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Inflationary Perturbations

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Abstract

Lectures on the general slow-roll, $\delta\mathcal{N}$ and δN formalisms for calculating the perturbations produced during inflation.

1 General slow-roll in single field inflation [1]

1.1 Perturbation equations

The evolution equation for

$$\varphi \equiv a\left(\delta\phi - \frac{\dot{\phi}}{H}\mathcal{R}\right) \tag{1}$$

is

$$\frac{d^2\varphi}{d\xi^2} + k^2\varphi - \left[\left(\frac{H}{a\dot{\phi}}\right)\frac{d^2}{d\xi^2}\left(\frac{a\dot{\phi}}{H}\right)\right]\varphi = 0 \tag{2}$$

where ξ is minus the conformal time

$$\xi \equiv -\eta = \int_t^\infty \frac{dt}{a} = \frac{1}{aH}\left[1 + \mathcal{O}\left(\frac{\dot{H}}{H^2}\right)\right] \tag{3}$$

Defining

$$f(\ln\xi) \equiv \frac{2\pi a\xi\dot{\phi}}{H} \simeq \frac{2\pi\dot{\phi}}{H^2} \tag{4}$$

we can rearrange the equation as

$$\frac{d^2\varphi}{d\xi^2} + k^2\varphi - \frac{2}{\xi^2}\varphi = \left(\frac{f'' - 3f'}{f}\right)\frac{1}{\xi^2}\varphi \tag{5}$$

where

$$f' \equiv \frac{df}{d\ln\xi} \simeq -\frac{1}{H}\frac{df}{dt} \tag{6}$$

1.2 General slow-roll approximation

Our equation is

$$\frac{d^2\varphi}{d\xi^2} + k^2\varphi - \frac{2}{\xi^2}\varphi = \left(\frac{f'' - 3f'}{f}\right)\frac{1}{\xi^2}\varphi \tag{7}$$

The homogeneous equation

$$\frac{d^2\varphi_0}{d\xi^2} + k^2\varphi_0 - \frac{2}{\xi^2}\varphi_0 = 0 \tag{8}$$

has solution

$$\varphi_0 = \frac{1}{\sqrt{2k}}\left(1 + \frac{i}{k\xi}\right)e^{ik\xi} \tag{9}$$

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which gives the scale invariant spectrum

$$P = \frac{1}{f^2} \simeq \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \tag{10}$$

Thus we see that

$$\frac{f'}{f}, \frac{f''}{f} \ll 1 \quad \leftrightarrow \quad \text{scale invariance} \tag{11}$$

However, unlike in standard slow-roll, we will not make any further assumptions

$$\frac{f''}{f} \not\ll \frac{f'}{f} \quad \leftrightarrow \quad \frac{dn}{d\ln k} \not\ll n-1 \tag{12}$$

1.3 General slow-roll formula

Solving

$$\frac{d^2\varphi}{d\xi^2} + k^2\varphi - \frac{2}{\xi^2}\varphi = \left(\frac{f'' - 3f'}{f}\right)\frac{1}{\xi^2}\varphi \tag{13}$$

perturbatively, and taking the late time limit $\xi \to 0$, we get

$$\ln P(k) = \int_0^\infty \frac{d\xi}{\xi}\left[-k\xi\, W'(k\xi)\right]\left(\ln\frac{1}{f^2} + \frac{2}{3}\frac{f'}{f}\right) \tag{14}$$

where ξ and $f(\ln\xi)$ were defined in Eqs. (3) and (4). The window function has the window property

$$\int_0^\infty \frac{dx}{x}\left[-x\, W'(x)\right] = 1 \tag{15}$$

and is generated from

$$W(x) = \frac{3\sin(2x)}{2x^3} - \frac{3\cos(2x)}{x^2} - \frac{3\sin(2x)}{2x} - 1 \tag{16}$$

which has the asymptotic behavior

$$\lim_{x\to 0} W(x) = \frac{2}{5}x^2 + \mathcal{O}(x^4) \tag{17}$$

1.4 Inverse

The general slow-roll formula for the spectrum

$$\ln P = \int_0^\infty \frac{d\xi}{\xi}\left[-k\xi\, W'(k\xi)\right]\left(\ln\frac{1}{f^2} + \frac{2}{3}\frac{f'}{f}\right) \tag{18}$$

can be inverted to give

$$\ln\frac{1}{f^2} = \int_0^\infty \frac{dk}{k}\, m(k\xi)\, \ln P \tag{19}$$

where

$$\frac{1}{f(\ln\xi)} \equiv \frac{H}{2\pi a\xi\dot{\phi}} \simeq \frac{H^2}{2\pi\dot{\phi}} \tag{20}$$

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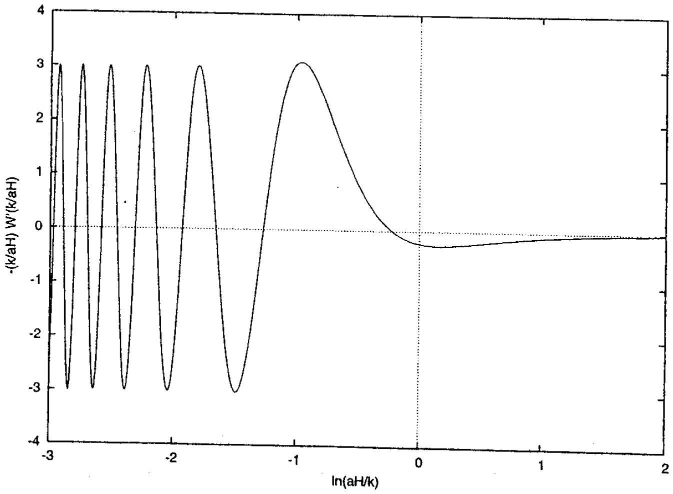

Figure 1: Window function $-\frac{k}{aH}W'\left(\frac{k}{aH}\right)$

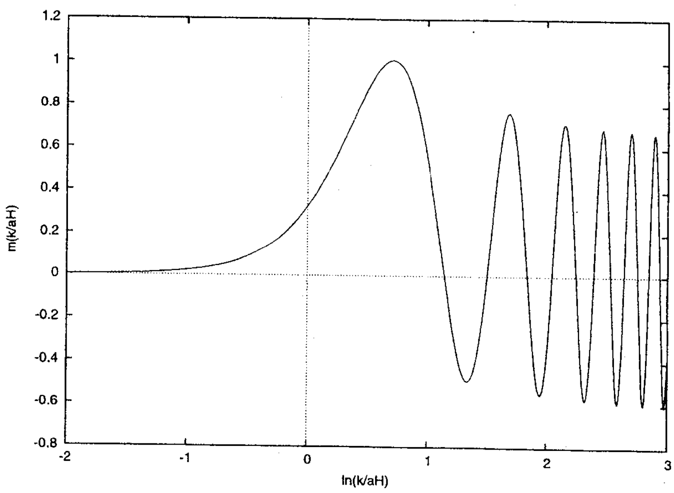

Figure 2: Inverse window function $m(k/aH)$

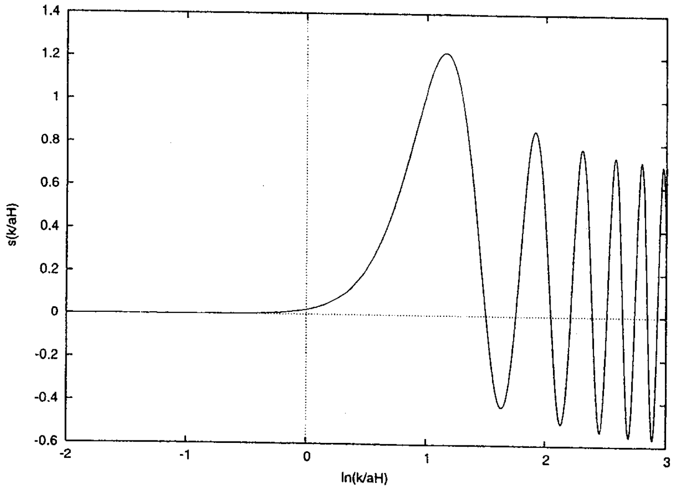

Figure 3: Inverse window function $s(k/aH)$

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The inverse window function

$$m(x) = \frac{2}{\pi}\left[\frac{1}{x} - \frac{\cos(2x)}{x} - \sin(2x)\right] \tag{21}$$

has the window property

$$\int_0^\infty \frac{dx}{x} m(x) = 1 \tag{22}$$

and the asymptotic behavior

$$\lim_{x\to 0} m(x) = \frac{4}{3\pi}x^3 + \mathcal{O}(x^5) \tag{23}$$

2 $\delta\mathcal{N}$ formalism for multi-field inflation [2]

2.1 N

The homogeneous background metric is

$$ds^2 = dt^2 - a(t)^2\delta_{ij}\, dx^i dx^j \tag{24}$$

We define the e-folding number as

$$N(t_{\rm fin}, t) \equiv \int_{t_{\rm fin}}^t H\, dt \tag{25}$$

where $H \equiv \dot{a}/a$ and $t_{\rm fin}$ is a late time when all trajectories have converged, i.e. a time after complete reheating when the curvature perturbation $\mathcal{R}_{\rm c}$ has become constant.

2.2 $\mathcal{N}$

We write the scalar part of the perturbed metric as

$$ds^2 = (1+2A)dt^2 - 2\partial_i B\, dt\, dx^i - a^2\left[(1+2\mathcal{R})\delta_{ij} + 2a^{-2}\partial_i\partial_j E\right] dx^i dx^j \tag{26}$$

The perturbed e-folding number is defined by

$$\mathcal{N}(t_{\rm fin}, t) \equiv \int_{t_{\rm fin}}^t \frac{1}{3}\theta\, d\tau \tag{27}$$

where τ is the proper time, $d\tau = (1+A)\, dt$, and θ is the volume expansion rate of the constant time hypersurfaces

$$\frac{1}{3}\theta = H\left(1 + \frac{1}{H}\dot{\mathcal{R}} - A - \frac{q^2}{3H}\mathcal{S}\right) \tag{28}$$

$q^2 = k^2/a^2$ is the physical wave number and

$$\mathcal{S} = \dot{E} - 2HE - B \tag{29}$$

is the shear of the unit vector normal to the constant time hypersurfaces. Therefore

$$\mathcal{N}(t_{\rm fin}, t) = \int_{t_{\rm fin}}^t H\left(1 + \frac{1}{H}\dot{\mathcal{R}} - A - \frac{q^2}{3H}\mathcal{S}\right)(1+A)\, dt \tag{30}$$

$$= N(t_{\rm fin}, t) + \mathcal{R}(t) - \mathcal{R}(t_{\rm fin}) - \frac{1}{3}\int_{t_{\rm fin}}^t q^2\mathcal{S}\, dt \tag{31}$$

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2.3 $\delta\mathcal{N}$

$$\delta\mathcal{N}(t_{\rm fin},t) \equiv \mathcal{N}(t_{\rm fin},t) - N(t_{\rm fin},t) \quad (32)$$

$$= \mathcal{R}(t) - \mathcal{R}(t_{\rm fin}) - \frac{1}{3}\int_{t_{\rm fin}}^{t} q^2\mathcal{S}\,dt \quad (33)$$

Taking the initial hypersurface to be flat and the final hypersurface to be comoving

$$\delta\mathcal{N}(t_{\rm fin},t_{\rm ini}) = -\mathcal{R}_{\rm c}(t_{\rm fin}) - \frac{1}{3}\int_{t_{\rm fin}}^{t_{\rm ini}} q^2\mathcal{S}\,dt \quad (34)$$

The Einstein equation gives

$$\dot{\mathcal{S}} + H\mathcal{S} = A + \mathcal{R} + \pi \quad (35)$$

Thus $q^2\mathcal{S}$ decays rapidly outside the horizon, and this decaying behavior is independent of the choice of gauge. So, taking the initial time $t_{\rm ini}$ sufficiently late, so that scales are sufficiently greater than the horizon, we get our central result

$$\mathcal{R}_{\rm c}(t_{\rm fin}) \simeq -\delta\mathcal{N}(t_{\rm fin},t_{\rm ini}) \quad (36)$$

This relation is valid irrespective of whether the background universe is dominated by scalar field or not.

3 A new δN formalism for multi-component inflation [3]

3.1 Scalar fields during inflation

We assume that for $t \leq t_{\rm ini}$, i.e. while modes are leaving the horizon during inflation, we can take the action to be

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}R + \frac{1}{2}h_{\rm ab}g^{\mu\nu}\partial_\mu\phi^{\rm a}\partial_\nu\phi^{\rm b} - V(\phi)\right] \quad (37)$$

and the scalar field perturbations on flat hypersurfaces satisfy

$$\delta\ddot{\phi}^{\rm a}_{\rm f} + 3H\delta\dot{\phi}^{\rm a}_{\rm f} - R^{\rm a}{}_{\rm bcd}\dot{\phi}^{\rm b}\dot{\phi}^{\rm c}\delta\phi^{\rm d}_{\rm f} + q^2\delta\phi^{\rm a}_{\rm f} + h^{\rm ab}V_{\phi^{\rm b}\phi^{\rm c}}\delta\phi^{\rm c}_{\rm f} = \frac{1}{a^3}\frac{D}{dt}\left(\frac{a^3\dot{\phi}^{\rm a}\dot{\phi}^{\rm b}}{H}\right)h_{\rm bc}\delta\phi^{\rm c}_{\rm f} \quad (38)$$

We do not require this effective description to continue to be valid for $t > t_{\rm ini}$. Indeed, we expect that in most cases it will break down some time before $t_{\rm fin}$.

For $t \leq t_{\rm ini}$, we represent N in phase space $(\phi,\dot{\phi})$ as

$$N(t_{\rm fin};\phi,\dot{\phi}) \equiv \int_{t_{\rm fin}}^{t(\phi,\dot{\phi})} H\,dt \quad (39)$$

where the integral is performed along the trajectory that passes through $(\phi,\dot{\phi})$.

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3.2 δN

We define

$$\delta N \equiv \mathbb{N}_{\phi^a}\delta\phi^a + \mathbb{N}_{\dot\phi^a}\delta\dot\phi^a \quad (40)$$

$$\equiv N_{\phi^a}\delta\phi^a + N_{\dot\phi^a}\left(\delta\dot\phi^a - \dot\phi^a A\right) + \frac{1}{3}q^2\left(a^3\int_{t_{fin}}^t \frac{dt}{a^3}\right)\mathcal{S}_f \quad (41)$$

Note that we may interpret

$$\delta\dot\phi^a - \dot\phi^a A = \delta\left(\frac{d\phi^a}{d\tau}\right) \quad (42)$$

$\mathbb{N}_{\phi^a}$ and $\mathbb{N}_{\dot\phi^a}$ are given explicitly by

$$\mathbb{N}_{\phi^a} = -a^3\frac{D}{dt}\left(\frac{\mathbb{N}_{\dot\phi^a}}{a^3}\right) \quad (43)$$

and

$$\mathbb{N}_{\dot\phi^a} = N_{\dot\phi^a} - \frac{1}{6}\left(a^3\int_{t_{fin}}^t \frac{dt}{a^3}\right)h_{ab}\frac{\dot\phi^b}{H} \quad (44)$$

Choosing flat hypersurfaces, we have

$$\delta N_f = \delta N - \mathcal{R} \quad (45)$$

3.3 Evolution of δN_f

Taking the time derivative of δN_f, we find

$$\delta\dot N_f = -q^2\mathbb{N}_{\dot\phi^a}\delta\phi_f^a \quad (46)$$

Thus δN_f is constant on super-horizon scales.

In the special case that our scalar field description is still valid at $t = t_{fin}$, i.e. if it is valid beyond the point where convergence of trajectories occurs, we can evaluate the constant by inserting the super-horizon adiabatic growing mode into the definition of δN to get

$$\delta N_f(t_{ini}) = \delta N_f(t_{fin}) = \left.\frac{H\dot\phi_a\delta\phi_f^a}{\dot\phi_b\dot\phi^b}\right|_{t=t_{fin}} = -\mathcal{R}_c(t_{fin}) = \delta\mathcal{N}(t_{fin}, t_{ini}) \quad (47)$$

Returning to the general case, and taking the derivative again, we get the equation of motion for δN_f

$$\delta\ddot N_f + 5H\,\delta\dot N_f + q^2\delta N_f = -2q^2\dot{\mathbb{N}}_{\dot\phi^a}\delta\phi_f^a \quad (48)$$

3.4 $\delta\phi_\parallel^a$ and $\delta\phi_\perp^a$

Our equations are

$$\delta\dot N_f = -q^2\mathbb{N}_{\dot\phi^a}\delta\phi_f^a \quad (49)$$

and

$$\delta\ddot N_f + 5H\,\delta\dot N_f + q^2\delta N_f = -2q^2\dot{\mathbb{N}}_{\dot\phi^a}\delta\phi_f^a \quad (50)$$

We decompose $\delta\phi_f^a$ into relevant and irrelevant components

$$\delta\phi_f^a = \delta\phi_\parallel^a + \delta\phi_\perp^a \quad (51)$$

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with the relevant direction defined by $\mathbb{N}_{\dot\phi^{\mathrm{a}}}$

$$\delta\phi^{\mathrm{a}}_{\parallel} \equiv P^{\mathrm{a}}_{\mathrm{b}}\,\delta\phi^{\mathrm{b}}_{\mathrm{f}} \quad , \quad \delta\phi^{\mathrm{a}}_{\perp} \equiv Q^{\mathrm{a}}_{\mathrm{b}}\,\delta\phi^{\mathrm{b}}_{\mathrm{f}} \tag{52}$$

where

$$P^{\mathrm{a}}_{\mathrm{b}} \equiv \frac{h^{\mathrm{ac}}\mathbb{N}_{\dot\phi^{\mathrm{c}}}\mathbb{N}_{\dot\phi^{\mathrm{b}}}}{h^{\mathrm{de}}\mathbb{N}_{\dot\phi^{\mathrm{d}}}\mathbb{N}_{\dot\phi^{\mathrm{e}}}} \quad , \quad Q^{\mathrm{a}}_{\mathrm{b}} \equiv \delta^{\mathrm{a}}_{\mathrm{b}} - P^{\mathrm{a}}_{\mathrm{b}} \tag{53}$$

so that $\delta\phi^{\mathrm{a}}_{\parallel}$ can be expressed entirely in terms of δN_{f}. Then

$$\delta\ddot N_{\mathrm{f}} + 5H\,\delta\dot N_{\mathrm{f}} + q^2\delta N_{\mathrm{f}} = 2\left(\frac{h^{\mathrm{ab}}\dot{\mathbb{N}}_{\dot\phi^{\mathrm{a}}}\mathbb{N}_{\dot\phi^{\mathrm{b}}}}{h^{\mathrm{cd}}\mathbb{N}_{\dot\phi^{\mathrm{c}}}\mathbb{N}_{\dot\phi^{\mathrm{d}}}}\right)\delta\dot N_{\mathrm{f}} - 2q^2\dot{\mathbb{N}}_{\dot\phi^{\mathrm{a}}}\delta\phi^{\mathrm{a}}_{\perp} \tag{54}$$

3.5 General slow-roll equation

Our equation is

$$\delta\ddot N_{\mathrm{f}} + 5H\,\delta\dot N_{\mathrm{f}} + q^2\delta N_{\mathrm{f}} = 2\left(\frac{h^{\mathrm{ab}}\dot{\mathbb{N}}_{\dot\phi^{\mathrm{a}}}\mathbb{N}_{\dot\phi^{\mathrm{b}}}}{h^{\mathrm{cd}}\mathbb{N}_{\dot\phi^{\mathrm{c}}}\mathbb{N}_{\dot\phi^{\mathrm{d}}}}\right)\delta\dot N_{\mathrm{f}} - 2q^2\dot{\mathbb{N}}_{\dot\phi^{\mathrm{a}}}\delta\phi^{\mathrm{a}}_{\perp} \tag{55}$$

Changing time variable to $x \equiv k\xi \simeq k/aH$, where $\xi = -\int dt/a$ is minus the conformal time, we obtain

$$\frac{d^2\delta N_{\mathrm{f}}}{dx^2} - \frac{4}{x}\frac{d\delta N_{\mathrm{f}}}{dx} + \delta N_{\mathrm{f}} = \frac{2}{x}\frac{\Pi'}{\Pi}\frac{d\delta N_{\mathrm{f}}}{dx} - 2\dot{\mathbb{N}}_{\dot\phi^{\mathrm{a}}}\delta\phi^{\mathrm{a}}_{\perp} \tag{56}$$

where

$$\Pi^2(\ln\xi) \equiv \left(\frac{3}{2\pi a^2\xi^2}\right)^2 h^{\mathrm{ab}}\mathbb{N}_{\dot\phi^{\mathrm{a}}}\mathbb{N}_{\dot\phi^{\mathrm{b}}} \simeq \left(\frac{3H^2}{2\pi}\right)^2 h^{\mathrm{ab}}\mathbb{N}_{\dot\phi^{\mathrm{a}}}\mathbb{N}_{\dot\phi^{\mathrm{b}}} \tag{57}$$

and

$$\Pi' \equiv \frac{d\Pi}{d\ln\xi} \simeq -\frac{1}{H}\frac{d\Pi}{dt} \tag{58}$$

Neglecting the right hand side of Eq. (56) gives the scale invariant spectrum

$$P = \Pi^2 \tag{59}$$

3.6 General slow-roll formula

The general slow-roll solution of

$$\delta N''_{\mathrm{f}} - \frac{4}{x}\delta N'_{\mathrm{f}} + \delta N_{\mathrm{f}} = \frac{2}{x}\frac{\Pi'}{\Pi}\delta N'_{\mathrm{f}} - 2\dot{\mathbb{N}}_{\dot\phi^{\mathrm{a}}}\delta\phi^{\mathrm{a}}_{\perp} \tag{60}$$

is

$$\ln P(k) = \int_0^\infty \frac{d\xi}{\xi}\left[-k\xi\, W'(k\xi)\right]\left(\ln\Pi^2 + \frac{2}{3}\frac{\Pi'}{\Pi}\right) \tag{61}$$

$$-\frac{2}{\pi}\sum_{i=1}^{D}\frac{\Pi_i}{\Pi^2}\int_0^\infty \frac{d\xi}{\xi}\, z_\perp(k\xi)\,\dot{\mathbb{N}}_{\dot\phi^{\mathrm{a}}}(\xi)\,\mathrm{Im}\left[\delta\phi^{\mathrm{a}}_{i\perp}(k,\xi)\right] \tag{62}$$

$$+\frac{1}{\pi^2\Pi^2}\sum_{i=1}^{D}\left|\int_0^\infty \frac{d\xi}{\xi}\, z_\perp(k\xi)\,\dot{\mathbb{N}}_{\dot\phi^{\mathrm{a}}}(\xi)\,\delta\phi^{\mathrm{a}}_{i\perp}(k,\xi)\right|^2 \tag{63}$$

where $W(x)$ was given in Eq. (16), Π was defined in Eq. (57), and

$$z_\perp(x) = \frac{9\sin x}{x^3} - \frac{9\cos x}{x^2} - \frac{3\sin x}{x} \tag{64}$$

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3.7 Inverse

If we neglect the $\delta\phi_{\perp}^{\mathrm{a}}$ terms, the formula for the spectrum reduces to

$$\ln P(k) = \int_0^\infty \frac{d\xi}{\xi} \left[-k\xi\, W'(k\xi)\right] \left(\ln \Pi^2 + \frac{2}{3}\frac{\Pi'}{\Pi}\right) \tag{65}$$

which can be inverted to give

$$\ln \Pi^2 = \int_0^\infty \frac{dk}{k}\, s(k\xi) \ln P \tag{66}$$

where

$$\Pi^2(\ln \xi) \equiv \left(\frac{3}{2\pi a^2 \xi^2}\right)^2 h^{\mathrm{ab}} \mathrm{N}_{\dot\phi^{\mathrm{a}}} \mathrm{N}_{\dot\phi^{\mathrm{b}}} \simeq \left(\frac{3H^2}{2\pi}\right)^2 h^{\mathrm{ab}} \mathrm{N}_{\dot\phi^{\mathrm{a}}} \mathrm{N}_{\dot\phi^{\mathrm{b}}} \tag{67}$$

The inverse window function

$$s(x) \equiv \frac{2}{\pi}\left[\frac{3}{x^3} - \frac{3\cos(2x)}{x^3} - \frac{6\sin(2x)}{x^2} + \frac{2}{x} + \frac{4\cos(2x)}{x} + \sin(2x)\right] \tag{68}$$

has the window property

$$\int_0^\infty \frac{dx}{x}\, s(x) = 1 \tag{69}$$

and the asymptotic behavior

$$\lim_{x\to 0} s(x) = \frac{4}{45\pi} x^5 + \mathcal{O}(x^7) \tag{70}$$

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