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ISOLATED HORIZONS AND BLACK HOLE ENTROPY

A Thesis in
Physics
by
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Abstract

The aim of this work is to investigate symmetry based approaches to the problem of black hole entropy. Two sets of symmetries are presented. The first one is based on the new framework of isolated horizons which has emerged recently and which provides a local description of black holes. We describe this framework in 2+1 dimensions and then use it to investigate the question of black hole entropy. We show that the natural symmetries of isolated horizons do not suffice to explain the entropy of black holes. We then turn our attention to a different set of symmetries who are distinguished by the fact that they are only defined in a neighborhood of the horizon and do not have a well defined limit to the horizon. It is then shown that these symmetries provide an explanation of the black hole entropy. We then consider the significance of the results obtained for the search of a theory of quantum gravity.

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Chapter 1

Introduction

Black hole entropy has received so much attention from theoretical physics over the last decades since it seems to be the only clue to a theory of quantum gravity available. Quantum gravity is the attempt to unify the two theories that revolutionized physics in the beginning of the last century: Quantum Mechanics and General Relativity. This introduction will show why it is widely believed that black hole physics holds the clues to a synthesis of those two theories.

1.1 INSIGHTS FROM CLASSICAL GENERAL RELATIVITY

The clues coming from General Relativity are a series of results that are now known as the four laws of black hole mechanics (see table 1.1).

Lets begin the discussion of the classical laws of black hole mechanics with the zeroth law. We will not be too detailed here for we give a new derivation of this fact in chapter 3. For a more detailed exposition of the laws we refer the reader to the literature [6, 33, 50, 51]. Lets assume that we are dealing with a Killing horizon, i.e. we are given a Killing vector field ξ in our spacetime and the horizon is ruled by integral curves of this vector field. The surface gravity κ can then be defined locally to be

$$\kappa^2 \equiv -(\nabla^a|\xi|)(\nabla_a|\xi|). \quad (1.1)$$

Note here that ξ has been normalized to unity at infinity to fix the overall rescaling freedom. It is now a fact that this locally defined quantity is constant over the horizon. This is the content of the zeroth law of black hole mechanics.

There exist several proofs of the zeroth law. Some of them rely on the existence of a Killing horizon together with some symmetry assumptions on the spacetime [15, 16, 17, 39]. These proofs do not make use of the field equations. The law can also be proved using stationarity, the field equations, and the dominant energy condition [6, 23, 25].

0th law	$\kappa = \text{const.}$	The surface gravity κ is constant over the horizon of the black hole.
1st law	$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega \delta J + \Phi \delta Q$	Changes of the parameters of nearby stationary black hole spacetimes are not independent and are related by the first law.
2nd law	$\delta A \geq 0$	The horizon area never decreases.
3rd law		The surface gravity of a black hole can not be reduced to zero in a finite number of steps.

Table 1.1: This table shows the classical laws of black hole mechanics. In this table A denotes the area of the horizon, Ω is the angular velocity, J the angular momentum, Φ is the electric potential, and Q the charge of the horizon.

The first law of black hole mechanics can be given in two versions: The "physical process version" and the "equilibrium state version". The physical process version relates the parameters of a black hole in a given spacetime before and after a physical process bringing about a small change has occurred. The equilibrium state version relates the parameters of two distinct black hole spacetimes. Both versions lead to

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega \delta J + \Phi \delta Q, \quad (1.2)$$

where A is the surface area of the horizon, Ω is the angular velocity of the horizon, J is the horizons angular momentum, Φ the electric potential, and Q the horizon charge. The original proofs can be found in [6] (see also [46, 52, 32]). We will give a new derivation of this law in chapter 3 in the case of 2+1 dimensional gravity.

The next law is concerned with the horizon area A of a black hole. It states that the horizon area can never decrease.

$$\delta A \geq 0 \quad (1.3)$$

The original proof was given by Hawking [22] (See also [50]).

The third law of black hole mechanics has been formulated and proven by Israel [31]. It states that the surface gravity of a black hole can not be reduced to zero in a finite number of steps.

Comparison of these laws to the usual laws of thermodynamics reveals a striking resemblance. If some multiple of the surface gravity is identified with the temperature, some multiple of the area with the entropy, and the mass with the energy (see table 1.1) one finds an exact match.

It was Bekenstein [8] who suggested that this resemblance is more than a mere coincidence. He argued that the laws of black hole mechanics are nothing more than the actual laws of thermodynamics applied to black holes. He thus said that the entropy of a black hole is given by some multiple of its area

$$S_{\text{BH}} = \gamma A, \quad (1.4)$$

for some constant γ with the units of inverse area. By comparison with the usual first law of thermodynamics this means that the temperature of a black hole would have to be

$$T_{\text{BH}} = \frac{\kappa}{\gamma 8\pi G}, \quad (1.5)$$

with the same constant γ as above. The exact value of this constant γ can not be determined from these purely classical considerations.

	Thermodynamics	Black Hole Mechanics
0th law	T is constant in equilibrium	κ is constant over the horizon
1st law	$dE = TdS + \text{work terms}$	$dM = \frac{\kappa}{8\pi G}dA + \dots$
2nd law	$dS \geq 0$	$dA \geq 0$
3rd law	$T = 0$ can not be attained in a finite number of steps.	$\kappa = 0$ can not be attained in a finite number of steps.

Table 1.2: The laws of usual thermodynamics and the laws of black hole mechanics bear a striking resemblance if one identifies the surface gravity κ with the temperature T , the area A with the entropy, and the mass M with the energy E . This comparison does not fix the exact relation between S and A . If $S = \gamma A$, for some constant γ with units of inverse area, then $T = \kappa/\gamma 8\pi G$. This constant can not be fixed by looking at the classical theory alone.

Bekenstein then went on to propose a generalized second law of thermodynamics which said that the entropy of black holes which is determined by their areas together with the entropy of everything else can not decrease,

$$\delta S_{\text{BH}} + \delta S_{\text{everything else}} \geq 0. \quad (1.6)$$

As it stands this identification has several problems. The first problem is that classically the temperature of a black hole vanishes. The second problem is that entropy and area have different dimensions. Entropy is dimensionless whereas area has the dimension length squared. What is the length scale here that relates the two quantities? Is third problem is that this generalized second law can be violated. By lowering a box containing entropy to the horizon and dropping it into the black hole after almost all energy has been extracted at infinity one can decrease the outside entropy without increasing the area of the black hole. A last problem is the factor of γ that remains undetermined in the classical theory.

All these problems can be solved if Quantum Mechanics is taken into account.

1.2 QUANTUM THEORETICAL INSIGHTS

The key observation that helps to resolve the problems mentioned above was made by S. Hawking [24]. He realized that when quantum fields are brought into the picture a black hole emits particles at the *Hawking* temperature

$$T_{\text{Hawking}} = \frac{\hbar\kappa}{2\pi} \quad (1.7)$$

(See [33, 51] for a derivation of this result and [20] for an alternative approach). We will see how the introduction of this temperature gets rid of the problems mentioned above.

The introduction of the Hawking temperature shows that quantum mechanically black holes do have a temperature. The surface gravity κ that looks like a temperature in the classical laws of black hole thermodynamics actually is the physical temperature of the black hole.

Having an expression for the temperature we can now fix the constant γ relating the area A of the black hole and the entropy. We obtain

$$S_{\text{BH}} = \frac{A}{4\hbar G}. \quad (1.8)$$

The quantity $\hbar G$ is exactly the Planck length squared¹. The entropy of a black hole is thus just one fourth of the horizon area measured in units of the Planck length squared.

¹In units $c = 1$.

It can also be shown that the generalized second law of thermodynamics can not be violated any more if the quantum fields outside the horizon are taken into account. For a detailed discussion see [48, 49].

Through the inclusion of Quantum Mechanics into the picture we have thus removed the obstacles that prevented us from identifying the classical laws of black hole mechanics with the corresponding laws of thermodynamics.

Having identified the area of the black hole with its true thermodynamic entropy we are now ready to ask the central question that has guided the search for a theory of quantum gravity for the last decades:

Central question What are the microscopic degrees of freedom that give rise to the black hole entropy.

This thesis is concerned with a partial answer to this question.

1.3 ANSWERS

The last few years have seen a remarkable progress towards finding answers to the central question posed above. Answers have been given from two very different approaches to quantum gravity, namely string theory and loop quantum gravity. Lets review shortly how both these approaches arrive at a microscopical description of black hole entropy.

1.3.1 Black hole entropy and string theory

In string theory the counting of states that give rise to black hole entropy is done in the weak coupling limit. In this limit the states are quantum fluctuations around D-branes that reside in flat space. The immediate question is then: What does such a counting have to do with the problem of black hole entropy?

To understand this we have to look at the special character of the objects that are being counted and the fact that we are dealing with a supersymmetric theory.

Let us start with the special nature of the objects that we look at in the weak coupling limit. They are the so called D-branes. These objects are a rather recent addition to string theory [37, 38]. With the advent of string dualities it became clear that such an addition was needed since these dualities interchange Neveu-Schwartz – Neveu-Schwartz and Ramond–Ramond states and the string perturbation states only carry NS charges. D-branes carry these Ramond charges and allow for a flat space description in the weak field limit.

Next we look at the consequences of supersymmetry. A state which is invariant under some (usually half) of the supersymmetry generators is called a BPS state. The number of BPS states is essentially a topological invariant, and thus will not change if the coupling is changed continuously. Using this fact one can perform the counting of states in either the strong or the weak coupling regime. The black hole states that we are interested in are strong coupling states but it turns out that the counting is more easily performed for small values of the coupling constant.

The picture is thus the following. One starts counting BPS states in the weakly coupled flat regime and then increases the string coupling g . Since Newton's constant grows like g^2 the gravitational field becomes stronger and at some point leads to the formation of a black hole. The number of BPS states which have the same charges as the black hole is then identified with the exponential of the entropy of that black hole. The fact that we have to look at BPS states means that only extremal black holes can be dealt with.

To obtain extremal black holes with non-vanishing surface area one has to start with BPS states with several non-zero charges. The easiest example is given by a type IIB string theory on $T^5 \times \mathbb{R}^5$. Here only three non-zero charges are required to obtain an extremal black hole with a regular horizon and a non-zero surface area. Type IIB string theory is chosen here since it possesses fields carrying Ramond charges.

We begin with the weakly coupled string configurations. They are given by Q_5 D-5 branes wrapped on T^5 , Q_1 D-1 branes wrapped around one of the circles of T^5 , and $-n$ units of momentum around one of the S^1 making up the T^5 . The counting of the number of states can be done in several ways (see [10, 18, 35, 21]). We just give the answer. It is

$$\exp(2\pi\sqrt{Q_1 Q_2 n}). \quad (1.9)$$

All these string configurations lead to the same black hole in the strong coupling regime. The classical black hole metric can be written down as follows,

$$ds^2 = -\lambda^{-2/3} dt^2 + \lambda^{1/3} [dr^2 + r^2 d\Omega_3^2], \quad (1.10)$$

with

$$\lambda = \left(1 + \frac{r_1^2}{r^2}\right) \left(1 + \frac{r_5^2}{r^2}\right) \left(1 + \frac{r_n^2}{r^2}\right), \quad (1.11)$$

and the radii are related to Q_1 , Q_5 , and n in the following way

$$r_1^2 = (RV)^{2/3} g^{-1/2} Q_1 V \quad (1.12)$$

$$r_5^2 = (RV)^{2/3} g^{1/2} Q_5 \quad (1.13)$$

$$r_n^2 = (RV)^{2/3} n/R^2 V. \quad (1.14)$$

If $T^5 = T^4 \times S^1$, then $(2\pi)^4 V$ denotes the volume of T^4 , R is the radius of S^1 , and g is the string coupling. The Bekenstein-Hawking entropy of this solution is given by

$$S_{\text{BH}} = \frac{A}{4G} = \exp(2\pi\sqrt{Q_1 Q_5 n}), \quad (1.15)$$

in agreement with the value obtained above.

More details on Black Hole entropy in string theory can be found in a number of review articles [42, 36, 45].

1.3.2 Black hole entropy from loop quantum gravity

We will only give a rough sketch of the ideas leading to an explanation of the black hole entropy in loop quantum gravity. A more detailed analysis can be found in [1, 4]. One of the most striking features of loop quantum gravity is the fact that certain geometrical operators have a discrete spectrum. An example is the operator corresponding to a two dimensional surface in space. The Hilbert space of loop quantum gravity is spanned by spin network states which are graphs whose edges are labelled by representations of $SU(2)$, i.e. by spins $j = 1/2, 1, \dots$, and whose edges are labelled by intertwining operators. If such a spin network punctures a two dimensional surface transversely with an edge of spin j it contributes an amount of

$$8\pi\gamma l_P^2 \sqrt{j(j+1)} \quad (1.16)$$

to the area of the surface. The parameter γ is a positive real number known as the Immirzi parameter [28, 29, 30]. It reflects an ambiguity in the choice of conjugate variables that are used in the quantization. The idea is now to ask for the number of states that endow the horizon of the black hole with a given area A . The states that are counted are just the surface states of the theory. The states in the bulk are traced over. A detailed exposition of the counting can be found in [4]. We here use the result that the main contribution to the entropy comes from punctures with spin $1/2$. The number of states is then 2^N , and the entropy is just

$$S = N \ln 2. \quad (1.17)$$

The number N of punctures can be computed from equation (1.16). One obtains

$$S = \frac{\ln 2}{\pi\sqrt{3}} \frac{1}{\gamma} \frac{A}{4} l_P^2. \quad (1.18)$$

If one thus chooses γ to be $\ln 2 / \pi\sqrt{3}$, one obtains the Bekenstein - Hawking entropy.

1.3.3 Symmetry based approaches to black hole entropy

Recently a new approach to the problem of black hole entropy has emerged that does not rely on a specific theory of quantum gravity. The idea is that the space of states that classically represent a black hole carries a representation of a symmetry algebra and that the elements of this algebra act as approximate symmetries in the classical spacetime. Finding the microscopical entropy then reduces to a problem in the representation theory of that algebra. This is the approach that we will follow in this work. We give a detailed overview of this approach in chapter 2.

The remarkable aspect of this approach is the fact that it is so deeply rooted in the classical theory. This might shed some light on the importance of the problem of black hole entropy for the search of a quantum theory of gravity. We will discuss this point in chapter 6.

1.4 ORGANIZATION OF THE WORK

The organization of this thesis is as follows. Our aim is it to use the symmetry based approach to finding the black hole entropy. A detailed description of the method and the tools used is given in chapter 2. A key ingredient is the notion of symmetry used in the approach. One motivation for this work was to investigate the natural symmetries of isolated horizons. We thus introduce isolated horizons in chapter 3. This chapter also discusses new versions of the zeroth and first law of black hole mechanics. The next chapter 4 then deals with the natural symmetries of an isolated horizon. It is shown that this set of symmetries does not give an explanation of black hole entropy. We then discuss a new set of symmetries in chapter 5. These symmetries are distinguished from the previous ones by the fact that there are only given in a neighborhood of the horizon and do not possess a limit to the horizon. These symmetries allow us to complete the symmetry based approach to black hole entropy. In the last chapter 6 we discuss the results obtained. An appendix gives a comprehensive overview of the Newman - Penrose formalism in $2 + 1$ dimensions. This formalism has been extended here from its original $3 + 1$ dimensional form to be used in our treatment of $2 + 1$ dimensional isolated horizons.

Symmetry Approach to Black Hole Entropy

In this chapter we discuss an approach to black hole entropy that is based on symmetries of the spacetime. The hope is that the microstates of the black hole furnish a representation space for some symmetry algebra and that the elements of this algebra can be found as approximate symmetries of the classical spacetime. The problem of counting the states then becomes a problem in the representation theory of groups and algebras.

The organization of this chapter is thus as follows. In the first section we give a more detailed exposition of the strategy that we have just outlined. This strategy will be followed in the chapters 4 and 5 for two different sets of symmetries. In the following section we discuss the occurrence of central extensions in the theory of representations of groups and algebras. This is important for us since the group that we are looking at is the diffeomorphism group of the circle which may obtain a central extension when represented in a Hilbert space. The appearance of a central extension will be crucial for what follows. In the following section we then concentrate on the Virasoro algebra. Since the dimension of the representation spaces of this algebra is so crucial we spend the next section describing a formula that provides us with a convenient way of calculating this dimension.

2.1 THE NAME OF THE GAME

Given a black hole we want to count the possible quantum mechanical microstates representing it to find its entropy. This task is facilitated if we know that the microstates are related to each other by the action of some group or algebra. It would then suffice to know the group and its representation theory to obtain the desired number of states. If remnants of the group action survive in the classical theory as symmetries, only the classical spacetime needs to be looked at to find the black hole entropy.

The motivation behind this approach goes back to earlier work by Brown, Henneaux [9].

They calculated symmetries of spacetimes that are asymptotically anti de Sitter and found that the corresponding Hamiltonians give rise to a Virasoro algebra with non-vanishing central charge. These authors did not apply their result to the problem of black hole entropy though. The first one to do this was Strominger[44]. He applied the idea to 2+1 dimensions in the context of the BTZ black hole [5] (see also section 3.1). The symmetries, however, are taken from the previous analysis [9] which is tailored to *asymptotic infinity* rather than to the black hole horizon. Therefore, it is not apparent why these symmetries are relevant for the black hole in the spacetime interior. For example, in asymptotically flat, 4-dimensional space-times, the symmetry group at (null) infinity is always the Bondi-Metzner-Sachs group, irrespective of the interior structure of the space-time. Thus, the results of [44] are equally applicable to a star that has similar asymptotic behavior as that of the black hole. Subsequently, Carlip improved on this idea significantly by making the symmetry analysis in the near-horizon region. Conceptually this approach is much more satisfactory in that the black hole geometry is now at the forefront. However, at the technical level, Carlip's work [13] appears to have some important limitations which we will correct in chapter 5.

Let us review the strategy. The starting point is a phase space consisting of solutions to Einstein's equations that is defined through the boundary conditions on the inner and outer boundary. In the group of diffeomorphisms that preserve the boundary conditions we then look for a subgroup that is isomorphic to $\text{Diff}(S^1)$. Given a vector field ξ in this $\text{Diff}(S^1)$ we can assign to it the Hamiltonian H_ξ that generates the corresponding motion on phase space. We thus obtain a map from $\text{Diff}(S^1)$ into the Poisson algebra of our phase space. As we will see in the following section 2.2 this map might not be a homomorphism of Lie algebras and we might pick up a central charge. Since our $\text{Diff}(S^1)$ preserves the boundary conditions and is in this sense a symmetry it is natural to assume that it is unitarily implemented in the underlying quantum theory. This allows us to use Cardy's formula to calculate the dimension of this unitary representation (see also figure 2.1).

Let us conclude this section with a couple of remarks. First it has to be pointed out that the motivation for looking for a $\text{Diff}(S^1)$ comes mainly from the history of the subject. There is no a priori reason why this group should play such an important role. The best reason that can be given is a posteriori, namely that it gives the correct result.

It is also worth pointing out that the we look at $\text{Diff}(S^1)$ as a purely algebraic object. There is *no* geometric object isomorphic to a circle whose diffeomorphism group give rise to the group we are looking at. In 2 + 1 dimensions the horizon of a black hole has the

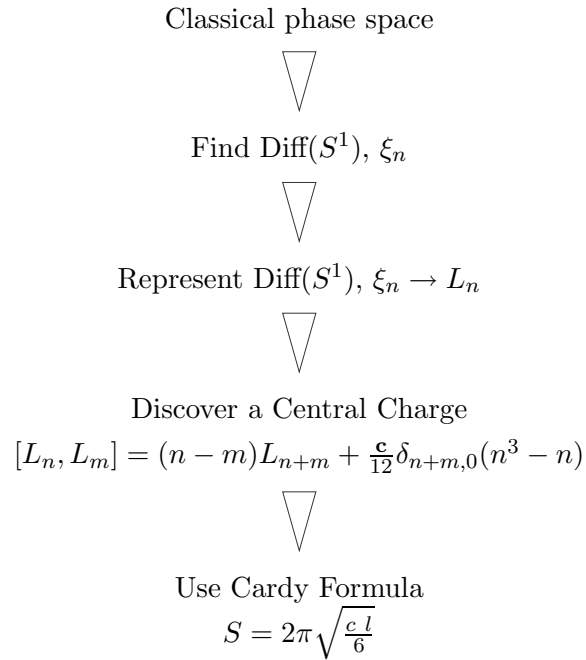


Figure 2.1: This figure gives an overview of the strategy that we will pursue to find the entropy of a black hole. We begin with a classical phase space together with boundary conditions on the inner and outer boundary. In the group of diffeomorphisms that preserve the boundary conditions we then look for a subgroup that is isomorphic to a $\text{Diff}(S^1)$. Next we calculate the Hamiltonians generating these symmetries together with their Poisson brackets. From the bracket we can read of the central charge c and use the Cardy formula to calculate the entropy.

geometry $S^1 \times \mathbb{R}$ and it is tempting to try to relate this S^1 appearing here to the $\text{Diff}(S^1)$ we are looking at. As we will see in later chapters no such relation exists.

Finally we want to point out that we arrive at a phase space, the starting point of our investigation, in two different ways. In our first approach we use the phase space of isolated horizons (see chapter 3). In our second approach we start with a BTZ spacetime (see section 2) and the phase space consists of those spacetimes that possess, in a sense to be made precise later, the same near-horizon geometry as BTZ.

2.2 THE APPEARANCE OF CENTRAL CHARGES

In this section we want to review a couple of facts about the appearance of a central charge for Lie algebras. As we have mentioned in the introduction to this chapter a central charge may arise when we try to represent a given Lie algebra \mathfrak{g} on some linear vector space V . We are thus looking for a map $T : \mathfrak{g} \rightarrow \text{Hom } V$, such that

$$T([X, Y]) = [T(X), T(Y)]. \quad (2.1)$$

It might now happen that we have found a map T that does not quite satisfy this relation but comes close in the sense that the elements

$$\Theta(X, Y) \equiv [T(X), T(Y)] - T([X, Y]) \quad (2.2)$$

form an abelian algebra that commutes with all elements in $T(\mathfrak{g})$. We will call the algebra formed by these elements \mathfrak{a} . Given this map T one might ask whether it is possible to redefine the map in such a way that the corresponding Θ vanishes. If such a choice could be made we would obtain a true representation of our algebra \mathfrak{g} .

If we replace $T(X)$ by

$$T(X) + \mu(X), \quad (2.3)$$

for some map $\mu : \mathfrak{g} \rightarrow \mathfrak{a}$, it is easy to see that $\Theta(X, Y)$ changes to

$$\Theta(X, Y) + \mu([X, Y]). \quad (2.4)$$

The function $\Theta(X, Y)$ can thus be made to vanish if and only if it can be written as $\mu([X, Y])$ for some function μ . In general this will not be the case.

Before proceeding let us introduce some notation that facilitates the discussion. It is easy to see that the map Θ satisfies the following properties:

1. Θ is bilinear
2. Θ is alternating, i.e.

$$\Theta(X, Y) = -\Theta(Y, X) \quad (2.5)$$

3. It satisfies the Jacoby identity:

$$\Theta(X, [Y, Z]) + \Theta(Y, [Z, X]) + \Theta(Z, [X, Y]) = 0 \quad (2.6)$$

To show the last property one has to use the fact that $[,]$ is a Lie bracket for which the Jacoby identity is valid. We denote by $\mathcal{Z}^2(\mathfrak{g}, \mathfrak{a})$ the set of all maps satisfying the above three properties. We want to identify maps in $\mathcal{Z}^2(\mathfrak{g}, \mathfrak{a})$ if they are related as in equation (2.4). We thus introduce the space $\mathcal{B}^2(\mathfrak{g}, \mathfrak{a})$ of maps

$$b : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{a} \quad (2.7)$$

$$(X, Y) \longmapsto b(X, Y) = \mu([X, Y])$$

for some function μ . We can then form the quotient of these two spaces

$$H^2(\mathfrak{g}, \mathfrak{a}) \equiv \mathcal{Z}^2(\mathfrak{g}, \mathfrak{a})/\mathcal{B}^2(\mathfrak{g}, \mathfrak{a}). \quad (2.8)$$

This space is called the second cohomology group of \mathfrak{g} with values in \mathfrak{a} .

So far we have seen that we can construct a true representation from our map T if and only if the class $[\Theta]$ corresponding to Θ in the second cohomology group $H^2(\mathfrak{g}, \mathfrak{a})$ vanishes. If $[\Theta]$ does not vanish we can obtain a representation not of \mathfrak{g} but of a Lie algebra closely related to it. Let

$$\mathfrak{h} \equiv \mathfrak{g} \oplus \mathfrak{a} \quad (2.9)$$

as vector spaces and let us define the following bracket on it

$$[,]_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h} \quad (2.10)$$

$$(X, A), (Y, B) \longmapsto ([X, Y]_{\mathfrak{g}}, \Theta[X, Y]).$$

Together with this bracket \mathfrak{h} becomes a Lie algebra and the following sequence of algebras is exact

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow 0, \quad (2.11)$$

where the second and third map are the natural injection and surjection respectively. The algebra \mathfrak{h} is called the central extension of \mathfrak{g} by \mathfrak{a} . It can further be shown that two

such extension are isomorphic if and only if the corresponding maps Θ belong to the same cohomology class. We thus find that the second cohomology group is isomorphic to the set of central extensions of the algebra \mathfrak{g} .

The new Lie algebra \mathfrak{h} is interesting because it is quite easy to construct a true representation for it, given the map T . For $(X, A) \in \mathfrak{h}$ let

$$\tilde{T}(X, A) \equiv T(X) + A. \quad (2.12)$$

It is easy to see that this provides us with a representation of the algebra \mathfrak{h} .

Lets summarize. We started with a map T that was almost a representation of the Lie algebra \mathfrak{g} on a vector space V . The failure Θ of this map to be a representation determines an element in the second cohomology of \mathfrak{g} with values in an abelian algebra \mathfrak{a} . Each such cohomology class gives rise to a central extension \mathfrak{h} of \mathfrak{g} by the abelian algebra \mathfrak{a} . This central extension \mathfrak{h} then possesses a natural true representation on V .

Before ending this rather technical section we give a well known example of what we have just encountered.

Example 1. *Let \mathfrak{g} be the Lie algebra of a semi simple finite dimensional Lie group (such as $SU(2)$) then its second cohomology group with values in \mathbb{R} , \mathbb{C} or any other finite dimensional \mathfrak{g} module vanishes. This was shown by Whitehead (see [27]). For a large class of algebras this result was established before by V. Bargman [7].*

2.3 THE VIRASORO ALGEBRA

In this section we will describe the Lie algebra of the diffeomorphism group of the circle and its central extension, the Virasoro algebra. The diffeomorphism group of the circle $\text{Diff}(S^1)$ is of interest because the conformal group of the plane can be written as the product of two copies of $\text{Diff}(S^1)$. The Lie algebra of $\text{Diff}(S^1)$ can be identified with the set of vector fields on \mathbb{S}

$$\text{Lie Diff}(S^1) = \text{Vect}(\mathbb{S}). \quad (2.13)$$

If we think of \mathbb{S} as being imbedded in \mathbb{C} as the set of complex numbers with modulus one we can represent a vector field on \mathbb{S} by a series

$$\sum_{n \in \mathbb{Z}} a_n z^{n+1} \frac{d}{dz}. \quad (2.14)$$

A basis of these fields is thus given by the operators

$$L_n \equiv z^{1-n} \frac{d}{dz}, n \in \mathbb{Z}. \quad (2.15)$$

It then easy to verify that these operators satisfy the following commutation relation

$$[L_n, L_m] = (n - m)L_{n+m}, n, m \in \mathbb{Z}. \quad (2.16)$$

we will denote this algebra by \mathfrak{f} . If these operators generate symmetries of our theory we are guaranteed the existence of hermitian operators $T(L_n)$ representing them in the given Hilbert space. These operators are uniquely determined up to the addition of a multiple of the identity operator $\mathbf{1}$. We are thus in the position discussed in the last section. The abelian algebra \mathfrak{a} is here spanned by the identity operator and is thus isomorphic to \mathbb{C} . We are thus led to discussing central extensions of \mathfrak{f} by \mathbb{C} . The set of these extensions is given by the second cohomology $H^2(\mathfrak{f}, \mathbb{C})$. The central result is now the following theorem.

Theorem 1. *The second cohomology $H^2(\mathfrak{f}, \mathbb{C})$ is isomorphic to \mathbb{C} .*

$$H^2(\mathfrak{f}, \mathbb{C}) \simeq \mathbb{C} \quad (2.17)$$

This second cohomology is generated by the cocycle

$$\theta(L_n, L_m) \equiv \delta_{n+m} \frac{1}{12} n(n^2 - 1). \quad (2.18)$$

A proof of this theorem can be found in [41]. The generic situation is thus that we encounter a representation of a central extension of \mathfrak{f} rather than of \mathfrak{f} itself. For the central extension corresponding to the cocycle $c\theta$ we arrive at the following commutation relation

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{n+m} \frac{c}{12} n(n^2 - 1), n, m \in \mathbb{Z}. \quad (2.19)$$

This algebra is commonly denoted as the *Virasoro Algebra* with central charge c .

2.4 COUNTING STATES: THE CARDY FORMULA

Given an irreducible representation of the Virasoro algebra we now would like to know what the dimensionality of the representation space is. The situation might be compared with that encountered when looking at $SU(2)$. If we know the highest weight l we know that the dimension of the representation space is $2l + 1$. An equivalent formula for the case of the

Virasoro algebra was given Cardy [11] (see also [14]). If we look at a representation of the Virasoro algebra with central charge c and highest weight l , i.e. we have

$$L_0|l\rangle = l|l\rangle, \tag{2.20}$$

and $L_n|l\rangle = 0$, for $n > 0$, the dimension d of the representation space is given by

$$d = \exp\left(2\pi\sqrt{\frac{cl}{6}}\right). \tag{2.21}$$

Chapter 3

Isolated Horizons

In this chapter we describe the concept of an isolated horizon. This concept has been introduced to deal with a number of restrictions that exist in the usual treatment of black holes which is centered around the notion of event horizons in globally stationary spacetimes.

An event horizon H^+ in a spacetime M is the boundary of the causal past $J^-(\mathcal{I}^+)$ of future null infinity \mathcal{I}^+ ,

$$H^+ = M \cap \partial J^-(\mathcal{I}^+). \quad (3.1)$$

The important point to realize here is that in order to find the event horizon the *whole* spacetime has to be known. In most situations this knowledge is not available. If one talks for example about a black hole in the center of a galaxy it is not the event horizon that is meant. Also in numerical relativity one usually does not evaluate the entire spacetime but only patches of it (see [19] for a more detailed discussion of the use of isolated horizons in numerical relativity). Still one would like to be able to talk about black holes and their properties. A more local notion of a black hole is thus desirable.

Another problem arises from the requirement of stationarity which implies the existence of a Killing field in the spacetime. Such a degree of symmetry will usually not be present in physically interesting situations. In a gravitational collapse e.g. the spacetime will contain gravitational radiation even if the black hole itself has long settled down. We would thus like to describe situations in which the black hole is in equilibrium while the rest of the universe might not be (see figures 3 and 3 for more discussion on these points).

In the first part of this chapter we will introduce the notion of an isolated horizon. It will be immediately clear from the definitions that isolated horizons are able to address the mentioned above. We then proceed to show that although the framework is rather general we are able to derive some of the laws of black hole mechanics that served as our motivation. Since the definition of an isolated horizon is so local we have to find new definitions for quantities that in the old framework made use of global structures. Examples

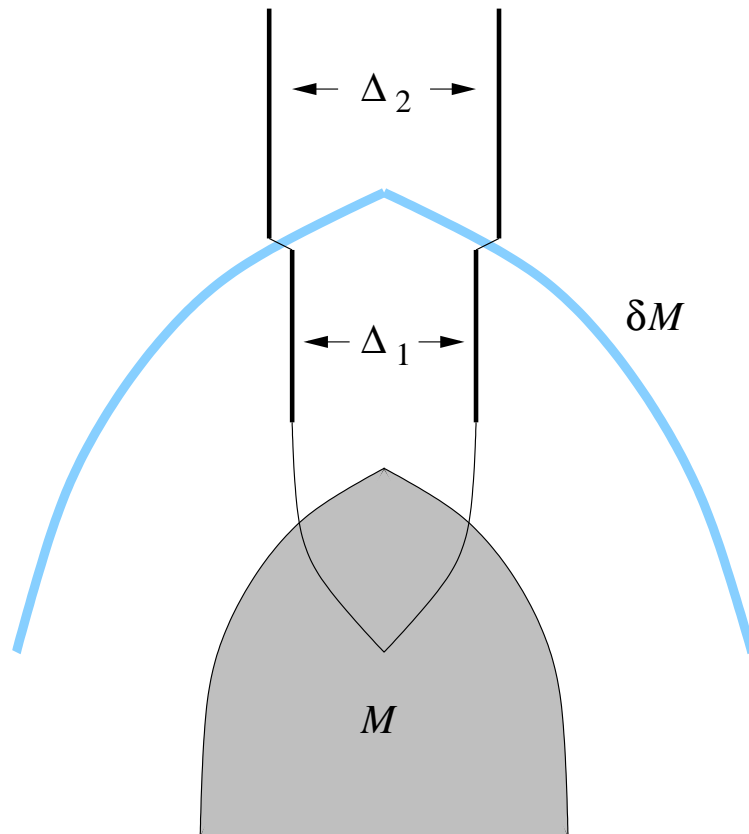


Figure 3.1: This figure shows why the concept of event horizons is not able to deal with some physically interesting situations. Imagine a black hole is formed through the gravitational collapse of a star M . After the collapse the situation will settle down and if nothing further would happen the event horizon would be at a position which we denoted by Δ_1 . If now a long time after the collapse a shell of matter falls into the black hole the mass of the hole will increase and ultimately the event horizon will be outside, at a position Δ_2 . Thus although the shell of matter might have been arbitrary far away after the collapse we still would have had no concept that allowed us to call Δ_1 the horizon of a black hole. Isolated horizons include both Δ_1 and Δ_2 are thus able to include more physical situations.

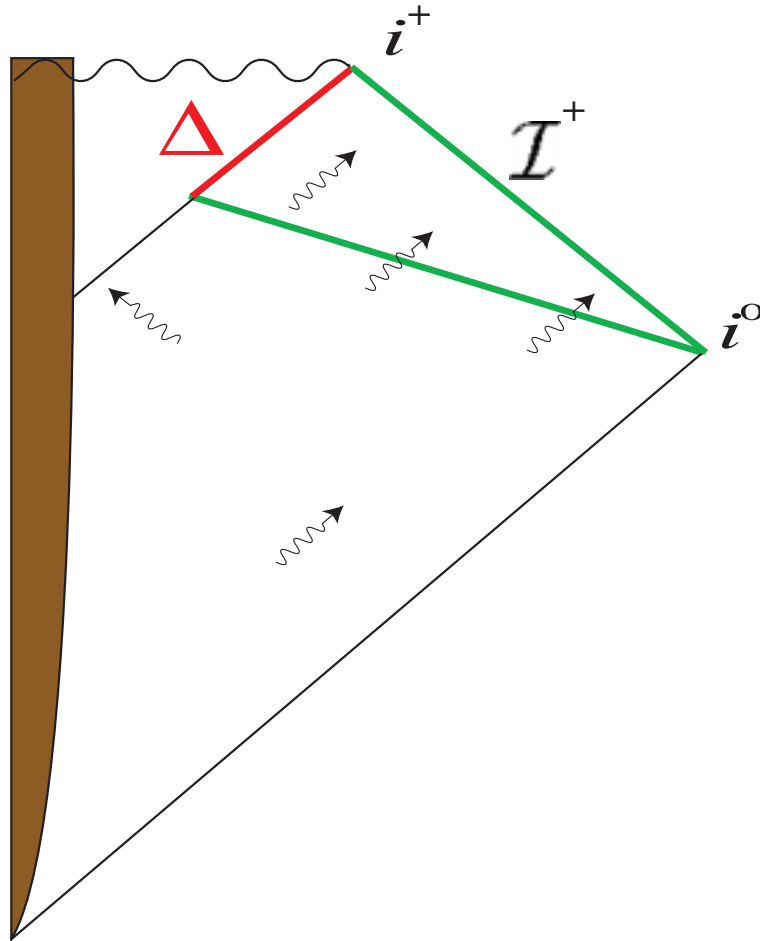


Figure 3.2: Black holes are usually discussed within the framework of event horizons in stationary spacetimes. This framework will not allow the discussion of a situation as the one depicted in this figure. After a gravitational collapse a black hole will settle down while the spacetime is still filled with gravitational radiation and thus will not admit a Killing field. The concept of an isolated horizon is built to describe the part Δ of the black hole that is in equilibrium while the rest of the spacetime may be far from equilibrium.

are the surface gravity and the mass.

3.1 DEFINITION

In this section we will give the definition of an isolated horizon. It is advantageous to proceed in several steps since preliminary results have to be established before the final definition can be given.

Let \mathcal{M} be a three dimensional manifold with metric tensor g_{ab} of signature $(- + +)$. We will be considering null hypersurfaces of \mathcal{M} , i.e. hypersurfaces whose tangent spaces at each point contain a null direction. It is easy to see that a null vector field tangent to the hypersurfaces is automatically a normal to the surface. We will denote a future pointing null normal vector field to the null hypersurface by l . This null normal l is ambiguous up to a rescaling by a positive function.

If ∇ is the covariant derivative compatible with g_{ab} then we can define the expansion $\Theta_{(l)}$ of l by $m^a m^b \nabla_a l_b$, where m^a is any spacelike vector field tangent to the hypersurface. It is easy to see that this definition does not depend on the choice of m .

3.1.1 Non-expanding horizons

After these preliminaries we will now start with the notion of a non-expanding horizon.

Definition 1. *Let $\Delta \subset \mathcal{M}$ be a null hypersurface of the spacetime (\mathcal{M}, g_{ab}) and l a future pointing null normal vector field on Δ . The hypersurface Δ is called a **non-expanding horizon** iff it satisfies the following conditions:*

NE1 Δ is isomorphic to $S^1 \times I$, where I is some interval in \mathbb{R} .

NE2 The expansion $\Theta_{(l)}$ of l vanishes.

NE3 The equations of motion hold at Δ . Furthermore the stress-energy tensor T_{ab} is such that $-T^a_b l^b$ is future directed and causal for any future directed null normal l .

The paradigm of a non-expanding horizon in three dimensions is best illustrated by the BTZ black hole[5]. We give here a slightly generalized version of the standard BTZ black hole:

Example 2. *We describe the generalized BTZ black hole in Eddington-Finkelstein-like coordinates. In these coordinates the metric is given by*

$$ds^2 = -(N^\perp)^2 dv^2 + 2dvdr + r^2 (d\phi + N^\phi dv)^2, \quad (3.2)$$

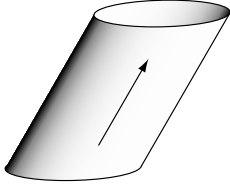
A		Δ non-expanding horizon	<ul style="list-style-type: none"> • Newman-Penrose ρ, κ_{NP} vanish. α, π coincide. • $\nabla_a l_b \hat{=} \omega_a l_b$. Surface gravity $\kappa = l^a \omega_a$. • Intrinsic derivative \mathcal{D}_a on Δ.
		<ul style="list-style-type: none"> • $\Delta \simeq S^1 \times \mathbb{R}$ • Expansion $\Theta_{(l)} = 0$ • Equations of motion hold. 	
B	$(\Delta, [l])$ weakly isolated horizon	<ul style="list-style-type: none"> • 0th law: $\kappa = \text{const.}$ • Natural foliation of Δ with $\pi = \text{const.}$ 	
	<ul style="list-style-type: none"> • Δ non-expanding horizon • $\mathcal{L}_l \omega_a = 0$ 		
C	$(\Delta, [l])$ isolated horizon	<ul style="list-style-type: none"> • Unique choice of $[l]$ in interesting situations. 	
	<ul style="list-style-type: none"> • Δ non-expanding horizon • $[\mathcal{L}_l, \mathcal{D}]v = 0$, for every vector v tangential to the horizon. 		

Figure 3.3: In this table we give a road map to section 3.1 of this article. Its main purpose is to provide an overview over the structures that are being introduced and also to show what the consequences are that can be derived from them. The first structure is given in **A** and is that of a **non-expanding horizon**. The fact that the Newman-Penrose like coefficients ρ and κ_{NP} vanish and that α and π coincide has the consequence that we can define a one-form ω_a by looking at the divergence of the null-normal l^a . This one-form then can be used to define the surface gravity κ of the horizon once a null normal l^a is chosen. Another consequence is that there exists an intrinsic derivative operator \mathcal{D}_a on the horizon. The one-form ω_a is now used in part **B** to define the notion of a **weakly isolated horizon**. For a weakly isolated horizon the surface gravity κ is constant over the horizon and once a class $[l]$ of null-normals is chosen there exists a natural foliation of the horizon with circles. The intrinsically given derivative operator \mathcal{D}_a is used in **C** to define an **isolated horizon**. Isolated horizons have the property that in many interesting situations they possess a unique class $[l]$ of null normals.

where

$$N^\perp = \left(f(r) + \frac{J^2}{4r^2} \right)^{1/2} \quad \text{and} \quad N^\phi = -\frac{J}{2r^2}, \quad (3.3)$$

for some function $f(r)$. The classical BTZ black hole is obtained when the function

$$f(r) = -M + \frac{r^2}{l^2} \quad (3.4)$$

is used. The length l here is related to the cosmological constant Λ through

$$\Lambda = -\frac{1}{l^2}. \quad (3.5)$$

The metric (3.2) is singular when N^\perp vanishes. For the classical BTZ black hole this happens for $r = r_\pm$, with

$$r_\pm^2 = \frac{Ml^2}{2} \left\{ 1 \pm \left[1 - \left(\frac{J}{Ml} \right)^2 \right]^{1/2} \right\}. \quad (3.6)$$

Sometimes it is advantageous to have the mass M and the angular momentum J expressed in terms of r_+ and r_- :

$$M = \frac{r_+^2 + r_-^2}{l^2} \quad J = \frac{2r_+r_-}{l} \quad (3.7)$$

The horizon of the BTZ black hole, given by $r = r_+$, will serve as our first example of a non-expanding horizon in 2+1 dimensions.

To check our conditions for a non-expanding horizon we have to give a null normal to the horizon. We choose

$$l = \partial_v - N^\phi(r_+) \partial_\phi. \quad (3.8)$$

To calculate the expansion of l we have to choose a spacelike vector to the horizon. Here we take

$$m = \frac{1}{r_+} \partial_\phi. \quad (3.9)$$

It is then easy to check that the expansion of l indeed vanishes. That the horizon has the right topology is easy to check and that the equations of motion hold is clear since we are in a vacuum solution to Einsteins equation.

Before we start discussing consequences of this definition a couple of remarks are in order that show how these conditions relate to the physical motivation given in the introduction.

The first part of the definition **NE1** just reflects the situation that is most commonly encountered. The horizons in the examples given here have this topology. More general topologies could be considered but this will be done elsewhere.

A consequence of condition **NE2** is that the horizon area is constant along the horizon. This is the precise meaning of the idea that the horizon is isolated. We will denote the area of the horizon by a_Δ . We also introduce the radius R_Δ of the horizon such that $a_\Delta = 2\pi R_\Delta$.

Given a boundary one usually requires that the metric approaches a certain fixed metric. This is what is usually done at infinity. In **NE3** we just impose the condition that the equations of motion are satisfied at the horizon. This requirement is much weaker than assuming a particular form of the metric. The energy condition is also very weak. It follows from the much stronger dominant energy condition.

Although the conditions that we have imposed seem rather weak they have a number of interesting consequences. To explore these consequences we introduce a Newman-Penrose-like (NP-like) triad consisting of the vectors l^a, n^a , and m^a in the neighborhood of the horizon Δ . The vectors l^a and n^a are null and we choose l^a such that it coincides with the null normal of the horizon. The vector m^a is normalized to unity. We further require $l^a n_a = -1$. All other contractions are zero. Having such a triad we can introduce NP-like coefficients as in the higher dimensional case. Appendix A gives the corresponding definitions and a summary of important relations for these coefficients. In what follows we further assume that we have chosen our triad such that $dn = 0$.

The expansion $\Theta_{(l)}$ coincides with the NP coefficient ρ . It thus follows that

$$\hat{\rho} \doteq 0, \tag{3.10}$$

where a hat denotes an equation that is valid on the horizon.

Since l^a is hypersurface orthogonal to the horizon it follows that the NP coefficient κ_{NP} vanishes:

$$\hat{\kappa}_{NP} \doteq 0 \tag{3.11}$$

From the fact that we have chosen our triad such that $dn = 0$ it follows that the NP coefficients α and π coincide:

$$\hat{\alpha} \doteq \hat{\pi} \tag{3.12}$$

We thus see that as a consequence of our boundary conditions the two NP coefficients ρ and κ_{NP} vanish and the two coefficients α and π coincide. These facts will allow us to introduce more structure on the horizon Δ . We begin with a one-form ω_a on the horizon.

If we now look at the covariant derivative of l along directions in the horizon we obtain (see equation (A.10) in appendix A)

$$\nabla_{\underline{a}} l_b := \omega_a l_b, \tag{3.13}$$

where the one-form ω_a is given by

$$\omega_a = \pi m_a - \epsilon n_a. \quad (3.14)$$

Given a choice of a null-normal l^a on the horizon we thus obtain a one-form ω_a . If we had chosen a different null-normal $l' = fl$, for some positive function f , we would have obtained

$$\omega'_a = \omega_a + \partial_a \ln f. \quad (3.15)$$

We can also look at the acceleration of l^a . It is given by

$$l^a \nabla_a l_b \hat{=} l^a \omega_a l_b = \epsilon l_b. \quad (3.16)$$

We introduce a new name for the coefficient of l_b appearing in this equation.

Definition 2. *Given a non-expanding horizon Δ with a choice of a null-normal l^a , following the standard terminology used in black hole physics, we denote the quantity $l^a \omega_a \equiv \epsilon$ by κ and call it the **surface gravity of Δ** .*

Like the one-form ω_a the surface gravity κ depends on l^a . If we change l^a as above by a positive function f we obtain

$$\kappa' = l^a \partial_a f + f \kappa. \quad (3.17)$$

Later in this section we will discuss how we deal with the freedom in the choice of l^a . We calculate the one-form ω_a and the surface gravity κ for the example given above.

Example 3. *In example 5 of appendix A we have calculated the NP coefficients for the BTZ black hole. We can thus read off the one-form ω_a :*

$$\omega_a = N^\phi m_a - \kappa n_a \quad (3.18)$$

where the surface gravity κ is given by

$$\kappa \equiv \epsilon = \frac{r}{l^2} - r(N^\phi)^2. \quad (3.19)$$

Another consequence of the fact that the NP coefficients ρ and κ_{NP} vanish is that there exists a unique derivative operator \mathcal{D} on Δ which is induced by the covariant derivative on \mathcal{M} . Since Δ is null there is no natural way to decompose $\nabla_a v^b$ into a component tangent to the horizon and a component perpendicular to the horizon. Generally the definition of a derivative operator would thus depend on the choice of such a decomposition. It turns out

that as a consequence of our boundary conditions no such decomposition is necessary. We have already seen in equation (3.13) that the derivative of the null-normal l^a restricted to directions in the horizon is tangential to the horizon provided ρ and κ_{NP} vanish. For the derivative of m^a we find (using again appendix A)

$$\nabla_{\underline{a}} m^b = (\kappa_{NP} n_a - \rho m_a) n^b + (\mu m_a - \pi n_a) l^b. \quad (3.20)$$

Again we find that the term involving components that are not tangential to the horizon vanishes since ρ and κ_{NP} are zero. We thus find that our boundary conditions already ensure that all components of the derivative operator, pulled back to the horizon, lie in the horizon. The need to decompose the derivative operator thus does not arise and no ambiguity exists. We can thus set

$$\mathcal{D}_a v^b := \nabla_{\underline{a}} v^b \quad (3.21)$$

and obtain an intrinsically defined derivative operator \mathcal{D}_a on the horizon Δ .

Next we discuss the restrictions on the Ricci tensor that follow from our boundary conditions. The first restriction can be read off from equation (A.34) in appendix A. We get

$$R_{ab} l^a l^b \hat{=} 0. \quad (3.22)$$

Now we are going to make use of the restrictions on the type of matter that we consider, i.e. we will make use of the energy condition contained in **NE3**, which says that

$$k^a = -T^a_b l^b \quad (3.23)$$

is causal, i.e. future pointing, and time-like or null on the horizon.

Using the field equations

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = 8\pi G T_{ab} \quad (3.24)$$

we see upon contraction with $l^a l^b$ and use of the fact that $R_{ab} l^a l^b$ vanishes that

$$T_{ab} l^a l^b = k_a l^a \hat{=} 0. \quad (3.25)$$

It follows from this that k must be of the form

$$k^a \hat{=} \alpha l^a + \beta m^a. \quad (3.26)$$

The only way for this vector to be time-like or null is when β vanishes. The component $T_b^a l^b$ is thus proportional to l^a . Using this fact it follows from the field equations contracted with $m^a l^b$ that the component $R_{ab} m^a l^b$ of the Ricci tensor vanishes on the horizon.

Making again use of appendix A we find that $R_{ab} m^a l^b$ can be expressed as

$$R_{ab} m^a l^b \hat{=} \mathcal{L}_l \pi - \mathcal{L}_m \kappa. \quad (3.27)$$

We thus have

$$\mathcal{L}_l \pi \hat{=} \mathcal{L}_m \kappa. \quad (3.28)$$

As a last step we further describe the intrinsic geometry of the horizon. We begin by noting that as a consequence of our boundary conditions it follows that

$$dm_{ab} = 0. \quad (3.29)$$

Now using Cartan's identity we find

$$\mathcal{L}_l m = d(l \cdot m) + l \cdot dm = 0. \quad (3.30)$$

Since the induced metric on Δ can be written as $q_{ab} = m_a m_b$ we also find

$$\mathcal{L}_l q_{ab} = 0. \quad (3.31)$$

On Δ we now introduce the following equivalence relation. We call two points on Δ equivalent if they lie on one integral curve of the null-normal l^a . The space of equivalence classes, i.e. the space of integral curves of l^a , will be denoted by $P\Delta$. Topologically $P\Delta$ is a circle. The canonical projection map from Δ onto $P\Delta$ will be called p . The fact that $\mathcal{L}_l q_{ab} = 0$ means that $P\Delta$ can naturally be endowed with a metric \hat{q}_{ab} . Since $P\Delta$ is a circle there exists a unique one-form \hat{m}_a such that

$$\hat{q}_{ab} = \hat{m}_a \hat{m}_b. \quad (3.32)$$

Using the projection p we can lift this one-form to the whole of Δ . It turns out that this one-form coincides with m . We thus have

$$m = p^* \hat{m} \quad (3.33)$$

We thus see that a non-expanding horizon is endowed with a *canonical* one-form m_a .

3.1.2 Weakly isolated horizons

We have seen that our boundary conditions gave rise to a rather rich structure on the horizon. We found that we are given an intrinsic derivative operator \mathcal{D} and a one-form ω_a . We will now use these structures to refine our definition. Our guiding principles will be that we want these additional structures to be time independent. We have already seen that the metric on Δ is time independent in the sense that $\mathcal{L}_l q_{ab}$ vanishes. We will impose similar conditions on ω and \mathcal{D} and in thus doing will arrive at our final definition of an isolated horizon. We have not started with this stronger definition precisely because we needed to introduce the additional structures first in order to formulate our conditions.

The next condition that we impose involves the one-form ω_a . As we have seen above the one-form ω_a depends on the choice of null-normal l^a . Thus if we want to formulate a condition using ω_a it is not enough just to talk about the surface Δ but we will have to include the chosen null-normal l^a .

Equation (3.15) shows that ω_a does not change if we rescale l^a by a constant. We now denote by $[l]$ the equivalence class of null-normals which differ from l only by a multiplicative constant. Given such an equivalence class $[l]$ we have a unique one-form ω_a which we can use to formulate further conditions. This situation should be compared with the situation encountered on Killing horizons. The Killing vector field on such horizons is also given only up to a multiplicative constant.

Definition 3. *Let $(\Delta, [l])$ be a non-expanding horizon Δ together with an equivalence class $[l]$ of null-normals. We call $(\Delta, [l])$ a **weakly isolated horizon** if and only if*

$$\mathcal{L}_l \omega_a = 0, \tag{3.34}$$

where ω_a is the one-form given by the equivalence class $[l]$.

The most important consequence of this condition is the fact that the surface gravity $\kappa = l \cdot \omega$ is constant over the horizon. We have

$$0 = \mathcal{L}_l \omega = dl \cdot \omega + l \cdot d\omega. \tag{3.35}$$

Using the boundary conditions we have

$$d\omega = (\mathcal{L}_l \pi - \mathcal{L}_m \kappa) m \wedge n. \tag{3.36}$$

In equation (3.28) we have seen that the term in brackets in the above equation vanishes and $d\omega$ is thus zero. This means that $dl \cdot \omega$ is zero or in other words

$$\kappa = \text{const.} \tag{3.37}$$

This is the zeroth law of black hole thermodynamics which states that the surface gravity is constant over the horizon. Weakly isolated horizons thus have constant surface gravity.

Weakly isolated horizons with $\kappa \neq 0$ possess additional structure. The equivalence class of null-normals allows for the construction of preferred foliation of the horizon by circles. We already know that a non-expanding horizon has a canonical one-form m_a given on it. Since dm is zero it generates a class in $H^1(\Delta)$, the first cohomology of Δ . Given that Δ is isomorphic to $S^1 \times \mathbb{R}$ we know that

$$H^1(\Delta) = H^0(S^1) = \mathbb{R}. \quad (3.38)$$

Integrating m_a over a cross section gives the circumference of the horizon and m_a is thus not in the zero class of $H^1(\Delta)$. The closed one-form ω_a must thus be of the form

$$\omega_a = C m_a + \partial_a \psi, \quad (3.39)$$

for some constant $C \in \mathbb{R}$ and some function ψ . This function ψ can now be used to define a preferred foliation of the horizon. We first notice that

$$l^a \omega_a = \kappa = l(\psi). \quad (3.40)$$

If we thus assume that we are in the case of non-vanishing surface gravity κ we see that the surfaces of constant ψ are circles and thus define a foliation of Δ . We will adjust our NP - triad in such a way that m^a is tangential to these circles.

Comparing equation (3.14) with equation (3.39) we see that

$$\pi \hat{=} \text{const}. \quad (3.41)$$

We thus obtain a foliation in which the NP - coefficient π is constant on the horizon.

Example 4. For the BTZ - black hole we have found an expression for ω_a in equation (3.18) of example 3. Since N^ϕ is constant we see that

$$\partial_a \psi = -\kappa n_a. \quad (3.42)$$

We have chosen n_a to be $-dv$ where v is one of the coordinates of the BTZ metric. The function ψ is thus just the rescaled coordinate v .

3.1.3 Isolated horizons

We now give the definition of an isolated horizon. To this end we will require that the intrinsically defined derivative operator \mathcal{D} is also time independent.

Definition 4. *A weakly isolated horizon $(\Delta, [l])$ is called an **isolated horizon** if and only if*

$$[\mathcal{L}_l, \mathcal{D}]v = 0, \quad (3.43)$$

for all vector fields v tangential to the horizon.

The nature of this condition is somewhat different from the ones encountered so far. While it is possible for every non-expanding horizon to choose $[l]$ such that it becomes a weakly isolated horizon this is not true for isolated horizons. For the rest of the paper we will not assume that we are dealing with isolated horizons though. We will only require the horizons to be weakly isolated.

3.2 ACTION PRINCIPLE.

The action for the 2+1 dimensional GR can be defined as follows.

$$S = \frac{1}{8\pi G} \int_{\mathcal{M}} \left(e^I \wedge F_I - \frac{\Lambda}{6} \varepsilon^{IJK} e_I \wedge e_J \wedge e_K \right) - \quad (3.44)$$

$$- \frac{1}{16\pi G} \int_{t_i}^{t_f} dt \int_{S_\infty} e^I \wedge A_I + C_\Delta \Delta t - C_\infty \Delta t.$$

Here, e^I is an orthonormal triad, A_I is a connection and F_I is its curvature. C_Δ is a function of the parameters of the horizon (like area, charge, angular momentum), which are constants for any given history. Adding a constant term has no influence for the lagrangian formulation of the theory. In the hamiltonian formulation, however, it does make a difference, and therefore we shall keep track of this term in what follows. Similarly, C_∞ is such that it does not vary between histories, but in order to obtain a differentiable Hamiltonian it should be kept non-zero, in general. Integral over S_∞ should be understood as a suitable limit of integrals evaluated at finite distances r . In order to define the coordinate r we demand that there exist coordinates (t, r, ϕ) on \mathcal{M} such that our variables approach those of the BTZ metric for large values of r .

In order to fully specify our phase-space we need to add boundary conditions. The pair (A, e) is subject to the isolated horizon conditions on the surface Δ . We also impose fall-off

conditions at spatial infinity. We require that the field approach their BTZ values. Details can be found in [2]. One can show that the action is differentiable with those conditions. Indeed,

$$\delta S = \text{bulk terms} + \frac{1}{8\pi G} \int_{\Delta} e^I \wedge \delta A_I. \quad (3.45)$$

Due to the above conditions, the boundary term at spatial infinity goes to zero like $1/r$. Also, using the form of the connection A derived in Appendix C.4., one can show that on the horizon

$$\delta A_a^I \hat{=} \delta[(-\kappa m^I + \pi l^I)n_a + (-\mu l^I + \pi m^I)m_a]. \quad (3.46)$$

Therefore

$$\delta S = \text{bulk terms} - \frac{1}{8\pi G} \int_{\Delta} e^I \wedge (\delta\kappa)m_I n, \quad (3.47)$$

where we used the fact that variations of l , n and m are proportional to themselves on Δ .

Let us call a cross-section of the surface of constant time with the horizon by S_{Δ} . Using $n = -dv$, we obtain

$$\delta S = \text{bulk terms} - \frac{1}{8\pi G} \int dv \int_{S_{\Delta}} (\delta\kappa)m. \quad (3.48)$$

Since κ is constant on Δ and is kept fixed at initial and final times, we have $\delta\kappa = 0$. The action is thus differentiable at the horizon.

3.3 LEGENDRE TRANSFORM, PHASE SPACE AND ANGULAR MOMENTUM.

In this section we perform the transformation to a Hamiltonian framework. This will allow us to introduce a mass for the horizon and ultimately to formulate the first law of black hole mechanics.

3.3.1 The Legendre transform.

In order to introduce the first law, we will pass to the Hamiltonian framework. First, we need to perform the Legendre transform. Let us use the convention $8\pi G = 1$. We arrive at

the following form of the action

$$\begin{aligned}
S = & \int_{t_i}^{t_f} dt \int_{\Sigma} [-e^I \wedge \mathcal{L}_t A_I + e^I \wedge D(t \cdot A_I) + \\
& + (t \cdot {}^3 e^I)(F_I - \frac{1}{2} \Lambda \epsilon^{IJK} e_J \wedge e_K) - \\
& - \frac{1}{2} \int_{t_i}^{t_f} dt \int_{S_{\infty}} ((t \cdot {}^3 e^I) A_I - (t \cdot {}^3 A^I) e_I) + \\
& + \int_{t_i}^{t_f} dt (C_{\Delta} - C_{\infty})
\end{aligned} \tag{3.49}$$

where N is the lapse function. In this chapter we shall use the convention in which the superscript 3 in front of a symbol refers to space-time quantities and the symbols without a superscript in front are pull-backs to the two-dimensional space-like slice.

From the above we obtain the Hamiltonian

$$\begin{aligned}
H = & \int_{\Sigma} \left[-(t \cdot {}^3 e^I) F_I - (t \cdot {}^3 A_I) D e^I + \frac{\Lambda}{2} \epsilon^{IJK} (t \cdot {}^3 e_I) e_J \wedge e_K \right] - C_{\Delta} + C_{\infty} + \\
& + \int_{S_{\Delta}} (t \cdot {}^3 A_I) e^I + \frac{1}{2} \int_{S_{\infty}} [(t \cdot {}^3 e^I) A_I + (t \cdot {}^3 A^I) e_I]
\end{aligned} \tag{3.50}$$

A variation of C_{Δ} is now not zero in general. In fact this term is needed if we want the Hamiltonian framework to make sense. We will show in the next paragraphs that the requirement of differentiability of the Hamiltonian restricts the form of this term. It can be then shown that the horizon terms are exactly analogous to the terms at infinity.

3.3.2 The phase space.

In what follows we will work in the canonical framework. Therefore it is useful to state here definition of the phase space of isolated horizons. Our variables will be fields on 2-dimensional manifold Σ such that $\mathcal{M} = \Sigma \times \mathbb{R}$. Topologically, the boundary of Σ consists of two circles, spatial infinity S_{∞} and a cross section of the horizon S_{Δ} . The pairs of canonically conjugate variables are one forms $(A_I, -e^I)$. In the space-time language, they are all just pull-backs of the respective 3-dimensional quantities to Σ . Our phase space consists of only those fields which satisfy the isolated horizon conditions on S_{Δ} and the asymptotic fall-off conditions in the neighborhood of S_{∞} .

Additionally, in order to have an unambiguous notion of angular momentum, we will impose further restrictions. Namely, we choose to work with so called *axi-symmetric* horizons. These are defined as the points in our phase space which posses a symmetry vector

field on the horizon φ . We choose to fix one such vector field for all of our phase space and a different choice of φ leads to a different choice of phase space. This vector field is required to have the following properties: 1) φ is tangent to S_Δ , 2) the affine length of its orbits is equal to 2π , and 3) it Lie-drags the intrinsic metric q_{ab} on the horizon.

The task of defining our phase space will be completed once we give the symplectic structure. Its form is given by

$$\Omega(\delta_1, \delta_2) = \int_{\Sigma} (\delta_1 A^I \wedge \delta_2 e_I - \delta_2 A^I \wedge \delta_1 e_I) + \tilde{\Omega}(\delta_1, \delta_2), \quad (3.51)$$

where the extra term $\tilde{\Omega}$ is given by a suitable integral over S_Δ . The reason for the boundary term is that the volume integral is not conserved, i.e. it depends on the spatial slice Σ . This can be seen in the following way. Using equations of motion, one can check that for the spacetime quantities

$$D(\delta_1 A^I \wedge \delta_2 e_I - \delta_2 A^I \wedge \delta_1 e_I) = 0. \quad (3.52)$$

Consider now a region of space-time \tilde{M} , boundary of which consists of spatial slices Σ_1, Σ_2 , the part of the horizon between the slices $\tilde{\Delta}$ and spatial infinity. Because of (3.52) and the fall-off conditions at spatial infinity, the difference between the volume terms on the slices Σ_2 and Σ_1 (as defined above) is given by the integral at the horizon:

$$\int_{\tilde{\Delta}} [\delta_1 A^I \wedge \delta_2 e_I - (1 \leftrightarrow 2)] = \int_{\tilde{\Delta}} [\delta_1 \omega \wedge \delta_2 m - (1 \leftrightarrow 2)]. \quad (3.53)$$

We know however that there exists a function ψ on the horizon such that

$$l \cdot d\psi \doteq l \cdot \omega. \quad (3.54)$$

Therefore, the above expression can also be written as a difference of two integrals over the circles $S_1 = \tilde{\Delta} \cap \Sigma_1$ and $S_2 = \tilde{\Delta} \cap \Sigma_2$

$$\int_{S_2} - \int_{S_1} [\delta_1 \psi \delta_2 m - (1 \leftrightarrow 2)]. \quad (3.55)$$

Thus the symplectic form is conserved and we have

$$\tilde{\Omega}(\delta_1, \delta_2) = \int_{S_\Delta} (\delta_1 \psi \delta_2 m - \delta_2 \psi \delta_1 m). \quad (3.56)$$

Moreover, we can fix the freedom in the choice of the function ψ in such a way that this boundary term vanishes on the initial slice Σ_1 .

3.3.3 Angular momentum.

Since our horizon has a symmetry φ we are able to find a conserved quantity associated with it. We will call it angular momentum of the horizon J_Δ . More precisely, consider any extension $\tilde{\varphi}$ of φ outside of the horizon which is an asymptotic rotational Killing vector field at spatial infinity. One can check that, on shell,

$$\Omega(\delta, X_{\tilde{\varphi}}) = \delta J_\infty - \delta J_\Delta, \quad (3.57)$$

where $X_{\tilde{\varphi}}$ is the vector field on the phase-space generated by $\tilde{\varphi}$ and J_∞ , J_Δ are given by integrals at spatial infinity and at the horizon, respectively. Thus J_Δ is the horizon term in the current generated by $\tilde{\varphi}$.

The above equation provides us with a formula for angular momentum of the horizon (up to an additive constant) in terms of basic variables and φ :

$$J_\Delta = - \int_{S_\Delta} (\varphi \cdot \omega) m \quad (3.58)$$

The additive constant in this formula was fixed by the requirement that J_Δ vanishes in the non-rotating BTZ solution.

3.4 THE FIRST LAW.

In order to formulate the first law of mechanics for isolated horizons we need to define an energy. Therefore we need to introduce the time evolution vector field t . Motivated by the stationary space-time examples, we consider only the following choices

$$t \hat{=} c_{(t)} l - \Omega_{(t)} \varphi, \quad (3.59)$$

where $c_{(t)}$ and $\Omega_{(t)}$ are constants on Σ , but they are not constant on the phase space. $c_{(t)}$ and $\Omega_{(t)}$ may depend on parameters of the horizon like area a_Δ , angular momentum J_Δ and charge Q_Δ . Following terminology used in [3] we call t a live vector field.

The vector field t defines a vector field on the phase space δ_t generating this time evolution. It is given by $\delta_t(e^I, A_J) \equiv (\mathcal{L}_t e^I, \mathcal{L}_t A_J)$. In the Hamiltonian framework it is natural to ask whether this evolution is Hamiltonian. In other words, is there a function on the phase space H_t such that

$$\delta H_t = \Omega(\delta, \delta_t)? \quad (3.60)$$

The answer is yes, if and only if, the one form X_t defined by

$$X_t(\delta) = \Omega(\delta, \delta_t) \quad (3.61)$$

is closed.

We can evaluate the right-hand side of the above equation using the equations of motion. The resulting expression involves only integrals at the boundary of the spacetime \mathcal{M} . On shell the bulk terms vanish.

$$X_t(\delta) = (\text{terms at infinity}) - \kappa_{(t)}\delta a_\Delta - \Omega_{(t)}\delta J_\Delta \quad (3.62)$$

where $\kappa_{(t)}$ is the surface gravity associated with $c_{(t)}l$. To derive this equation we used that $\delta l \sim l$ and $\delta n \sim n$. The evolution is thus Hamiltonian if and only if there exists a function E_Δ^t on phase space such that

$$\delta E_\Delta^t = \kappa_{(t)}\delta a_\Delta + \Omega_{(t)}\delta J_\Delta. \quad (3.63)$$

We thus conclude that the time evolution is Hamiltonian if and only if the first law of mechanics holds. Let us call t 's for which the first law holds admissible vector fields. The next section is devoted to the study of these fields.

3.4.1 Admissible t 's.

If a function E_Δ^t satisfying the first law exists, it must be a function of only the horizon parameters (a_Δ, J_Δ) . This is also true for $\kappa_{(t)}$ and $\Omega_{(t)}$. In order to construct admissible vector fields t we choose a function κ_0 on phase space only depends on a_Δ, J_Δ . Let us then ask that $\kappa_0 = \kappa_{(t)}$. This determines $c_{(t)}$ in (3.59) by $c_{(t)} = \kappa_{(t)}/\kappa$. Here κ is determined by l in the standard way. We then turn to $\Omega_{(t)}$ in (3.59). It follows from the first law that

$$\frac{\partial \kappa_{(t)}}{\partial J_\Delta} = \frac{\partial \Omega_{(t)}}{\partial a_\Delta}. \quad (3.64)$$

Integrating this condition gives

$$\Omega_{(t)} = \int_\infty^a \frac{\partial \kappa_{(t)}}{\partial J_\Delta} da_\Delta + F(J_\Delta), \quad (3.65)$$

where F is an arbitrary function of J_Δ . We can fix this function by imposing the condition

$$\lim_{\substack{J_\Delta = \text{const.} \\ a_\Delta \rightarrow \infty}} \Omega_{(t)} = 0 \quad (3.66)$$

By picking a function κ_0 of the horizon parameters that satisfies the condition mentioned above we thus arrive at a unique admissible vector field t . Since there is an infinite number of such functions we also have an infinite number of admissible vector fields t .

Once we have the functional form of $\kappa_{(t)}$ and $\Omega_{(t)}$ we can integrate the first law to derive a formula for the energy E_{Δ}^t . This will determine the energy only up to an additive constant. This can be fixed by imposing the physical condition

$$\lim_{\substack{J_{\Delta}, Q_{\Delta}=0 \\ a_{\Delta} \rightarrow 0}} E_{\Delta}^t = 0. \quad (3.67)$$

In the next section we demonstrate this procedure by choosing κ_0 to coincide with the corresponding function for the BTZ metric.

3.4.2 Energy of an isolated horizon.

As an illustration, we will use the above considerations to find E_{Δ}^t for a general isolated horizon with $T_{ab} \hat{=} 0$. We choose the constant $c_{(t)}$ to be unity and the surface gravity to be that of the BTZ black hole. Let

$$\kappa_{(t)} = -\frac{\Lambda a_{\Delta}}{2\pi} - \frac{2\pi J_{\Delta}^2}{a_{\Delta}^3}. \quad (3.68)$$

This implies that

$$\Omega_{(t)} = \frac{2\pi J_{\Delta}}{a_{\Delta}^2} + f(J_{\Delta}), \quad (3.69)$$

where f is any function which depends only on J_{Δ} and the cosmological constant. However, from physical considerations, we expect that

$$\lim_{\substack{J_{\Delta}=\text{const.} \\ a_{\Delta} \rightarrow \infty}} \Omega_{(t)} = 0. \quad (3.70)$$

This fixes the function f to be zero. Therefore the first law results in

$$\frac{\delta E_{\Delta}^t}{\delta a_{\Delta}} = -\frac{\Lambda a_{\Delta}}{2\pi} - \frac{2\pi J_{\Delta}^2}{a_{\Delta}^3}, \quad (3.71)$$

$$\frac{\delta E_{\Delta}^t}{\delta J_{\Delta}} = \frac{2\pi J_{\Delta}}{a_{\Delta}^2}. \quad (3.72)$$

These can be integrated to give

$$E_{\Delta}^t = -\frac{1}{4\pi} \Lambda a_{\Delta}^2 + \frac{\pi J_{\Delta}^2}{a_{\Delta}^2} + \text{const.} \quad (3.73)$$

Since we expect that for a non-rotating isolated horizon $\lim_{a_\Delta \rightarrow 0} E_\Delta^t = 0$, the constant should be chosen to be zero. The resulting expression is a general formula for the energy of an isolated horizon in terms of its intrinsic parameters.

After this digression into black hole mechanics we return in the following chapters to the topic of black hole entropy. We start by outlining the general procedure that we intend to follow.

Symmetries of Isolated Horizons

Given an isolated horizon it is natural to ask what we obtain if we apply the strategy outlined in chapter 2. The first step is to find the classical symmetries. Since an isolated horizon comes equipped with a number of structures there are natural candidates for symmetries of these horizons.

4.1 THE VECTOR FIELDS

We will say that a vector field ξ generates a symmetry of the horizon if the flow generated by ξ on the phase-space preserves the basic structure of the horizon, namely the equivalence class $[l]$ of null normals and the intrinsic metric q , i.e. $q_{ab} \equiv g_{ab}$. We thus demand

$$\mathcal{L}_\xi l \in [l], \tag{4.1}$$

$$\mathcal{L}_\xi q_{ab} = 0. \tag{4.2}$$

It is not difficult to check that any vector field ξ satisfying the above conditions can be written as

$$\xi^a = Al^a + Bm^a, \tag{4.3}$$

where the functions A and B are restricted to be of the form

$$A = C(v_-) + \text{const.} \cdot v \tag{4.4}$$

$$B = \text{const.} \tag{4.5}$$

The coordinates v and v_- are defined by the relations $n = -dv$, $m = \frac{1}{r_+} \frac{\partial}{\partial \phi}$, and $v_\pm = v \mp \phi/\Omega$. It is easy to see that the algebra of these vector fields (4.3) closes. These vector fields ξ are the symmetries of an isolated horizon.

4.2 HAMILTONIAN AND POISSON BRACKET ALGEBRA

Having found the symmetries of an isolated horizon we now want to find a representation of them. To this end we use the Hamiltonian theory of isolated horizons developed in chapter 3. Using these results we can construct the Hamiltonians corresponding to the symmetries together with their Poisson brackets.

As we have seen in chapter 3 there is a natural phase space together with a symplectic structure for isolated horizons. This symplectic structure on-shell is given by

$$\Omega(\delta_\xi, \delta) = -\frac{1}{\pi} \oint_{S_\Delta} [(\xi \cdot A_I) \delta e^I + (\xi \cdot e^I) \delta A_I] + \tilde{\Omega}(\delta_\xi, \delta), \quad (4.6)$$

where $\tilde{\Omega}$ is a gauge term which is not important for the present analysis. A and e are the connection one-form and the orthonormal triad, respectively. Using this expression we can find the Hamiltonian corresponding to the ξ 's as well as the Poisson bracket of two Hamiltonians. The obtain

$$H_\xi = -\frac{1}{\pi} \oint_{S_\Delta} (\xi \cdot A_I) e^I + C_\Delta, \quad (4.7)$$

$$\{H_{\xi_1}, H_{\xi_2}\} = -\frac{1}{\pi} \oint_{S_\Delta} [(\xi_1 \cdot A_I) \mathcal{L}_{\xi_2} e^I + (\xi_1 \cdot e^I) \mathcal{L}_{\xi_2} A_I], \quad (4.8)$$

where C_Δ is zero except when ξ contains a constant multiple of l . Then we have $C_\Delta[cl] = c(M + 2r_+ \kappa + J\Omega)$.

Subsequently, one can check that for any such symmetry vector fields

$$\{H_{\xi_1}, H_{\xi_2}\} \hat{=} H_{[\xi_1, \xi_2]}, \quad (4.9)$$

and therefore there is no central extension of the corresponding algebra of conserved charges.

This result is not quite unexpected since our analysis is entirely classical and a central charge often arises from the violation of classical symmetries in the quantized theory. Nonetheless, it shows that, in general, for symmetries represented by smooth vector fields on the horizon, the ideas of [44, 13] do not go through. If one wishes to use smooth fields—as is most natural at least in the classical theory—the central charge can arise only from quantization and the analysis would be sensitive to the details of the quantum theory, such as the regularization scheme used, etc. If the original intent of the ideas of [44, 13] is to be preserved, one must consider symmetries represented by vector fields which do not admit smooth limits to the horizon; in a consistent treatment, the use of “stretched horizons” [13]

is not optional but a necessity. Perhaps this is the price one has to pay to transform an essentially quantum analysis in the language of classical Hamiltonian theory.

Finally, note that any reasonable local definition of a horizon would lead to the above conclusions since we have made very weak assumptions in this sub-section.

A New Set of Symmetries

In the last chapter we found that the natural set of symmetries of an isolated horizon does not lead to a central charge. This chapter will introduce a new set of symmetries. They will be rather different from the ones encountered before. As we will see below the corresponding vector fields will not even allow for a smooth limit to the horizon. This is not too surprising since other symmetries have been covered in the last chapter and we found no central charge.

In this chapter we will not be as general as in the last two chapters. Thus far we have dealt entirely with the general notion of an isolated horizon. From now on we will be more concrete and deal with the near horizon geometry of the BTZ black hole. This is because we need specific properties of the BTZ black hole, such as the existence of a Killing vector, that are not available in general. For what follows it is useful to recall the definition of the BTZ black hole given in section 2 and the results about Newman-Penrose coefficients gathered in appendix A.

As pointed out in chapter 2 a strategy like ours has been followed in [13]. Since that analysis was flawed with technical problems we will point to these problems as we go along.

5.1 THE VECTOR FIELDS

The line-element of the BTZ black hole in the Eddington-Finkelstein like coordinates is given by

$$ds^2 = -N^2 dv^2 + 2dvdr + r^2(d\phi + N^\phi dv)^2, \quad N^2 = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}, \quad N^\phi = -\frac{J}{2r^2}. \quad (5.1)$$

For this metric (5.1) a convenient choice of the Newman-Penrose basis vector fields is

$$l = \partial_v + \frac{1}{2}N^2\partial_r - N^\phi\partial_\phi, \quad n = -\partial_r, \quad m = \frac{1}{r}\partial_\phi. \quad (5.2)$$

The BTZ space-time admits a global Killing vector

$$\chi = \partial_v - \Omega \partial_\phi, \quad \Omega = N^\phi(r_+). \quad (5.3)$$

As in [13] we now define another vector field ρ^a which is given by

$$\nabla_a \chi^2 = -2\kappa \rho_a, \quad \rho = \frac{r}{r_+} \left(\partial_v + N^2 \partial_r - N^\phi \partial_\phi \right). \quad (5.4)$$

It follows that $\chi \cdot \rho = 0$ and $\mathcal{L}_\chi \rho^a = 0$ everywhere. For convenience we express both vector fields χ, ρ in the Newman-Penrose basis up to order $(r - r_+)^2$ terms

$$\chi^a = l^a + (r - r_+)(\kappa n^a + 2\Omega m^a) + \mathcal{O}(r - r_+)^2, \quad \rho^a = \frac{r}{r_+} l^a - (r - r_+) \kappa n^a + \mathcal{O}(r - r_+)^2. \quad (5.5)$$

Clearly, at the horizon $\chi \hat{=} \rho \hat{=} l$. Two other useful identities are $\nabla_a \rho_b = \nabla_b \rho_a$ and $\chi^a \nabla_a \chi_b = \kappa \rho_b$ which follow from the definition (5.4) of ρ and the fact that χ is a Killing vector.

The classical phase-space can be taken to be the space of solutions of Einstein's equations of the form

$$g_{ab}^{\text{BTZ}} + \delta g_{ab}, \quad (5.6)$$

where g_{ab}^{BTZ} is the BTZ metric discussed above. To ensure that the near-horizon geometry of the metrics thus obtained remains close to that of the BTZ metric we will impose boundary conditions at the inner horizon. The Killing vector field χ^a becomes the null normal of the horizon. We want this to be true for all spacetimes in our phase space and we thus impose

$$\chi^a \chi^b \delta g_{ab} = \mathcal{O}(r - r_+)^2, \quad \chi^a m^b \delta g_{ab} \hat{=} 0, \quad (5.7)$$

where m^a is any space-like vector tangent to the inner boundary. The hat over the equality sign here means that the above equation holds on the horizon.

Clearly the vector field ξ which preserves these boundary conditions (5.7) under diffeomorphisms has to be tangent to the horizon. Keeping the same notation as in [13] let us take the vector field to be

$$\xi^a = T \chi^a + R \rho^a \quad (5.8)$$

where, to begin with, R and T are arbitrary functions. By demanding that (5.8) preserves (5.7) under diffeomorphisms one puts restrictions on R and T . These are derived in [13] (cf. eq (4.8))

$$R = \frac{1}{\kappa} \frac{\chi^2}{\rho^2} DT, \quad D \equiv \chi^a \nabla_a. \quad (5.9)$$

The vector field, satisfying (5.9), can then be said to generate symmetries in the precise sense of (5.7).

Before proceeding we want to emphasize a point that we made earlier in chapter 2. The vector fields ξ lie by construction in the plane spanned by the vectors χ^a and ρ^a . If these vector fields form a $\text{Diff}(S^1)$ it is a purely algebraic statement. There is no geometric object isomorphic to S^1 whose diffeomorphism group is responsible for our $\text{Diff}(S^1)$.

Let us now check the closure of the Lie-algebra of these vector fields. It is at this point that the analysis of [13] appears to be flawed. The errors arise at three levels:

- As noted in [13] the requirement that the Lie bracket of symmetry vector fields should close imposes a new condition

$$\mathcal{L}_\rho T \hat{=} 0. \quad (5.10)$$

In [13] this condition was imposed *at the horizon*. However, at the horizon $\rho^a \hat{=} \chi^a \hat{=} l^a$ and hence, (5.10) reads $DT \hat{=} 0$. Then the main steps in the calculations of [13] fail to go through. In particular, the central charge is expressed in terms of DT at the horizon and therefore vanishes identically. This in turn implies that the entropy also vanishes identically. While the restriction on DT has been noted explicitly in [13], its (obvious) consequences on the value of the central charge and entropy are overlooked.

- Furthermore, it is *not* sufficient to impose (5.10) only *at* the horizon; closure will fail unless it holds in a neighborhood.
- Later, for explicit calculations, a specific function T is chosen in [13] (cf. eq. (5.6)). Unfortunately, this function does not satisfy the condition (5.10) which is required in the earlier part of the analysis in [13].

In other words, although the boundary conditions (5.7) and (5.10) are reasonable, the technical implementation of them, as presented in [13], is incorrect. We will now propose an implementation of the boundary conditions that does not suffer from these problems.

Let us now consider symmetry vector fields defined in a neighborhood of the horizon. Thus, we will now use the stronger set of conditions (5.7) which requires that the closure condition (5.10) be satisfied *everywhere*¹. This guarantees that the Lie-algebra of the vector fields (5.8) closes

¹Strictly speaking, we only consider a neighborhood of the horizon where the vector field χ is Killing. In the BTZ example, however, it is globally Killing

$$[\xi_{T_1}, \xi_{T_2}] = \mathcal{L}_{\xi_{T_1}} \xi_{T_2} = \xi_{T_1} D T_2 - T_2 D T_1, \quad \xi_T = R\rho + T\chi \quad (5.11)$$

where R is determined in terms of T as in (5.9). One is to make use of the facts that $\mathcal{L}_\chi \rho = \mathcal{L}_\rho T = \mathcal{L}_\rho R = 0$. The condition (5.10), however, restricts the choice of the vector fields everywhere. To solve for the vector fields we consider a ‘stretched’-horizon at $r = r_+ + \varepsilon$ as the inner boundary. Consider solutions that are of the form

$$T_n \sim f_n(r) \exp(in\Omega v_+). \quad (5.12)$$

This choice is interesting because these solutions furnish a $\text{Diff}(S^1)$, provided $f_n f_m \sim f_{n+m}$. However, the condition (5.10) has to be imposed carefully because of the $(r - r_+)$ terms in the vector field ρ (5.4)

$$\rho^a \nabla_a T \sim \left(\partial_{v_+} + N^2 \partial_r \right) T = 0. \quad (5.13)$$

Clearly, the radial derivative of T blows up at the horizon. With the ansatz (5.12) there is a unique solution for T_n in the neighborhood of the horizon

$$T_n^\varepsilon = \frac{1}{2\Omega} \exp\left(-in\frac{\Omega}{\kappa} \log(r - r_+) + in\Omega v_+\right). \quad (5.14)$$

Note that the T_n^ε are functions of all three coordinates v, r , and ϕ .

The normalization of T is so chosen that the vector fields ξ form a $\text{Diff}(S^1)$ algebra

$$[\xi_{T_n}^\varepsilon, \xi_{T_m}^\varepsilon] = i(n - m)\xi_{T_{m+n}}^\varepsilon \quad (5.15)$$

in the neighborhood of the horizon.

Notice that because of (5.14) the vector fields ξ *do not have a well defined limit at the horizon*. They are defined only at the stretched horizon and oscillate wildly in the limit $r \rightarrow r_+$. Also the radial derivative of ξ blows up, as expected from the condition (5.10). So one has to take great care in evaluating the Poisson bracket and Hamiltonians – now *one cannot ignore terms which are of order $\mathcal{O}(r - r_+)$* especially in presence of radial derivatives in the Poisson brackets. Actually more terms will contribute to the Poisson bracket and a thorough examination of the entire calculation is needed.

5.2 HAMILTONIAN AND POISSON BRACKET ALGEBRA

The existence of the Hamiltonian under the boundary conditions (5.7) is shown in [13]. The surface Hamiltonian is

$$H_{\xi_n}^\varepsilon = \frac{1}{2\pi} \oint_{S_\Delta^\varepsilon} \epsilon_{abc} \nabla^b \xi_n^{\varepsilon c} . \quad (5.16)$$

The S_Δ^ε is a circle with a radius of $r_+ + \varepsilon$. We thus perform our calculations on a stretched horizon and later take a limit to the horizon. The phase space, described in the previous section is associated with a conserved symplectic current [32]. The corresponding symplectic structure may be used to evaluate the Poisson brackets between any two functions on phase-space. On shell, the symplectic structure can be written as the sum of boundary terms only. However, one may choose appropriate fall-off conditions of the fields at asymptotic infinity such that the contribution from the outer boundary vanishes. In the present example the fields ‘strongly’ approach the asymptotic AdS-values. In that case given two Hamiltonian vector fields ξ_1 and ξ_2 , the Poisson bracket between the two corresponding Hamiltonian functionals is given solely by the terms at the inner-boundary [32]

$$\{H_{\xi_1}, H_{\xi_2}\} = \oint_{S_\Delta} \left(\xi_2 \cdot \Theta[g, \mathcal{L}_{\xi_1} g] - \xi_1 \cdot \Theta[g, \mathcal{L}_{\xi_2} g] - \xi_2 \cdot (\xi_1 \cdot L) \right) \quad (5.17)$$

where $2\pi\Theta_a[g, \delta g] = \epsilon_{ab}[g^{bc}\nabla_c(g_{de}\delta g^{de}) - \nabla_c\delta g^{bc}]$ is the one-form symplectic potential and L is the three-form Lagrangian density. Making use of Einstein’s equations $R_{ab} = 2\Lambda g_{ab}$ we can express the Poisson bracket explicitly in terms of the vector fields

$$\{H_{\xi_1}^\varepsilon, H_{\xi_2}^\varepsilon\} = \frac{1}{2\pi} \oint_{S_\Delta^\varepsilon} \epsilon_{abc} \left[\xi_2^{\varepsilon b} \nabla_d (\nabla^d \xi_1^{\varepsilon c} - \nabla^c \xi_1^{\varepsilon d}) + 8\Lambda \xi_2^{\varepsilon b} \xi_1^{\varepsilon c} - (1 \leftrightarrow 2) \right] . \quad (5.18)$$

Our purpose is to find the terms proportional to n^3 in the Poisson bracket (5.18) which give rise to a non-trivial central extension to the Poisson bracket algebra. The Hamiltonian (5.16) contains terms only linear in n . The central charge can then be read off from the n^3 terms with appropriate normalizations. After a long calculation we arrive at the following expression

$$\lim_{\varepsilon \rightarrow 0} \left[\{H_{\xi_n}^\varepsilon, H_{\xi_m}^\varepsilon\} \right] = 4in^3 \delta_{m+n} \frac{a_\Delta \Omega}{2\pi\kappa} + \text{terms linear in } n . \quad (5.19)$$

As mentioned above the calculation was performed close to the horizon but not directly on the horizon. Only after we have performed the calculation did we take the limit to the

horizon. This procedure is essential for the result since it is the radial derivatives appearing in (5.18) that give rise to the non-vanishing n^3 term. Notice that although the vector fields (5.14) do not have a smooth limit as $r \rightarrow r_+$ the Hamiltonian and the Poisson bracket have well defined limits.

5.2.1 Entropy arguments

According to the standard normalization (up to linear order terms in n)

$$\lim_{\varepsilon \rightarrow 0} \left[\{H_{\xi_n}^\varepsilon, H_{\xi_m}^\varepsilon\} - i(n-m)H_{\xi_{n+m}}^\varepsilon \right] = i \frac{c}{12} n^3 \delta_{n+m} \quad (5.20)$$

the central charge can be read off from the n^3 -term in the Poisson bracket (5.18)

$$c = 24 \frac{a_\Delta \Omega}{\pi \kappa} . \quad (5.21)$$

In order for us to be able to use the Cardy formula we need to calculate the value of the Hamiltonian corresponding to the zero mode ξ_0 . Since the vector fields ξ_n are periodic in the variable ϕ the integration in (5.16) picks out only the zero mode. We thus obtain

$$\lim_{\varepsilon \rightarrow 0} \left[H_{\xi_n}^\varepsilon \right] = \frac{a_\Delta \kappa}{2\pi \Omega} \delta_{n,0} . \quad (5.22)$$

Here we see again that the order in which we performed the calculation is important. Had we taken the limit to the horizon, i.e. $\varepsilon \rightarrow 0$, before calculating the poisson bracket we would not have picked up the central charge. Now, using Cardy's formula [11], the entropy is

$$S = 2\pi \sqrt{\frac{cH_{\xi_0}}{6}} = 2\sqrt{2}a_\Delta, \quad (5.23)$$

which agrees with the Bekenstein-Hawking entropy (in units $8G = \hbar = 1$) up to a factor of $\sqrt{2}$.

It is worth noting here that Carlip's central extension (see formula 5.10 of [13]) and the zero-th mode Hamiltonian have the same numerical factor as ours. Nevertheless, he argues that one should use a different, so called effective central extension, and obtains the right numerical factor for the entropy. In our case this strategy fails since we have an extra factor of Ω/κ or its inverse in front of our expressions. It should be stressed, however, that this factor is rigidly fixed by the requirements that the symmetry algebra closes, that it gives a $\text{Diff}(S^1)$, and that the symmetry vector fields are periodic in the coordinate ϕ

with the period 2π . Moreover, following the arguments of [34, 12], since within a classical framework it is impossible to determine the value of the Hamiltonian in the ground state of the corresponding quantum theory, the right value of the central charge that is to be used in the Cardy formula is not determined classically.

The discrepancy between our result and the Bekenstein-Hawking entropy might thus point to a limitation in our purely classical approach. A quantum mechanical calculation of H_{ξ_0} could give rise to the right coefficient. We thus advocate the position that the factor appearing in front of the area should not be taken too seriously; whether it is one fourth or not. We will have more to say on this in the following chapter 6.

Chapter 6

Discussion

The entropy calculation of [44] faces certain conceptual limitations because the asymptotic symmetries may be completely different from the horizon symmetries. Both central charge (5.21) and Hamiltonian (5.16) are quite different from the ones found in [9] for asymptotic infinity. Thus, one needs an analysis restricted to the neighborhood of the horizon. In [13], Carlip recognized this limitation and carried out a Hamiltonian analysis using symmetries defined near the horizon. However, as we saw in section 2, the resulting analysis has certain technical flaws. In particular, the vector fields which correctly incorporate the ideas laid out in the beginning of that paper are quite different from the ones used in the detailed analysis later on.

In section 3 we made a proposal to overcome those technical problems and obtained a consistent formulation which implements the previous ideas. However, now the symmetry vector fields (5.8) do not have a well-defined limit at the horizon. Nonetheless both the Hamiltonians and their Poisson brackets are well-defined. Furthermore, there is a central charge which, following the reasoning of [44, 13], implies that the entropy is proportional to area. While the argument has attractive features, its significance is not entirely clear because the vector fields generating the relevant symmetries fail to admit well-defined limits to the horizon. Presumably, this awkward feature is an indication that, in a fully coherent and systematic treatment, the central charge would really be quantum mechanical in origin and could be sensitive to certain details of quantization, such as the regularization scheme used. Indeed, in the detailed analysis, we had to first evaluate the Poisson bracket and then take the limit $\lim \epsilon \rightarrow 0$ (see expressions (5.22) and also (5.20)), a step typical in quantum mechanical regularization schemes. Thus, it could well be that the awkwardness stems from the fact that, following [44, 13], we have attempted to give an essentially classical argument for a phenomenon that is inherently quantum mechanical.

This viewpoint is supported by our analysis of section 3.1 of symmetries corresponding

to smooth vector fields. If one requires that vector fields generating symmetries be smooth at the horizon—a most natural condition in a fully classical setting—we found that the central charge would be *zero*! Thus, the fact that the vector fields do not admit a smooth limit to the horizon is essential to the Carlip-type analysis. The fact that one has to ‘push’ the analysis an ϵ away from the horizon indicates that the procedure may be a ‘short-cut’ for a more complete quantum mechanical regularization¹.

This, however, raises some questions about the method in general: a) How satisfactory is the classical analysis and how seriously should one consider such vector fields? In particular, role of such vector fields in terms of space-time geometry is far from obvious since they are not even defined on the horizon. b) Why should this particular algebra be the focus of attention? c) Does the whole analysis suggest a rather transparent quantum mechanical regularization scheme and hence, systematically constrain the quantum theory?

The fact that our final expression of entropy differs from the standard Hawking-Bekenstein formula by a factor of $\sqrt{2}$ might also provide a test for theories of quantum gravity. The value H_{ξ_0} appearing in Cardy’s formula is of a quantum mechanical nature. A classical calculation may not give the right numerical value for it. It then follows that a quantum theory of gravity will give the correct value for the entropy provided it (a) has classical general relativity as its low energy limit, and, (b) the expectation value H_{ξ_0} is $a_{\Delta}\kappa/4\pi\Omega$ (assuming H_{ξ_0} is well defined in quantum theory).

Another point is worth mentioning here. For the longest time black hole entropy has been the paradigmatic problem of quantum gravity. It was believed that a correct derivation of the entropy would provide a key test for viable theories. It is correct that black hole entropy is a necessary test but the calculation that we have carried out in this work and the general idea on which it is based, proposed by others before, shows that black hole entropy is not that decisive a factor in deciding which theory of quantum gravity is right. If the

¹Sometimes it is argued that only a classical central charge can give rise to the standard expression $a_{\Delta}/4G\hbar$ of entropy and a central charge induced by a quantum anomaly can only give corrections to this expression. The reasoning goes as follows. To obtain the standard entropy expression, the central charge should go as $c \sim a_{\Delta}/G\hbar$. The presence of $1/\hbar$ in this expression implies that in the (naive) classical limit of the quantum Virasoro algebra,

$$[\hat{L}_n, \hat{L}_m] = (n - m)\hat{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{m+n}, \quad (6.1)$$

the central charge should survive. Note however, that in our detailed calculations, Hamiltonians, Poisson brackets etc. are defined on an ϵ -stretched horizon. Thus, if in the final picture, a quantum theory does lead to (6.1), its classical limit is likely to involve a delicate procedure, involving both ϵ and \hbar . Therefore, a priori it is unclear whether the central charge would survive in the calculation of classical Poisson brackets between quantities which are all well-defined at the horizon.

quantum theories have classical general relativity as their low energy limit the symmetries discussed in this article will be there and a central charge will be obtained.

In spite of the limitations of this calculation, the final result *is* of considerable interest because it is not a priori obvious that all the relevant subtleties of the full quantum mechanical analysis can be compressed in a classical calculation simply by stretching the physical horizon an ϵ distance away, performing all the Poisson brackets and then taking the limit $\epsilon \rightarrow 0$ in the *final* expressions. Note, however, that a careful treatment of technical issues that were overlooked in [13] was necessary to bring out these features. Indeed, our analysis provides the precise sense in which the original intention in [44, 13] of reducing the problem to a classical calculation is borne out in a technically consistent fashion.

Newman - Penrose formalism in 2+1 dimensions

A.0.2 Definition of the Coefficients

In this appendix we describe a Newman - Penrose like formalism for 2+1 dimensions. The triad we will be considering consists of the two null vectors l^a and n^a and the space - like vector m^a . The relations that we impose on these vectors are

$$l \cdot l = n \cdot n = 0, \quad m \cdot m = 1 \tag{A.1}$$

$$l \cdot m = n \cdot m = 0 \tag{A.2}$$

$$l \cdot n = -1. \tag{A.3}$$

Unlike in 3+1 dimensions the vector m^a is real. There will be no complex quantities appearing in the Newman - Penrose formalism for 2+1 dimensions.

Given the relations between the vectors of the triad we can express the metric in terms of the triad. We get

Lemma 1. *The metric g_{ab} can be written as follows:*

$$g_{ab} = -2l_{(a}n_{b)} + m_a m_b \tag{A.4}$$

Its inverse is given by

$$g^{ab} = -2l^{(a}n^{b)} + m^a m^b. \tag{A.5}$$

Proof. Using the relations satisfied by l , n , and m these relations can be easily verified. \square

We will now investigate the derivatives of the vectors in the triad. The requirement that

	l^b	n^b	m^b	
D	l^a	0	$-\epsilon$	$-\kappa_{NP}$
Δ	n^a	0	$-\gamma$	$-\tau$
δ	m^a	0	$-\alpha$	$-\rho$

Table A.1: The components of $\nabla_a l_b$.

l , n , and m form a triad of the above kind immediately leads to the following relations:

$$l^b \nabla_a l_b = n^b \nabla_a n_b = m^b \nabla_a m_b = 0 \quad (\text{A.6})$$

$$l^b \nabla_a m_b = -m^b \nabla_a l_b \quad (\text{A.7})$$

$$l^b \nabla_a n_b = -n^b \nabla_a l_b \quad (\text{A.8})$$

$$n^b \nabla_a m_b = -m^b \nabla_a n_b \quad (\text{A.9})$$

If we did not have any relations between l , n , and m we would have $3 \times 3 \times 3 = 27$ free parameters describing the derivatives of the basic vectors. The above equations (A.6) – (A.9) give $3 \times 3 + 3 + 3 = 18$ relations. The number of free parameters is thus brought down to 9. We use the notation of Stewart¹ to denote these free parameters and summarize them in the following tables A.1 – A.3.

The relations in these tables can be written as follows:

Lemma 2. *We have*

$$\begin{aligned} \nabla_a l_b &= -\epsilon n_a l_b + \kappa_{NP} n_a m_b - \gamma l_a l_b \\ &\quad + \tau l_a m_b + \alpha m_a l_b - \rho m_a m_b \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \nabla_a n_b &= \epsilon n_a n_b - \pi n_a m_b + \gamma l_a n_b \\ &\quad - \nu l_a m_b - \alpha m_a n_b + \mu m_a m_b \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} \nabla_a m_b &= \kappa_{NP} n_a n_b - \pi n_a l_b + \tau l_a n_b \\ &\quad - \nu l_a l_b - \rho m_a n_b + \mu m_a l_b \end{aligned} \quad (\text{A.12})$$

¹We have tried here to use the same notation. A problem arises because we only have m and not \bar{m} and \bar{m} . The factors of 2 that would occur because of this have been dropped.

		l^b	n^b	m^b
D	l^a	ϵ	0	π
Δ	n^a	γ	0	ν
δ	m^a	α	0	μ

Table A.2: The components of $\nabla_a n_b$.

		l^b	n^b	m^b
D	l^a	κ_{NP}	$-\pi$	0
Δ	n^a	τ	$-\nu$	0
δ	m^a	ρ	$-\mu$	0

Table A.3: The components of $\nabla_a m_b$.

Proof. These relations follow directly from the information that we have gathered in the tables A.1 to A.3. \square

As corollary we note the following relations:

Corollary 1. *We have*

$$\nabla_a l^a = \epsilon - \rho \tag{A.13}$$

$$\nabla_a n^a = \mu - \gamma \tag{A.14}$$

$$\nabla_a m^a = \pi - \tau \tag{A.15}$$

Proof. This follows from the equations in lemma 2 and the defining relations of the triad upon contraction of the indices. \square

We conclude this section with the example of the generalized BTZ black hole (see example 2 in section 3.1 for the definition of this example).

Example 5. *As an example we calculate the above coefficients for the generalized BTZ black hole. It is easy to verify that the following equations define a triad in the whole space-time:*

$$l^a = \partial_v + \frac{1}{2}(N^\perp)^2 \partial_r - N^\phi \partial_\phi \tag{A.16}$$

$$n^a = -\partial_r \tag{A.17}$$

$$m^a = \frac{1}{r} \partial_\phi \tag{A.18}$$

The corresponding one-forms are

$$l_a = -\frac{1}{2}(N^\perp)^2 dv + dr \tag{A.19}$$

$$n_a = -dr \tag{A.20}$$

$$m_a = rN^\phi dv + rd\phi. \tag{A.21}$$

With this triad the coefficients are:

$$\epsilon = \frac{f'(r)}{2} - r(N^\phi)^2 = \frac{r}{l^2} - r(N^\phi)^2 \quad (\text{A.22})$$

$$\gamma = 0 \quad (\text{A.23})$$

$$\alpha = N^\phi \quad (\text{A.24})$$

$$\kappa = 0 \quad (\text{A.25})$$

$$\tau = N^\phi \quad (\text{A.26})$$

$$\rho = -\frac{1}{2r}(N^\perp)^2 \quad (\text{A.27})$$

$$\pi = N^\phi \quad (\text{A.28})$$

$$\nu = 0 \quad (\text{A.29})$$

$$\mu = -\frac{1}{r} \quad (\text{A.30})$$

The second expression for ϵ is the one that one obtains if one uses the function $f(r)$ for the BTZ black hole, namely

$$f(r) = -M + \frac{r^2}{l^2}. \quad (\text{A.31})$$

A.0.3 Curvature expressions

In this section we will calculate the components of the curvature tensor. Since we are in $2+1$ dimensions all the information of the curvature tensor is contained in the Ricci tensor R_{ab} . We will thus calculate the different components of R_{ab} when contracted with l , n , and m .

To obtain these components we will make frequent use of the relation

$$\nabla_a \nabla_b t^c - \nabla_b \nabla_a t^c = -R_{abd}^c t^d. \quad (\text{A.32})$$

Contracting the indices b and c we obtain

$$\nabla_a \nabla_b t^b = \nabla_b \nabla_a t^b - R_{ad} t^d. \quad (\text{A.33})$$

Using the tables of the previous section we give here the general expressions for the components of the Ricci tensor.

Lemma 3. *The components of the Ricci tensor are:*

$$\begin{aligned} R_{abl}{}^al^b &= -\pi \kappa_{NP} + 2\alpha \kappa_{NP} - \epsilon \rho - \rho^2 + \kappa_{NP} \tau + \\ &\quad + \mathcal{L}_l \rho - \mathcal{L}_m \kappa_{NP} \end{aligned} \quad (\text{A.34})$$

$$\begin{aligned} R_{abl}{}^an^b &= \pi^2 - \pi \alpha + 2\gamma \epsilon - \epsilon \mu + \mu \rho - \pi \tau - \alpha \tau + \\ &\quad + \mathcal{L}_l \gamma - \mathcal{L}_l \mu - \mathcal{L}_n \epsilon + \mathcal{L}_m \pi \end{aligned} \quad (\text{A.35})$$

$$R_{abl}{}^am^b = 2\gamma \kappa_{NP} - \pi \rho - \rho \tau + \mathcal{L}_l \tau - \mathcal{L}_n \kappa_{NP} \quad (\text{A.36})$$

$$\begin{aligned} R_{abn}{}^al^b &= -\pi \alpha + 2\gamma \epsilon - \gamma \rho + \mu \rho - \pi \tau - \alpha \tau + \tau^2 + \\ &\quad + \mathcal{L}_l \gamma - \mathcal{L}_n \epsilon + \mathcal{L}_n \rho - \mathcal{L}_m \tau \end{aligned} \quad (\text{A.37})$$

$$R_{abn}{}^an^b = -\gamma \mu - \mu^2 + \pi \nu + 2\alpha \nu - \nu \tau - \mathcal{L}_n \mu + \mathcal{L}_m \nu \quad (\text{A.38})$$

$$R_{abn}{}^am^b = -\pi \mu + 2\epsilon \nu - \mu \tau + \mathcal{L}_l \nu - \mathcal{L}_n \pi \quad (\text{A.39})$$

$$\begin{aligned} R_{abm}{}^al^b &= -\pi \epsilon + \alpha \epsilon + \gamma \kappa_{NP} + \kappa_{NP} \mu - \pi \rho - \alpha \rho + \\ &\quad + \mathcal{L}_l \alpha - \mathcal{L}_m \epsilon \end{aligned} \quad (\text{A.40})$$

$$\begin{aligned} R_{abm}{}^an^b &= \alpha \gamma - \alpha \mu + \epsilon \nu + \nu \rho - \gamma \tau - \mu \tau - \\ &\quad - \mathcal{L}_n \alpha + \mathcal{L}_m \gamma \end{aligned} \quad (\text{A.41})$$

$$\begin{aligned} R_{abm}{}^am^b &= -\pi^2 + \epsilon \mu + 2\kappa_{NP} \nu + \gamma \rho - 2\mu \rho - \tau^2 + \\ &\quad + \mathcal{L}_l \mu - \mathcal{L}_n \rho - \mathcal{L}_m \pi + \mathcal{L}_m \tau \end{aligned} \quad (\text{A.42})$$

Because the Ricci tensor is symmetric we obtain the following relations:

Corollary 2. *The following relations hold:*

$$0 = \pi^2 - \epsilon \mu + \gamma \rho - \tau^2 - \mathcal{L}_l \mu - \mathcal{L}_n \rho + \mathcal{L}_m \pi + \mathcal{L}_m \tau \quad (\text{A.43})$$

$$\begin{aligned} 0 &= \pi \epsilon - \alpha \epsilon + \gamma \kappa_{NP} - \kappa_{NP} \mu + \alpha \rho - \rho \tau - \\ &\quad - \mathcal{L}_l \alpha + \mathcal{L}_l \tau - \mathcal{L}_n \kappa_{NP} + \mathcal{L}_m \epsilon \end{aligned} \quad (\text{A.44})$$

$$\begin{aligned} 0 &= -\alpha \gamma - \pi \mu + \alpha \mu + \epsilon \nu - \nu \rho + \gamma \tau + \\ &\quad + \mathcal{L}_l \nu - \mathcal{L}_n \pi + \mathcal{L}_n \alpha - \mathcal{L}_m \gamma \end{aligned} \quad (\text{A.45})$$

A.0.4 Behaviour under transformations

In this section we investigate how the Newman - Penrose coefficients change under Lorentz transformations. We begin with a boost in the plane spanned by l^a and n^a :

$$l^a \longrightarrow c l^a \quad (\text{A.46})$$

$$n^a \longrightarrow \frac{1}{c} n^a \quad (\text{A.47})$$

$$m^a \longrightarrow m^a \quad (\text{A.48})$$

The Newman Penrose coefficients now transform as follows:

$$\begin{aligned} \kappa'_{NP} &= c^2 \kappa_{NP} & \pi' &= \pi & \epsilon' &= c\epsilon + l^a \nabla_a c \\ \tau' &= \tau & \nu' &= \frac{1}{c^2} \nu & \gamma' &= \frac{1}{c} (\gamma + \frac{1}{c} n^a \nabla_a c) \end{aligned} \quad (\text{A.49})$$

$$\rho' = c\rho \quad \mu' = \frac{1}{c} \mu \quad \alpha' = \alpha + \frac{1}{c} m^a \nabla_a c$$

Next we look at a null rotation:

$$l^a \longrightarrow l^a \quad (\text{A.50})$$

$$n^a \longrightarrow \frac{1}{2} c^2 l^a + n^a + c m^a \quad (\text{A.51})$$

$$m^a \longrightarrow c l^a + m^a \quad (\text{A.52})$$

The coefficients now transform as follows:

$$\kappa'_{NP} = \kappa_{NP} \quad (\text{A.53})$$

$$\tau' = \tau + \frac{1}{2} c^2 \kappa_{NP} + c\rho \quad (\text{A.54})$$

$$\rho' = \rho + c\kappa_{NP} \quad (\text{A.55})$$

$$\pi' = \pi + \frac{1}{2} c^2 \kappa_{NP} + c\epsilon + l^a \nabla_a c \quad (\text{A.56})$$

$$\begin{aligned} \nu' &= \nu + \frac{1}{2} c^3 \epsilon + \frac{1}{4} c^4 \kappa_{NP} + c\gamma + \frac{1}{2} c^2 \tau + c^2 \alpha + c^3 \rho + \frac{1}{2} c^2 \pi \\ &\quad + \frac{1}{2} c^2 l^a \nabla_a c + n^a \nabla_a c + c m^a \nabla_a c \end{aligned} \quad (\text{A.57})$$

$$\begin{aligned} \mu' &= \mu + c^2 \epsilon + \frac{1}{2} c^3 \kappa_{NP} + c\alpha + c\pi + \frac{1}{2} c^2 \rho + \\ &\quad + c l^a \nabla_a c + m^a \nabla_a c \end{aligned} \quad (\text{A.58})$$

$$\epsilon' = \epsilon + c\kappa_{NP} \quad (\text{A.59})$$

$$\gamma' = \gamma + \frac{1}{2} c^2 \epsilon + \frac{1}{2} c^3 \kappa_{NP} + c\tau + c\alpha + c^2 \rho \quad (\text{A.60})$$

$$\alpha' = \alpha + c\epsilon + c^2 \kappa_{NP} + c\rho \quad (\text{A.61})$$

A.0.5 Newman Penrose form of the connection

We can express the covariant derivative operator ∇_a in terms of the connection one - form A_a^I . Using the relation

$$\nabla_a v_b = A_{aJ}^I v^J e_{Ib}, \quad (\text{A.62})$$

where e_{Ib} is the triad and using

$$A_a{}_{IJ} = \epsilon_{IJK} A_a^K \quad (\text{A.63})$$

we arrive at

$$\begin{aligned} A_a^K &= (\pi n_a + \nu l_a - \mu m_a) l^K \\ &\quad + (\kappa n_a + \tau l_a - \rho m_a) n^K \\ &\quad + (-\epsilon n_a - \gamma l_a + \alpha m_a) m^K \end{aligned} \quad (\text{A.64})$$

For the triad we obtain

$$e_a^I = -l_a n^I - n_a l^I + m_a m^I. \quad (\text{A.65})$$

A.0.6 The Maxwell's equations

The Newman-Penrose components of the field strength \mathbf{F} are defined by

$$\mathbf{F} = \Phi_0 n \wedge m + \Phi_1 l \wedge n + \Phi_2 l \wedge m. \quad (\text{A.66})$$

The Maxwell equations are then given by

$$D\Phi_1 - \delta\Phi_0 = (\pi - \alpha)\Phi_0 + \rho\Phi_1 - \kappa\Phi_2, \quad (\text{A.67})$$

$$2D\Phi_2 - \delta\Phi_1 = -\mu\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2, \quad (\text{A.68})$$

$$2\Delta\Phi_0 - \delta\Phi_1 = (2\gamma - \mu)\Phi_0 - 2\tau\Phi_1 + \rho\Phi_2, \quad (\text{A.69})$$

$$\Delta\Phi_1 - \delta\Phi_2 = \nu\Phi_0 - \mu\Phi_1 + (\alpha - \tau)\Phi_2. \quad (\text{A.70})$$

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His publications include

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