Journal of Nonlinear Mathematical Physics (2023) 30:92–113
https://doi.org/10.1007/s44198-022-00073-6

RESEARCH ARTICLE

Lie Symmetry Analysis and Conservation Laws for the (2 + 1)-Dimensional Dispersionless B-Type Kadomtsev–Petviashvili Equation

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Received: 21 May 2022 / Accepted: 11 July 2022 / Published online: 25 July 2022

Abstract

The Lie symmetry analysis is adopted to the (2 + 1)-dimensional dispersionless B-type Kadomtsev–Petviashvili (dBKP) equation. The combination of symmetry analysis and symbolic computing methods proves that Lie algebra of infinitesimal symmetry of the dBKP equation depends on four independent arbitrary functions and one arbitrary parameter. The Lie algebra is reduced to four classes for deriving commutative relations, group invariant solutions of dBKP equation and conservation laws, and the optimal system of 1-dimensional subalgebras from one class is constructed. Based on the optimal system and other particular infinitesimal symmetries, plentiful symmetry reductions and invariant solutions are computed by using Lie group method. Six successive symmetries and conserved quantities of the dBKP equation are linked by the new conservation theorem. Besides, exact solution of the dBKP equation is constructed according to a conservation vector.

Keywords DBKP equation · Lie symmetry analysis · Optimal system · Invariant solutions · Conservation laws

Mathematics Subject Classification 76M60 · 22E60 · 17B80

Abbreviations

NLPEDs	Non-linear partial differential equations
dBKP equation	(2 + 1)-Dimensional dispersionless B-type Kadomtsev–Petviashvili equation

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1 Introduction

Non-linear partial differential equations (NLPDEs) are widely used to explain nonlinear physical phenomena in engineering sciences such as fluid dynamics, plasma physics and oceanography. Analysing mathematical properties of NLPDEs such as symmetry reductions, exact closed-form solutions and dynamic behavior of solutions is crucial to predict and utilize these phenomena. Lie symmetry analysis, a valid and concise approach to comprehend these properties of NLPDEs, was first presented by Lie in 1881 [1]. Based on transformative invariance of one-parameter Lie group, this method can reduce the number of independent variables. Since a Lie algebra of NLPDEs almost always contains infinite subalgebras, an optimal system should be found to avoid getting the equivalent group invariant solutions. In recent six decades, scholars such as Ovsyannikov, Olver, Ibragimov, Miao and Hu et al. [2–6] have sought to, but not limited to improve the way of searching optimal system. The impact of Lie symmetry's thought simultaneously spread more widely [7–9].

Conservation law appears in NLPDEs related to areas such as water waves, foam, atmospheric flows, etc, its existence strongly proves the integrability of NLPDEs. Symmetries of NLPDEs could connect with conservation laws, Noether's theorem in [10], partial Noether's approach in [11] and multiplier approach in [12] are all highly effective for the derivation of conservation laws. The new conservation theorem is more widely utilized since there is no need for Lagrangian and relies only on the commutator table [13]. According to the fundamental notion of nonlinearly self-adjointness, corresponding conservation laws by self-adjointness can be generated by adjoint symmetry [14], which are examples of conservation laws by pairs of symmetries and adjoint symmetries [15]. In addition, exact solutions of NLPDEs could be constracted adopting particular conservation laws [16].

The significance has been realized recently to dispersionless limits of integrable hierarchies and equations, since they present in the research of various problems in applied mathematics and physics from the theory of conformal maps to the theory of quantum fields and strings [17–19]. In the quasi-classical $\overline{\partial}$–dressing scheme for dispersionless KP hierarchy, dispersionless B-type KP hierarchy is dispersionless KP hierarchy with even times frozen at zero plus symmetry, and the dBKP equation can be given by the compatibility condition for the first two Hamilton–Jacobi equations [20–22]

$$3w_t + 15w^2w_x - 5(w\partial^{-1}w_y)_x - \frac{5}{3}\partial^{-1}w_{yy} = 0, \tag{1}$$

where $w = w(x, y, t)$ denotes the unknown function of space variables x, y and time variable t, ∂^{-1} means to integrate x. Substituting $w = u_x$ into Eq. (1) to remove the integral symbol

$$(3u_t + 5u_x^3 - 5u_xu_y)_x - \frac{5}{3}u_{yy} = 0. \tag{2}$$

Quantum W-infinity algebra, classical quantum torus structure and other discussions of dBKP hierarchy have been detailed in [21, 23, 24], yet mathematical properties of dBKP equation is just beginning to explore.

This paper explores new group invariant solutions of the dBKP equation by the Lie symmetry approach, the Lie point symmetries of dBKP equation are presented and discussed which have not been studied in previous literature. Lie symmetry method was applied to the dBKP equation and eight symmetry reductions were derived in 2021, but we derive twenty-five symmetry reductions. The invariant solutions are dissimilar since our research involves the general Lie algebra and its four reductive classifications, whereas the previous discussion was based on one reduction of the general Lie algebra. In the construction of optimal system, adjoint representation was adopted by previous research, yet we rely only on commutator table and its derivation process is given. Depending on the constracted optimal system, four indirectly solvable symmetry reductions with infinite number of solutions are given and discussed, which were not available in previous study. Although the new conservation theorem is both applied, the conservation laws are distinct for distinct Lie point symmetries, we further construct exact solution of the dBKP equation by a conservation law. The aim of this paper is to analyze the Lie algebra classifications, symmetry reductions and exact invariant solutions of the dBKP equation based on the Lie point symmetry, and to derive the conservation laws according to the new conservation theorem.

The whole structure can be divided into five main sections. In next section, we review the connection between the dBKP hierarchy and equation. Section 3 describes the Lie algebra with one-parameter transformation and their reductions for the dBKP equation under Lie symmetry analysis. Section 4 constructs an one-dimensional optimal system of subalgebras of dBKP equation. In Sect. 5, plentiful reduced equations and invariant solutions are derived. In Sect. 6, the new conservation theorem is utilized to link six successive symmetries and conserved quantities of the dBKP equation, besides, exact solution is constructed according to a set of conservation law. The last section gives conclusions and discussions.

2 The dBKP Hierarchy and the dBKP Equation

We begin with a brief review to the connection between the dBKP equation and dBKP hierarchy. The Lax function of dispersionless BKP hierarchy

$$\varGamma = \kappa + u_{2n-1}\kappa^{2n-1}, \quad n = 1, 3, 5, \dots, \tag{3}$$

in which κ is a conjugate variable of x, and $\varGamma$ is odd Laurent series of κ. $\varGamma$ can be written as

$$\varGamma = e^{adY}(\kappa), \quad adY(\varPsi) = \{Y, \varPsi\} = \partial_\kappa Y \partial_x \varPsi - \partial_\kappa \varPsi \partial_x Y, \tag{4}$$

the form of the dressing function is as

$$\Upsilon = \Upsilon_2\kappa^{-1} + \Upsilon_4\kappa^{-3} + \cdots . \tag{5}$$

Definition 2.1 The dispersionless BKP hierarchy are composed of flows in the Lax pair [20]

$$\partial_{\Gamma_i} = \left\{(\Gamma^i)_+, \Gamma\right\} = \partial_\kappa(\Gamma^i)_+\partial_x\Gamma - \partial_\kappa\Gamma\partial_x(\Gamma^i)_+, \quad i = 1, 3, 5, \dots , \tag{6}$$

in which "+" represents the nonnegative projection about κ and "−" represents the negative projection. Eq. (1) is the simplest nontrivial flow in the dBKP hierarchy.

3 Lie Algebra Classifications

In this section, the Lie symmetry analysis to Eq. (2) starts at considering following one-parameter Lie group of infinitesimals transformation

$$\begin{cases} \tilde{x} : x + \varsigma\xi^x + O(\varsigma^2), \\ \tilde{t} : t + \varsigma\xi^t + O(\varsigma^2), \\ \tilde{y} : y + \varsigma\xi^y + O(\varsigma^2), \\ \tilde{u} : u + \varsigma\Phi + O(\varsigma^2), \end{cases} \tag{7}$$

where $\xi^x, \xi^t, \xi^y, \Phi$ are functions of x, t, y, u and $\varsigma > 0$ is a sufficiently small one-parameter, the vector field relevant to (7) is

$$V = \xi^x\partial_x + \xi^t\partial_t + \xi^y\partial_y + \Phi\partial_u. \tag{8}$$

The following second prolongation holds

$$pr^{(2)}\,V = V + \Phi^x\partial_{u_x} + \Phi^y\partial_{u_y} + \Phi^{yy}\partial_{u_{yy}} + \Phi^{xt}\partial_{u_{xt}} + \Phi^{xy}\partial_{u_{xy}} + \Phi^{xx}\partial_{u_{xx}}, \tag{9}$$

with

$$\begin{cases} \Phi^y = D_y(\Phi - \xi^x u_x - \xi^y u_y - \xi^t u_t) + \xi^x u_{xy} + \xi^y u_{yy} + \xi^t u_{ty}, \\ \Phi^x = D_x(\Phi - \xi^x u_x - \xi^y u_y - \xi^t u_t) + \xi^x u_{xx} + \xi^y u_{yx} + \xi^t u_{tx}, \\ \Phi^{yy} = D_y^2(\Phi - \xi^x u_x - \xi^y u_y - \xi^t u_t) + \xi^x u_{xyy} + \xi^y u_{yyy} + \xi^t u_{tyy}, \\ \Phi^{xy} = D_y D_x(\Phi - \xi^x u_x - \xi^y u_y - \xi^t u_t) + \xi^x u_{xxy} + \xi^y u_{xyy} + \xi^t u_{xty}, \\ \Phi^{xx} = D_x^2(\Phi - \xi^x u_x - \xi^y u_y - \xi^t u_t) + \xi^x u_{xxx} + \xi^y u_{xxy} + \xi^t u_{xxt}, \\ \Phi^{xt} = D_t D_x(\Phi - \xi^x u_x - \xi^y u_y - \xi^t u_t) + \xi^x u_{xxt} + \xi^y u_{xyt} + \xi^t u_{xtt}, \end{cases} \tag{10}$$

in which D_p, $(p = x, y, t)$ are the total derivatives respectively about p .

Above deterministic Eq. (10) can be produced under constant condition

$$pr^{(2)}\,V\big|_{\Delta=0} = 0, \tag{11}$$

with $\Delta = 3u_{xt} + 15u_x^2 u_{xx} - 5u_x u_{xy} - 5u_{xx} u_y - \frac{5}{3} u_{yy} = 0$. According to Eq. (11), the following equation holds

$$3\Phi^{xt} + \Phi^x(30u_x u_{xx} - 5u_{xy}) + \Phi^{xx}(15u_x^2 - 5u_y) - 5u_x \Phi^{xy} - 5u_{xx}\Phi^y - \frac{5}{3}\Phi^{yy} = 0 \tag{12}$$

The form of the coefficient function is obtained by calculating the standard symmetry group

$$\begin{cases} \xi^x = \frac{x}{5}(f_{1t} + 10c) + \frac{9}{50}y^2 f_{1tt} + \frac{3}{5}y f_{2t} + f_3, \\ \xi^y = y(\frac{3}{5}f_{1t} + c) + f_2, \\ \xi^t = f_1, \\ \Phi = u(3c - \frac{1}{5}f_{1t}) - \frac{3x}{25}(y f_{1tt} + \frac{5}{3}f_{2t}) - \frac{9}{250}y^3 f_{1ttt} - \frac{9}{50}y^2 f_{2tt} - \frac{3}{5}y f_{3t} + f_4, \end{cases} \tag{13}$$

where c is an arbitrary constant, f_1,f_2,f_3 and f_4 are arbitrary functions about t. Consequently, the vector field of Eq. (2) can be determined by below infinitesimal symmetries

$$\begin{cases} V_1(f_1) = f_1\partial_t + (\frac{x}{5}f_{1t} + \frac{9}{50}y^2 f_{1tt})\partial_x + \frac{3y}{5}f_{1t}\partial_y - (\frac{u}{5}f_{1t} + \frac{3x}{25}y f_{1tt} + \frac{9}{250}y^3 f_{1ttt})\partial_u, \\ V_2(f_2) = \frac{3y}{5}f_{2t}\partial_x + f_2\partial_y - (\frac{x}{5}f_{2t} + \frac{9y^2}{50}f_{2tt})\partial_u, \\ V_3(f_3) = f_3\partial_x - \frac{3}{5}y f_{3t}\partial_u, \\ V_4(f_4) = f_4\partial_u, \\ V_5 = 2x\partial_x + y\partial_y + 3u\partial_u. \end{cases} \tag{14}$$

This set of vectors form a Lie algebra under commutative operations $[V_i, V_j] = V_i V_j - V_j V_i$, which is skew symmetric with each diagonal term being zero. One can give the commutator table of system (2) (see Table 1).

The following four cases are convenient for calculating the commutative relations of infinitesimal symmetries and deriving invariant solutions of Eq. (2), if arbitrary functions $f_i\,(i = 1, \dots, 4)$ are defined as

Case 1. $f_1 = \cdots = f_4 = 1$ The infinitesimal symmetries of Eq. (2) form the five-dimensional Lie algebra L^5 are spanned by the following independent operators

Table 1 Commutation table of symmetries

$[V_i,V_j]$	V_1	V_2	V_3	V_4	V_5
V_1	0	$V_2(f_1 f_{2t} - \frac{3}{5}f_{1t}f_2)$	$V_3(f_1 f_{3t} - \frac{1}{5}f_{1t}f_3)$	$V_4(f_1 f_{4t} + \frac{1}{5}f_{1t}f_4)$	0
V_2	$V_2(-f_1 f_{2t} + \frac{3}{5}f_{1t}f_2)$	0	$V_4(-\frac{3}{5}f_2 f_{3t} + \frac{1}{5}f_{2t}f_3)$	0	$V_2(f_2)$
V_3	$V_3(-f_1 f_{3t} + \frac{1}{5}f_{1t}f_3)$	$V_4(\frac{3}{5}f_2 f_{3t} - \frac{1}{5}f_{2t}f_3)$	0	0	$V_3(2f_3)$
V_4	$V_4(-f_1 f_{4t} - \frac{1}{5}f_{1t}f_4)$	0	0	0	$V_4(3f_4)$
V_5	0	$V_2(-f_2)$	$V_3(-2f_3)$	$V_4(-3f_4)$	0

$$V_1 = \partial_t, \quad V_2 = \partial_y, \quad V_3 = \partial_x, \quad V_4 = \partial_u, \quad V_5 = 2x\partial_x + y\partial_y + 3u\partial_u. \tag{15}$$

Then the commutative relations of above operators can be given in Table 2.

The following one-parameter ς symmetry groups τ_i $(1, \ \ldots \ ,5)$ generated by the correlating infinitesimal symmetries V_i $(i = 1 \ \ldots \ ,5)$ hold

$$\begin{cases} \tau_1 : (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) \to (x, y, \varsigma + t, u), \\ \tau_2 : (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) \to (x, y + \varsigma, t, u), \\ \tau_3 : (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) \to (x + \varsigma, y, t, u), \\ \tau_4 : (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) \to (x, y, t, u + \varsigma), \\ \tau_5 : (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) \to (e^{2\varsigma}x, e^{\varsigma}y, t, e^{3\varsigma}u), \end{cases} \tag{16}$$

where τ_1 is a translation of time, $\tau_2, \ \tau_3$ are translations about space, τ_4 is a dependent variable translation, τ_5 is a translation of scale. (16) implies that if $u = f(x, y, t)$ is a solution of Eq. (2), so are $u^{(j)}$ $(1 \le j \le 5)$

$$\begin{cases} u^{(1)} = f(x, y, t - \varepsilon), \quad u^{(2)} = f(x, y - \varepsilon, t), \quad u^{(3)} = f(x - \varepsilon, y, t), \\ u^{(4)} = \varepsilon + f(x, y, t), \quad u^{(5)} = e^{-3\varepsilon} f(xe^{-2\varepsilon}, ye^{-\varepsilon}, t). \end{cases} \tag{17}$$

Case 2. $f_1 = \cdots = f_4 = t + 1$

The infinitesimal symmetries are shown as follows

$$\begin{cases} V_1^2 = (t+1)\partial_t + \frac{x}{5}\partial x + \frac{3y}{5}\partial y - \frac{u}{5}\partial u, \quad V_2^2 = \frac{3y}{5}\partial x + (t+1)\partial_y - \frac{x}{5}\partial u, \\ V_3^2 = (t+1)\partial_x - \frac{3y}{5}\partial u, \quad V_4^2 = (t+1)\partial_u, \quad V_5 = 2x\partial_x + y\partial_y + 3u\partial_u. \end{cases} \tag{18}$$

Case 3. $f_1 = f_2 = t + 1$, $f_3 = f_4 = (t+1)^2$

With the infinitesimal symmetries

$$\begin{cases} V_1^2 = (t+1)\partial_t + \frac{x}{5}\partial x + \frac{3y}{5}\partial y - \frac{u}{5}\partial u, \quad V_2^2 = \frac{3y}{5}\partial x + (t+1)\partial_y - \frac{x}{5}\partial u, \\ V_3^3 = (t+1)^2\partial_x - \frac{6y}{5}(t+1)\partial u, \quad V_4^3 = (t+1)^2\partial_u, \quad V_5 = 2x\partial_x + y\partial_y + 3u\partial_u. \end{cases} \tag{19}$$

Case 4. $f_1 = f_2 = 1$, $f_3 = f_4 = e^t$

With the infinitesimal symmetries

$$V_1 = \partial_t, \quad V_2 = \partial_y, \quad V_3^4 = e^t\partial_x - \frac{3y}{5}e^t\partial u, \quad V_4^4 = e^t\partial_u, \quad V_5 = 2x\partial_x + y\partial_y + 3u\partial_u. \tag{20}$$

Table 2 Commutation table of symmetries in case 1

$[V_i, V_j]$	V_1	V_2	V_3	V_4	V_5
V_1	0	0	0	0	0
V_2	0	0	0	0	V_2
V_3	0	0	0	0	$2V_3$
V_4	0	0	0	0	$3V_4$
V_5	0	$-V_2$	$-2V_3$	$-3V_4$	0

The commutative operations of infinitesimal symmetries in cases 2, 3, 4 are presented in the Appendix 1.

4 Optimal System

In this section, we construct an optimal system of one-dimensional subalgebras of case 1 for Eq. (2) with the method of Ibragimov in [6], which has the advantage of relying only on the commutator table (see Table 2), Supposing that any vector field can be written as

$$V = R_1V_1 + R_2V_2 + R_3V_3 + R_4V_4 + R_5V_5, \tag{21}$$

utilizing the following generators to find linear transformations of the vector $(R_1, R_2, R_3, R_4, R_5)$

$$\Xi_i = P^{\gamma}_{ij}R_j\partial_{R^\gamma}, \quad i = 1,\ 2,\ 3,\ 4, \tag{22}$$

where P^{γ}_{ij} are represented by the formula $[\mathrm{V}_i, \mathrm{V}_j] = \mathrm{P}^{\gamma}_{ij}$.

Theorem 4.1 *The following operators provide an optimal system of one-dimensional subalgebras of the Lie algebra spanned by* $V_1,\ V_2,\ V_3,\ V_4,\ V_5$

$$\begin{cases} V_1,\ V_2,\ V_3,\ V_4,\ V_5,\ V_1 \pm V_2,\ V_1 \pm V_3,\ V_1 \pm V_4,\ V_1 \pm V_5,\ V_2 \pm V_3,\ V_2 \pm V_4,\ V_3 \pm V_4, \\ V_1 \pm V_2 \pm V_3,\ V_1 \pm V_2 \pm V_4,\ V_1 \pm V_3 \pm V_4,\ V_2 \pm V_3 \pm V_4,\ V_1 \pm V_2 \pm V_3 \pm V_4. \end{cases} \tag{23}$$

Proof According to (22) and the Table 2, $\Xi_1, \Xi_2, \Xi_3, \Xi_4$ and Ξ_5 can be written as

$$\begin{cases} \Xi_1 = 0, \\ \Xi_2 = R^5\partial_{R^2}, \\ \Xi_3 = 2R^5\partial_{R^3}, \\ \Xi_4 = 3R^5\partial_{R^4}, \\ \Xi_5 = -R^2\partial_{R^2} - 2R^3\partial_{R^3} - 3R^4\partial_{R^4}, \end{cases} \tag{24}$$

for the generators Ξ_i, c_i with the initial condition $\tilde{R}|_{c_i=0} = R\,(i = 1, \dots, 5)$ are written as

$$\begin{cases} \frac{d\tilde{R}_1}{dc_1} = 0,\ \frac{d\tilde{R}_2}{dc_1} = 0,\ \frac{d\tilde{R}_3}{dc_1} = 0,\ \frac{d\tilde{R}_4}{dc_1} = 0,\ \frac{d\tilde{R}_5}{dc_1} = 0, \\ \frac{d\tilde{R}_1}{dc_2} = 0,\ \frac{d\tilde{R}_2}{dc_2} = \tilde{R}_5,\ \frac{d\tilde{R}_3}{dc_2} = 0,\ \frac{d\tilde{R}_4}{dc_2} = 0,\ \frac{d\tilde{R}_5}{dc_2} = 0, \\ \frac{d\tilde{R}_1}{dc_3} = 0,\ \frac{d\tilde{R}_2}{dc_3} = 0,\ \frac{d\tilde{R}_3}{dc_3} = 2\tilde{R}_5,\ \frac{d\tilde{R}_4}{dc_3} = 0,\ \frac{d\tilde{R}_5}{dc_3} = 0, \\ \frac{d\tilde{R}_1}{dc_4} = 0,\ \frac{d\tilde{R}_2}{dc_4} = 0,\ \frac{d\tilde{R}_3}{dc_4} = 0,\ \frac{d\tilde{R}_4}{dc_4} = 3\tilde{R}_5,\ \frac{d\tilde{R}_5}{dc_4} = 0, \\ \frac{d\tilde{R}_1}{dc_5} = 0,\ \frac{d\tilde{R}_2}{dc_5} = -\tilde{R}_2,\ \frac{d\tilde{R}_3}{dc_5} = -2\tilde{R}_3,\ \frac{d\tilde{R}_4}{dc_5} = -3\tilde{R}_4,\ \frac{d\tilde{R}_5}{dc_5} = 0, \end{cases} \tag{25}$$

and the solutions of Eq. (25) give the transformations

$$\begin{cases} T_1 \,:\, \tilde{R_1} = R_1,\ \tilde{R_2} = R_2,\ \tilde{R_3} = R_3,\ \tilde{R_4} = R_4,\ \tilde{R_5} = R_5, \\ T_2 \,:\, \tilde{R_1} = R_1,\ \tilde{R_2} = R_2 + c_2 R_5,\ \tilde{R_3} = R_3,\ \tilde{R_4} = R_4,\ \tilde{R_5} = R_5, \\ T_3 \,:\, \tilde{R_1} = R_1,\ \tilde{R_2} = R_2,\ \tilde{R_3} = R_3 + 2c_2 R_5,\ \tilde{R_4} = R_4,\ \tilde{R_5} = R_5, \\ T_4 \,:\, \tilde{R_1} = R_1,\ \tilde{R_2} = R_2,\ \tilde{R_3} = R_3,\ \tilde{R_4} = R_4 + 3c_2 R_5,\ \tilde{R_5} = R_5, \\ T_5 \,:\, \tilde{R_1} = R_1,\ \tilde{R_2} = e^{-c_5} R_2,\ \tilde{R_3} = e^{-2c_5} R_3,\ \tilde{R_4} = e^{-3c_5} R_4,\ \tilde{R_5} = R_5. \end{cases} \tag{26}$$

It is necessary to construct the optimal system to simplify the vector

$$\mathbf{R} = (R_1,\ R_2,\ R_3,\ R_4,\ R_5). \tag{27}$$

Since the rank of a matrix $\|\mathbf{P}^{\gamma}_{\mathbf{ij}}\mathbf{R_j}\|(\gamma = 1, \ldots, 5)$ of the coefficients of the operators (24) is three, which means that the matrix has two functionally independent invariants R^m $(m = 1,\ 2)$. Integrating the equation

$$\Xi_{\gamma}(\mathbf{R}) = 0, \quad \gamma = 1, \ldots, 5, \tag{28}$$

following invariants hold

$$\begin{cases} R^1 = R_1, \\ R^2 = R_5. \end{cases} \tag{29}$$

Since whether the invariant is 0 or not affects the combinations of generators, the process of constructed optimal system can only be divided into two cases. □

Case 1. $R_5 \neq 0$

Respectively taking $c_2 = -\frac{R_2}{R_5}$, $c_3 = -2\frac{R_3}{R_5}$, $c_4 = -3\frac{R_4}{R_5}$ in $T_3,\ T_4,\ T_5$ and simplify the vector (27) to the form

$$\mathbf{R} = (R_1,\ 0,\ 0,\ 0,\ R_5). \tag{30}$$

(1.1) $R_1 = 0$

Providing the operator

$$V_5. \tag{31}$$

(1.2) $R_1 \neq 0$

Without losing generality, assuming $R_1 = 1, R_5 = \pm 1$ in the vector (27), providing the operator

$$V_1 \pm V_5. \tag{32}$$

Case 2. $R_5 = 0$

(2.1) $R_1 \neq 0$

The vector (27) is reduced to the form

$$\mathbf{R} = (R_1,\ R_2,\ R_3,\ R_4,\ 0), \tag{33}$$

that cannot be reduced, taking all possible linear combinations, the following representatives are given

$$\begin{aligned} &V_1,\ V_1 \pm V_2,\ V_1 \pm V_3,\ V_1 \pm V_4,\ V_1 \pm V_2 \pm V_3,\ V_1 \pm V_2 \pm V_4,\\ &\quad V_1 \pm V_3 \pm V_4,\ V_1 \pm V_2 \pm V_3 \pm V_4. \end{aligned} \tag{34}$$

(2.2) $R_1 = 0$

Taking into account all combinations, we obtain the following representatives

$$V_2 \pm V_3 \pm V_4,\ V_2 \pm V_3,\ V_2 \pm V_4,\ V_3 \pm V_4,\ V_2,\ V_3,\ V_4. \tag{35}$$

The optimal system is provided by collecting subalgebras (31), (32), (34) and (35), thus the proof is completed.

5 Symmetry Reductions and Invariant Solutions

In this section, the Lie group of point transformation method is employed in the dBKP equation. The subalgebras in (14), the obtained optimal system and cases 2–4 are selected to acquire symmetry reductions and group invariant solutions of dBKP equation, which can been realized through solving associated Lagrange characteristic equation

$$\frac{dx}{\xi^x(x,y,t,u)} = \frac{dy}{\xi^y(x,y,t,u)} = \frac{dt}{\xi^t(x,y,t,u)} = \frac{du}{\phi(x,y,t,u)}. \tag{36}$$

Subalgebra $V_3(f_3) = f_3\partial_x - \frac{3}{5}yf_{3t}\partial_u$

Associated Lagrange equation is

$$\frac{dx}{f_3} = -\frac{5du}{3yf_{3t}}. \tag{37}$$

Then, the following similarity form holds

$$\alpha = t, \quad \beta = y, \quad u = \frac{5f(\alpha,\beta) - 3x\beta f_{3\alpha}}{5f_3}. \tag{38}$$

Substituting above variables from Eq. (38) into Eq. (2), then Eq. (2) is reduced to

$$27\beta f_{3\alpha\alpha} + 25 f_{\beta\beta} = 0, \tag{39}$$

therefore, solutions of Eq. (1) are

$$w = u_x = -\frac{3yf_{3t}}{5f_3}. \tag{40}$$

Subalgebra $V_3(f_3) + V_4(f_4) = f_3\partial_x - \frac{3}{5}yf_{3t}\partial_u + V_4(f_4) = f_4\partial_u$

By solving the associated characteristic equation, the similarity variables yield

$$\alpha = t, \quad \beta = y, \quad u = \frac{(f(\alpha,\beta) - x)(5f_4 + 3\beta f_{3t})}{5f_3}. \tag{41}$$

Substituting these variables into Eq. (2), one has

$$27y(f_{3\alpha\alpha} + f_{4\alpha}) + 25f_{\beta\beta} = 0, \tag{42}$$

therefore, we get solutions of Eq. (1)

$$w = u_x = \frac{-3yf_{3t} - 5f_4}{5f_3}. \tag{43}$$

The concrete results in optimal system and cases 2–4 are presented in Tables 3 and 4 by repeating the Lie group method

The exact solutions of Eq. (1) can be acquired by determining certain arbitrary functions.

Subalgebra $V_2 = \partial_y$ Taking $F_1(z) = \frac{1}{3}(c_1 z + c_2)$, where c_1 and c_2 are arbitrary constants, we get following binary equation

$$5z^2(x,t)t + c_1 z(x,t) + c_2 - 3x = 0, \tag{44}$$

solving Eq. (44), a solution of Eq. (1) is derived as

$$w = \frac{(-c_1 \pm \sqrt{60tx - 20c_2 t + c_1^2})}{10t}. \tag{45}$$

Subalgebra $V_2 + V_4 = \partial_y + \partial_u$

Taking $F_1(z) = lnz$, we get the following binary equation

$$15z^2(x,t)t + lnz(x,t) - 5t - 3x = 0, \tag{46}$$

a solution of Eq. (1) can be obtained by solving Eq. (46)

$$w = exp(-\frac{1}{2}LamberW(30t \cdot exp(10t + 6x)) + 5t + 3x). \tag{47}$$

Subalgebra $V_2 + V_3 = \partial_y + \partial_x$

Taking $F_1(z) = 0$, solving the following binary equation

$$45z^2(x,t)t + 30tz(x,t) - 5t - 9x - 9y = 0, \tag{48}$$

we acquire solutions of Eq. (1)

$$w = -\frac{1}{3} \pm \frac{\sqrt{50t^2 + 45t(x-y)}}{15t} \mp \int \frac{3}{2(\sqrt{50t^2 + 45t(x-y)})} dy. \tag{49}$$

Subalgebra $V_2 - V_3 + V_4 = \partial_y - \partial_x + \partial_u$

Taking $F_1(z) = 0$, solving the following binary equation

Table 3 Reductions of Eq. (2)

Vector field	Similarity variables	Reduced equation
V_1	$\alpha = y, \beta = x, u = f(\alpha, \beta)$	$3f_\beta^2 f_{\beta\beta} - f_\beta f_{\alpha\beta} - f_{\beta\beta} f_\alpha - \frac{1}{3} f_{\alpha\alpha} = 0$
$V_1 + V_4$	$\alpha = y, \beta = x, u = f(\alpha, \beta) + t$	$3f_\beta^2 f_{\beta\beta} - f_\beta f_{\alpha\beta} - f_{\beta\beta} f_\alpha - \frac{1}{3} f_{\alpha\alpha} = 0$
V_2	$\alpha = t, \beta = x, u = f(\alpha, \beta)$	$f_{\alpha\beta} + 5f_\beta^2 f_{\beta\beta} = 0$
$V_2 + V_4$	$\alpha = t, \beta = x, u = f(\alpha, \beta) + y$	$3f_{\alpha\beta} + 15f_\beta^2 f_{\beta\beta} - 5f_{\beta\beta} = 0$
$V_2 + V_3$	$\alpha = t, \beta = x - y, u = f(\alpha, \beta)$	$3f_{\alpha\beta} + 15f_\beta^2 f_{\beta\beta} + 10f_\beta f_{\beta\beta} - \frac{5}{3} f_{\beta\beta} = 0$
$V_2 - V_3 + V_4$	$\alpha = t, \beta = x + y, u = f(\alpha, \beta) + y$	$3f_{\alpha\beta} + 15f_\beta^2 f_{\beta\beta} - 10f_\beta f_{\beta\beta} - \frac{20}{3} f_{\beta\beta} = 0$
V_3	$\alpha = t, \beta = y, u = f(\alpha, \beta)$	$f_{\beta\beta} = 0$
$V_3 + V_4$	$\alpha = t, \beta = y, u = f(\alpha, \beta) + x$	$f_{\beta\beta} = 0$
V_5	$\alpha = \frac{x}{y^2}, \beta = t, u = y^3 f(\alpha, \beta)$	$3f_{\alpha\beta} + 15f_\alpha^2 f_{\alpha\alpha} - 5f_\alpha^2 + 20\alpha f_\alpha f_{\alpha\alpha} - 15ff_{\alpha\alpha} - 10f + 10\alpha f_\alpha - \dfrac{20}{3}\alpha^2 f_{\alpha\alpha} = 0$
$V_1 + V_5$	$\alpha = 2t - lnx, \beta = t - lny$	$-be^{2(\alpha-2\beta)} f_{\alpha\alpha} - 3e^{2(\alpha-2\beta)} f_{\alpha\beta} + 15e^{4(\alpha-2\beta)} f_\alpha^3 + 15e^{4(\alpha-2\beta)} f_\alpha^2 f_{\alpha\alpha} + e^{2(\alpha-2\beta)} f_\alpha f_\beta$
	$u = y^3 f(\alpha, \beta)$	$-15e^{2(\alpha-2\beta)} f_\alpha^2 - 15e^{2(\alpha-2\beta)} ff_\alpha + 5e^{2(\alpha-2\beta)} f_\beta f_{\alpha\alpha} - 15e^{2(\alpha-2\beta)} ff_{\alpha\alpha} + \frac{25}{3} f_\beta - 10f - \frac{5}{3} f_{\beta\beta} = 0$
$V_1 + V_2$	$\alpha = x, \beta = y - t, u = f(\alpha, \beta)$	$-3f_{\alpha\beta} + 15f_\alpha^2 f_{\alpha\alpha} - 5f_\alpha f_{\alpha\beta} - 5f_{\alpha\alpha} f_\beta - \frac{5}{3} f_{\beta\beta} = 0$
$V_1 - V_2 + V_4$	$\alpha = x, \beta = y + t, u = f(\alpha, \beta) + t$	$3f_{\alpha\beta} + 15f_\alpha^2 f_{\alpha\alpha} - 5f_\alpha f_{\alpha\beta} - 5f_{\alpha\alpha} f_\beta - \frac{5}{3} f_{\beta\beta} = 0$
$V_1 + V_3$	$\alpha = y, \beta = x - t, u = f(\alpha, \beta)$	$-3f_{\beta\beta} + 15f_\beta^2 f_{\beta\beta} - 5f_\beta f_{\alpha\beta} - 5f_{\beta\beta} f_\alpha - \frac{5}{3} f_{\alpha\alpha} = 0$
$V_1 - V_3 + V_4$	$\alpha = y, \beta = x + t, u = f(\alpha, \beta) + t$	$3f_{\beta\beta} + 15f_\beta^2 f_{\beta\beta} - 5f_\beta f_{\alpha\beta} - 5f_{\beta\beta} f_\alpha - \frac{5}{3} f_{\alpha\alpha} = 0$
$V_1 + V_2 + V_3$	$\alpha = y - t, \beta = x - t, u = f(\alpha, \beta)$	$-3(f_{\alpha\beta} + f_{\beta\beta}) + 15f_\beta^2 f_{\beta\beta} - 5f_\beta f_{\alpha\beta} - 5f_{\beta\beta} f_\alpha - \frac{5}{3} f_{\alpha\alpha} = 0$
$V_1 + V_2 - V_3 + V_4$	$\alpha = y - t, \beta = x + t, u = f(\alpha, \beta) + t$	$-3(f_{\alpha\beta} - f_{\beta\beta}) + 15f_\beta^2 f_{\beta\beta} - 5f_\beta f_{\alpha\beta} - 5f_{\beta\beta} f_\alpha - \frac{5}{3} f_{\alpha\alpha} = 0$
V_3^2	$\alpha = t, \beta = y, u = \frac{1}{\alpha}(f(\alpha, \beta) - \frac{3x\beta}{5})$	$\frac{18\beta}{5\alpha} + \frac{5}{3} f_{\beta\beta} = 0$
$V_3^2 + V_4^2$	$\alpha = t, \beta = y, u = (f(\alpha, \beta) - x)(1 + \frac{3\beta}{5(\alpha+1)})$	$27\beta + 25f_{\beta\beta} = 0$
V_1^3	$\alpha = x^5(t+1)^{-1}, \beta = y^{\frac{5}{3}}(t+1)^{-1},$	$-18\alpha f_\alpha - 15\alpha(\alpha f_{\alpha\alpha} + \beta f_{\beta\beta}) + 375f_\alpha^2(20\alpha^{\frac{12}{5}} f_\alpha + 15\alpha^{\frac{17}{5}} f_{\alpha\alpha} - \frac{625}{3}\alpha^{\frac{9}{5}}\beta^{\frac{2}{5}} f_\alpha f_{\alpha\beta}$
	$u = (t+1)^{-\frac{1}{5}} f(\alpha, \beta)$	$-\frac{25}{3} f_\beta(20\alpha^{\frac{4}{5}}\beta^{\frac{2}{5}} + 20\alpha^{\frac{9}{5}}\beta^{\frac{2}{5}} f_{\alpha\alpha}) - \frac{50}{27}\alpha^{\frac{1}{5}}\beta^{\frac{1}{5}} - \frac{125}{27}\alpha^{\frac{1}{5}}\beta^{\frac{4}{5}} f_{\beta\beta} = 0$

Table 3 (continued)

Vector field	Similarity variables	Reduced equation
V_3^3	$\alpha = t, \beta = y, u = \frac{6\beta}{5(\alpha+1)}(f(\alpha,\beta) - x)$	$\frac{9}{5(\alpha+1)} + 2f_\beta + \beta f_{\beta\beta} = 0$
$V_3^3 + V_4^3$	$\alpha = t, \beta = y, u = (f(\alpha,\beta) - x)(1 + \frac{6\beta}{5(\alpha+1)})$	$54\beta(2+\alpha) + 25f_{\beta\beta} = 0$
$V_1 + V_5$	$\alpha = x(t+1)^{-\frac{11}{5}}, \beta = y(t+1)^{-\frac{8}{5}}, u = (t+1)^{\frac{14}{5}} f(\alpha,\beta)$	$\frac{9}{5}f_\alpha + 3\alpha f_{\alpha\alpha} + 3\beta f_{\alpha\beta} + 15f_\alpha^2 f_{\alpha\alpha} - 5f_\alpha f_{\alpha\beta} - 5f_{\alpha\alpha}f_\beta - \frac{5}{3}f_{\beta\beta} = 0$
V_3^4	$\alpha = t, \beta = y, u = f(\alpha,\beta) - \frac{3x\beta}{5}$	$\frac{9\beta}{5} + \frac{5}{3}f_{\beta\beta} = 0$
$V_3^4 + V_4^4$	$\alpha = t, \beta = y, u = (f(\alpha,\beta) - x)(1 + \frac{3}{5}\beta)$	$54\beta e^\alpha + 25f_{\beta\beta} = 0$
$V_2 + V_3^4$	$\alpha = t, \beta = ye^t - x, u = \frac{3e^\alpha}{10}[f(\alpha,\beta) - y^2]$	$f_\beta + f_{\alpha\beta} + e^{2\alpha}f_\beta f_{\beta\beta} + \frac{5}{9}e^{2\alpha}f_{\beta\beta} - \frac{9}{20}e^{2\alpha}f_\alpha^2 f_{\beta\beta} - \frac{10}{9} = 0$

Table 4 Group invariant solutions of Eq. (2)

Vector field	Invariant solutions
V_1	$u = c_3 + c_4(c_1 y + c_2 x + c_3)$
$V_1 + V_4$	$u = t + c_3 + c_4(c_1 y + c_2 x + c_3)$
$V_1 + V_2$	$u = c_3 + c_4(c_1 x + c_2(y - t) + c_3)$
$V_1 - V_2 + V_4$	$u = t + c_3 + c_4(c_1 x + c_2(y + t) + c_3)$
$V_1 + V_3$	$u = c_3 + c_4(c_1 y + c_2(x - t) + c_3)$
$V_1 - V_3 + V_4$	$u = t + c_3 + c_4(c_1 y + c_2(x + t) + c_3)$
$V_1 + V_2 + V_3$	$u = c_3 + c_4(c_1(y - t) + c_2(x - t) + c_3)$
$V_1 + V_2 - V_3 + V_4$	$u = t + c_3 + c_4(c_1(y - t) + c_2(x + t) + c_3)$
V_2	$u = \int RootOf(5z^2 t + 3F_1(z) - 3x)dx + F_2(t), w = RootOf(5z^2 t + 3F_1(z) - 3x)$
$V_2 + V_4$	$u = y + \int RootOf(15z^2 t + 3F_1(z) - 5t - 3x)dx + F_2(t), w = RootOf(15z^2 t + 3F_1(z) - 5t - 3x)$
$V_2 + V_3$	$u = \int RootOf(45tz^2 + 30tz + 9F_1(z) - 5t - 9x + 9y)d(x - y) + F_2(t),$ $w = RootOf(45tz^2 + 30tz + 9F_1(z) - 5t - 9x + 9y) - [\int RootOf(45tz^2 + 30tz + 9F_1(z) - 5t - 9x + 9y)dy]_x$
$V_2 - V_3 + V_4$	$u = \int RootOf(45tz^2 - 30tz + 9F_1(z) - 20t - 9x - 9y)d(x + y) + F_2(t) + y,$ $w = RootOf(45tz^2 - 30tz + 9F_1(z) - 20t - 9x - 9y) + [\int RootOf(45tz^2 - 30tz + 9F_1(z) - 20t - 9x - 9y)dy]_x$
V_3	$u = F_1(t)y + F_2(t)$
$V_3 + V_4$	$u = x + F_1(t)y + F_2(t)$
V_3^2	$u = \frac{5(yF_1(t)+F_2(t))-3xy}{5(t+1)}, w = -\frac{3y}{5(t+1)}$
$V_3^2 + V_4^2$	$u = (-x - \frac{9y^3}{50} + yF_1(t) + F_2(t))(1 + \frac{3y}{5t}), w = -(1 + \frac{3y}{5t})$
V_3^3	$u = (t+1)^{-2}(-\frac{9y^3}{25} + yF_1(t) + F_2(t)) - \frac{6xy}{5(t+1)}, w = -\frac{6y}{5(t+1)}$
$V_3^3 + V_4^3$	$u = (-x - \frac{9}{25}y^3(t+2) + yF_1(t) + F_2(t))(1 + \frac{6y}{5(t+1)}), w = -(1 + \frac{6y}{5(t+1)})$
V_3^4	$u = -\frac{9}{50}y^3 + e^{-t}(yF_1(t) + F_2(t)) - \frac{3xy}{5}, w = -\frac{3y}{5}$
$V_3^4 + V_4^4$	$u = (-x - \frac{9}{25}y^3 e^t + yF_1(t) + F_2(t))(1 + \frac{3y}{5}), w = -(1 + \frac{3y}{5})$

$$45z^2(x,t)t - 30tz(x,t) - 20t - 9x - 9y = 0, \tag{50}$$

solutions of Eq. (1) can be given as

$$w = \frac{1}{3} \pm \frac{\sqrt{125t^2 + 45t(x+y)}}{15t} \pm \int \frac{3}{2(\sqrt{50t^2 + 45t(x+y)})} dy. \tag{51}$$

6 Construction of Conservation Laws of the dBKP System

6.1 Nonlinear Self-Adjointness

In this subsection, we will introduce certain symbols and theorems from [13, 14]. For a kth-order system of PDEs with n-independent variables $\mathbf{x} = (x^1, x^2, \dots, x^n)$ and m-dependent variables $\mathbf{u} = (u^1, u^2, \dots, u^m)$

$$F_\alpha(\mathbf{x}, \mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots, \mathbf{u}_{(k)}) = 0, \quad \alpha = 1, 2, \dots, m, \tag{52}$$

where $\mathbf{u}_{(r)}$ represents the set of derivatives of order r ($r = 1, \dots, m$), the system of adjoint equations for system (52) is as follows

$$F^*_\alpha(\mathbf{x}, \mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots, \mathbf{u}_{(k)}) = 0, \quad \alpha = 1, 2, \dots, m, \tag{53}$$

where

$$F^*_\alpha(\mathbf{x}, \mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots, \mathbf{u}_{(k)}) = \frac{\delta L}{\delta u^\alpha}. \tag{54}$$

The formal Lagrangian and Euler–Lagrange operator are defined as

$$L = v^\beta F_\beta(\mathbf{x}, \mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots, \mathbf{u}_{(k)}) = 0, \quad \mathbf{v{=}v(x)}. \tag{55}$$

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{i=1}^{\infty} (-1)^s D_{i_1} \dots D_{i_j} \frac{\partial}{u^\alpha_{i_1 \dots i_j}}, \quad \alpha = 1, 2, \dots, m, \tag{56}$$

in which D_i is total derivative operator with respect to x^i.

Definition 6.1 A nonlinear system is nonlinear self-adjoint if its adjoint system satisfies

$$F^*_\alpha(\mathbf{x}, \mathbf{v}, \mathbf{u}, \dots, \mathbf{v}_{(s)}, \mathbf{u}_{(s)}) \mid_{v^\alpha = \theta^\alpha(\mathbf{x},\mathbf{u})} = \gamma^\beta_\alpha F_\beta(\mathbf{x}, \mathbf{v}, \mathbf{u}, \dots, \mathbf{v}_{(s)}), \quad \alpha = 1, \dots, m. \tag{57}$$

where $\boldsymbol{\Theta}^\beta_\alpha(\mathbf{x}, \mathbf{u}) \neq \mathbf{0}$ is undecided, $\boldsymbol{\Theta} = (\theta^1, \dots, \theta^m)$ is a n-dimension vector.

Theorem 6.1 *For symmetry generator admitted by system* (52)

$$X_\alpha = \zeta^i(\mathbf{x}, \mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots)\frac{\partial}{\partial x^i} + \eta^\alpha(\mathbf{x}, \mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots)\frac{\partial}{\partial u^\alpha}, \tag{58}$$

the adjoint equations (53) conserve symmetry above, which means the following adjoint symmetry must be admitted by the system of adjoint equations (53),

$$Y = X + \eta_*^\alpha \frac{\partial}{\partial v^\alpha}, \quad \eta_*^\alpha = -[\lambda_\alpha^\beta + D_i(\zeta^i)]v^\beta, \tag{59}$$

where $Y(F_\alpha) = \lambda_\alpha^\beta F_\alpha$, with λ_α^β is a constant that needs to be determined.

Theorem 6.2 (New conservation theorem) *Every Lie point symmetry, Lie–Bäcklund symmetry and nonlocal symmetry X admitted by the system of (52) can give the conservation law of the system consisting of Eq. (52) and the adjoint equations (53), its conservation vector $\mathbf{C} = (C^1, C^2, \dots)$ has the form*

$$\begin{aligned} C^i =& \zeta^n L + W^\alpha \left[\frac{\partial L}{\partial u_i^\alpha} - D_j\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) + D_iD_k\left(\frac{\partial L}{u_{ijk}^\alpha}\right) - \cdots\right] \\ &+ D_j(W^\alpha)\left[\frac{\partial L}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) + \cdots\right] + D_jD_k(W^\alpha)\left[\frac{\partial L}{\partial u_{ijk}^\alpha} - \cdots\right], \end{aligned} \tag{60}$$

where $W^\alpha = \eta^\alpha - \zeta^j u_j^\alpha,\ \alpha = 1, 2, \dots, m$.

Theorem 6.3 *If a vector $\mathbf{C} = (C^x, C^y, C^t)$ satisfies the conservation equation*

$$D_x(C^x) + D_y(C^y) + D_t(C^t) = 0, \tag{61}$$

it is called a conserved vector for Eq. (2).

6.2 Construction of Conservation Laws Using Symmetries

According to the definition and theorems mentioned in Sect. 6.1, for system

$$F = 3u_{xt} + 15u_x^2 u_{xx} - 5u_x u_{xy} - 5u_{xx}u_y - \frac{5}{3}u_{yy}, \tag{62}$$

we write the formal Lagrangian in symmetric form

$$L = -3u_x v_t + 15u_x^2 u_{xx} v + 5u_x u_y v_x + \frac{5}{3}u_y v_y, \tag{63}$$

with L satisfies

$$\frac{\delta L}{\delta u} = F^*, \quad \frac{\delta L}{\delta v} = F. \tag{64}$$

Therefore

$$v = \rho_1(t)y + \rho_2(t), \tag{65}$$

here ρ_i are arbitrary functions of t.

The next step is to use particular symmetries obtained in Sect. 3 to construct conservation laws.

Symmetry generator $X_1 = \frac{\partial}{\partial x}$

We derive the corresponding Lie characteristic functions $W^1 = -u_x$ and $W^2 = -v_x$, hence the conserved vector is composed of

$$\begin{cases} C_1^x = \frac{5}{3}u_y v_y, \\ C_1^y = -\frac{5}{3}u_x v_y, \\ C_1^t = 0. \end{cases} \tag{66}$$

Symmetry generator $X_2 = \frac{\partial}{\partial y}$

The corresponding Lie characteristic functions are $W^1 = -u_y$ and $W^2 = -v_y$, then conserved vector is composed of

$$\begin{cases} C_2^x = 3u_y v_t - 5u_x u_y v_y - 15u_x^2 u_{xy} v, \\ C_2^y = L - \frac{10}{3}u_y v_y, \\ C_2^t = 3u_x v_y. \end{cases} \tag{67}$$

It can be calculated and verified

$$D_x(C_2^x) + D_y(C_2^y) + D_t(C_2^t) = v_y F = 0. \tag{68}$$

Symmetry generator $X_3 = \frac{\partial}{\partial t}$

In this case, we have the corresponding Lie characteristic functions $W^1 = -u_t$ and $W^2 = -v_t$, and conserved vector is composed of

$$\begin{cases} C_3^x = 3u_t v_t - 5u_x u_y v_t - 15u_x^2 u_{xt} v, \\ C_3^y = \frac{5}{3}u_t v_y - \frac{5}{3}u_y v_t, \\ C_3^t = 15u_x^2 u_{xx} v + \frac{5}{3}u_y v_y. \end{cases} \tag{69}$$

We can verify that

$$D_x(C_3^x) + D_y(C_3^y) + D_t(C_3^t) = v_t F = 0. \tag{70}$$

Symmetry generator $X_4 = (t+1)\frac{\partial}{\partial x} - \frac{3y}{5}\frac{\partial}{\partial u}$

We can get the Lie characteristic functions $W^1 = -\frac{3y}{5} - (t+1)u_x$ and $W^2 = -(t+1)v_x$, then following conserved component vectors hold

$$\begin{cases} C_4^x = \frac{5(t+1)}{3}u_y v_y + \frac{9y}{5}v_t, \\ C_4^y = -\frac{5(t+1)}{3}u_x v_y, \\ C_4^t = 0. \end{cases} \tag{71}$$

Symmetry generator $X_5 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 3u\frac{\partial}{\partial u}$

The coefficients of X_5's extension can be calculated

$$\lambda^5 = 1, \quad \eta_* = -4. \tag{72}$$

An adjoint symmetry is generated

$$Y_5 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 3u\frac{\partial}{\partial u} - 4v\frac{\partial}{\partial v}, \tag{73}$$

the corresponding Lie characteristic functions are $W^1 = 3u - 2xu_x - yu_y$ and $W^2 = -4v - yv_y - 2xv_x$, and following conserved component vectors hold

$$\begin{cases} C_5^x = \frac{10x}{3}u_y v_y + y(3u_y v_t - 5u_x u_y v_y - 15u_x^2 u_{xy} v) - 9uv_t - 20u_x u_y v + 15u_x^3 v, \\ C_5^y = y(-3u_x v_t + 15u_x^2 u_{xx} v - \frac{5}{3}u_y v_y) - \frac{10x}{3}u_x v_y + 5uv_y - \frac{20}{3}vu_y, \\ C_5^t = 12vu_x + 3yu_x v_y. \end{cases} \tag{74}$$

Conservation law can be verified

$$D_x(C_5^x) + D_y(C_5^y) + D_t(C_5^t) = (yv_y + 4v)F = 0. \tag{75}$$

Symmetry generator $X_6 = (t+1)\frac{\partial}{\partial t} + \frac{x}{5}\frac{\partial}{\partial x} + \frac{3y}{5}\frac{\partial}{\partial y} - \frac{u}{5}\frac{\partial}{\partial u}$

The coefficients of X_6's extension can be calculated

$$\lambda^6 = -\frac{7}{5}, \quad \eta_* = -\frac{2}{5}, \tag{76}$$

and one gives adjoint symmetry

$$Y_6 = (t+1)\frac{\partial}{\partial t} + \frac{x}{5}\frac{\partial}{\partial x} + \frac{3y}{5}\frac{\partial}{\partial y} - \frac{u}{5}\frac{\partial}{\partial u} - \frac{2v}{5}\frac{\partial}{\partial v}, \tag{77}$$

the corresponding Lie characteristic functions are $W^1 = -\frac{u}{5} - (t+1)u_t - \frac{x}{5}u_x - \frac{3y}{5}u_y$ and $W^2 = -\frac{2v}{5} - (t+1)v_t - \frac{3y}{5}v_y$, then following conserved component vectors

$$\begin{cases} C_6^x = \frac{x}{3}u_y v_y + \frac{3y}{5}(3u_y v_t - 5u_x u_y v_y - 15u_x^2 u_{xy} v) + (t+1)(3u_t v_t - 5u_x u_y v_t - 15u_x^2 u_{xt} v), \\ C_6^y = \frac{3y}{5}(-3u_x v_t + 15u_x^2 u_{xx} v - \frac{5}{3}u_y v_y) + (t+1)(-\frac{5}{3}u_t v_y - \frac{5}{3}u_t v_t) - \frac{x}{3}u_x v_y - \frac{u}{3}v_y - \frac{2}{3}u_y v, \\ C_6^t = (t+1)(15u_x^2 u_{xx} v + \frac{5}{3}u_y v_y) + \frac{6}{5}vu_x + \frac{9y}{5}u_x v_y \end{cases} \tag{78}$$

can be obtained. Conservation law can be verified

$$D_x(C_6^x) + D_y(C_6^y) + D_t(C_6^t) = \frac{2v}{5}F = 0. \tag{79}$$

6.3 Constructing Solution of the dBKP Equation by Conserved Vector (66)

We know the conserved component vectors obtained of symmetry generator X_1

$$\begin{cases} C_1^x = \frac{5}{3}u_y v_y, \\ C_1^y = -\frac{5}{3}u_x v_y, \\ C_1^t = 0. \end{cases} \tag{80}$$

Taking

$$D_x(C_1^x) = D_y(C_1^y) = D_t(C_1^t) = 0, \tag{81}$$

and assuming

$$\begin{cases} \frac{5}{3}u_y\rho_1 = G(t,y), \\ \frac{5}{3}u_x\rho_1 = H(t,x), \end{cases} \tag{82}$$

then

$$\begin{cases} u_y = \frac{3G}{5\rho_1}, \\ w(x,y,t) = u_x = -\frac{3H}{5\rho_1}. \end{cases} \tag{83}$$

Substituting the relationships of (81) into Eq. (2), one has

$$\frac{9}{5}\left(\frac{H\rho_{1t} - H_t\rho_1}{\rho_1}\right) - \frac{81H^2H_x}{25\rho_1} + 9GH_x - G_y = 0, \tag{84}$$

only the last two terms $9GH_x, -G_y$ may contain independent variable y, thus the calculations can be shorten significantly if we consider two special cases.

Case 1. $H_x = 0$

Under this circumstance, the following relationship holds

$$\begin{cases} H(x,t) = h(t), \\ G_y(y,t) = g(t). \end{cases} \tag{85}$$

Then Eq. (84) becomes

$$h_t - \frac{\rho_{1t}}{\rho_1}h = -\frac{5}{9}g, \tag{86}$$

using the constant coefficient variation method, we obtain

$$h = \rho_1(a - \int \frac{5}{9}\frac{g}{\rho_1}dt), \tag{87}$$

where a is an arbitrary constant, Eq. (83) can be written as

$$\begin{cases} u_y = \frac{3}{5}\frac{yg+l}{\rho_1}, \\ w(x,y,t) = u_x = \int \frac{g}{3\rho_1}dt - \frac{3}{5}c. \end{cases} \tag{88}$$

Integrating the system (88), we get

$$u(x,y,t) = \frac{3}{5\rho_1}\left(\frac{y^2 g}{2} + yl\right) + x\left(\int \frac{g}{3\rho_1}dt - \frac{3}{5}c\right) + m, \tag{89}$$

where l, m are arbitrary functions of t.

Case 2. $G(t,y) = g(t)$

Equation (84) becomes

$$H_t + \frac{9H_x H^2 - 5H\rho_{1t}}{5\rho_1} - GH_x = 0, \tag{90}$$

if taking

$$\begin{cases} \rho_1 = 1, \\ G = t, \end{cases} \tag{91}$$

the following formula is calculated

$$H(t,x) = RootOf(18z^2 t - 5t^2 - 10k(z) - 10x), \tag{92}$$

where k is an arbitrary functions of z, a solution of Eq. (84) can be acquired if we make $k(z) = lnz$.

$$H(x,t) = exp\Big(-\frac{1}{2}LambertW(-36t \cdot exp(-10t^2 - 20x)) - 5t^2 - 10x\Big). \tag{93}$$

Hence one of solutions of Eq. (1) is

$$w = u_x = -\frac{3}{5}H(x,t) = -\frac{3}{5}exp\Big(-\frac{1}{2}LambertW(-36t \cdot exp(-10t^2 - 20x)) - 5t^2 - 10x\Big). \tag{94}$$

7 Conclusions and Discussions

In summary, the authors study the dBKP equation using the Lie symmetry method and Ibragimov's adjoint symmetry approach. The important results are that the Lie point symmetries of the dBKP equation are reduced to four classes, the rich symmetry reductions and new group invariant solutions are derived based on the above reduced symmetries. Except for particular complex symmetries themselves tend to generate still complex reduced equations, these main results prove that the reduction of the infinite-dimensional Lie algebra of symmetries to four classes is effective. For the symmetry reductions of subalgebras V_2, $V_2 + V_4$, $V_2 + V_3$ and $V_2 - V_3 + V_4$ contain infinite solutions which can not be obtained directly, we obtain the exact solutions by determining arbitrary functions. The traveling wave solutions can not be acquired since the dBKP system has the characteristic that each term contains the first power factor of the same order without dispersion term. Compared with other published papers on the study related to B-type equations [25–28], the current paper adds the application of symmetry analysis on the dispersionless B-type equation. Further research on obtained Lie point symmetries and special symmetry reductions by μ-symmetry [29], Laplace transform [30] and PT-symmetry [31] are worth trying in the future.

Appendix 1: Commutation Tables of Cases 2–4 in Sect. 3

See Tables 5, 6 and 7.

Table 5 Commutation table of symmetries in case 2

$[V_i^2, V_j^2]$	V_1^2	V_2^2	V_3^2	V_4^2	V_5
V_1^2	0	$\frac{2}{5}V_2^2$	$\frac{4}{5}V_3^2$	$\frac{6}{5}V_4^2$	0
V_2^2	$-\frac{2}{5}V_2^2$	0	$-\frac{2}{5}V_4^2$	0	V_2^2
V_3^2	$-\frac{4}{5}V_3^2$	$\frac{2}{5}V_4^2$	0	0	$2V_3^2$
V_{24}	$-\frac{6}{5}V_4^2$	0	0	0	$3V_4^2$
V_5	0	$-V_2^2$	$-2V_3^2$	$-3V_4^2$	0

Table 6 Commutation table of symmetries in case 3

$[V_i^3, V_j^3]$	V_1^2	V_2^2	V_3^3	V_4^3	V_5
V_1^2	0	$\frac{2}{5}V_2^2$	$\frac{9}{5}V_3^3$	$\frac{11}{5}V_4^3$	0
V_2^2	$-\frac{2}{5}V_2^2$	0	$-V_4^3$	0	V_2^2
V_3^3	$-\frac{9}{5}V_3^3$	V_4^3	0	0	$2V_3^3$
V_4^3	$-\frac{11}{5}V_4^3$	0	0	0	$3V_4^3$
V_5	0	$-V_2^2$	$-2V_3^3$	$-3V_4^3$	0

Table 7 Commutation table of symmetries in case 4

	V_1	V_2	V_3^4	V_4^4	V_5
V_1	0	0	V_3^4	V_2	0
V_2	0	0	$-\frac{3}{5}V_4^4$	0	V_2
V_3^4	$-V_3^4$	$\frac{2}{5}V_4^4$	0	0	$2V_3^4$
V_4^4	$-V_4^4$	0	0	0	$3V_4^4$
V_5	0	$-V_2$	$-2V_3^4$	$-3V_4^4$	0

Author Contributions QZ conceived the study and the manuscript design as well as classified the Lie algebras of the dBKP equation. HW constructed the one-dimensional optimal system and computed the symmetry reductions of dBKP equation. XL constructed conservation laws and acquired a solution of the dBKP equation from a constructed conservation vector. CL introduced the applications and theories of the dispersionless hierarchies and equations.

Funding Partial financial support was received from the National Nature Science Foundation of China (no. 11701334) and the "Jingying" Project of Shandong University of Science and Technology.

Data Availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no competing interests.

Ethics approval and consent to participate The authors approve and consent to participate.

Consent for publication The authors agree to publication.

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