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A SIMPLE CONNECTION BETWEEN COVARIANT FEYNMAN FORMALISM AND TIME-ORDERED PERTURBATION THEORY IN THE INFINITE MOMENTUM FRAME*

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ABSTRACT

We introduce a parametrization of loop momenta which allows us to perform one of the Feynman integrations in a very transparent way. This leads to expressions which can easily be related to terms resulting from time-ordered perturbation theory in the infinite momentum frame. To exemplify our method we consider some simple Feynman integrals. As another example we discuss the covariant expressions of Landshoff, Polkinghorne, and Short for the scaling graph and the electromagnetic form factor. We indicate how to substitute the Sudakov parametrization in their work in order to simplify their discussion and to make comparisons with the work of Gunion, Brodsky, and Blankenbecler more convenient. Finally we derive an elegant form of the bound state Bethe-Salpeter equation in which one of the Feynman integrations is performed.

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I. INTRODUCTION

The infinite momentum frame in connection with time-ordered perturbation theory[1] was a very powerful tool in the discussion of numerous problems, e.g., of electromagnetic scaling,[2] of the question of fixed poles,[3] of quantum electrodynamical calculations,[4] of the eikonal approximation,[5] and recently of high energy fixed angle scattering in exclusive and inclusive reactions.[6] Whereas Lorentz covariance and the connection to the corresponding Feynman diagrams is easily established for low order diagrams — especially because the infinite momentum limit restricts the number of diagrams, the bookkeeping becomes complicated in higher orders. The covariance structure of nonperturbative insertions, e.g., Bethe-Salpeter type vertex functions obtained from an integral equation in the time-ordering formalism, is not obvious.

On the other hand successful attempts have been made to handle the type of problems enumerated above in a manifestly covariant way.[7, 10] Normally the results of these works are very similar to what one obtained in the previous formalism. It seems desirable to have a method which allows establishment of systematical relations between the two approaches.

Our method in this work will be to perform one of the Feynman loop integrations with a special choice of the loop momentum k. We propose the following parametrization of k:

$$k = \left(x\mathbb{P} + \frac{k^2+k_\perp^2}{2\mathbb{P}x},\ k_\perp,\ x\mathbb{P} \right) \tag{1}$$

with arbitrary x and $P \to \infty$. The loop integration then transforms as

$$\int d^4k = \int d^2k_\perp \int_{-\infty}^{+\infty} \frac{dx}{2|x|} \int_{-\infty}^{+\infty} dk^2 \tag{2}$$

The first three integrations look very similar to the integration variables of time-ordered perturbation theory in the infinite momentum frame. Indeed the k^2 integration can be easily done as a Cauchy contour integral closed with a semicircle at infinity picking up propagator poles. We end up with an integration range in x as in time-ordered perturbation theory if we parametrize the outer particle momenta in the usual way in the infinite momentum frame.

Actually all our calculations do not really depend on the infinite momentum limit $P \to \infty$. If we parametrize k as[8]

$$k = \left(x\mathbb{P} + \frac{k^2 + k_\perp^2}{4x\mathbb{P}},\ k_\perp,\ x\mathbb{P} - \frac{k^2 + k_\perp^2}{4x\mathbb{P}} \right) \tag{3}$$

and similarly the outer momenta (e.g., $p = (\mathbb{P} + m^2/4\mathbb{P},\ 0,\ \mathbb{P} - m^2/4\mathbb{P})$) one can have arbitrary $\mathbb{P}$ and the relation (2) is unchanged as all relations we are going to write down. In the following we will use the parametrization (1) and the usual form of outer momenta which saves us some writing. But we should always keep in the background of our mind that the calculation looks quite the same for arbitrary finite $\mathbb{P}$ (which is not a difficult point since our final expressions do not contain $\mathbb{P}$ anymore).

We exemplify the method in Section II in the case of the triangle graph corresponding to the electromagnetic form factor and for the crossed box graph which plays an important role in the work of Gunion, Brodsky and Blankenbecler (BBG).[6] We indicate how covariant vertex functions should be handled.

In Section III we demonstrate that the discussion of Landshoff, Polkinghorne and Short (LPS) of the scaling graph and the electromagnetic form factor in a covariant formalism can be considerably simplified with the new parametrization substituting their choice of Sudakov variables. Again a comparison with infinite momentum calculations is very transparent.

In Section IV we formulate a new form of composite state Bethe-Salpeter equation. With a suitable choice of outer particle momenta we are able to perform one of the loop integrations. The resulting equation for a spin 0 bound state wave function has a very elegant form. We can check that the known solutions of the Bethe-Salpeter equation fulfill this equation.

We restrict to spin 0 intermediate states in all examples. The application of our method to graphs with, e.g., spin 1/2, is straightforward in principle though technically more involved.[9]

II. EXAMPLES FOR THE INTEGRATION PRESCRIPTION

As our first example we calculate the electromagnetic form factor graph of Fig. 1 for scalar particles. With

$$p = \left(\mathbb{P}+\frac{m^2}{2\mathbb{P}},\ 0,\ \mathbb{P}\right) ,$$

$$q = \left(\frac{q_\perp^2}{2\mathbb{P}},\ q_\perp,\ 0\right) \tag{4}$$

and k as given in (1), we get for the large 0- or 3-components according to Feynman rules

$$2\mathbb{P}\ F(q^2) = \frac{i}{(2\pi)^4} \int d^2k_\perp\ \frac{dx}{2|x|}\ dk^2\ 2\mathbb{P}x$$

$$\left\{k^2-m_1^2+i\epsilon\right\}^{-1} \left\{x\left(q_\perp^2 + \frac{k^2+k_\perp^2}{x}\right) - (k_\perp+q_\perp)^2 - m_1^2+i\epsilon\right\}^{-1}$$

$$\times\left\{(1-x)\left(m^2 - \frac{k^2+k_\perp^2}{x}\right) - k_\perp^2 - m_2^2+i\epsilon\right\}^{-1} \tag{4a}$$

where the first two denominators correspond to the propagators in k^2 and $(k+q)^2$ and represent poles in k^2 in the lower k^2 half plane, whereas the third one

— connected to the pole in $(p-k)^2$ — leads to a k^2 pole in the upper or lower half plane depending if $(1-x)/x$ is positive or negative. In the second case we can close the integration contour in the upper half plane pushing a semicircle to infinity and we end up with zero. Thus x is restricted to the interval $0 \leq x \leq 1$. If we close the contour in this case we can do it in the upper or lower half plane picking up one pole or two poles respectively. Both expressions of course have to be equal. The result is

$$F(q^2) = \frac{1}{(2\pi)^3} \int d^2k_\perp \int_0^1 \frac{dx}{x(1-x)} \frac{x(1-x)}{m^2\, x(1-x) - k_\perp^2 - m_2^2 x - m_1^2(1-x)}$$

$$\frac{x(1-x)}{m^2\, x(1-x) - \left(k_\perp - (1-x)\, q_\perp\right)^2 - m_1^2(1-x) - m_2^2 x} \tag{5}$$

This exactly agrees with the form given by time-ordered perturbation theory in the infinite momentum frame.

The rearrangement graph of Fig. 2, discussed extensively in the work of BBG, where the variables are parametrized as

$$\begin{aligned}
p_1 &= \left(\mathbb{P} + \frac{m^2}{2\,\mathbb{P}},\ \vec{0},\ \mathbb{P}\right) \\
p_2 &= \left(\mathbb{P} + \frac{m^2 + r_\perp^2 + q_\perp^2}{2\mathbb{P}},\ r_\perp + q_\perp,\ \mathbb{P}\right) \\
p_3 &= \left(\mathbb{P} + \frac{m^2 + q_\perp^2}{2\mathbb{P}},\ q_\perp,\ \mathbb{P}\right) \\
p_4 &= \left(\mathbb{P} + \frac{m^2 + r_\perp^2}{2\mathbb{P}},\ r_\perp,\ \mathbb{P}\right)
\end{aligned} \tag{6}$$

with $t=-q_\perp^2$, $u=-r_\perp^2$, $r_\perp \cdot q_\perp = 0$, can be written as

$$\frac{i}{(2\pi)^4}\int d^4k \frac{1}{k^2-m_1^2+i\epsilon}\,\frac{1}{(k-p_1)^2-m_2^2+i\epsilon}\,\frac{1}{(k+p_3-p_1)^2-m_1^2+i\epsilon}\,\frac{1}{(k-p_4)^2-m_2^2+i\epsilon}$$

$$= \frac{i}{(2\pi)^4}\int d^2k_\perp \int_{-\infty}^{+\infty} \frac{dx}{2|x|}\, dk^2 \left\{k^2-m_1^2+i\epsilon\right\}^{-1} \left\{\frac{1}{x}\left(m^2x(1-x)-k_\perp^2-k^2(1-x)\right)-m_2^2+i\epsilon\right\}^{-1}$$

$$\left\{k^2-(1-x)q_\perp^2-2k_\perp\cdot q_\perp-m_1^2+i\epsilon\right\}^{-1} \left\{\frac{1}{x}\left(m^2x(1-x)-k_\perp^2-k^2(1-x)-x^2r_\perp^2+2r_\perp\cdot k_\perp x\right)-m_2^2+i\epsilon\right\}^{-1} \tag{6a}$$

There are two pairs of poles in k^2 now, one pair corresponding to the propagators in k^2 and $(k+p_3-p_1)^2$ and always in the lower half plane and another pair which is in the upper half plane for $0 \le x \le 1$. For x outside this region we again can close the integration contour in the upper half plane without picking up a pole and get zero. Closing in the upper or lower half plane for $0 \le x \le 1$ we pick up a pair of poles and obtain the result

$$\frac{1}{(2\pi)^3}\int d^2k_\perp \int_0^1 \frac{dx}{2x^2(1-x)^2}\, \frac{2m^2 - S\left(x,k_\perp+(1-x)q_\perp\right)-S(x,k_\perp-xr_\perp)}{\left(m^2-S(x,k_\perp)\right)\left(m^2-S\left(x,k_\perp+(1-x)q_\perp\right)\right)\left(m^2-S(x,k_\perp-xr_\perp)\right)}$$

$$\times\left(m^2-S\left(x,k_\perp+(1-x)q_\perp-xr_\perp\right)\right) \tag{7}$$

with

$$S(x,k_\perp) = \frac{k_\perp^2+m_1^2(1-x)+m_2^2x}{x(1-x)}$$

This is the same expression one calculates as the sum of four time orderings in time-ordered perturbation theory.

We remark that $\mathbb{P}$ does not appear anymore in our final expressions (5) and (7). As discussed in the introduction we could have equally well calculated with finite $\mathbb{P}$ and the parametrization (3) and with analogous changes in Eqs. (4) and (6).

Up to now we did not introduce vertex functions ϕ. In a covariant calculation one would introduce Bethe-Salpeter vertex functions $\phi(k^2, (p-k)^2)$ dependent on the off shell masses of the parton constituents. Performing the k^2 integration in (4a) or (6a) one has to take into account also singularities in k^2 of ϕ. This can be done by an DGS type ansatz for ϕ.[11, 12] Then, however, the integration becomes rather complicated — e.g., in our second example one has to introduce four ϕ representations (though the calculation in some cases of interest simplifies if one takes the relevant asymptotic ϕ behavior for large off shell masses outside the integral).[13] A different (but related) possibility would be to use the ansatz $\phi(\kappa^2)$ of Drell and Lee[10] where

$$\kappa^2 = \left(\frac{m_2 k - m_1(p-k)}{m_1 + m_2}\right)^2 = \eta k^2 + (1-\eta)(p-k)^2 - m^2\eta(1-\eta)$$

$(\gamma = m_2/m_1 + m_2)$ is the continued c.m.s. momentum square.

If one neglects the k^2 singularities inside ϕ, the result of the analog of the calculations given above depends on if we close the integration contour in k^2 above or below the real axis — which of course is very bad.

From the relation

$$\frac{k^2 + k_\perp^2}{x} + \frac{(p-k)^2 + k_\perp^2}{1-x} = m^2 \tag{8}$$

we see that fixing $k^2 = m_1^2$ or $(p-k)^2 = m_2^2$ leads to

$$(p-k)^2 = \frac{m^2 x(1-x) - k_\perp^2 - m_1^2(1-x)}{x} \quad \text{or} \quad k^2 = \frac{m^2 x(1-x) - k_\perp^2 - m_2^2 x}{1-x}$$

respectively. This is very similar to the arguments of ϕ functions in the formalism of BBG, differing by a factor 1-x and x only. It indicates that a detailed comparison of both methods will have to deal especially with the factors x and (1-x) appearing in both approaches.

III. SCALING GRAPH AND ELECTROMAGNETIC FORM FACTOR IN THE EXPLICIT COVARIANT FORMULATION OF LPS

Landshoff, Polkinghorne, and Short[7] have derived scaling from the graph of Fig. 3. The loop there is formed by partons.

LPS make the following ansatz

$$T^{-}_{\mu\nu} = \frac{i}{(2\pi)^4} \int d^4k \; \Delta_F\left((k+q)^2 + i\epsilon\right) \; (2k+q)_\mu \; (2k+q)_\nu \; T_-(s'+i\epsilon, k^2+i\epsilon) \tag{9}$$

where Δ_F is the parton propagator $\Delta_F\left((k+q)^2\right) = \int \frac{\rho(\sigma)\, d\sigma}{(k+q)^2 - \sigma}$ with $\int \rho(\sigma)\, d\sigma = 1$ and where $T_-(s', k^2)$ is the untruncated parton-particle amplitude depending on the parton mass k^2, by assumption strongly peaked at small k^2, and the energy variable $s' = (p-k)^2$. With

$$p = \left(P + \frac{m^2}{2P},\ 0,\ P\right)$$

$$q = \left(\frac{2\nu}{2P},\ q_\perp,\ 0\right)$$

where

$$\nu = \frac{s - m^2 + q_\perp^2}{2} \quad \text{and} \quad s = (p+q)^2 \quad ,$$

and with k as given by Eq. (1) we deduce

$$\begin{aligned} s' &= (p-k)^2 = (1-x)\left(m^2 - \frac{k^2}{x}\right) - \frac{k_\perp^2}{x} \\ u' &= (p+k)^2 = (1+x)\left(m^2 + \frac{k^2}{x}\right) + \frac{k_\perp^2}{x} \\ \nu' &= \frac{s'-u'}{2} = -\frac{k^2 + k_\perp^2}{x} - m^2 x \\ (k+q)^2 &= 2x\nu - q_\perp^2 + \left(k^2 - 2k_\perp \cdot q_\perp\right) \quad , \end{aligned} \tag{10}$$

We can now perform the k^2 integration with our parametrization of k instead of the Sudakov parameters in LPS. The singularities in k^2 and $(k+q)^2$ are in the lower k^2 half plane. For $0 \leq x \leq 1$ and $-1 \leq x \leq 0$ the s' and u' cuts in T_- respectively are in the upper half plane in k^2 and we have to take the contour around them, giving an integral over this absorptive part of T_-, if we want to close the integration contour with an semicircle in the upper half plane.

In the scaling limit $\nu \to \infty$, $q^2 = -q_\perp^2 \to -\infty$, $2\nu/-q^2 = \omega$ fixed and for $|k^2| = |\mu^2| \ll q_\perp^2$ we have $(k+q)^2 \sim xq_\perp^2(\omega - x^{-1})$ and we can take it out of the integral in σ and integrate over σ. For the component T_{00} we obtain in this limit

$$T_{00}^- = \mathbb{P}^2 T_2^- = \frac{-2}{(2\pi)^4} \int d^2k_\perp \int \frac{dx}{2|x|} \, d\mu^2 \, \frac{4\mathbb{P}^2 x^2}{xq^2(\omega - x^{-1}) + i\epsilon}$$

$$\left\{ \theta(x)\, \theta(1-x)\, \mathrm{Im}\, T_-^R \left(s' = (1-x)\left(m^2 - \frac{\mu^2}{x}\right) - \frac{k_\perp^2}{x},\ \mu^2 \right) \right. \tag{11}$$

$$\left. + \, \theta(-x)\, \theta(1+x)\, \mathrm{Im}\, T_-^L \left(u' = (1+x)\left(m^2 - \frac{\mu^2}{x}\right) + \frac{k_\perp^2}{x},\ \mu^2 \right) \right\}.$$

Here the μ^2 integration is constrained by $s' \geq s_0$ for T^R and by $u' \geq u_0$ for T^L. The imaginary part W_2^- is given by substituting a δ-function for the denominator $(\omega - x^{-1})^{-1}$

$$\nu W_2^- = \frac{2\nu}{q_\perp^2} \frac{1}{(2\pi)^3} \int d^2k_\perp \, dx \, d\mu^2 x^2 \, \delta(\omega^{-1} - x)$$

$$\left\{ \theta(x)\, \theta(1-x)\, \mathrm{Im}\, T_-^R(s', \mu^2) + \theta(-x)\, \theta(1+x)\, \mathrm{Im}\, T_-^L(u', \mu^2) \right\} \tag{12}$$

$$= \frac{\omega^{-1}}{(2\pi)^3} \int d^2k_\perp \, d\mu^2 \left\{ \mathrm{Im}\, T_-^R(s', \mu^2)\, \theta(\omega - 1) + \mathrm{Im}\, T_-^L(u', \mu^2)\, \theta(-\omega - 1) \right\}$$

In taking out of the integration the denominator in (11) we had to assume that the μ^2 integration is rapidly convergent. The $k_\perp$ integration is limited because of relations (10) and $s' \geq s_0$, $u' \geq u_0$.

For $x \to 0$, i.e., large ω, ν' becomes large, and with the assumption

$$\mathrm{Im}\, T_-^R(s', \mu^2) \sim f(\mu^2)\, \nu'^{\alpha} \tag{13}$$

for large ν', we can evaluate Eq. (12) as

$$F_2^-(\omega) = \nu W_2^- \approx \frac{\omega^{\alpha - 1}}{(2\pi)^3} \int_C d^2k_\perp\, d\mu^2\, f(\mu^2) \left(-\mu^2 - k_\perp^2\right)^{\alpha} \tag{14}$$

where C refers to the constraint

$$-\left(\mu^2 + k_\perp^2\right) \geq \frac{\nu_0}{\omega} .$$

For $x \to 1$, i.e., $\omega \to 1$, the condition $s' \geq s_0$ results in $\mu^2 \leq \frac{-1}{1-x} s_0$. We therefore have large $|\mu^2|$. In order not to invalidate the derivation of (11) we have to fulfill $|\mu|^2 \ll q_\perp^2$, on the other hand, i.e., we have to increase $q_\perp^2$ to get scaling.

With the ansatz

$$\mathrm{Im}\, T_-^R(s', \mu^2) \sim g(s') \left(\frac{-\mu^2}{\mu_0^2}\right)^{-\gamma} \tag{15}$$

for large $|\mu|^2$, we derive ($s' \geq s_0$)

$$\nu W_2^- \approx \frac{(\omega - 1)^{\gamma - 1}}{(2\pi)^3} \int d^2k_\perp\, ds'\, g(s') \left(\frac{s' + k_\perp^2}{\mu_0^2}\right)^{-\gamma} . \tag{16}$$

The form factor can be written in an analogous way. According to Fig. 4 we have

$$F(q^2) = \frac{2}{(2\pi)^4} \int d^2k_\perp \, dx \, d\mu^2 \Big\{ \theta(x) \, \theta(1-x)$$

$$\mathrm{Im}\, T_-^R\left(s', \mu^2, \tilde{\mu}^2 = x \left(\frac{\mu^2 + k_\perp^2}{x} + q_\perp^2\right) - \left(k_\perp + q_\perp\right)^2, \; t = -q_\perp^2\right)$$

$$- \theta(-x) \, \theta(1+x) \, \mathrm{Im}\, T_-^L\left(u', \mu^2, \tilde{\mu}^2, \; t = -q_\perp^2\right)\Big\} \tag{17}$$

where T_- is now the parton particle amplitude with parton masses μ^2 and $\tilde{\mu}^2 = (k+q)^2$ and momentum transfer q^2, and where the integration range again is given by the cuts in s' and u'.

For $q^2 = 0$ we can use the crossing relation $\mathrm{Im}\, T_-^L(u') = \mathrm{Im}\, T_+^R(s')$ and write

$$F(0) = \int_0^1 dx \left\{\phi_-(x) - \phi_+(x)\right\} \tag{18}$$

with

$$\pi x \, \phi_\mp(x) = \nu W_{2\mp}(x = \omega^{-1}) \quad .$$

Here T_+ is the antiparton-particle amplitude appearing in the crossed graph analog of Eq. (11) and νW_{2+} the corresponding structure function.

For large q^2 the Regge behavior in ν' near x=0 analog to Eq. (13) with a dependence $\left\{-\left(\mu^2 + k_\perp^2\right)/x\right\}^{\alpha(-q_\perp^2)}$ or some reasonable large angle behavior for very large $q_\perp^2$) suppresses strongly a contribution of this region to the integral (17). With an ansatz

$$\mathrm{Im}\, T_-(s', \mu^2, \tilde{\mu}^2, t) \sim g(s', t, \mu^2) \left(\frac{-\tilde{\mu}^2}{\mu_0^2}\right)^{-\gamma/2} \tag{19}$$

which is a natural generalization of ansatz (15), we obtain

$$F(q^2) \sim \left(q_\perp^2\right)^{-\gamma/2} \int d\mu^2 \int d^2k_\perp \int_0^{1-\frac{|\mu^2|}{q_\perp^2}} dx\, g\left(s', t=-q_\perp^2, \mu^2\right) \left(\frac{1-x}{\mu_0^2}\right)^{-\gamma/2} . \quad (20)$$

In Eq. (20) the contribution near $x=1$ should be suppressed, e.g., by a $(\mu^2)^{-\gamma/2}$ behavior for large μ^2. Assuming further the independence of g on t in leading order in the interesting region of small s' the comparison of (16) and (20) leads us to the Drell-Yan relation between the power behavior of $F(q^2)$ (up to eventual logarithmic factors) for large q^2 and the behavior of $F_2(\omega)$ for $\omega \to 1$.

Our discussion of the form factor differs from the one given by LPS since we argue that the Regge behavior is not important in explaining the limiting behavior for $q^2 \to -\infty$.[17] The limit of large μ^2 is the same which is responsible for $q^2 \to -\infty$ in composite particle models where

$$T_- = \Psi(k^2, s')\, \Psi\left((k+q)^2, s'\right) (s'-\mu^2) \quad . \quad (21)$$

IV. DERIVATION OF A NEW FORM OF THE BETHE-SALPETER EQUATION

The most direct approach in modifying the bound state Bethe-Salpeter equation given graphically in Fig. 5 with $p^2 = m^2$ as bound state mass would be to perform the k^2 integration in the k-loop on the r.h.s. of the equation. However in general one then has to encounter singularities of the unknown function and thus runs into complications. In some cases of physical interest one can handle the dependence on one of the constituent of shell masses in a simplified way by truncating the kernel as done by Brodsky.[14] In this kind of approach one introduces an asymmetry in the constituent masses which is easiest seen from Eq. (8) fixing k^2 or $(p-k)^2$ respectively.

Let us consider a different approach. We look at the graph, of Fig. 5 from a different side and in a different notation (Fig. 6). Our procedure will be to write the equation first for negative q^2 and then to continue it to positive bound state masses. Though the actual bound state q^2 of course has some positive value m^2 it is totally permissible to formulate the equation for negative q^2. Another equivalent way to arrive at our equation would be to write first the Bethe-Salpeter equation for the scattering amplitude T of Fig. 7 at negative q^2, to continue this equation to $q^2 > 0$ and to factor out the bound state equation afterwards. Let us now write down the equation corresponding to Fig. 6. With the choice

$$(p \pm q/2) = \left(\mathbb{P} + \frac{q_\perp^2/4 + (p\pm q/2)^2}{2\mathbb{P}} \, , \, \pm \frac{q_\perp}{2} \, , \, \mathbb{P} \right)$$

$$q = \left(\frac{(p+q/2)^2 - (p-q/2)^2}{2\mathbb{P}} \, , \, q_\perp \, , \, 0 \right)$$

$$p = \left(\mathbb{P} + \frac{q_\perp^2/4 + \frac{(p+q/2)^2 + (p-q/2)^2}{2}}{2\mathbb{P}} \, , \, 0, \, \mathbb{P} \right)$$

the equation for a scalar bound state wave function of scalar constituents reads

$$\phi\left((p+q/2)^2, (p-q/2)^2, q^2\right) = \frac{i\lambda}{(2\pi)^4} \int d^2k_\perp \, \frac{dx}{2|x|} \, dk^2 \, \frac{1}{(k+q/2)^2 - m_1^2 + i\epsilon}$$

$$\frac{1}{(k-q/2)^2 - m_2^2 + i\epsilon} \, \frac{1}{(p-k)^2 - m_3^2 + i\epsilon} \, \phi\left((k+q/2)^2, (k-q/2)^2, q^2\right) \tag{22}$$

where

$$(p-k)^2 = (1-x) \left(\frac{q_\perp^2}{4} + \frac{(p+q/2)^2 + (p-q/2)^2}{2} - \frac{k^2+k_\perp^2}{x} \right) - k_\perp^2$$

$$= (1-x) \left(p^2 - \frac{k^2+k_\perp^2}{x} \right) - k_\perp^2$$

$$(k \pm q/2)^2 = x \left(\frac{k^2+k_\perp^2}{x} \pm \frac{(p+q/2)^2 - (p-q/2)^2}{2} \right) - \left(k_\perp \pm \frac{q_\perp}{2} \right)^2 \quad . \tag{22a}$$

From (22) and (22a) one can conclude that the singularity in p^2 of the left-hand ϕ and hence consistently the singularity in k^2 of the $\phi\left((k+q/2)^2, (k-q/2)^2, q^2\right)$ inside the integral is in the lower half plane if $x \leq 1$. In this case however we have $0 \leq x \leq 1$ because otherwise all singularities are in the lower half k^2 plane and we could close the integration contour with a semicircle in the upper half plane without picking up a singularity. This proves consistency for a solution where the x-integration is in the range $0 \leq x \leq 1$ and where we can close the integration contour around one single pole given by the "potential" $1/(p-k)^2 - m_3^2 + i\epsilon$. Equation (22) then results in

$$\phi\left((p+q/2)^2, (p-q/2)^2, q^2\right) = \frac{\lambda}{(2\pi)^3} \int_0^1 \frac{dx}{2(1-x)} \int d^2k_\perp$$

$$\left\{ \frac{1}{1-x} \left((p+q/2)^2 \; x(1-x) - \left(k_\perp + (1-x) \frac{q_\perp}{2} \right)^2 - m_3^2 x - m_1^2 (1-x) \right) \right\}^{-1}$$

$$\left\{ q \rightarrow -q, \; m_1 \rightarrow m_2 \right\}^{-1}$$

$$\phi \left\{ \frac{1}{1-x} \left((p+q/2)^2 \; x(1-x) - \left(k_\perp + (1-x) \frac{q_\perp}{2} \right)^2 - m_3^2 x \right); \; q \rightarrow -q, \; m_1 \rightarrow m_2 \right\} \tag{23}$$

We now continue Eq. (23) in $q_\perp$:

$$q_\perp \to i q_\perp \tag{24}$$

thus arriving in $k_\perp$ region of positive $q^2 = q_\perp^2 = m^2$. A simultaneous rotation of the integration path in $k_\perp$

$$k_\perp \to i k_\perp \tag{25}$$

keeps real arguments of ϕ in the integrand. With the replacement of ϕ by a wave function ψ defined by

$$\psi\left\{(p+q/2)^2, (p-q/2)^2, m^2\right\} = \frac{\phi\left\{(p+q/2)^2, (p-q/2)^2, m^2\right\}}{\left\{(p+q/2)^2-m_1^2\right\}\left\{(p-q/2)^2-m_2^2\right\}} \tag{26}$$

we obtain the final equation

$$\begin{aligned}
&\left\{(p+q/2)^2-m_1^2\right\}\left\{(p-q/2)^2-m_2^2\right\} \psi\left\{(p+q/2)^2, (p-q/2)^2, m^2\right\} \\
&\quad = \frac{-\lambda}{(2\pi)^3} \int_0^1 \frac{dx}{2(1-x)} \int d^2k_\perp \, \psi\left(\frac{1}{1-x}\left((p+q/2)^2 x(1-x) + \left(k_\perp + (1-x)\frac{q_\perp}{2}\right)^2 - m_3^2 x\right), \right. \\
&\quad\quad \left. q \to -q, \; m_1 \to m_2 \right)
\end{aligned} \tag{27}$$

with $q_\perp^2 = m^2$ as the bound state $(\text{mass})^2$. Singularities in the integrand are passed with obvious $i\epsilon$ prescriptions. We remark that a potential $\int d\mu^2 \rho(\mu^2)/(p-k)^2 - \mu^2 + i\epsilon$ would allow a very similar derivation. The corresponding generalization of Eq. (27) would contain another integration in μ^2.

Equation (27) looks quite different from the usual form of the Bethe-Salpeter equation. It does not contain the ladder potential explicitly anymore and involves only three integrations. We can check that the known solutions of the spin zero bound state Bethe-Salpeter equation fulfill (27). In the case of vanishing mass of the exchanged particle m_3 exact Wick[15] solutions are known. For the bound state mass $m^2=0$ and $m_1^2=m_2^2=\mu^2$, we have the equation

$$\left(\Pi^2-\mu^2\right)^2 \psi(\pi^2) = \frac{-\lambda}{(2\pi)^3} \int_0^1 \frac{dx}{2(1-x)} \int d^2k_\perp \; \psi\left(\frac{\Pi^2 x(1-x)+k_\perp^2}{1-x}\right) \tag{28}$$

with

$$(p\pm q/2)^2 = \Pi^2$$

which has the solution

$$\psi(\pi^2) \sim \left(\Pi^2-\mu^2\right)^{-3} \tag{29}$$

easily checked by integration together with the eigenvalue equation

$$\frac{\lambda}{(2\pi)^3} = \frac{4\mu^2}{\pi} \quad . \tag{30}$$

In the case $m^2 \neq 0$ an ansatz

$$\psi \sim \int dx \int_0^1 dy \frac{g(\alpha, y)}{\left(y(p+q/2)^2 + (1-y)(p-q/2)^2 - \alpha\right)^3} \tag{31}$$

leads to the one variable integral equation for $g(\alpha,y) = \delta(\alpha-\mu^2)\, g(y)$ first derived by Wick.[15] The variable x seems to play the role of a Feynman parameter.

We have also checked that in the more general case of exchange particle mass $m_3 \neq 0$ the ansatz (31) leads to the two variable integral equation for $g(\alpha, y)$ given by Wanders.[16]

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8. This remark is due to S. Brodsky. After completing this work we realized that the approach of Chang and Ma, Phys. Rev. 180, 1506 (1968) is very similar in spirit though their parametrization with light cone variables $k_0 \pm k_3$ is somewhat different. We thank Ling Fong Li for drawing our attention to this paper.

9. Spin effects and complicated Z graph contributions in the time-ordered perturbation theory can be discussed in a very effective manner with our method. For a technically more involved field of applications see a recent paper by S. J. Brodsky, R. Z. Roskies, R. Suaya (PITT 108, May 1973). In an appendix they also discuss methods in case of the appearance of divergent integrals.

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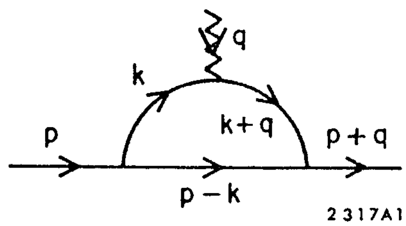

FIG. 1

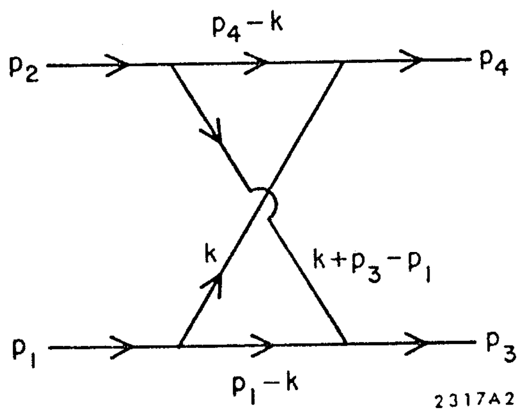

FIG. 2

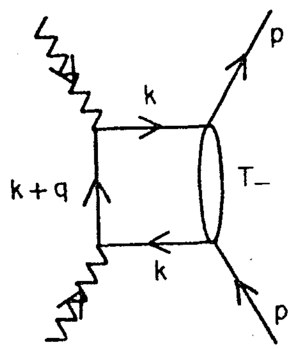

FIG. 3

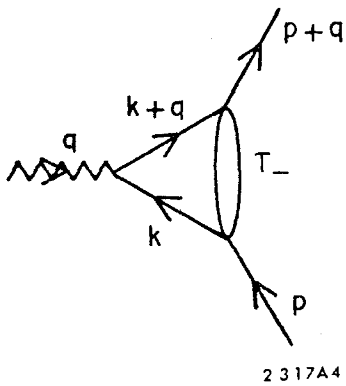

FIG. 4

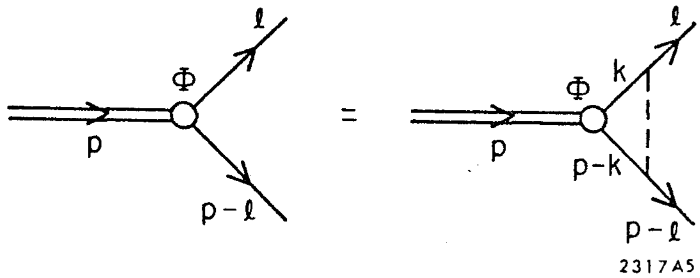

FIG. 5

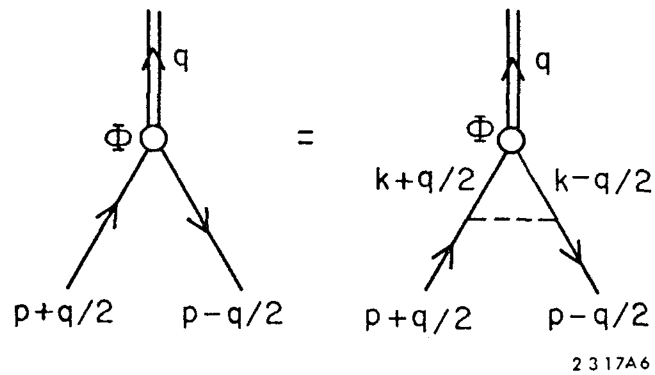

FIG. 6

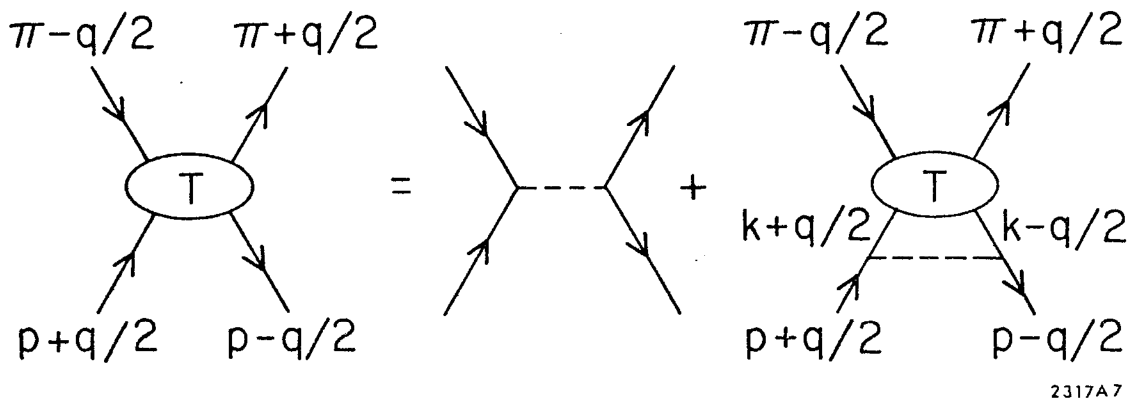

FIG. 7