise it corresponds to the naked singularity, which is not the subject of the current research.

The problem of light propagation in Kerr geometry was studied thoroughly by many authors and one can for reference find profound investigation of isotropic geodesics, per example, in [1]. The point is that every infinite light trajectory that reaches a distant observer with angular coordinate θ_0 is fully determined by two conserving integrals of motion ξ and η [1], and creates a bright point in the celestial sphere of the observer with coordinates $(x; y)$ (see figure 1) related to these integrals through equations

$$x = -\frac{\xi}{\sin \theta_0} \quad y = \pm \sqrt{\eta + a^2 \cos^2 \theta_0 - \xi^2 \operatorname{ctg}^2 \theta_0}. \quad (4)$$

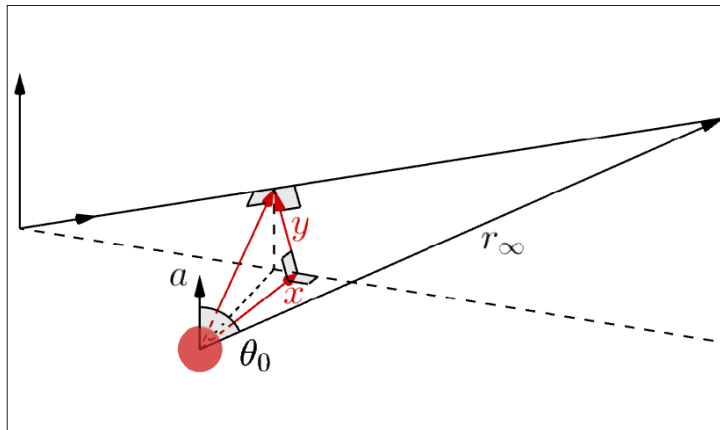


Figure 1. An illustration to the introduction of celestial coordinates. Quantities r_∞ and θ_0 stand for the position of the observer.

The very important fact is that infinite isotropic geodesics can be divided into three categories. If all turning points of a trajectory lie beneath the event horizon or they do not exist, such trajectory is classified as a capture trajectory, and, otherwise, if a trajectory has a turning point outside the event horizon, such trajectory is classified as a scattering one. Scattering and capture trajectories are separated by the critical trajectories, which form the third category. The tricky thing is that conserving integrals of motion corresponding to critical trajectories also correspond to unstable finite trajectories of a constant radii, that are often referred to as unstable circular orbits. So, starting at some distance a critical trajectory, as it comes closer to the black hole, spirals around it infinite amount of times merging with unstable circular orbit with the same values of ξ and η .

Therefore, if a black hole is situated between a source of light with a bigger angular size and a distant observer, the observer will see a dark area in the celestial sphere formed by capture trajectories and referred to as the black hole shadow, and a brighter background formed by scattering trajectories. Critical trajectories then define a closed curve that represents the contour of the shadow. Bardeen [2] was the first to find the formulas that describe dependencies of conserving integrals of motion for critical trajectories on the corresponding radii of unstable circular orbits. In terms of dimensionless (divided by M) celestial coordinates, dimensionless radii r and dimensionless angular momentum a they are equivalent to

$$\left\{ \begin{array}{l} x = \frac{1}{as(r-1)} \left[r^3 - 3r^2 + a^2r + a^2 \right] \\ y^2 = \frac{1}{a^2s^2(r-1)^2} \left[4a^2s^2r^2(r^2 - 2r + a^2) - [r^3 - 3r^2 + a^2(1+s^2)r + a^2(1-s^2)]^2 \right] \end{array} \right. \quad (5)$$

Here we introduce $s = \sin \theta_0$ and r takes values within the range

$$4 \cos^2 \left(\frac{1}{3} \arccos a + \frac{2\pi}{3} \right) \leq r \leq 4 \cos^2 \left(\frac{1}{3} \arccos a \right). \quad (6)$$

So, each unstable circular orbit radius r satisfying (6) corresponds to a point of the boundary of the shadow with celestial coordinates that can be calculated by (5). Therefore it is useful to view a system (5) as a parametrization for the shadow contour with r employed as the parameter.

2. Evaluation of the angular momentum

Examples of Kerr black hole shadows for different values of angular momentum and for different positions of a distant observer are presented in figure 2. From the panel (a) one can see, that the contour of the shadow of a black hole is not a circle and has an asymmetric shape and the discrepancy from the circular shape grows as parameter a increases. From the panel (b) it follows that the asymmetry of the shadow also depends on the position of a distant observer and is as more distinguishable as closer the observer gets to the equatorial plane.

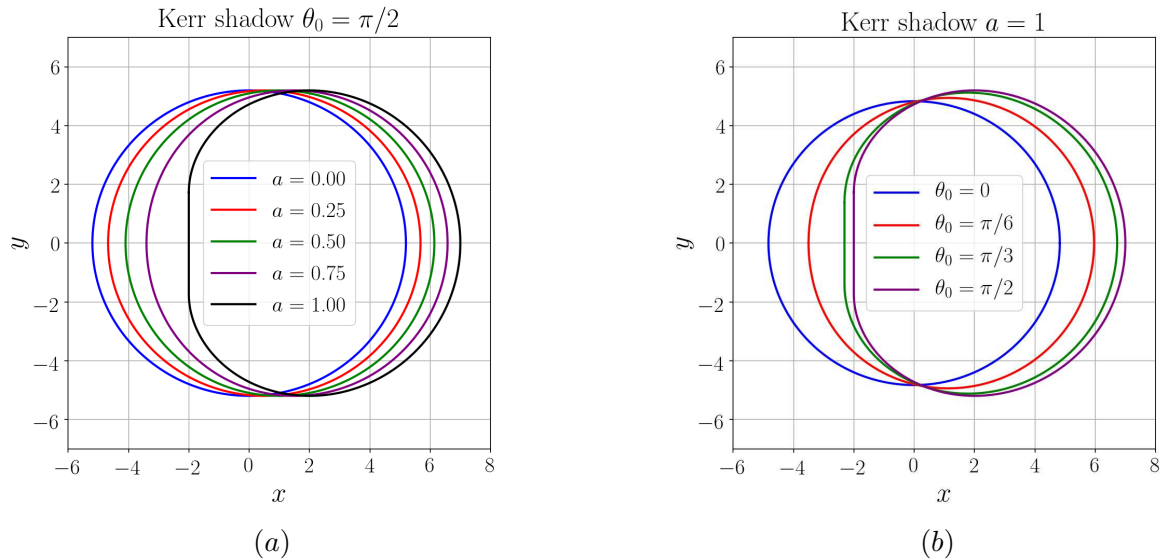


Figure 2. Kerr black hole shadows

So far as deformation of the contour is maximal for equatorial observer, now we are going to focus on this specific scenario. Let the coordinates of the contour's top point be $(x_m; y_m)$ and the x -coordinates of extreme right and left points of the contour be x_R and x_L correspondingly. Values of x_m and y_m can be obtained from the equation for extremum $y'(x) = 0$, and values for x_R and x_L represent two solutions of the equation $y(x) = 0$. Solutions of these two equations must correspond to r satisfying (6). As a result we come to following formulas:

$$x_m = 2a \quad y_m = 3\sqrt{3}r_m = 3 \quad (7)$$

$$x_R = a + 6 \cos \left(\frac{1}{3} \arccos a \right) \quad x_L = a + 6 \cos \left(\frac{1}{3} \arccos a + \frac{2\pi}{3} \right). \quad (8)$$

The shape of the shadow then may be characterized by the observable quantity

$$\zeta = \frac{x_R - x_L}{2y_m} \quad (9)$$

that represents the horizontal size of the shadow divided by the vertical size of the shadow. Using (7)-(8) we find

$$\zeta = \cos\left(\frac{1}{3}\arccos a - \frac{\pi}{6}\right) \quad a = \sqrt{1 - \zeta^2(4\zeta^2 - 3)^2}. \quad (10)$$

Curve $a(\zeta)$ is plotted in the panel (a) of figure 3.

The first method for angular momentum evaluation from the black hole shadow was proposed by Hioki and Maeda in pioneering work [3]. They compare the shadow to the circle that passes through extreme right, top and bottom points of the shadow. Let D stand for the distance between extreme left points of the fitting circle and the shadow, and let R be the radius of the fitting circle. Then the observable introduced in [3] is defined as $\delta = D/R$. The dependency of angular momentum on parameter δ is supposed to be calculated numerically. However, now we will show, that in the case of the equatorial observer the equation describing $a(\delta)$ -curve can be easily derived. If the center of the fitting circle has coordinates $(x_C; 0)$, then

$$R = x_R - x_C \quad R = \sqrt{y_m^2 + (x_m - x_C)^2} \quad D = 2R - (x_R - x_L). \quad (11)$$

Excluding x_C and using (7)-(8) we find that

$$\delta = 2\left[1 - \frac{(x_R - x_L)(x_R - 2a)}{27 + (x_R - 2a)^2}\right]. \quad (12)$$

As quantities x_R and x_L are known functions of the angular momentum, the equation (12) provides an analytic description for the dependency of the parameter δ on the black hole's angular momentum a . From the panel (b) of figure 3 one can see, that the curve plotted with (12) fits very well the points obtained with numeric calculations.

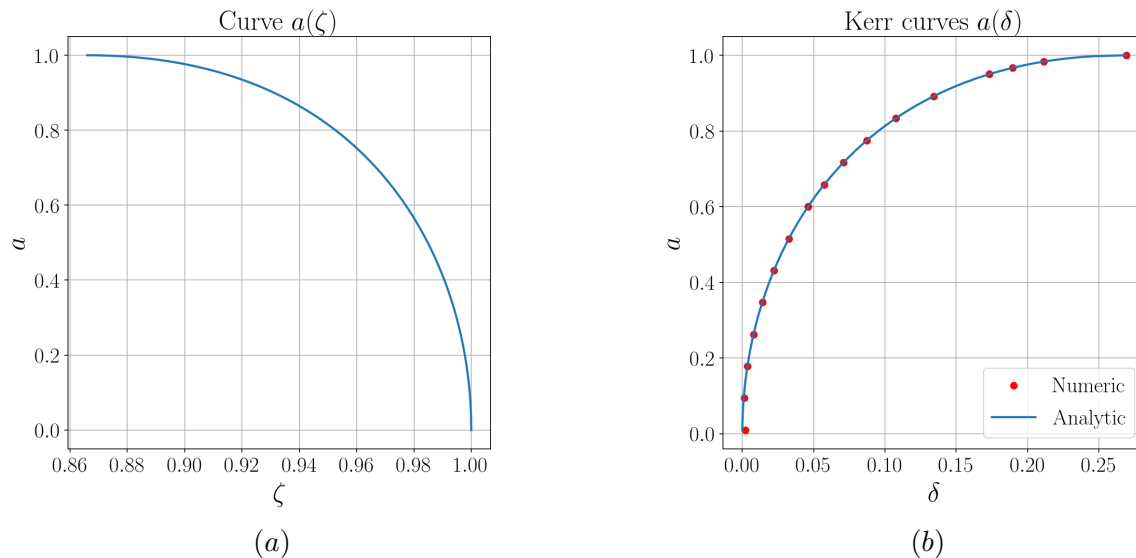


Figure 3. Curves of angular momentum evaluation. Solid blue lines are used for analytic curves, whereas red dots show values that were obtained numerically.

3. Non-parametric description of the shadow

Sometimes it is not useful to deal with the curve that is defined with the use of parametrization, and instead the formula that straightly expresses y as a function of x can be more preferable.

In order to derive such a formula for a black hole shadow we mention, that according to (5) celestial coordinate x and parameter r satisfy the relation

$$r^3 - 3r^2 + a(a - sx)r + a(a + sx) = 0, \quad (13)$$

that can be regarded as a cubic equation for parameter r considered as a function of x . Solving this equation we find

$$r(x) = \begin{cases} 1 + 2\sqrt{q(x)} \cos\left(\frac{1}{3} \arccos\left[\frac{1 - a^2}{q^{3/2}(x)}\right]\right), & x \geq x_2 \\ 1 + 2\sqrt{q(x)} \operatorname{ch}\left(\frac{1}{3} \operatorname{arch}\left[\frac{1 - a^2}{q^{3/2}(x)}\right]\right), & x_1 \leq x \leq x_2, \\ 1 + 2\sqrt{|q(x)|} \operatorname{sh}\left(\frac{1}{3} \operatorname{arsh}\left[\frac{1 - a^2}{|q^{3/2}(x)|}\right]\right), & x \leq x_1 \end{cases} \quad (14)$$

with

$$q(x) = 1 - \frac{a(a - sx)}{3} \quad x_1 = \frac{3}{as} \left[\frac{a^2}{3} - 1 \right] \quad x_2 = \frac{3}{as} \left[(1 - a^2)^{2/3} - \left(1 - \frac{a^2}{3}\right) \right]. \quad (15)$$

The next step is to notice, that

$$\frac{d}{dx} [y^2 + (x - as)^2] = \frac{4as}{r(x) - 1}. \quad (16)$$

The integration of this equation leads to a formula

$$y^2 + (x - as)^2 = \mathcal{F}(x) + 12 \quad \mathcal{F}(x) = \frac{6|q(x)|}{r(x) - 1} \left[1 + \frac{(r(x) - 1)^2}{|q(x)|} \right]. \quad (17)$$

Integration constant was found by substitution of parameters of the top point of the shadow.

4. Conclusion

Using the formulas describing the Kerr black hole shadow we plot examples of the contours for different values of angular momentum a and positions of a distant observer θ_0 . The shape of the shadow differs from circular one and the difference increases as the angular momentum grows and as the observer comes closer to the equatorial plane. As an observable quantitative characteristic of the contour's asymmetry we use the Hioki-Maeda parameter δ and also we introduce a new parameter ζ that represents the ratio between horizontal and vertical sizes of the shadow. We show, that for the case of the equatorial observer the dependency of the angular momentum on the observable ζ takes a quite simple form. Also within this case we derive the equation, that provides the analytical description of the $\delta(a)$ dependency that before has always been calculated only numerically. Finally we present the derivation of the formula, that describes points of the contour of the shadow without using parametrization through radii of unstable circular orbits.

Acknowledgements

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References

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