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Geometry of Torsion Gerbes and Flat Twisted Vector Bundles

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Abstract: Gerbes and higher gerbes are geometric cocycles representing higher degree cohomology classes, and are attracting considerable interest in differential geometry and mathematical physics. We prove that a 2-gerbe has a torsion Dixmier–Douady class if and only if the gerbe has locally constant cocycle data. As an application, we give an alternative description of flat twisted vector bundles in terms of locally constant transition maps. These results generalize to n-gerbes for $n = 1$ and $n \geq 3$, providing insights into the structure of higher gerbes and their applications to the geometry of twisted vector bundles.

Keywords: gerbe; 2-gerbe; smooth Deligne cohomology; Dixmier–Douady class; twisted vector bundles

MSC: primary 53C08; secondary 55R65; 55N05

1. Introduction

In modern differential geometry, the study of higher categorical structures has led to significant advancements in our understanding of manifolds and their invariants. Gerbes and higher gerbes, as geometric realizations of such structures, play a crucial role in this landscape, connecting diverse areas such as algebraic topology, complex geometry, and mathematical physics.

$U(1)$-gerbes are geometric objects representing degree 3 integral cohomology classes, just as line bundles represent degree 2 integral cohomology classes. Gerbes were originally introduced by Giraud [1], and began to be used more often in the context of algebraic topology and differential geometry after Brylinski [2]. In particular, Murray [3] conceived and constructed an explicit and geometric model of a gerbe, called a bundle gerbe, as opposed to a description as a certain *sheaf of groupoids*. This model by Murray has been further developed by several authors. Most notably, Stevenson has developed a geometric model of a 2-gerbe, and the 2-stack structure of gerbes was considered [4,5], which was further studied by Waldorf [6], and equivariant refinements were studied in [7,8]. Gerbes and higher gerbes have been applied to several problems in mathematics and physics. For example, twisted K-theory and Ramond–Ramond field classifications [9–12], local formulas for $2d$ Wess–Zumino (WZ) action [13] and its Feynman amplitude interpreted as a bundle gerbe holonomy [14,15], geometric string structures [16], and even topological insulators [17–19].

As mentioned above, there are several models for higher gerbes with connection. To list a few, there are bundle n-gerbes with connection, sheaves of higher groupoids, and a map into a classifying ∞-stack $\mathbb{B}^{n+2}_{\nabla}$. However, one of the most classical and elementary models would be the Deligne cocycle model, consisting of Čech cocycles and local differential form data. Indeed, the Deligne complex is the natural home for studying differential geometric cocycles such as line bundles with connections and (higher) gerbes with connection.

This article is a brief technical report on differential geometry of torsion gerbes. Namely, we prove that a necessary and sufficient condition for the Dixmier–Douady class of a 2-gerbe to be torsion is that its cocycle data consist of locally constant maps, and its proof essentially generalizes for the case of n-gerbes with $n = 1$ or $n \geq 3$. The idea comes

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from a well-known fact on flat vector bundles, i.e., a necessary and sufficient condition for a vector bundle to admit a flat connection is that the Čech-cocycle data of the underlying vector bundle consists of locally constant maps. Using our results on torsion 2-gerbes, we also prove a generalization of this fact to flat twisted vector bundles.

As is well-known, a gerbe being torsion or not is crucial in studying geometric cocycles of twisted K-theory. Indeed, if a geometric cocycle admits a nontorsion twist, it has to be an infinite dimensional construction (see [9,12]). Therefore, we expect that our results will be useful in studying finite-dimensional constructions such as twisted vector bundles or bundle gerbe modules with finite-dimensional fibers.

This paper is organized as follows. In Section 2, we review the $U(1)$-gerbe with connections and its higher analogues. This section also serves the purpose of setting up notations and terminologies we will be using throughout this paper. In Section 3, we prove that a 2-gerbe is torsion if and only if its cocycle data consists of locally constant functions. In Section 4, we apply our main theorem to prove a twisted analogue of a classical fact that a vector bundle is flat if and only if there exist local trivializations whose transition maps are locally constant.

2. Preliminaries

In this section, we review (higher) gerbes with connection. Throughout this paper, all of our manifolds are smooth manifolds, and all of our maps are smooth maps, unless specified otherwise. In particular, X always denotes a manifold. By gerbes, we will always mean $U(1)$-gerbes. We will use the notation $U_{i_1\cdots i_n}$ to denote an n-fold intersection $U_{i_1} \cap \cdots \cap U_{i_n}$. If an open cover is locally finite and every n-fold intersection is contractible for all $n \in \mathbb{Z}^+$, we will call it a good cover. On a smooth manifold, a good cover always exists. A Čech cocycle $\zeta = (\zeta_{i_1\cdots i_n})$ is said to be *completely normalized* if $\zeta_{i_1\cdots i_n} \equiv 1$ whenever there is a repeated index, and $\zeta_{\sigma(i_1)\cdots\sigma(i_n)} = (\zeta_{i_1\cdots i_n})^{\mathrm{sign}(\sigma)}$ for any $\sigma \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on n letters.

2.1. gerbes with connection

In this subsection, we shall review a Čech cocycle description of a gerbe with connections. See Gawędzki and Reis [15] and Hitchin [20] for a broader account.

Definition 1. *Let X be a manifold and $\mathcal{U} := \{U_i\}_{i\in\Lambda}$ an open cover of X. A **gerbe** over X subordinate to $\mathcal{U}$ is a $U(1)$-valued completely normalized Čech 2-cocycle $\{\lambda_{kji}\} \in \check{Z}^2(\mathcal{U}, U(1))$. A **connection** on a gerbe $\{\lambda_{kji}\}$ on $\mathcal{U}$ is a pair $(\{A_{ji}\}, \{B_i\})$ consisting of a family of differential 1-forms $\{A_{ji} \in \Omega^1(U_{ij}; \sqrt{-1}\mathbb{R})\}_{i,j\in\Lambda}$, and a family of differential 2-forms $\{B_i \in \Omega^2(U_i; \sqrt{-1}\mathbb{R})\}_{i\in\Lambda}$, satisfying the following relations:*

- $\lambda_{kji}\lambda_{lji}^{-1}\lambda_{lki}\lambda_{lkj}^{-1} = 1;$
- $d\log\lambda_{kji} = A_{ji} + A_{ik} + A_{kj};$
- $B_j - B_i = dA_{ji}.$

*From $dB_i = dB_j$ for all $i, j \in \Lambda$, the family of exact 3-forms $\{dB_i\}_{i\in\Lambda}$ defines a global closed differential 3-form H. The differential form H is called the **curvature** of the gerbe, or the **Neveu–Schwarz 3-form**.*

A gerbe with connections on $\mathcal{U}$ is therefore a Deligne cocycle of degree 2. Notice that our total differential is $D = d + (-1)^q\delta$ on $\check{C}^p(\mathcal{U}, \Omega^q)$. Throughout the rest of this paper, $\widehat{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ always denotes a gerbe with connections defined on an open cover $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ of X, and H denotes the 3-curvature form of $\widehat{\lambda}$.

Definition 2. *Two gerbes with connections $\widehat{\lambda}$ and $\widehat{\lambda}'$ are **isomorphic** if $\widehat{\lambda}'$ is obtained by adding a total degree 2 Deligne coboundary to $\widehat{\lambda}$, i.e., $\widehat{\lambda}' = \widehat{\lambda} + D\widehat{\mu}$ for some $\widehat{\mu} \in \check{C}^1(\mathcal{U}, \Omega^0) \oplus \check{C}^0(\mathcal{U}, \Omega^1)$.*

Remark 1. *Let $\{\lambda_{kji}\} \in \check{Z}^2(\mathcal{U}, U(1))$ be a gerbe, and $\delta : \check{H}^2(\mathcal{U}, U(1)) \to H^3(X; 2\pi i\mathbb{Z})$ be the connecting map. The image in $H^3_{dR}(X; \sqrt{-1}\mathbb{R})$ of the cohomology class $\delta([\lambda]) \in H^3(X; 2\pi i\mathbb{Z})$ coincides with the cohomology class of $H \in H^3_{dR}(X; \sqrt{-1}\mathbb{R})$ (see Brylinski ([2] p. 175) Corollary 4.2.8.). Here, the cohomology class $\delta([\lambda])$ is a topological invariant of a gerbe, called the* Dixmier–Douady class.

2.2. Higher gerbes with connection

In the previous subsection, we have seen that a gerbe with connections is a degree 2 Deligne cocycle. It is possible to generalize it to higher degrees for a cocycle definition of an n-gerbe with connections. Compare Stevenson [4,5] and Gajer [21].

Definition 3. *Let X be a manifold, and $\mathcal{U} := \{U_i\}_{i\in\Lambda}$ be an open cover of X. An **n-gerbe** over X subordinate to $\mathcal{U}$ is a $U(1)$-valued completely normalized Čech $(n+1)$-cocycle $\{\lambda_{i_{n+2}\cdots i_1}\} \in \check{Z}^{n+1}(\mathcal{U}, U(1))$. A **connection** on an n-gerbe $\{\lambda_{i_{n+2}\cdots i_1}\}$ on $\mathcal{U}$ is an $(n+1)$-tuple $(\{A^{(1)}_{i_{n+1}\cdots i_1}\}, \{A^{(2)}_{i_n\cdots i_1}\}, \cdots, \{A^{(n+1)}_{i_1}\})$, consisting of a family of differential k-forms $\{A^{(k)}_{i_{n+k-2}\cdots i_1} \in \Omega^k(U_{i_{n+k-2}\cdots i_1}; \sqrt{-1}\mathbb{R})\}_{i_{n+k-2},\cdots,i_1\in\Lambda}$, satisfying that the $(n+2)$-tuple $\widehat{\lambda} = (\lambda, A^{(1)}, \cdots, A^{(n+1)})$ is a degree $(n+1)$-Deligne cocycle, i.e., $D\widehat{\lambda} = 0$. The differential $(n+1)$-forms $\{A^{(n+1)}_i\}$ defined on each open set satisfy $dA^{(n+1)}_i = dA^{(n+1)}_j$ for all $i, j \in \Lambda$; the family of exact $(n+2)$-forms $\{dA^{(n+1)}_i\}_{i\in\Lambda}$ defines a global closed differential $(n+2)$-form $\mathcal{H}$. The differential form $\mathcal{H}$ is called the **curvature** of the n-gerbe.*

Definition 4. *Two n-gerbes with connection $\widehat{\lambda}$ and $\widehat{\lambda}'$ are **isomorphic** if $\widehat{\lambda}'$ is obtained by adding a total degree $n+1$ Deligne coboundary to $\widehat{\lambda}$, i.e., $\widehat{\lambda}' = \widehat{\lambda} + D\widehat{\mu}$ for some $\widehat{\mu} \in \check{C}^n(\mathcal{U}, \Omega^0) \oplus \check{C}^{n-1}(\mathcal{U}, \Omega^1) \oplus \cdots \oplus \check{C}^0(\mathcal{U}, \Omega^n)$*

Similarly for gerbes, an n-gerbe $\lambda \in \check{Z}^{n+1}(\mathcal{U}, U(1))$ has an higher analogue of the Dixmier–Douady class in $H^{n+2}(X; 2\pi i\mathbb{Z})$ as its topological invariant. Its image in $H^{n+2}_{\mathrm{dR}}(X; \sqrt{-1}\mathbb{R})$ coincides with the curvature $\mathcal{H}$ of n-gerbe (Cf. Stevenson [4], Chapter 11).

Remark 2. *For later use, we give explicit formula of the cocycle condition for a* 2*-gerbe with connection $(\{\lambda_{lkji}\}, \{A_{kji}\}, \{B_{ji}\}, \{C_i\})$.*

C1. $\lambda_{kji}\lambda^{-1}_{lji}\lambda_{lki}\lambda^{-1}_{lkj} = 1$;
C2. $d\log\lambda_{lkji} = A_{kji} - A_{lji} + A_{lki} - A_{lkj}$;
C3. $dA_{kji} = -B_{ji} + B_{ki} - B_{kj}$;
C4. $dB_{ji} = C_i - C_j$.

3. Main Theorems

In this section, we shall state and prove our main theorems on a necessary and sufficient condition for a 2-gerbe having a torsion Dixmier–Douady class. We state and prove the sufficiency and then the necessity.

Theorem 1. *Let X be a manifold, $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ be an open cover of X, and $\lambda = \{\lambda_{lkji}\}$ be a* 2*-gerbe on X. If each λ_{lkji} is a locally constant map, then this* 2*-gerbe determines a torsion class $\delta([\lambda])$ in $H^4(X; 2\pi i\mathbb{Z})$.*

Proof. Suppose that $(\{A_{kji}\}, \{B_{ji}\}, \{C_i\})$ is a connection on the given 2-gerbe λ. Since λ_{lkji} are locally constant maps, it follows that $A_{kji} - A_{lji} + A_{lki} - A_{lkj} = \lambda^{-1}_{lkji}d\lambda_{lkji} = 0$. Accordingly, we could have chosen a connection with $A_{kji} \equiv 0$, $B_{ji} \equiv 0$, and $C_i := \zeta|_{U_i}$ for some $\zeta \in \Omega^3(X; \sqrt{-1}\mathbb{R})$, since the quadruple $(\{\lambda_{kji}\}, \{0\}, \{0\}, \{\zeta|_{U_i}\})$ satisfies the cocycle conditions **C1** to **C4** in Remark 2. Moreover, since the curvature 4-form of this 2-gerbe with connections is exact, it follows that $\delta([\lambda]) \otimes \mathbb{R} = [d\zeta] = 0$, i.e., $\delta([\lambda])$ is a torsion class in $H^4(X, 2\pi i\mathbb{Z})$. Here, $\delta : \check{H}^3(\mathcal{U}, U(1)) \to H^4(X; 2\pi i\mathbb{Z})$ is the connecting map. □

Proceeding similarly as in the above proof, a similar theorem also holds for n-gerbes for $n = 1$ or ≥ 3, as stated in the following corollary.

Corollary 1. *Let X be a manifold,* $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ *be an open cover of X, and* $\lambda = \{\lambda_{i_{n+2}\cdots i_1}\}$ *be an n-gerbe on X. If each* $\lambda_{i_{n+2}\cdots i_1}$ *is a locally constant map, then this n-gerbe determines a torsion class* $\delta([\lambda])$ *in* $H^{n+2}(X; 2\pi i\mathbb{Z})$.

Theorem 2. *Let X and* λ *be as above. Suppose that the* 2*-gerbe* λ *is defined on a good cover* $\mathcal{U} = \{U_i\}_{i\in\Lambda}$*, and also that* λ *determines a torsion class* $\delta([\lambda])$ *in* $H^4(X; 2\pi i\mathbb{Z})$*. Then, given any connection* $(\{A_{kji}\}, \{B_{ji}\}, \{C_i\})$ *on this* 2*-gerbe, there exists a* 2*-gerbe with connection* $(\widetilde{\lambda}, \widetilde{A}, \widetilde{B}, \widetilde{C})$ *that has an underlying* 2*-gerbe consisting of a family of locally constant maps* $\widetilde{\lambda}_{lkji} : U_{ijkl} \to U(1)$*, such that the difference between* (λ, A, B, C) *and* $(\widetilde{\lambda}, \widetilde{A}, \widetilde{B}, \widetilde{C})$ *is a Deligne coboundary of degree* 3*.*

Proof. Suppose a 2-gerbe λ determines a torsion class $\delta([\lambda])$ in $H^4(X; 2\pi i\mathbb{Z})$. We first choose an arbitrary connection $(\{A_{kji}\}, \{B_{ji}\}, \{C_i\})$ on the 2-gerbe λ. For the curvature $\mathcal{H}$ of the 2-gerbe, $\delta([\lambda]) \otimes \mathbb{R} = [\mathcal{H}]$ is satisfied, and since the 2-gerbe is a torsion, $[\mathcal{H}]$ has a representative $d\zeta$, where ζ is a differential 3-form on X. Now, from $dC_i = \mathcal{H}|_{U_i} = d\zeta|_{U_i}$, we have $d(\zeta|_{U_i} - C_i) = 0$, and since U_i is contractible, by Poincaré's Lemma, $\zeta|_{U_i} - C_i = d\Pi_i$ for some $\Pi_i \in \Omega^2(U_i; i\mathbb{R})$. We define

$$\widetilde{C}_i := C_i + d\Pi_i = \zeta|_{U_i}.$$

Applying **C4**, we see that $d(B_{ji} + \Pi_i - \Pi_j) = 0$. Again by Poincaré's Lemma, there exists $\xi_{ji} \in \Omega^1(U_{ij}; \sqrt{-1}\mathbb{R})$, such that

$$B_{ji} + \Pi_i - \Pi_j = d\xi_{ji}. \tag{1}$$

We set

$$\widetilde{B}_{ji} := B_{ji} + \Pi_i - \Pi_j - d\xi_{ji} = 0.$$

Applying **C3** and Equation (1), we have $d(A_{kji} + \xi_{ji} - \xi_{ki} + \xi_{kj}) = 0$. Again, there exists $\chi_{kji} \in \Omega^0(U_{ijk}; U(1))$ such that $A_{kji} + \xi_{ji} - \xi_{ki} + \xi_{kj} = d\log\chi_{kji}$, so we define

$$\widetilde{A}_{kji} := A_{kji} + \xi_{ji} - \xi_{ki} + \xi_{kj} - d\log\chi_{kji} = 0$$
$$\widetilde{\lambda}_{lkji} := \lambda_{lkji}\chi_{kji}^{-1}\chi_{lji}\chi_{lki}^{-1}\chi_{lkj}.$$

It can be readily seen that $\widehat{\widetilde{\lambda}} = (\{\widetilde{\lambda}_{lkji}\}, \{\widetilde{A}_{kji}\}, \{\widetilde{B}_{ji}\}, \{\widetilde{C}_i\})$ satisfies conditions from **C1** to **C4**, where $\widetilde{A}_{kji} \equiv 0 \equiv \widetilde{B}_{ji}$ and $\widetilde{C}_i$ is a restriction of a global 3-form ζ to U_i. The 2-gerbe cocycles being locally constant follows from the cocycle condition **C2** for $\widehat{\widetilde{\lambda}}$. In addition, $\widehat{\lambda} = (\{\lambda_{lkji}\}, \{A_{kji}\}, \{B_{ji}\}, \{C_i\})$ satisfies $\widehat{\widetilde{\lambda}} = \widehat{\lambda} + D\widehat{\chi}$ where $\widehat{\chi} = (\{\chi_{kji}^{-1}\}, \{-\xi_{ji}\}, \{\Pi_i\})$. □

Proceeding similarly as in the above proof, a similar theorem also holds for torsion n-gerbes for $n = 1$ or ≥ 3, as stated in the following corollary.

Corollary 2. *Let X and* λ *be as above. Suppose that the n-gerbe* λ *is defined on a good cover* $\mathcal{U} = \{U_i\}_{i\in\Lambda}$*, and also that* λ *determines a torsion class* $\delta([\lambda])$ *in* $H^{n+2}(X; 2\pi i\mathbb{Z})$*. Then, given any connection* $(\{A^{(1)}_{i_{n+1}\cdots i_1}\}, \{A^{(2)}_{i_n\cdots i_1}\}, \cdots, \{A^{(n+1)}_{i_1}\})$ *on this n-gerbe, there exists an n-gerbe with connection* $(\widetilde{\lambda}, \widetilde{A}^{(1)}, \cdots, \widetilde{A}^{(n+1)})$ *that has an underlying n-gerbe consisting of a family of locally constant maps* $\widetilde{\lambda}_{i_{n+2}\cdots i_1} : U_{i_{n+2}\cdots i_1} \to U(1)$*, such that the difference between* $(\lambda, A^{(1)}, \cdots, A^{(n+1)})$ *and* $(\widetilde{\lambda}, \widetilde{A}^{(1)}, \cdots, \widetilde{A}^{(n+1)})$ *is a Deligne coboundary of degree* $n+1$*.*

4. Application: Flatness of Twisted Vector Bundle

In this section, we briefly review what a twisted vector bundle with connections is. After that, we recall an alternative characterization of a flat vector bundle via locally constant transition maps. We apply Corollary 2 to state and prove its twisted analogue.

Definition 5. *Let $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ be an open cover of X, and λ be a $U(1)$-valued completely normalized Čech 2-cocycle. A **λ-twisted vector bundle** E of rank n over X consists of a family of product bundles $\{U_i \times \mathbb{C}^n : U_i \in \mathcal{U}\}_{i\in\Lambda}$ together with transition maps*

$$g_{ji} : U_{ij} \to U(n)$$

satisfying

$$g_{ii} = \mathbf{1}, \quad g_{ji} = g_{ij}^{-1}, \quad g_{kj}g_{ji} = g_{ki}\lambda_{kji}.$$

The gerbe λ in this definition is also called a *twist*. A λ-twisted vector bundle is *smooth* if all transition maps and gerbe cocycle data are smooth maps. We shall write a λ-twisted vector bundle E over X of rank n as a triple $(\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$.

Definition 6. *Let $\widehat{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ be a gerbe with connections, and $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ be a smooth λ-twisted vector bundle of rank n. A **connection** on E compatible with $\widehat{\lambda}$ is a family $\Gamma = \{\Gamma_i \in \Omega^1(U_i; \mathfrak{u}(n))\}_{i\in\Lambda}$ satisfying*

$$\Gamma_i - g_{ji}^{-1}\Gamma_j g_{ji} - g_{ji}^{-1}dg_{ji} = -A_{ji} \cdot \mathbf{1}, \tag{2}$$

where $A_{ji} \in \Omega^1(U_{ij}; i\mathbb{R})$. Here, $\mathfrak{u}(n)$ denotes the Lie algebra of $U(n)$, and $\mathbf{1}$ the identity matrix.

It is easy to see that Equation (2) is compatible with the cocycle condition of gerbes with connection, i.e., $\delta(A)_{kji} \cdot \mathbf{1} = d\log\lambda_{kji} \cdot \mathbf{1}$. A standard argument using partitions of unity shows that, for any λ-twisted vector bundle E, there exists a connection on E compatible with $\widehat{\lambda}$.

Definition 7. *Let $\widehat{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ be as above, and (E, Γ) be a λ-twisted vector bundle $(\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ of rank n with a connection Γ compatible with $\widehat{\lambda}$. The **curvature form** of Γ is the family $R = \{R_i \in \Omega^2(U_i; \mathfrak{u}(n))\}_{i\in\Lambda}$, where $R_i := d\Gamma_i + \Gamma_i \wedge \Gamma_i$.*

The following proposition is a well-known characterization of a flat vector bundle.

Proposition 1. *If a vector bundle E over X admits a flat connection ∇, then there exists a cocycle consisting of locally constant transition maps. Conversely, if a cocycle (g_{ji}) of a vector bundle E over X defined on an open cover $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ consists of locally constant maps, then E admits a flat connection.*

Proof. Since ∇ is a flat connection, there exists a locally trivial open cover $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ such that the connection form ω_i on U_i is identically zero. Let $\{g_{ji}\}$ be a cocycle of the vector bundle E over X defined on the open cover $\mathcal{U}$. Connection forms satisfy the following gauge transformation formula:

$$\omega_i = g_{ji}^{-1}\omega_j g_{ji} + g_{ji}^{-1}dg_{ji}.$$

It follows that $dg_{ji} = 0$, and hence each g_{ji} is a locally constant map. Conversely, if each g_{ji} is locally constant, then $dg_{ji} = 0$. So, we can take $\omega \equiv 0$ for each $i \in \Lambda$. □

A λ-twisted vector bundle admits only torsion twists. By Corollary 2, a torsion gerbe with connections is always isomorphic to a gerbe with connection $\widehat{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$

where all λ_{kji} are locally constant, $A_{ji} \equiv 0$, and $B_i = \zeta|_{U_i}$ for a globally defined differential form $\zeta \in \Omega^2(X; i\mathbb{R})$.

Theorem 3. *Let $\widehat{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ be a gerbe with connections, provided that every λ_{kji} is locally constant, and $A_{ji} = 0$ for all $i, j \in \Lambda$. $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ is a λ-twisted vector bundle that admits a connection $\Gamma = \{\Gamma_i\}_{i\in\Lambda}$ compatible with the connection of $\widehat{\lambda}$ such that $R_i \equiv 0$ for each $i \in \Lambda$, if and only if each g_{ji} is locally constant.*

Proof. Suppose a λ-twisted vector bundle with connection (E, Γ) is flat, i.e., $R_i \equiv 0$. Then, over each $U_i \in \mathcal{U}$, it admits a parallel framing such that the connection form $\Gamma_i \equiv 0$. By Equation (2), we obtain $dg_{ji} = 0$ and, hence, g_{ji} is locally constant. Suppose each g_{ji} is locally constant. The family $\Gamma_{ii\in\Lambda}$ with $\Gamma_i \equiv 0$ is a connection on E. The corresponding curvature form $R_i \equiv 0$. □

5. Discussion

In this paper, we have investigated the differential geometry of torsion gerbes, focusing on providing a necessary and sufficient condition for the Dixmier–Douady class of a 2-gerbe to be torsion. Our primary result demonstrates that a 2-gerbe is torsion if and only if its cocycle data consists of locally constant functions. This insight extends to n-gerbes for $n = 1$ and $n \geq 3$, offering a generalized perspective on the structure of higher gerbes.

We drew upon the well-established understanding of flat vector bundles, wherein the existence of a flat connection is characterized by locally constant Čech cocycles. This analogy underscored the significance of locally constant cocycle data in the context of gerbes. We extended this result for the case of flat twisted vector bundles, thereby broadening the applicability of our findings.

In summary, this paper contributes to the deeper understanding of the geometry and topology of torsion gerbes and their higher analogues, offering new tools and perspectives for future research in both mathematics and theoretical physics. For example, our results can be applied to investigating the role of locally constant cocycle data in the differential geometry of twisted vector bundles over orbifolds, and more general stratified spaces.

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