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Nonsingular Bernoulli actions & quantum symmetries

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

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Für Ute, Walter und Alexa

It’s over now
And I’m a little bit petrified
Of what’s to come
Yes, my head’s a bit stir-fried
It’s over now
And I feel a little unqualified
But fire up the rockets!

Now that my time as a PhD student is approaching an end after four turbulent years of fun, frustration and fries, I would like to thank the people with whom I had the fun, who pulled me out of frustration and who stole my fries during lunch time (I am looking at you, Thibault).

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Abstract

This thesis contains the results obtained in the research articles [VW17] [AdLW16] [AdLW17] and [TW16]. We study four types of mathematical phenomena.

1. We prove existence and nonexistence results for type III Bernoulli actions of discrete groups. We show that the existence of a type III Bernoulli action of a given discrete group G depends on the existence of a nonzero element inside its first L^2-cohomology $H^1(G, \ell^2(G))$. We also provide concrete examples of type III Bernoulli actions of $\mathbb{Z}$ and of free product groups [VW17].

2. We define and examine the Fourier algebra, the Fourier-Stieltjes algebra and the algebra of completely bounded multipliers of a rigid C^*-tensor category [AdLW17].

3. We prove that the representation categories of q-deformations of connected compact simple Lie groups with trivial center, $0 < q \leq 1$ have the Howe-Moore property [AdLW16].

4. We describe a correspondence between the free product of planar algebras and the free wreath product of compact quantum groups and examine its consequences [TW16].

Beknopte samenvatting

In deze thesis bestuderen we vier soorten van wiskundige fenomenen en beschrijven de in de artikels [VW17] [AdLW16] [AdLW17] en [TW16] gepubliceerde resultaten.

1. We onderzoeken het bestaan en niet-bestaan van type III Bernoulli-acties van discrete groepen. We tonen aan dat de existentie van een type III Bernoulli-actie voor een groep G afhangt van het bestaan van een niet-triviaal element in de eerste L^2-cohomologie $H^1(G, \ell^2(G))$. We geven bovendien concrete voorbeelden van type III Bernoulli-acties van $\mathbb{Z}$ en van vrije producten van groepen [VW17].

2. We definiëren en bestuderen de Fourier-algebra, de Fourier-Stieltjes-algebra en de algebra van volledig begrensde multiplicatoren van een eindige C^*-tensor-categorie [AdLW17].

3. We tonen aan dat de representatiecategorieën van q-deformatie's van samenhangende compacte eenvoudige Lie-groepen met een triviaal centrum, $0 < q \leq 1$, de Howe-Moore-eigenschap hebben [AdLW16].

4. We beschrijven de relatie tussen het vrije product van planare algebra's en het vrije kransproduct van compacte kwantumgroepen. We onderzoeken ook de gevolgen van deze correspondentie [TW16].

Contents

Introduction

The purpose of this introductory chapter is to familiarize the reader with the main results of this thesis and its context without diving all too deep into mathematical subtleties.

The thesis is divided into two parts which are essentially independent of each other although both share a home in the overarching field of operator algebras. The first part treats the subject of nonsingular Bernoulli actions of infinite discrete groups and constitutes a more detailed version of the article [VW17], written jointly with my PhD advisor Stefaan Vaes. The second part focuses on representation categories arising from several types of 'quantum symmetries', most notably from compact quantum groups. This part is based on the articles [AdLW16], [AdLW17], which are the result of a collaboration with Yuki Arano, Tim de Laat, and the article [TW16] which is the product of a collaboration with Pierre Tarrago.

Due to their independent nature, both parts come with a preliminary chapter of their own.

Nonsingular Bernoulli actions

Bernoulli actions are specific instances of group actions on measure spaces. These are in turn the main object of study in a branch of mathematics called *ergodic theory*. Initially, ergodic theory was brought about as a mathematical framework to formalize problems in *statistical physics*. This is the branch of physics that studies the macroscopic behaviour of physical entities formed by a large number of particles whose individual behaviour is not, or only partially known. In this spirit, group actions on measure spaces can be understood as an abstract way to describe how a physical system consisting of a large number of particles evolves. In some situations, the evolution of the system is best described discretely. In others, it is more practical to make use of a model that is continuous in time instead.

The actions that concern us in this thesis, the Bernoulli actions, are generic but nevertheless highly nontrivial examples of group actions. Their generic nature lies in the fact that they can be defined for any discrete group G. In addition, many transformations of classical interest, such as the famous Baker's transformation, are Bernoulli actions in disguise [Sh73]. It is therefore not surprising that these actions have been of interest to mathematicians for a long time. The question of how to classify the most canonical Bernoulli actions, namely those preserving a probability measure, dates back to von Neumann in the 1940's. This particular classification problem has been a central motivation for Kolmogorov's theory of dynamical entropy [Kol58], [Kol59] and stunning answers to this problem have been found by Kolmogorov himself, Ornstein [Orn70] and Bowen [B10a], [B10b].

However, in this thesis, we are interested in a more general class of Bernoulli actions: we drop the assumption that our Bernoulli actions preserve the measure. While the class of such actions is a large potential source for interesting behaviour from the viewpoint of both ergodic theory and operator algebras, results in this direction have been scarce, although not completely absent, c.f. [Kre70] [Ha81] [Ko09] [Ko10] [Ko12] [DL16].

At this point, let us describe Bernoulli actions more formally. To that end, fix an infinite discrete group G and a standard Borel space X_0 which we will henceforth call the *base space*. In addition, for any group element $g \in G$, let μ_g be a probability measure on X_0. The Bernoulli action of G with base space X_0 is the action on the product space

$$(X, \mu) = \prod_{g \in G} (X_0, \mu_g)$$

by shifting the indices, that is to say

$$g\big(x_h\big)_{h \in G} = \big(x_{g^{-1}h}\big)_{h \in G}, \quad g \in G.$$

Although we do not ask for the measure μ to be preserved under the action, we need to put a minimal requirement on the interplay between the action and the measure, namely that the action preserves sets of measure zero. This property of the action is called *nonsingularity*. Naturally, this leads to the need of a characterization of nonsingularity that makes it easier to decide in concrete examples whether a given Bernoulli action is nonsingular or not. Luckily, such a characterization is an immediate consequence of a result of Kakutani in [Ka48] on the equivalence of product measures. When the base space X_0 is $\{0, 1\}$ and when the masses of the component measures μ_g in 0 and 1 are uniformly bounded away from zero, the nonsingularity criterion becomes particularly easy to state: $G \curvearrowright (X, \mu)$ is nonsingular if and only if for any $g \in G$, the map

$$c_g : G \to \mathbb{R}, \quad c_g(h) = \mu_{gh}(0) - \mu_h(0)$$

lies in $\ell^2(G)$. This is discussed in more detail in the preliminary section 1.8.1.

Once we have characterized nonsingularity, there is a natural follow-up question which will be the central question of the first part of this thesis:

Which discrete groups admit nonsingular Bernoulli actions that do not preserve the measure in a genuine sense?

This deserves some explanation: While it is easy to find a nonsingular Bernoulli action $G \curvearrowright (X, \mu)$ by which the measure is not literally preserved, in many cases we can replace the measure μ by an equivalent measure μ' which is preserved under the action. Here, by equivalence of μ and μ' we mean that the measures share the same null sets. If the equivalence class of μ does not contain a measure which is preserved, the action is called of *type* III and hence our question becomes:

Which discrete groups admit nonsingular Bernoulli actions of type III?

Usually, we are particularly interested in actions that are minimal in the sense that they have no non-trivial invariant subsets and such actions are called *ergodic*.

Let us motivate this question, that is to say, let us explain why Bernoulli actions of type III deserve our attention. From my perspective, the main motivation to discuss actions of type III comes from *von Neumann algebras*. These are the $*$-subalgebras of the algebra of operators on some Hilbert space that are closed in the weak or, equivalently, the strong operator topology. The study of von Neumann algebras of type III has a long history dating back to the seminal works of Murray and von Neumann [MvN36], [vN41]; a history full of beautiful but difficult-to-prove results. However, von Neumann algebras of type III are not easy to find and even more difficult to study. Therefore, producing group actions of type III, which in turn produce von Neumann algebras of type III that can be studied in terms of the underlying action, has its merits. In addition, group actions of type III have turned out to be useful in the context of general ergodic theorems, see [BN11], or to characterize properties of discrete groups such as property (T), see [BHT16].

We should point out here that, previous to our article [VW17], the only infinite discrete group known to admit Bernoulli actions of type III was the group of integers $\mathbb{Z}$; a result which was obtained by Hamachi [Ha81] and further refined by Kosloff [Ko09]. However, while Hamachi and Kosloff proved the existence of such actions, their construction was rather inexplicit and hence difficult to work with.

Our answer to the question above is based on the fundamental observation that Kakutani's nonsingularity criterion intimately relates the Bernoulli action $G \curvearrowright (X, \mu)$ to the *first* L^2*-cohomology* $H^1(G, \ell^2(G))$ of G through the simple fact that the map

$$G \to \ell^2(G), \quad g \mapsto c_g$$

defines a so-called *1-cocycle* into $\ell^2(G)$ and therefore an element in $H^1(G, \ell^2(G))$. Since the first L^2-cohomology of a discrete group is a well studied object in group theory, see e.g. [L02], this observation hands us several tools in the form of deep theorems from group cohomology. As a first result, we obtain the following.

Theorem A. *If G is an infinite discrete group with $H^1(G, \ell^2(G)) = \{0\}$, then G does not admit nonsingular Bernoulli actions of type III.*

A more general and precise version of Theorem A and its proof will be the focus of Section 2.1. Examples of groups satisfying the condition in Theorem A include property (T) groups, products of infinite and nonamenable groups and nonamenable groups with infinite amenable normal subgroups.

On the other hand, as soon as we consider groups G with $H^1(G, \ell^2(G)) \neq \{0\}$, we can almost always prove the existence of nonsingular ergodic Bernoulli actions of type III:

Theorem B. *Let G be an infinite discrete group with $H^1(G, \ell^2(G)) \neq \{0\}$. Assume that G satisfies one of the following conditions*

- *G is amenable;*
- *G is nonamenable and satisfies one of the following:*
  - *G contains an element of infinite order;*
  - $\beta_1^{(2)}(G) \geq 1$*;*
  - *G is residually finite.*

Then G admits a nonsingular free ergodic Bernoulli action of type III.

We can in fact say much more about these Bernoulli actions (specify the subtype, discuss mixing properties, etc.) but we will refer to Sections 2.3 and 2.4 for more general statements, which together imply Theorem B, and for the proofs of these statements. We would like to point out though, that one of the crucial ingredients to the proof of Theorem B is a result relating the type of a nonsingular Bernoulli action $G \curvearrowright (X, \mu)$ to the growth of the norms of the 1-cocycle $c : G \to \ell^2(G)$ defined above. This result is explained and proven in Section 2.2. Also, we do not know any examples of infinite discrete groups with nontrivial first L^2-cohomology that are not covered by Theorem B.

We will end part I of this thesis by working out several completely explicit examples of nonsingular ergodic Bernoulli actions of type III for the group of integers and for free product groups, which demonstrate a wide range of possible behaviours. For instance, we will describe nonsingular Bernoulli actions of subtype III_1 of the nonabelian free groups for which we can arbitrarily prescribe the Connes invariants of their orbit equivalence relations. These constructions will make up Section 2.5. We will also discuss examples of nonsingular ergodic Bernoulli actions of type III_1 for $\mathbb{Z}$, which are much simpler than all previously known examples (see Section 2.5.1).

C^*-tensor categories arising from compact quantum groups

As we have seen above, in the first part of this thesis we encode symmetries of probability spaces in the form of groups. These group structures are the most straightforward and oldest way to represent symmetries; they are present in every branch of mathematics and every aspiring mathematician encounters them in the first days of their studies. However, due to the developments in theoretical physics in the beginning of the 20th century and in particular to the appearance of quantum mechanics, it became apparent that other symmetries were needed to describe the new quantum world. *Von Neumann algebras*, which were already shortly mentioned in the previous section, are one example of what can quite literally be called the aftermath of quantum mechanics [MvN36]. In the second part of this thesis, three modern descendants of the search for quantum symmetries will take center stage, namely

- compact quantum groups,
- planar algebras,
- and (rigid) C^*-tensor categories.

Compact quantum groups were introduced by Woronowicz in [Wo87a] [Wo87b] as a generalization of compact groups. The basic idea is the following: instead of studying a compact group G directly, one has a look at the algebra of continuous functions $C(G)$ on G and translates the group multiplication into a map from $C(G)$ into the minimal tensor product $C(G) \otimes C(G)$. Next, one drops the commutativity assumptions on $C(G)$ and considers arbitrary unital C^*-algebras in its place. Hence one arrives at a pair of a unital C^*-algebra A and a morphism $\Delta : A \to A \otimes A$. Of course one also needs to find appropriate compatibility conditions between A and Δ and this is part of Woronowicz's achievements. In addition, the q-deformations of compact Lie groups found by Drinfel'd [Dr86] and, independently, by Jimbo [Ji85] fit perfectly into Woronowicz's theory [Ro90] and serve as beautiful examples of compact quantum groups that show genuinely different behaviour than classical compact groups, see also [Wo88].

Planar algebras appear first in the work of Vaughan Jones [J99] and are derived from Jones's theory of *subfactors*. This theory studies how a von Neumann algebra with trivial center N (a *factor*) can sit inside another factorial von Neumann algebra M. There are several obstructions that limit the possibilities to nest a factor inside another one; the most famous such obstruction is given by the range of the possible values of the index $[M : N]$ in Jones's index theorem [J83]. Other obstructions are of a more combinatorial nature and can be collected in the *standard invariant* of the subfactor inclusion. There are several abstract descriptions of this standard invariant available[1], such as Popa's λ-lattices [Po95], Longo's Q-systems [Lo94] and the planar algebras of Jones [J99] to name a few. In a nutshell, the latter are towers of vector spaces whose morphisms can be described by planar diagrams such as the one below.

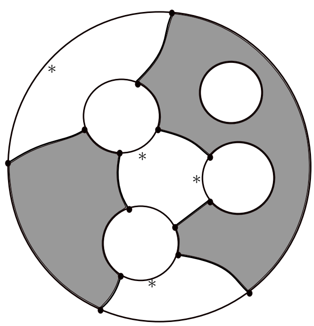

Figure 1: A planar diagram or *tangle*.

Subfactor theory and the theory of compact quantum groups are far from independent notions and mathematicians started to explore their interplay soon after both theories had appeared [Xu98] [B99b] [B01]. Their interaction

[1] abstract in the sense that it does not need to refer to the subfactor itself anymore

is based on the fact that both subfactors and compact quantum groups give rise to canonical categories that structurally look very much alike. A subfactor $N \subset M$ yields categories of N-N- respectively M-M-bimodules while a compact quantum group, like a compact group, yields a natural category of representations. Both types of categories were therefore incorporated into a common abstract framework, the so-called *rigid C^*-tensor categories.*

The study of rigid C^*-tensor categories has brought several advantages to the table including the obvious one of allowing uniform proofs of results that appear in both theories.

1. Rigid C^*-tensor categories admit a representation theory of their own thanks to results of Popa and Vaes et al. [PV15] [NY15a] [GJ16]. In particular, this allows for a systematic study of analytic properties of such categories. A systematic approach to analytic properties was long yearned for in the subfactor setting [Po99], and finally achieved in [PV15].

2. From a quantum group perspective, studying representation categories abstractly pointed towards new and very succesful approaches to problems that can per se be formulated without using any categorical language. This is most apparent in the computation of L^2-Betti numbers of the *quantum permutation groups* in [KRVV17] which is based on the study of the L^2-cohomology of a rigid C^*-tensor category in [PSV15].

3. C^*-tensor categories serve as a relay between subfactor theory and quantum group theory. Often, it is possible to study concrete examples on either the subfactor side or the quantum group side and then transfer these examples to the respective other side by means of tensor categories. An example of this are Arano's results on property (T) of representation categories of q-deformed compact semisimple Lie groups of higher rank [Ar15] [Ar16] that imply the existence of property (T) subfactors.

Having these advantages in mind, we will now move on to describe the main results in the second part of this thesis and relate them to what we have just summarized.

In Chapter 4, based on the joint article [AdLW17], we initiate the study of *abstract harmonic analysis* for rigid C^*-tensor categories. In the context of locally compact groups, abstract harmonic analysis has been an active field of research ever since Eymard introduced the Fourier algebra and the Fourier–Stieltjes algebra in his seminal article [Ey64]. These algebras can be regarded as a condensed form of the representation theory of locally compact groups: they contain the matrix coefficients of all unitary representation (in the case of the Fourier–Stieltjes algebra $B(G)$) or the coefficients of the left regular representation (in the case of the Fourier algebra $A(G)$) of the corresponding group G. The analytic flavour of these algebras, that is to say, the fact that they admit norms that turn them into Banach algebras, makes them useful tools when discussing group properties at the border of analysis and representation theory such as amenability [Le68] and property (T). A Banach algebra closely related to the previously mentioned ones, is the *algebra of completely bounded multipliers* $M_0A(G)$, see e.g. [BoFe84]. This algebra has been recently used to define property (T*) in [HdL12] which meant significant progress in the study of the *approximation property* or rather the lack thereof, see also [HdL13] [HKdL14].

Continuing the systematic study of the representation theory of rigid C^*-tensor categories initiated by Popa and Vaes [PV15], we define the algebras $A(\mathcal{C})$, $B(\mathcal{C})$ and $M_0A(\mathcal{C})$ for a rigid C^*-tensor category $\mathcal{C}$. We show that, as in the group case, they form (dual) Banach algebras and discuss their relationship. Next, we give some remarks on property (T) and prove a characterization of amenability for rigid C^*-tensor categories analogous to the one for locally compact groups due to Leptin [Le68]. Since the conceptual tools for the study of matrix coefficients of representations of rigid C^*-tensor categories were already in place thanks to the work of Popa, Vaes et al., none of these results are particularly surprising nor difficult to prove. Nevertheless, we hope that they will form useful tools in further research on rigid C^*-tensor categories.

In Chapter 5 which is based on the joint article [AdLW16] we turn to yet another classical property of locally compact groups categorical: the *Howe–Moore property.*

A locally compact group is said to have the Howe-Moore property if for every unitary representation without invariant vectors, the matrix coefficients of the representation vanish at infinity. This property was established for connected non-compact simple Lie groups with finite center by Howe and Moore [HM79] and Zimmer [Zi84]. Howe and Moore also showed the property for certain subgroups of simple algebraic groups over non-Archimedean local fields. Other important examples of groups with the Howe-Moore property were produced in [LuMo91] and [BuMo00]. Classically, the most important application of the Howe-Moore property are geometrical: the proofs of several rigidity results, most notably Margulis's superrigidity theorem [Mar91] and Mostow's rigidity theorem [Mos73], use it in a fundamental way. We refer to [BaG14] and [Ci15] for an overview on and a unified approach to locally compact groups with the Howe-Moore property.

Let $\mathcal{C}$ be a rigid C^*-tensor category, and let $\mathrm{Irr}(\mathcal{C})$ denote the set of equivalence classes of irreducible objects in $\mathcal{C}$. We then say that $\mathcal{C}$ has the Howe-Moore property if every completely positive multiplier $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ (see Definition 3.4.3) has a limit at infinity.

In contrast to the content of Chapter 4, Chapter 5 will not focus on an abstract study of the Howe–Moore property as we will address a specific class of rigid C^*-tensor categories, the representation categories of Drinfel'd and Jimbo's q-deformations of connected compact Lie groups [Dr86] [Ji85]. Our main result is the following.

Theorem C. *Let $q \in (0,1]$, and let K_q be a q-deformation of a connected compact simple Lie group K with trivial center. Then every completely bounded multiplier on $\mathrm{Rep}(K_q)$ (see Definition 3.4.3) has a limit at infinity. In particular, the representation category $\mathrm{Rep}(K_q)$ has the Howe-Moore property.*

From the same result that will imply Theorem C in Chapter 5, namely from Theorem 5.1.12, we will also deduce the Howe–Moore property for several C^*-tensor categories that are central to subfactor theory, as for instance the bimodule categories associated to Temperley-Lieb-Jones subfactors (Example 5.1.13). This is a satisfying example of the third motivation for the study of rigid C^*-tensor categories listed above.

For the proof of Theorem 5.1.12 (and hence of Theorem C), we relate the character algebra of K_q to the character algebra of K, which can in turn be identified with the algebra of continuous functions on T/W, where T is a maximal torus and W is the associated Weyl group. A crucial ingredient of our proof is a certain general asymptotic behaviour of the characters of highest weight representations of compact Lie groups (see Proposition 5.1.9).

In Chapter 6 we explore the connection between compact quantum groups and planar algebra based on the joint work [TW16]. As a starting point to explain the results obtained in this chapter, we would like to recall a classical theorem in quantum group theory, the Tannaka-Krein duality theorem of Wornowicz [Wo88], see also Theorem 3.2.5. Roughly speaking, the theorem states the following: If one is handed a rigid C^*-tensor category $\mathcal{C}$ and a "realisation" of $\mathcal{C}$ inside the category of finite-dimensional Hilbert spaces, one can find a compact quantum group $\mathbb{G}$ whose representation category equals $\mathcal{C}$. Moreover, if one uses appropriate notions of equivalence, the correspondence established in the theorem is bijective.

The first major result in Chapter 6 is a similar correspondence between a huge class of actions of compact quantum groups on one side and planar algebras together with an appropriate "realisation on Hilbert spaces" on the other. More concretely, this realisation of a planar algebra $\mathcal{P}$ on Hilbert spaces manifests itself as an embedding of $\mathcal{P}$ inside a so-called *graph planar algebra* [J98]. The conditions to put on the class of actions of compact quantum groups and even more so the definition of graph planar algebras are somewhat technical to describe. Instead of doing so, we would rather like to refer to Section 3.2.5 for a description of the former and to Section 3.3.4 for a description of the latter. Nevertheless, we would like to state the main result.

Fix a finite-dimensional C^*-algebra A and denote the Markov trace of the inclusion $\mathbb{C}1_A \subset A$ by tr.

Theorem D. *There is a natural bijective correspondance between (conjugacy classes of) actions of compact quantum groups $\alpha : A \to A \otimes \mathbb{C}[\mathbb{G}]$ that are faithful, centrally ergodic and preserve* tr *and (isomorphism classes of) subfactor planar subalgebras of the graph planar algebra $\mathcal{P}^A$ associated to the inclusion $\mathbb{C}1_A \subset A$.*

We should point out here that it was already proven by Banica in [B05a] that quantum group actions as in the theorem give rise to subfactor planar algebras of $\mathcal{P}^A$, see Theorem 6.1.1.

We apply Theorem D to the study of *free wreath product quantum groups*. The free wreath product is a construction due to Bichon [Bi04] that takes two compact quantum groups $\mathbb{G}$ and $\mathbb{F}$ as input and produces a new compact quantum group $\mathbb{G} \wr_* \mathbb{F}$. In order for Bichon's construction to be well defined, the compact quantum group $\mathbb{F}$ has to act on the C^*-algebra $\mathbb{C}^n \cong C(\{1, \ldots, n\})$ for some $n \in \mathbb{N}$ in a way that preserves the trace given by integration against the uniform probability on $\{1, \ldots, n\}$. Bichon's construction resembles both the free product and the classical wreath product of groups, hence its name. During recent years, the free wreath product has been studied quite extensively by numerous authors, see e.g. [LT16] [FP16] [FS15] [W15].

In an influential paper [BJ95], Bisch and Jones introduced a free product operation for subfactors that can also be reformulated for planar algebras, see e.g. [La02]. Let us formulate the next result somewhat informally.

Theorem E. *Let $\mathbb{G}$ and $\mathbb{F}$ be compact quantum groups acting on B respectively $A = \mathbb{C}^n$ as in Theorem D. Denote the action of $\mathbb{G}$ by β and the action of $\mathbb{F}$ by α. The free wreath product $\mathbb{G} \wr_* \mathbb{F}$ admits a natural action $\beta \wr_* \alpha$ on $A \otimes B$ satisfying the assumption of Theorem D and its corresponding planar algebra $\mathcal{P}(\beta \wr_* \alpha)$ agrees with the free product $\mathcal{P}(\alpha) * \mathcal{P}(\beta)$ of the planar algebras associated with α and β.*

Bichon's construction has been partially generalized in [FP16] and we prove a version of Theorem E for this partial generalization as well. We then end Chapter 6 with an application of our theorem in the spirit of the transfer principle lined out in point 3 of the list of motivations for C^*-tensor category above. More precisely, we discuss approximation properties for free wreath products and mention a result of Kyed, Raum, Vaes and Valvekens [KRVV17] on their L^2-cohomology, which is a consequence of Theorem E.

Part I

Bernoulli actions of type III and L^2-cohomology

Overview of part I

As explained in the introductory chapter of this thesis, its first part deals with the question which discrete groups admit nonsingular Bernoulli actions of type III. It comes with an extensive preliminary chapter (Chapter 1) before discussing the main results in Chapter 2. Additional results and useful facts on the L^2-cohomology of discrete groups are summarized in Appendix A.

The preliminary chapter is divided into eight sections. It contains information on the measure theoretic conventions and notations of this thesis (Section 1.1), on the nature of group actions (Section 1.2), von Neumann algebras (Section 1.3) and Borel equivalence relations (Section 1.4), on the concepts of conservativity (Section 1.5), mixing (1.6) and amenability (1.7) and on nonsingular Bernoulli actions (Section 1.8).

Chapter 2 is structured in the following way. Section 2.1 is devoted to groups with trivial first L^2-cohomology and to a proof of a more general version of Theorem A. In Section 2.2, we discuss a criterion for conservativity/dissipativity of nonsingular Bernoulli actions (Theorem 2.2.1), which links these properties to the growth of the 1-cocycle associated to the action. Section 2.3 deals with groups with positive first L^2-Betti numbers and the construction of nonsingular Bernoulli actions of type III for many of these groups (Theorem 2.3.1). We will then turn our attention to amenable groups in Section 2.4 and prove the existence of nonsingular ergodic Bernoulli actions of type III_1 for amenable groups (Theorem 2.4.1). Together, Theorem 2.3.1 and Theorem 2.4.1 imply Theorem B. Lastly, we describe concrete examples of nonsingular ergodic Bernoulli actions of type III for the group of integers (Subsection 2.5.1) and free product groups (Subsection 2.5.2). The latter include Bernoulli actions of type III_1 with prescribed Connes invariants of their orbit equivalence relations (Example 2.5.4), Bernoulli actions of type $\mathrm{III}_\lambda, 0 < \lambda < 1$ (Example 2.5.4) and Bernoulli actions with exotic dissipativity properties (Proposition 2.5.8 and Proposition 2.5.11).

Appendix A consists of two sections. In Section A.1, we discuss basic facts on the first L^2-cohomology of discrete groups and in Section A.2, we prove the existence of 1-cocycles of arbitrarily slow growth for amenable groups.

Chapter 1

Preliminaries

1.1 Notations and Conventions

We will assume that the reader is familiar with the basic ideas and concept of measure theory as taught in a standard undergraduate course. However, for the convenience of the reader and in order to clarify our notation from the start, we will at least recall some basic definitions in this section.
Let us remind the reader that a Borel space is a set endowed with a σ-algebra and that an isomorphism between two Borel spaces X, Y is a bijective, bimeasurable map $X \to Y$. Let us also mention here that whenever we talk about Borel spaces throughout this chapter, we will usually suppress the σ-algebra at hand in the notation. The typical letters by which we will denote Borel spaces will be X and Y. Measures will be generically denoted by the letters μ, ν, η and, if not explicitly specified otherwise, all measures will be assumed to be σ-finite throughout this thesis.

Recall the following definitions.

Definition 1.1.1.
- A Borel space is called *standard* if it is isomorphic to a separable complete metric space endowed with its Borel σ-algebra;
- A *standard measure space* (X, μ) is (the completion of) a pair consisting of a standard Borel space X and a measure μ on X;
- A *standard probability space* (X, μ) is a standard measure space satisfying $\mu(X) = 1$.

Identification of standard measure spaces works like this:

Definition 1.1.2. By an isomorphism between two standard measure spaces (X, μ) and (Y, ν), we mean a nonsingular Borel isomorphism mod 0, that is to say a Borel isomorphism $X' \to Y'$, where $X' \subset X$ and $Y' \subset Y$ are conegligible subsets, which moreover preserves null-sets (*nonsingularity*).

Taking this definition into account, the special merit of standard probability spaces is that they are isomorphic in a measure preserving way to the interval $[0, 1]$ equipped with its natural Borel structure and a convex combination of the Lebesgue measure and a discrete probability measure (see e.g [Kec95, Theorem 17.41]).
Now, let us also rehash that a Polish space is a topological space which is homeomorphic to a separable complete metric space.

Notation 1.1.3. Let (X, μ) be a standard measure space and let E be a Polish space.

- We denote the set of equivalence classes of measurable functions from X to E by $L^0(X, E)$. Here, two functions are considered equivalent if they agree up to a set of measure zero.
- For $f \in L^0(X, E)$, we denote the push-forward measure with respect to f by $f_*\mu$.

For later reference, let us note that $L^0(X, E)$ becomes a Polish space if equipped with the topology of convergence in measure. Before defining the essential range of a measurable function, recall that an *atom* of a standard measure space (X, μ) is a Borel subset $A \subset X$ such that $\mu(A) > 0$ and such that for any Borel subset $B \subset A$, $\mu(A) > \mu(B)$ implies that $\mu(B) = 0$.

Definition 1.1.4. Let (X, μ) be a standard measure space and let E be a Polish space.

- The *essential range* $\operatorname{ran}(f)$ of a measurable function $f \in L^0(X, E)$ is the smallest closed subset B of E with the property that $\mu(X \setminus f^{-1}(B)) = 0$, i.e. $\operatorname{ran}(f)$ is the topological support of $f_*\mu$.
- The *atomic range* $\operatorname{aran}(f)$ of a measurable function $f \in L^0(X, E)$ is the set of atoms of $f_*\mu$.

1.2 Group actions

In this section, we will gather the basic definitions on group actions on measure spaces with which we work throughout this chapter. The letters G, H, Γ, Λ

will generically denote groups, which we will always assume to be discrete and countable unless explicitly stated otherwise. Let us start by defining what it means for a group to act on a standard Borel space.

Definition 1.2.1.
- A group action $G \curvearrowright^{\alpha} X$ on a standard Borel space X is a homomorphism $\alpha : G \to \mathrm{Aut}(X)$ into the group of Borel automorphisms on X;
- If the standard Borel space X comes with a measure μ, we speak of a measurable group action and write $G \curvearrowright^{\alpha} (X, \mu)$. A measurable group action $G \curvearrowright^{\alpha} (X, \mu)$ is called *nonsingular* if α_g is a nonsingular automorphism of (X, μ) for all $g \in G$.

Whenever there is no ambiguity in doing so, we suppress the letter α and write gx instead of $\alpha_g(x)$. In this notation, $G \curvearrowright (X, \mu)$ is nonsingular if and only if $\mu(A) = 0$ implies $\mu(gA) = 0$ for all Borel sets $A \subset X$ and all $g \in G$. Another way to phrase nonsingularity is to ask that, for all $g \in G$, the measure μ is equivalent to the pushforward measure $g_*\mu$ defined by $g_*\mu(A) = \mu(g^{-1}A)$ for $A \subset X$ Borel. If this is the case, by the Radon-Nykodim theorem, we can define a measurable function $\omega(g, \cdot) : X \to (0, +\infty)$ such that

$$\int_X F(x)\, \omega(g, x)\, d\mu(x) = \int_X F(g^{-1}x)\, d\mu(x) \tag{1.2.1}$$

for all positive measurable functions $F : X \to [0, +\infty)$ and all $g \in G$. If for all $g \in G$, the pushforward $g_*\mu$ is identical to μ, we call the action *measure preserving (mp)*. If μ is moreover a probability measure, we call the action *probability measure preserving (pmp)*. Let us also recall when two nonsingular group actions are the same.

Definition 1.2.2. Two nonsingular group actions $G \curvearrowright (X, \mu)$, $H \curvearrowright (Y, \nu)$ are *conjugate*, if there exists a group isomorphism $\Phi : G \to H$ and a nonsingular isomorphism $\Delta : X \to Y$ such that $\Delta(gx) = \Phi(g)\Delta(x)$ for every $g \in G$ and almost all $x \in X$.

Lastly, let us define two fundamental properties that a nonsingular group action can have, namely *freeness* and *ergodicity*. Free group actions are those without fixed points and these actions are particularly well behaved when connecting the theories of group actions, Borel equivalence relations and von Neumann algebras. Considering that we currently find ourselves within the mathematical area of *ergodic theory*, the reader might not be stunned to hear that the property of ergodicity is fundamental. This is both true from a physical and mathematical standpoint: most dynamical systems that appear in statistical physics are ergodic and general nonsingular group actions decompose into ergodic ones, which means that the entire theory can be build from ergodic actions. The formal definitions are the following ones.

Definition 1.2.3. A nonsingular group action $G \curvearrowright (X, \mu)$ is

- *(essentially) free* if $\mu(\{x \in X\ ;\ gx = x\}) = 0$ for all $g \in G \backslash \{e\}$;
- *ergodic*, if any invariant measurable sets $A \subset X$ is either null or conull, that is to say, whenever $GA = A$ then either $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.

1.3 von Neumann algebras

In this part of the preliminary chapter, we will remind the reader of the definition of a von Neumann algebra. The interested reader can find a detailed discussion on the topic in one of the classical textbooks [Ta03], [KR92].
Let $\mathcal{H}$ be a complex Hilbert space. Recall that the *commutant* M' of a subset $M \subset B(\mathcal{H})$ is defined as

$$M' = \{T \in B(\mathcal{H})\ ;\ ST = TS \text{ for all } S \in M\}.$$

and that the commutant of the commutant of M is called the *bicommutant* of M and denoted by M''.

Definition 1.3.1. A *von Neumann algebra* on $\mathcal{H}$ is a $*$-subalgebra M of $B(\mathcal{H})$ containing the identity operator $\mathbb{1}$ and satisfying one of the following equivalent conditions.

- M is closed in $B(\mathcal{H})$ with respect to the weak operator topology;
- M is closed in $B(\mathcal{H})$ with respect to the strong operator topology;
- $M = M''$.

The assertion that the three conditions above are in fact equivalent is called the bicommutant theorem of von Neumann [vN29]. A von Neumann algebra satisfying $\mathcal{Z}(M) := M \cap M' = \mathbb{C}\mathbb{1}$ is called a *factor* and any von Neumann algebra is build out of factors, making them the 'fundamental building blocks' of the theory, see e.g. [KR92].
It is a powerful result by Murray and von Neumann [MvN36] that the class of factors can be subdivided into the classes of factors of *types* I, II_1, II_∞ and III. More generally, any von Neumann algebra can be decomposed into a direct sum

$$M = M_I \oplus M_{II_1} \oplus M_{II_\infty} \oplus M_{III}$$

of von Neumann algebra of the *types* I, II_1, II_∞ and III with any of these summands possibly degenerated to $\{0\}$, see [KR92]. In order to present the definition of these types, we need to recall some facts on projections in von Neumann algebras and on how to compare them.

First of all, a *projection* inside a von Neumann algebra M is an element $p \in M$ satisfying $p = p^* = p^2$. Secondly, for projections $p, q \in M$, we write $p \leq q$ if p is a subprojection of q, that is to say, if $q - p$ is a projection as well. A *partial isometry* in M is an element $v \in M$ such that v^*v and vv^* are projections. Now, we call two projections $p, q \in M$ equivalent and write $p \sim q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. Moreover, we call a projection $z \in M$ *central* if $z \in \mathcal{Z}(M)$. Then, for any projection $p \in M$, there exists a smallest central projection $z \in \mathcal{Z}(M)$ such that $p \leq z$ and we call $z = z(p)$ the central support of p. Lastly, a projection $p \in M \subset B(\mathcal{H})$ is called

- *abelian*, if the corner $pMp \subset B(p\mathcal{H})$ is an abelian von Neumann algebra;
- *finite*, if for any subprojection $q \leq p$ with $p \sim q$, we have $p = q$;
- *infinite*, if p is not finite;
- *properly infinite*, if zp is either zero or infinite for all $z \in \mathcal{Z}(M)$.

We are now ready to state what it means for a von Neumann algebra M to be of a certain type. Namely, M is of type

- I if M contains a non-zero abelian projection with central support $\mathbb{1}$;
- II if M has no non-zero abelian projection but contains a non-zero finite projection with central support $\mathbb{1}$;
- III if M does not contain non-zero finite projections.

Moreover, a von Neumann algebra M of type II is of subtype II_1 if the identity $\mathbb{1}$ is finite and of subtype II_∞ if $\mathbb{1}$ is properly infinite. Also type I and type III von Neumann algebras allow for more sophisticated subtypes (see [KR92, Section 6.5]): M is of type I_n, $n \in \{1, 2, \ldots, \infty\}$ if $\mathbb{1}$ is the sum of n equivalent abelian projections. In the type III case, we postpone the discussion of subtypes to Section 1.4.3.

We now turn towards examples of von Neumann algebras coming from group actions on measure spaces.

1.3.1 The group measure space construction

Any nonsingular action $G \curvearrowright (X,\mu)$ on a standard measure space (X,μ) also induces an action of G on the abelian von Neumann algebra $L^\infty(X,\mu)$ of essentially bounded functions on (X,μ) through the formula

$$G \curvearrowright^\alpha L^\infty(X,\mu),\ \alpha_g(f)(x) = f(g^{-1}x) \quad x \in X. \tag{1.3.1}$$

In fact, one can show that any group action on $L^\infty(X,\mu)$ is of this form, see e.g. [Bo07, Theorem 9.5.1]. Using this, one can associate a von Neumann algebra to $G \curvearrowright (X,\mu)$ called the *group measure space construction* [MvN36]. To do so, consider the Hilbert spaces $\mathcal{H} = L^2(X,\mu)$ and

$$\ell^2(G,\mathcal{H}) = \{\xi : G \to \mathcal{H}\ ;\ \sum_{g\in G} \|\xi(g)\|_{\mathcal{H}}^2 < \infty\}$$

with their canonical norms. Every $f \in L^\infty(X,\mu)$ acts on $\ell^2(G,\mathcal{H})$ as a bounded linear operator through the formula

$$m_f : \ell^2(G,\mathcal{H}) \to \ell^2(G,\mathcal{H}),\ (m_f\xi)(s) = \alpha_{s^{-1}}(f)\xi(s), \quad s \in G. \tag{1.3.2}$$

On the other hand, every group element $g \in G$ also acts on $\ell^2(G,\mathcal{H})$ as a unitary operator $u_g : \ell^2(G,\mathcal{H}) \to \ell^2(G,\mathcal{H})$ defined by

$$(u_g\xi)(s) = \xi(g^{-1}s), \quad s \in G. \tag{1.3.3}$$

The group measure space construction or crossed product von Neumann algebra $L^\infty(X,\mu) \rtimes G$ is then defined as the von Neumann algebra generated by the operators described above, that is to say,

$$L^\infty(X,\mu) \rtimes G := \big(\{m_f\ ;\ f \in L^\infty(X,\mu)\} \cup \{u_g\ ;\ g \in G\}\big)'' \subset B(\ell^2(G,\mathcal{H})). \tag{1.3.4}$$

Note that in $L^\infty(X,\mu) \rtimes G$, we have $u_g m_f u_g^* = m_{\alpha_g(f)}$ for all $f \in L^\infty(X,\mu),\ g \in G$. There is a lot more to be said about the group measure space construction and how it relates to the nonsingular action it belongs with. However, we will postpone this discussing to Section 1.4.1, where we will associate a von Neumann algebra to any nonsingular Borel equivalence relation. Luckily, whenever a nonsingular action $G \curvearrowright (X,\mu)$ is essentially free, the group measure space construction presented here will coincide with the von Neumann algebra associated to the orbit equivalence relation of $G \curvearrowright (X,\mu)$ as described in Section 1.4.

1.3.2 Some remarks on modular theory

While the study of von Neumann algebras of type I and II has progressed quickly after their introduction by Murray and von Neumann, their type III counterparts were more difficult to get a grip on. It was not until 1970 that Tomita, Takesaki and Connes developed their *modular theory* as a conceptual toolset to handle the type III situation, see [Ta70] [Co72]. Although modular theory will be present in the background whenever we deal with von Neumann algebras, explicit appearances will be kept to a minimum in this thesis. Nevertheless, in view of our discussion on the Connes invariants in Section 1.4.4, we will quickly recall some of the major terminology and results in this area. We refer to [Ta03] for details.

Modular theory assigns to any *weight* on a von Neumann algebra M an $\mathbb{R}$-action $\mathbb{R} \curvearrowright M$ called the *modular action*. Let us recall that a weight is a map $\varphi : M_+ \to [0,\infty]$ on the positive cone $M_+ = \{y^*y \ ; \ y \in M\}$ of M such that $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M_+$ and $\varphi(\lambda x) = \lambda\varphi(x)$ for all $\lambda \in [0, \infty)$, $x \in M_+$ with the convention that $0 \cdot \infty = 0$. Such a weight is

- *normal* if for every bounded increasing net $(x_i)_{i\in I}$ in M_+, we have $\varphi(\sup_{i\in I} x_i) = \sup_{i\in I} \varphi(x_i)$;
- *faithful* if for $x \in M_+$, $\varphi(x) = 0$ implies $x = 0$;
- *semifinite* if $\{x \in M_+ \ ; \ \varphi(x) < \infty\}$ is σ-strongly dense in M_+.

Note that every von Neumann algebra M admits a normal semifinite faithful (n.s.f) weight by [Ta03, Theorem VII.2.7]. So let us fix a von Neumann algebra M and a n.s.f. weight φ with GNS-representation $\pi_\varphi : M \to B(L^2(M, \varphi))$ (see [Ta03, Chapter VII] for details on this representation). The involution on M defines a (generically unbounded) preclosed operator

$$S_0 : \mathcal{D}(S_0) \subset L^2(M, \varphi) \to L^2(M, \varphi), \quad x \mapsto x^*,$$

where $\mathcal{D}(S_0) = \{x \in M \ ; \ \varphi(x^*x) < \infty \text{ and } \varphi(xx^*) < \infty\}$. The closure of S_0 is typically denoted by S_φ and can be used to define the linear positive self-adjoint operator $\Delta_\varphi = S_\varphi^* S_\varphi$ which is called the *modular operator* of φ. [Ta03, Theorem VI.1.19] asserts that

$$\Delta_\varphi^{it} \pi_\varphi(M) \Delta_\varphi^{-it} = \pi_\varphi(M) \quad \text{for all } t \in \mathbb{R}$$

and therefore, after identifying $M \cong \pi_\varphi(M)$, we obtain the modular action

$$\mathbb{R} \curvearrowright^{\sigma^\varphi} M, \quad \sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it},$$

respectively the *modular automorphism group* $(\sigma_t^{\varphi})_{t\in\mathbb{R}}$. Thanks to a celebrated theorem of Connes [Co72],[Ta03, Theorem VIII.3.3], the modular action does not depend on the choice of the weight up to conjugation by a σ-strongly continuous one parameter family of unitaries in M. More precisely, given two n.s.f. weights φ and ψ on M, there is a a σ-strongly continuous one parameter family $(u_t)_{t\in\mathbb{R}}$ of unitaries in M such that $\sigma_t^{\varphi}(x) = u_t\sigma_t^{\psi}(x)u_t^*$ for $x \in M$ and $t \in \mathbb{R}$. In other words, the modular action is a quantity *intrinsic to M* and therefore a very useful tool to study von Neumann algebras of type III. For instance, using the theory of crossed products with locally compact groups, one can define a new von Neumann algebra $c(M) = M \rtimes_{\varphi} \mathbb{R}$ called the *(Connes-Takesaki) continuous core* of M, which does not depend of the choice of the n.s.f. weight φ up to isomorphism. The continuous core is the analogue of the *Maharam extension* of a nonsingular Borel equivalence relation/group action, which we will meet in the following section. It is in terms of the continuous core that we can define the subtypes $\mathrm{III}_\lambda,\ \lambda \in [0,1]$ for factorial von Neumann algebras. However, we prefer not to go into detail here and refer the reader to Subsection 1.4.3 where we will discuss the types of nonsingular Borel equivalence relations and group actions and where we will sketch the connection between the continuous core and the Maharam extension.

1.4 Nonsingular Borel equivalence relations

Any group action $G \curvearrowright X$ on a standard Borel space gives rise to an equivalence relation

$$\mathcal{R}(G \curvearrowright X) = \{(x, gx)\ ;\ x \in X,\ g \in G\}$$

on X which is a Borel subset of $X \times X$. Motivated by work of Dye [D59], [D63], Krieger [Kri70] and others, Feldman and Moore axiomatized and studied this type of equivalence relation without refering to the group G. Since we will only recall those definitions and results within the theory that will fulfill the specific needs of this thesis later on, we would like to refer to the original work of Feldman and Moore [FM77a], [FM77b] and the book [KM04] for a more thorough introduction to the topic.

Definition 1.4.1 (Feldman-Moore)**.** Let X be a standard Borel space. A Borel equivalence relation on X is an equivalence relation $\mathcal{R} \subset X \times X$ that is a Borel subset.

- $\mathcal{R}$ is called *countable* if every orbit $\mathcal{R}[x] = \{y \in X\ ;\ (x,y) \in \mathcal{R}\}$ is countable.
- If μ is a measure on X, then $\mathcal{R}$ is called *μ-nonsingular*, if for each μ-null subset $A \subset X$, the *saturation* $\mathcal{R}(A) := \bigcup_{x\in A} \mathcal{R}[x]$ is also μ-null.

Whenever there is no ambiguity in doing so, we simply call $\mathcal{R}$ nonsingular instead of μ-nonsingular.

Remark 1.4.2. Note that if $G \curvearrowright X$ is a group action, then $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$ is countable if and only if the group G is countable. Unless explicitly specified otherwise, any Borel equivalence relation in this thesis will always be assumed to be countable. Let us also remark that, given a measure μ, $\mathcal{R}$ is μ-nonsingular if and only if the action $G \curvearrowright (X, \mu)$ is nonsingular.

There are two important notions of isomorphism between countable Borel equivalence relations.

Definition 1.4.3. Two nonsingular Borel equivalence relations $\mathcal{R}$ on (X, μ) and $\mathcal{S}$ on (Y, ν) are

- *isomorphic* if there exists a nonsingular Borel isomorphism $\phi : X \to Y$ such that $\phi(\mathcal{R}(x)) = \mathcal{S}(\phi(x))$ for a.e. $x \in X$.
- *weakly* or *stably isomorphic*, if there exist Borel subsets $X' \subset X$ and $Y' \subset Y$ of positive measure that meet the class of almost every point and there exists a nonsingular isomorphism $\phi : X' \to Y'$ such that $\phi(\mathcal{R}(x) \cap X') = \mathcal{S}(\phi(x)) \cap Y'$ for a.e. $x \in X'$.

The notion of ergodicity for group actions has an obvious counterpart within the setting of Borel equivalence relations.

Definition 1.4.4. A countable nonsingular Borel equivalence relation $\mathcal{R}$ on (X, μ) is called (μ-)ergodic if, for any Borel subset $A \subset X$, $\mathcal{R}(A) = A$ implies that A is either μ-null or μ-conull.

The following theorem due to Feldman and Moore [FM77a, Theorem 1] clarifies the relationship between nonsingular group actions and nonsingular Borel equivalence relations.

Theorem 1.4.5 (Feldman-Moore)**.** *Let $\mathcal{R}$ be a countable nonsingular Borel equivalence relation on (X, μ). There exists a countable discrete group G, a nonsingular group action $G \curvearrowright (X, \mu)$ and a μ-conull subset $U \subset X$ such that $\mathcal{R} \cap U \times U = \mathcal{R}(G \curvearrowright X) \cap U \times U$.*

Now, fix a countable Borel equivalence relation $\mathcal{R}$ on a standard measure space (X, μ). Define two Borel measures μ_l and μ_r on $\mathcal{R} \subset X \times X$ as follows:

$$\mu_l(A) := \int_X |\{y \in X \ : \ (x, y) \in A\}| \ d\mu(x),$$

$$\mu_r(A) := \int_X |\{x \in X \ : \ (x, y) \in A\}| \ d\mu(y)$$

for each Borel set $A \subset \mathcal{R}$. If $\mathcal{R}$ is μ-nonsingular then μ_l is equivalent to μ_r as shown in [FM77a, Theorem 2]. In this case, we can define the Radon-Nykodim derivative

$$D = D_\mu : \mathcal{R} \to (0, \infty); \ D(x, y) = \frac{d\mu_l}{d\mu_r}(x, y) \quad \text{for a.e. } (x, y) \in \mathcal{R}.$$

Whenever $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$ for some nonsingular group action $G \curvearrowright (X, \mu)$, D agrees with the Radon-Nykodim derivative $\omega : G \times X \to (0, \infty)$ of the action in the sense that

$$D(gy, y) = \omega(g, y) \quad \text{for a.e. } y \in X, \ g \in G.$$

Definition 1.4.6. We say that a countable nonsingular Borel equivalence relation $\mathcal{R}$ on (X, μ) preserves the measure if one of the following equivalent conditions hold.

- $D_\mu(x, y) = 1$ for μ_l-a.e. $(x, y) \in \mathcal{R}$;
- $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$ for some measure preserving group action $G \curvearrowright (X, \mu)$.

The equivalence of the two conditions in Definition 1.4.6 is shown in [FM77a, Corollary 1].

1.4.1 The von Neumann algebra of a nonsingular equivalence relation

In [FM77b], Feldman and Moore associated a von Neumann algebra to any countable nonsingular Borel equivalence relation. Let us quickly go through their construction.
So, fix a countable nonsingular Borel equivalence relation $\mathcal{R}$ on a standard measure space (X, μ).

Definition 1.4.7. A function $f \in L^\infty(\mathcal{R}, \mu_l)$ is called *left finite* if there exists $n \in \mathbb{N}$ such that for μ_l-a.e. $(x, y) \in \mathcal{R}$ we have

$$|\{z \in X \ ; \ f(x, z) \neq 0\}| + |\{z \in X \ ; \ f(z, y) \neq 0\}| \leq n.$$

Two left finite functions e and f can be multiplied in a manner resembling matrix multiplication, namely

$$(ef)(x,y) := \sum_{z \in \mathcal{R}[x]} e(x,z)f(z,y)$$

and we can define an adjoint f^* of a left finite function f by $f^*(x,y) = \overline{f(y,x)}$ for $(x,y) \in \mathcal{R}$. In addition, any left finite function f acts as a bounded linear operator L_f on $L^2(\mathcal{R}, \mu_l)$, again by the formula

$$(L_f \psi)(x,y) = \sum_{z \in \mathcal{R}[x]} f(x,z)\psi(z,y) \quad \psi \in L^2(\mathcal{R}, \mu_l).$$

Using that $L_e L_f = L_{ef}$ and $L_f^* = L_{f^*}$, it was shown in [FM77b] that $\{L_f \ ; \ f \text{ left finite}\}$ forms a $*$-subalgebra of $B(L^2(\mathcal{R}, \mu_l))$, which leads us to the following definition.

Definition 1.4.8. The von Neumann algebra of the nonsingular countable Borel equivalence relation $\mathcal{R}$ is defined as

$$L(\mathcal{R}) = \{L_f \ ; \ f \text{ left finite}\}'' \subset B(L^2(\mathcal{R}, \mu_l)).$$

As was the case for the group measure space construction, $L(\mathcal{R})$ contains a canonical copy of $L^\infty(X, \mu)$. The embedding is defined by identifying $f \in L^\infty(X, \mu)$ with the operator associated to the left finite function $(x,y) \mapsto f(x)\mathbb{1}_\Delta(x,y)$ where $\mathbb{1}_\Delta$ is the characteristic function of the diagonal $\Delta \subset \mathcal{R}$.

$L(\mathcal{R})$ also contains the *full pseudogroup* $[[\mathcal{R}]]$ of $\mathcal{R}$, that is to say, the set of partial Borel isomorphisms $\phi : A \to B$ between Borel subsets of X whose graph is contained in $\mathcal{R}$, i.e $(x, \phi(x)) \in \mathcal{R}$ for all $x \in A$. Any $\phi : A \to B$ in $[[\mathcal{R}]]$ gives rise to a left finite function $S_\phi = \mathbb{1}_{\text{graph}(\phi)}$, the characteristic function of its graph. The operator $u_\phi = L_{S_\phi} \in L(\mathcal{R})$ is a partial isometry with $u_\phi^* u_\phi = \mathbb{1}_A \in L^\infty(X, \mu)$ and $u_\phi u_\phi^* = \mathbb{1}_B \in L^\infty(X, \mu)$. Moreover, if (ϕ_n) is a sequence in $[[\mathcal{R}]]$ with pairwise disjoint graphs and $\mathcal{R} = \sqcup_n \text{graph}(\phi_n)$, then $L(\mathcal{R})$ is generated by the copy of $L^\infty(X, \mu)$ and the operators u_{ϕ_n}.

As pointed out before, if $G \curvearrowright (X, \mu)$ is a nonsingular essentially free action, the von Neumann algebra $L(\mathcal{R}(G \curvearrowright X))$ is isomorphic to the group measure space construction defined in Definition 1.3.1. Another convenient fact is that $M = L(\mathcal{R})$ is a factor if and only if the equivalence relation $\mathcal{R}$ is ergodic, see [FM77b, Proposition 2.9].

1.4.2 Cocycles, skew products and the Maharam extension

In this section, we will introduce the notion of cocycles on Borel equivalence relations and explain how to build a new equivalence relation from a given one and a cocycle. We will then apply this construction to the Radon-Nykodim cocycle. We refer to the introduction of the article [HMV17] for a nice review of these ideas.
Let us fix a countable nonsingular Borel equivalence relation $\mathcal{R}$ on a standard measure space (X, μ). Recall that we denote the set of all equivalence classes of Borel functions from X to a Polish space E by $L^0(X, E)$, see Notation 1.1.3.

Definition 1.4.9. Let Σ be a locally compact second countable group. A *(1-)cocycle* into Σ is a Borel map $Z \in L^0(\mathcal{R}, \Sigma)$ such that $Z(x, y)Z(y, z) = Z(x, z)$ for a.e. $x, y, z \in X$ with $(x, y), (y, z) \in \mathcal{R}$.

Example 1.4.10. The Radon-Nykodim derivative $D_\mu : \mathcal{R} \to \mathbb{R}_+^*$ defined in the previous section is easily seen to be a cocycle into $\mathbb{R}_+^*$.

Most properties of measurable cocycles are preserved under the following notion of equivalence.

Definition 1.4.11. Two cocycles $Z_1, Z_2 :\in L^0(\mathcal{R}, \Sigma)$ are called *cohomologous* if there exists a Borel map $\phi : X \to \Sigma$ such that

$$Z_1(x, y) = \phi^{-1}(x)Z_2(x, y)\phi(y) \quad \text{for a.e. } x, y \in X \text{ such that } (x, y) \in \mathcal{R}.$$

Moreover, a cocycle which is cohomologous to the trivial cocycle $\mathcal{R} \to \Sigma, (x, y) \mapsto e$ is called *inner*.

Example 1.4.12. Let μ' be a measure on X equivalent to μ. Using the Borel map $\phi = \frac{d\mu'}{d\mu} : X \to \mathbb{R}_+^*$, we see that the Radon-Nykodim derivative $D_{\mu'}$ is cohomologous to D_μ.

In analogy with the discussion on L^2-cohomology in the appendix (see A.1.3), we will denote the set of 1-cocycles into Σ by $Z^1(\mathcal{R}, \Sigma)$ and the set of inner 1-cocycles by $B^1(\mathcal{R}, \Sigma)$. The quotient $H^1(\mathcal{R}, \Sigma) = Z^1(\mathcal{R}, \Sigma)/B^1(\mathcal{R}, \Sigma)$ is called the *1-cohomology* of $\mathcal{R}$ with coefficients in Σ. For $Z \in Z^1(\mathcal{R}, \Sigma)$ we will denote its equivalence class in $H^1(\mathcal{R}, \Sigma)$ by $[Z]$.

Definition 1.4.13. Let Σ be a locally compact second countable abelian group with left Haar measure λ and let $Z : \mathcal{R} \to \Sigma$ be a cocycle. The *skew product* equivalence relation $\mathcal{R} \times_Z \Sigma$ is defined on the product space $(X \times \Sigma, \mu \times \lambda)$ by

$$(x, g) \sim (y, h) \quad \text{if and only if} \quad (x, y) \in \mathcal{R} \text{ and } g^{-1}h = Z(x, y).$$

Remark 1.4.14. Note that $\mathcal{R} \times_Z \Sigma$ is a countable $(\mu \times \lambda)$-nonsingular Borel equivalence relation. Moreover, if $Z_1, Z_2 : \mathcal{R} \to \Sigma$ are cohomologous cocycles, then $\mathcal{R} \times_{Z_1} \Sigma$ and $\mathcal{R} \times_{Z_2} \Sigma$ are isomorphic, see e.g. [HMV17, Section 2.5].

Example 1.4.15. The skew product of $\mathcal{R}$ by the cocycle $\log(D_\mu) : \mathcal{R} \to \mathbb{R}$ is called the *Maharam extension* ([Ma64]) of $\mathcal{R}$ and denoted by $c(\mathcal{R})$. We rephrase this slighty by noting that $c(\mathcal{R})$ is isomorphic to the countable nonsingular Borel equivalence relation on $(X \times \mathbb{R}, \mu \times \nu)$ where $d\nu = \exp(-t)\ dt$ and

$$(x, s) \sim (y, t) \quad \text{if and only if} \quad (x, y) \in \mathcal{R} \text{ and } t - s = \log(D_\mu(x, y)).$$

The reason that we replaced the Haar measure on $\mathbb{R}$, i.e. the Lebesgue measure, by the equivalent measure ν, is the convenient fact that $\mu \times \nu$ is invariant under $c(\mathcal{R})$. Note that when $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$ for a nonsingular action $G \curvearrowright (X, \mu)$, then $c(\mathcal{R})$ is implemented by the action

$$G \curvearrowright (X \times \mathbb{R}, \mu \times \nu), \quad g(x, t) = (gx, t + \log(\omega(g, x)))$$

and consequently we will call this action the Maharam extension of $G \curvearrowright (X, \mu)$.

1.4.3 Type classification of orbit equivalence relations and group actions

We will now proceed to classify nonsingular ergodic Borel equivalence relations respectively group actions into types I, II_1, II_∞ and III, see [MvN36], [vN41], [FM77a]. The latter type can be classified further into subtypes III_λ, $\lambda \in [0, 1]$ ([Co73], see also [Kri70]) and this is where the Maharam extension will demonstrate its usefulness. Let us however start by defining the overtypes I, II_1, II_∞ and III, following the work of Murray and von Neumann for group actions, and its translation to the setting of Borel equivalence relations in [FM77a].
Again, fix a countable μ-nonsingular Borel equivalence relation $\mathcal{R}$ on a standard Borel space X. Let us also assume that $\mathcal{R}$ is ergodic. Then $\mathcal{R}$ is said to be of type

- I — if there exists an atom $A \subset X$ such that $\mathcal{R}(A) = X$ up to measure zero;
- II_1 — if $\mathcal{R}$ is not of type I and if there exists a finite measure on X that is equivalent to μ and $\mathcal{R}$-invariant;
- II_∞ — if $\mathcal{R}$ is not of type I or II_1 and if there exists an infinite measure on X that is equivalent to μ and $\mathcal{R}$-invariant;
- III — if there is no $\mathcal{R}$-invariant μ-equivalent measure on X.

Consequently a nonsingular, free and ergodic group action $G \curvearrowright (X,\mu)$ is defined to be of a certain type if and only if $\mathcal{R}(G \curvearrowright X)$ is.
Let us now assume further that $\mathcal{R}$ is of type III and consider the action $\mathbb{R} \curvearrowright (X \times \mathbb{R}, \mu \times \nu)$ where $d\nu = \exp(-t)\ dt$ and where $\mathbb{R}$ acts by translation on the second component. Since this action preserves the Maharam extension $c(\mathcal{R})$, it induces an action of $\mathbb{R}$ on the abelian von Neumann algebra of $c(\mathcal{R})$-invariant functions $L^\infty(X \times \mathbb{R}, \mu \times \nu)^{c(\mathcal{R})}$ called the associated *flow*. Since $\mathcal{R}$ was assumed ergodic, the only invariant functions with respect to the action $\mathbb{R} \curvearrowright L^\infty(X \times \mathbb{R}, \mu \times \nu)^{c(\mathcal{R})}$ are the constant ones. We define the subtype of $\mathcal{R}$ in terms of this action as follows. $\mathcal{R}$ is said to be of type

- III_1 if the Maharam extension is ergodic and thus the flow is trivial;
- III_λ, for $0 < \lambda < 1$ if the flow is $-\log(\lambda)$ periodic;
- III_0 otherwise.

Remark 1.4.16. The original interest of Murray and von Neumann when introducing the different types above was the study of von Neumann algebras rather than group actions, let alone Borel equivalence relations which were only properly defined by Feldman and Moore a mere 40 years later. The type classification of group actions was therefore rather a byproduct of their type classification of factorial von Neumann algebras (see Section 1.3) and their early realization of the worth of group actions as a rich source of examples of von Neumann algebras. Similarly, Connes introduced the subtypes $\mathrm{III}_\lambda,\ \lambda \in [0,1]$ for factorial von Neumann algebras using the modular theory developed by Tomita, Takesaki and himself, see Subsection 1.3.2. This type classification for von Neumann algebras and its reformulation for Borel equivalence relations of course agree in the sense that the nonsingular ergodic Borel equivalence relation $\mathcal{R}$ is of a specific type if and only if its associated von Neumann algebra $L(\mathcal{R})$ is of that type. We also already hinted at a connection between the continuous core of a von Neumann algebra and the Maharam extension of a nonsingular Borel equivalence relation in Subsection 1.3.2. This relation is nothing but the observation that the von Neumann algebra associated to the Maharam extension $c(\mathcal{R})$ of a nonsingular Borel equivalence relation $\mathcal{R}$ on (X,μ) is canonically isomorphic to the continuous core $c(L(\mathcal{R}))$ of $L(\mathcal{R})$ with respect to the canonical weight associated to the measure μ. In the language of nonsingular free group actions $G \curvearrowright (X,\mu)$ we have

$$L^\infty(X \times \mathbb{R}, \mu \times \nu) \rtimes G \cong \big(L^\infty(X,\mu) \rtimes G\big) \rtimes_{\sigma^\varphi} \mathbb{R},$$

where the G-action on the left is the Maharam extension of the G-action $G \curvearrowright (X,\mu)$ on the right and where $\mathbb{R} \curvearrowright^\sigma L^\infty(X,\mu) \rtimes G$ is the modular action with respect to the weight $\varphi(x_g u_g) = \delta_{g,e} \int_X x_e \ d\mu$. For a detailed explanation of how this translation works, we refer to [FM77a] and [Ta03].

1.4.4 Connes' invariants for orbit equivalence relations and von Neumann algebras

In [Co74], Connes introduced two new invariants in order to study *full* factors of type III. The notion of fullness of von Neumann algebra is a strong factoriality property, which is intimitely related to the notion of *strong ergodicity* in the context of Borel equivalence relations/group actions. The analogues of Connes invariants for strongly ergodic Borel equivalence relations have been defined recently in [HMV17]. The aim of this section is to recall the definitions of strong ergodicity and fullness and to introduce Connes' invariants in both contexts. We require a minimum of modular theory for von Neumann algebras in this section and we refer the reader to Subsection 1.3.2 and [Ta03] for an introduction on this subject. We also refer to [Co74] for a more detailed discussion of full factors.

Let us now work our way up to a definition of fullness.

Let $M \subset B(\mathcal{H})$ be a factorial von Neumann algebra with separable predual M_*. The restriction of the strong topology to the group of unitaries in M, denoted $\mathcal{U}(M)$, turns $\mathcal{U}(M)$ into a Polish group. Similarly, we can turn the group $\mathrm{Aut}(M)$ of $*$-automorphisms of M into a Polish group by equipping it with the topology of pointwise norm convergence on M_*. More precisely, this means that a net (θ_i) in $\mathrm{Aut}(M)$ converges to $\mathrm{id} \in \mathrm{Aut}(M)$ if and only if

$$\|\varphi(\theta^{-1}(\cdot)) - \varphi\| \to 0 \quad \text{for all } \varphi \in M_*.$$

With respect to the two topologies defined above, the group homomorphism

$$\mathrm{Ad} : \mathcal{U}(M) \to \mathrm{Aut}(M), \quad \mathrm{Ad}(u)(x) = uxu^*$$

is continuous. We denote its image by $\mathrm{Inn}(M)$.

Definition 1.4.17 ([Co74])**.** The factor M is called *full* if $\mathrm{Inn}(M)$ is closed in $\mathrm{Aut}(M)$.

By [Co74, Theorem 3.1], M is full if and only if every uniformly bounded net (x_i) in M which is *centralizing*, meaning that

$$\|\varphi(x_i \,\cdot) - \varphi(\cdot\, x_i)\| \to 0 \quad \text{for all } \varphi \in M_*,$$

must be trivial in the sense that there exists a bounded net (z_i) in $\mathbb{C}$ such that $x_i - z_i\mathbb{1} \to 0$ $*$-strongly. Therefore, fullness is considered a strong version of factoriality. For a full factor M, the outer automorphism group $\mathrm{Out}(M) := \mathrm{Aut}(M)/\mathrm{Inn}(M)$ is a Polish group as well. Denote by $p : \mathrm{Aut}(M) \to \mathrm{Out}(M)$ the canonical quotient map. Let ψ be a weight on M with modular automorphism group $(\sigma_t^\psi)_{t\in\mathbb{R}}$. We obtain a map

$$\delta : \mathbb{R} \to \mathrm{Out}(M), \quad t \mapsto p(\sigma_t^\psi)$$

which thanks to Connes' cocycle derivative theorem, see [Ta03, Theorem VIII.3.3], does not depend on the choice of the weight and therefore is a quantity intrinsic to M.

Definition 1.4.18. Let M be a full factor with separable predual. The *τ-invariant* $\tau(M)$ of M is the weakest topology making the map $\delta : \mathbb{R} \to \mathrm{Out}(M)$ continuous.

Connes defined another invariant for so called *almost periodic* factors in [Co74], and this invariant bears quite some resemblence to the type while still being able to distinguish almost periodic factors of type III_1. A factorial von Neumann algebra is called almost periodic, if it admits an almost periodic weight, that is to say, a weight ψ whose modular operator Δ_ψ is diagonalizable. In this case, we denote the point spectrum of Δ_ψ by $\mathrm{Sp}_p(\Delta_\psi)$.

Definition 1.4.19. The *Sd-invariant* of a factor M is defined as

$$\mathrm{Sd}(M) = \bigcap_{\psi \text{ almost periodic weight on } M} \mathrm{Sp}_p(\Delta_\psi).$$

We will now take a turn towards orbit equivalence relations and introduce the concepts of strong ergodicity and the Connes invariants in this setting. Since these are of more central interest in this thesis, our explanations will be more extensive.

Fix a standard measure space (X, μ).

Definition 1.4.20. A nonsingular group action $G \curvearrowright (X, \mu)$ is called *strongly ergodic* if for any sequence of measurable subsets $A_n \subset X$ such that $\lim_{n\to\infty} \mu(A_n \triangle g A_n) = 0$ for every $g \in G$, we have $\lim_{n\to\infty} \mu(A_n)\mu(X \backslash A_n) = 0$.

Similarly, a nonsingular Borel equivalence relation $\mathcal{R}$ is strongly ergodic if $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$ for some nonsingular strongly ergodic group action $G \curvearrowright (X, \mu)$.

Strong ergodicity is related to fullness in that a nonsingular Borel equivalence relation $\mathcal{R}$ whose associated von Neumann algebra $L(\mathcal{R})$ is full, is always strongly ergodic. The converse is however not true in general, as was shown by Connes and Jones in [CJ81].

By [Sc79, Proposition 2.3], a nonsingular Borel equivalence relation $\mathcal{R}$ is strongly ergodic if and only if the group $B^1(\mathcal{R}, \mathbb{T})$ of inner 1-cocycles with coefficients in $\mathbb{T}$ is closed in $Z^1(\mathcal{R}, \mathbb{T}) \subset L^0(\mathcal{R}, \mathbb{T})$ with respect to the topology of convergence in measure. In this case the quotient $H^1(\mathcal{R}, \mathbb{T})$ becomes a Hausdorff Polish group. We are now ready to define Connes' invariant τ for Borel equivalence relations.

Definition 1.4.21 ([Co74], [HMV17])**.** The τ*-invariant* $\tau(\mathcal{R})$ of a nonsingular strongly ergodic Borel equivalence relation $\mathcal{R}$ on (X, μ) is the weakest topology on $\mathbb{R}$ that makes the map $\mathbb{R} \to H^1(\mathcal{R}, \mathbb{T}),\ t \mapsto [D_\mu^{it}]$ continuous.

The second invariant we would like to introduce is the Sd-invariant and it can be used to distinguish so-called almost periodic Borel equivalence relations.

Definition 1.4.22. Let $\mathcal{R}$ be a nonsingular ergodic Borel equivalence relation on (X, μ).

- A measure $\nu \sim \mu$ on X is called *almost periodic* if the push-forward of μ_l by $D_\nu \in L^0(\mathcal{R}, \mathbb{R}_*^+)$ is purely atomic.
- $\mathcal{R}$ is called *almost periodic* if there exists an almost periodic measure $\nu \sim \mu$ on X.

We can now proceed to the definition of the Sd-invariant.

Definition 1.4.23 ([Co74], [HMV17])**.** Let $\mathcal{R}$ be a nonsingular ergodic and almost periodic Borel equivalence relation on (X, μ). The Sd-invariant of $\mathcal{R}$ is defined as

$$\mathrm{Sd}(\mathcal{R}) = \bigcap_{\nu \text{ a.p.}} \mathrm{aran}(D_\nu) \ < \ \mathbb{R}_*^+.$$

Remark 1.4.24.
- $\mathrm{Sd}(\mathcal{R})$ is in fact a countable subgroup of $\mathbb{R}_*^+$ by [HMV17, Proposition 5.2].
- If $\mathcal{R}$ is not of type III, i.e. it is of type I or II, then $\mathcal{R}$ is almost periodic and $\mathrm{Sd}(\mathcal{R}) = \{1\}$.

Now assume that $\mathcal{R}$ is strongly ergodic.

- There exists an almost periodic measure ν such that $\ker(D_\nu)$ is an ergodic subrelation of $\mathcal{R}$. Moreover, for any such measure we have $\mathrm{Sd}(\mathcal{R}) = \mathrm{aran}(D_\nu)$, see [HMV17].
- If $\mathcal{R}$ is of type $\mathrm{III}_\lambda, 0 < \lambda < 1$, then $\mathrm{Sd}(\mathcal{R}) = \lambda^{\mathbb{Z}}$ by [HMV17, Theorem D].
- If $\mathcal{R}$ is of type III_1, then $\mathrm{Sd}(\mathcal{R})$ is a dense countable subgroup of $\mathbb{R}_*^+$ by [HMV17, Theorem D].

1.5 Conservativity/dissipativity

In this section, we introduce the notion of conservative and dissipative group actions and gather/prove some preliminary results. We mostly base ourselves on the textbook [Aa97].

Definition 1.5.1. A nonsingular essentially free group action $G \curvearrowright (X, \mu)$ is called

- *conservative*, if for all Borel sets $A \subset X,\ \mu(A) > 0$, there exists $g \in G \backslash \{e\}$ such that $\mu(gA \cap A) > 0$;
- *dissipative*, if there exists a Borel set $A \subset X,\ \mu(A) > 0$ such that all translates $gA,\ g \in G$ are pairwise disjoint (up to measure zero) and $\bigcup_{g \in G} gA = X$ (up to measure zero).

Note that every nonsingular action $G \curvearrowright (X, \mu)$ splits into a conservative and a dissipative part. More precisely, consider the collection

$$W = \{A \subset X\ ;\ \mu(A) > 0,\ (gA)_{g \in G} \text{ p.w. disjoint}\}.$$

By [Aa97, Lemma 1.0.7] there exists a measurable set D with the properties that

- $A \subset D$ (up to measure zero) for all $A \in W$ and
- for all measurable subsets $B \subset D,\ \mu(B) > 0$, there exists $A \in W$ such that $A \subset B$.

It follows from the discussion preceding [Aa97, Lemma 1.0.7] that D is uniquely determined by these properties up to measure zero. We set $C = X \backslash D$. Then $X = C \sqcup D$, both C and D are G-invariant and if $\mu(C) > 0$ the restriction $G \curvearrowright (C, \mu|_C)$ is conservative. Similarly, if $\mu(D) > 0$, the restriction $G \curvearrowright (D, \mu|_D)$ is dissipative. In particular, if $G \curvearrowright (X, \mu)$ is ergodic, the action is either conservative or dissipative.

Remark 1.5.2. Let $G \curvearrowright (X, \mu)$ be nonsingular and essentially free of an infinite discrete group G. If $G \curvearrowright (X, \mu)$ is in addition ergodic, it follows directly that the action is dissipative if and only if the Borel equivalence relation $\mathcal{R}(G \curvearrowright X)$ is of type I in the sense of Subsection 1.4.3. More generally, it follows that $G \curvearrowright (X, \mu)$ is dissipative if and only if $\mathcal{R}(G \curvearrowright X)$ is of type I_∞ in the sense of [FM77a, Definition 3.4], that is to say, $\mathcal{R}(G \curvearrowright X)$ is isomorphic to a Borel equivalence relation $\mathcal{S}$ on $(\mathbb{N} \times Y, \text{counting} \times \nu)$, of the form

$$(n, x) \sim (m, y) \quad \text{if and only if} \quad x = y.$$

Here (Y,ν) denotes a standard probability space. Note also that one obtains the definition of type I_n, $n<\infty$, by repeating the definition of type I_∞ verbatim but with $\mathbb{N}$ replaced by a set of cardinality n. Then, a countable nonsingular Borel equivalence relation $\mathcal{R}$ on (X,μ) is of type I if X decomposes measurably into a sequence (X_n) of $\mathcal{R}$-invariant subsets such that $\mathcal{R}\cap X_n\times X_n$ is of type $\mathrm{I}_n, \quad n\in\{1,2,\ldots,\infty\}$.

We have the following quantitive criterium to determine whether a nonsingular, essentially free action is conservative/dissipative. For invertible nonsingular transformations, i.e. actions of $\mathbb{Z}$, this result is stated in [Aa97, Proposition 1.3.1]. In the general case, the proof is identical but we nevertheless include it for the sake of completeness.

Proposition 1.5.3. *Let (X,μ) be a standard probability space, let $G \curvearrowright (X,\mu)$ be nonsingular and essentially free with Radon-Nykodim cocycle $\omega : G\times X\to(0,+\infty)$. The dissipative part $D\subset X$ satisfies*

$$D=\{x\in X\ ;\ \sum_{g\in G}\omega(g,x)<+\infty\}\quad \textit{mod } \mu. \tag{1.5.1}$$

Equivalently, the conservative part $C\subset X$ satisfies

$$C=\{x\in X\ ;\ \sum_{g\in G}\omega(g,x)=+\infty\}\quad \textit{mod } \mu. \tag{1.5.2}$$

Proof. Let $A\subset X$ be Borel. [Aa97, Proposition 1.6.2] states that if

$$\sum_{g\in G}1_A(gx)<+\infty \qquad \text{for a.e. } x\in A,$$

then $A\subset D$. Set $B=\{x\in X\ ;\ \sum_{g\in G}\omega(g,x)<+\infty\}$ and note that B is a G-invariant Borel set. By G-invariance of B and the monotone convergence theorem, it follows that

$$\begin{aligned}\int_B\sum_{g\in G}1_B(gx)\ d\mu(x)&=\sum_{g\in G}\int_{g^{-1}B}1_B(gx)\ d\mu(x)\\&=\sum_{g\in G}\int_B\omega(g,x)\ d\mu(x)\\&=\int_B\sum_{g\in G}\omega(g,x)\ d\mu(x)<+\infty.\end{aligned}$$

Hence $B \subset D$ as desired. To prove the converse inclusion, consider a Borel set $A \subset X$ such that all gA, $g \in G$ are pairwise disjoint. Since by [Aa97, Proposition 1.6.2], D is the countable disjoint union of sets of this form, it suffices to proof that

$$\sum_{g \in G} \omega(g, x) < +\infty \quad \text{for a.e. } x \in A. \tag{1.5.3}$$

Indeed, by the monotone convergence theorem it follows that

$$\begin{aligned} \int_A \sum_{g \in G} \omega(g, x)\ d\mu(x) &= \sum_{g \in G} \int_A \omega(g, x)\ d\mu(x) \\ &= \sum_{g \in G} \mu(gA) \leq 1, \end{aligned}$$

which implies 1.5.3. □

1.5.1 A theorem by Maharam

The following theorem was proven by Maharam in [Ma64] for nonsingular $\mathbb{Z}$-actions. We therefore refere to it as Maharam's theorem in this thesis although we are aware of the ambiguity of this name choice.

Theorem 1.5.4. *Let (X, μ) denote a standard probability space. A nonsingular and essentially free action $G \curvearrowright (X, \mu)$ is conservative if and only if its Maharam extension is conservative.*

The purpose of this section is to provide a proof of Maharam's theorem in the general case. We will therefore need the following result which is a standard result within the theory of Borel equivalence relations. We will include a proof for the sake of completeness as we could not find one in the literature.

Proposition 1.5.5. *Let $\mathcal{R}$ be a nonsingular countable Borel equivalence relation on a standard measure space (X, μ). Then $\mathcal{R}$ is of type I_n, $n \in \{1, 2, \ldots, \infty\}$ in the sense of Remark 1.5.2 if and only if $M = L(\mathcal{R})$ is of type I_n in the sense of Section 1.3.*

In particular, a nonsingular essentially free action $G \curvearrowright (X, \mu)$ of a countable infinite discrete group is dissipative if and only if the group measure space construction $M = L^\infty(X, \mu) \rtimes G$ is of type I.

Proof. Assume first that $\mathcal{R}$ is of type I_n for $n \in \{1, 2, \ldots, \infty\}$. It suffices to show that the von Neumann algebra of the Borel equivalence relation $\mathcal{S}$ on

$(S \times Y, \text{counting} \times \nu)$ defined by $(k,x) \sim (m,y)$ if and only if $x = y$ is of type I_n where $S = \{1,\dots,n\}$ and (Y,ν) is a standard probability space. Define the sets $A_k := \{k\} \times Y$ and the partial isomorphisms

$$\phi_{k,m} : A_k \to A_m, \quad \phi_{k,m}(k,y) = (m,y).$$

The projections $p_k := u_{\phi_{k,k}} = \mathbb{1}_{A_k} \in L(\mathcal{S})$ are pairwise orthogonal and sum up to $\mathbb{1}$. Since $p_k u_{\phi_{j,l}} p_k$ and $p_k u_{\phi_{s,t}} p_k$ commute for all j,k,l,s,t and since the family $\phi_{k,m} \in [[\mathcal{S}]]$ has pairwise disjoint graphs that cover all of $\mathcal{S}$, all p_k are abelian projections. Lastly, since $p_k = u^*_{\phi_{k,m}} u_{\phi_{k,m}}$ and $p_m = u_{\phi_{k,m}} u^*_{\phi_{k,m}}$, all our projections p_k are pairwise equivalent and we are done.

Now, assume that $M = L(\mathcal{R})$ is of type I_n. Consider a family $(\phi_k)_{k\in\mathbb{N}}$ in $\mathcal{R}$ such that $\mathcal{R}$ is the disjoint union of the graphs of ϕ_k up to measure zero. We may assume that $\phi_1 = \mathrm{id}$ such that its graph is the diagonal Δ. Applying [KR92, Exercise 6.9.23] (if $n < \infty$) and [KR92, Exercise 6.9.24][1] (if $n = \infty$) to the maximal abelian subalgebra $L^\infty(X,\mu) \subset M$, one obtains a non-zero projection $p \in L^\infty(X,\mu)$ that is abelian and has central support $\mathbb{1}$. Take a Borel set $A \subset X$ of positive measure such that $p = \mathbb{1}_A$. We need to show that the restriction $\mathcal{R} \cap A \times A$ yields the trivial relation or in other words, that $\mu_l(\mathrm{graph}(\phi_k) \cap (A \times A)) = 0$ for $k \neq 1$. Since p is abelian, by [KR92, Proposition 6.4.2], we have $pMp = \mathcal{Z}(M)p$. Further for any $k \geq 1$, the multiplier associated to the left finite function $\mathbb{1}_{\mathrm{graph}(\phi_k) \cap (A\times A)} : \mathcal{R} \to \mathbb{C}$ equals $pu_{\phi_k}p \in \mathcal{Z}(M)p \subset L^\infty(X,\mu)$ such that

$$\mathbb{1}_{\mathrm{graph}(\phi_k)\cap(A\times A)} = f\mathbb{1}_\Delta \quad \text{for some} \quad f \in L^\infty(X,\mu).$$

Since $\mathrm{graph}(\phi_k)$ and Δ are measurably disjoint it follows that $\mu_l(\mathrm{graph}(\phi_k) \cap (A \times A)) = 0$ for $n \geq 2$ and hence $\mathcal{R}$ restricts to the trivial relation on A. It then follows by a standard maximality argument and [KR92, Theorem 6.3.11] that $\mathcal{R}$ is of type I_n. □

Corollary 1.5.6. *Let (X,μ) denote a standard measure space. A nonsingular and essentially free action $G \curvearrowright (X,\mu)$ is conservative if and only if the crossed product $L^\infty(X,\mu) \rtimes G$ has no direct summand of type I.*

Proof. Splitting $X = C \sqcup D$ into its conservative and dissipative parts induces a decomposition

$$L^\infty(X,\mu) \rtimes G = M_1 \oplus M_2$$

where $M_1 \cong L^\infty(C,\mu|_C) \rtimes G,\ M_2 \cong L^\infty(D,\mu|_D) \rtimes G$. Now, let us assume that the action is not conservative, so that $\mu(D) > 0$. Then, the previous proposition implies that M_2 is of type I.

[1] These exercises come with solutions in [KR92]

Now, assume that M has a direct summand of type I, meaning that there is a nonzero central projection $z \in \mathcal{Z}(M)$ such that zM is of type I. In particular, we have that $z \in L^\infty(X, \mu)$ and therefore there exists a G-invariant Borel set $\tilde{D} \subset X$ of positive measure such that $z = 1_{\tilde{D}}$. It then follows that zM is isomorphic to the crossed product $L^\infty(\tilde{D}, \mu|_{\tilde{D}}) \rtimes G$ and by the previous proposition we get that the dissipative part of $G \curvearrowright (X, \mu)$ is nontrivial. Hence the action is not conservative. $\square$

We are now ready to prove Maharam's theorem.

Proof of Theorem 1.5.4. Suppose first that the nonsingular essentially free action $G \curvearrowright (X, \mu)$ is not conservative that is to say, there exists a Borel set $A \subset X$ of positive measure such that all translates $gA, g \in G$ are pairwise disjoint. Setting $B = A \times \mathbb{R}$, it follows that the translates of B under the Maharam extension are also pairwise disjoint. Hence the Maharam extension is not conservative.

Now, assume that $G \curvearrowright (X, \mu)$ is conservative. Then, by Corollary 1.5.6, the von Neumann algebra $M = L^\infty(X, \mu) \rtimes G$ does not have a direct summand of type I. By [Ta03, Theorem XII.1.1], the continuous core $c(M)$ of M does not have a direct summand of type I. Since $c(M)$ identifies with the group measure space construction of the Maharam extension

$$c(M) \cong L^\infty(X \times \mathbb{R}, \mu \times \nu) \rtimes G,$$

we can apply Corollary 1.5.6 once more to get that the Maharam extension of $G \curvearrowright (X, \mu)$ does not have a direct summand of type I. $\square$

1.6 Mixing phenomena and invariant functions

One of our central objectives in Chapter 2 will be to prove that the nonsingular Bernoulli actions we construct, and/or their Maharam extensions, are ergodic. In other words, our goal will be to show that the only functions invariant under these actions are the constant ones. The purpose of this subsection is to gather several useful tools from ergodic theory that enable us to do so or that help us at the very least to reduce the 'number of variables' these functions depend on. First let us introduce the notions of mild and weak mixing which are strenghtenings of ergodicity.

Definition 1.6.1 ([SW81]). Let (X, μ) be a nonatomic standard probability space. A nonsingular conservative group action $G \curvearrowright (X, \mu)$ is called *mildly mixing* if one of the following equivalent conditions hold.

- For every nonsingular conservative ergodic action $G \curvearrowright (Y, \nu)$ on a standard probability space, the diagonal action

$$G \curvearrowright (X \times Y, \mu \times \nu),\ g(x, y) = (gx, gy), \quad g \in G,\ x \in X,\ y \in Y,$$

  remains ergodic;

- For every nonsingular conservative action $G \curvearrowright (Y, \nu)$, any function in $L^\infty(Y \times X, \nu \times \mu)$ invariant under the diagonal action, only depends on Y, that is to say

$$L^\infty(Y \times X, \nu \times \mu)^G = L^\infty(Y, \nu)^G \otimes 1.$$

Schmidt and Walters also provided the following useful characterization of mild mixing in [SW81, Theorem 2.3].

Theorem 1.6.2 ([SW81])**.** *A nonsingular conservative group action $G \curvearrowright (X, \mu)$ is mildly mixing if and only if for all Borel sets $A \subset X,\ 0 < \mu(A) < 1$, we have* $\liminf_{g \to \infty} \mu(gA \Delta A) > 0$*;*

As a consequence of this theorem we obtain that pmp Bernoulli actions are mildly mixing since they are mixing in the ordinary sense by [KL16, Section 2.3.1].

Another interesting results of [SW81] is that mildly mixing actions are always pmp.

Theorem 1.6.3 ([SW81])**.** *If a nonsingular conservative group action $G \curvearrowright (X, \mu)$ is mildly mixing, then μ is equivalent to a G-invariant probability measure on X.*

Let us turn towards weak mixing.

Definition 1.6.4. Let (X, μ) be a standard measure space. A nonsingular group action $G \curvearrowright (X, \mu)$ is called *weakly mixing* if one of the following equivalent conditions holds.

- For every ergodic pmp action $G \curvearrowright (Y, \nu)$ the diagonal action $G \curvearrowright (X \times Y, \mu \times \nu)$ remains ergodic;
- For every pmp action $G \curvearrowright (Y, \nu)$, any function in $L^\infty(Y \times X, \nu \times \mu)$ invariant under the diagonal action, only depends on Y.

While this notion of weak mixing is the only one we will work with in this thesis, one should note that, for general nonsingular actions there are several

other versions of weak mixing around (see [GW16] for a discussion on their relationship). Whenever the action is measure preserving, all these versions are equivalent and there is no ambiguity.
Let us end this section with a consequence of a lemma due to Silva and Theuillen ([STh94, Lemma 4.3]) on functions invariant under nonsingular $\mathbb{Z}$-actions. We will phrase this lemma in the language of nonsingular transformations, that is to say, instead of considering a $\mathbb{Z}$-action $\mathbb{Z} \curvearrowright (X, \mu)$ as a whole, we discuss its canonical generator $T : X \to X$, $T(x) = 1 \cdot x$ which determines the action uniquely. In this context, we say that a nonsingular transformation is conservative, if the $\mathbb{Z}$-action it generates is conservative. The lemma of Silva and Theuillen directly implies the following.

Lemma 1.6.5 ([STh94])**.** *Assume that*

- (Z, ζ) *and* (Z_0, ζ_0) *are standard measure spaces, with* σ*-algebras of measurable sets* $\mathcal{B}$ *and* $\mathcal{B}_0$*;*
- $T : Z \to Z$ *is a measure preserving, conservative automorphism and* $T_0 : Z_0 \to Z_0$ *is a measure preserving endomorphism;*
- $\pi : (Z, \zeta) \to (Z_0, \zeta_0)$ *is a measure preserving factor map. In particular,* $\pi \circ T = T_0 \circ \pi$ *a.e.;*
- $\mathcal{B}$ *is, up to measure zero, generated by* $\{T^k(\pi^{-1}(\mathcal{B}_0)) \mid k \in \mathbb{Z}\}$*.*

Then, every T*-invariant function* $Q \in L^\infty(Z)$ *factors through* π*, that is to say, there exists a* T_0*-invariant function* $P \in L^\infty(Z_0)$ *such that* $Q(z) = P(\pi(z))$ *for a.e.* $z \in Z$*.*

1.7 Amenability in the sense of Zimmer

Let G be a countable group. The *amenability* of an essentially free nonsingular action $G \curvearrowright (X, \mu)$ was defined in [Zi76a, Definition 1.4] through a fixed point property. When μ is an invariant probability measure, this notion is equivalent with the amenability of G. In general, this notion is equivalent with the injectivity of the crossed product von Neumann algebra $L^\infty(X) \rtimes G$ by [Zi76b] and [Zi76c, Theorem 2.1]. Denote by $\lambda : G \to \mathcal{U}(\ell^2(G))$ the left regular representation. The following definition is equivalent to the original one of Zimmer by [AD01, Theorem 3.1.6].

Definition 1.7.1. A nonsingular action $G \curvearrowright (X, \mu)$ is said to be *amenable* if there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ of Borel maps $\xi_n : X \to \ell^2(G)$ with the following properties.

- For all $n \in \mathbb{N}$ and a.e. $x \in X$, we have that $\|\xi_n(x)\|_2 = 1$;
- for all $g \in G$ and $P \in L^1(X, \mu)$, we have that

$$\lim_n \int_X \langle \lambda_g \xi_n(g^{-1} \cdot x), \xi_n(x) \rangle \, P(x) \, d\mu(x) = \int_X P(x) \, d\mu(x). \tag{1.7.1}$$

In order to prove nonamenability of the Bernoulli actions in Theorem 2.3.1, we will use the following result which is implicitly contained in the proof of [DN10, Theorem 7].

Proposition 1.7.2 ([DN10, Theorem 7]). *Let $G \curvearrowright (X, \mu)$ be a nonsingular action. Denote by $\omega : G \times X \to (0, +\infty)$ the Radon-Nikodym cocycle. If there exists a finite subset $\mathcal{F} \subset G$ such that*

$$\sum_{g \in \mathcal{F}} \int_X \sqrt{\omega(g, x)} d\mu(x) > \Big\| \sum_{g \in \mathcal{F}} \lambda_g \Big\| \, ,$$

then the action $G \curvearrowright (X, \mu)$ is nonamenable in the sense of Zimmer.

Proof. Assume that $G \curvearrowright (X, \mu)$ is amenable in the sense of Zimmer and fix a finite subset $\mathcal{F} \subset G$. Since $G \curvearrowright (X, \mu)$ is amenable, we can take a sequence $\xi_n \in L^2(X, \ell^2(G))$ such that $\|\xi_n(x)\|_2 = 1$ for a.e. $x \in X$ and such that for all $g \in G$ and $P \in L^1(X, \mu)$ we have

$$\lim_n \int_X \langle \lambda_g \xi_n(g^{-1} \cdot x), \xi_n(x) \rangle \, P(x) \, d\mu(x) = \int_X P(x) \, d\mu(x).$$

Set $\mathcal{K} = L^2(X, \ell^2(G))$ and define the unitary representation

$$\pi : G \to \mathcal{U}(\mathcal{K}) : (\pi(g)\xi)(x) = \sqrt{\omega(g^{-1}, x)} \, \lambda_g \xi(g^{-1} \cdot x) \, .$$

We view ξ_n as a sequence of unit vectors in $\mathcal{K}$ and find that

$$\lim_n \langle \pi(g) \xi_n, \xi_n \rangle = \lim_n \langle \xi_n, \pi(g^{-1}) \xi_n \rangle = \int_X \sqrt{\omega(g, x)} \, d\mu(x) \, .$$

It follows that

$$\sum_{g \in \mathcal{F}} \int_X \sqrt{\omega(g, x)} \, d\mu(x) \leq \Big\| \sum_{g \in \mathcal{F}} \pi(g) \Big\| \, .$$

We identify $\mathcal{K} \cong L^2(X, \mu) \otimes \ell^2(G)$ and define the unitary

$$U : \mathcal{K} \to \mathcal{K}, \quad U(f \otimes \delta_g) = \big(\sqrt{\omega(g, \,\cdot\,)} f(g \,\cdot\,) \big) \otimes \delta_g$$

to obtain

$$\pi(g) = U^*(\mathrm{id} \otimes \lambda_g)U \quad \text{for all } g \in G.$$

Therefore, π is unitarily equivalent with a multiple of the regular representation of G and hence

$$\Big\|\sum_{g\in\mathcal{F}} \pi(g)\Big\| = \Big\|\sum_{g\in\mathcal{F}} \lambda_g\Big\|,$$

so the proposition is proved. □

1.8 Nonsingular Bernoulli actions

In this section, we will introduce the specific class of actions this thesis is concerned with, the *Bernoulli shifts* or *Bernoulli actions*. Let us fix the following notation. As before, G will always denote a countable infinite discrete group. Moreover, X_0 will denote a standard Borel space, I will denote a countable infinite set and $(\mu_i)_{i\in I}$ will denote a family of probability measures on X_0, so that we can consider the product probability space

$$(X,\mu) = \prod_{i\in I}(X_0,\mu_i).$$

Assume further that $G \curvearrowright I$ is a free action, that is to say for all $g \in G\backslash\{e\}$, we have $gi \neq i$ for all $i \in I$.

Definition 1.8.1. The action $G \curvearrowright (X,\mu)$ given by

$$(gx)_i = (x_{g^{-1}i}), \quad g \in G,\ i \in I,\ x \in X_0,$$

is called the *generalized Bernoulli action* of G on (X,μ) (with respect to $G \curvearrowright I$). If $I = G$ and $G \curvearrowright G$ is given by left multiplication, we simply speak of the *Bernoulli action* of G on (X,μ).

In almost all situations the base space X_0 will be the two point set $\{0,1\}$, a notable exception being Theorem 2.1.1. Moreover, our true interest in this thesis lies in non-generalized Bernoulli actions rather than in generalized ones. However, since we will often restrict Bernoulli actions of G to a subgroup $\Lambda < G$, considering the latter ones as well comes in handy at times.

1.8.1 Kakutani's nonsingularity criterion

In this subsection we will present an explicit criterion to determine whether a (generalized) Bernoulli action $G \curvearrowright (X,\mu) = \prod_{i\in I}(X_0,\mu_i)$ is nonsingular. In

other words, we need to determine when the product probabilities μ and $g_*\mu$ for $g \in G$ are equivalent.
Let

$$\nu = \prod_{j\in J} \nu_j, \quad \nu' = \prod_{j\in J} \nu'_j$$

be two product probability measures on the product space $Y = \prod_{j\in J} Y_j$, where J is a countable infinite set and $(Y_j)_{j\in J}$ is a family of Borel spaces. We observe first that if $\nu \sim \nu'$, then we must have

$$\nu_j \sim \nu'_j \quad \text{for all } j \in J. \tag{1.8.1}$$

The following satisfying answer to the question whether two product probabilities are equivalent, is due to Kakutani.

Theorem 1.8.2 ([Ka48])**.** *Assume that 1.8.1 holds and for every $j \in J$ and choose another probability measure κ_j on Y_j such that $\nu_j \sim \nu'_j \sim \kappa_j$. Set*

$$\xi_j = \sqrt{\frac{d\nu_j}{d\kappa_j}}, \quad \xi'_j = \sqrt{\frac{d\nu'_j}{d\kappa_j}} \in L^2(Y_j, \kappa_j).$$

Then, the infinite product probabilities ν, ν' are equivalent if and only if

$$\sum_{j\in J} \|\xi_j - \xi'_j\|_2^2 < \infty. \tag{1.8.2}$$

For our purposes, the following corollary of Kakutani's theorem is central.

Corollary 1.8.3. *Let $X_0 = \{0,1\}, (X,\mu) = \prod_{i\in I}(X_0, \mu_i)$ and $0 < \mu_i(0) < 1$ for all $i \in I$.The generalized Bernoulli action $G \curvearrowright (X,\mu)$ (with respect to $G \curvearrowright I$) is nonsingular if and only if*

$$\sum_{i\in I}\left(\sqrt{\mu_{gi}(0)} - \sqrt{\mu_i(0)}\right)^2 + \sum_{i\in I}\left(\sqrt{\mu_{gi}(1)} - \sqrt{\mu_i(1)}\right)^2 < \infty\,. \tag{1.8.3}$$

If, in addition there exists $\delta > 0$ such that $\delta \le \mu_i(0) \le 1-\delta$ for all $i \in I$, condition 1.8.3 is equivalent to

$$\sum_{i\in G}(\mu_{gi}(0) - \mu_i(0))^2 < \infty. \tag{1.8.4}$$

Proof. The first assertion is a direct consequence of Theorem 1.8.2, if, for $g \in G$, we put $\nu = \mu, \nu' = g_*\mu$ and $\kappa_j(0) = 1/2$ for all $j \in J = I$ in the statement of the theorem. The second assertion follows from the inequalities

$$(x-y)^2 \leq 2\Big((\sqrt{x}-\sqrt{y})^2 + (\sqrt{1-x}-\sqrt{1-y})^2\Big), \qquad \forall x,y \in [0,1]$$

$$(x-y)^2 \geq 2\delta^2\Big((\sqrt{x}-\sqrt{y})^2 + (\sqrt{1-x}-\sqrt{1-y})^2\Big), \quad \forall x,y \in [\delta, 1-\delta].$$

□

Proposition 1.8.4. *Let* $X_0 = \{0,1\}, (X,\mu) = \prod_{i\in I}(X_0,\mu_i)$ *and* $0 < \mu_i(0) < 1$ *for all* $i \in I$.

(i) The measure μ *is nonatomic if and only if*

$$\sum_{i\in I} \min\{\mu_i(0), \mu_i(1)\} = +\infty. \tag{1.8.5}$$

(ii) Let $G \curvearrowright (X,\mu)$ *be a nonsingular generalized Bernoulli action. If there exists* $\delta > 0$ *such that* $\delta \leq \mu_i(0) \leq 1-\delta$ *for all* $i \in I$*, then* $G \curvearrowright (X,\mu)$ *is essentially free.*

The first part of the Proposition follows from the following easy lemma.

Lemma 1.8.5. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers* $0 \leq a_n < 1$*. The infinite sum* $\sum_{n=1}^\infty \log(1-a_n)$ *converges if and only if the infinite sum* $\sum_{n=1}^\infty a_n$ *converges.*

Proof. Clearly, we may assume that $a_n \to 0$ as $n \to \infty$. In this case, since $\lim_{x\to 0} \frac{\log(1-x)}{-x} = 1$, we can choose $m \in \mathbb{N}$ such that $\log(1-a_n) > -\frac{a_n}{2}$ for all $n \geq m$. Hence $\sum_{n=1}^\infty a_n < \infty$ implies $\sum_{n=1}^\infty \log(1-a_n) > -\infty$. On the other hand, it follows from $x \leq \exp(x-1)$ for all $x \in \mathbb{R}$ that

$$\sum_{n=1}^\infty \log(1-a_n) \leq -\sum_{n=1}^\infty a_n. \tag{1.8.6}$$

□

Proof of Proposition 1.8.4. (i) If μ is nonatomic, it follows in particular that $\mu(\{x\}) = 0$ for all $x = (x_i)_{i\in I} \in X$. By Lemma 1.8.5, this is equivalent to $\sum_{i\in I}(1-\mu_i(x_i)) = \infty$ for all $x = (x_i)_{i\in I} \in X$, which is equivalent to

$$\sum_{i\in I} \min\{\mu_i(0), \mu_i(1)\} = +\infty.$$

On the other hand, if $A \subset X$, $\mu(A) > 0$ is an atom, by inductively intersecting with basic cylinders we find a point $x = (x_i)_{i\in I} \in X$ such that $\mu(\{x\}) = \mu(A) > 0$ and by the same argument as before we obtain

$$\sum_{i\in I} \min\{\mu_i(0), \mu_i(1)\} < +\infty.$$

(ii) Let $g \in G\backslash\{e\}$. Since I is infinite and $G \curvearrowright I$ is free, we can inductively construct a sequence $J_1 \subset \cdots \subset J_n \subset J_{n+1} \subset \cdots \subset I$ of finite sets with $|J_n| = n$ and $gJ_n \cap J_n = \emptyset$. By setting $J = \bigcup_{n\in\mathbb{N}} J_n$ we obtain an infinite set $J = \{j_1, j_2, \dots\} \subset I$ such that $gJ \cap J = \emptyset$. Using this set J, we get

$$\begin{aligned}
\mu(\mathrm{Fix}(g)) &\leq \mu(\{x \in \{0,1\}^I,\ x_{g^{-1}j} = x_j\ \forall j \in J\}) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \mu(\{x \in \{0,1\}^I,\ x_{g^{-1}j_k} = x_{j_k}\}) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} (\mu_{g^{-1}j_k}(0)\mu_{j_k}(0) + \mu_{g^{-1}j_k}(1)\mu_{j_k}(1)),
\end{aligned}$$

and since $\mu_{g^{-1}i}(0)\mu_i(0) + \mu_{g^{-1}i}(1)\mu_i(1) \leq 1 - 2(\delta - \delta^2)$ for all $g \in G, i \in I$, we can conclude that the limit equals zero.

$\square$

In [Ka48, Lemma 6], Kakutani also computes the Radon-Nikodym derivative of two equivalent infinite product measures as the infinite product of the Radon-Nikodym derivatives of the component measures. For a nonsingular Bernoulli action $G \curvearrowright (\{0,1\}^I, \prod_{i\in I} \mu_i)$, $I = \{i_1, i_2, \dots\}$, Kakutani's computation implies that the Radon-Nykodim derivative

$$\omega(g, \cdot) = \frac{dg_*\mu}{d\mu} : \{0,1\}^I \to (0, +\infty)$$

is given by the formula

$$\omega(g, x) = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{\mu_{g\cdot i_k}(x_{i_k})}{\mu_{i_k}(x_{i_k})} \quad \text{for a.e. } x \in X. \tag{1.8.7}$$

Moreover, setting $\omega_n(g, x) = \prod_{k=1}^{n} \frac{\mu_{g\cdot i_k}(x_{i_k})}{\mu_{i_k}(x_{i_k})}$, it follows from [Ka48, Lemma 4] that

$$\sqrt{\omega_n(g, \cdot)} \to \sqrt{\omega(g, \cdot)} \quad \text{as } n \to \infty \text{ in } L^2(X, \mu).$$

Chapter 2

Main results

2.1 Groups with trivial first L^2-cohomology

The following theorem says that for groups with vanishing first L^2-cohomology, a nonsingular free Bernoulli action is either probability measure preserving (pmp) or has a dissipative part, and thus, is never of type III.

Theorem 2.1.1. *Let G be a countable infinite group with $H^1(G, \ell^2(G)) = \{0\}$. Assume that $(\mu_g)_{g \in G}$ is a family of probability measures on a standard Borel space X_0. If the Bernoulli action $G \curvearrowright (X, \mu) = \prod_{g \in G}(X_0, \mu_g)$ is nonsingular and free, then there exists a partition $X_0 = Y_0 \sqcup Z_0$ into Borel sets such that, writing $Y = Y_0^G \subset X$, we have*

1. *$\mu(Y) > 0$ and $\mu|_Y \sim \nu^G$ for some probability measure ν on Y_0, so that $G \curvearrowright (Y, \mu|_Y)$ is an ergodic Bernoulli action of type II_1;*
2. *$\sum_{g \in G} \mu_g(Z_0) < \infty$, so that the action $G \curvearrowright (X \setminus Y, \mu)$ is dissipative.*

Note that there are large classes of groups for which $H^1(G, \ell^2(G)) = \{0\}$, so that all their free ergodic nonsingular Bernoulli actions must be of type II_1. As summarized in Facts A.1.7 this holds true for all infinite groups with property (T), for all nonamenable groups that admit an infinite amenable normal subgroup, and for all direct product groups $G = G_1 \times G_2$ with G_1 infinite and G_2 nonamenable.

Proof. Since $G \curvearrowright (X, \mu)$ is nonsingular, all measures μ_g are in the same measure class. We fix a probability measure μ_0 on X_0 such that $\mu_g \sim \mu_0$ for all $g \in G$.

Define the unit vectors $\xi_g \in L^2(X_0, \mu_0)$ given by $\xi_g = \sqrt{d\mu_g/d\mu_0}$. By Theorem 1.8.2, we get that $\sum_{k\in G} \|\xi_{gk} - \xi_k\|_2^2 < \infty$ for all $g \in G$. So, the map

$$c : G \to \ell^2(G) \otimes L^2(X_0, \mu_0) : c_g = \sum_{k\in G} \delta_k \otimes (\xi_k - \xi_{g^{-1}k})$$

is a well defined 1-cocycle.

Write $\mathcal{K} = L^2(X_0, \mu_0)$. Since $H^1(G, \ell^2(G)) = \{0\}$ and G is infinite, the group G is nonamenable. It follows that the inner 1-cocycles form a closed subspace of the space of 1-cocycles $Z^1(G, \ell^2(G) \otimes \mathcal{K})$ equipped with the topology of pointwise convergence, see e.g [Sh04, Proposition 2.4.1]. Fix a sequence of finite rank projections P_n on $\mathcal{K}$ that converge to 1 strongly. Since $H^1(G, \ell^2(G)) = \{0\}$, every $g \mapsto (1 \otimes P_n)c_g$ is an inner 1-cocycle. Since $\lim_n (1 \otimes P_n)c_g = c_g$ for every $g \in G$, it then follows that also c is inner. This means that there exists a $\xi_0 \in \mathcal{K}$ such that

$$\sum_{k\in G} \|\xi_k - \xi_0\|_2^2 < \infty \ . \tag{2.1.1}$$

We get in particular that $\xi_k \to \xi_0$ as $k \to \infty$ in G. So, ξ_0 is positive a.e. and $\|\xi_0\|_2 = 1$. Denote by ν the unique probability measure on X_0 such that $\nu \prec \mu_0$ and $\xi_0 = \sqrt{d\nu/d\mu_0}$. Write $X_0 = Y_0 \sqcup Z_0$ such that $\nu(Z_0) = 0$ and $\nu \sim \mu_0|_{Y_0}$.

Since $\nu(Z_0) = 0$, we have

$$\|\xi_k - \xi_0\|_2^2 \geq \int_{Z_0} \xi_k(x)^2 \, d\mu_0(x) = \mu_k(Z_0) \ .$$

It follows that $\sum_{k\in G} \mu_k(Z_0) < \infty$. Writing $Y = Y_0^G \subset X$, we conclude from Lemma 1.8.5 that $\mu(Y) > 0$. Since $\mu_k|_{Y_0} \sim \nu$ for all $k \in G$, it follows from (2.1.1) and Theorem 1.8.2 that $\mu|_Y \sim \nu^G$. As $G \curvearrowright (X, \mu)$ is essentially free, (Y_0, ν) is not the probability space consisting of one atom.

Write $Z = \{x \in X \mid x_e \in Z_0\}$. It follows that

$$\sum_{k\in G} \mu(k \cdot Z) = \sum_{k\in G} \mu_k(Z_0) < \infty \ .$$

Since $X \setminus Y = \bigcup_{k\in G} k \cdot Z$, it follows from Proposition 1.5.3 that the action $G \curvearrowright (X \setminus Y, \mu)$ is dissipative. □

We will now move forward to construct nonsingular Bernoulli actions of type III for groups with nontrivial first L^2-cohomology. Our constructions will almost exclusively deal with the base space $X_0 = \{0, 1\}$. In view of Corollary 1.8.3 and Proposition 1.8.4 it is useful to make the following assumptions.

Assumptions 2.1.2. Let I be a countable infinite set and let $\Lambda \curvearrowright I$ be a free action. Given any function $F : I \to (0,1)$, we define the product probability space $(X,\mu) = \prod_{i\in I}(\{0,1\},\mu_i)$ where $\mu_i(0) = F(i)$ and we assume the following:

$$\begin{aligned}&\text{there exists a } \delta > 0 \text{ such that } \ \delta \leq F(i) \leq 1-\delta \ \text{ for all } i \in I,\\ &\text{for every } g \in \Lambda, \text{ we have that } \ \sum_{i\in I}(F(g\cdot i) - F(i))^2 < \infty \, .\end{aligned} \tag{2.1.2}$$

Note that under Assumptions 2.1.2 the generalized Bernoulli action $\Lambda \curvearrowright (X,\mu)$ associated to $\Lambda \curvearrowright I$ is nonsingular and essentially free by Corollary 1.8.3 and Proposition 1.8.4. In addition, we can associate with $F : I \to (0,1)$ the 1-cocycle

$$c : \Lambda \to \ell^2(I) : c_g(i) = F(i) - F(g^{-1}\cdot i) \, . \tag{2.1.3}$$

2.2 A conservativity criterion for Bernoulli actions

Recall from Definition 1.5.1 that a nonsingular essentially free action $\Lambda \curvearrowright (X,\mu)$ is called *conservative* if there is no nonnegligible Borel set $A \subset X$ such that all $g \cdot A$, $g \in \Lambda$ are disjoint in a measurable sense.

The key ingredient to prove Theorems 2.3.1 and 2.4.1 is the following criterion to ensure that a Bernoulli action is conservative. The criterion says that it suffices that the 1-cocycle c given by (2.1.3) has logarithmic growth in at least one direction, thus providing an answer to [DL16, Question 10.5]. The second point of the theorem is easier and is a straightforward generalization of [Ko12, Lemma 2.2] to Bernoulli actions of arbitrary countable groups.

Theorem 2.2.1. *Let $\Lambda \curvearrowright I$ be a free action of the countable group Λ on the countable set I and let $F : I \to (0,1)$ be a function satisfying* (2.1.2)*, in particular $\delta \leq F(i) \leq 1-\delta$ for all $i \in I$. Denote by $\Lambda \curvearrowright (X,\mu)$ the associated Bernoulli action and by $c : \Lambda \to \ell^2(I)$ the associated* 1*-cocycle as in* (2.1.3)*.*

1. *If $\sum_{g\in\Lambda}\exp(-\kappa\|c_g\|_2^2) = +\infty$ for some $\kappa > \delta^{-2} + \delta^{-1}(1-\delta)^{-2}$, then the action $\Lambda \curvearrowright (X,\mu)$ is conservative.*
2. *If $\sum_{g\in\Lambda}\exp(-\frac{1}{2}\|c_g\|_2^2) < +\infty$, then the action $\Lambda \curvearrowright (X,\mu)$ is dissipative.*

In particular, if $1/3 \leq F(i) \leq 2/3$ for all $i \in I$ and if $\sum_{g\in\Lambda}\exp(-16\|c_g\|_2^2) = +\infty$, then the action $\Lambda \curvearrowright (X,\mu)$ is conservative.

We will first prove the following basic lemma, that will become helpful in the proof of the theorem.

Lemma 2.2.2. *For all* $0 \leq a, b \leq 1$*, we have*

$$\sqrt{ab} + \sqrt{(1-a)(1-b)} \leq 1 - \frac{1}{2}(b-a)^2. \tag{2.2.1}$$

Moreover, given $\delta \in (0, 1/2]$*, for all* $\delta \leq a, b \leq 1-\delta$*, it follows that*

$$0 \leq \frac{a + 2b - 2ab - b^2}{b^2\,(1-b)^2} \leq \delta^{-2} + \delta^{-1}(1-\delta)^{-2}. \tag{2.2.2}$$

Lastly, for all $1/3 \leq a, b \leq 2/3$*, we have*

$$\sqrt{ab} + \sqrt{(1-a)(1-b)} \geq 1 - \frac{9}{16}(b-a)^2. \tag{2.2.3}$$

Proof. To prove the first inequality 2.2.1, note that by the standard geometric/arithmetic mean estimate we obtain

$$2\sqrt{a(1-a)b(1-b)} \leq a + b - a^2 - b^2.$$

The desired inequality then follows from the computation

$$\begin{aligned} \left(\sqrt{ab} + \sqrt{(1-a)(1-b)}\right)^2 &= 1 + 2ab - a - b + 2\sqrt{a(1-a)b(1-b)} \\ &\leq 1 + 2ab - a^2 - b^2 \\ &\leq 1 - (a-b)^2 + \frac{1}{4}(a-b)^4 \\ &= \left(1 - \frac{1}{2}(b-a)^2\right)^2. \end{aligned}$$

The positivity assertion in the second inequality 2.2.2 is immediate. The computation of the upper bound follows from a short analysis of the function

$$f : [\delta, 1-\delta] \times [\delta, 1-\delta] \to \mathbb{R}, \quad f(x,y) = \frac{x + 2y - 2xy - y^2}{y^2\,(1-y)^2},$$

which yields that f takes its maximal value in $(x,y) = (\delta, 1-\delta)$. Since $f(\delta, 1-\delta) = \delta^{-2} + \delta^{-1}(1-\delta)^{-2}$, we are done.
To prepare for the proof of the last inequality, set $c = \sin^{-1}(\sqrt{1/3}), d = \sin^{-1}(\sqrt{2/3})$. Substituting $a = \sin^2(\alpha), b = \sin^2(\beta)$ and using the identity

$$\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) = \cos(\alpha - \beta),$$

we observe that 2.2.3 is equivalent to

$$1 - \cos(\alpha - \beta) \leq \frac{9}{16}(\sin^2(\alpha) - \sin^2(\beta)) \tag{2.2.4}$$

for all $\alpha, \beta \in [c, d]$. An application of the trigonometric identities

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

then yields

$$2\sin^2\left(\frac{\alpha-\beta}{2}\right) \leq \frac{9}{4}\sin^2(\alpha+\beta)\cos^2\left(\frac{\alpha-\beta}{2}\right)\sin^2\left(\frac{\alpha-\beta}{2}\right).$$

Dividing by the common term on both sides and applying the square root, we arrive at the insight that proving 2.2.3 is equivalent to proving the equality

$$\frac{2\sqrt{2}}{3} \leq \sin(\alpha+\beta)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{2.2.5}$$

for all $\alpha, \beta \in [c, d]$. So, let us consider the function

$$f : [c,d] \times [c,d] \to \mathbb{R},\ \ f(\alpha,\beta) = \sin(\alpha+\beta)\cos\left(\frac{\alpha-\beta}{2}\right).$$

Computing the partial derivatives of f, we see that the only extremal point in the interior of the domain is (α_0, α_0) satisfying $\sin^2(\alpha_0) = 1/2$, and in this case we have $f(\alpha_0, \alpha_0) = 1 \geq \frac{2\sqrt{2}}{3}$. Hence, it suffices to show that $f(\alpha, \beta) \geq \frac{2\sqrt{2}}{3}$ on the boundary of the domain. Since cos is an even function, we only have to show that the boundary functions

$$f_c, f_d : [c,d] \to \mathbb{R},\ \ f_c(\beta) = f(c,\beta),\ \ f_d(\beta) = f(d,\beta)$$

majorize $\frac{2\sqrt{2}}{3}$ everywhere. However, differentiating f_c and f_d together with an application of the trigonometric identities

$$\cos(x)\cos(y) = \frac{1}{2}\Big(\cos(x-y) + \cos(x+y)\Big)$$

$$\sin(x)\sin(y) = \frac{1}{2}\Big(\cos(x-y) - \cos(x+y)\Big)$$

leads to

$$\frac{d^2 f_s}{d^2\beta}(\beta) = -\frac{1}{8}\left(9\sin\left(\frac{s+3\beta}{2}\right) + \sin\left(\frac{3s+\beta}{2}\right)\right) < 0$$

for all $\beta \in (c, d)$, where $s = c, d$. Therefore we can conclude that f takes its minimum in one of the corners of the domain. But we have

$$f(c,d) = f(d,c) > f(c,c) = f(d,d) = \frac{2\sqrt{2}}{3}$$

and the lemma is proven. □

We can now proceed to show the main result of the section.

Proof of Theorem 2.2.1. Denote by $\omega : \Lambda \times X \to (0, +\infty)$ the Radon-Nikodym cocycle given by (1.2.1). Recall from Proposition 1.5.3 that the essentially free nonsingular action $\Lambda \curvearrowright (X, \mu)$ is conservative if and only $\sum_{g \in \Lambda} \omega(g, x) = +\infty$ for a.e. $x \in X$, while it is dissipative if and only if $\sum_{g \in \Lambda} \omega(g, x) < +\infty$ for a.e. $x \in X$.

Write $\kappa_0 = \delta^{-2} + \delta^{-1}(1-\delta)^{-2}$. We start by proving that

$$\int_X \omega(g,x)^{-2}\, d\mu(x) \leq \exp(\kappa_0 \, \|c_g\|_2^2) \quad \text{for all } g \in \Lambda. \tag{2.2.6}$$

To prove (2.2.6), not that for all $0 < a, b < 1$,

$$\frac{a^3}{b^2} + \frac{(1-a)^3}{(1-b)^2} = 1 + \frac{a + 2b - 2ab - b^2}{b^2\,(1-b)^2}\,(a-b)^2$$

and that by the Lemma 2.2.2 we have

$$0 \leq \frac{a + 2b - 2ab - b^2}{b^2\,(1-b)^2} \leq \kappa_0 \quad \text{for all } \; \delta \leq a, b \leq 1 - \delta \; .$$

Fix an enumeration $I = \{i_1, i_2, \ldots\}$ and define the functions

$$\omega_n : \Lambda \times X \to (0, +\infty) : \omega_n(g, x) = \prod_{k=1}^{n} \frac{\mu_{g \cdot i_k}(x_{i_k})}{\mu_{i_k}(x_{i_k})} \; .$$

Fix $g \in \Lambda$. By (1.8.7), we have that $\omega_n(g,x) \to \omega(g,x)$ a.e. $x \in X$. By Fatou's lemma, we get that

$$\begin{aligned}\int_X \omega(g,x)^{-2}\, d\mu(x) &\leq \liminf_n \int_X \omega_n(g,x)^{-2}\, d\mu(x) \\ &= \liminf_n \prod_{k=1}^{n} \Big(\frac{F(i_k)^3}{F(g\cdot i_k)^2} + \frac{(1-F(i_k))^3}{(1-F(g\cdot i_k))^2} \Big) \\ &\leq \liminf_n \prod_{k=1}^{n} \big(1+\kappa_0(F(i_k)-F(g\cdot i_k))^2\big) \\ &\leq \liminf_n \exp\Big(\kappa_0 \sum_{k=1}^{n} (F(i_k)-F(g\cdot i_k))^2\Big) \\ &= \exp\big(\kappa_0 \|c_g\|_2^2\big)\ .\end{aligned}$$

So, (2.2.6) is proved.

Assume that $\kappa > \kappa_0$ and that $\sum_{g\in\Lambda} \exp(-\kappa\|c_g\|_2^2) = +\infty$. We have to prove that $\Lambda \curvearrowright (X,\mu)$ is conservative. Write $\kappa_1 = \frac{1}{2}(\kappa_0+\kappa)$ and $\kappa_2 = \frac{3}{4}\kappa + \frac{1}{4}\kappa_0$. Note that $\kappa_0 < \kappa_1 < \kappa_2 < \kappa$. We claim that there exists an increasing sequence $s_k \in (0,+\infty)$ such that $\lim_k s_k = +\infty$ and

$$\#\big\{g \in \Lambda \mid \|c_g\|_2^2 \leq s_k\big\} \geq \exp(\kappa_2 s_k) \quad \text{for all } k \geq 1. \tag{2.2.7}$$

Define, for every $s \geq 0$,

$$\varphi(s) = \#\big\{g \in \Lambda \mid \|c_g\|_2^2 \leq s\big\}\ .$$

Then,

$$+\infty = \frac{1}{\kappa} \sum_{g\in\Lambda} \exp(-\kappa\|c_g\|_2^2) = \int_0^{+\infty} \varphi(s)\, \exp(-\kappa s)\, ds\ .$$

If there exists an $s_0 \geq 0$ such that $\varphi(s) \leq \exp(\kappa_2 s)$ for all $s \geq s_0$, the integral on the right hand side is finite. So such an s_0 does not exist and the claim is proven. We fix the sequence s_k as in the claim.

Choose finite subsets $\mathcal{F}_k \subset \Lambda$ such that $\|c_g\|_2^2 \leq s_k$ for all $g \in \mathcal{F}_k$ and

$$|\mathcal{F}_k| \in \big[\exp(\kappa_2 s_k) - 1, \exp(\kappa_2 s_k)\big]\ .$$

For every k and every $g \in \mathcal{F}_k$, define

$$\mathcal{U}_{g,k} = \{x \in X \mid \omega(g,x) \leq \exp(-\kappa_1 s_k)\}\ .$$

When $x \in \mathcal{U}_{g,k}$, we have $\omega(g,x)^{-2} \geq \exp(2\kappa_1 s_k)$. It thus follows from (2.2.6) that

$$\mu(\mathcal{U}_{g,k}) \leq \exp((\kappa_0 - 2\kappa_1)s_k)$$

for all k and all $g \in \mathcal{F}_k$. Defining $\mathcal{V}_k = \bigcup_{g \in \mathcal{F}_k} \mathcal{U}_{g,k}$, we get that

$$\mu(\mathcal{V}_k) \leq \exp((\kappa_2 + \kappa_0 - 2\kappa_1)s_k) = \exp(-\varepsilon s_k) \; ,$$

where $\varepsilon = (\kappa - \kappa_0)/4 > 0$. So, $\mu(\mathcal{V}_k) \to 0$ when $k \to \infty$.

When $x \in X \setminus \mathcal{V}_k$, we have $\omega(g,x) \geq \exp(-\kappa_1 s_k)$ for all $g \in \mathcal{F}_k$. Therefore,

$$\sum_{g \in \Lambda} \omega(g,x) \geq |\mathcal{F}_k| \, \exp(-\kappa_1 s_k) \geq \exp((\kappa_2 - \kappa_1)s_k) - 1 \; .$$

Since the right hand side tends to infinity as $k \to \infty$, it follows that $\sum_{g \in \Lambda} \omega(g,x) = +\infty$ for a.e. $x \in X$. So, $\Lambda \curvearrowright (X,\mu)$ is conservative.

To prove the second statement, we claim that

$$\int_X \sqrt{\omega(g,x)} \, d\mu(x) \leq \exp\bigl(-\frac{1}{2} \, \|c_g\|_2^2\bigr) \quad \text{for all } g \in \Lambda. \tag{2.2.8}$$

To prove (2.2.8) we use that

$$\sqrt{ab} + \sqrt{(1-a)(1-b)} \leq 1 - \frac{1}{2}(b-a)^2 \quad \text{for all } 0 \leq a,b \leq 1,$$

by Lemma 2.2.2, to obtain

$$\begin{aligned} \int_X \sqrt{\omega(g,x)} \, d\mu(x) &\leq \liminf_n \prod_{k=1}^n \Bigl(\sqrt{F(g \cdot i_k)F(i_k)} + \sqrt{(1 - F(g \cdot i_k))(1 - F(i_k))} \Bigr) \\ &\leq \liminf_n \exp\Bigl(-\frac{1}{2} \sum_{k=1}^n (F(g \cdot i_k) - F(i_k))^2\Bigr) \\ &= \exp\bigl(-\frac{1}{2} \|c_g\|_2^2\bigr) \; . \end{aligned}$$

Assuming that $\sum_{g \in \Lambda} \exp(-\frac{1}{2}\|c_g\|_2^2) < +\infty$, it follows from (2.2.8) and the monotone convergence theorem that

$$\int_X \Bigl(\sum_{g \in \Lambda} \sqrt{\omega(g,x)} \Bigr) \, d\mu(x) < +\infty \; .$$

So for a.e. $x \in X$, we have $\sum_{g \in \Lambda} \sqrt{\omega(g,x)} < +\infty$ and, a fortiori, $\sum_{g \in \Lambda} \omega(g,x) < \infty$. So, $\Lambda \curvearrowright (X,\mu)$ is dissipative. $\square$

2.3 Groups with positive first L^2-Betti number

We prove that "almost all" groups with positive first L^2-Betti number admit a nonsingular Bernoulli action of type III_1. We actually do not know of any example of a group G with $\beta_1^{(2)}(G) > 0$ that is not covered by the following theorem.

Theorem 2.3.1. *Let G be a countable group with $\beta_1^{(2)}(G) > 0$. Assume that one of the following conditions holds.*

1. *G has at least one element of infinite order.*
2. *G admits an infinite amenable subgroup.*
3. *$\beta_1^{(2)}(G) \geq 1$.*
4. *G is residually finite; or more generally, G admits a finite index subgroup $G_0 < G$ such that $[G : G_0] \geq \beta_1^{(2)}(G)^{-1}$.*

Then G satisfies the assumptions of Lemma 2.3.4 below and thus, G admits a nonsingular Bernoulli action that is essentially free, ergodic, of type III_1 and nonamenable in the sense of Zimmer and that has a weakly mixing Maharam extension.

For completeness, let us recall here that a discrete group G is *residually finite*, if the intersection of all normal finite index subgroups of G is trivial. In particular, any residually finite group contains finite index normal subgroups with arbitrarily high index which justifies statement 4 in the theorem.
Theorem 2.3.1 is deduced from a series of lemmas, the first of which is the following.

Lemma 2.3.2. *Let $G \curvearrowright I$ be a free action of a countable group G on a countable set I and let $F : I \to (0,1)$ be a function satisfying Assumptions 2.1.2 with $\delta = 1/3$. Denote by $G \curvearrowright (X, \mu)$ the associated Bernoulli action, by $\omega : G \times X \to (0, +\infty)$ its Radon-Nikodym cocycle and by $c : G \to \ell^2(I)$ the associated 1-cocycle as in (2.1.3). Then,*

$$\int_X \sqrt{\omega(g,x)}\, d\mu(x) \geq \exp\Big(-\frac{3}{5}\|c_g\|_2^2\Big) \quad \text{for all } g \in G. \tag{2.3.1}$$

Proof. Let $I = \{i_1, i_2, \ldots\}$ be an enumeration of I. Define

$$\omega_n : G \times X \to (0, +\infty) : \omega_n(g,x) = \prod_{k=1}^{n} \frac{\mu_{g \cdot i_k}(x_{i_k})}{\mu_{i_k}(x_{i_k})} .$$

Fix $g \in G$. As pointed out in Section 1.8.1, we know that $\omega_n(g,x) \to \omega(g,x)$ for a.e. $x \in X$ and that $\sqrt{\omega_n(g,\cdot)} \to \sqrt{\omega(g,\cdot)}$ in $L^2(X,\mu)$. Therefore,

$$\int_X \sqrt{\omega(g,x)}\, d\mu(x) = \lim_n \prod_{k=1}^n \left(\sqrt{F(i_k)F(g\cdot i_k)} + \sqrt{(1-F(i_k))(1-F(g\cdot i_k))}\right) . \tag{2.3.2}$$

By Lemma 2.2.2, for all $1/3 \leq a, b \leq 2/3$, we have that

$$\sqrt{ab} + \sqrt{(1-a)(1-b)} \geq 1 - \frac{9}{16}(b-a)^2 .$$

Moreover, for every $0 \leq t \leq 1/16$, we have that $\log(1-t) \geq -(16/15)t$. Since $\frac{9}{16}(b-a)^2$ lies between 0 and 1/16, we get that

$$\log(\sqrt{ab} + \sqrt{(1-a)(1-b)}) \geq -\frac{3}{5}(b-a)^2 .$$

It then follows from (2.3.2) that

$$\int_X \sqrt{\omega(g,x)}\, d\mu(x) \geq \exp\Big(-\frac{3}{5}\sum_{i\in I}(F(i) - F(g\cdot i))^2\Big) = \exp\big(-\frac{3}{5}\|c_g\|_2^2\big) .$$

So (2.3.1) holds and the lemma is proved. $\square$

The next lemma reveals another sufficient condition on our group G for the conclusion of Theorem 2.3.1 to hold. This condition is not quite strong enough yet to include all the conditions presented in Theorem 2.3.1 but it already covers a quite large range of examples. In fact, the essence of the proof of Theorem 2.3.1 is to be found in the proof of the next lemma.

Lemma 2.3.3. *Let G be a countable infinite group. Assume that G admits an infinite subgroup $\Lambda < G$ such that $\beta_1^{(2)}(\Lambda) < \beta_1^{(2)}(G)$. Then G admits a nonsingular Bernoulli action that is essentially free, ergodic, of type III_1 and nonamenable in the sense of Zimmer and that has a weakly mixing Maharam extension.*

Proof. Denote by $\lambda : G \to \mathcal{U}(\ell^2(G))$ the left regular representation. Since $\beta_1^{(2)}(G) > 0$, we have that G is nonamenable and therefore we can fix a finite subset $\mathcal{F} \subset G$ and $\varepsilon_0 > 0$ such that

$$\Big\|\sum_{g\in\mathcal{F}} \lambda_g\Big\| \leq (1-\varepsilon_0)|\mathcal{F}| , \tag{2.3.3}$$

see e.g [BO08, Theorem 2.6.8]. By [PT10, Theorem 2.2], we have that $\beta_1^{(2)}(\Lambda)$ equals the $L(G)$-dimension of $H^1(\Lambda, \ell^2(G))$. So, the kernel of the restriction map $H^1(G,\ell^2(G)) \to H^1(\Lambda,\ell^2(G))$ has positive $L(G)$-dimension. Therefore, we can choose a non-inner 1-cocycle $b : G \to \ell^2(G)$ with the property that $b_g = 0$ for all $g \in \Lambda$.

Denote by $H : G \to \mathbb{C}$ the function given by $H(k) = b_k(k)$ for all $k \in G$. Then, $H(e) = 0$ and

$$b_g(k) = H(k) - H(g^{-1}k) \quad \text{for all } \; g,k \in G \; .$$

Since b vanishes on Λ, the function H is invariant under left translation by Λ. Since b is not identically zero, H is not the zero function. Replacing b by ib if needed, we may assume that the real part $\operatorname{Re} H$ is not identically zero. At the end of the proof, we explain that the 1-cocycle b may be chosen so that $\operatorname{Re} H$ takes at least three different values.

For any fixed $\kappa_1, \kappa_2 > 0$, we define the function

$$\mathcal{F} : \mathbb{R} \to [-\kappa_1,\kappa_2] : \mathcal{F}(t) = \begin{cases} -\kappa_1 & \text{if } t \leq -\kappa_1, \\ t & \text{if } -\kappa_1 \leq t \leq \kappa_2, \\ \kappa_2 & \text{if } t \geq \kappa_2. \end{cases}$$

Note that $|\mathcal{F}(t) - \mathcal{F}(s)| \leq |t-s|$ for all $s,t \in \mathbb{R}$.

We define $K : G \to [-\kappa_1,\kappa_2] : K(k) = \mathcal{F}(\operatorname{Re} H(k))$. Since $\operatorname{Re} H$ takes at least three different values, we can fix $\kappa_1,\kappa_2 > 0$ so that the range of K generates a dense subgroup of $\mathbb{R}$, meaning that there is no $a > 0$ such that $K(k) \in \mathbb{Z}a$ for all $k \in G$. Note that K is invariant under left translation by Λ.

We then fix $\varepsilon_1 > 0$ such that

$$\exp(\varepsilon_1 \kappa_i) \leq 2 \quad \text{for } i = 1,2, \text{ and} \quad \exp\Big(-\frac{3}{5}\,\varepsilon_1^2\,\|b_g\|_2^2\Big) > 1 - \varepsilon_0 \quad \text{for all } g \in \mathcal{F}.$$

Define the function

$$F : G \to [1/3, 2/3] : F(k) = \frac{1}{1 + \exp(\varepsilon_1 K(k))} \; .$$

Associated with F, we have the product probability measure μ on $X = \{0,1\}^G$ given by $\mu = \prod_{k \in G} \mu_k$ with $\mu_k(0) = F(k)$.

For every $g \in G$, we have that

$$\sum_{k\in G} |F(gk) - F(k)|^2 \leq \varepsilon_1^2 \sum_{k \in G} |K(gk) - K(k)|^2 \leq \varepsilon_1^2 \sum_{k\in G} |H(gk) - H(k)|^2 = \varepsilon_1^2\,\|b_g\|_2^2 \; .$$

So, the Bernoulli action $G \curvearrowright (X,\mu)$ is essentially free, nonsingular and the 1-cocycle $c : G \to \ell^2(G)$ given by $c_g(k) = F(k) - F(g^{-1}k)$ satisfies $\|c_g\|_2 \leq \varepsilon_1\,\|b_g\|_2$ for all $g \in G$.

Denote by $\omega : G \times X \to (0, +\infty)$ the Radon-Nikodym cocycle given by (1.2.1) and consider the Maharam extension $G \curvearrowright (X \times \mathbb{R}, \mu \times \nu)$. Let $G \curvearrowright (Y, \eta)$ be any pmp action and consider the diagonal action $G \curvearrowright (Y \times X \times \mathbb{R}, \eta \times \mu \times \nu)$. We prove that $L^\infty(Y \times X \times \mathbb{R})^G = L^\infty(Y)^G \otimes 1 \otimes 1$. Once this statement is proved, it follows from the discussion on mixing properties in Section 1.6 that $G \curvearrowright (X, \mu)$ is ergodic and of type III_1 and that its Maharam extension is weakly mixing.

Since F is invariant under left translation by Λ, we have that $\omega(g, x) = 1$ for all $g \in \Lambda$, $x \in X$ and we have that the action $\Lambda \curvearrowright (X, \mu)$ is isomorphic with a probability measure preserving Bernoulli action of Λ. In particular, it is ergodic. So, a G-invariant (and hence Λ-invariant) function $Q \in L^\infty(Y \times X \times \mathbb{R})$ is of the form $Q(y, x, s) = P(y, s)$ for some $P \in L^\infty(Y \times \mathbb{R})$ satisfying

$$P(g \cdot y, s + \log(\omega(g, x))) = P(y, s) \quad \text{for all } g \in G \text{ and a.e. } (y, x, s) \in Y \times X \times \mathbb{R}.$$

It follows that

$$P(y, s + \log(\omega(g, x)) - \log(\omega(g, x'))) = P(y, s)$$

for all $g \in G$ and a.e. $(y, x, x', s) \in Y \times X \times X \times \mathbb{R}$. For every $g \in G$, denote by R_g the essential range of the map

$$X \times X \to \mathbb{R} : (x, x') \mapsto \log(\omega(g, x)) - \log(\omega(g, x')) \; .$$

To conclude that $P \in L^\infty(Y) \otimes 1$, it suffices to prove that $\bigcup_{g \in G} R_g$ generates a dense subgroup of $\mathbb{R}$. So it suffices to prove that there is no $a > 0$ such that $\log(\omega(g, x)) - \log(\omega(g, x')) \in \mathbb{Z}a$ for all $g \in G$ and a.e. $(x, x') \in X \times X$. Assume the contrary.

Fix $g, k \in G$ and define the measure preserving factor map

$$\pi : (\{0,1\} \times X, \mu_k \times \mu) \to (X, \mu) : (\pi(z, x))_h = \begin{cases} x_h & \text{if } h \neq k, \\ z & \text{if } h = k. \end{cases}$$

By our assumption, $\log(\omega(g, \pi(z, x))) - \log(\omega(g, x)) \in \mathbb{Z}a$ for a.e. $z \in \{0, 1\}$, $x \in X$. Since

$$\log(\omega(g, x)) = \sum_{h \in G} \bigl(\log(\mu_{gh}(x_h)) - \log(\mu_h(x_h))\bigr)$$

with convergence a.e., we find that

$$\begin{aligned}(\omega(g, \pi(z, x))) &- \log(\omega(g, x)) \\ &= \bigl(\log(\mu_{gk}(z)) - \log(\mu_k(z))\bigr) - \bigl(\log(\mu_{gk}(x_k)) - \log(\mu_k(x_k))\bigr)\end{aligned}$$

for all $g \in G$ and a.e. $z \in \{0,1\}$, $x \in X$. Taking $z = 1$ and $x_k = 0$, it follows that

$$\log\Big(\frac{\mu_{gk}(1)}{\mu_{gk}(0)}\Big) - \log\Big(\frac{\mu_k(1)}{\mu_k(0)}\Big) \in \mathbb{Z}a \; .$$

But the left hand side equals $\varepsilon_1(K(gk) - K(k))$. Since $g, k \in G$ were arbitrary and $K(e) = 0$, we conclude that $K(g) \in \mathbb{Z}(a/\varepsilon_1)$ for all $g \in G$, contrary to our choice of K.

So, we have proven that $P \in L^\infty(Y) \otimes 1$ and thus, $Q \in L^\infty(Y) \otimes 1 \otimes 1$. This means that $G \curvearrowright (X, \mu)$ is ergodic, of type III_1 and with weakly mixing Maharam extension.

By Lemma 2.3.2 and (2.3.3), we get that

$$\sum_{g\in\mathcal{F}} \int_X \sqrt{\omega(g,x)}d\mu(x) \geq \sum_{g\in\mathcal{F}} \exp\Big(-\frac{3}{5}\|c_g\|_2^2\Big) \geq \sum_{g\in\mathcal{F}} \exp\Big(-\frac{3}{5}\varepsilon_1^2\|b_g\|_2^2\Big)$$
$$> (1-\varepsilon_0)|\mathcal{F}| \geq \Big\|\sum_{g\in\mathcal{F}} \lambda_g\Big\| \; .$$

So by Proposition 1.7.2, we conclude that the action $G \curvearrowright (X, \mu)$ is nonamenable.

It remains to prove that we may choose a 1-cocycle $c : G \to \ell^2(G)$ with $c_g = 0$ for all $g \in \Lambda$ and such that the associated function $\operatorname{Re} H : G \to \mathbb{R}$, determined by $H(e) = 0$ and $c_g = H - g \cdot H$ for all $g \in G$, takes at least three different values. The space of 1-cocycles $c : G \to \ell^2(G)$ that vanish on Λ is an $L(G)$-module of positive $L(G)$-dimension. It is in particular an infinite dimensional vector space. So we can choose 1-cocycles $c, c' : G \to \ell^2(G)$ that vanish on Λ and such that the associated functions $\operatorname{Re} H : G \to \mathbb{R}$ and $\operatorname{Re} H' : G \to \mathbb{R}$ are $\mathbb{R}$-linearly independent and, in particular, nonzero. If either $\operatorname{Re} H$ or $\operatorname{Re} H'$ takes at least three values, we are done. Otherwise, after multiplying c and c' with nonzero real numbers, we may assume that $\operatorname{Re} H = 1_A$ and $\operatorname{Re} H' = 1_{A'}$, where A, A' are distinct nonempty subsets of G. But then the function $\operatorname{Re} H + 2\operatorname{Re} H'$, associated with the 1-cocycle $c + 2c'$, takes at least three different values.

□

We will now generalize Lemma 2.3.3 further to arrive at a sufficient condition for the existence of a type III Bernoulli action that covers all the conditions of Theorem 2.3.1.

Lemma 2.3.4. *Let G be a countable infinite group. Assume that G admits subgroups $\Lambda < G_0 < G$ such that Λ is infinite, $G_0 < G$ has finite index and $\beta_1^{(2)}(\Lambda) < \beta_1^{(2)}(G_0)$. Then G admits a nonsingular Bernoulli action that is* *essentially free, ergodic, of type III_1 and nonamenable in the sense of Zimmer and that has a weakly mixing Maharam extension.*

Proof. Since $\beta_1^{(2)}(G_0) = [G : G_0]\, \beta_1^{(2)}(G)$, we also have that $\beta_1^{(2)}(G) > 0$. So if Λ is amenable, we have $\beta_1^{(2)}(\Lambda) = 0 < \beta_1^{(2)}(G)$ and we can apply Lemma 2.3.3. So we may assume that Λ is nonamenable.

Choose a finite subset $\mathcal{F} \subset G$ and $\varepsilon_0 > 0$ such that (2.3.3) holds. Since $\beta_1^{(2)}(\Lambda) < \beta_1^{(2)}(G_0)$, we can proceed exactly as in the the proof of Lemma 2.3.3 and find $\kappa_1, \kappa_2 > 0$ and a function $K : G_0 \to [-\kappa_1, \kappa_2]$ satisfying the following properties.

- The range of K generates a dense subgroup of $\mathbb{R}$.
- K is invariant under left translation by Λ.
- Writing $c_g(k) = K(k) - K(g^{-1}k)$ for all $g, k \in G_0$, we have that $c_g \in \ell^2(G_0)$ for all $g \in G_0$.

Write $G = \sqcup_{i=1}^{\kappa} g_i G_0$. Define

$$F : G \to [-\kappa_1, \kappa_2] : F(g_i h) = K(h) \quad \text{for all } i \in \{1, \ldots, \kappa\} \text{ and } h \in G_0.$$

For every $g, h \in G$, define $b_g(h) = F(h) - F(g^{-1}h)$. By construction, $b_g \in \ell^2(G)$ for every $g \in G$ and $G \to \ell^2(G) : g \mapsto b_g$ is a cocycle. Note however that b need not vanish on Λ.

For every $i \in \{1, \ldots, \kappa\}$, define the nonamenable group $\Lambda_i = g_i \Lambda g_i^{-1}$. By Schoenberg's theorem (see e.g. [BO08, Theorem D.11]), for every $\varepsilon > 0$ and $i \in \{1, \ldots, \kappa\}$, the map

$$\varphi_{\varepsilon,i} : \Lambda_i \to \mathbb{R} : h \mapsto \exp(-8\,\varepsilon^2\, \|b_h\|_2^2)$$

is a positive definite function on Λ_i. When $\varepsilon \to 0$, we get that $\varphi_{\varepsilon,i} \to 1$ pointwise. Since Λ_i is nonamenable, it follows that $\varphi_{\varepsilon,i} \notin \ell^2(\Lambda_i)$ for ε small enough. So we can choose $\varepsilon_1 > 0$ small enough such that

$$\exp(\varepsilon_1 \kappa_i) \leq 2 \quad \text{for } i = 1, 2, \quad \exp\Bigl(-\frac{3}{5}\,\varepsilon_1^2\, \|b_g\|_2^2\Bigr) > 1 - \varepsilon_0 \quad \text{for all } g \in \mathcal{F}, \text{ and}$$

$$\sum_{h \in \Lambda_i} \exp\bigl(-16\,\varepsilon_1^2\, \|b_h\|_2^2\bigr) = +\infty \quad \text{for all } i \in \{1, \ldots, \kappa\}. \tag{2.3.4}$$

For every $g \in G$, define the probability measure μ_g on $\{0, 1\}$ given by

$$\mu_g(0) = \frac{1}{1 + \exp(\varepsilon_1 F(g))}\ .$$

Note that $\mu_g(0) \in [1/3, 2/3]$ for all $g \in G$. Defining $d_g(h) = \mu_h(0) - \mu_{g^{-1}h}(0)$, we find that $\|d_g\|_2 \leq \varepsilon_1 \|b_g\|_2$. So, $d_g \in \ell^2(G)$ and the Bernoulli action $G \curvearrowright (X, \mu) = \prod_{g \in G}(\{0,1\}, \mu_g)$ is nonsingular and essentially free.

Choose an arbitrary pmp action $G \curvearrowright (Y, \eta)$ and consider the diagonal action $G \curvearrowright Y \times X \times \mathbb{R}$ of $G \curvearrowright Y$ and the Maharam extension $G \curvearrowright X \times \mathbb{R}$. Let $Q \in L^\infty(Y \times X \times \mathbb{R})$ be G-invariant. We have to prove that $Q \in L^\infty(Y) \otimes 1 \otimes 1$.

For every subset $J \subset G$, define $(X_J, \mu_J) = \prod_{g \in J}(\{0,1\}, \mu_g)$ and view $L^\infty(X_J) \subset L^\infty(X)$. Fix $i \in \{1, \ldots, \kappa\}$. We prove that

$$Q \in L^\infty(Y \times X_{G \setminus g_i G_0} \times \mathbb{R}).$$

Since the map $K : G_0 \to \mathbb{R}$ is Λ-invariant, we get that $\Lambda_i \curvearrowright (X_{g_i G_0}, \mu_{g_i G_0})$ is a pmp Bernoulli action. By (2.3.4), the inequality $\|d_g\|_2 \leq \varepsilon_1 \|b_g\|_2$ and Proposition 2.2.1, the action $\Lambda_i \curvearrowright X$ is conservative. By Proposition 1.5.3, this means that $\sum_{g \in \Lambda_i} \omega(g, x) = +\infty$ for a.e. $x \in X$, so that also the diagonal action $\Lambda_i \curvearrowright Y \times X$ is conservative. A fortiori, the factor action $\Lambda_i \curvearrowright Y \times X_{G \setminus g_i G_0}$ is conservative and therefore its Maharam extension $\Lambda_i \curvearrowright Y \times X_{G \setminus g_i G_0} \times \mathbb{R}$ is also conservative by Maharam's theorem 1.5.4. Since we can view $\Lambda_i \curvearrowright Y \times X \times \mathbb{R}$ as the diagonal product of $\Lambda_i \curvearrowright Y \times X_{G \setminus g_i G_0} \times \mathbb{R}$ and the mixing pmp action $\Lambda_i \curvearrowright X_{g_i G_0}$, it follows from the discussion on mildly mixing actions in Section 1.6 that the Λ_i-invariant functions in $L^\infty(Y \times X \times \mathbb{R})$ belong to $L^\infty(Y \times X_{G \setminus g_i G_0} \times \mathbb{R})$. So, $Q \in L^\infty(Y \times X_{G \setminus g_i G_0} \times \mathbb{R})$.

Since this holds for every $i \in \{1, \ldots, \kappa\}$, it follows that $Q \in L^\infty(Y) \overline{\otimes} 1 \overline{\otimes} L^\infty(\mathbb{R})$. We now proceed as in the proof of Lemma 2.3.3. Since the range of K generates a dense subgroup of $\mathbb{R}$, the same holds for F and we conclude that $Q \in L^\infty(Y) \otimes 1 \otimes 1$.

The fact that $G \curvearrowright (X, \mu)$ is nonamenable in the sense of Zimmer follows exactly as in the proof of Lemma 2.3.3. □

We now deduce Theorem 2.3.1 from Lemma 2.3.4 by proving that a group satisfying the assumptions of Theorem 2.3.1 automatically admits subgroups $\Lambda < G_0 < G$ as in Lemma 2.3.4.

Proof of Theorem 2.3.1. Let G be a countable group with $\beta_1^{(2)}(G) > 0$, satisfying one of the properties in 1–4. Since $\mathbb{Z}$ is amenable, case 1 follows from case 2. In case 2, if $\Lambda < G$ is an infinite amenable group, we have $\beta_1^{(2)}(\Lambda) = 0 < \beta_1^{(2)}(G)$ and taking $G_0 = G$, the assumptions of Lemma 2.3.4 or even Lemma 2.3.3 are satisfied.

Case 3 follows from case 4 by taking $G_0 = G$. So it remains to prove the theorem in case 4, i.e. in the presence of a finite index subgroup $G_0 < G$ with

$[G : G_0] \geq \beta_1^{(2)}(G)^{-1}$. Then,

$$\beta_1^{(2)}(G_0) = [G : G_0]\, \beta_1^{(2)}(G) \geq 1.$$

Since we already proved the theorem in cases 1 and 2, we may assume that G_0 is a torsion group without infinite amenable subgroups. We claim that there exist $a, b \in G_0$ such that the subgroup $\Lambda = \langle a, b\rangle$ generated by a and b is infinite. Indeed, if all two elements $a, b \in G_0$ generate a finite subgroup, it follows from [St66, Theorem 7] that G_0 contains an infinite abelian subgroup, contrary to our assumptions. So the claim is proved and we fix $a, b \in G_0$ generating an infinite subgroup $\Lambda = \langle a, b\rangle$.

We prove that $\beta_1^{(2)}(\Lambda) < 1$. Since $\beta_1^{(2)}(G_0) \geq 1$, the subgroups $\Lambda < G_0 < G$ then satisfy the assumptions of Lemma 2.3.4. Assume that a has order n and b has order m. We note that any cocycle $\gamma : \Lambda \to \ell^2(\Lambda)$ is cohomologous to a cocycle that vanishes on the finite subgroup generated by a. To see this, implement γ by a function $H : \Lambda \to \mathbb{C}$ such that $\gamma_g(k) = H(gk) - H(k),\ g, k \in \Lambda$ and average H by the finite subgroup $\Gamma = \langle a\rangle$ to obtain a new Γ-invariant function

$$H' : \Lambda \to \mathbb{C}, \quad H'(k) = \frac{1}{n}\sum_{i=0}^{n} H(a^i k).$$

Since Γ is finite, we have $H' - H \in \ell^2(\Lambda)$ and therefore the cocycle generated by H' is cohomologous to γ. In particular this means that γ is entirely determined by its value on b and thus we find that

$$\begin{aligned}\beta_1^{(2)}(\Lambda) = \dim_{L(\Lambda)}\big(\{\xi \in \ell^2(\Lambda) \mid \text{there is a 1-cocycle } \gamma : \Lambda \to \ell^2(\Lambda) \text{ with}\\ \gamma_a = 0 \text{ and } \gamma_b = \xi\ \}\big)\\ - \dim_{L(\Lambda)}\big(\{\eta - b\cdot\eta \mid \eta \in \ell^2(\Lambda)\ ,\ a\cdot\eta = \eta\}\big)\ .\end{aligned}$$

The first term is bounded by 1. Because Λ is infinite and a has order n, the second term equals

$$\dim_{L(\Lambda)}\big(\{\eta \in \ell^2(\Lambda) \mid a\cdot\eta = \eta\}\big) = \frac{1}{n}\ .$$

So, $\beta_1^{(2)}(\Lambda) \leq 1 - 1/n < 1$. This concludes the proof of the theorem. □

2.4 Amenable groups

We now turn our attention away from nonamenable groups and towards amenable ones. For a reminder of the (or rather a) definition of an amenable group, we refer the reader to appendix A. We would also like to recall at this point that for any infinite amenable group G, we always have $H^1(G, \ell^2(G)) \neq \{0\}$, see A.1.7. The following theorem is the main result of this section.

Theorem 2.4.1. *Let G be an amenable countable infinite group. Then G admits a nonsingular Bernoulli action $G \curvearrowright (X, \mu) = \prod_{g \in G}(X_0, \mu_g)$ that is essentially free, ergodic and of type III_1 and that has a weakly mixing Maharam extension.*

In the following two cases, we can choose as a base space X_0 the two point set $\{0, 1\}$:

- *when G has at least one element of infinite order;*
- *when G admits an infinite subgroup of infinite index.*

The only amenable groups G that do not satisfy any of the extra assumptions in Theorem 2.4.1 are the amenable torsion groups with the property that every subgroup is either finite or of finite index. While it is unknown whether there are finitely generated groups of this kind, the locally finite Prüfer p-groups, for p prime, given as the direct limit of the finite groups $\mathbb{Z}/p^n\mathbb{Z}$, have the property that every proper subgroup is finite. We do not know whether these groups admit a nonsingular Bernoulli action of type III with base space $X_0 = \{0, 1\}$.

In order to prove Theorem 2.4.1, we will first need to develop some machinery. We will do so in the following three subsections before presenting the proof of the theorem in Section 2.4.4.

2.4.1 Determining the type: removing inessential subsets of I

Fix a countable infinite group Λ acting freely on a countable set I and fix a function $F : I \to (0, 1)$ satisfying Assumptions 2.1.2. Define the probability measures μ_i on $\{0, 1\}$ given by $\mu_i(0) = F(i)$. Denote by $\Lambda \curvearrowright (X, \mu) = \prod_{i \in I}(\{0, 1\}, \mu_i)$ the associated Bernoulli action with Radon-Nikodym cocycle $\omega : \Lambda \times X \to (0, +\infty)$ given by (1.8.7) and Maharam extension $\Lambda \curvearrowright (X \times \mathbb{R}, \mu \times \nu)$. Fix an arbitrary pmp action $\Lambda \curvearrowright (Y, \eta)$ and consider the diagonal action $\Lambda \curvearrowright (Y \times X \times \mathbb{R}, \eta \times \mu \times \nu)$.

For every subset $J \subset I$, we consider

$$(X_J, \mu_J) = \prod_{j \in J}(\{0, 1\}, \mu_j).$$

We denote by $x \mapsto x_J$ the natural measure preserving factor map $(X, \mu) \to (X_J, \mu_J)$. Given $0 < \lambda < 1$, we denote by ν_λ the probability measure on $\{0, 1\}$ given by $\nu_\lambda(0) = \lambda$. We also use the notation

$$\varphi_{\lambda,i} : \{0,1\} \to \mathbb{R} : \varphi_{\lambda,i}(x) = \log \frac{\mu_i(x)}{\nu_\lambda(x)} = \begin{cases} \log(F(i)) - \log(\lambda) & \text{if } x = 0, \\ \log(1 - F(i)) - \log(1 - \lambda) & \text{if } x = 1. \end{cases} \tag{2.4.1}$$

We introduce the following ad hoc terminology.

Definition 2.4.2. Given $0 < \lambda < 1$, we call a subset $J \subset I$ λ-*inessential* if the following two conditions hold.

1. $\mu_j = \nu_\lambda$ for all but finitely many $j \in J$.
2. For every Λ-invariant function $Q \in L^\infty(Y \times X \times \mathbb{R})^\Lambda$, there exists a $P \in L^\infty(Y \times X_{I \setminus J} \times \mathbb{R})$ such that $\|P\|_\infty \leq \|Q\|_\infty$ and
$$Q(y, x, s) = P\Big(y, x_{I \setminus J}\ ,\ s - \sum_{j \in J} \varphi_{\lambda,j}(x_j)\Big) \quad \text{for a.e. } (y, x, s) \in Y \times X \times \mathbb{R}\ .$$

Note that the sum over $j \in J$ is actually a finite sum since $\varphi_{\lambda,j}$ is the zero map for all but finitely many $j \in J$ by condition 1. The terminology "inessential" is motivated by the fact that these subsets "do not contribute" to the type of the action $\Lambda \curvearrowright (X, \mu)$.

Assuming that condition 1 holds, we can reformulate condition 2 in the following way. Denote by $\mu' \sim \mu$ the measure given by $\mu'_i = \mu_i$ for all $i \in I \setminus J$ and $\mu'_j = \nu_\lambda$ for all $j \in J$ and consider the diagonal action

$$\Lambda \curvearrowright (Y \times X \times \mathbb{R}, \eta \times \mu' \times \nu)$$

of $\Lambda \curvearrowright (Y, \eta)$ and the Maharam extension for $\Lambda \curvearrowright (X, \mu')$. Then, condition 2 is equivalent to the condition

2'. Every Λ-invariant function in $L^\infty(Y \times X \times \mathbb{R})^\Lambda$ w.r.t. the above action belongs to $L^\infty(Y \times X_{I \setminus J} \times \mathbb{R})$.

In particular, it follows that the union of two inessential subsets is again inessential.

We provide two criteria for subsets $J \subset I$ to be inessential.

Proposition 2.4.3. *Assume that $\Lambda \curvearrowright (X, \mu)$ is conservative (see Section 1.5). Let $0 < \lambda < 1$. If $i_0 \in I$ is such that $F(g \cdot i_0) = \lambda$ for all but finitely many $g \in \Lambda$, then $\Lambda \cdot i_0 \subset I$ is λ-inessential.*

Proof. Write $J = \Lambda \cdot i_0$ and replace μ by the equivalent measure μ' satisfying $\mu'_j = \nu_\lambda$ for all $j \in J$. We have to prove that every Λ-invariant function $Q \in L^\infty(Y \times X \times \mathbb{R})$ for the diagonal product of the fixed pmp action $\Lambda \curvearrowright (Y, \eta)$ and the Maharam extension $\Lambda \curvearrowright (X \times \mathbb{R}, \mu' \times \nu)$ belongs to $L^\infty(Y \times X_{I \setminus J} \times \mathbb{R})$.

Since a nonsingular action $\Lambda \curvearrowright (X, \mu)$ is conservative if and only if $\sum_{g \in \Lambda} \omega(g, x) = +\infty$ for a.e. $x \in X$ by Proposition 1.5.3, it follows that the diagonal action $\Lambda \curvearrowright (Y \times X, \eta \times \mu')$ is conservative as well. Note that $\Lambda \curvearrowright (X_J, \mu_J)$ is a probability measure preserving Bernoulli action and that $\Lambda \curvearrowright Y \times X \times \mathbb{R}$ can be viewed as the product of the action $\Lambda \curvearrowright Y \times X_{I \setminus J} \times \mathbb{R}$ and the action $\Lambda \curvearrowright X_J$. The action $\Lambda \curvearrowright (Y \times X_{I \setminus J}, \eta \times \mu_{I \setminus J})$ is a factor of the action $\Lambda \curvearrowright (Y \times X, \eta \times \mu)$ and is therefore conservative. By Maharam's theorem 1.5.4, also its Maharam extension $\Lambda \curvearrowright (Y \times X_{I \setminus J} \times \mathbb{R}, \eta \times \mu_{I \setminus J} \times \nu)$ is conservative. Since the probability measure preserving Bernoulli action $\Lambda \curvearrowright (X_J, \mu_J)$ is mixing and therefore mildly mixing, it follows from the third characterization in 1.6.1 that the Λ-invariant functions in $L^\infty(Y \times X \times \mathbb{R})$ belong to $L^\infty(Y \times X_{I \setminus J} \times \mathbb{R})$. □

Our next criterion for being inessential is a consequence of Lemma 1.6.5 on $\mathbb{Z}$-invariant functions.

Proposition 2.4.4. *Assume that $\Lambda = \mathbb{Z}$ and let $0 < \lambda < 1$. Assume that $\mathbb{Z} \curvearrowright (X, \mu)$ is conservative. If $i_0 \in I$ is such that $F(n \cdot i_0) = \lambda$ for all $n \geq 0$, then $\{n \cdot i_0 \mid n \geq n_0\}$ is λ-inessential for every $n_0 \in \mathbb{Z}$.*

Similarly, if $i_0 \in I$ such that $F(n \cdot i_0) = \lambda$ for all $n \leq 0$, then $\{n \cdot i_0 \mid n \leq n_0\}$ is λ-inessential for every $n_0 \in \mathbb{Z}$.

Proof. By symmetry, it suffices to prove the first statement. Fix $n_0 \in \mathbb{Z}$. Replace i_0 by $n_0 \cdot i_0$ and replace μ by the equivalent measure μ' satisfying $\mu'_j = \nu_\lambda$ for all $j \in J := \{n \cdot i_0 \mid n \geq 0\}$. Write $J' = I \setminus J$. We have to prove that every $\mathbb{Z}$-invariant function $Q \in L^\infty(Y \times X \times \mathbb{R})$ for the diagonal product of the fixed pmp action $\mathbb{Z} \curvearrowright (Y, \eta)$ and the Maharam extension $\mathbb{Z} \curvearrowright (X \times \mathbb{R}, \mu' \times \nu)$ belongs to $L^\infty(Y \times X_{J'} \times \mathbb{R})$.

Denote by

$$T : Y \times X \times \mathbb{R} \to Y \times X \times \mathbb{R} : T(y, x, s) = 1 \cdot (y, x, s) = (1 \cdot y, 1 \cdot x, \log(\omega(1, x)) + s)$$

the $\eta \times \mu \times \nu$-preserving transformation given by $1 \in \mathbb{Z}$. Define the $\eta \times \mu_{J'} \times \nu$-preserving endomorphism

$$T_0 : Y \times X_{J'} \times \mathbb{R} \to Y \times X_{J'} \times \mathbb{R} : T_0(y, x, s) = \ (1 \cdot y, x', \log(\omega(1, x)) + s),$$

where $x'_i = x_{(-1)\cdot i}$ for all $i \in J'$, which is well defined because $x \mapsto \omega(1,x)$ factors through $X_{J'}$ by (1.8.7).

As in the proof of Proposition 2.4.3, since $\mathbb{Z} \curvearrowright (X,\mu)$ is conservative, also the diagonal action $\mathbb{Z} \curvearrowright (Y \times X, \eta \times \mu')$ is conservative. So, the Maharam extension T is conservative as well. Applying Lemma 1.6.5 as in the discussion before the proposition to the natural, measure preserving factor map $Y \times X \times \mathbb{R} \to Y \times X_{J'} \times \mathbb{R}$, it follows that the Λ-invariant functions in $L^\infty(Y \times X \times \mathbb{R})$ belong to $L^\infty(Y \times X_{J'} \times \mathbb{R})$. □

2.4.2 Determining the type: reduction to the tail

Fix a countable infinite group Λ acting freely on a countable set I and fix a function $F : I \to (0,1)$ satisfying Assumptions 2.1.2. Denote by $\Lambda \curvearrowright (X,\mu)$ the associated Bernoulli action.

Proposition 2.4.5. *Assume that* $0 < \lambda < 1$ *such that*

$$\lim_{i\to\infty} F(i) = \lambda \quad \textit{and} \quad \sum_{i\in I} (F(i) - \lambda)^2 = +\infty \, .$$

Assume further that there exists a sequence of λ-inessential subsets $J_n \subset I$ (see Section 2.4.1) such that $\bigcup_n J_n = I$. Then, $\Lambda \curvearrowright (X,\mu)$ is ergodic and of type III_1, and has a weakly mixing Maharam extension.

The proof of the proposition is based on the following result due to Danilenko and Lemańczyk, see [DL16, Proposition 1.5], on the *tail equivalence relation* $\mathcal{R}$ on (X,μ). This relation $\mathcal{R}$ is given by $(x,x') \in \mathcal{R}$ if and only if $x_i = x'_i$ for all but finitely many $i \in I$.

Lemma 2.4.6 ([DL16])**.** *The* 1*-cocycle*

$$\alpha : \mathcal{R} \to \mathbb{R} : \alpha(x,x') = \sum_{i\in I} (\varphi_{\lambda,i}(x_i) - \varphi_{\lambda,i}(x'_i))$$

on the tail equivalence relation $\mathcal{R}$ is ergodic, that is to say the associated skew product $\mathcal{R}(\alpha)$ is an ergodic equivalence relation.

Proof of Proposition 2.4.5. Enumerate $I = \{i_1, i_2, \ldots\}$. Define $I_n = \{i_1, \ldots, i_n\}$. Since the union of two λ-inessential subsets is inessential and since subsets of λ-inessential sets are again λ-inessential, it follows that I_n is λ-inessential for every n. Write $I'_n = I \setminus I_n$.

Fix a pmp action $\Lambda \curvearrowright (Y,\eta)$. We have to prove that every Λ-invariant element $Q \in L^\infty(Y \times X \times \mathbb{R})$ for the diagonal product of $\Lambda \curvearrowright (Y,\eta)$ and the Maharam extension $\Lambda \curvearrowright (X \times \mathbb{R}, \mu \times \nu)$ belongs to $L^\infty(Y) \otimes 1 \otimes 1$. Using Definition 2.4.2, we find $Q_n \in L^\infty(Y \times X_{I'_n} \times \mathbb{R})$ with $\|Q_n\|_\infty \leq \|Q\|_\infty$ for all n and

$$Q(y,x,s) = Q_n\Big(y, x_{I'_n}\ ,\ s - \sum_{j \in I_n} \varphi_{\lambda,j}(x_j)\Big) \quad \text{for a.e. } (x,s) \in X \times \mathbb{R}\ . \qquad (2.4.2)$$

Define $S_n \in L^\infty(Y \times X_{I_n} \times \mathbb{R})$ as the conditional expectation of Q onto $L^\infty(Y \times X_{I_n} \times \mathbb{R})$. By the martingale convergence theorem, we have that

$$S_n(y,x,s) \to Q(y,x,s) \quad \text{for a.e. } (x,s) \in X \times \mathbb{R}.$$

Define $P_n \in L^\infty(Y \times \mathbb{R})$ such that $(P_n)_{13}$ is the conditional expectation of Q_n onto $L^\infty(Y) \mathbin{\overline{\otimes}} 1 \mathbin{\overline{\otimes}} L^\infty(\mathbb{R})$. Then $\|P_n\|_\infty \leq \|Q_n\|_\infty \leq \|Q\|_\infty$ and it follows from (2.4.2) that

$$S_n(y,x,s) = P_n\Big(y, s - \sum_{i \in I_n} \varphi_{\lambda,i}(x_i)\Big)\ . \qquad (2.4.3)$$

Consider the skew product equivalence relation $\mathcal{R}(\alpha)$ of the tail equivalence relation $\mathcal{R}$ by the cocycle α as in Lemma 2.4.6. Denote by $\mathcal{S}(\alpha)$ the equivalence relation on $Y \times X \times \mathbb{R}$ given by $\mathrm{id} \times \mathcal{R}(\alpha)$, i.e. with $(y,x,s) \sim_{\mathcal{S}(\alpha)} (y',x',t)$ if and only if $y = y'$ and $(x,s) \sim_{\mathcal{R}(\alpha)} (x',t)$.

Claim: $Q \in L^\infty(Y \times X \times \mathbb{R})$ is $\mathcal{S}(\alpha)$-invariant.

To prove the claim, define $\sigma : \{0,1\} \to \{0,1\}$ given by $\sigma(0) = 1$ and $\sigma(1) = 0$. For every $i \in I$, define

$$\sigma_i : Y \times X \times \mathbb{R} \to Y \times X \times \mathbb{R} : \sigma_i(y,x,s) = \ (y, x', s - \varphi_{\lambda,i}(x_i) + \varphi_{\lambda,i}(\sigma(x_i))),$$

where $x'_j = x_j$ if $j \neq i$ and $x'_i = \sigma(x_i)$.

Since the graphs of the automorphisms $(\sigma_i)_{i \in I}$ generate the equivalence relation $\mathcal{S}(\alpha)$, to prove the claim, it suffices to prove that

$$Q(\sigma_i(y,x,s)) = Q(y,x,s) \quad \text{for all } i \in I \text{ and a.e. } (y,x,s) \in Y \times X \times \mathbb{R}.$$

Whenever $i \in I_n$, it follows from (2.4.3) that $S_n(\sigma_i(y,x,s)) = S_n(y,x,s)$ for all (y,x,s). Since $S_n \to Q$ a.e. and $i \in I_n$ for n large enough, the claim is proven.

By Lemma 2.4.6 $\mathcal{R}(\alpha)$ is an ergodic equivalence relation. So every $\mathcal{S}(\alpha)$-invariant element $Q \in L^\infty(Y \times X \times \mathbb{R})$ belongs to $L^\infty(Y) \otimes 1 \otimes 1$. Therefore, $Q \in L^\infty(Y) \otimes 1 \otimes 1$ and the proposition is proven. $\square$

2.4.3 On Bernoulli actions of the group $\mathbb{Z}$

Combining Propositions 2.2.1, 2.4.4 and 2.4.5, we get the following result that we use to construct numerous concrete examples of type III_1 Bernoulli actions of $\mathbb{Z}$.

Proposition 2.4.7. *Let I be a countable set and $\mathbb{Z} \curvearrowright I$ a free action. Let $0 < \delta < 1$ and $\kappa > \delta^{-2} + \delta^{-1}(1-\delta)^{-2}$. Assume that $F : I \to [\delta, 1-\delta]$ is a function satisfying the following conditions.*

1. *There exists a $0 < \lambda < 1$ such that $\lim_{i\to\infty} F(i) = \lambda$ and $\sum_{i\in I}(F(i) - \lambda)^2 = +\infty$.*

2. *For every $k \in \mathbb{Z}$, the function $c_k : I \to \mathbb{R} : c_k(i) = F(i) - F((-k)\cdot i)$ belongs to $\ell^2(I)$.*

3. *We have $\sum_{k\in\mathbb{Z}} \exp(-\kappa\|c_k\|_2^2) = +\infty$.*

4. *For every $i \in I$, there exist $n_i \in \mathbb{Z}$ and $\varepsilon_i \in \{1, -1\}$ such that $F(n \cdot i) = \lambda$ for all $n \in \mathbb{Z}$ with $\varepsilon_i n \le n_i$.*

Then, the Bernoulli action $\mathbb{Z} \curvearrowright (X, \mu) = \prod_{i\in I}(\{0,1\}, \mu_i)$ with $\mu_i(0) = F(i)$ is nonsingular, essentially free, ergodic and of type III_1, and has a weakly mixing Maharam extension.

Proof. By 2, the Bernoulli action $\mathbb{Z} \curvearrowright (X, \mu)$ is nonsingular. Since $\delta \le \mu_i(0) \le 1-\delta$ for all $i \in I$, the action is essentially free. By 3 and Proposition 2.2.1, the action $\mathbb{Z} \curvearrowright (X, \mu)$ is conservative. By 4 and Proposition 2.4.4, the subset $\{n \cdot i \mid n \in \mathbb{Z}, \varepsilon_i n \le m\} \subset I$ is λ-inessential for every $i \in I$ and every $m \in \mathbb{Z}$. Since these subsets cover I, it follows from 1 and Proposition 2.4.5 that $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic and of type III_1, and that its Maharam extension is weakly mixing. $\square$

Lemma 2.4.8. *Let $a_0 \ge a_1 \ge a_2 \ge \cdots$ be a decreasing sequence of strictly positive real numbers. Let $\lambda > 0$ and $n_0 \in \mathbb{Z}$. Define the function*

$$F : \mathbb{Z} \to (0, +\infty) : F(n) = \begin{cases} \lambda + a_{n-n_0} & \text{if } n \ge n_0, \\ \lambda & \text{if } n < n_0. \end{cases}$$

For every $k \in \mathbb{Z}$, define the function $c_k : \mathbb{Z} \to \mathbb{R} : c_k(n) = F(n) - F(n-k)$. Then, $c_k \in \ell^2(\mathbb{Z})$ and

$$\sum_{n=0}^{|k|-1} a_n^2 \le \|c_k\|_2^2 \le 2\sum_{n=0}^{|k|-1} a_n^2 \quad \text{for every } k \in \mathbb{Z}.$$

Proof. Changing λ or n_0 does not change the value of $\|c_k\|_2$, so that we may assume that $\lambda = 0$ and $n_0 = 0$. Fix $k \geq 1$. For every $n_1 \geq k$, we have

$$\sum_{n=-\infty}^{n_1} |c_k(n)|^2 = \sum_{n=0}^{k-1} a_n^2 + \sum_{n=k}^{n_1} (a_{n-k} - a_n)^2 \; ,$$

so that $\|c_k\|_2^2 \geq \sum_{n=0}^{k-1} a_n^2$ and

$$\begin{aligned}\sum_{n=-\infty}^{n_1} |c_k(n)|^2 &\leq \sum_{n=0}^{k-1} a_n^2 + \sum_{n=k}^{n_1} |a_{n-k}^2 - a_n^2| = \sum_{n=0}^{k-1} a_n^2 + \sum_{n=k}^{n_1} (a_{n-k}^2 - a_n^2) \\ &= \sum_{n=0}^{k-1} a_n^2 + \sum_{n=0}^{k-1} a_n^2 - \sum_{n=n_1-k+1}^{n_1} a_n^2 \leq 2 \sum_{n=0}^{k-1} a_n^2 \; .\end{aligned}$$

Since this holds for all $n_1 \geq k$, we find that $c_k \in \ell^2(\mathbb{Z})$ and

$$\|c_k\|_2^2 \leq 2 \sum_{n=0}^{k-1} a_n^2 \; .$$

Since $c_0 = 0$ and $\|c_{-k}\|_2 = \|c_k\|_2$, the lemma is proven. □

2.4.4 Proof of Theorem 2.4.1

We first prove that in the following two cases, the group G admits a nonsingular Bernoulli action $G \curvearrowright \prod_{g \in G}(\{0,1\}, \mu_g)$ with base space $\{0,1\}$ satisfying the conclusions of Theorem 2.4.1.

Case 1. G is an amenable group that admits an infinite subgroup Λ of infinite index.

Case 2. G admits a copy of $\mathbb{Z}$ as a finite index subgroup.

Proof in case 1. We start by proving that G admits a Følner sequence $A_n \subset G$ for which all the sets ΛA_n are disjoint. To prove this claim, let $B_n \subset G$ be an arbitrary Følner sequence and define the left invariant mean m on G as a limit point of the means $m_n(C) = |C \cap B_n| / |B_n|$. Since $\Lambda < G$ has infinite index, we can fix a sequence $g_n \in G$ such that the sets $g_n \Lambda$ are disjoint. It follows that for every fixed $h \in G$, the sets $g_n \Lambda h$ are disjoint. By left invariance, this forces $m(\Lambda h) = 0$. So, for every finite subset $\mathcal{F} \subset G$, we get that $m(\Lambda \mathcal{F}) = 0$. This implies that after passing to a subsequence of B_n, we may assume that $|\Lambda \mathcal{F} \cap B_n| / |B_n| \to 0$ for every finite subset $\mathcal{F} \subset G$.

Write G as an increasing union of finite subsets $\mathcal{F}_n \subset G$ and choose $\mathcal{F}_n$ such that $B_k \subset \mathcal{F}_n$ for all $k < n$. Choose inductively $s_1 < s_2 < \cdots$ such that

$$\frac{|\Lambda \mathcal{F}_{s_{n-1}} \cap B_{s_n}|}{|B_{s_n}|} < \frac{1}{n}$$

for all $n \geq 1$. Defining $A_n = B_{s_n} \setminus \Lambda \mathcal{F}_{s_{n-1}}$, we have found a Følner sequence $A_n \subset G$ for which all the sets ΛA_n are disjoint.

Let $G = \{g_0, g_1, g_2, \ldots\}$ be an enumeration of the group G such that $g_0 = e$ and $\{g_0, g_2, g_4, \ldots\}$ is an enumeration of the infinite subgroup Λ. By Proposition A.2.1, we can pass to a subsequence of A_n and choose $\varepsilon_n \in (0, 1/6)$ such that $\varepsilon_n \to 0$, $\sum_n \varepsilon_n^2 = +\infty$ and such that the function F defined by

$$F : G \to [0, 1/6) : F(g) = \begin{cases} \varepsilon_n / \sqrt{|A_n|} & \text{if } g \in A_n \text{ for some } n, \\ 0 & \text{if } g \notin \bigcup_n A_n, \end{cases} \tag{2.4.4}$$

has the property that the associated 1-cocycle

$$c : G \to \ell^2(G), \quad c_g(k) = F(k) - F(g^{-1}k)$$

satisfies

$$\|c_{g_n}\|_2^2 \leq \frac{1}{16} \log(1 + n)$$

for all $n \geq 0$.

Define the probability measures μ_k on $\{0, 1\}$ given by $\mu_k(0) = F(k) + 1/2$ and note that $1/2 \leq \mu_k(0) \leq 2/3$ for all $k \in G$. Consider the associated Bernoulli action $G \curvearrowright (X, \mu)$, which is nonsingular since $c_g \in \ell^2(G)$ for all $g \in G$. Then,

$$\sum_{g \in \Lambda} \exp\bigl(-16\|c_g\|_2^2\bigr) \geq \sum_{n=0}^{\infty} \exp\bigl(-\log(1 + n)\bigr) = +\infty \, .$$

It follows from Proposition 2.2.1 that the action $\Lambda \curvearrowright (X, \mu)$ is conservative. By construction, for every $k \in G$, there is at most one A_n that intersects Λk. It then follows from Proposition 2.4.3 that $\Lambda k \subset G$ is 1/2-inessential, for every $k \in G$. So by Proposition 2.4.5, the action $\Lambda \curvearrowright (X, \mu)$ is ergodic and of type III_1, and has a weakly mixing Maharam extension. A fortiori, the same holds for $G \curvearrowright (X, \mu)$.

Proof in case 2. We note first that G also admits a copy of $\mathbb{Z}$ as a finite index *normal* subgroup. Indeed, viewing the action of G on the finite set $E = G/\mathbb{Z}$ by left multiplication as a group homomorphism $\pi : G \to \mathrm{Sym}(E)$, we obtain a finite index normal subgroup $N = \ker(\pi)$ of infinite order which is contained in $\mathbb{Z}$. Hence $N \cong \mathbb{Z}$. Denote $\kappa = [G : \mathbb{Z}]$ and fix $g_1, \dots, g_\kappa$ such that G is the disjoint union of the $g_i\mathbb{Z}$. Define the function

$$F_0 : \mathbb{Z} \to (0,1) : F_0(n) = \begin{cases} \frac{1}{2} & \text{if } n \leq 3, \\ \frac{1}{2} + \frac{1}{\sqrt{n\log(n)}} & \text{if } n \geq 4, \end{cases}$$

and then define the function $F : G \to (0,1)$ given by $F(g_i n) = F_0(n)$ for all $i \in \{1, \dots, \kappa\}$ and $n \in \mathbb{Z}$. For every $g \in G$, define the function $c_g : G \to \mathbb{R}$ given by $c_g(h) = F(h) - F(g^{-1}h)$.

Since $\sum_{n=4}^{k} (n\log(n))^{-1}$ grows like $\log(\log(k))$, it follows from Lemma 2.4.8 that for every $k \in \mathbb{Z}$, the function $F_0 - k \cdot F_0$ belongs to $\ell^2(\mathbb{Z})$ and that $\|F_0 - k \cdot F_0\|_2^2 / \log(|k|)$ tends to zero as $|k| \to \infty$ in $\mathbb{Z}$. It then also follows that $c_g \in \ell^2(G)$ for every $g \in G$ and that

$$\frac{\|c_k\|_2^2}{\log(|k|)} \to 0 \quad \text{as } |k| \to \infty.$$

Defining the probability measures μ_h on $\{0,1\}$ given by $\mu_h(0) = F(h)$, the associated Bernoulli action $G \curvearrowright (X, \mu)$ is nonsingular and essentially free. Applying Proposition 2.4.7 to the left action $\mathbb{Z} \curvearrowright G$, it follows that $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic and of type III_1, and has a weakly mixing Maharam extension. A fortiori, the same holds for $G \curvearrowright (X, \mu)$.

To conclude the proof of Theorem 2.4.1, let G be an arbitrary amenable group. Applying the proof of case 1 to the amenable group $G \times \mathbb{Z}$ with the infinite subgroup $G \times \{0\}$ of infinite index, we find for every $(g, n) \in G \times \mathbb{Z}$ a probability measure $\mu_{(g,n)}$ on $\{0,1\}$ such that the Bernoulli action $G \curvearrowright (X, \mu) = \prod_{(g,n) \in G \times \mathbb{Z}} (\{0,1\}, \mu_{(g,n)})$ is nonsingular and satisfies all the conclusions of the theorem. Defining $X_0 = \prod_{n \in \mathbb{Z}} \{0,1\}$ and $\mu_g = \prod_{n \in \mathbb{Z}} \mu_{(g,n)}$ for every $g \in G$, the Bernoulli action $G \curvearrowright (X, \mu)$ can be identified with the Bernoulli action $G \curvearrowright \prod_{g \in G} (X_0, \mu_g)$. This concludes the proof of Theorem 2.4.1.

2.5 Nonsingular Bernoulli actions of the free groups

While all our considerations on Bernoulli actions so far, have been abstract results, this section of the thesis focuses on concrete examples of nonsingular Bernoulli actions of type III. We will first give examples of such actions with some additional properties for the group of integers before turning our attention to nonabelian free groups.

2.5.1 Examples of Bernoulli actions of $\mathbb{Z}$

In [Ko10, Theorem 7], it is proven that there exist nonsingular Bernoulli shifts T that are ergodic, of type III_1 and *power weakly mixing* in the sense that all transformations $T^{a_1} \times \cdots \times T^{a_k}$ remain ergodic. Our proof of Theorem 2.4.1 also gives the following concrete examples.

Corollary 2.5.1. *Let $0 < \lambda < 1$ and put $n_0 = \lceil (1-\lambda)^{-2} \rceil$. Define for every $n \in \mathbb{Z}$, the probability measure μ_n on $\{0,1\}$ given by*

$$\mu_n(0) = \begin{cases} \lambda + \frac{1}{\sqrt{n\log(n)}} & \text{if } n \geq n_0, \\ \lambda & \text{if } n < n_0. \end{cases}$$

The associated Bernoulli shift T on $(X,\mu) = \prod_{n\in\mathbb{Z}}(\{0,1\},\mu_n)$ is essentially free, ergodic, of type III_1 and with weakly mixing Maharam extension. Moreover, for all $k \geq 1$ and $a_1,\ldots,a_k \in \mathbb{Z}\setminus\{0\}$, the nonsingular transformation

$$T^{a_1} \times \cdots \times T^{a_k} : X^k \to X^k : (x_1,\ldots,x_k) \mapsto (T^{a_1}(x_1),\ldots,T^{a_k}(x_k))$$

remains ergodic, of type III_1 and with weakly mixing Maharam extension.

Proof. It suffices to note that each of the transformations $T^{a_1} \times \cdots \times T^{a_k}$ can be viewed as a Bernoulli action associated with some free action $\mathbb{Z} \curvearrowright I$ having finitely many orbits. Since $\sum_{n=n_0}^{k}(n\log(n))^{-1}$ grows like $\log(\log(k))$, it follows from Lemma 2.4.8 that the associated 1-cocycle $c : \mathbb{Z} \to \ell^2(I)$ satisfies $\lim_{|k|\to\infty} \|c_k\|_2^2/\log(|k|) = 0$. By Proposition 2.4.7, the transformation $T^{a_1} \times \cdots \times T^{a_k}$ is ergodic and of type III_1, and has a weakly mixing Maharam extension. $\square$

As another application of our methods, we give the following concrete example of an ergodic type III_1 Bernoulli shift that is not power weakly mixing. As far as we know, such examples were not given before.

Corollary 2.5.2. *Define for every $n \in \mathbb{Z}$, the probability measure μ_n on $\{0,1\}$ given by*

$$\mu_n(0) = \begin{cases} \frac{1}{2} + \frac{1}{6\sqrt{n}} & \text{if } n \geq 1, \\ \frac{1}{2} & \text{if } n \leq 0. \end{cases}$$

The associated Bernoulli shift T on $(X,\mu) = \prod_{n\in\mathbb{Z}}(\{0,1\},\mu_n)$ is essentially free, ergodic, of type III_1 and with weakly mixing Maharam extension, but for m large enough (e.g. $m \geq 73$), the m-th power transformation

$$T \times \cdots \times T : X^m \to X^m : (x_1,\ldots,x_m) \mapsto (T(x_1),\ldots,T(x_m))$$

is dissipative.

Proof. By Lemma 2.4.8, the associated 1-cocycle $c : \mathbb{Z} \to \ell^2(\mathbb{Z})$ defined by (2.1.3) satisfies

$$\frac{1}{36} \sum_{n=1}^{|k|} \frac{1}{n} \;\leq\; \|c_k\|_2^2 \;\leq\; \frac{1}{18} \sum_{n=1}^{|k|} \frac{1}{n}$$

so that

$$\frac{1}{36} \log(1 + |k|) \;\leq\; \|c_k\|_2^2 \;\leq\; \frac{1}{18}(1 + \log |k|)$$

whenever $|k| \geq 2$. It follows that

$$\sum_{k\in\mathbb{Z}} \exp(-16\|c_k\|_2^2) \geq \sum_{k=2}^{\infty} \exp\bigl(-\frac{16}{18}(1 + \log(k))\bigr) = \exp(-8/9) \sum_{k=2}^{\infty} \frac{1}{k^{8/9}} = +\infty \;.$$

Since $1/3 \leq \mu_k(0) \leq 2/3$, it follows from Proposition 2.4.7 that T is ergodic, of type III_1, with weakly mixing Maharam extension.

Write $m = 73$. The m-fold power of T is a Bernoulli action associated with $\mathbb{Z} \curvearrowright I$, where I is the disjoint union of m copies of $\mathbb{Z}$. The associated 1-cocycle $d : \mathbb{Z} \to \ell^2(I)$ satisfies $\|d_k\|_2^2 = m\,\|c_k\|_2^2$ for every $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \exp\bigl(-\frac{1}{2}\|d_k\|_2^2\bigr) &= 1 + 2\sum_{k=1}^{\infty} \exp\bigl(-\frac{m}{2}\|c_k\|_2^2\bigr) \\
&\leq 1 + 2\sum_{k=1}^{\infty} \exp\bigl(-\frac{m}{72}\log(1+k)\bigr) \\
&= 1 + 2\sum_{k=2}^{\infty} \frac{1}{k^{m/72}} < +\infty \;.
\end{aligned}$$

So by Proposition 2.2.1, the m-fold power of T is dissipative. □

2.5.2 Examples of Bernoulli actions of $\mathbb{F}_n, n \geq 2$

In this section, we will concretize the construction in the proof of Theorem 2.3.1 in the special case of a free product group $G = \Lambda * \mathbb{Z}$. In this way we will obtain a wide range of nonsingular Bernoulli actions such as nonsingular Bernoulli actions of type III_λ for any $0 < \lambda < 1$ as well as nonsingular Bernoulli actions whose orbit equivalence relation can have any prescribed Connes invariant as defined in Section 1.4.4.

Before we can properly state the next proposition, we need to quickly recall the notion of inner amenability. An infinite discrete group G is called *inner* *amenable* if there exists a finitely additive measure m on the subsets of $G\backslash\{e\}$ such that $m(gAg^{-1}) = m(A)$ for all $A \subset G\backslash\{e\}$. In analogy to the relationship between amenability and the left regular representation, inner amenability of a group G with infinite conjugacy classes (ICC) is equivalent to the trivial representation being weakly contained in the representation

$$\pi : G \to \mathcal{U}(\ell^2(G\backslash\{e\})), \quad \pi(g)\delta_h = \delta_{ghg^{-1}}.$$

Proposition 2.5.3. *Let $G = \Lambda * \mathbb{Z}$ be any free product of an infinite group Λ and the group of integers $\mathbb{Z}$. Define $W \subset G$ as the set of reduced words whose last letter is a strictly positive element of $\mathbb{Z}$. Let μ_0 and μ_1 be Borel probability measures on a standard Borel space X_0. Assume that $\mu_0 \sim \mu_1$ and that μ_0, μ_1 are not supported on a single atom.*

The Bernoulli action $G \curvearrowright (X, \mu)$ with $(X, \mu) = \prod_{g \in G}(X_0, \mu_g)$ and

$$\mu_g = \begin{cases} \mu_1 & \text{if } g \in W, \\ \mu_0 & \text{if } g \notin W, \end{cases}$$

is nonsingular, essentially free, ergodic and nonamenable in the sense of Zimmer.

Denote by $T = d\mu_1/d\mu_0$ the Radon-Nikodym derivative. Define $\tau(T)$ as the weakest topology on $\mathbb{R}$ that makes the map

$$\pi : \mathbb{R} \to \mathcal{U}(L^\infty(X_0, \mu_0)) : \pi(t) = (x \mapsto T(x)^{it}) \tag{2.5.1}$$

continuous, where $\mathcal{U}(L^\infty(X_0, \mu_0))$ is equipped with the strong topology. We say that T is almost periodic if there exists a countable subset $S \subset \mathbb{R}^+_*$ such that $T(x) \in S$ for a.e. $x \in X_0$. In that case, we denote by $\mathrm{Sd}(T)$ the subgroup of $\mathbb{R}^+_*$ generated by the smallest such $S \subset \mathbb{R}^+_*$.

1. *The type of $G \curvearrowright (X, \mu)$ is determined as follows: the action is of type II_1 if and only if $T(x) = 1$ for a.e. $x \in X_0$; the action is of type III_λ with $0 < \lambda < 1$ if and only if the essential range of T generates the subgroup $\lambda^{\mathbb{Z}} < \mathbb{R}^+_*$; and the action is of type III_1 if and only if the essential range of T generates a dense subgroup of $\mathbb{R}^+_*$.*

2. *If Λ is nonamenable, the action $G \curvearrowright (X, \mu)$ is strongly ergodic. Then, the τ-invariant of the orbit equivalence relation $\mathcal{R}$ of $G \curvearrowright (X, \mu)$ equals $\tau(T)$. In particular, $\mathcal{R}$ is almost periodic if and only if T is almost periodic and in that case, $\mathrm{Sd}(\mathcal{R}) = \mathrm{Sd}(T)$.*

3. *If Λ has infinite conjugacy classes and is non inner amenable, then the crossed product factor $M = L^\infty(X, \mu) \rtimes G$ is full and its τ-invariant (in the sense of [Co74]) equals $\tau(T)$. Also, M is almost periodic (in the sense of [Co74]) if and only if T is almost periodic and in that case, $\mathrm{Sd}(M) = \mathrm{Sd}(T)$.*

For a Bernoulli action $G \curvearrowright (X, \mu)$ as in Proposition 2.5.3, the weak mixing of the Maharam extension and the stable type, i.e. the type of a diagonal action $G \curvearrowright (Y \times X, \eta \times \mu)$ given a pmp action $G \curvearrowright (Y, \eta)$, will be discussed later on in Proposition 2.5.5. Moreover, we will derive several concrete examples from Proposition 2.5.3 in Example 2.5.4.

Proof of Proposition 2.5.3. Since $\Lambda W = W$, the action $\Lambda \curvearrowright (X, \mu)$ is a probability measure preserving generalized Bernoulli action and in particular ergodic. Denote by $a \in \mathbb{Z}$ the generator $a = 1$. The measure $a_*^{-1} \cdot \mu$ given by $(a_*^{-1}\mu)(\mathcal{U}) = \mu(a \cdot \mathcal{U})$ equals the product measure

$$a_*^{-1}\mu = \prod_{g \in G} \mu_{ag} .$$

Since $a^{-1}W \triangle W = \{e\}$, we get that $a_*^{-1}\mu \sim \mu$ and that

$$\frac{d(a_*^{-1}\mu)}{d\mu}(x) = \frac{d\mu_1}{d\mu_0}(x_e)$$

by [Ka48, Lemma 6]. So, a acts nonsingularly on (X, μ) and the Radon-Nikodym cocycle is given by $\omega(a, x) = T(x_e)$. It follows that $G \curvearrowright (X, \mu)$ is nonsingular and ergodic.

Since μ_0 and μ_1 are equivalent and not supported on a single atom, we can choose a Borel set $A \subset X_0$ and $\delta > 0$ such that $\delta < \mu_0(A), \mu_1(A) < 1 - \delta$. We can then show that $G \curvearrowright (X, \mu)$ is essentially free as in the proof of Proposition 1.8.4: For $g \neq e$ choose $J = \{j_1, j_2, \dots\} \subset G$ such that $gJ \cap J = \emptyset$ and compute

$$\begin{aligned} \mu(\mathrm{Fix}(g)) &\leq \lim_{n \to \infty} \prod_{k=1}^{n} \mu(\{x \in X_0^G \; ; \; x_{g^{-1}j_k} = x_{j_k}\}) \\ &= \lim_{n \to \infty} \prod_{k=1}^{n} \mu_s(A)\mu_t(A) + \mu_s(X \backslash A)\mu_t(X \backslash A) \\ &\leq \lim_{n \to \infty} \prod_{k=1}^{n} 1 - 2(\delta - \delta^2) = 0 \end{aligned}$$

for an appropriate choice of $s, t \in \{0, 1\}$. So G acts freely on (X, μ).

To determine the type of $G \curvearrowright (X, \mu)$, consider the Maharam extension $G \curvearrowright (X \times \mathbb{R}, \mu \times \nu)$. Let $Q \in L^\infty(X \times \mathbb{R})$ be a G-invariant function. Since $\Lambda \curvearrowright (X, \mu)$ is measure preserving and ergodic, it follows that $Q(x, s) = P(s)$, where

$P \in L^\infty(\mathbb{R})$ is invariant under translation by t for every t in the essential range of one of the maps

$$x \mapsto \log(\omega(g,x)), \quad g \in G.$$

The union of these essential ranges equals the subgroup of $\mathbb{R}$ generated by $\log(\operatorname{ran} T)$. So our statement on the type of $G \curvearrowright (X,\mu)$ follows.

To prove that $G \curvearrowright (X,\mu)$ is nonamenable in the sense of Zimmer, denote by $\Lambda_1 < G$ the subgroup generated by Λ and $a\Lambda a^{-1}$. Note that Λ_1 is the free product of these two subgroups. Both Λ and $a\Lambda a^{-1}$ act on (X,μ) as a probability measure preserving Bernoulli action, although they do not preserve the same probability measure. In particular, the actions of Λ and $a\Lambda a^{-1}$ on (X,μ) are conservative. Since the action of their free product Λ_1 is essentially free, it follows from [HV12, Corollary F] that $\Lambda_1 \curvearrowright (X,\mu)$ is nonamenable in the sense of Zimmer. A fortiori, $G \curvearrowright (X,\mu)$ is nonamenable.

Now assume that Λ is nonamenable. Since $\Lambda \curvearrowright (X,\mu)$ is a probability measure preserving Bernoulli action, the action $\Lambda \curvearrowright (X,\mu)$ is strongly ergodic, see e.g. [KT08, Theorem 1.2]. A fortiori, the equivalence relations $\mathcal{S} := \ker D_\mu$ and $\mathcal{R} := \mathcal{R}(G \curvearrowright X)$ are strongly ergodic, where $D_\mu : \mathcal{R} \to (0,\infty)$ denotes the Radon-Nykodim cocycle. We adapt the proof of [HMV17, Theorem 6.4] to our setting to obtain that the τ-invariant of the orbit equivalence relation $\mathcal{R}(G \curvearrowright X)$ is the weakest topology on $\mathbb{R}$ that makes the map in (2.5.1) continuous. To see this, we first point out that $D(ax,x) = T(x_e)$ for all $a \in \mathbb{Z}$ and almost all $x \in X$ and that for arbitrary $g \in G$, we can find finitely many $h_1,\dots,h_k \in G$ and $n_1,\dots,n_k \in \mathbb{Z}$ such that $D(gx,x) = \prod_{j=1}^k T(x_{h_j})^{n_j}$ for a.e. $x \in X$. Therefore, the weakest topology making the map

$$\pi : \mathbb{R} \to \mathcal{U}(L^\infty(X_0,\mu_0)) : \quad \pi(t) = (x \mapsto T(x)^{it})$$

continuous equals the weakest topology making the map

$$\mathbb{R} \to Z^1(\mathcal{R}) : \quad t \mapsto D_\mu^{it}$$

continuous with respect to the topology of convergence in measure. Let us name this topology $\tau(\pi)$. Since the quotient map $Z^1(\mathcal{R}) \to H^1(\mathcal{R})$ is continuous by definition, we have $\tau(\mathcal{R}) \subset \tau(\pi)$. To show the other inclusion, we need to prove that for any sequence (t_n) in $\mathbb{R}$ such that $t_n \to 0$ with respect to $\tau(\mathcal{R})$, we have that $D_\mu^{it_n} \to 1$ in $Z^1(\mathcal{R})$ for the convergence in measure. Thus, let (t_n) in $\mathbb{R}$ be such that $t_n \to 0$ with respect to $\tau(\mathcal{R})$. Then, there exists a sequence $u_n \in L^0(X,\mathbb{T})$ such that $\partial u_n D_\mu^{it_n} \to 1$ in $Z^1(\mathcal{R})$ for the convergence in measure. Here, by $\partial : L^0(X,\mathbb{T}) \to Z^1(\mathcal{R})$, we denote the boundary map defined by $(\partial u)(x,y) = u(x)u^{-1}(y)$. Restricting our attention to the subrelation $\mathcal{S}$, we see that $\partial u_n \to 1$ in $Z^1(\mathcal{S})$ for the convergence in measure. The strong ergodicity of $\mathcal{S}$ then yields a sequence of complex numbers (z_n) in $\mathbb{T}$ such that

$z_n u_n \to 1$ in $L^0(X, \mathbb{T})$ for the convergence in measure, which implies $D_\mu^{it_n} \to 1$ in $Z^1(\mathcal{R})$ for the convergence in measure.

Finally assume that Λ has infinite conjugacy classes and that Λ is non inner amenable. Denote by $(u_g)_{g\in G}$ the canonical unitary operators in $M = L^\infty(X) \rtimes G$ and denote by φ the canonical faithful normal state on M given by $\varphi(F) = \int_X F(x)d\mu(x)$ and $\varphi(Fu_g) = 0$ for all $F \in L^\infty(X)$ and $g \in G \setminus \{e\}$. Denote by $\mathcal{H}$ the Hilbert space completion of M w.r.t. the scalar product given by $\langle c, d\rangle = \varphi(d^*c)$ for all $c, d \in M$. View $M \subset \mathcal{H}$. Since the action $\Lambda \curvearrowright (X, \mu)$ is measure preserving, both left and right multiplication by u_g, $g \in \Lambda$, defines a unitary operator on $\mathcal{H}$. To prove that the factor M is full and that the same topology as above is the τ-invariant of M, it suffices to prove that the unitary representation

$$\theta : \Lambda \to \mathcal{U}(\mathcal{H} \ominus \mathbb{C}1) : (\theta(g))(d) = u_g d u_g^*$$

does not weakly contain the trivial representation of Λ.

But θ is the direct sum of the subrepresentations θ_i on $\mathcal{H}_i$ where $\mathcal{H}_1$ is the closed linear span of $\{u_g F \mid g \in G, \int_X F\, d\mu = 0\}$, where $\mathcal{H}_2$ is the closed linear span of $\{u_g \mid g \in G \setminus \Lambda\}$, and where $\mathcal{H}_3$ is the closed linear span of $\{u_g \mid g \in \Lambda \setminus \{e\}\}$. Because $\Lambda \curvearrowright (X, \mu)$ is a probability measure preserving Bernoulli action, the representation

$$\phi : \Lambda \to \mathcal{U}(L^2(X, \mu) \ominus \mathbb{C}1), \quad (u_g f)(x) = f(g^{-1}x)$$

is a multiple of the left regular representation, see [KL16, Section 2.3]. On the other hand, we can identify the representation θ_1 with the tensor product representation $\phi \otimes \pi$, where $\pi : \Lambda \to \ell^2(G)$ acts by conjugation. Thus, by Fell's absorption principle, θ_1 is a multiple of the left regular representation as well. Next, since G is the free product of Λ and $\mathbb{Z}$, also θ_2 is a multiple of the left regular representation of Λ. To see this, decompose $G\backslash\Lambda$ in conjugacy classes with respect to Λ and note that for any conjugacy class $[d] = \{gdg^{-1},\ g \in \Lambda\}$, the associated conjugation representation $\Lambda \to \ell^2([d])$ is equivalent to the left regular one. Since Λ is nonamenable, θ_1 and θ_2 do not weakly contain the trivial representation of Λ. Finally, θ_3 does not weakly contain the trivial representation of Λ because Λ has infinite conjugacy classes and Λ is not inner amenable. $\square$

We arrive at the following concrete examples.

Example 2.5.4. We use the same notations as in the formulation of Proposition 2.5.3.

1. Take $0 < \lambda < 1$ and put $X_0 = \{0, 1\}$ with $\mu_0(0) = (1 + \lambda)^{-1}$ and $\mu_1(0) = \lambda(1 + \lambda)^{-1}$. It follows that $G \curvearrowright (X, \mu)$ is of type III_λ. So all free product

groups $G = \Lambda * \mathbb{Z}$ with Λ infinite admit nonsingular, essentially free, ergodic Bernoulli actions of type III_λ. Note that by [DL16, Corollary 3.3], the group $\mathbb{Z}$ does not admit nonsingular Bernoulli actions of type III_λ, at least under the assumption that all μ_n, $n < 0$, are identical.

2. Using the construction of [Co74, Section 5], we obtain the following examples of strongly ergodic, nonsingular Bernoulli actions whose orbit equivalence relation has an arbitrary countable dense subgroup of $\mathbb{R}^+_*$ as Sd-invariant or has any topology coming from a unitary representation of $\mathbb{R}$ as τ-invariant. This holds for any free product group $G = \Lambda * \mathbb{Z}$ with Λ nonamenable, and in particular for any free group $\mathbb{F}_n$ with $3 \leq n \leq +\infty$. So this provides an answer to [HMV17, Problem 3].

   Let η be any nonzero finite Borel measure on $\mathbb{R}^+_*$ with $\int_{\mathbb{R}^+_*} x \, d\eta(x) < \infty$. Define $X_0 = \mathbb{R}^+_* \times \{0,1\}$ and define the probability measures μ_0 and μ_1 on X_0 determined by

$$\kappa = \int_{\mathbb{R}^+_*} (1+x) \, d\eta(x) \; ,$$

$$\int_{X_0} F \, d\mu_0 = \kappa^{-1} \int_{\mathbb{R}^+_*} \bigl(F(x,0) + xF(x,1)\bigr) \, d\eta(x) \; ,$$

$$\int_{X_0} F \, d\mu_1 = \kappa^{-1} \int_{\mathbb{R}^+_*} \bigl(xF(x,0) + F(x,1)\bigr) \, d\eta(x) \; ,$$

   for all positive Borel functions F on X_0. Then, $\mu_0 \sim \mu_1$ and the Radon-Nikodym derivative $T = d\mu_1/d\mu_0$ is given by $T(x,0) = x$ and $T(x,1) = 1/x$ for all $x \in \mathbb{R}^+_*$.

   So when Λ is nonamenable, the nonsingular Bernoulli action associated with μ_0, μ_1 in Proposition 2.5.3 is strongly ergodic and the τ-invariant of the orbit equivalence relation is the weakest topology on $\mathbb{R}$ that makes the map

$$\mathbb{R} \to \mathcal{U}(L^\infty(X_0,\mu_0)), t \mapsto (x_0 \mapsto T(x_0)^{it})$$

   continuous. By construction of the measure μ_0, this topology is the weakest topology that makes the map

$$\mathbb{R} \to \mathcal{U}(L^\infty(\mathbb{R}^+_*,\eta)) : t \mapsto (x \mapsto x^{it})$$

   continuous. By identifying $\mathbb{R}^+_*$ with the unitary dual of $\mathbb{R}$ through the dual pairing $\mathbb{R} \times \mathbb{R}^+_* \to \mathbb{C}, \langle t, x\rangle = x^{it}$, we get that any unitary representation of $\mathbb{R}$ is equivalent to one of the above form. In particular, we see that the weakest topology which makes a given unitary representation π continuous, is the weakest topology which makes

$$\mathbb{R} \to \mathcal{U}(L^\infty(\mathbb{R}^+_*,\eta)) : t \mapsto (x \mapsto x^{it})$$

continuous for some finite Borel measure η with finite first moment. So, it follows that any topology on $\mathbb{R}$ induced by a unitary representation of $\mathbb{R}$ arises as the τ-invariant of the orbit equivalence relation of a strongly ergodic, nonsingular Bernoulli actions of a free product $G = \Lambda * \mathbb{Z}$ with Λ nonamenable.

In particular, taking an atomic measure η, we obtain strongly ergodic, nonsingular Bernoulli actions of $G = \Lambda * \mathbb{Z}$ with any prescribed Sd-invariant. More concretely, when $S < \mathbb{R}_*^+$ is a given countable dense subgroup, we enumerate $S \cap (0,1) = \{t_n \mid n \geq 1\}$ and define the finite atomic measure η on $\mathbb{R}_*^+$ given by

$$\eta = \sum_{n=1}^{\infty} \frac{1}{2^n(1+t_n)}\, \delta_{t_n} \; .$$

The orbit equivalence relation of $G \curvearrowright (X,\mu)$ is then almost periodic with Sd-invariant equal to S.

Proposition 2.5.5. *Let $G = \Lambda * \mathbb{Z}$ be any free product of an infinite group Λ and the group of integers $\mathbb{Z}$. Let $G \curvearrowright (X,\mu)$ be a Bernoulli action as in Proposition 2.5.3. Choose an ergodic pmp action $G \curvearrowright (Y,\eta)$. Then, the diagonal action $G \curvearrowright Y \times X$ is ergodic and its type is determined as follows.*

Using the same notations as in Proposition 2.5.3, denote $T = d\mu_1/d\mu_0$. Denote by $L < \mathbb{R}$ the subgroup generated by the essential range of the map $X_0 \times X_0 \to \mathbb{R} : (x,x') \mapsto \log(T(x)) - \log(T(x'))$.

1. *If $L = \{0\}$, then μ is G-invariant and the actions $G \curvearrowright X$ and $G \curvearrowright Y \times X$ are of type II_1.*

2. *If $L < \mathbb{R}$ is dense, then the Maharam extension of $G \curvearrowright (X,\mu)$ is weakly mixing and the diagonal action $G \curvearrowright Y \times X$ is of type III_1.*

3. *If $L = a\mathbb{Z}$, take the unique $b \in [0,a)$ such that $\log(T(x)) \in b + a\mathbb{Z}$ for a.e. $x \in X_0$. Denote by $\pi : G \to \mathbb{Z}$ the unique homomorphism given by $\pi(g) = 0$ if $g \in \Lambda$ and $\pi(n) = n$ if $n \in \mathbb{Z}$. The set*

$$\begin{aligned} H = \{k \in \mathbb{Z} \mid & \text{ there exists a Borel map } V : Y \to \mathbb{R}/a\mathbb{Z} \text{ s.t.} \\ & V(g\cdot y) = V(y) + k\pi(g)b \text{ for all } g \in G \text{ and a.e. } y \in Y\} \end{aligned} \tag{2.5.2}$$

   is a subgroup of $\mathbb{Z}$. Write $H = k_0\mathbb{Z}$ with $k_0 \geq 0$. If $k_0 = 0$, the action $G \curvearrowright Y \times X$ is of type III_1. If $k_0 \geq 1$, the action $G \curvearrowright Y \times X$ is of type III_λ with $\lambda = \exp(-a/k_0)$.

4. *If $L = a\mathbb{Z}$ and $b \in [0,a)$ is defined as in 3, then the following holds.*

- *If b is of finite order $k_1 \geq 1$ in $\mathbb{R}/a\mathbb{Z}$ (with the convention that $k_1 = 1$ if $b = 0$), varying the action $G \curvearrowright (Y, \eta)$, the possible types of $G \curvearrowright Y \times X$ are III_λ with $\lambda = \exp(-a/k_0)$ where $k_0 \geq 1$ is an integer dividing k_1. Given such a k_0, this type is realized by taking the transitive action of G on $Y = \mathbb{Z}/(k_1/k_0)\mathbb{Z}$ given by $g \cdot y = y + \pi(g)$, or any other pmp action $G \curvearrowright Y$ that is induced from a weakly mixing pmp action of the finite index normal subgroup $\pi^{-1}((k_1/k_0)\mathbb{Z}) < G$.*
- *If b is of infinite order in $\mathbb{R}/a\mathbb{Z}$, varying the action $G \curvearrowright (Y, \eta)$, the possible types of $G \curvearrowright Y \times X$ are III_1 and III_λ with $\lambda = \exp(-a/k_0)$ where $k_0 \geq 1$ is any integer. Given k_0, the latter is realized by taking $Y = \mathbb{R}/(a/k_0)\mathbb{Z}$ and $g \cdot y = y + \pi(g)b$, while the former is realized by taking $G \curvearrowright (Y, \eta)$ to be the trivial action, or any other weakly mixing action.*

By varying the probability measures μ_0 and μ_1 in the construction of Proposition 2.5.3, all values of $0 \leq b < a$ in Proposition 2.5.5 occur; see Example 2.5.6.

Proof. Fix $G \curvearrowright (X, \mu)$ as in Proposition 2.5.3 and fix an arbitrary ergodic pmp action $G \curvearrowright (Y, \eta)$. Since $\Lambda \curvearrowright (X, \mu)$ is a pmp Bernoulli action, a Λ-invariant element of $L^\infty(Y \times X)$ belongs to $L^\infty(Y) \otimes 1$. It follows that $G \curvearrowright Y \times X$ is ergodic.

Define $L < \mathbb{R}$ as in the formulation of the proposition. If $L = \{0\}$, we have that T is constant a.e. Since $\int_{X_0} T(x) d\mu_0(x) = 1$, this constant must be 1. So, $T(x) = 1$ for a.e. $x \in X_0$. This means that $\mu_0 = \mu_1$, so that $G \curvearrowright (X, \mu)$ is a pmp Bernoulli action. This proves point 1.

To prove the remaining points of the proposition, let $Q \in L^\infty(Y \times X \times \mathbb{R})$ be a G-invariant element for the diagonal action of $G \curvearrowright Y$ and the Maharam extension $G \curvearrowright X \times \mathbb{R}$ of $G \curvearrowright X$. A fortiori, Q is Λ-invariant. Since $\Lambda \curvearrowright (X, \mu)$ is a pmp Bernoulli action, it follows that $Q \in L^\infty(Y) \mathbin{\overline{\otimes}} 1 \mathbin{\overline{\otimes}} L^\infty(\mathbb{R})$.

As in the proof of Theorem 2.3.1, it follows that $Q(y, x, s) = P(y, s)$, where $P \in L^\infty(Y \times \mathbb{R})$ satisfies

$$P(g \cdot y, s + \log(\omega(g, x))) = P(y, s) \quad \text{for all } g \in G \text{ and a.e. } (y, x, s) \in Y \times X \times \mathbb{R}. \tag{2.5.3}$$

Note that L equals the subgroup of $\mathbb{R}$ generated by the essential ranges of the maps

$$X \times X \to \mathbb{R} : (x, x') \mapsto \log(\omega(g, x)) - \log(\omega(g, x')) \ , \ g \in G \ .$$

It then follows from (2.5.3) that $P(y, s + t) = P(y, s)$ for all $t \in L$ and a.e. $(y, s) \in Y \times \mathbb{R}$.

If $L < \mathbb{R}$ is dense, we conclude that $Q \in L^\infty(Y) \otimes 1 \otimes 1$ and thus, by ergodicity of $G \curvearrowright Y$, that Q is constant a.e., so that $G \curvearrowright Y \times X \times \mathbb{R}$ is ergodic. This means that $G \curvearrowright Y \times X$ is of type III_1. Since $G \curvearrowright (Y, \eta)$ was an arbitrary ergodic pmp action, it follows that the Maharam extension $G \curvearrowright X \times \mathbb{R}$ is weakly mixing. This proves point 2.

Next assume that $L = a\mathbb{Z}$ with $a > 0$ and take the unique $0 \leq b < a$ such that $\log(T(x)) \in b + a\mathbb{Z}$ for a.e. $x \in X_0$. Denote by $\pi : G \to \mathbb{Z}$ the unique homomorphism given by $\pi(g) = 0$ if $g \in \Lambda$ and $\pi(n) = n$ if $n \in \mathbb{Z}$. Since $\omega(g, x) = 1$ for all $g \in \Lambda$ and $\omega(1, x) = T(x_e)$, it follows that

$$\log(\omega(g, x)) \in \pi(g)b + a\mathbb{Z} \quad \text{for all } g \in G \text{ and a.e. } x \in X.$$

We conclude that an element $Q \in L^\infty(Y \times X \times \mathbb{R})$ is G-invariant if and only if $Q(y, x, s) = P(y, s)$ where $P \in L^\infty(Y \times \mathbb{R}/a\mathbb{Z})$ is invariant under the action $G \curvearrowright Y \times \mathbb{R}/a\mathbb{Z}$ given by $g \cdot (y, s) = (g \cdot y, \pi(g)b + s)$.

If $k \in \mathbb{Z}$ and $V : Y \to \mathbb{R}/a\mathbb{Z}$ is a Borel map satisfying

$$V(g \cdot y) = V(y) + k\pi(g)b \quad \text{for all } g \in G \text{ and a.e. } y \in Y,$$

the map $P(y, s) = \exp(2\pi i(V(y) - ks)/a)$ is G-invariant. Using a Fourier decomposition for $\mathbb{R}/a\mathbb{Z} \cong \widehat{\mathbb{Z}}$, it follows that these functions P densely span the space of all G-invariant functions in $L^2(Y \times \mathbb{R}/a\mathbb{Z})$. Define $H < \mathbb{Z}$ as in (2.5.2). If $H = \{0\}$, it follows that

$$L^\infty(Y \times X \times \mathbb{R})^G = \mathbb{C}1$$

and that $G \curvearrowright Y \times X$ is of type III_1. When $H = k_0\mathbb{Z}$ with $k_0 \geq 1$, we identified $L^\infty(Y \times X \times \mathbb{R})^G$ with $L^\infty(\mathbb{R}/(a/k_0)\mathbb{Z})$ and it follows that $G \curvearrowright Y \times X$ is of type III_λ with $\lambda = \exp(-a/k_0)$. This concludes the proof of point 3.

To prove point 4, first assume that b is of finite order k_1 in $\mathbb{R}/a\mathbb{Z}$. Using the map $V(y) = 0$ for all $y \in Y$, it follows that k_1 belongs to the subgroup $H < \mathbb{Z}$ defined in (2.5.2). Therefore, k_0 must divide k_1. Conversely, assume that $k_0 \geq 1$ divides k_1 and that $G \curvearrowright Y$ is induced from a weakly mixing pmp action of $G_0 := \pi^{-1}((k_1/k_0)\mathbb{Z})$ on Y_0. Denote by $H < \mathbb{Z}$ the subgroup defined in (2.5.2). We have to prove that $H = k_0\mathbb{Z}$. If $k \in \mathbb{Z}$ and $V : Y \to \mathbb{R}/a\mathbb{Z}$ is a Borel function satisfying $V(g \cdot y) = V(y) + k\pi(g)b$, it follows that V is invariant under $\pi^{-1}(k_1\mathbb{Z})$. Since $G_0 \curvearrowright Y_0$ is weakly mixing, G_0 is normal in G and $\pi^{-1}(k_1\mathbb{Z}) < G_0$ has finite index, it follows that V is G_0-invariant. This forces k to be a multiple of k_0. So, $H \subset k_0\mathbb{Z}$. By construction of the induced action, there is a Borel map $W : Y \to G/G_0$ satisfying $W(g \cdot y) = gW(y)$. Identifying G/G_0 with $\mathbb{Z}/((k_1/k_0)\mathbb{Z})$ through π and composing W with the map

$$\mathbb{Z}/((k_1/k_0)\mathbb{Z}) \to \mathbb{R}/a\mathbb{Z} : n \mapsto k_0 n b \; ,$$

we have found a Borel map $V : Y \to \mathbb{R}/a\mathbb{Z}$ satisfying $V(g \cdot y) = V(y) + k_0\pi(g)b$. So, $k_0 \in H$ and the equality $H = k_0\mathbb{Z}$ follows. By point 3, the action $G \curvearrowright Y \times X$ is of type III_λ with $\lambda = \exp(-a/k_0)$.

Finally assume that b is of infinite order in $\mathbb{R}/a\mathbb{Z}$. When $G \curvearrowright (Y, \eta)$ is weakly mixing, the subgroup of $H < \mathbb{Z}$ defined in (2.5.2) is trivial, so that $G \curvearrowright Y \times X$ is of type III_1. When $Y = \mathbb{R}/((a/k_0)\mathbb{Z}$ with $g \cdot y = y + \pi(g)b$, one checks that $H = k_0\mathbb{Z}$, so that $G \curvearrowright Y \times X$ is of type III_λ with $\lambda = \exp(-a/k_0)$. □

Example 2.5.6. Given $0 < b < a$, define the probability measures μ_0 and μ_1 on $\{0, 1\}$ given by

$$\mu_0(0) = \exp(-b)\frac{1 - \exp(b - a)}{1 - \exp(-a)} \quad \text{and} \quad \mu_1(0) = \frac{1 - \exp(b - a)}{1 - \exp(-a)} \; .$$

Denote $T = d\mu_1/d\mu_0$. We get that $T(0) = \exp(b)$ and $T(1) = \exp(b - a)$. So, the map $(x, x') \mapsto \log(T(x)) - \log(T(x'))$ generates the subgroup $a\mathbb{Z} < \mathbb{R}$ and $\log(T(x)) \in b + a\mathbb{Z}$ for all $x \in \{0, 1\}$.

Given $a > 0$ and $b = 0$, define the probability measures μ_0 and μ_1 on $\{0, 1, 2\}$ given by

$$\mu_0(0) = \frac{1}{2} \quad , \quad \mu_0(1) = \frac{1}{2(1 + \exp(a))} \quad , \quad \mu_0(2) = \frac{\exp(a)}{2(1 + \exp(a))} \quad ,$$

$$\mu_1(0) = \frac{1}{2} \quad , \quad \mu_1(1) = \frac{\exp(a)}{2(1 + \exp(a))} \quad , \quad \mu_1(2) = \frac{1}{2(1 + \exp(a))} \quad .$$

The range of $T = d\mu_1/d\mu_0$ equals $\{1, \exp(a), \exp(-a)\}$. Therefore, the range of the map $(x, x') \mapsto \log(T(x)) - \log(T(x'))$ generates the subgroup $a\mathbb{Z} < \mathbb{R}$ and $\log(T(x)) \in a\mathbb{Z}$ for all $x \in \{0, 1, 2\}$.

So all values $0 \leq b < a$ really occur in Proposition 2.5.5.

This means that given any $0 < \lambda < 1$, Proposition 2.5.5 provides concrete examples of nonsingular, weakly mixing Bernoulli actions $G \curvearrowright (X, \mu)$ of a free product group $G = \Lambda * \mathbb{Z}$ such that the type of $G \curvearrowright (Y \times X, \eta \times \mu)$ ranges over III_μ with $\mu \in \{1\} \cup \{\lambda^{1/k} \mid k \geq 1\}$.

Given any $0 < \lambda < 1$ and an integer $k_1 \geq 1$, Proposition 2.5.5 also provides concrete examples of nonsingular, weakly mixing Bernoulli actions $G \curvearrowright (X, \mu)$ such that the type of a diagonal action $G \curvearrowright (Y \times X, \eta \times \mu)$ ranges over III_μ with $\mu \in \{\lambda^{1/k} \mid k \geq 1 \; , \; k|k_1\}$. In particular, we find nonsingular Bernoulli actions of stable type III_λ.

Remark 2.5.7. In Corollary 2.5.2, we constructed explicit nonsingular Bernoulli actions $\mathbb{Z} \curvearrowright (X, \mu)$ of type III_1 such that the m-th power diagonal action $\mathbb{Z} \curvearrowright (X^m, \mu^m)$ is dissipative. However, as we explain now, this phenomenon does not always occur for nonamenable groups.

Let G be a nonamenable group, $G \curvearrowright I$ a free action and $F : I \to (0,1)$ a function satisfying Assumptions 2.1.2. Consider the associated nonsingular Bernoulli action $G \curvearrowright (X,\mu)$ and the 1-cocycle $c : G \to \ell^2(I)$ given by (2.1.3). If the 1-cocycle is not proper, meaning that there exists a $\kappa > 0$ such that $\|c_g\|_2 \leq \kappa$ for infinitely many $g \in G$, it follows from Proposition 2.2.1 that $G \curvearrowright (X,\mu)$ and all its diagonal actions $G \curvearrowright (X^m, \mu^m)$ are conservative.

So, if the group G has no proper 1-cocycles into $\ell^2(G)$, e.g. because G does not have the Haagerup property, then all its nonsingular Bernoulli actions are conservative.

On the other hand, the free group $\mathbb{F}_2$ admits proper 1-cocycles into $\ell^2(\mathbb{F}_2)$. We use this to construct the following peculiar example of a nonsingular Bernoulli action of $\mathbb{F}_2$. In Proposition 2.5.11, we use a 1-cocycle with faster growth to give an example of a dissipative Bernoulli action of $\mathbb{F}_2$.

Proposition 2.5.8. *Let $G = \mathbb{F}_2$ be freely generated by the elements a and b. Define the subset $W_a \subset \mathbb{F}_2$ consisting of all reduced words in a, b that end with a strictly positive power of a. Similarly define $W_b \subset \mathbb{F}_2$ and put $W = \mathbb{F}_2 \backslash (W_a \cup W_b)$. The Bernoulli action $G \curvearrowright (X,\mu)$ with $(X,\mu) = \prod_{g\in G}(\{0,1\},\mu_g)$ and*

$$\mu_g(0) = \begin{cases} 3/5 & \textit{if } g \in W_a, \\ 2/5 & \textit{if } g \in W_b, \\ 1/2 & \textit{if } g \in W, \end{cases}$$

is nonsingular, essentially free, ergodic, nonamenable in the sense of Zimmer and of type III$_1$.

For every $g \in G \setminus \{e\}$, the transformation $x \mapsto g \cdot x$ is dissipative. For $m \geq 220$, the m-th power diagonal action $G \curvearrowright (X^m, \mu^m)$ is dissipative.

The stable type of the Bernoulli actions $\mathbb{F}_2 \curvearrowright (X,\mu)$ in Proposition 2.5.8 is discussed in Remark 2.5.9.

Proof. Denote $F : G \to (0,1) : F(g) = \mu_g(0)$ and define $c_g(h) = F(h) - F(g^{-1}h)$. We find that

$$c_a = \frac{1}{10}\delta_a \quad \text{and} \quad c_b = -\frac{1}{10}\delta_b \; .$$

Since c is a 1-cocycle, it follows that $c_g \in \ell^2(G)$ for all $g \in G$. So, the action $G \curvearrowright (X, \mu)$ is nonsingular. Using the 1-cocycle relation, we find that

$$c_{a^n} = \begin{cases} \frac{1}{10} \sum_{k=1}^{n} \delta_{a^k} & \text{if } n \geq 1, \\ -\frac{1}{10} \sum_{k=n+1}^{0} \delta_{a^k} & \text{if } n \leq -1, \\ 0 & \text{if } n = 0, \end{cases}$$

and

$$c_{b^n} = \begin{cases} -\frac{1}{10} \sum_{k=1}^{n} \delta_{b^k} & \text{if } n \geq 1, \\ \frac{1}{10} \sum_{k=n+1}^{0} \delta_{b^k} & \text{if } n \leq -1, \\ 0 & \text{if } n = 0. \end{cases}$$

When $g = a^{n_0} b^{m_1} a^{n_1} \cdots a^{n_{k-1}} b^{m_k} a^{n_k}$ is a reduced word, with $k \geq 0$, $n_0, n_k \in \mathbb{Z}$ and $n_i, m_j \in \mathbb{Z} \setminus \{0\}$, the 1-cocycle relation implies that

$$c_g = c_{a^{n_0}} + a^{n_0} \cdot c_{b^{m_1}} + a^{n_0} b^{m_1} \cdot c_{a^{n_1}} + \cdots + a^{n_0} b^{m_1} a^{n_1} \cdots a^{n_{k-1}} b^{m_k} \cdot c_{a^{n_k}} \; . \quad (2.5.4)$$

All the terms at the right hand side of (2.5.4) are orthogonal, except two consecutive terms whose scalar product equals $1/100$ when $n_i \geq 1$ and $m_{i+1} \leq -1$, and also when $m_i \geq 1$ and $n_i \leq -1$. Denote by $|g|$ the word length of $g \in \mathbb{F}_2$. We conclude that

$$\|c_g\|_2^2 = \frac{1}{100}|g| + \frac{1}{50} \cdot \text{number of sign changes in } n_0, m_1, n_1, \ldots, m_k, n_k \; . \quad (2.5.5)$$

Denote by $\omega : G \times X \to (0, \infty)$ the Radon-Nikodym cocycle. Define $\mathcal{F} = \{a, a^{-1}, b, b^{-1}\}$. Denote by $\lambda : G \to \mathcal{U}(\ell^2(G))$ the left regular representation.

Combining Lemma 2.3.2 with (2.5.5) and Kesten's [Ke58], we find that

$$\sum_{g \in \mathcal{F}} \int_X \sqrt{\omega(g, x)} \, d\mu(x) \geq 4 \exp\big(-\frac{3}{500}\big) > 2\sqrt{3} = \Big\| \sum_{g \in \mathcal{F}} \lambda_g \Big\| \; .$$

So by Proposition 1.7.2, the action $G \curvearrowright (X, \mu)$ is nonamenable in the sense of Zimmer.

When $g_0 \in G \setminus \{e\}$, there exist integers α, β with $\alpha \geq 1$ and $\beta \geq 0$ such that $|g_0^n| = \alpha |n| + \beta$ for all $n \in \mathbb{Z} \setminus \{0\}$. It then follows from (2.5.5) that

$$\sum_{n \in \mathbb{Z}} \exp\big(-\frac{1}{2} \, \|c_{g_0^n}\|_2^2\big) \leq \sum_{n \in \mathbb{Z}} \exp\big(-\frac{1}{200} \, |g_0^n|\big) \leq 1 + 2 \sum_{n=1}^{\infty} \exp\big(-\frac{\alpha}{200} n\big) < +\infty \; .$$

So by Proposition 2.2.1, the transformation $x \mapsto g_0 \cdot x$ is dissipative.

Let $m \geq 220$. The m-th power diagonal action $G \curvearrowright (X^m, \mu^m)$ is a Bernoulli action whose corresponding 1-cocycle $(c_{m,g})_{g\in G}$ satisfies $\|c_{m,g}\|_2^2 = m\,\|c_g\|_2^2$. Define $B_n = \{g \in G \mid |g| = n\}$. For every $n \geq 1$, we have $|B_n| = 4 \cdot 3^{n-1}$. Therefore, using (2.5.5), we get that

$$\sum_{g\in G} \exp\Bigl(-\frac{1}{2}\,\|c_{m,g}\|_2^2\Bigr) \leq \sum_{g\in G} \exp\Bigl(-\frac{m}{200}\,|g|\Bigr) = 1+\sum_{n=1}^{\infty} \exp\Bigl(-\frac{m}{200}\,n\Bigr)\cdot 4\cdot 3^{n-1} < +\infty$$

because $m > 200 \cdot \log 3$. It follows from Proposition 2.2.1 that the m-th power diagonal action $G \curvearrowright (X^m, \mu^m)$ is dissipative.

It remains to prove that $G \curvearrowright (X, \mu)$ is ergodic and of type III_1. Denote by $G \curvearrowright (X \times \mathbb{R}, \mu \times \nu)$ the Maharam extension. Let $Q \in L^\infty(X \times \mathbb{R})$ be a G-invariant function. The main point is to prove that $Q \in 1 \otimes L^\infty(\mathbb{R})$.

Recall that we denoted by $W_a, W_b \subset G$ the sets of reduced words that start with a strictly positive power of a respectively b. Similarly define $W_{a^{-1}}$ and $W_{b^{-1}}$. Note that

$$\mathbb{F}_2 = \{e\} \sqcup W_a \sqcup W_{a^{-1}} \sqcup W_b \sqcup W_{b^{-1}} \ .$$

Whenever $U \subset G$, we denote $(X_U, \mu_U) = \prod_{g\in U}(\{0,1\}, \mu_g)$ and we identify $(X, \mu) = (X_U \times X_{U^c}, \mu_U \times \mu_{U^c})$. Define $\Lambda = \langle b, a^{-1}ba\rangle$ and note that Λ is freely generated by b and $a^{-1}ba$. The concatenation wv of a reduced word $w \in \Lambda$ and a reduced word $v \in W_a$ remains reduced. In particular, for all $w \in \Lambda$ and $v \in W_a$, the last letter of wv equals the last letter of v. Therefore, the restriction of F to $U := \Lambda W_a$ is Λ-invariant. It follows that $\Lambda \curvearrowright (X_U, \mu_U)$ is a probability measure preserving Bernoulli action.

We claim that the action $\Lambda \curvearrowright (X, \mu)$ is conservative. Whenever $k \geq 1$ and $n_i, m_j \geq 1$, the element

$$g = (a^{-1}ba)^{n_1}\, b^{m_1} \cdots (a^{-1}ba)^{n_k}\, b^{m_k} = a^{-1}b^{n_1}a\, b^{m_1}\ a^{-1}b^{n_2}a\, b^{m_2} \cdots a^{-1}b^{n_k}a\, b^{m_k}$$

belongs to Λ and by (2.5.5), we have

$$\|c_g\|_2^2 = \frac{1}{100}\Bigl(2k + \sum_{i=1}^{k}(n_i + m_i)\Bigr) + \frac{2k-1}{50} < \frac{1}{100}\sum_{i=1}^{k}(n_i + m_i) + \frac{3k}{50}\ .$$

It follows that

$$\begin{aligned}\sum_{g\in\Lambda}\exp\big(-16\|c_g\|_2^2\big) &\geq \sum_{k=1}^{\infty}\sum_{n_1,m_1,\dots,n_k,m_k=1}^{\infty}\exp\big(-\frac{24}{25}k\big)\prod_{i=1}^{k}\exp\big(-\frac{4}{25}(n_i+m_i)\big)\\ &=\sum_{k=1}^{\infty}\exp\big(-\frac{24}{25}k\big)\left(\sum_{n=1}^{\infty}\exp\big(-\frac{4}{25}n\big)\right)^{2k}\\ &=\sum_{k=1}^{\infty}\left(\frac{\exp\big(-\frac{32}{25}\big)}{\big(1-\exp\big(-\frac{4}{25}\big)\big)^2}\right)^{k}\\ &=+\infty\,.\end{aligned}$$

From Proposition 2.2.1, the claim that $\Lambda \curvearrowright (X,\mu)$ is conservative follows.

Since $\Lambda \curvearrowright (X_{U^c},\mu_{U^c})$ is a factor action of $\Lambda \curvearrowright (X,\mu)$, it is also conservative, and by Maharam's theorem, its Maharam extension $\Lambda \curvearrowright (X_{U^c}\times\mathbb{R},\mu_{U^c}\times\nu)$ is conservative as well. Since the action $\Lambda \curvearrowright (X_U,\mu_U)$ preserves the probability measure μ_U, we can view $\Lambda \curvearrowright (X\times\mathbb{R},\mu\times\nu)$ as the diagonal product of the mixing, probability measure preserving $\Lambda \curvearrowright (X_U,\mu_U)$ and the conservative $\Lambda \curvearrowright (X_{U^c}\times\mathbb{R},\mu_{U^c}\times\nu)$. By the equivalent characterizations of mild mixing in Definition 1.6.1, it follows that $Q\in L^\infty(X_{U^c}\times\mathbb{R})$. In particular, $Q\in L^\infty(X_{W_a^c}\times\mathbb{R})$.

We make the same reasoning for $W_{a^{-1}}$ and the group $\langle b, aba^{-1}\rangle$, for W_b and the group $\langle a, b^{-1}ab\rangle$ and for $W_{b^{-1}}$ and the group $\langle a, bab^{-1}\rangle$. Since $W_a\cup W_{a^{-1}}\cup W_b\cup W_{b^{-1}} = G\setminus\{e\}$, it follows that $Q\in L^\infty(X_{\{e\}}\times\mathbb{R})$.

We finally use the group $\Lambda = \langle aba^{-1}, a^2ba^{-2}\rangle$. We have $\Lambda\subset W$, so that $\Lambda \curvearrowright (X_\Lambda,\mu_\Lambda)$ is a probability measure preserving Bernoulli action. For all $k\geq 1$ and $n_i,m_j\geq 1$, we have that

$$\begin{aligned}g &= (aba^{-1})^{n_1}\,(a^2ba^{-2})^{m_1}\,\cdots\,(aba^{-1})^{n_k}\,(a^2ba^{-2})^{m_k}\\ &= a\,b^{n_1}\,a\,b^{m_1}\,a^{-1}\,b^{n_2}\,a\,b^{m_2}\,a^{-1}\,\cdots\,a^{-1}\,b^{n_k}\,a\,b^{m_k}\,a^{-2}\end{aligned}$$

and thus, using (2.5.5),

$$\|c_g\|_2^2 = \frac{1}{100}\Big(2k+2+\sum_{i=1}^{k}(n_i+m_i)\Big)+\frac{2k-1}{50}\,.$$

The same computation as above shows that $\Lambda \curvearrowright (X,\mu)$ is conservative. As above, it follows that $Q\in L^\infty(X_{\Lambda^c}\times\mathbb{R})$. Altogether, we have proved that $Q\in 1\otimes L^\infty(\mathbb{R})$.

So we get that $G \curvearrowright (X,\mu)$ is ergodic. To prove that the action is of type III_1, it suffices to show that the essential range of the map $x \mapsto \omega(a,x)$ generates a dense subgroup of $\mathbb{R}^+_*$. But using (1.8.7), we get that

$$\omega(a,x) = \prod_{g \in G} \frac{\mu_{ag}(x_g)}{\mu_g(x_g)} = \frac{\mu_a(x_e)}{\mu_e(x_e)} = \begin{cases} 6/5 & \text{if } x_e = 0, \\ 4/5 & \text{if } x_e = 1. \end{cases}$$

Since $6/5$ and $4/5$ generate a dense subgroup of $\mathbb{R}^+_*$, the proposition is proved.

□

Remark 2.5.9. The stable type of the nonsingular Bernoulli action $\mathbb{F}_2 \curvearrowright (X,\mu)$ constructed in Proposition 2.5.8 is given as follows. The essential ranges of the maps $(x,x') \mapsto \omega(g,x)/\omega(g,x')$, $g \in \mathbb{F}_2$, generate the subgroup $(2/3)^{\mathbb{Z}}$ of $\mathbb{R}^+_*$ and $\omega(g,x) \in (4/5)\cdot(2/3)^{\mathbb{Z}}$ for all $g \in \mathbb{F}_2$ and a.e. $x \in X$. Combining the proofs of Proposition 2.5.3 and 2.5.8, it follows that for every ergodic pmp action $\mathbb{F}_2 \curvearrowright (Y,\eta)$, the diagonal action $\mathbb{F}_2 \curvearrowright Y \times X$ is ergodic and that, varying $\mathbb{F}_2 \curvearrowright (Y,\eta)$, the type of this diagonal action ranges over III_μ with $\mu \in \{1\} \cup \{(2/3)^{1/k} \mid k \geq 1\}$.

Taking a slight variant of the action in Proposition 2.5.8, by putting

$$\mu_g(0) = \begin{cases} 3/5 & \text{if } g \in W_a, \\ 5/12 & \text{if } g \in W_b, \\ 1/2 & \text{if } g \in W, \end{cases}$$

all the conclusions of Proposition 2.5.8 remain valid – except that we have to take $m \geq 317$ to get a dissipative diagonal action $\mathbb{F}_2 \curvearrowright X^m$ – and moreover, the Maharam extension of $\mathbb{F}_2 \curvearrowright (X,\mu)$ is weakly mixing, so that all diagonal actions $\mathbb{F}_2 \curvearrowright Y \times X$ have type III_1. This follows because now, the essential ranges of the maps $(x,x') \mapsto \omega(g,x)/\omega(g,x')$, $g \in \mathbb{F}_2$, generate a dense subgroup of $\mathbb{R}^+_*$, namely the subgroup generated by $2/3$ and $5/7$.

The Bernoulli action $\mathbb{F}_2 \curvearrowright (X,\mu)$ constructed in Proposition 2.5.8 has the property that the diagonal action $\mathbb{F}_2 \curvearrowright (X^m,\mu^m)$ is dissipative for m large enough. This diagonal action is a Bernoulli action associated with $\mathbb{F}_2 \curvearrowright I$, where I consists of m disjoint copies of $\mathbb{F}_2$. This operation multiplies $\|c_g\|_2^2$ with a factor m, up to the point of satisfying the dissipative criterion in Proposition 2.2.1. It is however remarkably more delicate to produce a plain Bernoulli action $\mathbb{F}_2 \curvearrowright \prod_{g \in \mathbb{F}_2}(\{0,1\},\mu_g)$ that is dissipative. We do this in Proposition 2.5.11, based on Lemma 2.5.10 below, which provides a 1-cocycle for $\mathbb{Z}$ with large growth, but bounded "implementing function".

Lemma 2.5.10. *Let $D > 0$. There exists a function $H : \mathbb{Z} \to [0,1]$ such that $H(n) = 0$ for all $n \leq 0$ and such that the formula $c_k(n) = H(n) - H(n-k)$ defines a 1-cocycle $c : \mathbb{Z} \to \ell^2(\mathbb{Z})$ satisfying $\|c_k\|_2^2 \geq D|k|^{3/2}$ for all $k \in \mathbb{Z}$.*

Proof. For every integer $n \geq 1$, define the function

$$H_n : \mathbb{Z} \to [0,1] : H_n(k) = \begin{cases} k/n & \text{if } 0 \leq k \leq n, \\ (2n-k)/n & \text{if } n \leq k \leq 2n, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $(a_n)_{n\geq 0}$ be an increasing sequence of integers with $a_n \geq 1$ for all n and $\sum_{n=0}^{\infty} a_n^{-1} < +\infty$. A concrete sequence a_n will be chosen below. Put $b_0 = 0$ and $b_n = \sum_{k=0}^{n-1} 2a_k$ for all $n \geq 1$. Define the function

$$H : \mathbb{Z} \to [0,1] : H(k) = \begin{cases} H_{a_n}(k - b_n) & \text{if } n \geq 0 \text{ and } b_n \leq k \leq b_n + 2a_n = b_{n+1}, \\ 0 & \text{elsewhere.} \end{cases}$$

Note that we can view H as a "concatenation" of translates of H_{a_n}, in such a way that their supports become disjoint. By construction, $H(k) = 0$ for all $k \leq 0$.

Define $c_k(n) = H(n) - H(n-k)$. We have

$$\|c_1\|_2^2 = \sum_{m=1}^{\infty} |H(m) - H(m-1)|^2 = \sum_{n=0}^{\infty} \sum_{m=b_n+1}^{b_n+2a_n} |H(m) - H(m-1)|^2$$

$$= \sum_{n=0}^{\infty} \sum_{m=b_n+1}^{b_n+2a_n} \frac{1}{a_n^2} = 2\sum_{n=0}^{\infty} \frac{1}{a_n} < +\infty \, .$$

So, $c_1 \in \ell^2(\mathbb{Z})$. Since c satisfies the 1-cocycle relation, we have that $c_k \in \ell^2(\mathbb{Z})$ for all $k \in \mathbb{Z}$.

For every $k \geq 1$, define $\mathcal{F}_k = \{n \in \mathbb{Z} \mid n \geq 0 \text{ and } a_n \geq k\}$. For $k \geq 1$, we then have

$$\|c_k\|_2^2 \geq \sum_{n\in\mathcal{F}_k} \sum_{m=b_n+k}^{b_n+a_n} |c_k(m)|^2 = \sum_{n\in\mathcal{F}_k} \sum_{m=b_n+k}^{b_n+a_n} \frac{k^2}{a_n^2}$$

$$= \sum_{n\in\mathcal{F}_k} \frac{k^2(a_n - k + 1)}{a_n^2} \geq \frac{k^2}{2} \sum_{n\in\mathcal{F}_{2k}} \frac{1}{a_n} \, ,$$

where the last inequality follows because $\mathcal{F}_{2k} \subset \mathcal{F}_k$ and $a_n - k + 1 \geq a_n/2$ when $n \in \mathcal{F}_{2k}$.

Let $D > 0$. Take $0 < \delta \leq 1$ such that $12\sqrt{\delta} \leq D^{-1}$. Put $a_0 = 1$ and $a_n = \lceil \delta n^2 \rceil$ for all $n \geq 1$. We prove that $\|c_k\|_2^2 \geq D|k|^{3/2}$ for all $k \in \mathbb{Z}$. Since $\|c_{-k}\|_2 = \|c_k\|_2$, it suffices to prove this inequality for every $k \geq 1$.

Fix $k \geq 1$ and put $n_0 = \left\lceil \sqrt{2k/\delta} \right\rceil$. Note that $n_0 \geq 1$ and $\sqrt{\delta} n_0 \geq \sqrt{2k} \geq 1$. When $n \geq n_0$, we have $a_n \geq 2k$ and thus, $n \in \mathcal{F}_{2k}$. Therefore,

$$\begin{aligned} \|c_k\|_2^2 &\geq \frac{k^2}{2} \sum_{n=n_0}^{\infty} \frac{1}{a_n} \geq \frac{k^2}{2} \sum_{n=n_0}^{\infty} \frac{1}{1+\delta n^2} \\ &\geq \frac{k^2}{2} \int_{n_0}^{\infty} \frac{1}{1+\delta x^2}\, dx = \frac{k^2}{2\sqrt{\delta}} \left(\frac{\pi}{2} - \arctan(\sqrt{\delta} n_0) \right) . \end{aligned}$$

Since $\sqrt{\delta} n_0 \geq 1$ and $\frac{\pi}{2} - \arctan(x) \geq 1/(2x)$ for all $x \geq 1$, we get that

$$\|c_k\|_2^2 \geq \frac{k^2}{4\delta n_0} \geq \frac{k^2}{4\delta(\sqrt{2k/\delta}+1)} = \frac{k^{3/2}}{4\sqrt{\delta}} \frac{1}{\sqrt{2}+\sqrt{\delta/k}} \geq \frac{k^{3/2}}{12\sqrt{\delta}} \geq D\, k^{3/2} ,$$

because $\sqrt{\delta/k} \leq 1$ and $12\sqrt{\delta} \leq D^{-1}$. □

Proposition 2.5.11. *Let $G = \mathbb{F}_2$. There exists a function $F : G \to [1/4, 3/4]$ such that the Bernoulli action $G \curvearrowright (X, \mu) = \prod_{g \in G}(\{0,1\}, \mu_g)$ with $\mu_g(0) = F(g)$ is nonsingular, essentially free and dissipative.*

Proof. Again, denote the free generators of $\mathbb{F}_2$ by a and b and denote the set of reduced words that end with a nonzero power of a by $E_a \subset G$. Similarly define E_b and note that $G = \{e\} \sqcup E_a \sqcup E_b$. An element $g \in E_a$ is either a nonzero power of a or can be uniquely written as $g = ha^n$ with $h \in E_b$ and $n \in \mathbb{Z} \setminus \{0\}$. We can therefore define a map $\pi_a : E_a \to \mathbb{Z}$ by

$$\pi_a(a^n) = n \ \text{ when } n \in \mathbb{Z} \setminus \{0\}, \text{ and } \ \pi_a(ha^n) = n \ \text{ when } h \in E_b \text{ and } n \in \mathbb{Z} \setminus \{0\}.$$

We similarly define $\pi_b : E_b \to \mathbb{Z}$.

Fix $D > 0$ such that $D > 32 \log 3$. Using Lemma 2.5.10, fix a function $H : \mathbb{Z} \to [0,1]$ such that $H(n) = 0$ for all $n \leq 0$ and such that the formula $\gamma_k(n) = H(n) - H(n-k)$ defines a 1-cocycle $\gamma : \mathbb{Z} \to \ell^2(\mathbb{Z})$ satisfying $\|\gamma_k\|_2^2 \geq D|k|$ for all $k \in \mathbb{Z}$.

We define

$$F : G \to [1/4, 3/4] : F(g) = \begin{cases} 1/2 + H(\pi_a(g))/4 & \text{if } g \in E_a, \\ 1/2 - H(\pi_b(g))/4 & \text{if } g \in E_b, \\ 1/2 & \text{if } g = e. \end{cases}$$

Define $c_g(h) = F(h) - F(g^{-1}h)$. Define the isometries

$$\theta_a : \ell^2(\mathbb{Z}) \to \ell^2(G) : \theta_a(\delta_n) = \delta_{a^n} \quad \text{and} \quad \theta_b : \ell^2(\mathbb{Z}) \to \ell^2(G) : \theta_b(n) = \delta_{b^n} \; .$$

We then have $c_a = \theta_a(\gamma_1)/4$ and $c_b = -\theta_b(\gamma_1)/4$. So, $c_g \in \ell^2(G)$ for every $g \in G$. It follows that the Bernoulli action $G \curvearrowright (X,\mu) = \prod_{g\in G}(\{0,1\},\mu_g)$ with $\mu_g(0) = F(g)$ is nonsingular and essentially free.

We prove that $\sum_{g\in G} \exp(-\|c_g\|_2^2/2) < \infty$. It then follows from Proposition 2.2.1 that $G \curvearrowright (X,\mu)$ is dissipative.

When
$$g = a^{n_0}\; b^{m_1}\; a^{n_1}\; \cdots\; b^{m_k}\; a^{n_k}$$
is a reduced word, with $k \geq 0$, $n_0, n_k \in \mathbb{Z}$ and $n_i, m_j \in \mathbb{Z} \setminus \{0\}$, the 1-cocycle relation implies that

$$4c_g = \theta_a(\gamma_{n_0}) - a^{n_0}\cdot\theta_b(\gamma_{m_1}) + a^{n_0}b^{m_1}\cdot\theta_a(\gamma_{n_1}) - \cdots + a^{n_0}b^{m_1}a^{n_1}\cdots b^{m_k}\cdot\theta_a(\gamma_{n_k}) \; .$$

All terms in the sum on the right hand side are orthogonal, except possibly consecutive terms, whose scalar products are equal to

$$-\langle \theta_a(\gamma_{n_i}), a^{n_i}\cdot\theta_b(\gamma_{m_{i+1}})\rangle = -\gamma_{n_i}(n_i)\,\overline{\gamma_{m_{i+1}}(0)} = H(n_i)\,H(-m_{i+1}) \geq 0 \; ,$$

or equal to

$$-\langle \theta_b(\gamma_{m_i}), b^{m_i}\cdot\theta_a(\gamma_{n_i})\rangle = -\gamma_{m_i}(m_i)\,\overline{\gamma_{n_i}(0)} = H(m_i)\,H(-n_i) \geq 0 \; .$$

We conclude that

$$16\,\|c_g\|_2^2 \geq \sum_{i=0}^{k} \|\gamma_{n_i}\|_2^2 + \sum_{j=1}^{k} \|\gamma_{m_j}\|_2^2 \geq D \sum_{i=0}^{k} |n_i| + D \sum_{j=1}^{k} |m_j| = D\,|g| \; ,$$

where $|g|$ denotes the word length of $g \in \mathbb{F}_2$. So we have proved that $\|c_g\|_2^2 \geq (D/16)\,|g|$ for all $g \in G$.

Since for $n \geq 1$, there are precisely $4 \cdot 3^{n-1}$ elements in $\mathbb{F}_2$ with word length equal to n, it follows that

$$\sum_{g\in G} \exp(-\|c_g\|_2^2/2) \leq \sum_{g\in G} \exp(-D\,|g|/32) = 1 + 4 \sum_{n=1}^{\infty} \exp(-D\,n/32)\; 3^{n-1} < +\infty \; ,$$

because $D/32 > \log 3$. So the proposition is proved. □

The function $H = 1_{[1,+\infty)}$ implements a 1-cocycle $c : \mathbb{Z} \to \ell^2(\mathbb{Z})$ satisfying $\|c_k\|_2^2 = |k|$ for all $k \in \mathbb{Z}$. Multiplying H by a constant $D > 0$, we obviously obtain a 1-cocycle c with growth $\|c_k\|_2^2 = D^2\,|k|$. It is however more delicate to attain this growth while keeping $\|H\|_\infty \leq 1$. In particular, the easy construction of Lemma 2.4.8 does not give such large growth. We need a more intricate construction with an oscillating function H, giving examples where $\|c_k\|_2^2 \geq D\,|k|^{3/2}$, while $H : \mathbb{Z} \to [0,1]$.

2.6 A question of Bowen and Nevo

In this section, we give a positive answer to [BN11, Question 4.6] by proving the following theorem. The proof is independent of our results on type III Bernoulli actions, but uses several ingredients that we have encountered previously in this part of the thesis.

Theorem 2.6.1. *Let G be an arbitrary countable infinite group and let $\lambda \in (0,1]$. Then G admits an essentially free, amenable, weakly mixing action of stable type III$_\lambda$.*

We will prove this theorem in several steps. First, we will prove it for a very specific choice of the group G.

Lemma 2.6.2. *Let $G_1 = \left(\oplus_{n\in\mathbb{Z}}\mathbb{Z}/3\mathbb{Z}\right) \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on the direct sum by shifting. Then G_1 admits an essentially free, amenable, weakly mixing action of stable type III$_\lambda$.*

Proof. Let μ_0 be a non uniform probability measure on $\mathbb{Z}/3\mathbb{Z}$ with $\mu_0(i) > 0$ for every $i \in \mathbb{Z}/3\mathbb{Z}$. Define $(X,\mu) = (\mathbb{Z}/3\mathbb{Z}, \mu_0)^{\mathbb{Z}}$ and consider the nonsingular action $G_1 \curvearrowright (X,\mu)$ where each $\mathbb{Z}/3\mathbb{Z}$ acts by translation on $\mathbb{Z}/3\mathbb{Z}$ and where $\mathbb{Z}$ acts by Bernoulli shift. If the ratios $\mu_0(0)/\mu_0(1)$ and $\mu_0(1)/\mu_0(2)$ generate a dense subgroup of $\mathbb{R}^+_*$, put $\lambda = 1$ and otherwise define $\lambda \in (0,1)$ so that this subgroup is given by $\lambda^{\mathbb{Z}}$.

Clearly, the action $G_1 \curvearrowright (X,\mu)$ is essentially free. Next, since the group G_1 is amenable, the action is amenable as well. As its restriction to $\mathbb{Z}$ is a pmp Bernoulli shift, it is weakly mixing and for any essentially free ergodic pmp action $G_1 \curvearrowright (Y,\nu)$, every G_1-invariant function $Q \in L^\infty(X \times Y \times \mathbb{R})^{G_1}$ is of the form $Q(x,y,s) = P(s)$ for a.e. $(x,y) \in X \times Y$ and $P \in L^\infty(\mathbb{R})$. In addition, this function P is translation invariant under the essential image of the Radon-Nykodim cocycle $X \to \mathbb{R}, x \mapsto \log(\frac{dg_*\mu}{d\mu}(x))$ which equals $\log(\lambda)\mathbb{Z}$. Hence the action $G_1 \curvearrowright (X,\mu)$ is of stable type III$_\lambda$.

$\square$

Let us introduce the following ad-hoc terminology which adapts the notion of measure equivalent groups, see [Fu99, Fu11] to our setting.

Definition 2.6.3. We say that G_1 is a *weakly mixing measure equivalence (ME) subgroup* of G if there exists a measure space (Ω, ν) and an essentially free measure preserving action $G_1 \times G \curvearrowright (\Omega, \nu)$ with the following properties.

- Both the restriction of the action to G_1 and the restriction of the action to G are dissipative.
- The restriction of the action to G admits a fundamental domain of finite measure.
- For every ergodic pmp action $G \curvearrowright (Y, \eta)$, the induced action $G_1 \curvearrowright (\Omega \times Y)/G$ is ergodic. Here, G acts diagonally on $\Omega \times Y$ and since this action is dissipative, the quotient is well defined.

Now we prove that admitting an action as in Theorem 2.6.1 passes through our notion of weakly mixing measure equivalence (ME) subgroup in the following sense.

Lemma 2.6.4. *If G_1 is a weakly mixing ME subgroup of G and if G_1 admits an essentially free, amenable, weakly mixing action of stable type III$_\lambda$* ($\lambda \in (0, 1]$), *then also G admits such an action.*

Proof. Take (Ω, ν) as in the previous definition and let $G_1 \curvearrowright (Z, \eta)$ be an essentially free, amenable, weakly mixing action of stable type III_λ. Consider the essentially free nonsingular action $G \curvearrowright (Z \times \Omega)/G_1$, where G_1 acts diagonally on $Z \times \Omega$. Fix an ergodic pmp action $G \curvearrowright (Y, \rho)$. We have to prove that the diagonal action $G \curvearrowright (Z \times \Omega)/G_1 \times Y$ is amenable, ergodic and of type III_λ. We can now argue as in the proof that shows that measure equivalence implies stable orbit equivalence [Fu11]: The orbit equivalence relation of the action $G \curvearrowright (Z \times \Omega)/G_1 \times Y$ is isomorphic to the restriction of the orbit equivalence relation of $G_1 \times G \curvearrowright Z \times \Omega \times Y$ to a nonnegligible subset. So, the diagonal action $G \curvearrowright (Z \times \Omega)/G_1 \times Y$ is stably orbit equivalent to the diagonal action $G_1 \curvearrowright Z \times (\Omega \times Y)/G$. We therefore have to prove that the latter is amenable, ergodic and of type III_λ. The amenability follows because $G_1 \curvearrowright Z$ is amenable. Since $G_1 \curvearrowright (\Omega \times Y)/G$ is ergodic and pmp and since $G_1 \curvearrowright Z$ is weakly mixing and of stable type III_λ, we get that $G_1 \curvearrowright Z \times (\Omega \times Y)/G$ is ergodic and of type III_λ. □

For the proof of the following lemma, we use an induction procedure to arbitrary infinite groups along a weakly mixing cocycle introduced in [BN11].

Lemma 2.6.5. *$\mathbb{Z}$ is a weakly mixing ME subgroup of any countable nonamenable group G.*

Proof. Fix a symmetric probability measure μ_0 on G whose support generates the group G. Put $(X, \mu) = (G, \mu_0)^{\mathbb{Z}}$ and let $\mathbb{Z} \curvearrowright (X, \mu)$ be the Bernoulli shift, given by $(n \cdot x)_k = x_{k-n}$. Denote by $\omega : \mathbb{Z} \times X \to G$ the 1-cocycle introduced in [BN11, Theorem 6.1] and uniquely determined by $\omega(1, x) = x_0$. Denote by λ the counting measure on G and define $(\Omega, \nu) = (X \times G, \mu \times \lambda)$. Define the action $\mathbb{Z} \times G \curvearrowright \Omega$ given by

$$(n, g) \cdot (x, h) = (n \cdot x, \omega(n, x)hg^{-1}) \quad \text{for all } (n, g) \in \mathbb{Z} \times G \;,\; (x, h) \in X \times G \;.$$

This action is essentially free and measure preserving. Also, the restriction of the action to G has $X \times \{e\}$ as a finite measure fundamental domain. In [BN11, Theorem 6.1], it is proven that $\mathbb{Z} \curvearrowright (\Omega \times Y)/G$ is ergodic for every pmp ergodic action $G \curvearrowright (Y, \eta)$. So we only have to prove that the action $\mathbb{Z} \curvearrowright \Omega$ is dissipative.

Fix $g_0 \in G$ and define

$$V_{g_0} = \{(x, g) \in \Omega \mid \forall k \geq 0 : x_{-k} \cdots x_{-1} x_0 g \neq g_0\} \;.$$

Defining $\pi : \Omega \to G : \pi(x, g) = g$, we have that

$$V_{g_0} = \{(x, g) \in \Omega \mid \forall k \geq 1 : \pi(k \cdot (x, g)) \neq g_0\} \;.$$

So, $1 \cdot V_{g_0} \subset V_{g_0}$.

For every fixed $g \in G$, the measure

$$\mu\big(\{x \in X \mid \text{ there are infinitely many } k \geq 0 \text{ with } x_{-k} \cdots x_{-1} x_0 g = g_0 \;\}\big) \tag{2.6.1}$$

equals the probability that the invariant random walk on G with transition probabilities given by μ_0 and starting at g visits infinitely often the element g_0. Since the group G is nonamenable and the support of μ_0 generates G, this random walk is transient and the measure in (2.6.1) is zero for every $g \in G$. This means that $\bigcup_{k \in \mathbb{Z}} k \cdot V_{g_0}$ has a complement of measure zero for every $g_0 \in G$. Since $1 \cdot V_{g_0} \subset V_{g_0}$, it follows that the action of $\mathbb{Z}$ on $\Omega_{g_0} = \Omega \setminus \bigcap_{k \in \mathbb{Z}} k \cdot V_{g_0}$ is dissipative with fundamental domain $A_{g_0} := V_{g_0} \setminus 1 \cdot V_{g_0}$. Since $X \times \{g_0\} \subset \Omega_{g_0}$ and $g_0 \in G$ is arbitrary, it follows that $\mathbb{Z} \curvearrowright \Omega$ is dissipative with fundamental domain $\bigcup_{g_0 \in G} A_{g_0} \cap (X \times \{g_0\})$. ☐

Proof of Theorem 2.6.1. By [OW80], all essentially free ergodic pmp actions of infinite amenable groups are orbit equivalent. Applying this to a pmp Bernoulli action of G_1, it follows as in the proof of [Fu11, Theorem 2.5] that any infinite amenable group G_1 is a weakly mixing ME subgroup of any other infinite amenable group G. Therefore, by combining Lemma 2.6.2 and Lemma 2.6.4, the theorem follows for any amenable group and in particular for $\mathbb{Z}$. Applying Lemma 2.6.5 and using Lemma 2.6.4 once more, we get that the theorem also holds in the nonamenable case and thus for all groups.

□

Remark 2.6.6. For a countable nonamenable group G, the proof of Theorem 2.6.1 provides an explicit essentially free, amenable, weakly mixing action of stable type III_1. Indeed, it suffices to combine the explicit action $\mathbb{Z} \curvearrowright (Z, \eta)$ of stable type III_1 given by Corollary 2.5.1 with the explicit 1-cocycle $\omega : \mathbb{Z} \times X \to G$ of [BN11, Theorem 6.1].

Note that the resulting amenable, weakly mixing and stable type III_1 action of G on $\Xi = (Z \times X \times G)/\mathbb{Z}$ has the property that the diagonal action $G \curvearrowright \Xi \times \Xi$ is dissipative, contrary to the action of G on its Poisson boundary, which is doubly ergodic, see e.g. [Mo06]. To prove that $G \curvearrowright \Xi \times \Xi$ is dissipative, we write $\Lambda = \mathbb{Z} \times \mathbb{Z}$ and note that it is sufficient to prove that the action of Λ on $(Z \times Z \times X \times X \times G \times G)/G$ is dissipative. So, it suffices to prove that $\Lambda \curvearrowright (X \times X \times G \times G)/G$ is dissipative. This means that we have to prove that the action

$$\Lambda \curvearrowright X \times X \times G \quad \text{given by} \quad (k,l) \cdot (x,y,g) = (k \cdot x, l \cdot y, \omega(k,x) g \omega(l,y)^{-1})$$

for all $(k,l) \in \mathbb{Z}^2$ and $(x,y,g) \in X \times X \times G$, is dissipative.

For all $(k,l) \in \Lambda$, denote

$$V_{k,l} = \{(x,y,e) \in X \times X \times G \mid \omega(k,x) \neq \omega(l,y)\} \, .$$

For every $n \geq 1$, denote $\Lambda_n = n\mathbb{Z} \times n\mathbb{Z}$ and write

$$V_n = \bigcap_{(k,l) \in \Lambda_n \setminus \{(0,0)\}} V_{k,l} \, .$$

For $k, l \geq 1$, we have

$$\begin{aligned} V_{k,l} &= \{(x,y,e) \mid x_{-k+1} \cdots x_{-1} x_0 \neq y_{-l+1} \cdots y_{-1} y_0\} \\ &= \{(x,y,e) \mid x_{-k+1} \cdots x_0 y_0^{-1} \cdots y_{-l+1}^{-1} \neq e\} \, , \end{aligned}$$

so that $(\mu \times \mu \times \lambda)(V_{k,l}) = 1 - \mu^{*(k+l)}(e)$. When $k \geq 1$ and $l \leq -1$, we similarly find

$$V_{k,l} = \{(x,y,e) \mid x_{-k+1} \cdots x_0 y_{-l}^{-1} \cdots y_1^{-1} \neq e\}$$

and conclude that $(\mu \times \mu \times \lambda)(V_{k,l}) = 1 - \mu^{*(|k|+|l|)}(e)$ for all $k, l \in \mathbb{Z}$. Since G is nonamenable, we can fix $0 < \rho < 1$ so that $\mu^{*m}(e) \leq \rho^m$ for all $m \geq 1$. It follows that $(\mu \times \mu \times \lambda)(V_n) \to 1$, so that $\bigcup_{n\geq 1} V_n$ equals $X \times X \times \{e\}$, up to measure zero.

By construction, $(k,l) \cdot V_n \cap (X \times X \times \{e\}) = \emptyset$ for all $(k,l) \in \Lambda_n \setminus \{(0,0)\}$. In particular, $(k,l) \cdot V_n \cap V_n = \emptyset$, so that the action $\Lambda_n \curvearrowright \Lambda_n \cdot V_n$ is dissipative. Since $\Lambda_n < \Lambda$ has finite index, also the action $\Lambda \curvearrowright \Lambda \cdot V_n$ is dissipative. Since the union of all V_n equals $X \times X \times \{e\}$ up to measure zero, we conclude that the action $\Lambda \curvearrowright \Lambda \cdot (X \times X \times \{e\}) = X \times X \times G$ is dissipative.

2.7 A question of Monod

Generalizing the action of the wreath product group $\big(\oplus_{n\in\mathbb{Z}}\mathbb{Z}/3\mathbb{Z}\big) \rtimes \mathbb{Z}$ that we used in Lemma 2.6.2, we can also provide a negative answer to a problem posed by Monod [Mo06, Problem H]. To state Monod's problem properly, we first recall the definition of the *amenability degree* of a discrete group, see [Mo06, Definition 3.2].

Definition 2.7.1. Let G be a countable infinite discrete group. The *amenability degree* $a(G) \in \{1, 2, \ldots, \infty\}$ of G is the supremum over all integers n for which there exists an amenable action $G \curvearrowright (X, \mu)$ such that the diagonal action $G \curvearrowright (X^n, \mu^n)$ has only finitely many ergodic components.

For an infinite amenable group G, one has $a(G) = \infty$ as one can choose $G \curvearrowright (X, \mu)$ to be the trivial action on one point. Monod asked the question whether, conversely, $a(G) = \infty$ implied the amenability of G. We answer this in the negative.

Proposition 2.7.2. *Let Γ and Λ be countable groups with Γ nonamenable and Λ infinite amenable. The nonamenable group $G = \big(\bigotimes_{g\in\Lambda} \Gamma\big) \rtimes \Lambda$ has infinite amenability degree.*

Proof. Choose a nonsingular amenable action $\Gamma \curvearrowright (X_0, \mu_0)$. Define $(X, \mu) = (X_0, \mu_0)^\Lambda$ and consider the action $G \curvearrowright (X, \mu)$, where Λ acts by Bernoulli shift and where each copy of Γ acts on the corresponding copy of (X_0, μ_0) in the infinite product. We get that $G \curvearrowright (X, \mu)$ is amenable and that all its power actions $G \curvearrowright X^n = X \times \cdots \times X$ are ergodic, because the restriction of $G \curvearrowright (X, \mu)$ to the subgroup Λ is a pmp Bernoulli action. So G is a nonamenable group with infinite amenability degree. $\square$

Appendix A

The first L^2-cohomology of a discrete group

This appendix on L^2-cohomology is made up of two parts. Since L^2-cohomology is only used in a very ad-hoc manner throughout this thesis, the first part will consist of a rather short introduction to the basic notations and definitions we need. We will also collect a number of facts on L^2-cohomology, several of which are or at least rely on quite deep results within the theory. The proofs of these facts will be referred to but will not be included.
In the second part of this appendix, we will discuss the first L^2-cohomology of amenable discrete groups. More precisely, we will prove that amenable groups admit ℓ^2-cocycles of arbitrarily small growth, a result which will help us to construct Bernoulli action of type III of these groups in Chapter 2. Although this appeared as an original result in our article [VW17], we find it appropriate to include this result here, since it is somewhat independent of the rest of Chapter 2.

A.1 The first L^2-cohomology of a discrete group

Let G be a countable discrete group and let $\lambda : G \to \ell^2(G)$, $\lambda_g f(h) = f(g^{-1}h)$, $g, h \in G, f \in \ell^2(G)$ be its left regular representation. The objects at the heart of the study of the first L^2-cohomology of G are 1-cocycles and 1-coboundaries into $\ell^2(G)$ whose definition we will recall first.

Definition A.1.1. Let $\mathcal{H}$ be a Hilbert space and let $\pi : G \to B(\mathcal{H})$ be a unitary representation.

(i) A *1-cocycle* with respect to π is a map $c : G \to \mathcal{H}$ satisfying the cocycle identity

$$c(gh) = \pi(g)c(h) + c(g), \quad g, h \in G. \tag{A.1.1}$$

(ii) A 1-cocycle $c : G \to \mathcal{H}$ with respect to π is called a *1-coboundary* or *inner 1-cocycle* if there exists $\xi \in \mathcal{H}$ such that

$$c(g) = \pi(g)\xi - \xi \quad \text{for all } g \in G. \tag{A.1.2}$$

In this thesis, we are primarily interested in the case where $\mathcal{H} = \ell^2(G)$ and where π is the left regular representation. We usually write c_g instead of $c(g)$ for 1-cocycles c and $g \in G$. We should note the following for later use.

Lemma A.1.2. *Let $c : G \to \ell^2(G)$ be a 1-cocycle.*

(i) There exists a function $F : G \to \mathbb{C}$ satisfying $F(e) = 0$ and

$$c_g(h) = F(g^{-1}h) - F(h) \quad \text{for all } g, h \in G.$$

(ii) c is inner if and only if $\sup_{g \in G} \|c_g\|_2 < +\infty$.

Proof. (i) Define $F(g) = -c_g(g)$ for $g \in G$. The cocycle identity A.1.1 implies that $c_e(h) = c_e(h) - c_e(h) = 0$ for all $h \in G$ and in particular $F(e) = 0$. Another application of the cocycle identity yields

$$F(g^{-1}h) - F(h) = -c_{g^{-1}}(g^{-1}h) = -c_e(h) + c_g(h) = c_g(h)$$

for all $g, h \in G$.

(ii) The proof of this assertion can be found in [BdlHV08]. We include it for the sake of completeness. If $c_g = \lambda_g \xi - \xi$ for $g \in G, \xi \in \ell^2(G)$, then $\sup_{g \in G} \|c_g\|_2 \leq 2\|\xi\|_2$.

Conversely, let $\sup_{g \in G} \|c_g\|_2 < +\infty$ and define $K = \{c_g \, ; \, g \in G\} \subset \ell^2(G)$. Let $g \in G$, $\eta \in \ell^2(G)$, $r > 0$ and denote the closed ball of radius r centered at η by $\bar{B}(\eta, r)$. The computation

$$\|c_{gh} - (\lambda_g \eta + c_g)\| = \|(\lambda_g c_h + c_g) - (\lambda_g \eta + c_g)\| = \|c_h - \eta\|$$

for all $h \in G$, implies that

$$K \subset B(\eta, r) \quad \text{if and only if} \quad K \subset B(\lambda_g \eta + c_g, r). \tag{A.1.3}$$

By the lemma of the center, there exists a unique closed ball $\bar{B}(\xi, r_0)$ of minimal radius containing K. Hence by A.1.3, the center $\xi \in \ell^2(G)$ satisfies $\xi = \lambda_g \xi + c_g$ for all $g \in G$ and the proof is finished.

□

We are ready to define first L^2-cohomology of G.

Definition A.1.3. Consider the vector spaces

$$Z^1(G, \ell^2(G)) = \{c : G \to \ell^2(G) \ ; \ c \text{ 1-cocycle w.r.t } \lambda\}$$

$$B^1(G, \ell^2(G)) = \{b : G \to \ell^2(G) \ ; \ b \text{ 1-coboundary w.r.t } \lambda\} \subset Z^1(G, \ell^2(G)).$$

The quotient vector space

$$H^1(G, \ell^2(G)) = Z^1(G, \ell^2(G))/B^1(G, \ell^2(G))$$

is called the *first L^2-cohomology* of G.

The main purpose of studying $H^1(G, \ell^2(G))$ (or higher order L^2-cohomology) is to produce invariants of G by measuring the dimension of $H^1(G, \ell^2(G))$. However, as $H^1(G, \ell^2(G))$ is an infinite dimensional vector space for many groups, one needs to consider a much more subtle notion of dimension to obtain useful invariants. To be more precise, thanks to the work of Lück [L98], one can define the dimension $\dim_M$ of an arbitrary module over a tracial von Neumann algebra (M, τ) and then view $H^1(G, \ell^2(G))$ as a module over the group von Neumann algebra $L(G)$.

Definition A.1.4 ([CG85], [L98])**.** The quantity

$$\beta_1^{(2)}(G) := \dim_{L(G)} H^1(G, \ell^2(G))$$

is called the *first L^2-Betti number* of G.

We will neither discuss Lück's dimension function nor L^2-Betti numbers in detail and instead refer ot the monograph [L02]. However, we would like summarize several facts on the first L^2-cohomology related to amenability and nonamenability. Let us therefore give a definition of amenability that will also be useful in the second part of the appendix.

Definition A.1.5 ([F55])**.** A countable discrete group G is called *amenable* if there exists a sequence of finite, nonempty sets $A_n \subset G$ satisfying

$$\lim_n \frac{|gA_n \triangle A_n|}{|A_n|} = 0 \quad \text{for all } \ g \in G.$$

Such a sequence is called a *Følner sequence* for G.

Amenability is related to the notion of proper 1-cocycles and we will make this statement more precise in Facts A.1.7.

Definition A.1.6. (i) A function $\varphi : I \to [0, +\infty)$ on a countable infinite set I is called *proper* if $\{i \in I \mid \varphi(i) \leq \kappa\}$ is finite for every $\kappa > 0$.

(ii) A 1-cocycle $c : G \to \ell^2(G)$ is called *proper* if the function $G \to [0, +\infty),\ g \mapsto \|c_g\|_2$ is proper.

Let us finish this section by gathering the following facts.

Facts A.1.7. *Let G be an infinite discrete group.*

- *If $\Lambda < G$ is a finite index subgroup, its L^2-Betti number scales as*

$$\beta_1^{(2)}(\Lambda) = [G : \Lambda]\ \beta_1^{(2)}(G),$$

  see [CG85, Proposition 2.6].

- *If G is amenable, then $\beta_1^{(2)}(G) = 0$, see [CG85] [L02, Theorem 7.2], but $H^1(G, \ell^2(G)) \neq \{0\}$, see [PT10]. In addition, every 1-cocycle of G into $\ell^2(G)$ is either bounded or proper, [PT10].*

- *If G is nonamenable, then $\beta_1^{(2)}(G) = 0$ if and only if $H^1(G, \ell^2(G)) = \{0\}$, [PT10].*

Moreover, G satisfies $H^1(G, \ell^2(G)) = \{0\}$ if

- *G has Kazhdans property (T), see [BdlHV08];*
- *G is nonamenable and has an amenable normal subgroup, see [L02, Theorem 7.2];*
- *$G = G_1 \times G_2$ where G_1 is infinite and G_2 is nonamenable.*

A.2 Amenable groups have 1-cocycles of arbitrarily small growth

A countable group G has the *Haagerup property* if there exists a proper 1-cocycle $c : G \to \mathcal{H}$ into *some* unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$. In [CTV05, Proposition 3.10], it is proven that a group with the Haagerup property admits such proper 1-cocycles $c : G \to \mathcal{H}$ of arbitrarily slow growth. In [BCV93], it is proven that all amenable groups have the Haagerup property. Mimicking that proof, we show that an amenable group G admits a proper 1-cocycle $c : G \to \ell^2(G)$ of arbitrarily slow growth. We will apply this result in Section 2.4 to construct nonsingular Bernoulli actions of type III for a class of amenable groups.

Proposition A.2.1. *Let G be an amenable countable infinite group and $\varphi : G \to [0, +\infty)$ a proper function with $\varphi(g) > 0$ for all $g \neq e$. Then there exists a 1-cocycle $c : G \to \ell^2(G)$ such that $\|c_g\|_2 \leq \varphi(g)$ for every $g \in G$ and such that $g \mapsto \|c_g\|_2$ is proper.*

More concretely, given φ, given any Følner sequence $A_n \subset G$ with all A_n being disjoint and given $\delta > 0$, we can pass to a subsequence and choose $\varepsilon_n \in (0, \delta)$ such that

- $\lim_n \varepsilon_n = 0$ *and* $\sum_n \varepsilon_n^2 = +\infty$,
- *the function*

$$F : G \to [0, \delta) : F(g) = \begin{cases} \varepsilon_n/\sqrt{|A_n|} & \text{if } g \in A_n \text{ for some } n, \\ 0 & \text{if } g \notin \bigcup_n A_n, \end{cases} \tag{A.2.1}$$

is such that $c_g(k) = F(k) - F(g^{-1}k)$ defines a 1-cocycle $c : G \to \ell^2(G)$ with the properties that $\|c_g\|_2 \leq \varphi(g)$ for every $g \in G$ and that $g \mapsto \|c_g\|_2$ is proper.

Proof. Enumerate $G = \{g_0, g_1, g_2, \ldots\}$ with $g_0 = e$. Choose a sequence $\varepsilon_n \in (0, \delta)$ such that $\lim_n \varepsilon_n = 0$, $\sum_n \varepsilon_n^2 = +\infty$ and

$$\sum_{n=1}^{k} \varepsilon_n^2 \leq \frac{1}{2}\varphi(g_k)^2 \quad \text{for all } \; k \geq 1 \; .$$

After passing to a subsequence of A_n, we may assume that

$$\varepsilon_n^2 \, \frac{|g_k A_n \triangle A_n|}{|A_n|} \leq \varepsilon_k^2 \, 2^{-n} \quad \text{for all } \; 1 \leq k \leq n \; .$$

Define the function F as in (A.2.1). For every $k \geq 1$, we have

$$\|g_k \cdot F^2 - F^2\|_1 \leq 2 \sum_{n=1}^{k-1} \varepsilon_n^2 + \sum_{n=k}^{\infty} \varepsilon_n^2 \, \frac{|g_k A_n \triangle A_n|}{|A_n|} \leq 2 \sum_{n=1}^{k-1} \varepsilon_n^2 + \sum_{n=k}^{\infty} \varepsilon_k^2 \, 2^{-n}$$

$$\leq 2 \sum_{n=1}^{k-1} \varepsilon_n^2 + 2\varepsilon_k^2 = 2 \sum_{n=1}^{k} \varepsilon_k^2 \leq \varphi(g_k)^2 \; .$$

Since $\|g_k \cdot F - F\|_2^2 \leq \|g_k \cdot F^2 - F^2\|_1 \leq \varphi(g_k)^2$, we indeed find that the 1-cocycle $c : G \to \ell^2(G)$ defined by $c_g(k) = F(k) - F(g^{-1}k)$ satisfies $\|c_g\|_2 \leq \varphi(g)$ for all $g \in G$.

Since $\lim_{g\to\infty} F(g) = 0$ and $\sum_g F(g)^2 = +\infty$, the 1-cocycle c is not inner. By [PT10, Theorem 2.5], the 1-cocycle c is proper. $\square$

Part II

Representation theory of quantum symmetries

Overview of part II

This part of the thesis rests on the joint articles [AdLW16] [AdLW17] and [TW16], all of which explore quantum symmetries in the form of rigid C^*-tensor categories, compact quantum groups and planar algebras.

The first chapter (Chapter 3) contains an extensive set of preliminaries on all of these notions. Next, we discuss the results obtained in [AdLW17] on harmonic analysis in rigid C^*-tensor categories in Chapter 4. Chapter 5 contains the proof of Theorem C mentioned in the introduction and more generally a discussion of the Howe-Moore property of rigid C^*-tensor categories based on [AdLW16]. In the final chapter (Chapter 6), we describe the results on free wreath product quantum groups and free product of planar algebras obtained in [TW16]. A connection between planar tangles and non-crossing partitions that is exploited in Chapter 6 is discussed in Appendix B.

Chapter 3

Preliminaries

3.1 Rigid C^*-tensor categories

We will now introduce one of the main notions of study in this part of the thesis, the notion of a *rigid C^*-tensor category*. Its definition is somewhat long and drags several technical subleties along with it that we will not be able to discuss here. For such discussions we refer the reader to the excellent book [NT13] on which the definitions in this section are based. For a detailed introduction to general tensor categories, we refer to [EGNO15]. We will assume every category in this thesis to be *small*, that is to say, its class of objects is assumed to be a set.

Definition 3.1.1. A *C^*-category* is a tupel $(\mathcal{C}, *)$ of a $\mathbb{C}$-linear category $\mathcal{C}$ and a contravariant antilinear functor $* : \mathcal{C} \to \mathcal{C}$ (the *involution*) satisfying the following conditions:

1. For any two objects $X, Y \in \mathcal{C}$ the morphism space $\mathrm{Mor}(X, Y)$ is a Banach space and for any objects $X, Y, Z \in \mathcal{C}$ the composition map

$$\mathrm{Mor}(Y, Z) \times \mathrm{Mor}(X, Y) \to \mathrm{Mor}(X, Z), \quad (S, T) \mapsto ST$$

   is bilinear and satisfies $\|ST\| \leq \|S\|\|T\|$.

2. The functor $* : \mathcal{C} \to \mathcal{C}$ is the identity on objects and for morphisms $T \in \mathrm{Mor}(X, Y)$, it satisfies

   - $T^{**} = T$,

- $\|T^*T\| = \|T\|^2$, so that $\mathrm{End}(X)$ is a C^*-algebra,
- T^*T is positive in $\mathrm{End}(X)$.

The notion of functors between C^*-categories is the same as for $\mathbb{C}$-linear categories with the additional requirement that the $*$-structure is preserved:

Definition 3.1.2. A $\mathbb{C}$-linear functor $F : \mathcal{C} \to \mathcal{D}$ between C^*-categories $\mathcal{C}$ and $\mathcal{D}$ is called *unitary* if $F(T)^* = F(T^*)$ for all morphisms T in $\mathcal{C}$.

Recall that a monoidal or tensor category is a sextupel $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ of a category $\mathcal{C}$, a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $\mathbb{1}$, associativity isomorphisms a and left and right unit isomorphisms l, r respecting the pentagon and triangle diagrams, see [EGNO15, Section 2.1]. By Mac Lane's theorem [EGNO15, Section 2.1], every tensor category is monoidally equivalent to a *strict* one, that is to say, one where a, r and l are trivial, i.e.

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \quad \text{and} \quad X \otimes \mathbb{1} = X = \mathbb{1} \otimes X$$

for all objects $X, Y, Z \in \mathcal{C}$. Therefore, for our convenience, we will assume all tensor categories appearing in this thesis to be strict, meaning that we can neglect the associativity and unit constraints in our definitions. Let us now recall the compatibility conditions between the monoidal structure and the C^*-structure.

Definition 3.1.3. A *C^*-tensor category* is a quadrupel $(\mathcal{C}, *, \otimes, \mathbb{1})$ such that $(\mathcal{C}, *)$ is a C^*-category, $(\mathcal{C}, *, \otimes, \mathbb{1})$ is a (strict) tensor category and such that the following hold.

1. The tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear.
2. $(S \otimes T)^* = S^* \otimes T^*$ for all morphisms S, T.
3. The category admits finite direct sums over which the tensor operation distributes.
4. The category admits subobjects.
5. The unit object is simple that is to say $\mathrm{End}(\mathbb{1}) \cong \mathbb{C}$.

Recall that a *monoidal* or *tensor functor* between two tensor categories $\mathcal{C}$ and $\mathcal{D}$ consists of a tupel (F, F_1) of an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{1}_{\mathcal{D}}$ and natural isomorphisms

$$F_1^{X,Y} : F(X \otimes Y) \cong F(X) \otimes F(Y)$$

satisfying the monoidal structure axiom, see [EGNO15, Definition 2.4.1]. The right notion of functor between C^*-tensor categories is the following.

Definition 3.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be C^*-tensor categories. A tensor functor (F, F_1) between $\mathcal{C}$ and $\mathcal{D}$ is called a *unitary tensor functor* if $F : \mathcal{C} \to \mathcal{D}$ is unitary in the sense of Definition 3.1.2 and if the natural isomorphisms

$$F_1^{X,Y} : F(X \otimes Y) \cong F(X) \otimes F(Y)$$

are unitary as well.

A unitary tensor functor which is also an equivalence of categories is called a *unitary monoidal equivalence*.

In almost all situations, we will also require our C^*-tensor categories to admit a notion of duality between objects. This notion should be regarded as a very weak form of taking inverses of objects and it has many pleasant structural consequences as we will remark right after the next definition.

Definition 3.1.5. Let $\mathcal{C}$ be a C^*-tensor category and let X be an object in $\mathcal{C}$. An object $\bar{X}$ is said to be *conjugate* to X if there exist morphisms $R \in \mathrm{Mor}(\mathbb{1}, \bar{X} \otimes X)$ and $\bar{R} \in \mathrm{Mor}(\mathbb{1}, X \otimes \bar{X})$ such that

$$X \xrightarrow{1 \otimes R} X \otimes \bar{X} \otimes X \xrightarrow{\bar{R}^* \otimes 1} X \quad \text{and} \quad \bar{X} \xrightarrow{1 \otimes \bar{R}} \bar{X} \otimes X \otimes \bar{X} \xrightarrow{R^* \otimes 1} \bar{X}$$

are the identity morphisms. The equations

$$(\bar{R}^* \otimes 1)(1 \otimes R) = 1 \quad \text{and} \quad (R^* \otimes 1)(1 \otimes \bar{R}) = 1$$

are called the *conjugate equations*. If every object has a conjugate object, then $\mathcal{C}$ is called a *rigid*.

Whenever they exist, conjugate objects are uniquely determined up to isomorphism [NT13, Proposition 2.2.5]. Moreover, if an object X admits a conjugate, the C^*-algebra $\mathrm{End}(X)$ is finite-dimensional and as a consequence, X can be decomposed as a finite direct sum of *irreducible* objects. Recall here that an object Y is called irreducible if $\dim \mathrm{End}(Y) = 1$ and that the set of equivalence classes of irreducible objects of $\mathcal{C}$ is usually denoted by $\mathrm{Irr}(\mathcal{C})$. In particular, rigid C^*-tensor category have the pleasant property that all objects admit a decomposition into irreducibles [NT13, Corollary 2.2.9].
Let X be an object in the rigid C^*-tensor category $\mathcal{C}$ and let $(R, \bar{R})$ be a solution to the conjugate equations for X. It turns out that there is a particularly useful choice of solution to the conjugate equations for X. More precisely one can choose $(R, \bar{R})$ in such a way that the identity

$$R^*(1 \otimes T)R = \bar{R}^*(T \otimes 1)\bar{R}$$

is satisfied for all $T \in \mathrm{End}(X)$. Such a solution to the conjugate equations is called *standard* and for a standard solution, the functional

$$\mathrm{Tr}_X : \mathrm{End}(X) \to \mathrm{End}(\mathbb{1}) \cong \mathbb{C}, \quad \mathrm{Tr}_X(T) = R^*(1 \otimes T)R$$

is a positive faithful trace [NT13, Theorem 2.2.16], to which we refer as the *categorical trace* on $\mathrm{End}(X)$. It is independent of the choice of standard solution for X. The quantity $d(X) = \mathrm{Tr}_X(1)$ is called the *(categorical) dimension* of X. When X is irreducible, any standard solution of the conjugate equations satisfies $\|R\| = \|\bar{R}\| = d(X)^{\frac{1}{2}}$ [NT13, Theorem 2.2.19]. In addition, the dimension function is multiplicative and additive and hence can be computed as

$$d(X) = \sum_{i=1}^{m} d(X_i)$$

where $X = \oplus_{i=1}^{m} X_i$ is a decomposition of X into irreducible objects.

The primary example of a rigid C^*-tensor category is the category of finite-dimensional Hilbert spaces Hilb_f. For a detailed discussion of the conjugate equations for Hilb_f, we refer to [NT13, Example 2.2.2]. Other sources of rigid C^*-tensor category include representation categories of compact quantum groups, which we will discuss next, and categories of bimodules associated to subfactors, see [PV15].

3.2 Compact quantum groups

Compact quantum groups have been introduced by Woronowicz in his seminal work [Wo87b] as a generalization of compact groups that, among other things, provides a natural framework for extending Pontryagin duality. Later, Kustermans and Vaes fused the locally compact setting with Woronowicz's theory by introducing the notion of locally compact quantum groups in [KuV00]. These will however not play a significant role in this thesis and we will restrict our attention to compact quantum groups and their dual counterparts, the discrete quantum groups.

Definition 3.2.1 (Woronowicz)**.** A *compact quantum group* $\mathbb{G} = (A, \Delta)$ is a pair of a unital C^*-algebra A and a unital $*$-homomorphism $A \to A \otimes A$ (comultiplication) satisfying

1. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \quad$ (coassociativity).

2. The spaces $\operatorname{span}\{(A \otimes 1)\Delta(A)\}$ and $\operatorname{span}\{(1 \otimes A)\Delta(A)\}$ are dense in $A \otimes A$.

Here, by $\otimes$ we mean the minimal tensor product of C^*-algebras. Also, it is common to denote A by $C(\mathbb{G})$ instead.

Every compact group G naturally defines a compact quantum group by setting $A = C(G)$ and by defining the comultiplication in terms of the group operation through

$$\Delta(f)(g,h) = f(gh) \quad \text{for} \quad g,h \in G,\ f \in C(G)$$

such that $\Delta(f) \in C(G \times G) \cong A \otimes A$. In fact, all compact quantum groups for which the underlying C^*-algebra A is commutative are of this form [NT13, Example 1.1.2]. We will see concrete and less trivial examples of compact quantum groups in Subsection 3.2.4 and Subsection 3.2.5.

A fundamental structural result of Woronowicz is that compact quantum groups always come with a distinguished state, the *Haar state*, which generalizes the notion of Haar measure for compact groups [NT13, Theorem 1.2.1]. The Haar state of $\mathbb{G}$ is the unique state h on $C(\mathbb{G})$ satisfying the identity

$$\varphi * h = h * \varphi = \varphi(1)h \quad \forall \varphi \in C(\mathbb{G})^*$$

where $\varphi_1 * \varphi_2(a) = \varphi_1 \otimes \varphi_2(\Delta(a)),\ a \in C(\mathbb{G})$ denotes the convolution product. We now move towards the theory of finite-dimensional unitary representations of compact quantum groups.

Definition 3.2.2. A representation of a compact quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta)$ on a finite-dimensional vector space H is an invertible element u of $L(H) \otimes C(\mathbb{G})$ satisfying

$$(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13},$$

where we use the leg numbering notation. When H is a Hilbert space and u is a unitary element of $B(H) \otimes C(\mathbb{G})$, we call it a unitary finite-dimensional representation.

If we choose a basis of H and write u in terms of matrix units e_{ij} w.r.t. this basis, i.e. $u = \sum_{i,j} e_{ij} \otimes u_{ij}$, then the condition $(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13}$ becomes $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, which is often more convenient for computations.

Definition 3.2.3. An *intertwiner* between two finite-dimensional representations u and v is a linear map $T : H_u \to H_v$ satisfying

$$(T \otimes 1)u = v(T \otimes 1).$$

We denote the space of intertwiners by $\mathrm{Mor}(u, v)$ and note that, if u is unitary then $\mathrm{End}(u) := \mathrm{Mor}(u, u)$ is a C^*-algebra. More generally, the category of finite-dimensional unitary representations of $\mathbb{G}$ is a C^*-category. We will denote it by $\mathrm{Rep}(\mathbb{G})$. It carries a natural monoidal structure: we define the tensor product representation $u \otimes v$ of two finite-dimensional representations u and v as

$$u_{12}v_{23} \in L(H_u \otimes H_v) \otimes C(\mathbb{G}).$$

If both u and v are unitary, the natural inner product on $H_u \otimes H_v$ turns $u \otimes v$ into a unitary representation as well. In a similar way, one defines direct sums and subrepresentations, so that $\mathrm{Rep}(\mathbb{G})$ carries the structure of a C^*-tensor category.

Defining the conjugate object of a finite-dimensional unitary representation is much more subtle, since the obvious candidate, the so-called *contragredient representation* need not be unitary anymore.

To define the contragredient representation of a finite-dimensional unitary representation u on a Hilbert space H, denote by $\bar{H} = \{\bar{\xi},\ \xi \in H\}$ the conjugate Hilbert space with scalar multiplication $\lambda\bar{\xi} = \overline{\bar{\lambda}\xi}$ and by $J : B(H) \to B(\bar{H})$, the $*$-anti-homomorphism given by $J(T)\bar{\xi} = \overline{T^*\xi}$. The contragredient representation is the representation

$$u^c = (J \otimes \mathrm{id})(u^{-1}) \in B(\bar{H}) \otimes C(\mathbb{G}).$$

If we write $u = (u_{ij})$ with respect to some basis of H, the contragredient representation has the form $u^c = (u_{ij}^*)$ with respect to the dual basis. Although u^c need not be unitary itself, it is always equivalent to a unitary representation $\bar{u}$ on $\bar{H}$ which is a conjugate object to u in the C^*-tensor category $\mathrm{Rep}(\mathbb{G})$. It is therefore called the *conjugate representation* of u. The invertible intertwiner in $\mathrm{Mor}(u^c, \bar{u})$ implementing the equivalence can be made very explicit and it intimately relates to the modular properties of the Haar state on $\mathbb{G}$, see for example [NT13, Theorem 1.4.3]. We do not need to spell this out explicitly for the purposes of this thesis, but we will record the following theorem/definition.

Theorem 3.2.4. *Let $\mathbb{G}$ be a compact quantum group. The following are equivalent.*

(i) The Haar state on $\mathbb{G}$ is tracial;

(ii) the contragredient of any unitary representation u is unitary, i.e $u^c = \bar{u}$;

(iii) for any unitary representation u on a Hilbert space H_u, we have $d(u) = \dim H_u$, where d denotes the dimension of u inside the category $\mathrm{Rep}(\mathbb{G})$ in the sense of Section 3.1.

If one and hence all of these conditions are satisfied, $\mathbb{G}$ *is said to be of* Kac type*.*

The equivalence between the conditions (i) and (ii) follows from [NT13, Theorem 1.4.3], while the equivalence between (ii) and (iii) is discussed in [NT13, Section 1.4] and [NT13, Example 2.2.13].

A useful feature of irreducible finite-dimensional unitary representations of a compact quantum group $\mathbb{G}$ is that their matrix coefficients span a dense Hopf-$*$-algebra $\mathbb{C}[\mathbb{G}]$ inside $C(\mathbb{G})$. For a definition of Hopf-$*$-algebras, we refer to [NT13, Definition 1.6.1]. More precisely

$$\mathbb{C}[\mathbb{G}] := \mathrm{span}\{u^{\alpha}_{ij} \ ; \ \alpha \in \mathrm{Irr}(\mathbb{G})\},$$

where $\mathrm{Irr}(\mathbb{G}) := \mathrm{Irr}(\mathrm{Rep}(\mathbb{G}))$ and where for any $\alpha \in \mathrm{Irr}(\mathbb{G})$, we chose a representative $u^{\alpha} = (u^{\alpha}_{ij})$ in matrix form. The *coinverse* $S : \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{G}]$ is uniquely determined by the property $S(u^{\alpha}_{ij}) = (u^{\alpha}_{ji})^*$ and the *counit* $\epsilon : \mathbb{C}[\mathbb{G}] \to \mathbb{C}$ is uniquely determined by $\epsilon(u^{\alpha}_{ij}) = \delta_{ij}$.

Along with the Haar state h on $\mathbb{G}$ comes a natural representation of the C^*-algebra $C(\mathbb{G})$ on the GNS-space with respect to h. We denote the GNS-Hilbert space by $L^2(\mathbb{G})$ and the representation itself by $\pi_h : C(\mathbb{G}) \to B(L^2(\mathbb{G}))$. The C^*-algebra $C_r(\mathbb{G}) := \pi_h(C(\mathbb{G}))$ generated by this representation is called the *reduced C^*-algebra* of $\mathbb{G}$ and its enveloping von Neumann algebra is denoted by $L^{\infty}(\mathbb{G}) := \pi_h(C(\mathbb{G}))'' \subset B(L^2(\mathbb{G}))$. $C_r(\mathbb{G})$ also carries the structure of a compact quantum group with comultiplication $\Delta_r : C_r(\mathbb{G}) \to C_r(\mathbb{G}) \otimes C_r(\mathbb{G})$. This comultiplication extends normally to $L^{\infty}(\mathbb{G})$ and satisfies $\Delta_r \circ \pi_h = (\pi_h \otimes \pi_h) \circ \Delta$. We denote the quantum group $(C_r(\mathbb{G}), \Delta_r)$ by $\mathbb{G}_r$.

In addition, there also exists a universal version of $\mathbb{G}$, that is to say, there exists a universal C^*-envelope $C_u(\mathbb{G})$ of $\mathbb{C}[\mathbb{G}]$ to which the comultiplication can be extended. We denote the comultiplication on $C_u(\mathbb{G})$ by Δ_u and denote the universal version as a whole by $\mathbb{G}_u = (C_u(\mathbb{G}), \Delta_u)$. It comes with a canonical surjective homomorphism $C_u(\mathbb{G}) \to C(\mathbb{G})$ of compact quantum groups. Although the polynomial algebras of $\mathbb{G}_u$ and $\mathbb{G}_r$ always coincide, this canonical homomorphism in general need not be injective.

We refer to [Ti08, Section 5.4.2] for proofs of our statements on reduced and universal compact quantum groups.

3.2.1 Tannaka-Krein duality

An immensely useful characterization of compact quantum groups is given by the Tannaka-Krein duality theorem of Woronowicz [Wo88]. Note that a compact quantum group $\mathbb{G}$ does not only give rise to a rigid C^*-tensor category $\mathrm{Rep}(\mathbb{G})$ but also to a unitary tensor functor $\mathrm{Rep}(\mathbb{G}) \to \mathrm{Hilb}_f,\ u \mapsto H_u$. The Tannaka-Krein duality theorem states that, starting from an abstract rigid C^*-tensor category $\mathcal{C}$ and a C^*-tensor functor $\mathcal{C} \to \mathrm{Hilb}_f$, one can construct a compact quantum group $\mathbb{G}$ with $\mathrm{Rep}(\mathbb{G}) \cong \mathcal{C}$ and that this construction is even unique up to norm closure. The precise statement is the following, see [NT13, Theorem 2.3.2].

Theorem 3.2.5 (Tannaka-Krein reconstruction theorem of Woronowicz)**.** *Let $\mathcal{C}$ be a rigid C^*-tensor category and let (F, F_1) be a unitary tensor functor $\mathcal{C} \to \mathrm{Hilb}_f$. There exists a compact quantum group $\mathbb{G}$ and a unitary monoidal equivalence $(E, E_1) : \mathcal{C} \to \mathrm{Rep}(\mathbb{G})$ such that (F, F_1) is naturally unitarily monoidally isomorphic to the composition of (E, E_1) with the canonical unitary tensor functor $\mathrm{Rep}(\mathbb{G}) \to \mathrm{Hilb}_f$. Moreover, the Hopf algebra $(\mathbb{C}[\mathbb{G}], \Delta)$ is uniquely determined up to isomorphism.*

3.2.2 The dual picture: discrete quantum groups

One striking feature of the theory of compact quantum groups is the existence of a completely equivalent dual theory of discrete quantum groups: Every compact quantum group $\mathbb{G}$ comes with a *discrete dual* $\hat{\mathbb{G}}$ which in turn completely determines $\mathbb{G}$. The notion of discrete quantum group can be axiomatized [vD96] without refering to compact quantum groups at all. However, due to the duality between both theories, we will choose to work in the setting of compact quantum groups and to formulate properties of discrete ones as properties of the 'dual of $\mathbb{G}$'.

Let us first recall the simple case where Γ is a discrete group. Then Γ yields a compact quantum group $\mathbb{G} = (C^*(\Gamma), \Delta)$ whose comultiplication is determined by $\Delta(u_g) = u_g \otimes u_g$ where $u_g \in C^*(\Gamma)$ is the canonical unitary corresponding to $g \in \Gamma$. Since Γ is completely determined by $\mathbb{G}$ and vice versa, we should consider $\Gamma = \hat{\mathbb{G}}$ the dual of $\mathbb{G}$. Moreover, note that the family of one-dimensional irreducible unitary representations $(u_g)_{g\in\Gamma}$ is a full set of representatives of $\mathrm{Irr}(\mathbb{G})$ with the tensor product operation $u_g \otimes u_h = u_{gh}$ representing the group law. This example is the reason why rigid C^*-tensor categories are often considered 'discrete group-like' objects.

Now, let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be an arbitrary compact quantum group. On the dual $\mathcal{U}(\mathbb{G}) := \mathbb{C}[\mathbb{G}]^*$ of the polynomial algebra, we can put a $*$-algebra structure by defining multiplication as convolution, i.e.

$$\omega \cdot \nu = \omega * \nu = (\omega \otimes \nu) \circ \Delta$$

and by defining involution as $\omega^* = \bar{\omega} \circ S$, where $S : \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{G}]$ denotes the coinverse. $\mathcal{U}(\mathbb{G})$ also possesses a comultiplication although its range is slightly bigger than $\mathcal{U}(\mathbb{G}) \otimes \mathcal{U}(\mathbb{G})$. More precisely, we can define

$$\hat{\Delta} : \mathcal{U}(\mathbb{G}) \to (\mathbb{C}[\mathbb{G}] \otimes \mathbb{C}[\mathbb{G}])^*, \quad \hat{\Delta}(\omega)(a \otimes b) = \omega(ab).$$

The pair $\hat{\mathbb{G}} = (\mathcal{U}(\mathbb{G}), \hat{\Delta})$ is what one considers the *discrete dual* of $\mathbb{G}$. If $\mathbb{G} = (C^*(\Gamma), \Delta)$ for some discrete group Γ as above, then $\mathcal{U}(\mathbb{G})$ equals the space of functions on Γ with pointwise multiplication.

For every equivalence class $\alpha \in \mathrm{Irr}(\mathbb{G})$, let us choose a representative $u^\alpha \in B(H_\alpha) \otimes \mathbb{C}[\mathbb{G}]$. It is useful to note that for any $\alpha \in \mathrm{Irr}(\mathbb{G})$ we can define a $*$-homomorphism

$$\pi_\alpha : \mathcal{U}(\mathbb{G}) \to B(H_\alpha), \quad \pi_\alpha(\omega) = (\mathrm{id} \otimes \omega)(u^\alpha).$$

Using these $*$-homomorphism, one obtains a $*$-isomorphism

$$\mathcal{U}(\mathbb{G}) \to \prod_{\alpha \in \mathrm{Irr}(\mathbb{G})} B(H_\alpha), \quad \omega \mapsto (\pi_\alpha(\omega))_{\alpha \in \mathrm{Irr}(\mathbb{G})}.$$

This interpretation of $\mathcal{U}(\mathbb{G})$ will be quite useful to have around in Subsection 3.4.4.

3.2.3 Compact matrix quantum groups

Most concrete examples of compact quantum groups are defined in terms of one fundamental finite-dimensional unitary representation that generates all other unitary representations. These are referred to as compact matrix quantum groups.

Definition 3.2.6. A compact quantum group $\mathbb{G}$ is called *compact matrix quantum group* if there exists a finite-dimensional unitary representation u whose matrix coefficients generate $\mathbb{C}[\mathbb{G}]$ as a $*$-algebra. The representation u will be referred to as *generating* or *fundamental*.

Note that compact quantum group $\mathbb{G}$ is a compact matrix quantum group if and only if the rigid C^*-tensor category $\mathrm{Rep}(\mathbb{G})$ is finitely generated in the following sense.

Definition 3.2.7. A rigid C^*-tensor category is called *finitely generated*, if there exists an object $X \in \mathcal{C}$ such that any other object $Y \in \mathcal{C}$ is a subobject of some tensor product whose factors consist of X and $\bar{X}$.

Famous instances of compact matrix quantum groups are Wang's free unitary and free orthogonal quantum groups, see [NT13, Examples 1.1.6 and 1.1.7]. Another one is the quantum permutation group S_n^+ which will play a major role in Chapter 6.

Example 3.2.8 (Wang)**.** For $n \in \mathbb{N}$, denote by A the universal unital C^*-algebra generated by n^2 elements $u_{ij},\ i,j = 1 \dots n$ satisfying the following relations.

- The matrix $u = (u_{ij}) \in M_n \otimes A$ is unitary,
- all entries u_{ij} are projections, i.e. $u_{ij} = u_{ij}^* = u_{ij}^2$,
- $\sum_{i=1}^n u_{ij} = \mathbb{1}$ for all j and similarly $\sum_{j=1}^n u_{ij} = \mathbb{1}$ for all i.

Then, the map $\Delta : A \to A \otimes A,\ u_{ij} \mapsto \sum_{k=1}^n u_{ik} \otimes u_{kj}$ extends to a comultiplication on A, turning it into a compact quantum group $S_n^+ = (C(S_n)^+, \Delta)$ with canonical generator u. S_n^+ is called the *quantum permutation group* on n points.

S_n^+ will also reappear as an instance of a quantum automorphism group in Subsection 3.2.5.

3.2.4 q-Deformations of Lie groups

In this subsection, we would like to recall the definitions of what might be the most well-known class of examples of compact quantum groups, namely the q-deformations of compact Lie groups. The q-deformations were introduced independently by Drinfel'd [Dr86] and Jimbo [Ji85] and integrated into Woronowicz's setting of compact quantum groups by Rosso in [Ro90]. We will mainly follow the textbook [NT13, Section 2.4] here but additional details can also be found in [KlSch97]. We will need to assume that the reader is familiar with the basic ideas within representation theory for Lie groups and Lie algebras.

Let K be a connected simply connected compact Lie group with complexification $G := K_{\mathbb{C}}$. Let $\mathfrak{g}$ be the Lie algebra of G and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Note that $\mathfrak{g}$ is a complex semisimple Lie algebra thanks to our assumptions on K. Let $R \subset \mathfrak{h}^*$ be the associated set of roots, let $Q \subset \mathfrak{h}^*$ be the root lattice and let $P \subset \mathfrak{h}^*$ be the weight lattice. We also fix a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ which is invariant under the adjoint action of K and which is negative definite when restricted to the real Lie algebra of K. We assume this form to be normalized in a standard way, that is to say we assume that $(H^\alpha, H^\alpha) = 2$ for every short root α in every simple summand of $\mathfrak{g}$ where $H^\alpha \in \mathfrak{h}$ is the element defined by $(H^\alpha, H) = \alpha(H)$ for $H \in \mathfrak{h}$. For each $\alpha \in R$, define the coroot as $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ and denote the corresponding element of $\mathfrak{h}$ by H_α. Choose a set $\Pi = \{\alpha_1, \dots, \alpha_r\}$ of simple roots, and let R_+ denote the set of positive roots, Q_+ the positive elements in the root lattice and P_+ the positive weights with respect to our choice of simple roots. Put $d_i = \frac{(\alpha_i,\alpha_i)}{2}$, and denote by $a_{ij} = \frac{(\alpha_i,\alpha_j)}{d_i}$ the entries of the Cartan matrix.

Fix $q \in (0,1)$. Define $q_i = q^{d_i}$, and set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]_q! = [n]_q[n-1]_q \dots [1]_q$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}.$$

We will first define a deformation of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ depending on the parameter q. Recall that for every simple root $\alpha_i \in R$, we can find an element E_i in the root space

$$\mathfrak{g}_{\alpha_i} = \{X \in \mathfrak{g} \ ; \ [H, X] = \alpha_i(H)X \text{ for all } H \in \mathfrak{h}\}$$

such that $F_i := E_i^* \in \mathfrak{g}_{-\alpha_i}$ and $(E_i, F_i) = d_i^{-1}$. Then, $[E_i, F_i] = H_{\alpha_i}$ and E_i, F_i and $H_i := H_{\alpha_i}$ together generate a copy of $\mathfrak{sl}(2, \mathbb{C})$ inside $\mathfrak{g}$.

Definition 3.2.9. The quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ is the unital algebra defined by the generators $\{K_i^{\pm 1}, E_i, F_i \mid i \in I\}$ and the relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \qquad K_i K_j = K_j K_i,$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} E_i^r E_j E_i^{1-a_{ij}-r} = 0,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} F_i^r F_j F_i^{1-a_{ij}-r} = 0.$$

The quantized enveloping algebra $U_q(\mathfrak{g})$ can be turned into a Hopf $*$-algebra by defining a comultiplication $\hat{\Delta}_q$ and involution * by

$$\hat{\Delta}_q(K_i) = K_i \otimes K_i, \qquad \hat{\Delta}_q(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$

$$\hat{\Delta}_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \qquad K_i^* = K_i, \qquad E_i^* = F_i K_i, \qquad F_i^* = K_i^{-1} E_i.$$

In this case, the counit $\hat{\varepsilon}_q$ and the antipode $\hat{S}_q$ are given by the formulas

$$\hat{\varepsilon}_q(K_i) = 1, \qquad \hat{\varepsilon}_q(E_i) = \hat{\varepsilon}_q(F_i) = 0,$$

$$\hat{S}_q(K_i) = K_i^{-1}, \qquad \hat{S}_q(E_i) = -K_i^{-1} E_i, \qquad \hat{S}_q(F_i) = -F_i K_i.$$

We will now construct a compact quantum group that is 'dual' to $U_q(\mathfrak{g})$ from a well-behaved class of $U_q(\mathfrak{g})$-modules.

Definition 3.2.10. Let V be an $U_q(\mathfrak{g})$-module, and let $\mu \in P$. The *weight space* V_μ is defined as

$$V_\mu = \{v \in V \mid K_i v = q_i^{(\mu, \alpha_i^\vee)} v \ \forall i \in I\}.$$

The module V is said to be *admissible* if V decomposes as a direct sum of its weight spaces, i.e. $V = \bigoplus_{\mu \in P} V_\mu$.

If a vector v in an an admissible module V is an element of the direct summand V_μ, we refer to the weight μ as $\mathrm{wt}(v)$. In analogy to the classical case, the admissible $U_q(\mathfrak{g})$-modules are classified by their highest weight.

Definition 3.2.11. A vector ξ in an admissible $U_q(\mathfrak{g})$-module $V \neq 0$ is called a *highest weight vector* of weight $\lambda \in P_+$ if $\xi \in V_\lambda$, $E_i \xi = 0$ for $i = 1, \ldots, r$ and $V = U_q(\mathfrak{g})\xi$. If such a vector exists, then λ is determined uniquely and V is called *highest weight module* of weight λ.

We refer to [KlSch97, Chapter 7] for a proof of the following theorem.

Theorem 3.2.12. *The following statements hold true.*

1. *For every $\lambda \in P_+$, there exists a unique irreducible highest weight module $V(\lambda)$ of weight λ. $V(\lambda)$ is finite dimensional and unitarizable, i.e. it admits an inner product $\langle \cdot, \cdot \rangle$ with $\langle av, w \rangle = \langle v, a^* w \rangle$ for $v, w \in V(\lambda), a \in U_q(\mathfrak{g})$.*

2. *Any finite dimensional admissible $U_q(\mathfrak{g})$-module decomposes into a direct sum of heighest weight modules.*

3. *The dimensions of $V(\lambda)$, as well as the dimensions of its weight spaces, are the same as in the classification of irreducible representations of $\mathfrak{g}$.*

4. *The multiplicities $m^{\eta}_{\lambda,\mu}$ in the tensor product decomposition*

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\eta \in P_+} V(\eta)^{\oplus m^{\eta}_{\lambda,\mu}}$$

   also agree with the classical case.

There are now two possible ways to introduce the q-deformation K_q that yield the same result. Let us start with the algebraic one.

For $\lambda \in P_+$ and $v, w \in V(\lambda)$, we define the matrix coefficent $u^{\lambda}_{vw} \in U_q(\mathfrak{g})^*$ by $u^{\lambda}_{vw}(x) = \langle xv, w \rangle$. Let $\mathbb{C}[K_q]$ be the quantum coordinate algebra, i.e. the subspace of $U_q(\mathfrak{g})^*$ consisting of matrix coefficients of finite-dimensional admissible unitary modules or equivalently the subspace spanned by the matrix coefficients u^{λ}_{vw} of heighest weight modules. The algebra $\mathbb{C}[K_q]$ admits a unique Hopf $*$-algebra structure that turns the pairing $\mathbb{C}[K_q] \times U_q(\mathfrak{g}) \to \mathbb{C}$ into a Hopf $*$-algebra pairing. More concretely, multiplication, involution and comultiplication on $\mathbb{C}[K_q]$ are given by the formulas

$$(ab)(x) = (a \otimes b)(\hat{\Delta}_q(x)), \qquad a^*(x) = \overline{a(\hat{S}_q(x)^*)},$$

and

$$\Delta_q(a)(x \otimes y) = a(xy)$$

for $a, b \in \mathbb{C}[K_q]$ and $x, y \in U_q(\mathfrak{g})$. Thanks to [NT13, Theorem 2.7.14], there is only one way to complete $\mathbb{C}[K_q]$ to a compact quantum group $K_q = (C(K_g), \Delta_q)$. In particular, one can choose $C(K_q)$ to be the universal C^*-completion of the $*$-algebra $\mathbb{C}[K_q]$.

Definition 3.2.13 ([Dr86], [Ji85], [Ro90])**.** The compact quantum group $K_q = (C(K_g), \Delta_q)$ is called the *Drinfel'd-Jimbo q-deformation* of K.

The second way to introduce K_q is to show that the category $\mathcal{C}_q(\mathfrak{g})$ of finite dimensional unitary admissible $U_q(\mathfrak{g})$-modules is a concrete rigid C^*-tensor category and therefore arises as the representation category of a compact quantum group $\mathbb{H}$. Since one can identify $\mathrm{Rep}(K_q)$ with $\mathcal{C}_q(\mathfrak{g})$ as concrete rigid C^*-tensor categories, this yields the same compact quantum group, i.e. $\mathbb{H} \cong K_q$. For a proof of these identifications, see [NT13, Section 2.4].

3.2.5 Actions and quantum automorphism groups

We will now discuss actions of compact quantum groups on finite-dimensional C^*-algebras. First, we will define what it means to act on a finite-dimensional vector space. Fix a compact quantum group $\mathbb{G}$.

Definition 3.2.14. An action of $\mathbb{G}$ on a finite-dimensional vector space V is a linear map $\alpha : V \to V \otimes \mathbb{C}[\mathbb{G}]$ satisfying

- $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$ (coassociativity),
- $(\mathrm{id} \otimes \varepsilon) \circ \alpha = \alpha$ (counitality).

Let V be a finite-dimensional vector space with basis $(v_i)_{i=1}^{\dim V}$ and corresponding matrix units $e_{ij},\ i,j = 1,\dots,\dim V$. When α is a linear map $\alpha : V \to V \otimes \mathbb{C}[\mathbb{G}]$, we find uniquely determined elements $u_{ij}^\alpha \in \mathbb{C}[\mathbb{G}]$ such that $\alpha(v_j) = \sum_{i=1}^{\dim V} v_i \otimes u_{ij}^\alpha$ and thus an element $u^\alpha = (u_{ij}^\alpha) \in L(V) \otimes \mathbb{C}[\mathbb{G}]$. By [Ti08, Proposition 3.1.7], α is an action on V if and only if u^α is a representation in the sense of Definition 3.2.2 and the correspondence $\alpha \mapsto u^\alpha$ between actions and representations is bijective.

Now, let A be a finite-dimensional C^*-algebra with multiplication map

$$m : A \otimes A \to A,\ a \otimes b \mapsto ab \quad \text{and unit map} \quad \eta : \mathbb{C} \to A,\ z \mapsto z1_A.$$

Moreover, let $\varphi : A \to \mathbb{C}$ be a positive functional on A.

Definition 3.2.15. An action of $\mathbb{G}$ on A is an action $A \to A \otimes \mathbb{C}[\mathbb{G}]$ on A as a vector space that is also a $*$-homomorphism. Such an action is called *φ-preserving* if it satisfies the identity

$$(\varphi \otimes \mathrm{id}) \circ \alpha = \varphi(\cdot)\mathbb{1}_{\mathbb{C}[\mathbb{G}]}.$$

Let us again translate the properties of α into properties of u^α, see [Ti08, Proposition 3.1.7] and [B99a, Lemma 1.2].

Lemma 3.2.16. *Let $\alpha : A \to A \otimes \mathbb{C}[\mathbb{G}]$ be an action on A as a vector space and let* tr *be a faithful positive tracial state on A. Then, α is*

1. *multiplicative if and only if $m \in \mathrm{Mor}(u^\alpha \otimes u^\alpha, u^\alpha)$,*
2. *unital if and only if $\eta \in \mathrm{Mor}(\mathbb{1}, u^\alpha)$,*
3. tr*-preserving if and only if $\mathrm{tr} \in \mathrm{Mor}(u^\alpha, \mathbb{1})$.*

If these conditions are satisfied, α is

(4) involutive if and only if u_α is unitary (w.r.t. the inner product on A given by $\langle x, y\rangle = \mathrm{tr}(y^*x),\ x, y \in A$).

In [Wa98], it was shown that, given a faithful state φ on the finite-dimensional C^*-algebra A, there exists a compact quantum group $\mathbb{G} = \mathbb{G}_{aut}(A, \varphi)$ acting φ-preservingly on A that is universal in the following sense: For any other compact quantum group $\mathbb{H}$ acting φ-preservingly on A there is a surjective unital $*$-homomorphism $\pi : C(\mathbb{G}) \to C(\mathbb{H})$ intertwining the actions. We call $\mathbb{G} = \mathbb{G}_{aut}(A, \varphi)$ the *quantum automorphism group* of the pair (A, φ).

Turn A into a Hilbert space $\mathcal{H}$ by defining an inner product on A through $\langle x, y\rangle = \varphi(y^*x)$ for all $x, y \in A$. In view of Lemma 3.2.16, $C(\mathbb{G})$ can be described as the universal C^*-algebra generated by the coefficients of $u \in B(\mathcal{H}) \otimes C(\mathbb{G})$ subject to the relations which make u unitary, $m \in \mathrm{Mor}(u \otimes u, u)$, and $\eta \in \mathrm{Mor}(\mathbb{1}, u)$.

Example 3.2.17. The quantum permutation group $S_n^+,\ n \geq 1$ is the quantum automorphism group of the pair $(C(X), \mathrm{tr})$, where X is a set of n points and tr is obtained by integrating against the uniform probability measure on X.

A fixed point of an action $\alpha : A \to A \otimes \mathbb{C}[\mathbb{G}]$ is an element $x \in A$ such that $\alpha(x) = x \otimes \mathbb{1}$. We will refer to the set of fixed points of α as $\mathrm{Fix}(\alpha)$.

Definition 3.2.18. Let $\alpha : A \to A \otimes \mathbb{C}[\mathbb{G}]$ be an action on a finite-dimensional C^*-algebra A. We say that α is

- *ergodic*, if $\mathrm{Fix}(\alpha) = \mathbb{C}1_A$;
- *centrally ergodic*, if $Z(A) \cap \mathrm{Fix}(\alpha) = \mathbb{C}1_A$;
- *faithful*, if the coefficients of u_α generate $C(\mathbb{G})$.

Given an inclusion $A_0 \subset A_1$ of finite-dimensional C^*-algebras such that $Z(A_0) \cap Z(A_1) = \mathbb{C}1_{A_1}$, there exists a 'particularly good' choice of trace tr on A_1 which is commonly referred to as the *Markov trace* of the inclusion. This trace extends to the basic construction $A_2 = \langle A_1, e_1\rangle$ with Jones projection e_1 and is uniquely determined under mild irreducibility assumptions. For a thorough discussion of inclusions of finite-dimensional C^*-algebras, their traces and the basic construction, we recommend the book [GdlHJ89]. The following observation is due to Banica [B01, Proposition 5.1].

Proposition 3.2.19. *Let $A_0 \subset A_1$ be an inclusion of finite-dimensional C^*-algebras such that $Z(A_0) \cap Z(A_1) = \mathbb{C}1_{A_1}$ with Markov trace* tr*. Let*

$$A_0 \subset A_1 \subset A_2 = \langle A_1, e_1 \rangle \subset \cdots$$

be its Jones tower. If $\alpha : A_1 \to A_1 \otimes \mathbb{C}[\mathbb{G}]$ is an action leaving A_0 globally invariant and tr *invariant, there is a unique sequence $(\alpha_i)_{i\geq 0}$ of actions $\alpha_i : A_i \to A_i \otimes \mathbb{C}[\mathbb{G}]$ with $\alpha_1 = \alpha$ such that each α_i extends α_{i-1} and leaves the Jones projection e_{i-1} invariant.*

The following example shows that every compact matrix quantum group of Kac type admits a faithful, centrally ergodic action on some finite-dimensional C^*-algebra A preserving the Markov trace of $\mathbb{C} \subset A$.

Example 3.2.20. Let $\mathbb{G}$ be a compact quantum group whose Haar state is a trace and let $u \in B(\mathcal{H}_u) \otimes \mathbb{C}[\mathbb{G}]$ be a finite-dimensional unitary representation. Consider the conjugation action

$$\alpha_c(u) : B(\mathcal{H}_u) \to B(\mathcal{H}_u) \otimes \mathbb{C}[\mathbb{G}], \quad x \mapsto u(x \otimes \mathbb{1})u^*.$$

The factoriality of $B(H_u)$ implies that $\alpha_c(u)$ is centrally ergodic and it is clear that $\alpha_c(u)$ preserves the unique normalized trace on $B(\mathcal{H}_u)$. In addition, if u contains the trivial representation as a subobject and if its coefficients generate $\mathbb{C}[\mathbb{G}]$, the action $\alpha_c(u)$ is faithful since every coefficient of u (and its adjoint) appears as a coefficient of $\alpha_c(u)$.

3.2.6 Free wreath products

The free wreath product of a compact quantum group $\mathbb{G}$ with a quantum subgroup $\mathbb{F}$ of Wang's quantum permutation group S_n^+ is a non-commutative analogue of the classical wreath product of groups and was introduced by Bichon in [Bi04]. Recall that $\mathbb{F}$ is a quantum subgroup of $\mathbb{G}$ if there exists a surjective Hopf-$*$-algebra morphism $\mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{F}]$. Thanks to Example 3.2.17, the quantum subgroups of S_n^+ are exactly those compact quantum groups that act faithfully on n points.

Definition 3.2.21 (Bichon)**.** Let $\mathbb{G}$ be a compact quantum group and let $(\mathbb{F}, u)$ be a quantum subgroup of S_n^+ with fundamental representation $u = (u_{ij})_{1\leq i,j\leq n}$. Define the unital C^*-algebra $C(\mathbb{G} \wr_* \mathbb{F})$ as the quotient of

$$C(\mathbb{G})^{*n} * C(\mathbb{F})$$

by the relations that make the i-th copy of $C(\mathbb{G})$ inside the free product commute with the i-th row of u. Then $C(\mathbb{G} \wr_* \mathbb{F})$ carries the structure of a compact quantum group whose comultiplication $\Delta_{\mathbb{G}\wr_*\mathbb{F}}$ is determined by the identities

$$\Delta_{\mathbb{G}\wr_*\mathbb{F}}(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$

$$\Delta_{\mathbb{G}\wr_*\mathbb{F}}(\nu_i(a)) = \sum_{k=1}^{n} (\nu_i \otimes \nu_k)(\Delta_{\mathbb{G}}(a))(u_{ik} \otimes \mathbb{1}).$$

Here $\nu_i : C(\mathbb{G}) \to C(\mathbb{G} \wr_* \mathbb{F})$ denotes the composition of the embedding $C(\mathbb{G}) \to C(\mathbb{G})^{*n} * C(\mathbb{F})$ into the i-th copy and the quotient map. $\mathbb{G}\wr_*\mathbb{F} = (C(\mathbb{G}\wr_*\mathbb{F}), \Delta_{\mathbb{G}\wr_*\mathbb{F}})$ is called the *free wreath product* of $\mathbb{G}$ and $\mathbb{F} = (C(\mathbb{F}), u)$.

Note that whenever $\mathbb{G}$ comes with generating unitary representation $v = (v_{kl})$, we also obtain a unitary generator of $\mathbb{G} \wr_* \mathbb{F}$ through the formula $w = (v^i_{kl} u_{ij})_{(i,k)(j,l)}$ where $v^i_{kl} = \nu_i(v_{kl})$.

When $\mathbb{F} = S_n^+$, the representation theory of $\mathbb{G} \wr_* S_n^+$ has been studied in [LT16]. Moreover, Bichon's construction was partially generalized in [FP16] to the situation in which the right input $\mathbb{F}$ is replaced by the universal compact quantum group $\mathbb{G}_{aut}(A, \mathrm{tr})$ acting on a finite-dimensional C^*-algebra A and preserving a faithful trace. In fact, the definition in [FP16] is slightly more general, see [FP16] for details.

Fix a compact quantum group $\mathbb{G}$ and a finite-dimensional C^*-algebra A with faithful trace tr. For any $x \in \mathrm{Irr}(\mathbb{G})$, choose a representative $u^x \in B(\mathcal{H}_x) \otimes \mathbb{C}[\mathbb{G}]$. Set $\mathcal{H} = L^2(A, \mathrm{tr})$.

Definition 3.2.22 (Fima-Pittau)**.** Define $C(\mathbb{G}\wr_*\mathbb{G}_{aut}(A, \mathrm{tr}))$ to be the universal unital C^*-algebra generated by the coefficients of

$$a(x) \in B(\mathcal{H} \otimes \mathcal{H}_x) \otimes \mathbb{C}[\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr})],\ x \in \mathrm{Irr}(\mathbb{G}),$$

satisfying the relations:

- $a(x)$ is unitary for any $x \in \mathrm{Irr}(\mathbb{G})$,
- For all $x, y, z \in \mathrm{Irr}(\mathbb{G})$ and all $S \in \mathrm{Mor}(u^x \otimes u^y, u^z)$

  $$(m \otimes S) \circ \Sigma_{23} \in \mathrm{Mor}(a(x) \otimes a(y), a(z)),$$

  where $\Sigma_{23} : \mathcal{H} \otimes \mathcal{H}_x \otimes \mathcal{H} \otimes \mathcal{H}_y \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}_x \otimes \mathcal{H}_y$ is the tensor flip on the second and third leg.
- $\eta \in \mathrm{Mor}(\mathbb{1}, a(\mathbb{1}_{\mathbb{G}}))$.

The compact quantum group $\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}) = (C(\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr})), \Delta)$ with comultiplication as defined in [FP16, Proposition 2.7] is called the *free wreath product* of $\mathbb{G}$ and $\mathbb{G}_{aut}(A, \mathrm{tr})$.

When A is chosen to be $\mathbb{C}^n$ and tr is the trace obtained by mapping the canonical basis vectors to $1/n$, we have already noted that $\mathbb{G}_{aut}(A, \mathrm{tr})$ is nothing but S_n^+. In this case, the definition of Fima and Pittau coincides with Bichon's definition which follows from the discussion of the representation theory of $\mathbb{G} \wr_* S_n^+$ in [LT16].

We recall some facts on $\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr})$ for later use. For any finite-dimensional unitary representation $u \in B(H_u) \otimes \mathbb{C}[\mathbb{G}]$, one can define a unitary representation $a(u) \in B(\mathcal{H} \otimes H_u) \otimes \mathbb{C}[\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr})]$ in the following way (see [FP16, Section 8]). For any $x \in \mathrm{Irr}(\mathbb{G})$ appearing as a subobject of u, choose a family of isometries $S_{x,k} \in \mathrm{Mor}(u^x, u)$, $1 \leq k \leq \dim \mathrm{Mor}(u^x, u)$ such that $S_{x,k} S_{x,k}^*$ are pairwise orthogonal projections with $\sum_{x \subset u} \sum_{k=1}^{\dim \mathrm{Mor}(u^x,u)} S_{x,k} S_{x,k}^* = \mathrm{id}_{\mathcal{H}_u}$. We have

$$u = \sum_{x \subset u} \sum_{k=1}^{\dim \mathrm{Mor}(u^x,u)} (S_{x,k} \otimes \mathbb{1}) u^x (S_{x,k}^* \otimes \mathbb{1}).$$

Set

$$a(u) = \sum_{x \subset u} \sum_{k} (\mathrm{id}_{\mathcal{H}} \otimes S_{x,k} \otimes \mathbb{1}) a(x) (\mathrm{id}_{\mathcal{H}} \otimes S_{x,k}^* \otimes \mathbb{1}) \in B(\mathcal{H} \otimes \mathcal{H}_u) \otimes \mathbb{C}[\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr})].$$

Thanks to [FP16, Proposition 8.1], $a(u)$ is a well defined finite-dimensional unitary representation. Moreover, for all finite-dimensional unitary representations u, v, w of $\mathbb{G}$ and for every $S \in \mathrm{Mor}(u \otimes v, w)$ define

$$a(S) : \mathcal{H} \otimes \mathcal{H}_u \otimes \mathcal{H} \otimes \mathcal{H}_v \to \mathcal{H} \otimes \mathcal{H}_w, \quad a(S) = (m \otimes S)\Sigma_{23}.$$

Now, let A and B be finite-dimensional C^*-algebras with faithful tracial states $\mathrm{tr}_A, \mathrm{tr}_B$ respectively.

Proposition 3.2.23. *Let $\beta : B \to B \otimes \mathbb{C}[\mathbb{G}]$ be a faithful, centrally ergodic tr_B-preserving action of $\mathbb{G}$ with associated representation $u := u_\beta$. Denote the universal action of $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$ on A by α. The linear map*

$$\beta \wr_* \alpha := \alpha_{a(u)} : A \otimes B \to A \otimes B \otimes \mathbb{C}[\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)]$$

defines a faithful, centrally ergodic $\mathrm{tr}_A \otimes \mathrm{tr}_B$-preserving action of $\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)$ on $A \otimes B$. It is called the free wreath action of β and α.

Proof. For the multiplication map $m_{A\otimes B}$ on $A\otimes B$, we have $m_{A\otimes B} = (m_A \otimes m_B)\Sigma_{23} = a(m_B) \in Mor(a(u)\otimes a(u), a(u))$ by [FP16, Proposition 8.1]. By Lemma 3.2.16, $\beta \wr_* \alpha$ is multiplicative. By definition, we have $\eta_A \in \mathrm{Mor}(\mathbb{1}, a(\mathbb{1}_{\mathbb{G}}))$ for the unit map of A and it follows again from [FP16, Proposition 8.1] that $\mathrm{id}_A \otimes \eta_B \in \mathrm{Mor}(a(\mathbb{1}_{\mathbb{G}}), a(u))$ and hence $\eta_{A\otimes B} \in \mathrm{Mor}(\mathbb{1}, a(u))$ and by Lemma 3.2.16, $\beta \wr_* \alpha$ is unital. Since the universal action α on A is centrally ergodic and tr_A-preserving, we have $\mathrm{tr}_A = \eta_A^* \in \mathrm{Mor}(a(\mathbb{1}_{\mathbb{G}}), \mathbb{1})$ and $\mathrm{id}_A \otimes \mathrm{tr}_B \in \mathrm{Mor}(a(u), a(\mathbb{1}_{\mathbb{G}}))$ by [FP16, Proposition 8.1]. It follows that $\beta \wr_* \alpha$ preserves the trace and is involutive by Lemma 3.2.16. By definition, $a(u)$ decomposes as $a(u) = \sum_{x\subset u} a(x)$ and by central ergodicity of β, the summand $a(\mathbb{1}_{\mathbb{G}})$ appears exactly once in this decomposition. Due to the central ergodicity of α, $a(\mathbb{1}_{\mathbb{G}})$ contains the trivial representation exactly once as well, see [FP16, Proposition 5.1]. Consequently, $\dim \mathrm{Mor}(\mathbb{1}, a(u)) = 1$ and $\beta \wr_* \alpha$ is centrally ergodic. Lastly, the fact that $\beta \wr_* \alpha$ is faithful follows from point 5 in [FP16, Proposition 8.1]. ☐

Remark 3.2.24. If we return to Bichon's setting, that is to say, we replace the input on the right by an arbitrary faithful, centrally ergodic action of $\mathbb{F}$ on a set X of n points preserving the uniform probability measure (i.e. $A = C(X)$), one can easily see that the vector space action α_w associated to the unitary representation w is a faithful, trace-preserving action on $C(X)\otimes A$. It is however not immediate that this action is centrally ergodic and this issue will reappear in Chapter 6.

3.3 The planar algebra formalism

Planar algebras were introduced by Jones in [J99] in order to provide a diagrammatical alternative to Popa's axiomatization of the standard invariant of a subfactor as a λ-lattice [Po95]. The introduction to planar algebras in this section follows the PhD thesis of P.Tarrago [T15] which is also the source for the drawings in the first two subsections.

3.3.1 Planar tangles

Let us start by defining the diagrams used in the planar calculus.

Definition 3.3.1 (Jones)**.** A *planar tangle* T of degree $k \geq 0$ consists of the following data.

- A disk D_0 in $\mathbb{R}^2$, called the outer disk.

- Some disjoint disks $D_1, \dots, D_n$ in the interior of D_0 which are called the inner disks.
- For each $0 \leq i \leq n$, a finite subset $S_i \in \partial D_i$ of cardinality $2k_i$ (such that $k_0 = k$) with a distinguished element $i_* \in S_i$. The elements of S_i are called the boundary points of D_i and numbered clockwise starting from i_*. k_i is called the degree of the inner disk D_i. Whenever we draw pictures, we will distinguish i_* by marking the boundary region of D_i preceding i_* by $*$.
- A finite set of disjoint smooth curves $\{\gamma_j\}_{1 \leq j \leq r}$ such that each $\mathring{\gamma}_j$ lies in the interior of $D_0 \setminus \bigcup_{i \geq 1} D_i$ and such that $\bigcup_{1 \leq j \leq r} \partial\gamma_j = \bigcup_{0 \leq i \leq n} S_i$; it is also required that each curve meets a disk boundary orthogonally, and that its endpoints have opposite (resp. same) parity if they both belong to inner disks or both belong to the outer disk (resp. one belongs to an inner disk and the other one to the outer disk).
- A region of P is a connected component of $D_0 \setminus (\bigcup_{i \geq 1} D_i \cup (\bigcup \gamma_j))$. We give a chessboard shading on the regions of P in such a way that the interval components of type $(2i-1, 2i)$ are boundaries of shaded regions.

The skeleton of T, denoted by ΓT, is the set $(\bigcup \partial D_i) \cup (\bigcup \gamma_j)$.

Planar tangles will always be considered up to isotopy.
If the degree of T is 0, then T is of degree $+$ (respectively $-$) if the boundary of the outer disk is the boundary of an unshaded (respectively shaded) region. An example of a planar tangle is given in Figure 3.1.

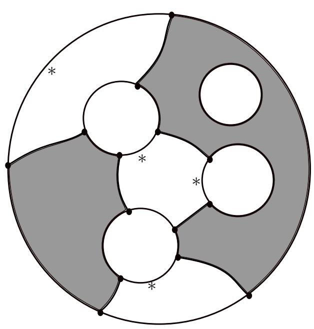

Figure 3.1: Planar tangle of degree 4 with 4 inner disks.

For clarity reasons, we will sometimes omit the numbering of the inner disks in our drawings.

Definition 3.3.2.

- A *connected planar tangle* is a planar tangle whose regions are simply connected; this implies that for any inner disk D and any element $x \in \partial D$, there is a path from x to the boundary of the outer disk which is contained in $(\bigcup \partial D_i) \cup (\bigcup \gamma_j)$.
- An *irreducible planar tangle* is a connected planar tangle such that each curve has an endpoint being a distinguished point of D_0 and the other one being on an inner disk.

An example of a connected planar tangle (resp. irreducible planar tangle) is given in Figure 3.2.

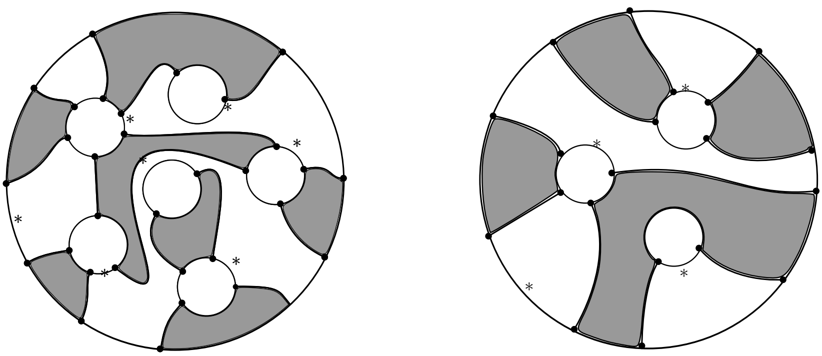

Figure 3.2: A connected and an irreducible planar tangle.

Composition and involution of planar tangles

Let T and T' be two planar tangles of respective degree k and k', and let D be an inner disk of T. We assume that the degree of D is also k'. Then the tangle T can be composed with the tangle T' by isotoping T' onto the inner disk D in such a way that the boundary of T' is mapped onto the boundary of D and that the i-th marked boundary point of T' is mapped onto the i-th marked boundary point of D. Hence, the strings adjacent to the boundary points of T' connect with those adjacent to the boundary points of D and by removing the boundary of D, we obtain a new planar tangle which is denoted by $T \circ_D T'$. It is called the *composition* of T and T' with respect to D.

The composition of the planar tangle depicted in Figure 3.1 with the second planar tangle shown in Figure 3.2 is given in Figure 3.3.

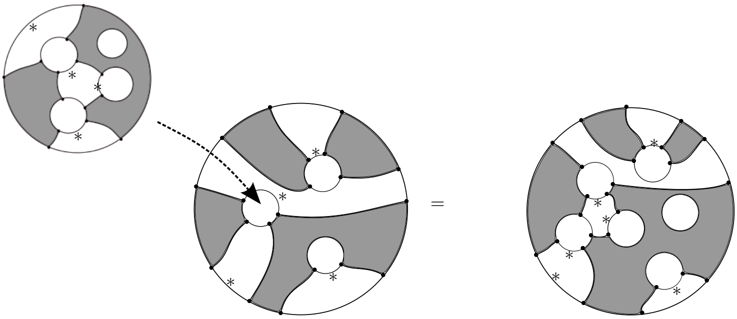

Figure 3.3: Composition of two planar tangles.

If $T_1, \dots, T_s$ are planar tangles and $D_{i_1}, \dots, D_{i_s}$ are distinct inner disks of T such that $\deg T_j = \deg D_{i_j}$ for all $1 \leq j \leq s$, we denote by $T \circ_{(D_{i_1}, \dots, D_{i_s})} (T_1, \dots, T_s)$ the planar tangle obtained by iterating the composition with respect to the different inner disks.

There also exists a natural involution operation on planar tangles. Number the regions adjacent to any disk of a tangle T starting with the region preceding the first boundary point of the disk. Reflect the tangle in a diameter of the outer disk passing through its first region. Renumber the boundary point of the reflected tangle in such a way that the image of a first region is a first region again to obtain the tangle T^*.

3.3.2 Planar algebras

We are now ready to define the notion of a planar algebra.

Definition 3.3.3 (Jones)**.** A planar algebra $\mathcal{P}$ is a collection of finite-dimensional vector spaces $(\mathcal{P}_n)_{n \in \mathbb{N}^* \cup \{-,+\}}$ such that each planar tangle T of degree k with n inner disks $D_1, \dots, D_n$ of respective degree $k_1, \dots, k_n$ yields a linear map

$$Z_T : \bigotimes_{1 \leq i \leq n} \mathcal{P}_{k_i} \longrightarrow \mathcal{P}_k.$$

We require the composition of such maps to be compatible with the composition of planar tangles. More precisely, if T' is another tangle of degree k_{i_0} for some $1 \leq i_0 \leq n$, then

$$Z_T \circ (\bigotimes_{i \neq i_0} \mathrm{Id}_{\mathcal{P}_{k_i}} \otimes Z_{T'}) = Z_{T \circ_{D_{i_0}} T'}.$$

If every $\mathcal{P}_n$ carries the structure of a C^*-algebra with multiplication induced by the natural multiplication tangles (see [J99]) whose involution commutes with the involution of planar tangles, it is called a C^*-planar algebra.

A planar algebra is *spherical* if the action of planar tangles is invariant under symmetries of the two-sphere (obtained by adding a point at infinity to $\mathbb{R}^2$). Note that the left and right trace tangles (as depicted in Figure 3.4) coincide up to a spherical symmetry.

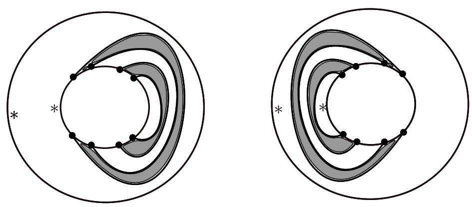

Figure 3.4: Trace tangles of degree 4.

By a theorem of Jones [J99], the properties in the following definition ensure that a planar algebra arises as the standard invariant of a subfactor.

Definition 3.3.4 (Jones)**.** A *subfactor planar algebra* $\mathcal{P}$ is a C^*-planar algebra such that

- $\mathcal{P}$ is spherical,
- $\dim \mathcal{P}_+ = \dim \mathcal{P}_- = 1$,
- For every n, the linear map Tr_n induced by the n-th trace tangles satisfies $\mathrm{Tr}(x^*x) > 0$ for all $0 \neq x \in \mathcal{P}_n$.

Here, it is worthwhile to recall that any finite index subfactor $N \subset M$ gives rise to a Jones tower

$$N \subset M := M_0 \subset M_1 = \langle M, e_1 \rangle \subset M_2 = \langle M_1, e_2 \rangle \subset \dots$$

by inductively adding the Jones projection e_i as a generator to M_{i-1}, $i = 1, 2, \dots$. If the normalized traces on N' and M coincide on the relative commutant $N' \cap M$, by [J99, Theorem 4.2.1], there is canonical planar action on the tower of finite-dimensional relative commutants

$$N' \cap N \subset N' \cap M \subset N' \cap M_1 \subset \dots,$$

yielding a subfactor planar algebra $\mathcal{P}^{N\subset M}$ where $\mathcal{P}_+^{N\subset M} = N' \cap N \cong \mathbb{C}$, $\mathcal{P}^{N\subset M} = M' \cap M \cong \mathbb{C}$ and $\mathcal{P}_i^{N\subset M} = N' \cap M_{i-1}$.

3.3.3 Annular tangles and Hilbert modules over a planar algebra

Since the annular category of a planar algebra will play a central role in Chapter 6, we need to introduce the relevant types of tangles. The definitions and results presented in this section are taken from the article [J01].

Definition 3.3.5. Let $m, n \in \mathbb{N}_* \cup \{+,-\}$. An *annular tangle* is a planar tangle together with a fixed choice of an inner disk which will henceforth be called the input disk. An annular (m,n)-tangle is an annular tangle whose outer disk has $2n$ boundary points and whose input disk has $2m$ boundary points.

Similar to arbitrary tangles, annular tangles come with a natural composition operation. Given an annular (m,n)-tangle T with distinguished disk D and an annular (l,m)-tangle S we obtain a natural composition tangle $T \circ_D S$ by isotoping S into the input disk of T such that boundary points meet and by erasing the common boundary afterwards. The input disk of $T \circ S$ is simply the image of the input disk of S under this isotopy. If T has internal disks other than the distinguished one, we can also compose it with an arbitrary tangle of the correct degree. More precisely, if $\tilde{D}$ is such an internal disk with $2k$ boundary points and $\tilde{S}$ is an arbitrary tangle with $2k$ outer boundary points, the composition $T \circ_{\tilde{D}} \tilde{S}$ yields another annular tangle.

There also exists a natural notion of involution on annular tangles. Starting from an annular (m,n)-tangle T, we obtain an annular (n,m)-tangle T^* by reflecting T in a circle halfway between the inner and outer boundaries (after isotoping the input disk to the center of T). The marked points of the inner and outer disks of T^* are defined as the images of the marked points of T under this reflection which is well-defined since the boundary of the input disk of T is mapped to the outer boundary of T^* and the other way around. Moreover, the reflection maps the boundary points of a non-distinguished internal disk of T to the boundary points of a non-distinguished internal disk of T^*. See [J01] for all this.

Let us quickly recall the definition of modules and Hilbert modules over a planar algebra.

Definition 3.3.6. Let $\mathcal{P} = (\mathcal{P}_n)_{n \in \mathbb{N}_* \cup \{+,-\}}$ be a planar algebra. A *left module* over $\mathcal{P}$ is a graded vector space $V = (V_n)_{n \in \mathbb{N}_* \cup \{+,-\}}$ such that, given an annular (m,n)-tangle T with input disk D_1 and other internal disks D_i, $i = 2, \ldots, s$ of degree k_i, there is a linear map

$$Z_T : V_m \otimes (\otimes_{p=2}^{s} P_{k_p}) \to V_n.$$

We require the family $(Z_T)_T$ to be compatible with both the composition of annular tangles and the composition with arbitrary tangles.

The Hilbert space version of this definition is the following.

Definition 3.3.7. Let $\mathcal{P} = (\mathcal{P}_n)_{n\in\mathbb{N}_*\cup\{+,-\}}$ be a C^*-planar algebra. A $\mathcal{P}$-module $V = (V_k)_{n\in\mathbb{N}_*\cup\{+,-\}}$ will be called a Hilbert $\mathcal{P}$-module if each V_n is a finite-dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle_n$ such that

$$\langle Z_T(v, x_2, \dots, x_s), w\rangle_n = \langle v, Z_{T^*}(w, x_2^*, \dots, x_s^*)\rangle_m$$

whenever T is an annular (m,n)-tangle with non-distinguished internal disks $D_i, i = 2, \dots, s$ of respective degree k_i, $x_i \in \mathcal{P}_{k_i}$ and $v \in V_m$, $w \in V_n$.

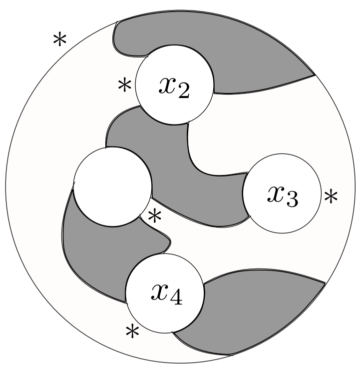

Figure 3.5: Labelled annular $(2,2)$-tangle.

Since the Hilbert spaces in the above definition are all assumed to be finite-dimensional, an annular (m,n)-tangle T with non-distinguished internal disks D_i, $i = 2, \dots, s$ of respective degree k_i which are labelled by $x_i \in \mathcal{P}_{k_i}$ induces a bounded linear map $Z_T(\cdot, x_2, \dots, x_s) : V_m \to V_n$. Lemma 3.12 in [J01] provides us with an estimate on the norm of this map. We have

$$\|Z_T(\cdot, x_2, \dots, x_s)\| \leq C_T \prod_{i=2}^{s} \|x_i\|$$

for some non-negative number $C_T \geq 0$ depending on T. To simplify notation, whenever we speak of labelled tangles we write Z_T instead of $Z_T(\cdot, x_2, \dots, x_s)$. In order to be coherent, whenever we speak of a labelled tangle T, Z_{T^*} means $Z_{T^*}(\cdot, x_2^*, \dots, x_p^*)$.

Example 3.3.8. Let $\mathcal{Q} = (\mathcal{Q}_n)_{n \in \mathbb{N}_* \cup \{+,-\}}$ be a spherical C^*-planar algebra with C^*-planar subalgebra $\mathcal{P}$. The inner product

$$\langle v, w \rangle = \mathrm{tr}_n(w^* v) \qquad v, w \in \mathcal{Q}_n,\ n \in \mathbb{N}_* \cup \{+,-\}$$

turns $\mathcal{Q}$ into a left Hilbert module over $\mathcal{P}$.

Let us single out a specific class of annular tangles which we will call special tangles.

Definition 3.3.9. A *special* (m,n)-tangle is an annular (m,n)-tangle having exactly one non-distinguished disk of degree $(m+n)$. We ask for the first $2n$ boundary points of this disk to be connected by strings to the boundary points of the outer disk in such a way that marked boundary points are connected. The other $2m$ boundary points are required to be connected to the input disk with the marked boundary point of the input disk being connected to the last boundary point of the non-distinguished disk.

Let $\mathcal{P} = (\mathcal{P}_n)_{n \in \mathbb{N}_* \cup \{+,-\}}$ be a C^*-planar algebra and let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}_* \cup \{+,-\}}$ be a Hilbert module over P. We say that a special $(m,n)-$tangle is labelled by $x \in \mathcal{P}_{m+n}$ if the non-distinguished disk is. Let T be such a special (m,n)-tangle labelled by $x \in \mathcal{P}_{m+n}$ and let S be a special (l,m)-tangle labelled by $y \in \mathcal{P}_{l+m}$. Figure 3.6 defines a special (l,n)-tangle $T \circ S$ such that $Z_T \circ Z_S = Z_{T \circ S} : \mathcal{H}_l \to \mathcal{H}_n$.

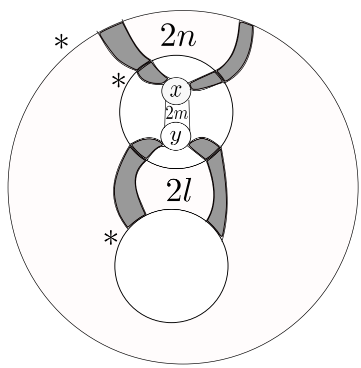

Figure 3.6: Composition $T \circ S$ of special tangles T and S.

Note that, since $\mathcal{H}$ is a Hilbert module, we have $Z_T^* = Z_{T^*}$. Given another special (i,j)-tangle T' labelled by z, we define a special $(m+i, n+j)$-tangle

$T \otimes T'$ (the *tensor tangle*) through Figure 3.7. Clearly, in general the maps $Z_{T\otimes T'}$ and $Z_T \otimes Z_{T'}$ will not coincide, however the tensor tangle will be useful in the specific case considered later on.

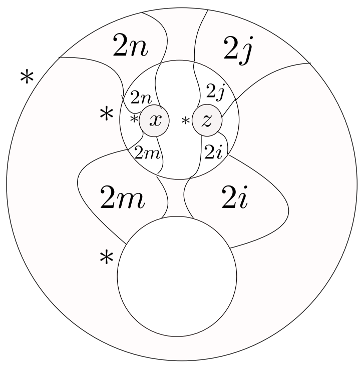

Figure 3.7: Tensor tangle $T \otimes T'$ (shading omitted in order to get a clear picture).

3.3.4 The planar algebra associated to a finite-dimensional C^*-algebra

In [J98], given a finite bipartite graph $\Gamma = (V, E)$ and an eigenvector $\mu : V \to \mathbb{C}$ of the adjacency matrix of Γ, Jones constructs a planar algebra which satisfies all conditions in the definition of subfactor planar algebras (Definition 3.3.4) except for the dimension condition. Every inclusion of finite dimensional C^*-algebras $A_0 \subset A_1$ induces a bipartite graph Γ with vertex set $V = V_+ \cup V_-$ where the vertices in V_+ are the irreducible representations of A_0 and the vertices in V_- are the irreducible representations of A_1. The number of edges between $a \in V_+$ and $b \in V_-$ is the multiplicity of the irreducible representation a in the restriction of b to A_0. Let $A_0 = \mathbb{C}$, let $\dim A_1 = d$ and decompose A_1 into simple components

$$A_1 = \oplus_{i=1}^{s} M_{m_i}(\mathbb{C}).$$

In this case, the bipartite graph Γ has one vertex $a \in V_+$ and s vertices $b_1, \dots, b_s \in V_-$ with m_i edges $\beta_i^1, \dots, \beta_i^{m_i}$ connecting a and b_i. The unique normalized Markov trace of the inclusion is given by $\mathrm{tr}(p_i) = \frac{m_i}{d}$ for a minimal projection $p_i \in M_{m_i}(\mathbb{C})$. We define the spin vector $\mu : V \to \mathbb{C}$ by setting

$$\mu(a) = 1, \quad \mu(b_i) = \frac{m_i}{\sqrt{d}} \quad i = 1, \dots, s.$$

This indeed defines an eigenvector μ of the adjacency matrix of Γ with non-zero positive entries and non-zero eigenvalue $\delta = \sqrt{d}$ satisfying the normalization condition of [J98, Definition 3.5]. Note that our notation differs from Jones's in the sense that the entries of our vector μ are the squares of the entries of the vector μ in [J98].

Let us analyze the planar algebra $\mathcal{P}^\Gamma = (\mathcal{P}^\Gamma_n)_{n\in\mathbb{N}_*\cup\{+,-\}}$ of this particular bipartite graph $\Gamma = \Gamma(A)$. In Chapter 6, we will often ease the notation a bit and write $\mathcal{P}^A$ instead of $\mathcal{P}^\Gamma$. Recall that $\mathcal{P}^\Gamma_n$ is defined as the vector space with basis

$$B_n \;=\; \{\xi;\ \xi = \text{ loop of length } 2n \text{ starting and ending at } a\}$$

for $n \in \mathbb{N}_{\geq 1} \cup \{+\}$, whereas $\mathcal{P}^\Gamma_-$ is the s-dimensional vector space spanned by $B_- = \{b_1, \ldots, b_s\}$. Note that $\mathcal{P}^\Gamma_-$ sits inside $\mathcal{P}^\Gamma_1$ if we identify b_i with the sum of all loops of length 2 passing through b_i. Let us represent a loop $\eta \in \mathcal{P}^\Gamma_n$ by a tuple

$$\eta = (\beta^{k_1}_{i_1}, \beta^{k_2}_{i_1}, \beta^{k_3}_{i_2}, \ldots, \beta^{k_{2n-1}}_{i_n}, \beta^{k_{2n}}_{i_n}).$$

Let T be a planar tangle of degree n with inner disks $D_1, \ldots, D_l$ of degree $n_i = \deg(D_i)$. Let $\eta_i \in B_{n_i},\ i = 1, \ldots, l$ be basis loops. To describe the action $Z_T : \bigotimes_{i=1}^l \mathcal{P}^\Gamma_{n_i} \to \mathcal{P}^\Gamma_n$, it suffices to specify the coefficients of $Z_T(\eta_1, \ldots, \eta_l)$ in the basis expansion

$$Z_T(\eta_1, \ldots, \eta_l) = \sum_{\eta_0 \in B_n} c(\eta_0, \eta_1, \ldots, \eta_l)\ \eta_0.$$

In order to do so, recall that a *state* on T is function σ which maps strings of T to edges of Γ, shaded regions of T to odd vertices of Γ and unshaded regions to even vertices of Γ (in our case the last part can be omitted since $V_+ = \{a\}$). A state must satisfy the following compatibility condition. If a string S is adjacent to a region R, then $\sigma(R)$ must be an endpoint of the edge $\sigma(S)$.
A state σ is compatible with a loop η at the disk D (outer or inner) if reading the output of σ clockwise around D starting from the first region produces η.

To define the coefficients $c(\eta_0, \eta_1, \ldots, \eta_l)$, we need to isotope T to a 'standard form' that is easier to depict if we replace our disks by rectangles. To be more precise, we isotope T so that all its strings are smooth, and so that all its disks become rectangles with the starred region on the left, the first half of the strings coming out of the top and the other half of the bottom of the rectangle. With any local maximum/minimum s of the y-coordinate function of any string S in T, we associate the value $\rho(s) = \sqrt{\frac{\mu(v)}{\mu(w)}}$, where v is the vertex labelling the convex side and w is the vertex labelling the concave side of the singularity. For

a more formal discussion of this procedure, see [JP10].
The coefficient $c(\eta_0, \eta_1, \ldots, \eta_l)$ is defined by the formula

$$c(\eta_0, \eta_1, \ldots, \eta_l) \quad = \quad \sum_{\sigma} \prod_{\substack{s \text{ singularity} \\ \text{in } T}} \rho(s).$$

Here, the first sum runs over all states σ which are compatible with η_i at disk D_i for every $i = 0, \ldots, l$. Note that the partition function $Z : V \to \mathbb{C}$ of $\mathcal{P}^\Gamma$ is given by $Z(a) = 1,\ Z(b_i) = \mu(b_i)^2 = \frac{m_i^2}{d}$. By [J98, Theorem 3.6], for every $k \geq 0$, the partition function induces a normalized trace Tr_k on $\mathcal{P}_k^\Gamma$ through the formula $\mathrm{Tr}_k(x) = (\sqrt{d})^{-k} Z(\hat{x}),\ x \in \mathcal{P}_k$, where $\hat{x}$ is the 0-tangle obtained by connecting the first k boundary points to the last k as in Figure 3.4.

As explained in [J98, Section 5], we can identify $\mathcal{P}_n^\Gamma$ with the n-th algebra A_n in the Jones tower of the inclusion $\mathbb{C} \subset A$. When $n = 1$, the identification is simply given by mapping the loop (β_i^k, β_i^l) to the (k, l)-matrix unit e_{kl}^i of the matrix summand of A represented by the vertex b_i. It is not hard to see that this mapping indeed identifies $\mathcal{P}_1^\Gamma$ and A as C^*-algebras. Moreover the trace Tr_1 coincides with the normalized Markov trace on A under this identification.

For a better understanding of the identification taking place here, let us work it out explicitly in the case $n = 2$. In this case, the basic construction $\mathbb{C} \subset A \subset A_2$ simply yields the algebra $B(A)$ of (bounded) linear operators on the Hilbert space A with the inner product induced by the Markov trace. The inclusion $A \subset B(A)$ is given by A acting on itself by left multiplication. Now let us identify $B(A)$ with $A \otimes A$ as a vector space: we map the tensor product of two matrix units $e_{k_1 l_1}^{i_1} \otimes e_{k_2 l_2}^{i_2} \in A \otimes A$ to the operator mapping $e_{l_2 k_2}^{i_2}$ to $e_{k_1 l_1}^{i_1}$ and mapping all other matrix units to zero. If we pull over the $*$-algebra structure on $B(A)$ to $A \otimes A$ through this map, we obtain the following formulas for multiplication and involution:

$$(e_{k_1 l_1}^{i_1} \otimes e_{k_2 l_2}^{i_2}) \cdot (e_{s_1 t_1}^{r_1} \otimes e_{s_2 t_2}^{r_2}) = \delta_{i_2 r_1} \delta_{l_2 s_1} \delta_{k_2 t_1}\ e_{k_1 l_1}^{i_1} \otimes e_{s_2 t_2}^{r_2}$$

$$(e_{k_1 l_1}^{i_1} \otimes e_{k_2 l_2}^{i_2})^* = (e_{l_2 k_2}^{i_2} \otimes e_{l_1 k_1}^{i_1}).$$

If we consider $A \otimes A$ equipped with this $*$-algebra structure, we denote it by $A \check{\otimes} A$ as this avoids confusion with the tensor product of C^*-algebras $A \otimes A$. The Markov trace on $B(A)$ is the unique normalized trace on $B(A)$ and hence given on $A \check{\otimes} A$ by the formula

$$\mathrm{tr}_2(e_{k_1 l_1}^{i_1} \otimes e_{k_2 l_2}^{i_2}) = \delta_{i_1 i_2} \delta_{k_1 l_2} \delta_{k_2 l_1} \frac{1}{\dim A}.$$

In this picture the identification between $A \check{\otimes} A$ with $\mathcal{P}_2^\Gamma$ becomes particularly easy as one simply needs to map $e_{k_1 l_1}^{i_1} \otimes e_{k_2 l_2}^{i_2} \in A \check{\otimes} A$ to the path

$(\beta_{i_1}^{k_1}, \beta_{i_1}^{l_1}, \beta_{i_2}^{k_2}, \beta_{i_2}^{l_2})$. In a similar manner, we can introduce a sequence of algebras $A^{\check{\otimes} n}$ that identify with the n-th algebra A_n in the Jones tower, see [B05a]. For an explicit description of the higher level isomorphisms $\mathcal{P}_n^\Gamma \cong A^{\check{\otimes} n} \cong A_n$, we refer to [J98, Theorem 5.1].

3.4 Representation theory of rigid C^*-tensor categories

Since standard invariants of subfactors and rigid C^*-tensor categories generalize discrete groups, it has been a long standing question whether analytical group properties still make sense in this larger context. A particularly interesting subproblem of this is to find non-group examples of group-like behaviourisms. In the subfactor setting, the first results in this direction were obtained by Popa in the mid 1990's [Po94b] [Po99] using his symmetric enveloping algebra as the central tool to study amenability and property (T). Much later and coming from a different angle, de Commer, Freslon and Yamashita [dCFY14] started studying central approximation properties of discrete quantum groups, which they found to be invariant under unitary monoidal equivalence of representation categories. These two approaches were finally united by Popa and Vaes in their seminal article [PV15] where they introduced the analogue of a unitary group representation for the subfactor setting as well as for the C^*-tensor category setting. The results of Popa and Vaes put the study of group-like properties for C^*-tensor categories on conceptual feet and they were quickly followed by equivalent approaches in [NY15a] and [GJ16]. Although being equivalent, all three approaches have distinct advantages when dealing with different aspects of representation theory. It is the purpose of this section to recapitulate all three of them.

Independent of the approach of our liking it is our aim to find 'good' representations of the *fusion algebra* of a rigid C^*-tensor category $\mathcal{C}$. This is the analogue of the group algebra $\mathbb{C}[\Gamma]$ of a discrete group Γ.

Definition 3.4.1. The *fusion algebra* $\mathbb{C}[\mathcal{C}]$ of a rigid C^*-tensor category $\mathcal{C}$ is the free vector space with basis $\mathrm{Irr}(\mathcal{C})$, multiplication

$$\alpha\beta = \sum_{\gamma \in \mathrm{Irr}(\mathcal{C})} \mathrm{mult}(\alpha \otimes \beta, \gamma)\gamma, \quad \alpha, \beta \in \mathrm{Irr}(\mathcal{C}).$$

and involution $\alpha^* = \bar{\alpha}$.

The class of $*$-representations $\mathbb{C}[\mathcal{C}] \to B(H)$ on Hilbert spaces is too large to give rise to a universal enveloping C^*-algebra. The approach of Popa and Vaes solves this problem by first determining the meaning of a positive function on $\mathcal{C}$ and then constructing the representations that yield these functions through their vector states.

3.4.1 Multipliers and representations of rigid C^*-tensor categories

The following two definitions were first given in [PV15]. Let $\mathcal{C}$ be a rigid C^*-tensor category.

Definition 3.4.2 (Popa-Vaes)**.** A multiplier on $\mathcal{C}$ is a family of linear maps

$$\theta_{X,Y} : \mathrm{End}(X \otimes Y) \to \mathrm{End}(X \otimes Y)$$

indexed by $X, Y \in \mathcal{C}$ such that

$$\theta_{X_2,Y_2}(UTV^*) = U\theta_{X_1,Y_1}(T)V^*,$$

$$\theta_{X_2 \otimes X_1, Y_1 \otimes Y_2}(1 \otimes T \otimes 1) = 1 \otimes \theta_{X_1,Y_1}(T) \otimes 1 \tag{3.4.1}$$

for all $X_i, Y_i \in \mathcal{C}, T \in \mathrm{End}(X_1 \otimes Y_1)$ and $U, V \in \mathrm{Mor}(X_1, X_2) \otimes \mathrm{Mor}(Y_1, Y_2)$.

Definition 3.4.3 (Popa-Vaes)**.** (i) A multiplier $(\theta_{X,Y})_{X,Y \in \mathcal{C}}$ is said to be *completely positive* (or a cp-multiplier) if for any $X, Y \in \mathcal{C}$, $\theta_{X,Y}$ is a completely positive map in the usual sense.

(ii) A multiplier $(\theta_{X,Y})_{X,Y \in \mathcal{C}}$ is said to be *completely bounded* (or a cb-multiplier) if all maps $\theta_{X,Y}$ are completely bounded in the usual sense and

$$\|\theta\|_{\mathrm{cb}} = \sup_{X,Y \in \mathcal{C}} \|\theta_{X,Y}\|_{\mathrm{cb}} < \infty.$$

Remark 3.4.4. Let us choose a representative X_α for any class of irreducible objects $\alpha \in \mathrm{Irr}(\mathcal{C})$. By [PV15, Proposition 3.6], every multiplier $(\theta_{X,Y})_{X,Y \in \mathcal{C}}$ is uniquely determined by a family of linear maps

$$\mathrm{Mor}(\mathbb{1}, X_\alpha \otimes X_{\bar{\alpha}}) \to \mathrm{Mor}(\mathbb{1}, X_\alpha \otimes X_{\bar{\alpha}})$$

where $\alpha \in \mathrm{Irr}(\mathcal{C})$. Since $\mathrm{Mor}(\mathbb{1}, X_\alpha \otimes X_{\bar{\alpha}})$ is one-dimensional due to the irreducibility of X_α, each of these linear maps is given by multiplication by a scalar $\varphi(\alpha) \in \mathbb{C}$, $\alpha \in \mathrm{Irr}(\mathcal{C})$. Hence every multiplier determines a function $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ and conversely every such function gives rise to a multiplier. In fact, this correspondance is one-to-one and therefore when we speak of a multiplier we will often mean the underlying function $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$. While this notational switch is often convenient since it brings us closer to the picture of positive functions on discrete groups, we should point out that we can not directly read off complete positivity/boundedness of the function φ without passing to the larger notion of multiplier.

With positivity at hand, Popa and Vaes defined the notion of admissible representation of $\mathbb{C}[\mathcal{C}]$ in the following way.

Definition 3.4.5 (Popa-Vaes)**.** An admissible representation of $\mathbb{C}[\mathcal{C}]$ is a unital $*$-representation $\Theta : \mathbb{C}[\mathcal{C}] \to B(\mathcal{H})$ such that for all $\xi \in \mathcal{H}$ the matrix coefficient

$$\mathrm{Irr}(\mathcal{C}) \to \mathbb{C},\ \alpha \to d(\alpha)^{-1}\langle\Theta(\alpha)\xi, \xi\rangle$$

is a cp-multiplier.

Note that, since $\mathrm{Irr}(\mathcal{C})$ forms a basis of the fusion algebra $\mathbb{C}[\mathcal{C}]$ as a vector space, there is a bijective correspondence $\varphi \mapsto \omega_\varphi$ between functions $\mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ and functionals $\mathbb{C}[\mathcal{C}] \to \mathbb{C}$ given by $\omega_\varphi(\alpha) = d(\alpha)\varphi(\alpha)$. If φ is a cp-multiplier, then by [PV15, Proposition 4.2] the functional ω_φ is positive in the sense that $\omega_\varphi(x^*x) \geq 0$ for all $x \in \mathbb{C}[\mathcal{C}]$. By the same proposition, the GNS-representation Θ_φ w.r.t. ω_φ is bounded with $\|\Theta_\varphi(\alpha)\| \leq d(\alpha)$ for all $\alpha \in \mathrm{Irr}(\mathcal{C})$.

Example 3.4.6.

1. Since every admissible representation is by its very definition unitarily isomorphic to the GNS-representation of some cp-multiplier, the norm bound $\|\Theta(\alpha)\| \leq d(\alpha)$ holds for any such admissible representation Θ. In other words, the class of admissible representations of $\mathcal{C}$ is small enough to admit a universal admissible representation. We denote the enveloping C^*-algebra of $\mathbb{C}[\mathcal{C}]$ in the universal representation by $C_u(\mathcal{C})$.

2. The function $\varphi_0 : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}, \varphi_0(\alpha) = \delta_{\alpha,\mathbb{1}}$ defines a cp-multiplier by [PV15, Corollary 4.4]. The GNS-representation Θ_{φ_0} is unitarily equivalent to the *left regular representation*

$$\lambda : \mathbb{C}[\mathcal{C}] \to B(\ell^2(\mathrm{Irr}(\mathcal{C}))), \quad \lambda(\alpha)\delta_\beta = \sum_{\gamma \in \mathrm{Irr}(\mathcal{C})} \mathrm{mult}(\gamma, \alpha \otimes \beta)\delta_\gamma.$$

The norm closure of $\lambda(\mathbb{C}[\mathcal{C}]) \subset B(\ell^2(\mathrm{Irr}(\mathcal{C})))$ is called the *reduced* C^*-algebra of $\mathcal{C}$ and denoted by $C_r^*(\mathcal{C})$.

3. Again by [PV15, Corollary 4.4], the one-dimensional $*$-representation $\epsilon : \mathbb{C}[\mathcal{C}] \to \mathbb{C},\ \alpha \mapsto d(\alpha)$ is admissible. It is equivalent to the GNS-representation of the constant function $\varphi_\epsilon : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}, \varphi_\epsilon(\alpha) = 1$.

3.4.2 The tube algebra and the Drinfel'd double

In [GJ16], Ghosh and Jones related the representation theory of rigid C^*-tensor categories to Ocneanu's tube algebra, which was introduced in [O93] and which contains the fusion algebra as a corner. In short, Ghosh and Jones proved that a representation of $\mathbb{C}[\mathcal{C}]$ is admissible in the sense of Popa and Vaes if and only if it is unitarily equivalent to the restriction of a $*$-representation of the tube algebra to $\mathbb{C}[\mathcal{C}]$. The tube algebra picture is convenient when studying completely bounded multipliers [AV15, Proposition 5.1] and also for the purpose of introducing cohomology for rigid C^*-tensor categories [PSV15].

Fix a rigid C^*-tensor category $\mathcal{C}$. For each equivalence class $\alpha \in \mathrm{Irr}(\mathcal{C})$, choose a representative $X_\alpha \in \alpha$, and choose the representative X_0 of the tensor unit to be the strict tensor unit. Moreover, let Λ be a countable set of equivalence classes of objects in $\mathcal{C}$ with distinct representatives $Y_\lambda \in \lambda$ for every $\lambda \in \Lambda$.

The *annular algebra* with weight set Λ is defined as a vector space as the algebraic direct sum

$$\mathcal{A}\Lambda = \bigoplus_{\lambda,\mu\in\Lambda,\ \alpha\in\mathrm{Irr}(\mathcal{C})} \mathrm{Mor}(X_\alpha \otimes Y_\lambda, Y_\mu \otimes X_\alpha).$$

An element of $\mathcal{A}\Lambda$ is thus of the form $x = (x^\alpha_{\lambda,\mu})$ with $x^\alpha_{\lambda,\mu} \in \mathrm{Mor}(X_\alpha \otimes Y_\lambda, Y_\mu \otimes X_\alpha)$ and with only finitely many non-zero entries. As foreshadowed by its name, $\mathcal{A}\Lambda$ is equipped with a natural $*$-algebra structure. Multiplication and involution have a nice diagrammatical expression which is explained in [GJ16]. Before we express multiplication and involution formally, we note that, if $X, Y \in \mathcal{C}$ and X is irreducible, then $\mathrm{Mor}(X, Y)$ admits a Hilbert space structure with inner product defined by $\eta^*\xi = \langle \xi, \eta\rangle 1_X$. For $\alpha, \beta, \gamma \in \mathrm{Irr}(\mathcal{C})$ choose an orthonormal basis $\mathrm{Onb}(X_\alpha, X_\beta \otimes X_\gamma)$ with respect to this inner product. Multiplication on $\mathcal{A}\Lambda$ is then defined by the formula

$$(x\cdot y)^\alpha_{\lambda,\mu} = \sum_{\substack{\beta,\gamma\in\mathrm{Irr}(\mathcal{C})\\ \nu\in\Lambda}} \sum_{V\in\mathrm{Onb}(X_\alpha, X_\beta\otimes X_\gamma)} (\mathrm{id}_\mu \otimes V^*)(x^\beta_{\nu,\mu} \otimes \mathrm{id}_\gamma)(\mathrm{id}_\beta \otimes y^\gamma_{\lambda,\nu})(V \otimes \mathrm{id}_\lambda).$$

To describe the involution, let us first choose standard solutions $(R_\alpha, \bar{R}_\alpha)$ for all irreducible objects $X_\alpha,\ \alpha \in \mathrm{Irr}(\mathcal{C})$. The involution $\sharp : \mathcal{A}\Lambda \to \mathcal{A}\Lambda$ is given by

$$(x^\sharp)^\alpha_{\lambda,\mu} = (\bar{R}^*_\alpha \otimes \mathrm{id}_\mu \otimes \mathrm{id}_\alpha)(\mathrm{id}_\alpha \otimes (x^{\bar{\alpha}}_{\mu,\lambda})^* \otimes \mathrm{id}_\alpha)(\mathrm{id}_\alpha \otimes \mathrm{id}_\lambda \otimes R_\alpha).$$

It turns out that $\mathcal{A}\Lambda$ only depends on the choices of representatives and orthonormal bases up to $*$-isomorphism. Moreover, we will always assume the weight set Λ to be *full*, i.e. every irreducible object is equivalent to a subobject of some element in Λ. We will often simply deal with the natural choice $\Lambda = \mathrm{Irr}(\mathcal{C})$, but in general it can be very useful to enlargen the set of weights as demonstrated by the proof of [AV15, Proposition 5.1] and various theorems in [PSV15].

Definition 3.4.7. The annular algebra with weight set $\Lambda = \mathrm{Irr}(\mathcal{C})$ is called the *tube algebra* of Ocneanu, and we write $\mathcal{A}\Lambda = \mathcal{A}\mathcal{C}$.

As mentioned before, a central feature of any annular algebra is that it contains the fusion algebra as a corner. To see this, one needs to consider the identity morphism $p_0 = \mathrm{id}_{X_0} \in \mathrm{Mor}(X_0 \otimes X_0, X_0 \otimes X_0) \subset \mathcal{A}\Lambda$, where as before $X_0 = \mathbb{1}$ is the tensor unit. Seen as an element of $\mathcal{A}\Lambda$, p_0 is a projection. We also single out the identity morphisms $\mathrm{id}_\alpha \in \mathrm{Mor}(X_\alpha, X_\alpha) = \mathrm{Mor}(X_\alpha \otimes X_0, X_0 \otimes X_\alpha)$. Then, $\mathbb{C}[\mathcal{C}]$ is isomorphic to the corner $p_0\mathcal{A}\Lambda p_0$ of $\mathcal{A}\Lambda$ as a $*$-algebra where the canonical isomorphism maps $\alpha \in \mathrm{Irr}(\mathcal{C})$ to $\mathrm{id}_\alpha \in \mathrm{Mor}(X_\alpha \otimes X_0, X_0 \otimes X_\alpha) \subset \mathcal{A}\Lambda$.

The annular algebra $\mathcal{A}\Lambda$ with weight set Λ comes with a canonical faithful positive tracial functional $\mathrm{Tr} : \mathcal{A}\Lambda \to \mathbb{C}$ defined by the formula

$$\mathrm{Tr}(x) = \sum_{\lambda \in \Lambda} \mathrm{Tr}_\lambda(x^{\mathbb{1}}_{\lambda,\lambda}),$$

where $x = (x^\alpha_{\lambda,\mu})$ and where Tr_λ refers to the categorical trace on $\mathrm{End}(Y_\lambda)$ defined in Section 3.1. We denote the GNS-Hilbert space with respect to Tr by $L^2(\mathcal{A}\Lambda)$.

The connection between the tube algebra and admissible representations in the sense of Popa and Vaes is spelled out in [GJ16, Corollary 6.7]: The admissible representations of the fusion algebra are exactly those that are unitarily equivalent to restrictions of $*$-representations of the tube algebra it sits in. We will postpone the exact statement until the end of the next subsection (Theorem 3.4.9), so that we can also include the yet-to-be-defined setting of Neshveyev and Yamashita.

However, we will already mention here a result due to Arano and Vaes [AV15, Proposition 5.1] which characterizes cb-multipliers in terms of completely bounded maps on full annular algebras. In order to state the result, let us recall that whenever $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ is a function on the irreducibles of $\mathcal{C}$ and Λ is a full family of objects, then there is a canonical linear map $M_\varphi : \mathcal{A}\Lambda \to \mathcal{A}\Lambda$ given by

$$M_\varphi(x) = \varphi(\alpha)x \quad \text{whenever} \quad x \in \mathrm{Mor}(X_\alpha \otimes Y_\lambda, Y_\mu \otimes X_\alpha).$$

Proposition 3.4.8 (Arano-Vaes)**.** *Let $\mathcal{C}$ be a rigid C^*-tensor category, let Λ be a full family of objects of $\mathcal{C}$, and let $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ be a function. Moreover, let $M_\varphi : \mathcal{A}\Lambda \to \mathcal{A}\Lambda$ be defined as above. Then $\|M_\varphi\|_{\mathrm{cb}} = \|\varphi\|_{\mathrm{cb}}$. If this cb-norm* *is finite, then* M_φ *extends uniquely to a normal completely bounded map on* $\mathcal{A}\Lambda'' \subset B(L^2(\mathcal{A}\Lambda))$.

We will close this section by discussing annular algebras when $\mathcal{C} = \mathrm{Rep}(\mathbb{G})$ for some compact quantum group $\mathbb{G}$. In this case, Neshveyev and Yamashita showed [NY15a, Theorem 2.4] that the annular algebra $\mathcal{A}\Lambda$ is isomorphic as a $*$-algebra to a well-known object from quantum group theory, the *Drinfel'd double* when choosing

$$\Lambda = \{\dim H_\alpha \alpha \ ; \ \alpha \in \mathrm{Irr}(\mathbb{G})\}.$$

Here, by $\dim H_\alpha \alpha$ we mean the direct sum of $\dim H_\alpha$ copies of α, where H_α is the associated Hilbert space to some chosen representative u^α of α. Let us quickly introduce the Drinfel'd double of $\mathbb{G}$.

Consider first the $*$-subalgebra $c_c(\hat{\mathbb{G}})$ of $\mathcal{U}(\mathbb{G}) \cong \prod_{\alpha\in\mathrm{Irr}(\mathbb{G})} B(H_\alpha)$ defined as the direct sum

$$c_c(\hat{\mathbb{G}}) = \bigoplus_{\alpha\in\mathrm{Irr}(\mathbb{G})} B(H_\alpha).$$

Next, define $\mathcal{D}$ to be the vector space $\mathcal{D} = \mathbb{C}[\mathbb{G}] \otimes c_c(\hat{\mathbb{G}})$. We write elementary tensors $a\otimes\omega$ as $a\omega$ and equip $\mathcal{D}$ with a multiplication that is uniquely determined by the exchange law

$$\omega(\cdot a_{(2)})a_{(1)} = a_{(2)}\omega(a_{(1)}\cdot), \quad a \in \mathbb{C}[\mathbb{G}], \ \omega \in c_c(\hat{\mathbb{G}})$$

in Sweedler's notation. Alternatively, when writing $u^\alpha = (u^\alpha_{ij})$ in matrix form with respect to some basis, multiplication in $\mathcal{D}$ is uniquely determined by

$$\omega \cdot u^\alpha_{ij} = \sum_{k,l} u^\alpha_{kl}\omega(u^\alpha_{ik} \cdot (u^\alpha_{lj})^*).$$

The involution on $\mathcal{D}$ is simply defined to be the canonical one that restricts to the involutions on $\mathbb{C}[\mathbb{G}]$ and $c_c(\hat{\mathbb{G}}) \subset \mathcal{U}(\mathbb{G})$ repectively. In fact, $\mathcal{D}$ admits a comultiplication that turns it into a $*$-algebraic quantum qroup that can be completed into a locally compact quantum group. The locally compact quantum group whose convolution algebra is $\mathcal{D}$ is then called the *Drinfel'd* or *quantum double* of $\mathbb{G}$ and denoted $D\mathbb{G}$.

While we will not work with the Drinfel'd double explicitly, it is a very important source of intuition. On the one hand, the identification of the double as annular algebra shows that the admissible representations of $\mathrm{Rep}(\mathbb{G})$ are restrictions of $*$-representations of $\mathcal{D}$, on the other hand the Drinfel'd double is usually considered a quantum version of the complexification of a compact Lie group. For instance, the (dual of) the Drinfel'd double of $SU_q(2)$ is the quantum Lorentz group $SL_q(2,\mathbb{C})$ for $q \neq 1$ [PWo90]. This motivation is crucial to the results in Chapter 5.

3.4.3 Unitary half braidings

The final approach to the representation theory of a rigid C^*-tensor category that we would like to discuss was developed in [NY15a] in terms of unitary half braidings on ind-objects. This approach is particularly well behaved when one is interested in taking tensor products of representations, a fact we will make use of in the proof of Theorem 4.1.4. Let us recall that, intuitively, an ind-object $Z \in \operatorname{ind}\mathcal{C}$ is a possibly infinite direct sum of objects in the rigid C^*-tensor category $\mathcal{C}$ and that $\operatorname{ind}\mathcal{C}$ is a C^*-tensor category containing $\mathcal{C}$, albeit generically not a rigid one. For a rigorous definition, we refer to [NY15a]. A unitary half braiding σ on an ind-object $Z \in \operatorname{ind}\mathcal{C}$ was defined in [NY15a] as a family of unitary morphisms $\sigma_X \in \operatorname{Mor}(X \otimes Z, Z \otimes X),\ X \in \mathcal{C}$ satisfying

- $\sigma_{\mathbb{1}} = \mathrm{id}$;
- $(1 \otimes V)\sigma_X = \sigma_Y(V \otimes 1)$ for all $V \in \operatorname{Mor}(X, Y)$;
- $\sigma_{X \otimes Y} = (\sigma_X \otimes 1)(1 \otimes \sigma_Y)$.

Every pair (Z, σ) consisting of an ind-object Z and a unitary half braiding σ on Z, defines a $*$-representation of $\mathbb{C}[\mathcal{C}]$ on the Hilbert space $\mathcal{H}_{(Z,\sigma)} = \operatorname{Mor}_{\operatorname{ind}\mathcal{C}}(\mathbb{1}, Z)$ with inner product $\langle \xi, \eta \rangle 1 = \eta^* \xi$. More concretely, if we choose a set of representatives X_α for $\alpha \in \operatorname{Irr}(\mathcal{C})$ with standard solution of the conjugate equations $(R_\alpha, \bar{R}_\alpha)$, then

$$\pi_{(Z,\sigma)} : \mathbb{C}[\mathcal{C}] \to B(\mathcal{H}_{(Z,\sigma)}), \quad \pi(\alpha)\xi = (1 \otimes \bar{R}_\alpha^*)(\sigma_{X_\alpha} \otimes 1)(1 \otimes \xi \otimes 1)\bar{R}_\alpha$$

defines a $*$-representation. Note that a different choice of representatives yields a unitarily equivalent $*$-representation. The following result connects unitary half braidings, tube algebra representations and admissible representations.

Theorem 3.4.9 ([GJ16] [NY15a])**.** *Let $\mathcal{C}$ be a rigid C^*-tensor category with tube algebra $\mathcal{AC}$. Also, let $\pi : \mathbb{C}[\mathcal{C}] \cong p_0\mathcal{AC}p_0 \to B(H)$ be a unital $*$-representation on a Hilbert space H. The following are equivalent.*

(i) π is admissible in the sense of Definition 3.4.5.

(ii) There exists a (non-degenerate) $*$-representation $\Pi : \mathcal{A} \to B(K)$ such that $\Pi|_{p_0\mathcal{AC}p_0}$ is unitarily equivalent to π.

(iii) There exists a unitary half braiding (Z, σ) such that π is unitarily equivalent to $\pi_{(Z,\sigma)}$.

Let us finish this discussion by remarking that the bijection between unitary half braidings on ind-objects and (non-degenerate) $*$-representations of the tube algebra, has been made very explicit in [PSV15, Proposition 3.14]. In [PSV15], the concept of annular algebras was also extended to the setting of quasi-regular inclusions.

3.4.4 Central functionals

The goal of this subsection is to analyse admissible representations, cp-multipliers and cb-multipliers in the case where $\mathcal{C} = \mathrm{Rep}(\mathbb{G})$ for some compact quantum group $\mathbb{G}$. The discussion on admissible representations and cp-multipliers does not contain anything new and is completely based on [dCFY14] and [PV15, Section 6].

We fix a compact quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta)$ and representatives $u^\alpha = (u^\alpha_{ij})_{i,j=1\dots \dim H_\alpha} \in B(H_\alpha) \otimes \mathbb{C}[\mathbb{G}]$ for all $\alpha \in \mathrm{Irr}(\mathbb{G})$. Recall the following terminology from [dCFY14].

Definition 3.4.10. A functional $\Omega : \mathbb{C}[\mathbb{G}] \to \mathbb{C}$ is called *central* if

$$(\Omega \otimes \psi) \circ \Delta = (\psi \otimes \Omega) \circ \Delta \quad \forall \psi \in \mathbb{C}[\mathbb{G}]^*.$$

In other words, a functional Ω is central if it literally lies in the center of the discrete dual algebra $\mathcal{U}(\mathbb{G})$. By recalling the isomorphism

$$\mathcal{U}(\mathbb{G}) \cong \prod_{\alpha\in\mathrm{Irr}(\mathbb{G})} B(H_\alpha)$$

from Subsection 3.2.2, we observe that $\Omega = (\Omega^\alpha) \in \prod_{\alpha\in\mathrm{Irr}(\mathbb{G})} B(H_\alpha)$ is central if for all $\alpha \in \mathrm{Irr}(\mathbb{G})$, the matrix $\Omega^\alpha \in B(H_\alpha)$ is a scalar multiple of the identity. This implies that there is a bijection $\varphi \mapsto \Omega_\varphi$ between functions $\varphi : \mathrm{Irr}(\mathbb{G}) \to \mathbb{C}$ and central functionals $\Omega_\varphi \in \mathcal{Z}(\mathcal{U}(\mathbb{G}))$ such that

$$\Omega_\varphi(u^\alpha_{ij}) = \delta_{ij}\varphi(\alpha), \quad i, j = 1 \dots, \dim H_\alpha,\ \alpha \in \mathrm{Irr}(\mathbb{G}).$$

We can also associate to $\varphi : \mathrm{Irr}(\mathbb{G}) \to \mathbb{C}$ a linear map

$$\Psi_\varphi : \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{G}], \quad \Psi_\varphi = (\mathrm{id} \otimes \Omega_\varphi) \circ \Delta.$$

The next proposition due to Popa and Vaes [PV15, Proposition 6.1] shows that the respective notions of positivity of φ, Ω_φ and Ψ_φ are compatible.

Proposition 3.4.11. *For $\varphi : \mathrm{Irr}(\mathbb{G}) \to \mathbb{C}$ the following are equivalent.*

(i) φ *is a cp-multiplier;*

(ii) Ω_φ *is positive in the sense that* $\Omega_\varphi(x^*x) \geq 0$ *for all* $x \in \mathbb{C}[\mathbb{G}]$*;*

(iii) Ψ_φ *extends to a completely positive map* $C_r(\mathbb{G}) \to C_r(\mathbb{G})$*.*

We will prove a similar result for completely bounded multipliers in Proposition 4.3.3.

3.4.5 Approximation properties and property (T)

Since the definition of amenability by von Neumann as a response to the Banach-Tarski paradox [vN29b], approximation properties of discrete groups have been a major focus of analytic group theory. The same can be said about a famous obstruction to amenability, the property (T) of Kazhdan [Kz67]. Amenability as well as property (T) also have natural analogues for discrete quantum groups (see e.g [To06], [Fi10]) and for rigid C^*-tensor categories [PV15]. In this subsection, we will recall the definitions of amenability, its cousin the Haagerup property and of property (T) in both situations and we will have a look at the relationship of both settings.

We will start start with amenability of discrete quantum groups, which we will again examine through the dual lense of compact quantum groups. There are many equivalent characterizations for amenability for discrete quantum groups and we choose the following ones.

Definition 3.4.12. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. We say that $\hat{\mathbb{G}}$ is amenable if one and hence all of the following equivalent conditions hold.

(i) The canonical surjective $*$-homomorphism $C_u(\mathbb{G}) \to C_r(\mathbb{G})$ is an isomorphism.

(ii) The counit $\epsilon : \mathbb{C}[\mathbb{G}] \to \mathbb{C}$ extends to a bounded linear functional $C_r(\mathbb{G}) \to \mathbb{C}$.

(iii) The left regular representation of the fusion algebra $\lambda : \mathbb{C}[\mathrm{Rep}(\mathbb{G})] \to \ell^2(\mathrm{Irr}(\mathbb{G}))$ satisfies $\|\lambda(\alpha)\| = \dim H_\alpha$ for all $\alpha \in \mathrm{Irr}(\mathbb{G})$. Here H_α is the Hilbert space associated to some representative of α.

A proof of the equivalence of the three conditions in the definition can for example be found in [NT13, Theorem 2.7.10]. Examples of amenable discrete quantum groups include amenable discrete groups or less trivially, duals of

q-deformations of simply connected semisimple compact Lie groups [NT13, Theorem 2.7.14].

For rigid C^*-tensor categories, we have the following equivalent formulations of amenability, which can for instance be found in [PV15, Proposition 5.3].

Definition 3.4.13. Let $\mathcal{C}$ be a rigid C^*-tensor category. We say that $\mathcal{C}$ is amenable if one and hence all of the following equivalent conditions hold.

(i) The canonical surjective $*$-homomorphism $C_u(\mathcal{C}) \to C_r(\mathcal{C})$ is an isomorphism.

(ii) There exists a net $(\varphi_i)_{i\in I}$ of finitely supported cp multipliers $\varphi_i : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ that converges pointwisely to 1.

(iii) The left regular representation of the fusion algebra $\lambda : \mathbb{C}[\mathcal{C}] \to \ell^2(\mathrm{Irr}(\mathcal{C}))$ satisfies $\|\lambda(\alpha)\| = d(\alpha)$ for all $\alpha \in \mathrm{Irr}(\mathcal{C})$. Here $d(\alpha)$ refers to the categorical dimension of α as in the end of Section 3.1.

If $\mathcal{C} = \mathrm{Rep}(\mathbb{G})$ for some compact quantum group, the amenability of $\mathcal{C}$ is a much stronger property than the amenability of $\hat{\mathbb{G}}$. In fact, if $\mathrm{Rep}(\mathbb{G})$ is amenable, the categorical dimension of α must coincide with the dimension $\dim H_\alpha$ of an associated Hilbert space [NT13, Corollary 2.7.9]. Combining this fact with the third characterization in both Definition 3.4.12 and Definition 3.4.13 and with Theorem 3.2.4, one obtains the following relation between the amenability of $\hat{\mathbb{G}}$ and the amenability of $\mathrm{Rep}(\mathbb{G})$.

Corollary 3.4.14. *Let $\mathbb{G}$ be a compact quantum group. Then* $\mathrm{Rep}(\mathbb{G})$ *is amenable if and only if $\mathbb{G}$ is of Kac type and $\hat{\mathbb{G}}$ is amenable.*

In particular, this corollary shows that $\mathrm{Rep}(K_q)$ is not amenable for $q \neq 1$ for any simply connected semisimple compact Lie group K.

Another approximation property of importance for discrete groups is the Haagerup property. It is strictly weaker than amenability and the most famous examples of nonamenable discrete groups with the Haagerup property are the free groups $\mathbb{F}_n$, $n \geq 2$ [Haa78]. Again, there is a generalization for discrete or even locally compact quantum groups, see e.g. [DFSW13] but we will only encounter the Haagerup property in this thesis in situations where the Haar state on $\mathbb{G}$ is tracial. In this case, we can define the Haagerup property for $\hat{\mathbb{G}}$ completely in terms of the von Neumann algebra $L^\infty(\mathbb{G})$.

Definition 3.4.15. (i) Let (M, ψ) be a von Neumann algebra with a faithful state ψ. We say that (M, ψ) has the *Haagerup property* if there exists a net $(\Psi_i)_{i\in I}$ of normal unital completely positive ψ-preserving maps $\Psi_i : M \to M$ such that

– The L^2-extension $\hat{\Psi}_i : L^2(M,\psi) \to L^2(M,\psi)$ is compact for all $i \in I$;
– $\|\hat{\Psi}_i(\hat{x}) - \hat{x}\|_2 \to 0$ as $i \to \infty$ for all $x \in M$.

Here $\hat{x}$ denotes the image of x under the natural embedding $M \hookrightarrow L^2(M,\psi)$.

(ii) Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group with tracial Haar state h. We say that $\hat{\mathbb{G}}$ has the *Haagerup property* if $(L^\infty(\mathbb{G}), h)$ has the Haagerup property.

For rigid C^*-tensor categories, the Haagerup property is defined as follows [PV15, Definition 5.1].

Definition 3.4.16. A rigid C^*-tensor category $\mathcal{C}$ is said to have the *Haagerup property* if there exists a net $(\varphi_i)_{i\in I}$ of cp-multipliers $\varphi_i : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ such that $\varphi_i \in c_0(\mathrm{Irr}(\mathcal{C}))$ for all $i \in I$ and such that $\varphi_i \to 1$ pointwise.

If $\mathcal{C} = \mathrm{Rep}(\mathbb{G})$ for some compact quantum group $\mathbb{G}$, the Haagerup property for $\mathcal{C}$ can be reformulated in terms of central states on $\mathbb{C}[\mathbb{G}]$ using Proposition 3.4.11. More precisely, it is the same as the Haagerup property of $(L^\infty(\mathbb{G}), h)$ plus the additional requirement that the completely positive maps involved are induced by central states as in Proposition 3.4.11. This reformulation was already implicitly present in the article [dCFY14] and is typically refered to as the *central Haagerup property* of $\hat{\mathbb{G}}$. It is thus clear that the central Haagerup property of $\hat{\mathbb{G}}$ implies the Haagerup property for $(L^\infty(\mathbb{G}), h)$ and in the case of a tracial Haar state, both are equivalent thanks to the presence of a well-behaved conditional expectation, see [KrR99, Theorem 5.14 and 5.15]. Nontrivial examples of discrete quantum groups with the Haagerup property include the duals of the free orthogonal and the free unitary quantum group [Bn12] and the quantum automorphism groups [Bn13].

A last approximation property that we would like to mention is the *almost completely positive approximation property.* We will only mention its definition for rigid C^*-tensor categories [PV15, Definition 9.1] and refer to [dCFY14, Definition 3] for a definition in the setting of discrete quantum groups. For duals of Kac type quantum groups, both notions coincide for the same reasons as for the Haagerup property.

Definition 3.4.17. A rigid C^*-tensor category $\mathcal{C}$ is said to have the *almost completely positive approximation property (ACPAP)* if there exists a net $(\varphi_i)_{i\in I}$ of cp-multipliers $\varphi_i : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ that converges pointwise to 1 and such that for every fixed $i \in I$, there exists a net of finitely supported cb-multipliers approximating φ_i in the cb-norm.

For locally compact groups, in Kazhdan's property (T) [Kz67] we have a well-known obstruction to amenability and the Haagerup property. It is an obstruction in the sense that the only groups having both (T) and the Haagerup property are the compact ones. Again, let us define property (T) for discrete quantum groups and rigid $\mathcal{C}^*$-tensor categories.

In the setting of discrete quantum groups, the first definition of property (T) was given in [Fi10], but we use the following equivalent characterizations from [Ky11] instead.

Definition 3.4.18. Let $\mathbb{G}$ be a compact quantum group. The discrete dual $\hat{\mathbb{G}}$ has property (T) if one (and hence all) of the following equivalent conditions is satisfied.

(i) If a net of states (ω_λ) in $C_u(\mathbb{G})^*$ converges pointwisely to the counit ϵ, then it converges in norm.

(ii) There exists a projection $p \in C_u(\mathbb{G})$ such that $xp = \epsilon(x)p$ for all $x \in C_u(\mathbb{G})$.

It was shown in [DSV17] that, also in the more general framework of locally compact quantum groups, condition (i) of the previous definition is equivalent to the conventional notion of property (T) in terms of (almost) invariant vectors.

The definition of property (T) in the context of rigid C^*-tensor categories by Popa and Vaes and two characterizations of this property obtained in [PV15], are given in the following definition.

Definition 3.4.19. A rigid C^*-tensor category $\mathcal{C}$ has property (T) if one (and hence all) of the following equivalent conditions is satisfied.

1. Every net (φ_λ) of cp-multipliers $\varphi_\lambda : \operatorname{Irr}\mathcal{C} \to \mathbb{C}$ converging to φ_ε pointwise converges uniformly, i.e. $\sup_{\alpha\in\operatorname{Irr}\mathcal{C}} |\varphi_\lambda(\alpha) - 1| \to 0$.
2. If (ω_λ) is a net of states on $C_u(\mathcal{C})$ converging to $\varepsilon : C_u(\mathcal{C}) \to \mathbb{C}, \varepsilon(\alpha) = d(\alpha)$ in the weak*-topology, it must already converge in norm.
3. There exists a unique nonzero projection $p \in C_u(\mathcal{C})$ such that $\alpha p = d(\alpha)p$ for all $\alpha \in \mathcal{C}$. Such a projection is the analogue of a Kazhdan projection in the setting of groups.

When $\mathcal{C} = \operatorname{Rep}(\mathbb{G})$ for some compact quantum group $\mathbb{G}$, one can again reformulate categorical property (T) in terms of central states. This yields the following theorem whose first part is [PV15, Proposition 6.3], and whose second part was proven in [Ar15].

Theorem 3.4.20. *Let $\mathbb{G}$ be a compact quantum group. The following conditions are equivalent:*

(i) the category $\operatorname{Rep}\mathbb{G}$ *has property (T) for rigid C^*-tensor categories,*

(ii) the discrete dual $\hat{\mathbb{G}}$ has central property (T), i.e. if a net (ω_λ) of central states on $C_u(\mathbb{G})^*$ converges to the counit in the weak-topology, then it converges in norm.*

Moreover, if we assume the Haar state on $\mathbb{G}$ to be tracial, this is equivalent to the discrete dual $\hat{\mathbb{G}}$ having (non-central) property (T).

We will now complete the picture by involving property (T) for von Neumann algebras. We use the following two characterizations of this property (see [BO08, Chapter 12] for the equivalence).

Definition 3.4.21. A finite von Neumann algebra (M, τ) has property (T) if one (and hence all) of the following equivalent conditions is satisfied.

(i) If $(\Phi_\lambda : M \to M)$ is a net of unital completely positive τ-preserving maps converging to the identity pointwise on $L^2(M)$, i.e. for all $x \in M$,

$$\|\Phi_\lambda(x) - x\|_2 \to 0, \quad \text{as} \quad \lambda \to \infty,$$

then it already converges in norm, i.e.

$$\sup_{x \in M_1} \|\Phi_\lambda(x) - x\|_2 \to 0 \quad \text{as} \quad \lambda \to \infty.$$

(ii) For any M-bimodule $\mathcal{H}$ and any net (ξ_λ) of unit vectors satisfying

$$\langle x\xi_\lambda y, \xi_\lambda\rangle_{\mathcal{H}} \to \tau(xy) \quad \text{as} \quad \lambda \to \infty$$

for all $x, y \in M$ and $\tau(x) = \langle x\xi_\lambda, \xi_\lambda\rangle = \langle \xi_\lambda x, \xi_\lambda\rangle$ for all λ, there exists a net of M-central vectors (μ_λ) with

$$\|\xi_\lambda - \mu_\lambda\| \to 0 \quad \text{as} \quad \lambda \to \infty.$$

The next theorem is a generalization of [Fi10, Theorem 3.1] and relates property (T) of $\hat{\mathbb{G}}$ to property (T) of the von Neumann algebra $L^\infty(\mathbb{G})$. The proof of the theorem is due to Stefaan Vaes and was included in the article [AdLW17] with his kind permission. For the sake of completeness, we also include it here.

Theorem 3.4.22. *Let $\mathbb{G}$ be a compact quantum group with a tracial Haar state. Then $\hat{\mathbb{G}}$ has (central) property (T) if and only if $L^\infty(\mathbb{G})$ has property (T).*

Proof. Suppose that $\hat{\mathbb{G}}$ has property (T), and let $\mathcal{H}$ be a $L^\infty(\mathbb{G})$-bimodule and (ξ_λ) a net of unit vectors in $\mathcal{H}$ such that $\langle x\xi_\lambda y, \xi_\lambda\rangle \to h(xy)\ \forall x, y \in L^\infty(\mathbb{G})$ and $h(x) = \langle x\xi_\lambda, \xi_\lambda\rangle = \langle \xi_\lambda x, \xi_\lambda\rangle$ for all λ and $x \in M$. We have to find a net (μ_λ) of $L^\infty(\mathbb{G})$-central vectors such that $\|\xi_\lambda - \mu_\lambda\| \to 0$. For every $\pi \in \mathrm{Irr}(\mathbb{G})$, choose a unitary matrix $u^\pi = (u^\pi_{ij})$ representing π. Since the Haar state is tracial, we can assume that $u^{\bar{\pi}} = \overline{u^\pi}$. Define the linear map

$$\Theta : \mathbb{C}[\mathbb{G}] \to B(\mathcal{H})\ ;\ \Theta(u^\pi_{ij})\xi = \sum_{k=1}^{d(\pi)} u^\pi_{ik}\xi(u^\pi_{jk})^*, \quad (\pi \in \mathrm{Irr}(\mathbb{G}))$$

and, denoting the the coinverse of $\mathbb{G}$ by S, observe that $\Theta = \vartheta \circ \Delta$ where $\vartheta : \mathbb{C}[\mathbb{G}] \otimes \mathbb{C}[\mathbb{G}] \to B(\mathcal{H})$ is the $*$-homomorphism defined by $\vartheta(a \otimes b)\xi = a\xi S(b),\ \xi \in \mathcal{H}$. Hence Θ is a $*$-homomorphism as well and therefore extends to $C_u(\mathbb{G})$. Moreover, the conditions on (ξ_λ) imply

$$\|\Theta(x)\xi_\lambda - \epsilon(x)\xi_\lambda\| \to 0 \quad \forall x \in C_u(\mathbb{G}).$$

Indeed, it suffices to show this for x being a coefficient of a irreducible corepresentation $\pi \in \mathrm{Irr}(\mathbb{G})$ and in that case one computes

$$\|\Theta(u^\pi_{ij})\xi_\lambda - \delta_{ij}\xi_\lambda\|^2 \xrightarrow{\lambda} \sum_{k,l=1}^{d(\pi)} h((u^\pi_{il})^* u^\pi_{ik}(u^\pi_{jk})^* u^\pi_{jl}) - 2\sum_{k=1}^{d(\pi)} h(u^\pi_{ik}(u^\pi_{jk})^*) + \delta_{ij} = 0.$$

Since $\hat{\mathbb{G}}$ has property (T), by Definition 3.4.18, we can find a projection $q \in C_u(\mathbb{G})$ such that $xq = \epsilon(x)q$ for all $x \in C_u(\mathbb{G})$ and in particular we have $\epsilon(q) = 1$. Defining $\mu_\lambda = \Theta(q)\xi_\lambda$, it follows that $\|\xi_\lambda - \mu_\lambda\| \to 0$. It only remains to prove that the vector μ_λ is $L^\infty(\mathbb{G})$-central for every λ. To see this, observe first that for $\pi \in \mathrm{Irr}(\mathbb{G})$, we have

$$\sum_{k=1}^{d(\pi)} u^\pi_{ik}\mu_\lambda(u^\pi_{jk})^* = \Theta(u^\pi_{ij})\mu_\lambda = \Theta(u^\pi_{ij}q)\xi_\lambda = \delta_{ij}\mu_\lambda.$$

Therefore, the computation

$$\mu_\lambda u^\pi_{il} = \sum_{j=1}^{d(\pi)} \delta_{ij}\mu_\lambda u^\pi_{jl} = \sum_{j,k=1}^{d(\pi)} u^\pi_{ik}\mu_\lambda(u^\pi_{jk})^* u^\pi_{jl} = u^\pi_{il}\mu_\lambda$$

for $\pi \in \mathrm{Irr}(\mathbb{G}),\ i, l = 1, \ldots, d(\pi)$, concludes the argument.

Let us now assume that $L^\infty(\mathbb{G})$ has property (T). We prove that $\mathrm{Rep}\,\mathbb{G}$ has property (T), which is equivalent to central property (T) by Theorem 3.4.20.

Let $(\varphi_\lambda)_\lambda$ be a net of cp-multipliers converging to ε pointwise. Without loss of generality, we can assume that $\varphi_\lambda(1) = 1$ for all λ. By Proposition 3.4.11, we obtain a net of h-preserving u.c.p. maps $\Psi_\lambda : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ such that $\Psi_\lambda(u^\pi_{ij}) = \varphi_\lambda(\pi)u^\pi_{ij}$ for all $\pi \in \mathrm{Irr}(\mathbb{G})$, $i, j = 1, \ldots, \dim \pi$. The pointwise convergence of the net $(\varphi_\lambda)_\lambda$ then implies that the unital completely positive maps $\Psi_\lambda : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ converge pointwise to the identity, i.e.

$$\|\Psi_\lambda(x) - x\|_2 \to 0, \quad \forall x \in L^\infty(\mathbb{G}) \quad \text{as} \quad \lambda \to \infty.$$

It follows from the assumption that $L^\infty(\mathbb{G})$ has property (T) that

$$\sup_{x \in L^\infty(\mathbb{G})_1} \|\Psi_\lambda(x) - x\|_2 \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Now, for all $\pi \in \mathrm{Irr}(\mathbb{G})$ and all λ, consider the unital completely positive map

$$\mathrm{id}_\pi \otimes \Psi_\lambda : B(H_\pi) \otimes L^\infty(\mathbb{G}) \to B(H_\pi) \otimes L^\infty(\mathbb{G})$$

and note that $(\mathrm{id}_\pi \otimes \Psi_\lambda)(u^\pi) = \varphi_\lambda(\pi)u^\pi$. Hence,

$$\begin{aligned}
\sup_{\pi \in \mathrm{Irr}(\mathbb{G})} |\varphi_\lambda(\pi) - 1| &= \sup_{\pi \in \mathrm{Irr}(\mathbb{G})} \|(\varphi_\lambda(\pi) - 1)u^\pi\|_2 \\
&= \sup_{\pi \in \mathrm{Irr}(\mathbb{G})} \|(\mathrm{id}_\pi \otimes \Psi_\lambda)(u^\pi) - u^\pi\|_2 \to 0,
\end{aligned}$$

which establishes property (T) in the categorial sense. □

Example 3.4.23. Although property (T) generalizes nicely to the setting of discrete quantum groups, it is very difficult to construct actual examples of discrete quantum groups with (T) that are not group-like. One can find examples in [Fi10] that are not exactly groups but bicharacter twists of property (T) groups and share the same von Neumann algebra. One can therefore not consider these examples to be 'genuinely quantum'.

For rigid C^*-tensor categories, however, the situation is very different: Arano [Ar15] [Ar16] has shown that the duals of q-deformations of connected simply connected compact Lie groups have central property (T) for $0 < q < 1$, or in other words, that the representation categories of such q-deformed Lie groups have (T).

Chapter 4

Harmonic analysis on rigid C^*-tensor categories

This chapter is based on the author's joint work [AdLW17] with Arano and de Laat. Its goal is to adapt some of the fundamental notions of classical harmonic analysis on locally compact groups to the setting of rigid C^*-tensor categories. More precisely, we will define the Fourier-Stieltjes algebra $B(\mathcal{C})$, the Fourier algebra $A(\mathcal{C})$ and the algebra $M_0A(\mathcal{C})$ of completely bounded multipliers of a rigid C^*-tensor category $\mathcal{C}$. Along with the basic definitions, we will prove some structural results on these algebras and characterize amenability in terms of the Fourier algebra $A(\mathcal{C})$ by proving a version of Leptin's theorem [Le68] in our setting. We will also give some remarks on the connection between $B(\mathcal{C})$ and property (T). While none of the results in this chapter are surprising, we hope to provide a useful toolset for further research on analytic aspects of rigid C^*-tensor categories.

4.1 The Fourier-Stieltjes algebra

Let $\mathcal{C}$ be a rigid C^*-tensor category, and let $\mathbb{C}[\mathcal{C}]$ be its fusion algebra. We would like to remind the reader that the definition of admissible $*$-representations and the universal admissible $*$-representation of $\mathcal{C}$, as introduced by Popa and Vaes in [PV15], were recalled in Section 3.4. We will use admissible $*$-representations now to define the Fourier-Stieltjes algebra of a C^*-tensor category.

Definition 4.1.1. The Fourier-Stieljes algebra $B(\mathcal{C})$ of $\mathcal{C}$ is the algebra of functions $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ of the form

$$\varphi(\alpha) = d(\alpha)^{-1}\langle \Theta(\alpha)\xi, \eta\rangle \quad (\alpha \in \mathrm{Irr}(\mathcal{C})),$$

where $\Theta : \mathbb{C}[\mathcal{C}] \to B(\mathcal{K})$ is an admissible $*$-representation of the fusion algebra and $\xi, \eta \in \mathcal{K}$. More precisely, the algebra operations are given by pointwise addition and multiplication.

Let us first confirm that $B(\mathcal{C})$ is well-defined, that is to say, let us check that the product of two functions in $B(\mathcal{C})$ stays in $B(\mathcal{C})$. To see this, we note that by the polarisation identity, we can write any element of $B(\mathcal{C})$ as a linear combination of four cp-multipliers. Therefore we get

$$\mathcal{B}(\mathcal{C}) = \left\{ \sum_{i=1}^{n} \lambda_i \varphi_i \;\middle|\; n \in \mathbb{N},\ \lambda_i \in \mathbb{C},\ \varphi_i \in CP(\mathcal{C}),\ i = 1, \ldots, n \right\},$$

where $CP(\mathcal{C})$ denotes the set of cp-multipliers on $\mathcal{C}$. Since it follows directly from the definition of cp-multipliers that the product of two cp-multipliers is a cp-multiplier again, $B(\mathcal{C})$ is indeed an algebra. Moreover, $B(\mathcal{C})$ is unital, since the constant function $\varphi = 1$ is a cp-multiplier by Example 3.4.6. We will mostly be satisfied with treating $B(\mathcal{C})$ as a unital algebra as opposed to turning it into a $*$-algebra. The reasons for this are that there are two involutions on $B(\mathcal{C})$ that can be considered canonical, namely $\varphi^*(\alpha) = \overline{\varphi(\alpha)}$ and $\varphi^\sharp(\alpha) = \overline{\varphi(\bar{\alpha})}$ and that for our purposes the algebra structure is simply sufficient.

We will now equip $\mathcal{B}(\mathcal{C})$ with a norm that turns it into a unital Banach algebra.

Proposition 4.1.2. *The map $\Phi_0 : CP(\mathcal{C}) \to C_u(\mathcal{C})^*_+,\ \Phi_0(\varphi)(\alpha) = \omega_\varphi(\alpha) = d(\alpha)\varphi(\alpha)$ extends linearly to an isomorphism of vector spaces $\Phi : \mathcal{B}(\mathcal{C}) \to C_u(\mathcal{C})^*$. Moreover, for an element $\varphi \in \mathcal{B}(\mathcal{C})$, we have the following equality of norms:*

$$\|\varphi\|_{\mathcal{B}(\mathcal{C})} := \|\Phi(\varphi)\| = \min\{\|\xi\|\|\eta\| \mid \varphi(\cdot) = d(\cdot)^{-1}\langle \Theta(\cdot)\xi, \eta\rangle,\ \Theta \text{ admissible}\}.$$

Proof. By definition of $C_u(\mathcal{C})$ and [PV15, Proposition 4.2], the map Φ_0 is well-defined, and so is Φ. It is clear that Φ defines a bijection. The second part follows directly from the following lemma. □

Lemma 4.1.3. *Let A be a unital C^*-algebra. For all $\omega \in A^*$, we have the following equality of norms:*

$$\|\omega\| = \min\{\|\xi\|\|\eta\| \mid \omega(\cdot) = \langle \Theta(\cdot)\xi, \eta\rangle,\ \Theta\ * -\text{representation of } A\}.$$

Although this result is well-known to experts, for the sake of completeness, we include a proof.

Proof. Since we can view A^* as the predual of the von Neumann algebra A^{**}, there exists a positive normal functional $|\omega| \in A^*_+$ and a partial isometry $V \in A^{**}$ such that $\omega = V|\omega|$ and $\|\omega\| = \|\ |\omega|\ \|$ [Ta79, Theorem III.4.2]. Consider the GNS-representation $\Theta : A \to B(\mathcal{K})$ of $|\omega|$, which has a cyclic vector, say η, such that $|\omega|(x) = \langle \Theta(x)\eta, \eta\rangle$ for all $x \in A$. As a consequence, we obtain that

$$\omega(x) = (V|\omega|)(x) = |\omega|(xV) = \langle \Theta(x)\Theta'(V)\eta, \eta\rangle \quad \forall x \in A,$$

where Θ' is the unique extension of Θ to A^{**}. Defining $\xi = \Theta'(V)\eta$, we have $\|\xi\| \le \|\eta\|$, since V is a partial isometry. Altogether, we have found ξ, η such that $\omega(\cdot) = \langle \Theta(\cdot)\xi, \eta\rangle$ and $\|\omega\| = \|\ |\omega|\ \| = \|\eta\|^2 \ge \|\xi\|\|\eta\|$.
On the other hand, for every $*$-representation $\Theta : A \to B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$ such that $\omega(\cdot) = \langle \Theta(\cdot)\xi, \eta\rangle$ we have

$$|\omega(x)| = |\langle \Theta(x)\xi, \eta\rangle| \le \|x\|\|\xi\|\|\eta\| \quad \forall x \in A$$

and therefore the lemma is proven. □

Theorem 4.1.4. *Let $\mathcal{C}$ be a rigid C^*-tensor category. Then $\mathcal{B}(\mathcal{C})$ is a Banach algebra with respect to the norm defined in the previous proposition.*

Proof. The definition of $\|\cdot\|_{\mathcal{B}(\mathcal{C})}$ directly implies that $(\mathcal{B}(\mathcal{C}), \|\cdot\|_{\mathcal{B}(\mathcal{C})})$ is a Banach space since it is isometrically isomorphic to the dual of $C_u(\mathcal{C})$ with its canonical norm. We are only left with showing that $\|\varphi_1\varphi_2\|_{\mathcal{B}(\mathcal{C})} \le \|\varphi_1\|_{\mathcal{B}(\mathcal{C})}\|\varphi_2\|_{\mathcal{B}(\mathcal{C})}$ for $\varphi_1, \varphi_2 \in \mathcal{B}(\mathcal{C})$. Now, by Proposition 4.1.2 and the discussion in Section 3.4.3, for $i = 1, 2$ we can find pairs (X_i, σ_i) of ind-objects $X_i \in \operatorname{ind}\mathcal{C}$ and unitary half braidings σ_i on X_i as well as $\xi_i, \eta_i \in \mathcal{H}_{(X_i,\sigma_i)}$ such that

$$\varphi_i(\alpha) = d(\alpha)^{-1}\langle \pi_{(X_i,\sigma_i)}(\alpha)\xi_i, \eta_i\rangle \quad \text{and} \quad \|\varphi_i\|_{\mathcal{B}(\mathcal{C})} = \|\xi_i\|\|\eta_i\|.$$

Following [NY15a, Section 2.1], $\sigma = (1 \otimes \sigma_2)(\sigma_1 \otimes 1)$ defines a unitary half braiding on $X = X_1 \otimes X_2 \in \operatorname{ind}\mathcal{C}$. Recall from [NY15a] that, in the same way as unitary half braidings are generalizations of group representations, this new half braiding is the proper analogue of the tensor product of the unitary half braidings (X_1, σ_1) and (X_2, σ_2). We have $\xi = (\xi_1 \otimes 1)\xi_2, \eta = (\eta_1 \otimes 1)\eta_2 \in \mathrm{Mor}_{\operatorname{ind}\mathcal{C}}(\mathbb{1}, X_1 \otimes X_2) = \mathcal{H}_{(X,\sigma)}$ with $\|\xi\| = \|\xi_1\|\|\xi_2\|, \|\eta\| = \|\eta_1\|\|\eta_2\|$. Choosing representatives Y_α for irreducible objects $\alpha \in \mathrm{Irr}(\mathcal{C})$ and using the fact that $\bar{R}^*_{Y_\alpha}\bar{R}_{Y_\alpha} = d(\alpha) \in \mathrm{Mor}(\mathbb{1}, \mathbb{1})$, we compute

$$\pi_{(X,\sigma)}(\alpha)\xi = d(\alpha)^{-1}(\pi_{(X_1,\sigma_1)}(\alpha)\xi_1 \otimes 1)(\pi_{(X_2,\sigma_2)}(\alpha)\xi_2),$$

and hence

$$\varphi_1(\alpha)\varphi_2(\alpha) = d(\alpha)^{-1}\langle \pi_{(X,\sigma)}(\alpha)\xi, \eta\rangle,$$

which finishes the proof. □

4.2 The Fourier algebra

Recall that by Example 3.4.6, the left regular representation of $\mathbb{C}[\mathcal{C}]$ given by

$$\lambda : \mathbb{C}[\mathcal{C}] \to B(\ell^2(\mathrm{Irr}(\mathcal{C}))),\ \lambda(\alpha)\delta_\beta = \sum_{\gamma \in \mathrm{Irr}(\mathcal{C})} \mathrm{mult}(\alpha \otimes \beta, \gamma)\delta_\gamma$$

is admissible and corresponds to the cp-multiplier defined by $\varphi_\lambda(\alpha) = \delta_{\alpha,\mathbb{1}}$ ($\alpha \in \mathrm{Irr}(\mathcal{C})$).

Definition 4.2.1. The Fourier algebra $A(\mathcal{C})$ of a rigid C^*-tensor category $\mathcal{C}$ is defined as the predual of the von Neumann algebra $\lambda(\mathbb{C}[\mathcal{C}])''$.

Recall that there is a one-to-one correspondence between functions on $\mathrm{Irr}(\mathcal{C})$ and functionals $\omega : \mathbb{C}[\mathcal{C}] \to \mathbb{C}$ given by $\varphi \mapsto \omega_\varphi$, where $\omega_\varphi(\alpha) = d(\alpha)\varphi(\alpha)$. By this correspondence, $A(\mathcal{C})$ can also be interpreted as an algebra of functions on $\mathrm{Irr}(\mathcal{C})$.

Proposition 4.2.2. *For every $\omega \in A(\mathcal{C})$, there exist $\xi, \eta \in \ell^2(\mathrm{Irr}(\mathcal{C}))$ such that $\omega(x) = \langle \lambda(x)\xi, \eta \rangle$. In addition,*

$$\|\omega\|_{A(\mathcal{C})} = \min\{\|\xi\|\|\eta\| \mid \omega(\cdot) = \langle \lambda(\cdot)\xi, \eta\rangle,\ \xi, \eta \in \ell^2(\mathrm{Irr}(\mathcal{C}))\}.$$

Proof. Since $M = \lambda(\mathbb{C}[\mathcal{C}])''$ is nothing but the GNS-representation with respect to ω_φ, where $\varphi_\lambda(\alpha) = \delta_{\alpha,\mathbb{1}}$ ($\alpha \in \mathrm{Irr}(\mathcal{C})$), we can represent every positive normal functional on M as a vector state on M by [Ta03, Lemma IX.1.6]. The result for a general normal functional follows as in the proof of Proposition 4.1.2 by polar decomposition. □

Remark 4.2.3. It is an immediate consequence of Proposition 4.2.2 that we have

$$\|\varphi\|_{B(\mathcal{C})} \leq \|\varphi\|_{A(\mathcal{C})}$$

for $\varphi \in A(\mathcal{C})$, and it is not hard to see that the norms are actually equal. Indeed, the dual $C_r(\mathcal{C})^*$ of the reduced C^*-algebra $C_r(\mathcal{C}) = \overline{\lambda(\mathbb{C}[\mathcal{C}])}$ identifies isometrically with the dual of a quotient of $C_u(\mathcal{C})$ and hence with the annihilator of a closed ideal in $C_u(\mathcal{C})$. Consequently,

$$\|\varphi\|_{A(\mathcal{C})} = \|\varphi\|_{C_r(\mathcal{C})^*} = \|\varphi\|_{C_u(\mathcal{C})^*}$$

for $\varphi \in A(\mathcal{C})$. This means that we could also have defined $A(\mathcal{C})$ as the closure of the coefficients of the left regular representation in $\mathcal{B}(\mathcal{C})$. Moreover, we will see in Corollary 4.3.1 that $A(\mathcal{C})$ is a closed ideal in $\mathcal{B}(\mathcal{C})$ and in particular a Banach algebra itself.

4.3 Completely bounded multipliers

In this section, we study completely bounded multipliers and in particular the algebra

$$M_0A(\mathcal{C}) = \{\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C} \mid \varphi \text{ cb-multiplier}\}.$$

While the Fourier algebra $A(\mathcal{C})$ is only defined in terms of the fusion algebra $\mathbb{C}[\mathcal{C}]$, the Fourier-Stieltjes algebra $B(\mathcal{C})$ and the algebra $M_0A(\mathcal{C})$ of completely bounded multipliers use considerably more information on the category $\mathcal{C}$. Therefore, there is no apparent reason why completely bounded maps on the von Neumann algebra $\lambda(\mathbb{C}[\mathcal{C}])''$ should yield completely bounded multipliers. In fact, we have seen in Proposition 3.4.8 and the discussion preceding it that the tube algebra is a more convenient tool to handle cb-multipliers. We record the following direct consequence of Proposition 3.4.8.

Corollary 4.3.1. *Let φ be a completely bounded multiplier. Then, the multiplication operator*

$$T_\varphi : A(\mathcal{C}) \to A(\mathcal{C}), \qquad \theta \mapsto \varphi\theta \quad (\theta \in A(\mathcal{C}))$$

is well defined and completely bounded with $\|T_\varphi\|_{\mathrm{cb}} \leq \|\varphi\|_{\mathrm{cb}}$.

Proof. The dual map of the multiplication operator T_φ is given by restricting the map M_φ in Proposition 3.4.8 to $A(\mathcal{C})^*$. By Proposition 3.4.8, the map T_φ is thus completely bounded with

$$\|T_\varphi\| = \|T_\varphi^*\| \leq \|M_\varphi\|_{\mathrm{cb}} = \|\varphi\|_{\mathrm{cb}}.$$

□

Recall that a Banach algebra A is called *dual Banach algebra* if $A = E^*$, as a Banach space, for some Banach space E and such that for all $x \in A$ the maps $A \to A, y \mapsto xy$ and $A \to A, y \mapsto yx$ are weak*-continuous. So, taking Sakai's theorem into account, we see that dual Banach algebras are to Banach algebras what von Neumann algebras (or rather W^*-algebras) are to C^*-algebras.

Corollary 4.3.2. *Let $\mathcal{C}$ be a rigid C^*-tensor category. Then $M_0A(\mathcal{C})$ carries the structure of a dual Banach algebra if we endow it with pointwise addition and multiplication and the cb-norm $\|\cdot\|_{\mathrm{cb}}$.*

Proof. Pick a full family of objects Λ, say $\Lambda = \mathrm{Irr}(\mathcal{C})$, and denote the reduced C^*-algebra of $\mathcal{A}\Lambda$ by A and its enveloping von Neumann algebra by $M = \mathcal{A}\Lambda''$. It follows from a well-known result in operator theory due to Effros and Ruan

[ER91] and, independently, due to Blecher and Paulsen [BP91], that the space of completely bounded maps $CB(A, M)$ is a dual operator space with predual $A\hat{\otimes}M_*$. Here, $\hat{\otimes}$ denotes the projective tensor product of operator spaces. While we will refer [Pi03, Chapter 4] for its precise definition, let us at least recall that on elementary tensors the duality between $CB(A, M)$ and $A\hat{\otimes}M_*$ is implemented by

$$\langle x \otimes \omega, \Psi \rangle \;=\; \omega(\Psi(x)) \quad \text{for } x \in A,\ \omega \in M_*,\ \Psi \in CB(A, M).$$

Let us show that the image of the isometric embedding $M_0A(\mathcal{C}) \to CB(A, M)$, $\varphi \mapsto \widetilde{M_\varphi}$ is w*-closed in $CB(A, M)$, where $\widetilde{M_\varphi}$ denotes the unique extension of M_φ to A. This will then imply that $M_0A(\mathcal{C})$ is isomorphic as a Banach space to the dual of a quotient of $A\hat{\otimes}M_*$. So, let (φ_i) be a net in $M_0A(\mathcal{C})$ such that $(\widetilde{M_{\varphi_i}})$ converges to a completely bounded map $\Psi \in CB(A, M)$. In particular, this means that

$$\omega(\widetilde{M_{\varphi_i}}(x)) \to \omega(\Psi(x)) \quad \text{as} \quad i \to \infty$$

for all $x \in A$, $\omega \in M_*$. By choosing $x \in \mathrm{Mor}(X_\gamma \otimes \mathbb{1}, \mathbb{1} \otimes X_\gamma)$ and $\omega \in M_*$ such that $\omega(x) \neq 0$ and by applying the definition of $\widetilde{M_{\varphi_i}}(x)$, we find that φ_i converges pointwise to a bounded function φ. It follows from a short computation that the restriction of Ψ to $\mathcal{A}\Lambda$ is equal to M_φ. As a consequence, φ is completely bounded by the previous proposition with $\Psi = \widetilde{M_\varphi}$. Lastly, the fact that the left (and analogously the right) multiplication map $\psi \mapsto \varphi\psi$ in $M_0A(\mathcal{C})$ is w*-continuous for $\varphi \in M_0A(\mathcal{C})$ follows from the computation

$$\omega(M_{\varphi\psi_i}(x)) = \varphi(\gamma)\omega(M_{\psi_i}(x)) \quad \to \quad \varphi(\gamma)\omega(M_\psi(x)) = \omega(M_{\varphi\psi}(x)),$$

where (ψ_i) converges to $\psi \in M_0A(\mathcal{C})$ in the w*-topology and where $\omega \in M_*, x \in \mathrm{Mor}(X_\gamma \otimes Y_\alpha, Y_\beta \otimes X_\gamma)$. Here, Y_α, Y_β denote chosen representatives of $\alpha, \beta \in \Lambda$. Hence. we conclude that $M_0A(\mathcal{C})$ is a dual Banach algebra. □

We will end this section by proving a result on cb-multipliers on compact quantum groups which is analogous to Proposition 3.4.11.

Proposition 4.3.3. *For a compact quantum group $\mathbb{G}$, choose representatives $u^\gamma = (u^\gamma_{ij})_{i,j=1}^{\dim\gamma}$ of $\gamma \in \mathrm{Irr}(\mathbb{G})$. For a function $\varphi : \mathrm{Irr}(\mathbb{G}) \to \mathbb{C}$ and $K \geq 0$, the following are equivalent.*

(i) The function φ is a completely bounded multiplier on $\mathrm{Rep}(\mathbb{G})$ with cb-norm K.

(ii) The linear map

$$\mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{G}], \quad u^\gamma_{ij} \mapsto \varphi(\gamma)u^\gamma_{ij}$$

extends to a normal completely bounded map $L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ with cb-norm K.

Moreover, if $\hat{\mathbb{G}}$ is amenable, (i) and (ii) are also equivalent to the following assertion.

(iii) The functional

$$\mathbb{C}[\mathbb{G}] \to \mathbb{C}, \quad u_{ij}^\gamma \mapsto \delta_{ij}\ \varphi(\gamma)$$

is bounded with norm K.

Proof. The implication (ii) to (i) is shown in Proposition 6.1 of [PV15].
To prove that (i) implies (ii), recall from Section 3.4.2 that in [NY15a, Theorem 2.4], it is shown that the convolution algebra $\mathcal{D}$ of the Drinfel'd double $D\mathbb{G}$ is an annular algebra of $\mathrm{Rep}(\mathbb{G})$ with weight set

$$\Lambda = \{\dim(\gamma)\gamma \mid \gamma \in \mathrm{Irr}(\mathbb{G})\},$$

Consider the normal completely bounded map $M_\varphi : \mathcal{A}\Lambda'' \to \mathcal{A}\Lambda''$ with cb-norm K discussed in Proposition 3.4.8. Then the restriction on $L^\infty(\mathbb{G}) \subset L^\infty(\hat{D}\mathbb{G}) = \mathcal{A}\Lambda''$ is the desired completely bounded map.

Assume that $\hat{\mathbb{G}}$ is amenable, i.e. $C_u(\mathbb{G}) \cong C_r(\mathbb{G})$ are canonically isomorphic and there exists a counit ϵ on $C_r(\mathbb{G}) \subset L^\infty(\mathbb{G})$. We have a one-to-one correspondence between bounded functionals on $C(\mathbb{G})$ and right invariant completely bounded maps on $L^\infty(\mathbb{G})$. More concretely, to every bounded functional $\omega \in C(\mathbb{G})^* \cong (C_u(\mathbb{G})^{**})_* \cong L^\infty(\mathbb{G})_*$, we associate the normal completely bounded map

$$\Psi_\omega : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}), \quad x \mapsto (\omega \otimes \mathrm{id})\Delta(x).$$

Moreover, if ω is as in (iii), that is to say $\omega(u_{ij}^\gamma) = \delta_{ij}\ \varphi(\gamma)$, then

$$\Psi_\omega(u_{ij}^\gamma) = (\omega \otimes \mathrm{id}) \sum_k u_{ik}^\gamma \otimes u_{kj}^\gamma = \varphi(\gamma) u_{ij}^\gamma$$

and $\|\Psi_\omega\|_{cb} \leq \|\omega\| = K$. On the other hand, given a completely bounded map Ψ on $L^\infty(\mathbb{G})$, we can define a bounded functional

$$\omega_\Psi \ = \ \epsilon \circ \Psi|_{C(\mathbb{G})}.$$

Again, if Ψ is as in (ii), we have that

$$\omega_\Psi(u_{ij}^\gamma) \ = \ \delta_{ij}\ \varphi(\gamma)$$

and $\|\omega_\Psi\| \leq \|\Psi\|_{cb}$. In other words we get a one-to-one map $\omega \mapsto \Psi_\omega$ between the central functionals in (iii) and the completely bounded maps in (ii) with inverse $\Psi \mapsto \omega_\Psi$ and preserving the relevant norms. □

4.4 Leptin's characterization of amenability

Recall from Definition 3.4.13 that a rigid C^*-tensor category $\mathcal{C}$ is said to be amenable if there exists a net of finitely supported cp-multipliers $\varphi_i : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ that converges to 1 pointwise.

In [Le68], Leptin proved that a locally compact group is amenable if and only if the Fourier algebra of the group admits a bounded approximate unit. We finish this section by proving a version of Leptin's theorem for rigid C^*-tensor categories. Before doing so, we note that, using the dimension function $d : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$, one can turn $\mathrm{Irr}(\mathcal{C})$ into a discrete hypergroup (see [Mu07] for the definition of a hypergroup and its Fourier algebra). When writing up this thesis, we realized that a version of Leptin's theorem for hypergroups was already proven in [Al14], although the terminology used in that article differs from ours. For the sake of completeness, we will nevertheless include a proof phrased in the setting of rigid C^*-tensor categories. In this section, we interpret elements of the Fourier algebra $A(\mathcal{C})$ as functions on $\mathrm{Irr}(\mathcal{C})$ as explained after Definition 4.2.1.

Theorem 4.4.1. *A rigid C^*-tensor category $\mathcal{C}$ is amenable if and only if $A(\mathcal{C})$ admits a bounded approximate unit, i.e. a net (φ_i) in $A(\mathcal{C})$ such that $\sup_i \|\varphi_i\|_{A(\mathcal{C})} < \infty$ and for all $f \in A(\mathcal{C})$,*

$$\|\varphi_i f - f\|_{A(\mathcal{C})} \to 0 \quad as \quad i \to \infty.$$

In order to prove this theorem, we first prove the following two lemmas that are certainly well-known to experts.

Lemma 4.4.2. *A rigid C^*-tensor category $\mathcal{C}$ is amenable in the sense of Definition 3.4.13 if and only if the counit given by $\epsilon : \mathbb{C}[\mathcal{C}] \to \mathbb{C},\ \epsilon(\alpha) = d(\alpha)$ extends continuously to $C_r(\mathcal{C})$.*

Proof. If $\mathcal{C}$ is amenable, Definition 3.4.13 states that the canonical *-epimorphism $C_u(\mathcal{C}) \to C_r(\mathcal{C})$ is an isomorphism. As ϵ is a one-dimensional admissible representation, it extends continuously to $C_u(\mathcal{C})$ and we are done. Conversely, if ϵ admits a continuous extension $\tilde{\epsilon}$ to $C_r(\mathcal{C})$, then this extension is automatically a character and in particular $\|\tilde{\epsilon}\| = 1$. Therefore, we obtain $\|\lambda(\alpha)\| \geq |\epsilon(\lambda(\alpha))| = d(\alpha)$ and by the last characterization in Definition 3.4.13, $\mathcal{C}$ is amenable. □

Lemma 4.4.3. *The space of finitely supported functions in the unit ball $A(\mathcal{C})_1$ is norm dense in $A(\mathcal{C})_1$, i.e. $A(\mathcal{C})_1 = \overline{c_c(\mathrm{Irr}(\mathcal{C})) \cap A(\mathcal{C})_1}^{A(\mathcal{C})}$.*

Proof. Note first that if $\xi, \eta \in c_c(\mathrm{Irr}(\mathcal{C}))$ are finitely supported functions, the same holds for the matrix coefficient $\varphi_{\xi,\eta}(\alpha) = d(\alpha)^{-1}\langle \lambda(\alpha)\xi, \eta\rangle,\ \alpha \in \mathrm{Irr}(\mathcal{C})$. Since we can approximate any function in $\ell^2(\mathrm{Irr}(\mathcal{C}))$ by finitely supported ones of smaller norm, every $\varphi \in A(\mathcal{C})_1$ can be approximated in norm by functions of the form $\varphi_{\xi,\eta}$ with $\xi, \eta \in c_c(\mathrm{Irr}(\mathcal{C}))$ and $\|\xi\|, \|\eta\| \le 1$. More precisely, this follows from the inequality

$$\|\varphi_{\xi_1,\eta_1} - \varphi_{\xi_2,\eta_2}\| \le \|\xi_1 - \xi_2\|\|\eta_1\| + \|\eta_1 - \eta_2\|\|\xi_2\|$$

for all $\xi_i, \eta_i \in A(\mathcal{C}),\ i = 1, 2$, which is easily established. □

Proof of Theorem 4.4.1. Assume first that $\mathcal{C}$ is amenable. By Lemma 4.4.2, this means that the trivial representation ϵ given by $\epsilon(\alpha) = d(\alpha),\ \alpha \in \mathrm{Irr}(\mathcal{C})$ extends to a character on $C_r(\mathcal{C})$ which we can extend to a (not necessarily normal) state on the von Neumann algebra $C_r(\mathcal{C})''$. Since the unit ball of every Banach space is w*-dense in the unit ball of its double dual, there exists a net of normal states (ω_i) on $C_r(\mathcal{C})''$ such that for all $x \in C_r(\mathcal{C})''$,

$$\omega_i(x) \to \epsilon(x) \quad \text{as} \quad i \to \infty.$$

Let $\varphi_i \in A(\mathcal{C})$ such that $\omega_i = \omega_{\varphi_i}$. By the previous lemma, it suffices to show that for all $f \in c_c(\mathrm{Irr}(\mathcal{C})) \cap A(\mathcal{C})_1$,

$$\|\omega_{\varphi_i f - f}\| \to 0 \quad \text{as} \quad i \to \infty.$$

Let $f \in c_c(\mathrm{Irr}(\mathcal{C})) \cap A(\mathcal{C})$. The operator given by

$$T_f : \mathbb{C}[\mathcal{C}] \to \mathbb{C}[\mathcal{C}],\ T_f(\alpha) = f(\alpha)\alpha \quad \alpha \in \mathrm{Irr}(\mathcal{C})$$

extends to a (completely) bounded finite rank operator on $C_r(\mathcal{C})''$ with norm $\|T_f\| = K$ for some $K > 0$ as a consequence of Proposition 3.4.8. We have

$$\|\omega_{\varphi_i f - f}\| = \sup_{\|x\|\le 1} |\omega_{\varphi_i - 1}(T_f(x))| \le \sup_{y \in \operatorname{ran} T_f,\ \|y\| \le K} |\omega_{\varphi_i - 1}(y)|.$$

But since $\operatorname{ran} T_f$ is finite-dimensional and $\omega_{\varphi_i - 1} \to 0$ as $i \to \infty$ in the w*-topology, the result follows.

Let us now assume that $A(\mathcal{C})$ admits a bounded approximate unit (φ_i) with $\|\varphi_i\|_{A(\mathcal{C})} \le K$ for all i and some $K > 0$. As the K-ball is w*-compact in the dual of $C_r(\mathcal{C})''$, we find a subnet (φ_j) of the approximate unit that converges in the w*-topology to an element $\omega \in (C_r(\mathcal{C})'')^*$. As $\varphi_j(\alpha) \to 1$ as $j \to \infty$ for all $\alpha \in \mathrm{Irr}(\mathcal{C})$, we obtain

$$\omega(\lambda(\alpha)) = \lim_{j\to\infty} \omega_{\varphi_j}(\lambda(\alpha)) = d(\alpha).$$

Hence, the restriction of ω to $C_r(\mathcal{C})$ is an extension of the counit and by Lemma 4.4.2, $\mathcal{C}$ is amenable. □

4.5 Remarks on property (T)

The material of Section 4.1 gives rise to some observations on property (T) in the setting of C^*-tensor categories that are motivated by Kazhdan's property (T) in the setting of groups. First, it is known that for every locally compact group G, the Fourier–Stieltjes algebra $B(G)$ has a unique invariant mean. This goes back to [Go48, Chapitre III]. This result was generalized to the setting of locally compact quantum groups in [DSV17]. The next proposition asserts the existence of an invariant mean on the Fourier-Stieltjes algebra of $\mathcal{C}$, but we formulate it in terms of the existence of a central projection on $W(\mathcal{C}) := B(\mathcal{C})^* = C_u(\mathcal{C})^{**}$.

Note that, since the multiplier $\varphi_\epsilon : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ given by $\varphi_\epsilon(\alpha) = 1$ is completely positive by Example 3.4.6, the counit $\epsilon : \mathbb{C}[\mathcal{C}] \to \mathbb{C}$ extends to a normal $*$-homomorphism on $W(\mathcal{C})$.

Proposition 4.5.1. *Let A be a unital C^*-algebra and let $\chi : A \to \mathbb{C}$ be a character on A. There exists a unique projection p in the von Neumann algebra A^{**} such that*

$$xp = px = \chi(x)p \quad \text{for all } x \in A^{**}.$$

In particular, setting $A = C_u(\mathcal{C})$, $\chi = \varepsilon$, we find a unique projection $p \in B(\mathcal{C})^* = W(\mathcal{C})$ such that $\varepsilon(p) = 1$ and $\langle \omega, \alpha p\rangle = \langle \omega, p\alpha\rangle = d(\alpha)\langle \omega, p\rangle$ for all $\alpha \in \mathrm{Irr}(\mathcal{C})$ and $\omega \in B(\mathcal{C})$.

Proof. Uniqueness of p is immediate. To prove the existence, note that, since χ is a normal $*$-homomorphism, its kernel $\ker(\chi)$ is weakly closed and therefore a von Neumann algebra itself. Denote its unit by e_χ. Then the central cover $p = 1 - e_\chi$ of χ is a projection in A^{**} satisfying $qp = pq = p$ for all q with $\chi(q) = 1$. On the other hand, if q is a projection in $\ker(\chi)$, we have $pq = 0$. Since every von Neumann algebra is the norm closure of the span of its projections, and χ is in particular norm continuous, the result follows. □

Remark 4.5.2. In the group case, it was shown in [AW81, Lemma 1] (see also [Va84, Lemma 3.1] and [HKdL14, Proposition 4.1]) that a locally compact group G has Kazhdan's property (T) if and only if the unique invariant mean on $B(G)$ is weak*-continuous, i.e. the mean is an element of $C^*(G)$ rather than just $C^*(G)^{**}$. In fact, under the natural map from $B(G)^*$ to $C^*(G)^{**}$, the mean is mapped to the Kazhdan projection, which is by weak*-continuity actually an element of $C^*(G)$.

By characterization (iii) in Definition 3.4.19 above, we see that the same thing happens for C^*-tensor categories: a rigid C^*-tensor category $\mathcal{C}$ has property (T) if and only if the mean on $B(\mathcal{C})$ is weak*-continuous.

Remark 4.5.3. In the group case, the unique invariant mean on $B(G)$ is the restriction to $B(G)$ of the unique invariant mean on the space $\mathrm{WAP}(G)$ of weakly almost periodic functions on G, which is well-known to have a unique invariant mean. Indeed, note that $B(G) \subset \mathrm{WAP}(G)$. Hence, the only thing one needs to show is that this restriction is the unique invariant mean on $B(G)$. In a similar fashion, it is shown (see [HKdL14, Theorem A]) that the space $M_0A(G)$ of completely bounded Fourier multipliers on G admits a unique invariant mean, using that $B(G) \subset M_0A(G) \subset \mathrm{WAP}(G)$. It is not known whether the space $M_0A(\mathcal{C})$ of a rigid C^*-tensor category admits a unique invariant mean, in particular because it is not known what the natural analogue of $\mathrm{WAP}(G)$ for rigid C^*-tensor categories should be. For locally compact quantum groups, WAP algebras were studied more thoroughly in [DD16]. However, to the author's knowledge, the existence of an invariant mean on the WAP algebra of a locally compact quantum group $\mathbb{G}$ is only known in the case where $\mathbb{G}$ is amenable [Ru09].

The unique invariant mean on $M_0A(G)$ leads in [HKdL14] to the notion of property (T*), defined in terms of the mean on $M_0A(G)$ being weak*-continuous, which obstructs the Approximation Property of Haagerup and Kraus (see [HK94]). The first examples of groups without the latter property were provided only recently (see [LdlS11], [HdL12], [HdL13], [HKdL14] and [Li15]). It is still an open problem to find an example of a quantum group without the analogue of the Approximation Property.

Chapter 5

Rigid C^*-tensor categories with the Howe-Moore property

This chapter is based on the joint article [AdLW16] of Arano, de Laat and myself, which is dedicated to the study of the Howe-Moore property of rigid C^*-tensor categories. Let us define this property in the following way.

Definition 5.1.4. A rigid C^*-tensor category $\mathcal{C}$ is said to have the Howe-Moore property if for every cp-multiplier $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$, we have $\varphi \in c_0(\mathrm{Irr}(\mathcal{C})) \oplus \mathbb{C}$.

Our goal is to prove that the representation categories of q-deformations of connected compact simple Lie groups with trivial center satisfy the Howe-Moore property. More generally, we aim to provide a proof of Theorem C in the introductory chapter, that is to say, we aim to show that every *completely bounded* multiplier on these categories has a limit at infinity. We then discuss the consequences of this result to categories arising from other contexts.

In order to prove the main result of this chapter, we will need to apply several well-known results from the theory of Lie groups and Lie algebras, most notably the Weyl character formula [He84, Theorem V.1.7] and the Weyl dimension formula [He84, Theorem V.1.8]. While we will have to assume some degree of familiarity with these results on the side of the reader, we will recall the relevant notions explicitly before putting them to use.

Before we prove that the representation categories of q-deformations of connected compact simple Lie groups with trivial center satisfy the Howe-Moore property, we would like to reformulate Definition 5.1.4 in the case when the rigid C^*-tensor category we are dealing with is the representation category of a compact quantum group $\mathbb{G}$.

In this case, it turns out that the Howe-Moore property for $\mathrm{Rep}(\mathbb{G})$ is equivalent to a central version of the Howe-Moore property, i.e. a version of the Howe-Moore property for the central states on the quantum coordinate algebra $\mathbb{C}[\mathbb{G}]$ of $\mathbb{G}$. In view of the results of [dCFY14] and [PV15] on the relationship between cp-multipliers and central states which were described in Section 3.4.4, this is hardly surprising. Similar to the approximation/rigidity properties discussed in Section 3.4.5, this central Howe-Moore property should be viewed as a property for the discrete dual $\widehat{\mathbb{G}}$ of $\mathbb{G}$. Let us formulate if formally.

Definition 5.1.5. Let $\mathbb{G}$ be a compact quantum group. The discrete quantum group $\widehat{\mathbb{G}}$ is said to have the central Howe-Moore property if $\omega \in c_0(\widehat{\mathbb{G}}) \oplus \mathbb{C}$ for every central state ω on $\mathbb{C}[\mathbb{G}]$.

Fix representatives $u^\gamma \in B(H_\gamma) \otimes \mathbb{C}[\mathbb{G}]$ for $\gamma \in \mathrm{Irr}(\mathbb{G})$. Then, $\omega \in c_0(\widehat{\mathbb{G}}) \oplus \mathbb{C}$ is equivalent to asking that there exists $z \in \mathbb{C}$ such that for every $\varepsilon > 0$ there is a finite set $F \subset \mathrm{Irr}(\mathbb{G})$ with

$$\|(\mathrm{id} \otimes \omega)(u^\gamma) - z1_{B(H_\gamma)}\| < \varepsilon \quad \text{for all } \gamma \in \mathrm{Irr}(\mathbb{G})\backslash F.$$

The following result relates the Howe-Moore property for representation categories to the central Howe-Moore property of the duals of the underlying quantum groups. It follows immediately from the fact that the completely positive multipliers on the category $\mathrm{Rep}(\mathbb{G})$ coincide with the central states of $\mathbb{G}$, which we stated in Section 3.4.4 as Proposition 3.4.11.

Proposition 5.1.6. *Let $\mathbb{G}$ be a compact quantum group. The representation category* $\mathrm{Rep}(\mathbb{G})$ *has the Howe-Moore property for rigid C^*-tensor categories if and only if the dual $\widehat{\mathbb{G}}$ of $\mathbb{G}$ has the central Howe-Moore property.*

The first step towards the proof of Theorem C is to prove a small observation on root systems. Let us recall from [Hl15, Definition 8.1] that a root system is a finite collection R of nonzero vectors inside a finite dimensional real vector space E with inner product $(\cdot,\cdot)$ satisfying the following properties:

- The vectors in R span E,
- If $\alpha \in R$, then $-\alpha \in R$ and these are the only multiples of α contained in R,

- For all $\alpha, \beta \in R$, the quantity $2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer and $\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ is contained in R as well.

Such a root system is *irreducible* if there is no orthogonal decomposition $E = E_0 \oplus E_1$ into proper subspaces such that all roots are contained in either E_0 or E_1. Lastly, the group generated by the orthogonal transformations $s_\alpha : v \mapsto v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha,\ \alpha \in R$ is called the *Weyl group* of the root system.

Lemma 5.1.7. *Let R be a root system with a root subsystem $R^0 \subset R$. Then for all $\alpha \in R$, we have the following: whenever there exists $\beta \in R \setminus R^0$ such that $(\alpha, \beta) \neq 0$, then $\alpha \in \operatorname{span}(R \setminus R^0)$.*

In particular, if R is irreducible and R^0 is a proper subsystem, then $\operatorname{span}(R \setminus R^0) = E$.

Proof. The lemma is true for trivial reasons when R is of rank 1. The root systems of rank 2 are completely classified and there are only four of them, in which the statement of the lemma is also easily verified. If R is of higher rank, let α, β be as in the lemma and consider the smallest root system S containing α and β, which is at most of rank 2. If it is of rank one then $\alpha \in \{\beta, -\beta\} \subset \operatorname{span}(R \setminus R^0)$. If S is of rank 2 and $S \cap R^0 = \emptyset$, there is nothing to prove. Otherwise, it follows from the rank 2 case that $\alpha \in \operatorname{span} S \setminus (S \cap R^0) \subset \operatorname{span}(R \setminus R^0)$.

Now, if R is irreducible with proper subsystem R^0, then $R^1 = R \cap \operatorname{span}(R \setminus R^0)$ is a nonempty root system and $R^2 = R \setminus R^1$ is a root system as well. By the first part of the lemma R^1 and R^2 are perpendicular to each other so that we must have $R^1 = R$ by irreducibility. $\square$

Let us add another small observation on root systems that follows directly from the definition of a root system and will be helpful later on.

Lemma 5.1.8. *Let $R \subset E$ be a root system with positive roots R_+ and Weyl group W. For any $v \in E$, the set $R^0 = \{\alpha \in R\ ;\ (\alpha, v) = 0\}$ is a root system (inside its span) with positive roots $R^0_+ = R_+ \cap R^0$ and Weyl group $W_0 = \{w \in W\ ;\ wv = v\}$.*

The following result constitutes a crucial ingredient of our approach. Let K be a connected compact simple Lie group (with center $Z(K)$), and fix a Cartan subalgebra $\mathfrak{t}$ of its Lie algebra. Let T be the associated maximal torus, R the set of roots (with positive part R_+) and P the weight lattice (with positive part P_+). Let $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$. For $\lambda \in P_+$, the highest weight representation is denoted by $V(\lambda)$. The character of $V(\lambda)$ is denoted by χ_λ. We refer to Section 3.2.4 for details on the aforementioned structures.

Proposition 5.1.9. *Let K be a connected compact simple Lie group, and let T, P_+ and $V(\lambda)$ be as above. For every $t \in T \setminus Z(K)$, we have*

$$\frac{1}{\dim(V(\lambda))}\chi_\lambda(t) \to 0 \qquad \textit{as} \qquad \lambda \to \infty.$$

Proof. We use the Weyl character formula (see e.g. [He84, Theorem V.1.7]), which computes the character χ_λ of a highest weight representation $V(\lambda)$ of highest weight λ of a connected compact simple Lie group:

$$\chi_\lambda(e^H) = \frac{\sum_{w\in W}\det(w)e^{w(\lambda+\rho)(H)}}{e^{\rho(H)}\prod_{\alpha\in R_+}(1-e^{-\alpha(H)})}$$

and the Weyl dimension formula (see e.g. [He84, Theorem V.1.8]), which computes the dimension of the highest weight representation $V(\lambda)$:

$$\dim(V(\lambda)) = \frac{\prod_{\alpha\in R_+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Delta_+}(\rho,\alpha)}.$$

Recall that we can write $t \in T$ as $t = e^H$ for some $H \in \mathfrak{t}$ and that both the Weyl character and the Weyl dimension formula do not depend on the choice of H. We write t^α for $e^{\alpha(H)}$.

Fix $t_0 \in T \setminus Z(K)$. Then, by Lemma 5.1.8, $R^0 = \{\alpha \in R \mid t_0^{-\alpha} = 1\} \neq R$ is a root system with positive roots $R^0_+ = \{\alpha \in R_+ \mid t_0^{-\alpha} = 1\}$ and Weyl group $W_0 = \{w \in W \mid wt_0 = t_0\}$. Put $\rho_0 = \frac{1}{2}\sum_{\alpha\in R^0_+}\alpha$. Fix representatives of $W_0\backslash W$ in W. Then

$$\chi_\lambda(t_0) = \lim_{t\to t_0}\sum_{w'\in W_0\backslash W}\det(w')\frac{1}{t^\rho\prod_{\alpha\in R_+\backslash R^0_+}(1-t^{-\alpha})}\frac{\sum_{w\in W_0}\det(w)t^{ww'(\lambda+\rho)}}{\prod_{\alpha\in R^0_+}(1-t^{-\alpha})}$$

$$= \lim_{s\to 1}\sum_{w'\in W_0\backslash W}\det(w')\frac{t_0^{w'(\lambda+\rho)}}{t_0^\rho s^{\rho-\rho_0}\prod_{\alpha\in R_+\backslash R^0_+}(1-(t_0s)^{-\alpha})}\frac{\sum_{w\in W_0}\det(w)s^{ww'(\lambda+\rho)}}{s^{\rho_0}\prod_{\alpha\in R^0_+}(1-s^{-\alpha})},$$

where we have used the invariance of t_0 under the action of W_0 in the second equality. Recall that the Weyl dimension formula is proven by computing the limit $\dim V(\lambda) = \lim_{s\to 1}\chi_\lambda(s)$ using the Weyl character formula. Taking the limit $s \to 1$ in the Weyl character formula for the subsystem R^0 in the same way as in the proof of the Weyl dimension formula, we obtain

$$\lim_{s\to 1}\frac{\sum_{w\in W_0}\det(w)s^{ww'(\lambda+\rho)}}{s^{\rho_0}\prod_{\alpha\in R^0_+}(1-s^{-\alpha})} = \frac{\prod_{\alpha\in R^0_+}(w'(\lambda+\rho),\alpha)}{\prod_{\alpha\in R^0_+}(\rho_0,\alpha)}.$$

Moreover, since $t_0^\rho s^{\rho-\rho_0}\prod_{\alpha\in R_+\setminus R_+^0}(1-(t_0s)^{-\alpha})$ is non-zero whenever s is sufficiently close to 1, it suffices to show that for every w', we have

$$\frac{\prod_{\alpha\in R_+^0}(w'(\lambda+\rho),\alpha)}{\prod_{\alpha\in R_+}(\lambda+\rho,\alpha)}\to 0 \qquad \text{as} \qquad \lambda\to\infty.$$

To this end, put

$$R_+^{w',0}=\{\beta\in R_+ \mid \pm\beta\in (w')^{-1}R_+^0\}.$$

Then

$$\frac{\prod_{\alpha\in R_+^0}(w'(\lambda+\rho),\alpha)}{\prod_{\alpha\in R_+}(\lambda+\rho,\alpha)}=\pm\prod_{\alpha\in R_+\setminus R_+^{w',0}}\frac{1}{(\lambda+\rho,\alpha)}.$$

Note that every factor $\frac{1}{(\lambda+\rho,\alpha)}$ is at most 1. Write λ as a linear combination of ϖ_i, where the ϖ_i denote the fundamental weights, i.e. $(\varpi_i,\alpha_j^\vee)=\delta_{ij}$. Then the maximum of the coefficients tends to infinity as λ tends to infinity. Hence, we only need to show that for all $i\in I$, there exists an $\alpha\in R_+\setminus R_+^{w',0}$ such that $(\varpi_i,\alpha)\neq 0$.

Suppose that this is not the case. Then there exists an $i\in I$ such that for all $\alpha\in R_+\setminus R_+^{w',0}$, we have $(\varpi_i,\alpha)=0$. This shows that $(w'\varpi_i,\alpha)=0$ for any $\alpha\in R\setminus R^0$. From Lemma 5.1.7 and the assumption that K is simple, we obtain that $(w'\varpi_i,\alpha)=0$ for all $\alpha\in\operatorname{span}(R_+\setminus R_+^0)=\mathfrak{h}^*$, which is a contradiction. ☐

To ease notation, we give the following definition.

Definition 5.1.10. Let K be a compact group. We say that a rigid C^*-tensor category $\mathcal{C}$ is of *K-type* if it satisfies the same fusion rules as K, i.e. if there exists a bijection $\operatorname{Irr}(\mathcal{C})\to\operatorname{Irr}(K)$ such that its linear extension $\mathbb{C}[\mathcal{C}]\to\mathbb{C}[\operatorname{Rep}K]$ is an isomorphism of $*$-algebras.

The following result is not hard to prove and well-known to experts. Nevertheless, it is crucial in the proof of our main result.

Proposition 5.1.11. *Let K be a connected compact Lie group with maximal torus T and Weyl group $W=N(T)/T$. For any rigid C^*-tensor category $\mathcal{C}$ of K-type, we have*

$$C_r^*(\mathcal{C})\cong C(T/W),$$

where T/W denotes the space of orbits under the natural action of the Weyl group.

Proof. Note first that the left regular representation $\lambda : \mathbb{C}[\mathcal{C}] \to B(\ell^2(\mathrm{Irr}(\mathcal{C})))$ only depends on the fusion rules of $\mathcal{C}$ and hence $C_r^*(\mathcal{C}) \cong C_r^*(\mathrm{Rep}(K))$. Now, denote by P_+ the positive weights of K such that $\mathrm{Irr}(K) = P_+$. For $\mu \in P_+$, let $\chi_\mu : K \to \mathbb{C}$ be the associated character, i.e.

$$\chi_\mu(g) = \mathrm{Tr}(\pi_\mu(g)), \qquad (g \in K),$$

where π_μ is the irreducible representation with highest weight μ. Since by the Peter-Weyl theorem [BrtD85, Theorem III.3.1], the characters $\chi_\mu,\ \mu \in P_+$ form an orthonormal basis of the Hilbert space $L^2(K)^K$ of conjugacy invariant square-integrable functions, it follows that the bijective map $\delta_\mu \mapsto \chi_\mu$ extends linearly to a unitary $U : \ell^2(P_+) \to L^2(K)^K$. It is easy to check that $U\lambda(\mu)U^*$ acts on $L^2(K)^K$ by left multiplication of the continuous conjugacy invariant function $\chi_\mu \in C(K)^K$. Since the characters generate $C(K)^K$, it follows that

$$C_r^*(\mathcal{C}) \cong C(K)^K.$$

By [BrtD85, Proposition IV.2.6], the compact set of conjugacy classes of K is homeomorphic to the space of orbits T/W and hence

$$C_r^*(\mathcal{C}) \cong C(T/W).$$

□

Theorem 5.1.12. *Let K be a connected compact simple Lie group, and let $\mathcal{C}$ be a rigid C^*-tensor category of K-type. For every completely bounded multiplier φ on $\mathcal{C}$, there exists a map $c : Z(K) \to \mathbb{C}$ such that*

$$\varphi(\mu) - \sum_{t \in Z(K)} c(t) t^\mu \to 0 \qquad \textit{as} \qquad \mu \to \infty.$$

In particular, if K has trivial center, then the category $\mathcal{C}$ has the Howe-Moore property.

Proof. Let $\varphi : P_+ \to \mathbb{C}$ be a completely bounded multiplier on $\mathcal{C}$. By Proposition 3.4.8, φ gives rise to a normal completely bounded map $M_\varphi : \mathcal{AC}'' \to \mathcal{AC}''$, on the tube algebra $\mathcal{AC}$. Restricting M_φ to the fusion algebra and using Proposition 5.1.11, we get a completely bounded map

$$\Psi \colon C(T/W) \to C(T/W), \chi_\mu \mapsto \varphi(\mu)\chi_\mu.$$

After composing Ψ with the evaluation map at the neutral element in T, we obtain a bounded functional

$$\omega_\varphi \colon C(T/W) \to \mathbb{C} \colon \chi_\mu \mapsto \dim(V(\mu))\varphi(\mu).$$

Since ω_φ is a bounded functional on $C(T/W)$, there exists a finite measure ν on T/W such that

$$\varphi(\mu) = \frac{1}{\dim(V(\mu))} \int_{t\in T/W} \chi_\mu(t) d\nu(t).$$

Now, by Proposition 5.1.9 and the dominated convergence theorem, we obtain that

$$\varphi(\mu) - \frac{1}{\dim(V(\mu))} \int_{t\in Z(K)} \chi_\mu(t) d\nu(t) \to 0 \qquad \text{as} \qquad \mu \to \infty.$$

Moreover, if $t \in Z(K), \mu \in P_+$, then $\pi_\mu(t) \in \mathrm{End}(V(\mu))$ must be a multiple of the identity, say $z1_{V(\mu)}$, so that the eigenvalue t^μ of $\pi_\mu(t)$ must equal $z = \frac{\chi_\mu(t)}{\dim(V(\mu))}$. Putting $c(t) = \nu(\{t\})$ for $t \in Z(K)$, we conclude

$$\varphi(\mu) - \sum_{t\in Z(K)} c(t) t^\mu \to 0 \qquad \text{as} \qquad \mu \to \infty.$$

$\square$

Examples 5.1.13. (i) If K is a connected compact simple Lie group and $q \in (0,1]$, then K_q is of K-type by Theorem 3.2.12 and hence satisfies the assumptions of Theorem 5.1.12. If, moreover, the Lie group has trivial center, then the result directly implies Theorem C.

(ii) For $q \in (-1,0)$, the q-deformation $\mathrm{SU}_q(2)$ of Woronowicz is of SU(2)-type (see e.g. [Ti08, Corollary 6.2.9]). Since every free orthogonal quantum group O_F^+ is monoidally equivalent to $\mathrm{SU}_q(2)$ (and hence has the same representation category) for some uniquely determined $q \in [-1,0) \cup (0,1]$, the representation categories of free orthogonal quantum groups also satisfy the conditions of Theorem 5.1.12.

(iii) The annular category of the Temperley-Lieb-Jones standard invariant $\mathrm{TLJ}(\lambda)$ is equivalent, as a rigid C^*-tensor category, to the representation category of the compact quantum group $\mathrm{PSU}_q(2)$, where q is the unique number $0 < q \leq 1$ such that $q + \frac{1}{q} = \lambda^{-\frac{1}{2}}$ (see e.g. [PV15]). Hence, it has the Howe–Moore property.

(iv) Other examples of rigid C^*-tensor categories with the same fusion ring as some connected compact simple Lie group are the Kazhdan-Wenzl categories [KzW93] (see also [Jor14]).

Chapter 6

Free wreath products of compact quantum groups

In this chapter, we discuss the results on the connection between planar algebras and free wreath products obtained in the joint work [TW16]. In Section 6.1, we prove Theorem D. Next, we discuss free products of planar algebras in Section 6.2 and prove Theorem E in Section 6.3. We finish with a discussion of analytic consequences of Theorem E in Section 6.5.

6.1 The reconstruction theorem

Let A be a finite-dimensional C^*-algebra and let α be a faithful, centrally ergodic action $\alpha : A \to A \otimes \mathbb{C}[\mathbb{G}]$ of a compact quantum group $\mathbb{G}$ which preserves the Markov trace of the inclusion $\mathbb{C} \subset A$. Recall from Proposition 3.2.19, that α induces a sequence of actions $(\alpha_n)_{n\geq 0}$ on the Jones tower

$$\mathbb{C} \subset A_1 = A \subset A_2 = \langle A_1, e_1 \rangle \subset \cdots$$

of $\mathbb{C}1_A \subset A$. We define $\alpha_+ := \alpha_0$ and $\alpha_- := \alpha|_{Z(A)} : Z(A) \to A \otimes \mathbb{C}[\mathbb{G}]$. Let $\Gamma(A)$ be the bipartite graph of the inclusion $\mathbb{C}1_A \subset A$ with spin vector μ and graph planar algebra $\mathcal{P}^A = (\mathcal{P}^A_n)_{n\in\mathbb{N}_*\cup\{+,-\}}$ as in Section 3.3.4 where we identify $\mathcal{P}^A_n$ with A_n through [J98, Theorem 5.1]. The next result is the main result of the article [B05a].

Theorem 6.1.1 (Banica)**.** *The graded vector space $\mathcal{P}(\alpha) = (\mathrm{Fix}(\alpha_n))_{n\in\mathbb{N}_*\cup\{+,-\}}$ is a subfactor planar subalgebra of $\mathcal{P}^A$.*

Let $(\mathbb{G}, \alpha)$ and $(\mathbb{F}, \beta)$ be pairs of conjugate actions on A, i.e. assume that there exists a tr-preserving unital $*$-automorphism $\phi : A \to A$ and a unital $*$-isomorphism of Hopf algebras $\Psi : \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{F}]$ satisfying $\beta \circ \phi = (\phi \otimes \Psi) \circ \alpha$. It is not hard to see, that ϕ extends to a whole tower of unital Markov trace-preserving $*$-automorphisms on the Jones tower: as ϕ preserves the Markov trace it gives rise to a unitary U_ϕ on the Hilbert space $\mathcal{H} = (A, \mathrm{tr}_A)$ on which the basic construction A_2 acts. We can thus define $\phi_2 : A_2 \to A_2$ through $\phi_2(X) = U_\phi x U_\phi^*$ which again gives a unital Markov trace-preserving $*$-automorphism on A_2 which restricts to ϕ on A. Continuing inductively, we obtain our tower of unital Markov trace-preserving $*$-automorphisms $(\phi_n : A_n \to A_n)$. By [Bu10, Theorem 3.1], this tower (ϕ_n) commutes with the planar action and thus yields a trace-preserving unital $*$-automorphism of $\mathcal{P}^A$ which maps $\mathcal{P}(\alpha)$ to $\mathcal{P}(\beta)$.

The following theorem is a reformulation of Theorem D in the introduction.

Theorem 6.1.2. *Let $\mathcal{Q}$ be a subfactor planar subalgebra of $\mathcal{P}^A$.*
There exists a compact quantum group $\mathbb{G}$ and a faithful, tr*-preserving, centrally ergodic action $\alpha : A \to A \otimes \mathbb{C}[\mathbb{G}]$ such that $\mathcal{Q} = \mathcal{P}(\alpha)$.*

Moreover, if there exists a planar automorphism of $\mathcal{P}^A$ mapping $\mathcal{Q}$ to another subfactor planar algebra $\tilde{\mathcal{Q}}$ with associated pair $(\tilde{\mathbb{G}}, \tilde{\alpha})$, then $(\mathbb{G}, \alpha)$ and $(\tilde{\mathbb{G}}, \tilde{\alpha})$ are conjugate.

In order to prove this theorem, we will apply Woronowicz's Tannaka-Krein duality (Theorem 3.2.5) to a concrete rigid C^*-tensor category whose morphisms are induced by special tangles.

Let $\mathcal{Q} = (\mathcal{Q}_n)_{n \in \mathbb{N}_* \cup \{+,-\}}$ be a subfactor planar subalgebra of $\mathcal{P}^A$. Recall from section 3.3.4 that, since $(\mathcal{H}_n = L^2(\mathcal{P}_n^A, \mathrm{tr}_n))_{n \in \mathbb{N}_* \cup \{+,-\}}$ is a Hilbert module over $\mathcal{Q}$, every special (k, l)-tangle T induces a bounded linear map $Z_T : \mathcal{H}_k \to \mathcal{H}_l$ such that $Z_T^* = Z_{T^*}$.
During the rest of this section, we identify the index value $+$ with 0 if not explicitly specified otherwise. Set $\mathcal{H} = \mathcal{H}_1$.
For $k, m \geq 0$, define the *concatenation tangle* $M_{k,m}$ as in Figure 6.1.

Lemma 6.1.3. *The maps $U_{k,m} := Z_{M_{k,m}} : \mathcal{H}_k \otimes \mathcal{H}_m \to \mathcal{H}_{k+m}$ define a family of unitary operators $U = (U_{k,m})_{k,m \geq 0}$ satisfying*

$$U_{l,n} \circ (Z_T \otimes Z_S) = Z_{T \otimes S} \circ U_{k,m},$$

whenever S is a special (m, n)-tangle, T is a special (k, l)-tangle and $T \otimes S$ is the tensor tangle defined in Figure 3.7.

Proof. By definition of the action of tangles on $\mathcal{P}^A$, the operator $U_{k,m}$ maps the tensor product $\xi \otimes \eta$ of a basis loop $\xi \in \mathcal{H}_k$ of length $2k$ and a basis loop

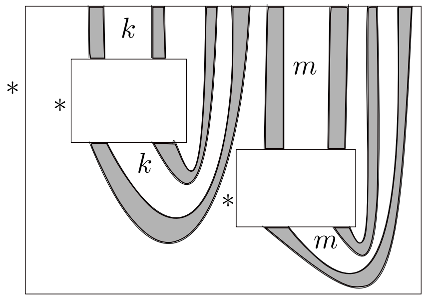

Figure 6.1: Concatenation tangle $M_{k,m}$ in standard form.

$\eta \in \mathcal{H}_m$ of length $2m$ to a non-zero scalar multiple of the concatenated loop $(\xi, \eta) \in \mathcal{H}_{k+m}$. On the other hand, since any basis loop of length $2(k+m)$ in $\mathcal{H}_{k+m}$ reaches the base vertex a after each even number of steps and in particular after $2k$ steps, it follows that $U_{k,m}$ is surjective. Recall that the square of the norm $\|x\|^2_{\mathcal{H}_i} = \mathrm{Tr}_i(x^*x),\ x \in \mathcal{H}_i$ can be implemented by the tangle shown in Figure 6.2. The fact that $U_{k,m}$ is isometric follows by composing this

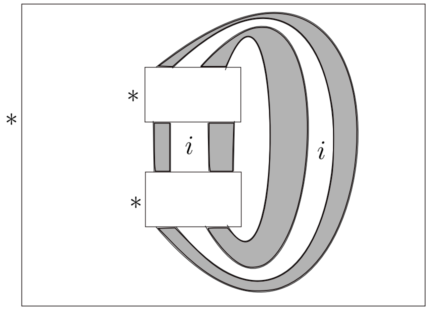

Figure 6.2: Tangle implementing the norm on $\mathcal{H}_i$.

tangle with $M_{k,m}$ and its adjoint. The fact that the family U conjugates the tensor product of maps into the tensor product of tangles follows from a short diagramatic computation as well. □

We will stand still for a moment to understand the unitaries in Lemma 6.1.3 in a less diagrammatic way by discussing the example $U_{1,1} : \mathcal{H}_1 \otimes \mathcal{H}_1 \to \mathcal{H}_2$

explicitly. To do so, recall from Section 3.3.4 that we obtain $\mathcal{H}_1 \otimes \mathcal{H}_1$ by turning the C^*-algebra $A \otimes A$ into a Hilbert space with inner product coming from its natural trace $\mathrm{tr}_A \otimes \mathrm{tr}_A$. On the other hand, $\mathcal{H}_2$ is obtained from the C^*-algebra $B(A) \cong A \check{\otimes} A$ by equipping it with the inner product coming from the unique trace on $B(A)$. Write $A = \oplus_{i=1}^{s} M_{m_i}(\mathbb{C})$ with matrix units e^i_{kl}, $i = 1, \ldots, s$, $k, l = 1, \ldots, m_i$. Then, the elementary tensors $e^{i_1}_{k_1 l_1} \otimes e^{i_2}_{k_2 l_2}$ form an orthogonal basis of both $A \otimes A$ and $A \check{\otimes} A$, but their norms are different in either spaces. These different normalizations are exactly the reason why the spin factors associated to the singularies on the strings of $M_{1,1}$ appear. In other words, $U_{1,1}$ is exactly given by the formula

$$U_{1,1} : A \otimes A \to A \check{\otimes} A, \quad e^{i_1}_{k_1 l_1} \otimes e^{i_2}_{k_2 l_2} \mapsto \sqrt{\frac{m_{i_1} m_{i_2}}{\dim A}}\; e^{i_1}_{k_1 l_1} \otimes e^{i_2}_{k_2 l_2}.$$

By the previous lemma, we inductively obtain a sequence of unitaries $(U_m)_{m \geq 0}$ such that $U_m : \mathcal{H}^{\otimes m} \to \mathcal{H}_m$ implements a natural isomorphism of Hilbert spaces $\mathcal{H}^{\otimes m} \cong \mathcal{H}_m$ and such that $U_{k+m} = U_{k,m}(U_k \otimes U_m)$.

Lemma 6.1.4. *The category $\mathcal{C}_{\mathcal{Q}}$ with $\mathrm{Obj}(\mathcal{C}_{\mathcal{Q}}) = \mathbb{N}$ and morphism spaces*

$$\begin{aligned}\mathrm{Mor}(k,l) &= U_l^* \;\mathrm{span}\{Z_T \;;\; T \textit{ labelled special } (k,l) - \textit{tangle}\}\; U_k \\ &\subset B(\mathcal{H}^{\otimes k}, \mathcal{H}^{\otimes l})\end{aligned}$$

is a rigid C^*-tensor category (up to completion w.r.t. direct sums and subobjects) with tensor operation given by $k \otimes l = k + l$ on objects and by the tensor product of linear maps on Hilbert spaces on morphisms. In particular, there exists a compact matrix quantum group $\mathbb{G}$ with fundamental representation $u \in B(\mathcal{H}) \otimes \mathbb{C}[\mathbb{G}]$ such that $\mathrm{Mor}(u^{\otimes k}, u^{\otimes l}) = \mathrm{Mor}(k,l)$.

Proof. The observations prior to Lemma 6.1.3 and the lemma itself show that $\mathcal{C}_{\mathcal{Q}}$ is a C^*-tensor category. The map $R = \bar{R} \in \mathrm{Mor}(0,2)$ induced by the special $(0,2)-$tangle shown in Figure 6.3 satisfies the conjugate equations for the generating object which is therefore self-conjugate. Therefore, the category is rigid and Tannaka-Krein duality applies after completing $\mathcal{C}_{\mathcal{Q}}$ w.r.t. direct sums and subobjects. □

We introduce the following notation for future reference.

Notation 6.1.5. For a labelled special (k,l)-tangle T, we denote the map $U_l^* Z_T U_k$ by Z'_T.

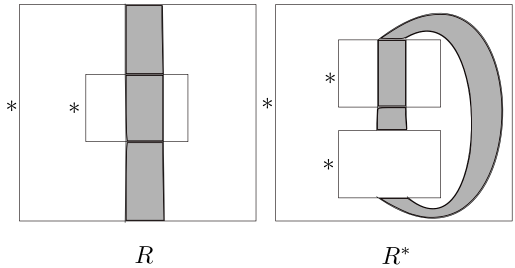

Figure 6.3: : The conjugate tangle R and its adjoint.

Proof of Theorem 6.1.2. Let $u \in B(\mathcal{H}) \otimes \mathbb{C}[\mathbb{G}]$ be the fundamental representation of the quantum group $\mathbb{G}$ obtained in the previous lemma. Consider the linear map $\alpha_u : A \to A \otimes \mathbb{C}[\mathbb{G}]$ as in Section 3.2.5. We have to prove that α_u is a faithful, tr-preserving, ergodic action such that $\mathcal{Q} = \mathcal{P}(\alpha_u)$. Note that, since the coefficients of u generate $\mathbb{C}[\mathbb{G}]$ by definition, the faithfulness of α_u is immediate. We invoke Lemma 3.2.16 to establish the missing properties of α_u. The multiplication map $m : A \otimes A \to A$, the unit map $\eta : \mathbb{C} \to A$ and the Markov trace $\mathrm{tr} : A \to \mathbb{C}$ are induced by the special tangles drawn in Figure 6.4. Hence, $m \in \mathrm{Mor}(2,1)$, $\eta \in \mathrm{Mor}(0,1)$ and $\mathrm{tr} \in \mathrm{Mor}(1,0)$ and by Lemma 3.2.16,

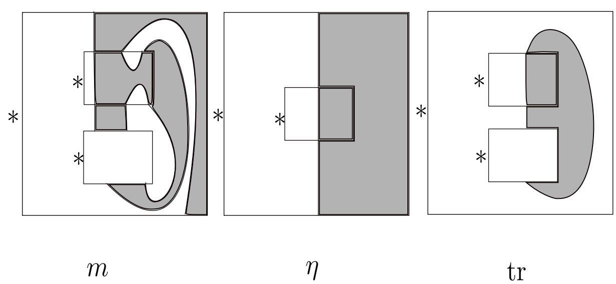

Figure 6.4: : Multiplication tangle m, unit tangle η and trace tangle tr.

α_u is multiplicative, unital and tr-preserving. Since $\mathcal{Q}$ is a subfactor planar algebra, we have $\dim Z(A) \cap \mathrm{Fix}(\alpha_u) = \dim \mathcal{Q}_- = 1$ which implies the central ergodicity of α_u.

If we start with a faithful, centrally ergodic, tr-preserving action α of a compact quantum group $\mathbb{G}$ on A, it is shown in [B05a, Section 2] that $\mathcal{P}(\alpha)_n = \mathrm{Mor}(\mathbb{1}, (u^\alpha)^{\otimes n})$ for all $n \geq 0$. As a consequence, if we consider another action $\tilde{\alpha}$ of $\tilde{\mathbb{G}}$ with $\mathcal{P}(\alpha) \cong \mathcal{P}(\tilde{\alpha})$, by Frobenius duality, we obtain a unitary tensor functor $\mathcal{C}_{\mathcal{P}(\alpha)} \to \mathcal{C}_{\mathcal{P}(\tilde{\alpha})}$ respecting the fibration over Hilb_f. Hence, by the unicity part of the Tannaka-Krein duality theorem, we get an isomorphism $\mathbb{C}[\mathbb{G}] \cong \mathbb{C}[\tilde{\mathbb{G}}]$ mapping u^α to $u^{\tilde{\alpha}}$. This implies that α and $\tilde{\alpha}$ are conjugate. $\square$

Example 6.1.6. If A is a finite-dimensional $C*$-algebra, every quantum group $\mathbb{G}$ acting faithfully and tr-preservingly on A is contained in $\mathbb{G}_{aut}(A, \mathrm{tr})$ as a quantum subgroup through a surjective morphism $C(\mathbb{G}_{aut}(A, \mathrm{tr})) \to C(\mathbb{G})$ which intertwines the universal action of $\mathbb{G}_{aut}(A, \mathrm{tr})$ and the action of $\mathbb{G}$. On the other hand, every subfactor planar algebra $\mathcal{P}^A$ contains the Temperley-Lieb-Jones TLJ algebra of modulus $\delta = \sqrt{\dim A}$ (see for instance [J99]) as a planar subalgebra. Therefore, the universal action of $\mathbb{G}_{aut}(A, \mathrm{tr})$ correponds to the Temperley-Lieb-Jones algebra through Theorem 6.1.2.

Remark 6.1.7. Recall that a Q-system inside a rigid C^*-category $\mathcal{C}$ (see [BYLR15, Section 3.1]) is an object $X \in \mathcal{C}$ together with two morphisms $\eta \in \mathrm{Hom}(\mathbb{1}_{\mathcal{C}}, X)$, $m \in \mathrm{Hom}(X \otimes X, X)$ such that (X, η, m) is an algebra, (X, η^*, m^*) is a coalgebra, and

$$(m \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes m^*) = m^* \circ m = (\mathrm{id}_X \otimes m) \circ (m^* \otimes \mathrm{id}_X).$$

A Q-system (X, η, m) is called *special* if $m \circ m^* = \lambda \mathrm{id}_X$ for some $\lambda \in \mathbb{R}$ and *standard* if $w^* w = \lambda \mathrm{id}_{\mathbb{1}_{\mathcal{C}}}$ with the same constant λ. Similar to planar algebras, Q-systems axiomatize the standard invariant of a finite index inclusion $N \subset M$ of a factor N inside a larger von Neumann algebra M, see [BYLR15, Theorem 3.11]. More precisely, every such inclusion yields a standard special Q-system $X = {}_N L^2(M)_N$ inside the category of N-bimodules associated to the Jones tower of $N \subset M$, and every standard special Q-system inside a rigid C^*-category $\mathcal{C}$ is of that form. Under this correspondence, the condition that M is a factor is equivalent to the condition that the Q-system X is simple as an X-bimodule inside $\mathcal{C}$.

Let us rephrase Theorem 6.1.2 in the setting of Q-systems. First, we observe that $\mathcal{H} = (A, \mathrm{tr}_A)$ together with its structure morphisms $m : A \otimes A \to A$ and $\eta : \mathbb{C} \to A$ is a special, standard (not necessarily simple) Q-system in the rigid C^*-category Hilb_f of finite-dimensional Hilbert spaces. Here the constant λ equals $\sqrt{\dim A}$. This Q-system is the analogue of the planar algebra associated to the bipartite graph of A. Now, let $\alpha : A \to A \otimes \mathbb{C}[\mathbb{G}]$ be a faithful, tr_A-preserving action on A. Then, the unitary representation u^α associated to α is a special standard Q-system in $\mathrm{Rep}\, \mathbb{G}$ which is simple if and only if α is centrally ergodic. Moreover, the canonical fiber functor $F : \operatorname{Rep}\mathbb{G} \to \mathrm{Hilb}_f$ maps u^α to $\mathcal{H}$.

On the other hand, if we start with a tripel $(X, \mathcal{C}, F)$ of a special standard simple Q-system inside a rigid C^*-category $\mathcal{C}$ and a unitary fiber functor $F : \mathcal{C} \to \mathrm{Hilb}_f$ such that $F(X) = \mathcal{H}$, the Tannaka-Krein duality theorem yields a compact quantum group $\mathbb{G}$ and a unitary monoidal equivalence $E : \mathcal{C} \to \operatorname{Rep}\mathbb{G}$ intertwining the fiber functors. The assumptions on X translate into the fact that the action α associated to the unitary representation $u^\alpha := E(X)$ is faithful, tr_A-preserving and centrally ergodic (with central ergodicity being a consequence of the simplicity of X). In other words, supressing the fiber functors, we obtain a correspondence between faithful, tr_A-preserving and centrally ergodic action on A and rigid C^*-tensor subcategories of Hilb_f containing $\mathcal{H}$ as a Q-system.

6.2 Tensor products and free products of planar algebras

Before relating the free wreath product to the free product of planar algebras, we need to introduce the latter and obtain some preliminary results on tensor products and free products of planar algebras.

Definition 6.2.1. Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. The tensor product planar algebra $\mathcal{P} \otimes \mathcal{Q}$ is the collection of vector spaces $(\mathcal{P} \otimes \mathcal{Q})_i = \mathcal{P}_i \otimes \mathcal{Q}_i$, with the action of any planar tangle being given by the tensor product of the action on each component. More precisely, for a planar tangle T,

$$Z_T(\bigotimes_{D_i}(x_i \otimes y_i)) = Z_T(\bigotimes_{D_i} x_i) \otimes Z_T(\bigotimes_{D_i} y_i),$$

where $x_i \in \mathcal{P}_{k_i}$ and $y_i \in \mathcal{Q}_{k_i}$.

Lemma 6.2.2. *Let A, B be finite-dimensional C^*-algebras. Then*

$$\mathcal{P}^A \otimes \mathcal{P}^B \cong \mathcal{P}^{A \otimes B}.$$

Proof. Let $\Gamma(A) = (V_1, E_1)$ and $\Gamma(B) = (V_2, E_2)$ be the bipartite graphs associated to A and B. Note first that the bipartite graph $\Gamma(A \otimes B) = (V, E)$ is given by glueing $\Gamma(B)$ (at the even vertex of $\Gamma(B)$) onto the uneven endpoints of every edge of $\Gamma(A)$ and deleting the glued vertices. Therefore, $\Gamma(A \otimes B)$ can be identified with the direct product $\Gamma(A) \times \Gamma(B)$, that is to say, $V_+ = V_{1,+} \times V_{2,+}$, $V_- = V_{1,-} \times V_{2,-}$ and $E = E_1 \times E_2$. As the Markov trace of $A \otimes B$

is the tensor product of the Markov traces of the components, the spin vector $\mu_{A\otimes B} : V \to \mathbb{C}$ is given by

$$\mu_{A\otimes B}((x,y)) \;=\; \mu_A(x)\mu_B(y), \qquad x \in V_1,\ y \in V_2.$$

For any $n \geq 0$, the map

$$\begin{array}{rcl} \Phi_n : \mathcal{P}_n^A \otimes \mathcal{P}_n^B & \to & \mathcal{P}_n^{A\otimes B} \\ (\beta_1, \dots, \beta_{2n}) \otimes (\gamma_1, \dots, \gamma_{2n}) & \mapsto & ((\beta_1, \gamma_1), \dots, (\beta_{2n}, \gamma_{2n})) \end{array}$$

is a linear isomorphism of vector spaces. If $n = -$, the corresponding isomorphism Φ_- is given by the identification $V_- = V_{1,-} \times V_{2,-}$. We have to show that the isomorphism $\Phi = (\Phi_n)_{n\in\mathbb{N}_*\cup\{+,-\}}$ of graded vector spaces commutes with the action of planar tangles. Let T be such a planar tangle and let σ be a state on T for $\Gamma(A \otimes B)$. We can decompose σ as $\sigma_1 \times \sigma_2$ where σ_1 is a state on T for $\Gamma(A)$ and σ_2 is a state on T for $\Gamma(B)$. Note further that σ is compatible with the loop $((\beta_1, \gamma_1), \dots, (\beta_{2n}, \gamma_{2n}))$ at a disk D if and only if σ_1 is compatible with $(\beta_1, \dots, \beta_{2n})$ at D and σ_2 is compatible with $(\gamma_1, \dots, \gamma_{2n})$ at D. This implies that, if T has outer disk D_0 and inner disks $D_1, \dots, D_s$ of degrees k_i and if $\xi_i \in \mathcal{P}_{k_i}^A,\ \eta_i \in \mathcal{P}_{k_i}^B,\ i = 0, \dots, s$ are basis loops, the coefficients of the action of T satisfy

$$c((\xi_0, \eta_0), \dots, (\xi_s, \eta_s)) = c(\xi_0, \dots, \xi_s)c(\eta_0, \dots, \eta_s).$$

Therefore, Φ commutes with the action of T. □

The free product of two planar algebras $\mathcal{P}$ and $\mathcal{Q}$ is a planar subalgebra of $\mathcal{P} \otimes \mathcal{Q}$ defined by the image of certain planar tangles. For planar algebras, the free product operation has appeared first in [J99]. On the level of subfactors, however, the construction goes back to Bisch and Jones's celebrated article [BJ95].

A pair (T, T') of planar tangles of degree k is called *free* if there exists a planar tangle R of degree $2k$ and two isotopies ϕ_1 and ϕ_2, respectively of T and T', such that

- $\Gamma R = \Gamma\phi_1(T) \cup \Gamma\phi_2(T')$, and the set of distinguished points of R is the image through ϕ_1 and ϕ_2 of the set of distinguished points of T and T'.
- $\phi_1(T \setminus D_0(T)) \cap \phi_2(T' \setminus D_0(T')) = \emptyset$. This means that a connected component of R is the image of a connected component of either T or T'.
- The distinguished point numbered i of $\partial D_0(T)$ is sent by ϕ_1 to the distinguished point numbered $2i - \delta(i)$ of $\partial D_0(R)$.

- The distinguished point numbered i of $\partial D_0(T')$ is sent by ϕ_2 to the distinguished point numbered $2i-(1-\delta(i))$, where $\delta(i)=i \mod 2$.
- A distinduished point of an inner disk coming from T is labelled as in T; a distinguished point i of an inner disk coming from T' is labelled $i-1$.

The last condition ensures that curves of R have endpoints with correct parities. If T and T' are connected planar tangles and R exists, then R is unique up to isotopy. This planar tangle is called the *free composition* of T and T' and denoted by $T*T'$. An example of a free pair of planar tangles, with the resulting free composition, is drawn in Figure 6.5.

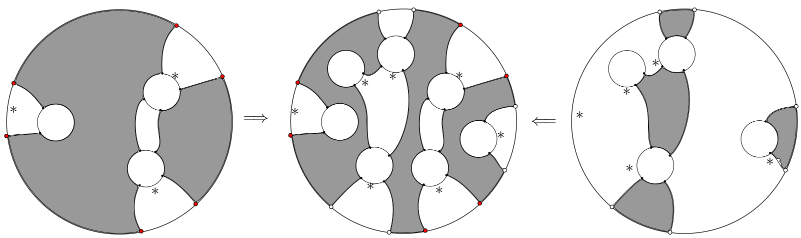

Figure 6.5: Free composition of two planar tangles.

Definition 6.2.3. Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. The free product planar algebra $\mathcal{P}*\mathcal{Q}$ is the collection of vector subspaces $(\mathcal{P}*\mathcal{Q})_k$ of $(\mathcal{P}\otimes\mathcal{Q})_k$ spanned by the image of the maps $Z_T\otimes Z_{T'}$ for all free pairs of planar tangles (T,T') of degree k.

The following result is certainly well known within the planar algebra community. Since we could not find a precise reference, we include a proof.

Lemma 6.2.4. *$\mathcal{P}*\mathcal{Q}$ is a planar subalgebra of $\mathcal{P}\otimes\mathcal{Q}$, that is to say $\mathcal{P}*\mathcal{Q}$ is stable under the action of planar tangles.*

Proof. It suffices to check the stability on the generating sets of the vector spaces $(\mathcal{P}*\mathcal{Q})_k$ given in Definition 6.2.3. Let T be a planar tangle, and for each inner disk D_i of T, let v_i be an element of $(\mathcal{P}*\mathcal{Q})_{k_i}$ of the form $(Z_{T_i}\otimes Z_{T_i'})((\bigotimes_{D_j(T_i)} v_j^i)\otimes(\bigotimes_{D_j(T_i')} w_j^i)$ where (T_i,T_i') is a free pair for any i. The compatibility condition on the composition of actions of planar tangles yields

$$Z_T(\bigotimes_{D_i(T)} v_i) = Z_{T\circ_{(D_1,\dots D_n)}(T_1,\dots,T_n)}(\bigotimes_{D_i(T)}\bigotimes_{D_j(T_i)} v_j^i)$$
$$\otimes Z_{T\circ_{(D_1,\dots D_n)}(T_1',\dots,T_n')}(\bigotimes_{D_i(T)}\bigotimes_{D_j(T_i')} w_j^i).$$

Thus, it is enough to prove that the tangles $S = T\circ_{(D_1,\dots D_n)}(T_1,\dots,T_n)$ and $S' = T\circ_{(D_1,\dots D_n)}(T_1',\dots,T_n')$ form a free pair. Let $\tilde{T}$ be the planar tangle of order $2k$ obtained from T by doubling all the curves of T and all the distinguished points (in such a way that the tangle still remains planar). By construction, a curve joining the point j of D_i to the point j' of $D_{i'}$ in T yields two curves in $\tilde{T}$: one joining the point $2j-1$ of D_i to the point $2j'-1$ of $D_{i'}$ and the other one joining the point $2j$ of D_i to the point $2j'$ of $D_{i'}$. Since T is a planar tangle, the conditions on the parities of j and j' yield that in $\tilde{T}$, the curves join points labelled 0 or 1 modulo 4 (resp. 2 or 3 $\mod 4$) to points labelled 0 or 1 modulo 4 (resp. 2 or 3 $\mod 4$). Reciprocally, by removing all the distinguished points labelled 0 and 1 modulo 4 and the curves joining them from $\tilde{T}$, we recover the planar tangle T. The same holds for the distinguished points labelled 2 and 3. Therefore, if we compose the tangle $T_i * T_i'$ inside each disk D_i, the resulting tangle is exactly $S * S'$ (after relabelling). Thus, $\tilde{T}\circ_{D_1,\dots,D_n}(T_1 * T_1',\dots,T_n * T_n') = S * S'$ and $\mathcal{P} * \mathcal{Q}$ is stable under the action of planar tangles.

□

In Appendix B, we describe a generating set of the free product of planar algebras that will turn out to be useful in the following section.

6.3 Free wreath products with $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$

In this section we show that, under the correspondence established in Theorem D, the free wreath product of a compact matrix quantum group $\mathbb{G}$ acting appropriately on a finite-dimensional C^*-algebra B with $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$, see Definition 3.2.22, is mapped to the free product of the associated planar algebras. Note that any trace appearing in this section is always assumed to be the Markov trace on its underlying finite-dimensional C^*-algebra.

To make our statement precise, recall from Proposition 3.2.23, that if $\beta : B \to B\otimes\mathbb{C}[\mathbb{G}]$ is a faithful, centrally ergodic tr_B-preserving action of $\mathbb{G}$ on a finite-dimensional C^*-algebra B and if α is the universal action of $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$ on A, the free wreath product $\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)$ admits a faithful, centrally ergodic, trace-preserving action $\beta \wr_* \alpha$ on $A \otimes B$. Consider the associated subfactor planar algebras $\mathcal{P}(\beta) \subset \mathcal{P}^B$, $\mathcal{P}(\alpha) = \mathrm{TLJ}_\delta \subset \mathcal{P}^{\mathrm{A}}$ with $\delta = \sqrt{\dim A}$ (see Example 6.1.6) and $\mathcal{P}(\beta \wr_* \alpha) \subset \mathcal{P}^{A\otimes B}$.

Theorem 6.3.1. *We have*

$$\mathcal{P}(\beta \wr_* \alpha) = \mathcal{P}(\alpha) * \mathcal{P}(\beta) = \mathrm{TLJ}_\delta * \mathcal{P}(\beta).$$

In order to prove Theorem 6.3.1, we will review some facts on the representation theory of $\mathbb{G}_{aut}(A, \mathrm{tr})$ and $\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)$. Recall first that the morphisms between tensor powers of the fundamental representation $u \in B(\mathcal{H}) \otimes \mathbb{C}[\mathbb{G}_{aut}(A, \mathrm{tr})]$ of $\mathbb{G}_{aut}(A, \mathrm{tr})$ are given by

$$\mathrm{Mor}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}\{Z_p \ ; \ p \in NC(k,l)\},$$

where $NC(k,l)$ denotes the set of non-crossing partitions with k upper points and l lower points with associated linear map $Z_p : \mathcal{H}^{\otimes k} \to \mathcal{H}^{\otimes l}$ which are defined in [FP16, Definition 1.11]. Instead of recalling these maps explicitly, we will identify them with maps induced by special tangles. Note that, as before $\mathcal{H}$ denotes the Hilbert space obtained by equipping A with the inner product induced by tr_A.

We use the 'fattening' isomorphism from [LT16, Proposition 5.2], see Figure 6.6, to identify any non-crossing partition $p \in NC(k,l)$ with a non-crossing pair partition $\psi(p) \in NC_2(2k, 2l)$ (i.e. a Temperley-Lieb-Jones diagram with first boundary point being the first upper point of $\psi(p)$) by drawing boundary lines around each block and by then deleting the block.

If T is the special (k,l)-tangle labelled by the diagram $\psi(p^*)$, then the map $\tilde{Z}_T := U_l^* Z_T U_k : \mathcal{H}^{\otimes k} \to \mathcal{H}^{\otimes l}$ introduced in Lemma 6.1.4 is exactly the map Z_p.

The representation category of $\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)$ is described in [FP16, Section 3]. Label the upper points of a partition $p \in NC(k,l)$ by unitary representations $u_1, \dots, u_k \in \mathrm{Rep}(\mathbb{G})$ and the lower points by $v_1, \dots, v_l \in \mathrm{Rep}(\mathbb{G})$. If b is a block of p, u_{U_b} will represent the tensor products of the representations labelling the upper part of b read from left to right and v_{L_p} will be the tensor product of the representations labelling the lower part of B. Similarly $\mathcal{H}_{U_b}$ and $\mathcal{H}_{L_b}$ will denote the tensor product of the Hilbert spaces belonging to these representations. If b does not have an upper (lower) part, we set $u_{U_b} = \mathbb{1}_{\mathbb{G}}$ ($v_{L_b} = \mathbb{1}_{\mathbb{G}}$) by convention. A non-crossing partition $p \in NC(k,l)$ is called well-decorated w.r.t $u_1, \dots, u_k; v_1, \dots, v_l$, if for every block b of p, we have $\mathrm{Mor}(u_{U_b}, v_{L_b}) \neq \{0\}$. The set of well-decorated partitions is denoted by $NC_{\mathbb{G}}(u_1, \dots, u_k; v_1, \dots, v_l)$.

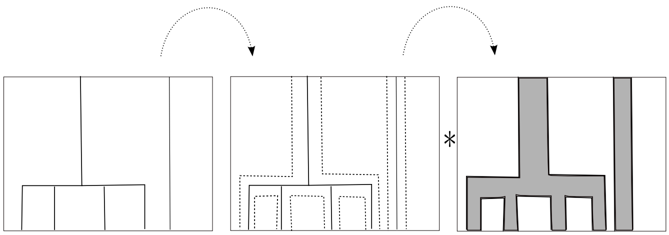

Figure 6.6: The fattening procedure.

If $p \in NC_{\mathbb{G}}(u_1, \dots, u_k; v_1, \dots, v_l)$ is a well-decorated partition and if $S_b \in \mathrm{Mor}(u_{U_b}, v_{L_b})$ is a morphism for any block b, we obtain a well-defined map

$$S = \otimes_{b \in p} S_b : \mathcal{H}_{u_1} \otimes \dots \otimes \mathcal{H}_{u_k} \to \mathcal{H}_{v_1} \otimes \dots \otimes \mathcal{H}_{v_l},$$

by applying the maps S_b iteratively on the legs of the tensor product belonging to b. To any well-decorated partition $p \in NC_{\mathbb{G}}(u_1, \dots, u_k; v_1, \dots, v_l)$ consisting of blocks b and any such morphism $S = \bigotimes_{b \in p} S_b \in \bigotimes_{b \in p} \mathrm{Mor}(u_{U_b}, v_{L_b})$, one associates a map

$$Z_{p,S} = s_{p,L}^{-1}(Z_p \otimes S)s_{p,U} : \bigotimes_{i=1}^{k} \mathcal{H} \otimes \mathcal{H}_{u_i} \to \bigotimes_{j=1}^{l} \mathcal{H} \otimes \mathcal{H}_{v_j}.$$

Here, $s_{p,U} : \bigotimes_{i=1}^{k} \mathcal{H} \otimes \mathcal{H}_{u_i} \to \mathcal{H}^{\otimes k} \otimes \bigotimes_{i=1}^{r} \mathcal{H}_{U_{b_i}}$ is the map reordering the spaces on the upper part. The map $s_{p,L}$ is defined analogously. It is shown in [FP16] that

$$\mathrm{Mor}\Big(\bigotimes_{i=1}^{k} a(u_i), \bigotimes_{j=1}^{l} a(v_j)\Big) =$$

$$\mathrm{span}\{Z_{p,S} \ ; \ p \in NC_{\mathbb{G}}(u_1, \dots, u_k; v_1, \dots, v_l), \ S \in \bigotimes_{i=1}^{r} \mathrm{Mor}(u_{U_{b_i}}, v_{L_{b_i}})\}, \tag{6.3.1}$$

whenever $u_1, \dots, u_k; v_1, \dots, v_l$ are irreducible.
Now, let $v = u_\beta$ be the unitary representation of $\mathbb{G}$ corresponding to the action β. Recall the definition of the unitary representation

$$a(v) = \sum_{x \subset v} \sum_{k} (\mathrm{id}_{\mathcal{H}} \otimes S_{x,k} \otimes \mathbb{1}) a(x) (\mathrm{id}_{\mathcal{H}} \otimes S_{x,k}^* \otimes \mathbb{1}) \tag{6.3.2}$$

from Section 3.2.6. We would like to give a description of the intertwiner spaces of tensor powers of $a(v)$.

Lemma 6.3.2. *The formula 6.3.1 holds whenever* $x_1, \dots, x_k, y_1, \dots, y_l \in \mathrm{Rep}(\mathbb{G})$ *are arbitrary unitary representations. In particular,*

$$\mathrm{Mor}(a(v)^{\otimes k}, a(v)^{\otimes l}) = \mathrm{span}\{Z_{p,S} \ ; \ p \in NC_{\mathbb{G}}(v^k; v^l), \ S \in \bigotimes_{b \in p} \mathrm{Mor}(v^{\otimes |U_b|}, v^{\otimes |L_b|})\}.$$

Proof. This result follows directly by decomposing the unitary representation using formula 6.3.2 and applying the equality 6.3.1 to the irreducible components. □

Proof of Theorem 6.3.1. Set $\mathcal{P} := \mathcal{P}(\beta)$ and $\mathcal{Q} = \mathrm{TLJ}_\delta * \mathcal{P}$. Consider the annular category $\mathcal{C}_{\mathcal{Q}}$ defined in Lemma 6.1.4 with morphism spaces $\mathrm{Mor}_{\mathcal{Q}}(k, l) \subset B((\mathcal{H} \otimes \mathcal{H}_v)^{\otimes k}, (\mathcal{H} \otimes \mathcal{H}_v)^{\otimes l})$. By Theorem 6.1.2 and its proof, we need to show that

$$\mathrm{Mor}(a(v)^{\otimes k}, a(v)^{\otimes l}) \ = \ \mathrm{Mor}_{\mathcal{Q}}(k, l).$$

First, we will start with a well-decorated partition $p \in NC_{\mathbb{G}}(v^k; v^l)$ with blocks labelled by $b \mapsto S_b \in \mathrm{Mor}(v^{\otimes |U_b|}, v^{\otimes |L_b|}) = \mathrm{Mor}_{\mathcal{C}_{\mathcal{P}}}(|U_b|, |L_b|)$ and construct a special (k, l)-tangle W such that $Z_{p,S} = Z'_W$, where $S = \bigotimes_{b \in p} S_b$. For any $b \in p$, choose an element $\eta_b \in \mathcal{P}_{|b|}$ such that $S_b = Z'_{T_b}$, where T_b is the special $(|U_b|, |L_b|)$-tangle labelled by η_b. Let ψ be the 'fattening' isomorphism from the discussion above, mapping a noncrossing partition $p \in NC(k, l)$ to a Temperley-Lieb-Jones diagram $\psi(p) \in NC_2(2k, 2l)$. In particular the map $\phi : p \mapsto \psi(p^*)$ maps a block of p to a shaded region of $\phi(p)$. Hence we can identify the pair $(p, S = \bigotimes_{b \in p} S_b)$ with the Temperley-Lieb-Jones diagram $\phi(p)$ together with a labelling of its shaded regions $r = \phi(b)$ by elements in $\mathrm{Mor}_{\mathcal{C}_{\mathcal{P}}}(|U_b|, |L_b|)$. Consider the unique irreducible planar tangle T of degree $k + l$ with $\pi_T = \mathrm{kr}'(\phi(p))$. Then, $(\phi(p), T)$ constitutes a free pair by Proposition B.2.9. Note that T has an inner disk D_b of degree $|b|$ for every block b in a canonical manner. Consider $\eta = Z_T((\eta_b)_{b \in p}) \in \mathcal{P}_{k+l}$. The special (k, l)-tangle T_η labelled by η exactly induces the map $S = Z'_{T_\eta}$. Since $\phi(p)$ and T form a free pair, we have $\phi(p) \otimes \eta \in \mathcal{Q}_{k+l}$. Therefore, the special (k, l)-tangle W labelled by $\phi(p) \otimes \eta \in \mathcal{Q}_{k+l}$ induces a morphism in $\mathrm{Mor}_{\mathcal{Q}}(k, l)$. A diagrammatical computation then shows that $Z'_W = Z_{p,S}$.

Next, we start with a special (k,l)-tangle labelled by $Z_{p'} \otimes \eta \in \mathcal{Q}_{k+l}$ with $p' \in \mathrm{TLJ}_{\delta,k+l}$ a Temperley-Lieb-Jones diagram and $\eta \in \mathcal{P}_{k+l}$. By Proposition B.2.4, we can write $\eta = Z_{T'} \circ_{D_1,\dots,D_r} (\eta_1, \dots, \eta_r)$ where (p', T') is a reduced free pair. In particular, $\pi_{T'} = \mathrm{kr}'(p')$ and T' has an inner disk D_i for any shaded region of p' or equivalently for every block b_i of $\phi^{-1}(p')$ labelled by $\eta_i \in \mathcal{P}_{|b_i|}$. In particular, we get a morphism $S_{b_i} = Z_{T_{\eta_i}} \in \mathrm{Mor}_{\mathcal{P}}(k,l)$ induced by the special (k,l)-tangle labelled by η_i and by construction $S := Z'_{T_\eta} = \bigotimes_{b \in \phi^{-1}(p')} S_b$. Hence, we have constructed a well-decorated partition $\phi^{-1}(p')$ with blocks labelled by S_b.

It is not hard to check that both constructions are inverse to each other and therefore identify $\mathrm{Mor}(a(v)^{\otimes k}, a(v)^{\otimes l}) = \mathrm{Mor}_{\mathcal{Q}}(k,l)$. As this identification respects composition of diagrams/ labelled partitions and the $*$-operation, we are indeed done.

□

6.4 Free wreath products with quantum subgroups of $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$

In the previous section, we restricted our findings on the free wreath product to the case where the right input is $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$, or in other words, the universal action on the pair (A, tr_A). In this section, we aim to prove an analogous result for arbitrary faithful, centrally ergodic, tr_A-preserving actions on A. When $A = \mathbb{C}^n$, so that $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$ becomes S_n^+, Definition 3.2.21 allows for free wreath products with quantum subgroups of S_n^+, that is to say with arbitrary actions on $\mathbb{C}^n$. The definition of Fima and Pittau (Definition 3.2.22) on the other hand does not. We will therefore first adapt Definition 3.2.22 to also accommodate more general actions on finite-dimensional C^*-algebras.

Let A be a finite-dimensional C^*-algebra with faithful tr_A and let $\mathbb{G}$ be a compact quantum group. We note first that there exists an injective morphism of compact quantum groups

$$\iota : C(\mathbb{G}_{aut}(A, \mathrm{tr}_A)) \to C(\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)),$$

mapping the canonical generator $u \in B(\mathcal{H}) \otimes \mathbb{C}[\mathbb{G}_{aut}(A, \mathrm{tr}_A)]$ (the one corresponding to the universal action) to the unitary representation $a(\mathbb{1}_{\mathbb{G}}) \in B(\mathcal{H}) \otimes \mathbb{C}[\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)]$, see Definition 3.2.22. This follows immediately from the description of the representation category of $\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)$ in [FP16, Section 3] as the intertwiner spaces $\mathrm{Mor}(u^{\otimes k}, u^{\otimes l})$ and $\mathrm{Mor}(a(\mathbb{1}_{\mathbb{G}})^{\otimes k}, a(\mathbb{1}_{\mathbb{G}})^{\otimes l})$ coincide.

Next, consider a quantum subgroup $\mathbb{F}$ of $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$, meaning that we have a surjective morphism of quantum groups $\pi : C(\mathbb{G}_{aut}(A, \mathrm{tr}_A)) \to C(\mathbb{F})$. Write $I = \ker \pi$ and note that I is a C^*-ideal in $C(\mathbb{G}_{aut}(A, \mathrm{tr}_A))$ satisfying $\Delta(I) \subset \ker \pi \otimes \pi$. Define $J := \langle \iota(I) \rangle$ to be the C^*-ideal in $C(\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A))$ generated by the image of I under the embedding ι. Define the canonical surjective $*$-homomorphism

$$\hat{\pi} : C(\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)) \to C(\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A))/J.$$

As ι is an embedding and $\iota(I)$ generates J, it is not hard to see that $J \subset \ker \hat{\pi} \otimes \hat{\pi}$. Therefore, the comultiplication on $C(\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A))$ factors through the quotient so that we obtain a comultiplication

$$\Delta_0 : C(\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A))/J \to (C(\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A))/J)^{\otimes 2}$$

Definition 6.4.1. Let A be a finite-dimensional C^*-algebra with faithful trace tr_A, let $\mathbb{G}$ be a compact quantum group and let $\mathbb{F} \subset \mathbb{G}_{aut}(A, \mathrm{tr}_A)$ be a quantum subgroup. The *free wreath product* $\mathbb{G} \hat{\wr}_* \mathbb{F}$ is defined as the compact quantum group $(C(\mathbb{G} \hat{\wr}_* \mathbb{G}_{aut}(A, \mathrm{tr}_A))/J, \Delta_0)$. If $\mathbb{G}$ is a compact matrix quantum group with fundamental representation v, the *fundamental representation* w of $\mathbb{G} \hat{\wr}_* \mathbb{F}$ is the image of the fundamental representation $a(v)$ of $\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)$ under the quotient map.

The following lemma follows directly from the definitions at hand. It allows us to return immediately to our regular notation $\wr_*$ for wreath products instead of using $\hat{\wr}_*$.

Lemma 6.4.2. *Consider a compact quantum group $\mathbb{G}$ and a quantum subgroup $\mathbb{F}$ of the quantum permutation group S_n^+. The Bichon free wreath product $\mathbb{G} \wr_* \mathbb{F}$ in the sense of Definition 3.2.21 coincides with the free wreath product $\mathbb{G} \hat{\wr}_* \mathbb{F}$ defined above.*

Circling back to the formulation of Definition 3.2.22, we can rephase the previous definition in the following way. Let $\mathbb{G}$ and $\mathbb{F} \subset \mathbb{G}_{aut}(A, \mathrm{tr}_A)$ be as in Definition 6.4.1. Let $\mathcal{H}$ be the Hilbert space obtained by equipping A with the inner product induced by tr_A and let $u \in B(\mathcal{H}) \otimes \mathbb{C}[\mathbb{F}]$ be the canonical fundamental representation of $\mathbb{F}$. Finally, choose for any $x \in \mathrm{Irr}(\mathbb{G})$ a representative $u^x \in B(\mathcal{H}_x) \otimes \mathbb{C}[\mathbb{G}]$. Then $C(\mathbb{G} \wr_* \mathbb{F})$ is the universal unital C^*-algebra generated by the coefficients of

$$a(x) \in B(\mathcal{H} \otimes \mathcal{H}_x) \otimes \mathbb{C}[\mathbb{G} \wr_* \mathbb{F}], \ x \in \mathrm{Irr}(\mathbb{G}),$$

satisfying the relations:

- $a(x)$ is unitary for any $x \in \mathrm{Irr}(\mathbb{G})$,
- For all $x, y, z \in \mathrm{Irr}(\mathbb{G})$ and all $S \in \mathrm{Mor}(u^x \otimes u^y, u^z)$

$$(m \otimes S) \circ \Sigma_{23} \in \mathrm{Mor}(a(x) \otimes a(y), a(z)),$$

  where $\Sigma_{23} : \mathcal{H} \otimes \mathcal{H}_x \otimes \mathcal{H} \otimes \mathcal{H}_y \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}_x \otimes \mathcal{H}_y$ is the tensor flip on the second and third leg.
- $\mathrm{Mor}(a(\mathbb{1}_{\mathbb{G}})^{\otimes k}, a(\mathbb{1}_{\mathbb{G}})^{\otimes l}) = \mathrm{Mor}_{\mathbb{F}}(u^{\otimes k}, u^{\otimes l}) \subset B(\mathcal{H}^{\otimes k}, \mathcal{H}^{\otimes l})$ for all $k, l \geq 0$.

Here, as before, $m : A \otimes A \to A$ denotes the multiplication morphism.

We are now ready to formulate the general version of Theorem 6.3.1. Let A and B be finite-dimensional C^*-algebras with respective Markov traces tr_A and tr_B. Let $\beta : B \to B \otimes \mathbb{C}[\mathbb{G}]$ be a faithful, centrally ergodic tr_B-preserving action of a compact quantum group $\mathbb{G}$ on B and put $v = u^\beta$. Similarly, let $\alpha : A \to A \otimes \mathbb{C}[\mathbb{F}]$ be a faithful, centrally ergodic, tr_A-preserving action of a compact quantum group $\mathbb{F}$. Note that, as in Proposition 3.2.23, we obtain that the action $\beta \wr_* \alpha := \alpha_w : A \otimes B \to A \otimes B \otimes \mathbb{C}[\mathbb{G} \wr_* \mathbb{F}]$ is faithful and preserves $\mathrm{tr}_A \otimes \mathrm{tr}_B$.

Theorem 6.4.3. *Consider the subfactor planar algebra $\mathcal{Q} = \mathcal{P}(\alpha) * \mathcal{P}(\beta) \subset \mathcal{P}^{A \otimes B}$ and let $\mathbb{H}$ and $\gamma = \alpha_{\mathcal{Q}} : A \otimes B \to A \otimes B \otimes \mathbb{C}[\mathbb{H}]$ be the pair corresponding to $\mathcal{Q}$ in the sense of Theorem 6.1.2. There is an isomorphism of compact matrix quantum groups*

$$(\mathbb{H}, u^\gamma) \cong (\mathbb{G} \wr_* \mathbb{F}, w)$$

or equivalently, a conjugacy of actions $\gamma \cong \alpha_w = \beta \wr_* \alpha$.

The reason that the formulation of Theorem 6.4.3 is somewhat inverse to the one of Theorem 6.3.1 is that we have not shown yet that the action α_w is centrally ergodic and therefore $\mathcal{P}(\beta \wr_* \alpha)$ is not yet well-defined. However, this will follow from the proof of the theorem and therefore a posteriori we can rephrase Theorem 6.4.3 in exactly the same way as Theorem 6.3.1.

Proof of Theorem 6.4.3. As before denote by $\mathcal{H}$ the Hilbert space obtained by equipping A with the inner product induced by tr_A and similarly denote by $\mathcal{K}$ the Hilbert space obtained by equipping B with the inner product induced by tr_B. The compact matrix quantum groups $(\mathbb{H}, u^\gamma)$ and $(\mathbb{G} \wr_* \mathbb{F}, w)$ both act on the same Hilbert space $\mathcal{H} \otimes \mathcal{K}$.

The proof of the theorem is done in two steps.

Step 1: Let us show that there exists a morphism of Hopf-$*$-algebras $\mathbb{C}[\mathbb{G} \wr_* \mathbb{F}] \to \mathbb{C}[\mathbb{H}]$ which maps the fundamental representation w onto u^γ by finding a family of unitary representations

$$\tilde{a}(x) \in B(\mathcal{H} \otimes \mathcal{H}_x) \otimes \mathbb{C}[\mathbb{H}],\ x \in \mathrm{Irr}(\mathbb{G}),$$

satisfying the relations displayed after Lemma 6.4.2. In particular, this will show that the free wreath action $\beta \wr_* \alpha$ is centrally ergodic.

By Theorem 6.3.1, we have $\mathrm{TLJ}_\delta * \mathcal{P}(\beta) = \mathcal{P}(\alpha_{\tilde{w}})$, where $\delta = \sqrt{\dim A}$ and where $\tilde{w}$ is the fundamental representation of the free wreath product $\mathbb{G} \wr_* \mathbb{G}_{aut}(A, \mathrm{tr}_A)$. Since $\mathrm{TLJ}_\delta \subset \mathcal{P}(\alpha)$ is a planar subalgebra, we obtain

$$\mathcal{P}(\alpha_{\tilde{w}}) = \mathrm{TLJ}_\delta * \mathcal{P}(\beta) \subset \mathcal{Q} = \mathcal{P}(\gamma).$$

Therefore, there exists a a family of unitary representations

$$\tilde{a}(x) \in B(\mathcal{H} \otimes \mathcal{H}_x) \otimes \mathbb{C}[\mathbb{H}],\ x \in \mathrm{Irr}(\mathbb{G}),$$

satisfying

- For all $x, y, z \in \mathrm{Irr}(\mathbb{G})$ and all $S \in \mathrm{Mor}(u^x \otimes u^y, u^z)$

$$(m_A \otimes S) \circ \Sigma_{23} \in \mathrm{Mor}(a(x) \otimes a(y), a(z)),$$

- for the unit morphism $\eta_A : \mathbb{C} \to A$, we have $\eta_A \in \mathrm{Mor}(\mathbb{1}, \tilde{a}(\mathbb{1}_\mathbb{G}))$.

We are left with showing that $\mathrm{Mor}_\mathbb{F}(u^{\otimes k}, u^{\otimes l}) \subset \mathrm{Mor}(\tilde{a}(\mathbb{1}_\mathbb{G})^{\otimes k}, \tilde{a}(\mathbb{1}_\mathbb{G})^{\otimes l})$. To do so, note that $\tilde{a}(\mathbb{1}_\mathbb{G})^{\otimes k}$ is a subrepresentation of $\tilde{a}(v)^{\otimes k}$ implemented by the projection $p_k = (\mathrm{id} \otimes \eta_B \eta_B^*)^{\otimes k} \in B((\mathcal{H} \otimes \mathcal{K})^{\otimes k})$. In the annular category of $\mathcal{Q}$, this morphism is induced by the annular tangle labelled by $\mathrm{id} \otimes S^{\mathcal{P}(\beta)} 2k \in \mathcal{Q}_{2k}$, see Figure B.6. As for any $k \geq 0$ and any tangle of degree k, $(T, S_k^{\mathcal{P}(\beta)})$ is a free pair , see Proposition B.2.13, it follows that

$$\mathrm{Mor}_\mathbb{F}(u^{\otimes k}, u^{\otimes l}) = \mathrm{Mor}_{\mathcal{C}_{\mathcal{P}(\alpha)}}(k, l) \subset p_l \,\mathrm{Mor}_\mathcal{Q}(k, l) p_k = \mathrm{Mor}(\tilde{a}(\mathbb{1}_\mathbb{G})^{\otimes k}, \tilde{a}(\mathbb{1}_\mathbb{G})^{\otimes l}).$$

Step 2: Since we know now that $\beta \wr_* \alpha$ is centrally ergodic, we can build the corresponding planar algebra $\mathcal{P}(\beta \wr_* \alpha)$. Let us prove that $\mathcal{P}(\gamma)$ is a planar subalgebra of $\mathcal{P}(\beta \wr_* \alpha)$. It suffices to prove that a generating subset of $\mathcal{P}(\gamma)$ is contained in $\mathcal{P}(\beta \wr_* \alpha)$. Since $\mathcal{P}(\gamma) = \mathcal{P}(\alpha) * \mathcal{P}(\beta)$, we can consider the generating subset given in Proposition B.2.13. For each $k \geq 1$, this subset is given by elements of two kinds:

- $U_{\mathcal{P}(\alpha)}(k) \otimes \mathcal{P}(\beta)_k$: note that $U_{\mathcal{P}(\alpha)}(k) \otimes \mathcal{P}(\beta)_k \subset (\mathrm{TLJ}_\delta * \mathcal{P}(\beta))_k$. By Theorem 6.3.1, $\mathrm{TLJ}_\delta * \mathcal{P}(\beta) = \mathcal{P}(\beta \wr_* \alpha')$, where α' is the universal action of $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$ on A. Since $\mathbb{F}$ is a quantum subgroup of $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$ we get $\mathcal{P}(\beta \wr_* \alpha') \subset \mathcal{P}(\beta \wr_* \alpha)$, and hence $U_{\mathcal{P}(\alpha)}(k) \otimes \mathcal{P}(\beta)_k \subset \mathcal{P}(w)_k$.

- $\mathcal{P}(\alpha)_k \otimes S_{\mathcal{P}(\beta)}(k)$: to discuss these generators, let us first recall some notation. Denote the unitaries introduced after Lemma 6.1.3 by
$$U_k^{\mathcal{H}} : \mathcal{H}^{\otimes k} \to \mathcal{H}_k, \quad U_k^{\mathcal{K}} : \mathcal{K}^{\otimes k} \to \mathcal{K}_k,$$
$$U_k^{\mathcal{H}\otimes\mathcal{K}} : (\mathcal{H} \otimes \mathcal{K})^{\otimes k} \to (\mathcal{H} \otimes \mathcal{K})_k = \mathcal{H}_k \otimes \mathcal{K}_k.$$
If we rearrange the tensor product through
$$\Xi_k : (\mathcal{H} \otimes \mathcal{K})^{\otimes k} \to \mathcal{H}^{\otimes k} \otimes \mathcal{K}^{\otimes k},$$
$$(x_1 \otimes y_1) \otimes \ldots \otimes (x_k \otimes y_k) \mapsto (x_1 \otimes \ldots \otimes x_k) \otimes (y_1 \otimes \ldots \otimes y_k),$$
we have $U_k^{\mathcal{H}\otimes\mathcal{K}} = (U_k^{\mathcal{H}} \otimes U_k^{\mathcal{K}}) \circ \Xi_k$. In $\mathrm{Rep}(\mathbb{G} \wr_* \mathbb{F})$ we have an embedding
$$\mathrm{Mor}_{\mathbb{F}}(\mathbb{1}_{\mathbb{F}}, u^{\otimes k}) = \mathrm{Mor}_{\mathbb{G}\wr_*\mathbb{F}}(\mathbb{1}_{\mathbb{G}\wr_*\mathbb{F}}, a(\mathbb{1}_{\mathbb{G}})^{\otimes k}) \to \mathrm{Mor}_{\mathbb{G}\wr_*\mathbb{F}}(\mathbb{1}_{\mathbb{G}\wr_*\mathbb{F}}, w^{\otimes k})$$
$$V \mapsto (V \otimes \mathrm{id})(\mathrm{id} \otimes \eta_B)^{\otimes k}.$$
Let $x \in \mathcal{P}(\alpha)_k$. Then
$$(U_k^{\mathcal{H}\otimes\mathcal{K}})^*(x \otimes S_{\mathcal{P}(\beta)}(k)) = \Xi_k^*((U_k^{\mathcal{H}})^* x \otimes (U_k^{\mathcal{K}})^* S_{\mathcal{P}(\beta)}(k)).$$
But since $(U_k^{\mathcal{H}})^* \mathcal{P}(\alpha) = \mathrm{Mor}_{\mathbb{F}}(\mathbb{1}_{\mathbb{F}}, u^{\otimes k})$ and $(U_k^{\mathcal{K}})^* S_{\mathcal{P}(\beta)}(k) = \eta_B^{\otimes k}$, we are done.

□

Remark 6.4.4. 1. Recall that by Example 3.2.20, every compact quantum group of Kac type whose representation category $\mathrm{Rep}(\mathbb{F})$ is finitely generated, admits a faithful, centrally ergodic, tr-preserving action on *some* finite-dimensional C^*-algebra. Moreover, due to the correspondence of actions and representations (see Section 3.2.5), choosing such an action is the same as choosing a well-behaved generator of $\mathrm{Rep}(\mathbb{F})$. However, typically, this generator will not be the canonical one, which often does not contain the trivial representation as a subrepresentation and hence does not correspond to an action. Therefore, free wreath products such as $\mathbb{G} \wr_* O_n^+$ or $\mathbb{G} \wr_* U_n^+$ are well-defined but very difficult to study in practice, as their study requires a reinterpretation of the representation categories $\mathrm{Rep}(O_n^+)$ and $\mathrm{Rep}(U_n^+)$ in terms of a non-canonical generating representation.

2. It is also possible to describe the representation theory of $\mathbb{G} \wr_* \mathbb{F}$, where $\mathbb{F} \subset \mathbb{G}_{aut}(A, \mathrm{tr}_A)$ (with faithful, centrally ergodic, tr_A-preserving action α) when $\mathbb{G}$ is a general compact quantum group. We will only sketch this.

   Let $\mathrm{Obj}(\mathcal{C}) = \langle \mathrm{Rep}(\mathbb{G}) \rangle$ be the set of words with letters in $\mathrm{Rep}(\mathbb{G})$. The tensor operation will be the concatenation of words and therefore we will denote such a word by $a(x_1) \otimes a(x_2) \ldots \otimes a(x_s),\ x_1, \ldots, x_m \in \mathrm{Rep}(\mathbb{G})$. To describe the morphism space

$$\mathrm{Mor}(a(x_1) \otimes a(x_2) \ldots \otimes a(x_m), a(y_1) \otimes a(y_2) \ldots \otimes a(y_n)),$$

   consider a planar tangle of degree $k_0 = m + n$ with inner disks D_i of degree k_i and elements $v_i \in \mathcal{P}(\alpha)_{k_i},\ i = 1, \ldots, s$. By definition, we have $\eta := Z_T(v_1, \ldots, v_s) \in \mathcal{P}(\alpha)_{k_0}$ and therefore we can consider the special (m, n)-tangle T_η labelled by η. If $\mathcal{H}$ is the Hilbert space associated to the action α of $\mathbb{F}$, this special tangle induces a map $Z(T, v_1, \ldots, v_s) := Z'_{T_\eta} : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes n}$. Now, mark the boundary intervals of the outer disk of T which are adjacent to shaded regions of the tangle clockwise by $y_1, \ldots, y_n, x_m, \ldots, x_1$. Every shaded region r of the tangle is now marked on its outer boundary by elements $x_{i_1}, \ldots x_{i_v}, y_{j_1}, \ldots, y_{j_w}, i_1 < i_2 \cdots < i_v,\ j_1 < j_2 \cdots < j_w$. Associate with any such region r a morphisms

$$S_r \ \in \ \mathrm{Mor}(x_{i_1} \otimes \ldots \otimes x_{i_v}, y_{j_1} \otimes \ldots \otimes y_{i_w})$$

   with the usual convention that we replace $x_{i_1} \otimes \ldots \otimes x_{i_v}$ by $\mathbb{1}_{\mathbb{G}}$ if no boundary interval of r is marked with a representation in $\{x_1, \ldots, x_m\}$ (with a similar convention for the y part). By ordering the morphisms S_r appropriately, we obtain a morphism

$$S = \bigotimes_{r \text{ shaded region}} S_r : \ \mathcal{H}_{x_1} \otimes \ldots \otimes \mathcal{H}_{x_m} \ \to \ \mathcal{H}_{y_1} \otimes \ldots \otimes \mathcal{H}_{y_n},$$

   and therefore a map

$$Z(T, v_1, \ldots, v_s) \otimes S : \mathcal{H}^{\otimes m} \otimes \mathcal{H}_{x_1} \otimes \ldots \otimes \mathcal{H}_{x_m} \ \to \ \mathcal{H}^{\otimes n} \otimes \mathcal{H}_{y_1} \otimes \ldots \otimes \mathcal{H}_{y_n}.$$

   If we reorder as in the beginning of this section, we obtain maps

$$Z(T, v_1, \ldots, v_s; S) : \bigotimes_{i=1}^{m} \mathcal{H} \otimes \mathcal{H}_{x_i} \ \to \ \bigotimes_{j=1}^{n} \mathcal{H} \otimes \mathcal{H}_{y_j}.$$

   We set $\mathrm{Mor}(a(x_1) \otimes a(x_2) \ldots \otimes a(x_m), a(y_1) \otimes a(y_2) \ldots \otimes a(y_n))$ to be the span of all maps obtained in this way. One can check that this defines a concrete rigid C^*-tensor category that coincides with $\mathrm{Rep}(\mathbb{G} \wr_* \mathbb{F})$ using the methods in [FP16].

6.5 Consequences of the main results

The following result is an immediate consequence of [PV15, Proposition 9.3]. We refer to Section 3.4.5 for definitions of the properties mentioned in the following corollary.

Corollary 6.5.1. *Let* $(\mathbb{F}, \alpha)$, $(\mathbb{G}, \beta)$ *be compact quantum groups with faithful, centrally ergodic,* tr- *preserving actions* α, β *on finite dimensional* C^**-algebras and let* $(\mathbb{G} \wr_* \mathbb{F}, \beta \wr_* \alpha)$ *be their free wreath product.*

- *If both* $\hat{\mathbb{F}}$ *and* $\hat{\mathbb{G}}$ *have the Haagerup property, so does the dual of* $\mathbb{G} \wr_* \mathbb{F}$*.*
- *If both* $\hat{\mathbb{F}}$ *and* $\hat{\mathbb{G}}$ *have the ACPAP, so does the dual of* $\mathbb{G} \wr_* \mathbb{F}$*.*

A last result to be mentioned here is due to Kyed, Raum, Vaes and Valvekens who computed the L^2-Betti numbers of several examples of discrete quantum groups in [KRVV17]. In particular, from Theorem 6.4.3 they derived a formula for the L^2-Betti numbers of free wreath products in [KRVV17, Theorem 5.2].

Corollary 6.5.2 ([KRVV17])**.** *Let* $\mathbb{G} = \mathbb{H} \wr_* \mathbb{F}$ *denote the free wreath product of a compact quantum group* $\mathbb{H}$ *and a quantum subgroup* $\mathbb{F}$ *of* $\mathbb{G}_{aut}(A, \mathrm{tr}_A)$*, acting on* A *in a centrally ergodic and* tr_A*-preserving manner. The* L^2*-Betti numbers of* $\hat{\mathbb{G}}$ *are given by the formula*

$$\beta_n^{(2)}(\hat{\mathbb{G}}) = \begin{cases} 0 & \text{if } n = 0, \\ \beta_1^{(2)}(\hat{\mathbb{H}}) + \beta_1^{(2)}(\hat{\mathbb{F}}) + 1 - (\beta_0^{(2)}(\hat{\mathbb{H}}) + \beta_0^{(2)}(\hat{\mathbb{F}})) & \text{if } n = 1, \\ \beta_n^{(2)}(\hat{\mathbb{H}}) + \beta_n^{(2)}(\hat{\mathbb{F}}) & \text{if } n \geq 2. \end{cases}$$

Appendix B

Tangles, partitions and generators of free products of planar algebras

In this appendix, we recapitulate useful facts from the study of non-crossing partitions and relate them to planar tangles. Using this partition calculus, we describe a generating subset of the the free product of planar algebras which we make use of in Chapter 6. The results of this appendix are of a rather technical nature. They also form part of the PhD thesis of my coauthor P. Tarrago [T15].

Non-crossing partitions are recurring tools in the theory of compact quantum groups (see for instance [TWe17]) and in free probability [NS06] as they emulate the absence of commutativity relations quite nicely. We will describe an underlying partition structure of planar tangles in order to express freeness of planar tangles in a simpler manner in Section 6.2.

B.1 Partial partitions

A partition p of order k is a partition of $[\![1,k]\!] := \{1,\dots,k\}$ into disjoint subsets. The subsets of $[\![1,k]\!]$ which appear in the partition are called the blocks of p. The number of blocks of p is denoted by $|p|$. A partition p of order k yields an equivalence relation $\sim_p$ on $[\![1,k]\!]$ by setting $x \sim_p y$ whenever x and y are in the same block of p. We get a bijection between equivalence relations on $[\![1,k]\!]$ and

partitions of order k. The set of partitions of order k is denoted by $P(k)$.
A partition of k is depicted by drawing k points on an horizontal line numbered from 1 to k and by linking two points whenever they belong to the same block of the partition. An example is drawn below.

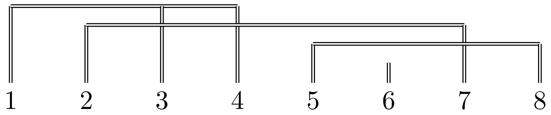

Figure B.1: Partition $\{\{1,3,4\},\{2,7\},\{5,8\},\{6\}\}$ with 4 blocks.

We define an order relation on $P(k)$ by setting $p \leq p'$ whenever each block of p is included in a block of p'. This order relation turns $P(k)$ into a lattice, and we denote by $p \wedge p'$ (resp. $p \vee p'$) the infimum (resp. the supremum) of two partitions p and p'. See [NS06] for details on this lattice structure.
A partition p of order n is called non-crossing whenever the following condition holds. If $i < j < k < l$ are integers between 1 and n such that $i \sim_p k$ and $j \sim_p l$, then $i \sim_p j \sim_p k \sim_p l$. The set of non-crossing partitions of order k is denoted by $NC(k)$. This set forms a sublattice of $P(k)$. A partition of k is called an interval partition if all its blocks are intervals of integers. For example, $\{\{1,2,3\},\{4,5\},\{6,7,8\}\}$ is an interval partition. The set of interval partitions is denoted by $\mathcal{I}(k)$.
Finally, we call a partition even if all its blocks have even cardinalities.

B.1.1 Partial partitions and Kreweras complement

Definition B.1.1. Let $k \geq 1$. A partial partition of $[\![1,k]\!]$ is a pair (p,S) consisting of a subset S of $[\![1,k]\!]$ and a a partition p of order $|S|$.

Equivalently, a partial partition is a set $\{B_1,\dots,B_r\}$ of disjoint subsets of $[\![1,n]\!]$, with $S = \bigcup B_i$. We denote by $P(S)$ ($NC(S)$) the set of partial partitions (non-crossing partial partitions) with support S. Two partial partitions (p,S) and (p',S^c) yield a partition $(p,S)\vee(p',S^c)$ of $[\![1,n]\!]$ by identifying $[\![1,|S|]\!]$ with S (resp. $[\![1,|S^c|]\!]$ with S^c) as ordered sets and by considering the union of p and p'.

Definition B.1.2. Let (p,S) be a non-crossing partial partition of S, the Kreweras complement $\mathrm{kr}(p)$ of p is the biggest partial partition (p',S^c) such that $(p,S)\vee(p',S^c)$ is non-crossing.

In the above definition, the order on $NC(S^c)$ is the refinement order coming from the canonical bijection between $NC(S^c)$ and $NC(|S^c|)$. The proof of the following lemma is straightforward.

Lemma B.1.3. *Let (p, S) be a partial partition of $[\![1, n]\!]$. Then $\mathrm{kr}(p)$ is the partial partition with support S^c defined by*

$$i \sim_{\mathrm{kr}(p)} j \Leftrightarrow k \not\sim_{(p,S)} l, \text{ for all } k \in [\![i, j]\!] \cap S, l \in S \setminus [\![i, j]\!].$$

B.2 Non-crossing partitions and irreducible planar tangles

In this subsection, we associate to each planar tangle T a non-crossing partition π_T and express the corresponding Kreweras complement in terms of shaded regions of T.

B.2.1 The non-crossing partition associated to a planar tangle

We start with the following definition.

Definition B.2.1. Let T be a planar tangle of degree k. The partition of T, denoted by π_T, is the set partition of $[\![1, 2k]\!]$ such that $i \sim_{\pi_T} j$ if and only if the distinguished points i and j of the outer disk D_0 belong to the closure of the same connected component of $\Gamma T \setminus \partial D_0$.

Note that π_T is well defined, since a distinguished point of the outer disk D_0 is in the closure of a unique connected component of $\Gamma T \setminus \partial D_0$.

Lemma B.2.2. *π_T is an even non-crossing partition of $2k$.*

Proof. Let $1 \leq i < j < k < l \leq 2k$ be such that $i \sim_T k$ and $j \sim_T l$. Thus, there exist a path ρ_1 in $\overline{\Gamma T \setminus D_0}$ between i and k and a path ρ_2 in $\overline{\Gamma T \setminus D_0}$ between j and l. Since the distinguished points on the outer disk are numbered clockwise, γ_1 and γ_2 intersect. Therefore, the four points are in the closure of the same connected component of $\Gamma T \setminus D_0$ and $i \sim_T j \sim_T k \sim_T l$. π_T is thus non-crossing.
Since each inner disk has an even number of distinguished points and each curve connects two distinguished points, a counting argument yields the parity of the size of the blocks. ☐

Reciprocally, an even non-crossing partition π of $2k$ yields an irreducible planar tangle T_π such that $\pi_{T_\pi} = \pi$. The construction is done recursively on the number of blocks as follows.

1. If π is the one block partition of $2k$, T_π is the planar tangle with one outer disk D_0 of degree $2k$, one inner disk D_1 of degree $2k$, and a curve between the point i of D_0 and the point i of D_1.

2. Let $r \geq 2$, and suppose that P_p is constructed for all even non-crossing partitions p having less than r blocks. Let π be an even non-crossing partition with r blocks. Let $B = [\![i_1, i_2]\!]$ be an interval block of π, and let π' be the non-crossing partition obtained by removing this block (and relabelling the points). π' has also even blocks. Let $T_{\pi,B}$ be the planar tangle consisting of an outer disk D_0 of degree $2k$ and two inner disks D_1 and D_2 of respective degree $i_2 - i_1 + 1$ and $2k - (i_2 - i_1 + 1)$, and curves connecting

   - i_{D_0} to i_{D_2} for $i < i_1$.
   - i_{D_0} to $(i - i_1 + 1)_{D_1}$ for $i_1 \leq i \leq i_2$ if i_1 is odd, and i_{D_0} to $(i - i_1)_{D_1}$ for $i_1 \leq i \leq i_2$ if i_1 is even.
   - i_{D_0} to $(i - (i_2 - i_1 + 1))$ for $i > i_2$.

   Set $T_\pi = T_{\pi,B} \circ_{D_2} T_{\pi'}$. Note that the resulting planar tangle does not depend on the choice of the interval block B in π.

By construction, T_π is irreducible and $\pi_{T_\pi} = \pi$. The inner disk of T_π corresponding to the block B of π is denoted by D_B. An outer point i is linked to D_B in T_π if and only if $i \in B$ in π. The distinguished boundary point of D_B is the one linked to the smallest odd boundary point on the outer disk. The next lemma is a straightforward deduction from the previous description of T_π.

Lemma B.2.3. *Suppose that π and π' are two even non-crossing partitions such that $\pi' \leq \pi$. Denotes by $B_1, \ldots, B_r$ the blocks of π. Then,*

$$T_{\pi'} = T_\pi \circ_{D_{B_1},\ldots,D_{B_r}} (T_{\pi'_{|B_1}}, \ldots, T_{\pi'_{|B_r}}).$$

Reciprocally, if π is an even non-crossing partition, B is a block of π and (τ, B) is an even non-crossing partial partition of B, then

$$T_\pi \circ_{D_B} T_\tau = T_{\pi'},$$

where π' is the non-crossing partition $\pi \setminus \{B\} \cup (\tau, B)$. In particular, $\pi' \leq \pi$.

The planar tangles $(T_\pi)_{\pi \in NC(k)}$ yield a decomposition of connected planar tangles of degree k. Let T be a connected planar tangle of degree k. Let $B_1, \ldots, B_r$ be the blocks of π_T ordered lexicographically and let $C_1, \ldots, C_r$ be the corresponding connected components of T. For $1 \leq i \leq r$, T_i is defined as the planar tangle $T \setminus (\bigcup_{j \neq i} C_j)$, where the distinguished points of the outer boundary of T which are in B_i have been clockwise relabeled in such a way that the first odd point is labeled 1. The planar tangle T_1 of the connected planar tangle of Figure 3.2 is depicted in Figure B.2.

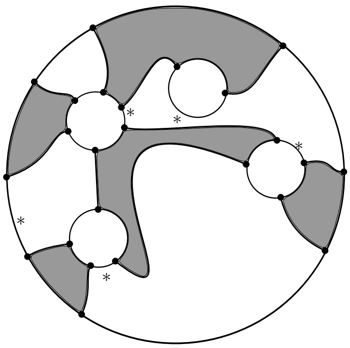

Figure B.2: First connected component of the first planar tangle of Figure 3.2.

Proposition B.2.4. *Let T be a connected planar tangle, and set $\pi = \pi_T$. Then*

$$T = T_\pi \circ_{D_{B_1}, \ldots, D_{B_r}} (T_1, \ldots, T_r).$$

Proof. It is possible to draw r disjoint Jordan curves $\{\rho_i\}_{1 \leq i \leq r}$ such that ρ_i intersects T $|B_i|$ times, once at each curve of C_i having an endpoint on the outer disk (or two times at a curve joining two distinguished points of the outer boundary). The intersection points are labeled clockwise around ρ_i, in such a way that the intersection point with the curve coming from the first odd point of B_i is labeled 1.

Let Γ_i be the bounded region whose boundary is ρ_i and set $\tilde{T}_i = (T \cap \Gamma_i) \cup \rho_i$, with the labeling of the distinguished points of ρ_i given above. Then $\tilde{T}_i$ is a planar tangle which is an isotopy of T_i. Figure B.3 shows a possible choice of Jordan curves for the connected planar tangle of Figure 3.2.

Let $\tilde{T}$ be the planar tangle whose inner disks are $(\Gamma_i)_{1 \leq i \leq r}$, with the distinguished points being the ones of ρ_i, and whose skeleton is $\Gamma T \setminus$

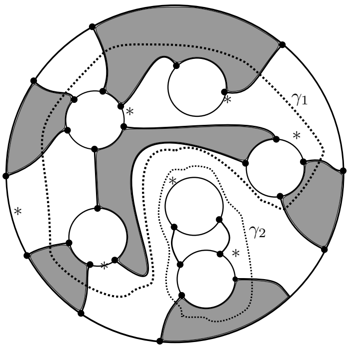

Figure B.3: Jordan curves surrounding the connected components of a planar tangle.

$\left(\Gamma T \cap (\bigcup \mathring{\Gamma}_i)\right)$. Then $\tilde{T}$ is isotopic to T_π and, by construction, $\tilde{T} \circ_{\Gamma_1,\dots,\Gamma_r} (\tilde{T}_1,\dots,\tilde{T}_r) = T$. $\square$

B.2.2 Shaded regions and Kreweras complement

If S is a set of cardinality k, $f : [\![1,k]\!] \longrightarrow S$ is a bijective function and $\pi \in P(k)$, then $f(\pi)$ is the partition of S defined by $f(i) \sim_{f(\pi)} f(j)$ if and only if $i \sim_\pi j$. For $i \geq 1$, set $\delta(i) = 1$ if i is odd and 0 else. A partial partition $(\tilde{\pi}, S)$ of $4k$ is associated to each $\pi \in NC(2k)$ as follows.

- $S = \{2i - \delta(i)\}_{1 \leq i \leq 2k}$.
- $\tilde{\pi} = f(\pi)$ where $f : [\![1,2k]\!] \to S \subset [\![1,4k]\!]$ given by $f(i) = 2i - \delta(i)$.

Note that S is the set $\{1,4,5,8,\dots,4k-3,4k\}$. Let $\tilde{f}$ be the map from $[\![1,2k]\!]$ to $[\![1,4k]\!] \setminus S$ defined by $\tilde{f}(i) = 2i - (1 - \delta(i))$.

Definition B.2.5. Let $\pi \in NC(2k)$. The nested Kreweras complement of π, denoted by $\mathrm{kr}'(\pi)$, is the partition of $2k$ such that $\tilde{f}(\mathrm{kr}'(\pi)) = \mathrm{kr}(\tilde{\pi}, S)$.

The nested Kreweras complement of the partition $\{\{1,3,4\},\{2\},\{5,6\}\}$ is given in Figure B.4.

Contrary to the usual Kreweras complement, the map $\pi \to \mathrm{kr}'(\pi)$ is not bijective, as we will see in the next lemma. Let π_0 (resp π_1) be the partition of $2k$ with k blocks being $\{1,2k\}$ and $\{2i,2i+1\}$ (resp. $\{2i-1,2i\}$) for $1 \leq i \leq k-1$.

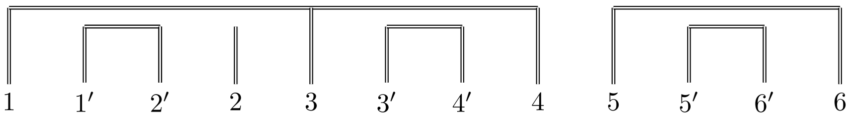

Figure B.4: The partition $\{\{1,3,4\},\{2\},\{5,6\}\}$ and its nested Kreweras complement $\{\{1,2\},\{3,4\},\{5,6\}\}$.

Lemma B.2.6. *We have* $\mathrm{kr}'(\pi) = \mathrm{kr}'(\pi \vee \pi_0)$.

Proof. Since $\pi \leq (\pi \vee \pi_0)$, it follows that $\mathrm{kr}'(\pi \vee \pi_0) \leq \mathrm{kr}'(\pi)$.
Suppose that $i \sim_{\mathrm{kr}'(\pi)} j$. Then $2i-(1-\delta(i)) \sim_{\mathrm{kr}(\tilde{\pi},S^c)} 2j-(1-\delta(j))$. By Lemma B.1.3, this implies that for all $k \in [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!] \cap S$, $l \in S \setminus [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!]$, $k \not\sim_\pi l$. If $k \in [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!] \cap S$ and $l \notin [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!]$, then $l \neq k \pm 1$ and thus $k \not\sim_{\pi_0} l$; therefore, for all $k \in [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!] \cap S$, $l \in S \setminus [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!]$, $k \not\sim_{\pi \vee \pi_0} l$. By Lemma B.1.3, this implies that $i \sim_{\mathrm{kr}'(\pi \vee \pi_0)} j$ and therefore $\mathrm{kr}'(\pi) \leq kr'(\pi \vee \pi_0)$. □

The nested Kreweras complement is involved in the description of planar tangles in the following way.

Proposition B.2.7. *Suppose that T is a planar tangle of degree k. Then, i and j are in the same block of* $\mathrm{kr}'(\pi_T)$ *if and only if i_{D_0} and j_{D_0} are boundary points of the same shaded region.*

Proof. Let T be a planar tangle. Relabel each distinguished point i_{D_0} with $2i-\delta(i)$.With this new labelling each interval of type $(4i+1, 4i+4)$ is the boundary interval of a shaded region, and π_T becomes a partial non-crossing partition of $4k$ with support S. Add two points $4i+2$ and $4i+3$ clockwise on each segment $(4i+1, 4i+4)$ of the outer disk. We define the equivalence relation $\sim$ on S^c as follows: $i \sim j$ if and only if the boundary points i and j are boundary points of a same shaded region. Let (π', S^c) be the partial partition associated to $\sim$. π' is non-crossing since two regions that intersect are the same. Let $\pi = ((\pi_T, S) \vee (\pi', S^c))$. Suppose that $1 \leq i < j < r < s \leq 4k$ are such that $i \sim_\pi r$ and $j \sim_\pi s$. Since π_T is non-crossing, if i, j, r, s are all in S then $i \sim_\pi j \sim_\pi r \sim_\pi s$. Assume from now on that they are not all in S, and suppose without loss of generality that $i \in S^c$. We have $i \sim_\pi r$, thus r is also in S^c and i and r are boundary points of a same shaded region σ. Since $j \in (i, r)$ and $s \in (r, i)$, any path on ΓT between j and s would cut σ in two distinct regions. Thus, if $j, s \in S$, then $j \not\sim_\pi s$. Therefore, the hypothesis $j \sim_\pi s$ yields that $j, s \in S^c$. As π' is non-crossing, i, j, r, s are in the same block of π. Finally,

$\pi_T \vee \pi'$ is non-crossing, which yields that $\pi' \leq \mathrm{kr}'(\pi_T)$.
Let π_2 be a partial partition with support S^c, such that $\pi_T \vee \pi_2$ is non-crossing. Suppose that $i \sim_{\pi_2} j$, with $i, j \in S^c$. Let σ_i (resp. σ_j) be the shaded region having i (resp. j) as boundary point. $\pi_T \vee \pi_2$ is non-crossing, thus for all $r, s \in S$ such that $i \leq r \leq j$ and $j \leq s \leq i$, $r \not\sim_{\pi_T} s$. Thus, there is no path in ΓT between (i, j) and (j, i), and $\sigma_i = \sigma_j$. This yields $i \sim_{\pi'} j$. Therefore, $(\pi_2, S^c) \leq (\pi', S^c)$ and $(\pi', S^c) = \mathrm{kr}(\pi_T, S)$.
Let $1 \leq i, j \leq 2k$. By the two previous paragraphs, i and j are in the same block of $\mathrm{kr}'(\pi_T)$ if and only if $2i - (1 - \delta(i))$ and $2j - (1 - \delta(j))$ are boundary points of the same shaded region. Since the points $2i - (1 - \delta(i))$ and $2i - \delta(i)$ both belong to the interval $(2(i + \delta(i)) - 3, 2(i + \delta(i)))$ (which is part of the boundary of a shaded region), $2i - (1 - \delta(i))$ and $2j - (1 - \delta(j))$ are boundary points of the same shaded region if and only if $2i - \delta(i)$ and $2j - \delta(j)$ are boundary points of the same shaded region which yields the result. $\square$

The remaining part of the appendix aims at simplifying the description of the free product of planar algebras, see Definition 6.2.3.

B.2.3 Reduced free pairs

We now introduce a set of pairs of planar tangles which is smaller than the set of free pairs and whose image in $\mathcal{P} \otimes \mathcal{Q}$ still spans $\mathcal{P} * \mathcal{Q}$. Let us begin by characterizing free pairs of planar tangles in terms of their associated non-crossing partitions, see Appendix B. In this subsection, all planar tangles are assumed to be connected.

Lemma B.2.8. *If (T, T') is a free pair, then $\pi_{T*T'} = (\pi_T, S) \vee (\pi_{T'}, S^c)$. In particular (T, T') is a free pair if and only if $(T_{\pi_T}, T_{\pi_{T'}})$ is a free pair.*

Proof. The first statement of the lemma is a direct consequence of the definition of S in the beginning of Section B.2.2, and the fact that $i \sim_T j$ if and only if $2i - \delta(i) \sim_{T*T'} 2j - \delta(j)$ and $i \sim_{T'} j$ if and only if $2i - (1 - \delta(i)) \sim_{T*T'} 2j - (1 - \delta(j)$. Thus, if (T, T') is a free pair, then $T_{\pi(T*T')}$ is exactly the free composition of T_{π_T} with $T_{\pi_{T'}}$ and $(T_{\pi_T}, T_{\pi_{T'}})$ is also a free pair.
Suppose that $(T_{\pi_T}, T_{\pi_{T'}})$ is a free pair. By Proposition B.2.4, there exist $T_1, \ldots, T_r, T'_1, \ldots, T'_{r'}$ such that $T = T_{\pi_T} \circ_{D_1, \ldots, D_R} (T_1, \ldots, T_r)$ and $T' = P_{\pi_{T'}} \circ_{D'_1, \ldots, D'_{r'}} (T'_1, \ldots, T'_{r'})$. Therefore, $(T_{\pi_T} * T_{\pi_{T'}}) \circ_{D_1, \ldots, D_r, D'_1, \ldots, D'_{r'}} (T_1, \ldots, T_r, T'_1, \ldots, T'_{r'})$ gives the free composition of T and T'. $\square$

Proposition B.2.9. *Let T and T' be two connected planar tangles. Then (T, T') is a free pair if and only if $\pi_{T'} \leq \mathrm{kr}'(\pi_T)$. In particular if T, U and*

T', U' satisfy $\pi_T = \pi_U$ and $\pi_{T'} = \pi_{U'}$, then (T, T') is a free pair if and only if (U, U') is a free pair.

Proof. If (T, T') is a free pair, then by the previous lemma $(\pi_T, S) \vee (\pi_{T'}, S^c)$ is non-crossing. Therefore $\pi_{T'} \leq \mathrm{kr}'(\pi_T)$.
If $\pi_{T'} \leq kr'(\pi_T)$, then $\tilde{\pi} = (\pi_T, S) \vee (\pi_{T'}, S^c)$ is even and non-crossing. Therefore, $T_{\tilde{\pi}}$ is a well-defined planar tangle. Let $\{C_i\}$ be the connected components of $T_{\tilde{\pi}}$ coming from blocks of (π_T, S) and $\{D_i\}$ the ones coming from blocks of $(\pi_{T'}, S^c)$. Then, $T_{\tilde{\pi}} \setminus \bigcup C_i$ is an irreducible planar tangle and $\pi(T_{\tilde{\pi}} \setminus \bigcup C_i) = \pi_{T'}$. Thus, $T_{\tilde{\pi}} \setminus \bigcup C_i = T_{\pi_{T'}}$ up to a relabelling of the distinguished points, and similarly $T_{\tilde{\pi}} \setminus \bigcup D_i = T_{\pi_T}$ up to relabelling. Therefore, $T_{\tilde{\pi}}$ is, up to a relabelling, the free composition of T_{π_T} with $T_{\pi_{T'}}$. Finally, $(T_{\pi_T}, T_{\pi_{T'}})$ is a free pair and by the previous lemma, (T, T') is also a free pair. □

Recall that π_0 (resp. π_1) is the pair partition of $2k$ with blocks $\{(2i, 2i+1)\}$ (resp. $\{2i+1, 2i+2)\}$).

Definition B.2.10. A free pair (T, T') of planar tangles is called reduced if $T = T_\pi$ and $T' = T_{\mathrm{kr}'(\pi)}$ for some non-crossing partition π such that $\pi \geq \pi_0$.

An example of reduced free pair is given in Figure B.5 with $\pi = \{\{1, 6\}, \{2, 3, 4, 5\}\}$.

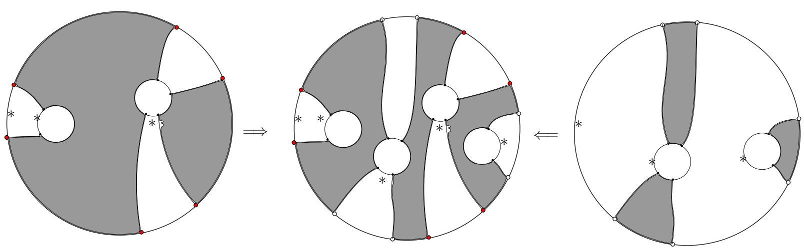

Figure B.5: Example of a reduced free pair.

Note that the number of reduced free pairs of degree k is exactly the cardinality of the set of non-crossing partitions of k. Despite the small number of reduced free pairs, only considering these pairs is nonetheless enough to describe free product of planar algebras.

Proposition B.2.11. *The free planar algebra $\mathcal{P} * \mathcal{Q}$ is spanned by the images of $Z_T \otimes Z_{T'}$, where (T, T') ranges through the set of reduced free pairs.*

Proof. It suffices to prove that the image of $Z_T \otimes Z_{T'}$ with (T, T') a free pair is contained in the image of (U, U') with (U, U') a reduced free pair.
The image of $Z_T \otimes Z_{T'}$ is contained in the image of $Z_{T_{\pi_T}} \otimes Z_{T_{\pi_{T'}}}$ by Proposition B.2.4 and by Lemma B.2.8 $(T_{\pi_T}, T_{\pi_{T'}})$ is again a free pair. We can therefore assume that $T = T_\pi$ and $T = T_{\pi'}$, with the condition $\pi' \leq \mathrm{kr}'(\pi)$ being given by Proposition B.2.9.
Suppose that $\mu \leq \nu$ are two noncrossing partitions of k. Let $B_1, \ldots, B_r$ be the blocks of ν in the lexicographical order. Since $\mu \leq \nu$, $\mu = \bigvee(\mu_{|B_i}, B_i)$. Therefore,

$$T_\mu = T_\nu \circ_{D_1,\ldots,D_r} (T_{\mu_{|B_1}}, \ldots, T_{\mu_{|B_r}}),$$

and the image of Z_{T_μ} is contained in the one of Z_{T_ν}.
Since $kr'(\pi) = \mathrm{kr}'(\pi_0 \vee \pi)$, we have $\pi' \leq \mathrm{kr}'(\pi_0 \vee \pi)$. As $\pi \leq \pi \vee \pi_0$, the image of Z_{T_π} is included in the image of $Z_{T_{\pi\vee\pi_0}}$ and as $\pi' \leq \mathrm{kr}'(\pi \vee \pi_0)$ the image of $Z_{T_{\pi'}}$ is included in the image of $Z_{T_{\mathrm{kr}'(\pi\vee\pi_0)}}$. By definition, $(T_{\pi\vee\pi_0}, T_{\mathrm{kr}'(\pi_0\vee\pi)})$ is a reduced pair and we are done. $\square$

Thanks to the previous proposition, there is a simpler way to describe the vector space $(\mathcal{P} * \mathcal{Q})_n$ for $n \geq 1$. Let $\pi = \{B_1, \ldots, B_r\}$ be an even non-crossing partition, such that its blocks are ordered lexicographically. For any planar algebra $\mathcal{P}$, we define the space $\mathcal{P}_\pi$ as the vector space $\mathcal{P}_{|B_1|/2} \otimes \ldots \mathcal{P}_{|B_r|/2}$. Proposition B.2.11 then precisely translates to the statement that $(\mathcal{P} * \mathcal{Q})_n$ is spanned by

$$\{Z_{T_\pi}(v) \otimes Z_{T_{\mathrm{kr}'(\pi)}}(w) | v \in \mathcal{P}_\pi, w \in \mathcal{Q}_{kr'(\pi)}, \pi \in NC(2n), \pi \geq \pi_0\}.$$

B.2.4 A generating subset of a free product of planar algebras

Let $\mathcal{P}$ be a planar algebra, and fix a particular subset X_n of $\mathcal{P}_n$ for each $n \in \mathbb{N}_* \cup \{+, -\}$. We say that $(X_n)_{n\in\mathbb{N}_*\cup\{+,-\}}$ *generates* $\mathcal{P}$ (or that $(X_n)_{n\in\mathbb{N}_*\cup\{+,-\}}$ is a *generating subset* of $\mathcal{P}$) if each $\mathcal{P}_n$ with $n \in \mathbb{N}_* \cup \{+, -\}$ is spanned by the union of all images $Z_T(X_{k_1}, \ldots, X_{k_m})$, where T is any planar tangle of degree n and $k_1, \ldots, k_m$ are the respective degrees of the inner disks of T.

The goal of this subsection is to introduce a simple generating subset of the free product $\mathcal{P} * \mathcal{Q}$. For this purpose let us introduce the planar tangle S_k (resp. U_k) which is the tangle without inner disk order k where $2i - 1$ is linked to $2i$ (resp. $2i$ is linked to $2i + 1$) for all $1 \leq i \leq k$, where as usual we identify 1 with $2k + 1$. Both tangles are drawn in Figure B.6 for $k = 4$. We simply denote by $S_{\mathcal{P}}(k)$ (resp. $U_{\mathcal{P}}(k)$) the image of the element Z_{S_k} (resp. Z_{U_k}) in $\mathcal{P}_k$.

Lemma B.2.12. *Let (T, T') be a reduced free pair of degree k. There exists a planar tangle R of degree k with r inner disks D_i of respective degree k_i,*

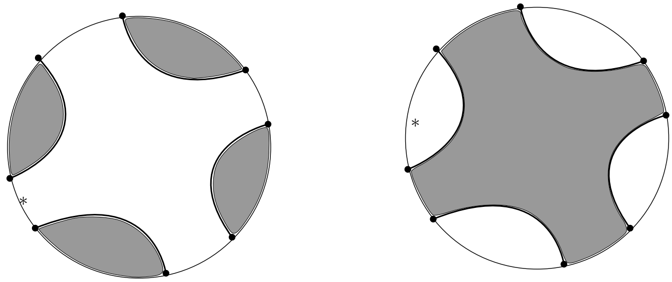

Figure B.6: Tangles S_4 and U_4.

such that $T = R_{D_1,\dots,D_r}(X_1,\dots,X_r)$ *and* $T' = R_{D_1,\dots,D_r}(\tilde{X}_1,\dots,\tilde{X}_r)$*, where for each* $1 \le i \le r$*,* $(X_i, \tilde{X}_i)$ *is either* (U_{k_i}, Id_{k_i}) *or* (Id_{k_i}, S_{k_i})*.*

Proof. Since (T, T') is a free pair, the free composition $T * T'$ exists; since this pair is reduced, each inner disk of $T * T'$ is only connected to the outer boundary, and, by definition, $\pi_T \ge \pi_0$ and $\pi_{T'} \ge \pi_1$. Thus, for each $1 \le i \le k$, both elements of $\{4i, 4i+1\}$ (resp. $\{4i-2, 4i-1\}$) are connected to the same inner disk coming from T (resp. T'). Color an inner disk D_i of $T * T'$ with 1 if it comes from T and with 2 if it comes from T'. We denote by γ_i the curve arriving on the boundary point i of the outer boundary and by $\bar{i}$ the boundary point of an inner disk which is connected to i.
We use the following operation on $T * T'$. For each interval $(4i-1, 4i)$, let σ be the region having $(4i-1, 4i)$ as a boundary interval. Add a curve $\tilde{\gamma}$ within this region connecting $\overline{4i-1}$ to $\overline{4i}$ and erase $\gamma_{4i-1}, \gamma_{4i}$ and the boundary points $4i-1$ and $4i$.
The degrees of the inner disks do not change and this yields a planar tangle R with $2k$ boundary points and r inner disks (where r is the sum of the number of inner disks in T and in T'). In the resulting planar tangle R, an odd point i of the outer boundary is still connected to the point $\bar{i}$ on a disk colored 1, and an even point i of the outer boundary is still connected to the point $\bar{i}$ on a disk colored 2. The construction of the tangle R is shown in Figure B.7.

Set $X_i = U_{k_i}, \tilde{X}_i = Id_{k_i}$ if D_i is colored 2, and $X_i = Id_{k_i}, \tilde{X}_i = S_{k_i}$ if D_i is colored 1. Consider $R_1 = R_{D_1,\dots,D_r}(X_1,\dots,X_r)$. Each disk of R colored 2 is replaced by a planar tangle without inner disk, and thus disappears in R_1. A disk of R colored 1 is composed with the identity, and thus remains the same in R_1. An odd point $4i+1$ is already connected to $\overline{4i+1}$. An even point $4i+2$ is connected to an odd point $\overline{4i+2}$ of a disk D colored 2. Therefore, since D is composed with U_{k_i}, $\overline{4i+2}$ is connected in R_1 by a curve to the following point

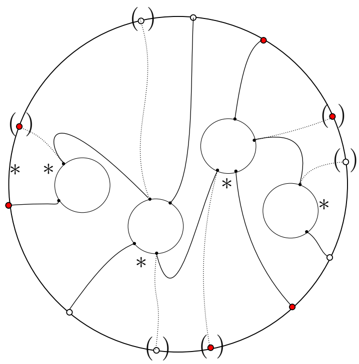

Figure B.7: Construction of the planar tangle R for the reduced free pair of Figure B.5.

of D in clockwise direction. Since $4i+2$ and $4i+3$ are in the same connected component, the following point is exactly $\overline{4i+3}$. By the modification we made on $P * Q$, $\overline{4i+3}$ is connected by a curve to the point $\overline{4i+4}$. Therefore, in R_1, $4i+2$ is connected to $\overline{4i+4}$. Thus, relabelling the outer boundary point $4i+1$ by $2i+1$ and $4i+2$ by $2i+2$ in R_1 yields exactly the image of T in $T * T'$. This reconstruction of T is shown in Figure B.8.

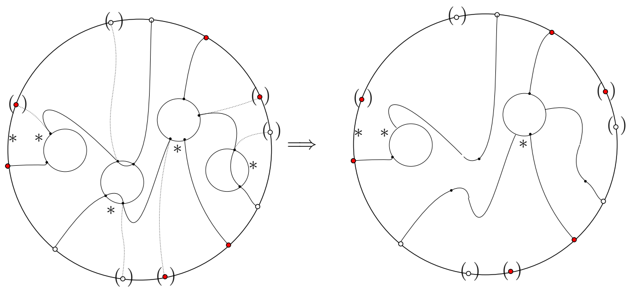

Figure B.8: Reconstruction of T from R.

Likewise, $R_2 = R_{D_1,\dots,D_r}(\tilde{X}_1, \dots, \tilde{X}_r)$ is equal to the image of T' in $T * T'$. $\square$

Proposition B.2.13. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. Then, $\{U_{\mathcal{P}}(k) \otimes \mathcal{Q}_k\}_{k\geq 1} \cup \{\mathcal{P}_k \otimes S_{\mathcal{Q}}(k)\}_{k\geq 1}$ is a generating subset of $\mathcal{P} * \mathcal{Q}$.*

Proof. First, note that (U_k, Id_k) and (Id_k, S_k) are two free pairs of planar tangles. Thus, for all $k \geq 1$, $U_{\mathcal{P}}(k) \otimes \mathcal{Q}_k$ and $\mathcal{P}_k \otimes S_{\mathcal{Q}}(k)$ are subspaces of

$(\mathcal{P} * \mathcal{Q})_k$. In particular the subplanar algebra generated by $\{U_{\mathcal{P}}(k) \otimes \mathcal{Q}_k\}_{k\geq 1} \cup \{\mathcal{P}_k \otimes S_{\mathcal{Q}}(k)\}_{k\geq 1}$ is also a subplanar algebra of $\mathcal{P} * \mathcal{Q}$.
Reciprocally, let (T, T') be a reduced free pair. By Lemma B.2.12, there exists a planar tangle R of degree k with r inner disks D_i of respective degree k_i, such that $T = R_{D_1,\ldots,D_r}(X_1, \ldots, X_r)$ and $T' = R_{D_1,\ldots,D_r}(\tilde{X}_1, \ldots, \tilde{X}_r)$, where for each $1 \leq i \leq r$, $(X_i, \tilde{X}_i)$ is either (U_{k_i}, Id_{k_i}) or (Id_{k_i}, S_{k_i}). Thus, the image $Z_T \otimes Z_{T'}$ of the reduced free pair (T, T') is equal to $Z_R(\chi_1, \ldots, \chi_r)$, where $\chi_i = Z_{X_i}(\mathcal{P}) \otimes Z_{\tilde{X}_i}(\mathcal{Q})$ is either equal to $U_{\mathcal{P}}(k_i) \otimes \mathcal{Q}_{k_i}$ or to $\mathcal{P}_{k_i} \otimes S_{\mathcal{Q}}(k_i)$. Since $\mathcal{P} * \mathcal{Q}$ is spanned by the images $Z_T \otimes Z_{T'}$ where (T, T') is any reduced free pair, the result follows. $\square$

Remark B.2.14. In [L02], the free product of two planar algebras $\mathcal{P} * \mathcal{Q}$ is directly defined as the subplanar algebra of $\mathcal{P} \otimes \mathcal{Q}$ generated by $\{U_{\mathcal{P}}(k) \otimes \mathcal{Q}_k\}_{k\geq 1}$ and $\{\mathcal{P}_k \otimes S_{\mathcal{Q}}(k)\}_{k\geq 1}$.

Conclusion

In this thesis, we have addressed several research questions in the fields of ergodic theory and operator algebras. First, we examined the question whether or not a given infinite discrete group G admits a nonsingular Bernoulli action of type III. In the specific case where $G = \mathbb{Z}$, the existence of a type III Bernoulli action $\mathbb{Z} \curvearrowright (X, \mu)$ was first proven by Hamachi [Ha81] in 1981. Much more recently, Hamachi's result was upgraded by Kosloff [Ko09] [Ko12] and Danilenko and Lemańczyk [DL16] but none of these findings left the realm of $\mathbb{Z}$-actions.

Our results in Chapter 2 constitute a significant improvement over all previous ones because of the following reasons:

- The methodology is entirely new and does not rely on the results of Hamachi, Kosloff et al. The connection between nonsingular ergodic theory and L^2-cohomology discussed in Chapter 2 is easy to formulate but has powerful consequences.
- We answer the problem uniformly for almost all discrete groups G.
- In the specific case where $G = \mathbb{Z}$, our methods yield new examples of type III Bernoulli actions that are very easy to write down explicitly. The examples of Hamachi et al. required an inductive construction of product measures and could not be written down in a closed formula.

The results obtained in the first part of this work also raise two follow-up question that we would like to state next.

1. First and foremost, all of our results point to the validity of the following conjecture.

   Conjecture. *An infinite discrete group G admits a nonsingular Bernoulli action of type III if and only if its first L^2-cohomology is nontrivial, i.e. $H^1(G, \ell^2(G)) \neq \{0\}$.*

Although the class of groups with $H^1(G, \ell^2(G)) \neq \{0\}$ that admit a Bernoulli action of type III is huge, a proof of the conjecture probably requires a genuinely new idea as we expect the methods used in Section 2.3 to be too ad-hoc to admit a generalization that would prove the conjecture.

2. Both the examples of Hamachi et al. and all of our examples of type III Bernoulli actions $G \curvearrowright \prod_{g \in G}(X_0, \mu_g)$ require the component measures μ_g to be identical on a large part of the group. In particular, when $G = \mathbb{Z}$, we always need to assume that there exists $m \in \mathbb{Z}$ such that $\mu_n = \mu_m$ for all $n \leq m$. When this requirement is removed, no results on the type of the Bernoulli action is known. However, in the recent article [Ko18], Kosloff proved that, under mild assumptions, conservative Bernoulli actions are automatically ergodic. His methods might be useful in order to arrive at conclusions on the type of these Bernoulli actions as well.

The second part of the thesis started with an examination of the analogues of the Fourier algebra, the Fourier-Stieltjes algebra and the algebra of completely bounded multipliers in the setting of rigid C^*-tensor categories. The results on the structures of these algebras are proven in a relatively straightforward way as the necessary tools were already in place thanks to the work of Popa and Vaes [PV15] and others [NY15a] [GJ16]. With these results we hope to have laid a solid groundwork for further research on abstract harmonic analysis for C^*-tensor categories. An interesting follow-up question is the following one: Does there exist an invariant mean on the algebra of completely bounded multipliers $M_0A(\mathcal{C})$? The existence of such a mean m would be a good basis for the study of property (T*) for rigid C^*-tensor categories which for locally compact groups, is defined in terms of a continuity property of m in [HKdL14]. Given a locally compact group G, the existence of a mean on $M_0A(G)$ follows from the existence of an invariant mean on the algebra of weakly almost periodic functions $\mathrm{WAP}(G)$ which contains $M_0A(G)$. Finding an analogue of $\mathrm{WAP}(G)$ in the categorial setting, might therefore be a good starting point for further research in this direction.

The next major result contained in part II of this thesis stated that q-deformations of connected compact simple Lie groups with trivial center have the Howe-Moore property. As pointed out in the introduction, for locally compact groups the Howe-Moore property has important implications in the form of geometric rigidity results. We hope for a similar success story in the categorical setting, although we admittedly have no concrete ideas of how to apply the Howe-Moore property in a fruitful manner at this moment. Nevertheless, we find that the strategy used in our proof of the Howe-Moore property is at the very least asthetically pleasing, as it reduces the proof to a statement that is phrased purely in terms of Lie group theory.

In the last chapter of this work, we investigated the relationship between the free wreath product of compact quantum groups and the free product of standard invariants of subfactors. Although the correspondence between these two operations enabled us to establish permanence properties of the free wreath product with respect to approximation properties, our results are far from hands on. In particular, the study of concrete examples such as $\mathbb{G} \wr_* O_n^+$ seems to involve highly advanced combinatorics.

In addition, even in the framework of Bichon's original definition, it is unknown whether the von Neumann algebra associated to a free wreath product has the same indecomposability properties as the free group factors. In particular, one could investigate whether these von Neumann algebras are *prime* (they can not be decomposed as a nontrivial tensor product) or *strongly solid* [OP07], which would imply that it is impossible to identify them with the von Neumann algebra of a Borel equivalence relation.

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- Free wreath product quantum groups and standard invariants of subfactors. With Pierre Tarrago. To appear in *Adv. in Math.* Preprint: arXiv:1609.01931 (2016).

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