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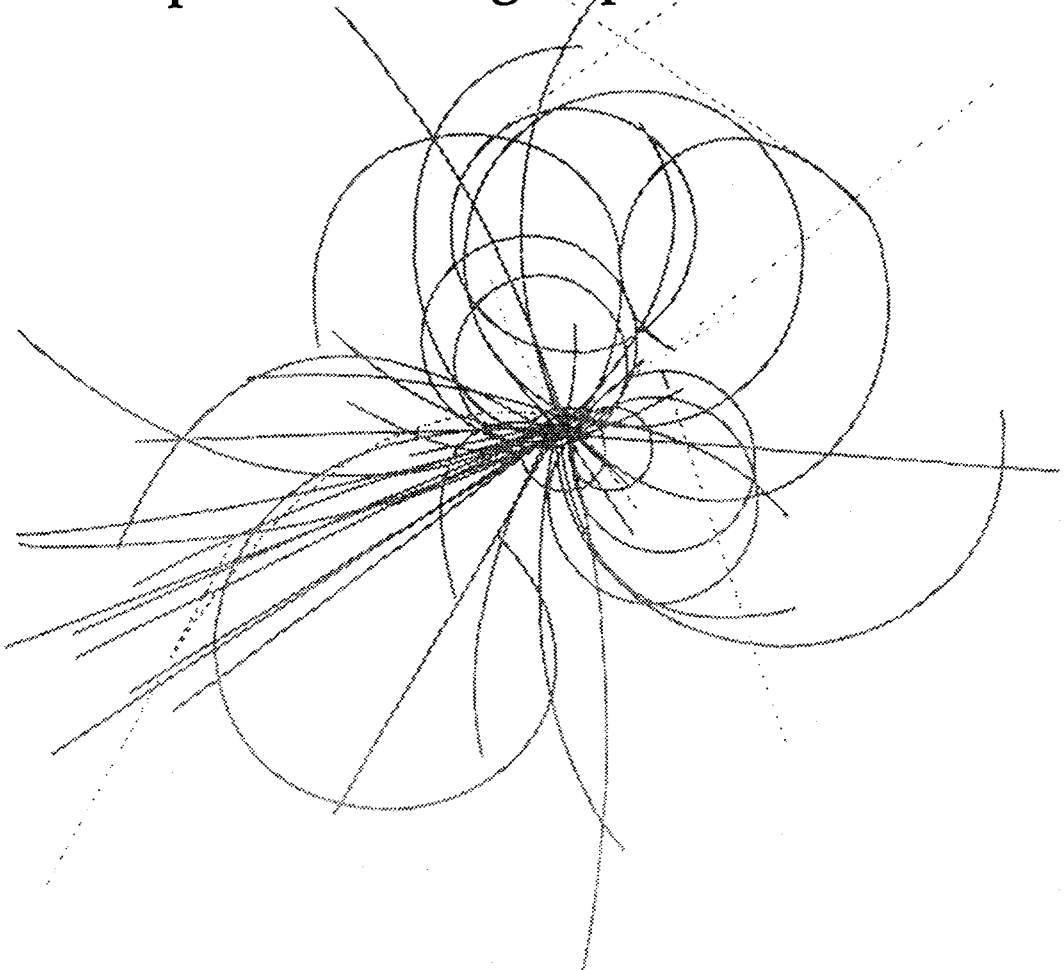

Constant of Motion, Hamiltonian, and Lagrangian for Autonomous Systems Defined in a Hyperbolic Flat Space

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Abstract

The Lagrangian, the Hamiltonian, and the Generalized Linear Momentum of an autonomous dynamical system defined in a hyperbolic flat space are studied in terms of a constant of motion associated with this system. Some restrictions in the symmetry of the Lagrangian are required in order for the Euler-Lagrange equations to be satisfied. The one-dimensional relativistic motion is given as an example of the approach.

I. INTRODUCTION

One of the major theoretical concerns since the foundation of Classical Mechanics has been the establishment of the Lagrangian of a system from the associated equations of motion.[1] This topic is called "the inverse problem of the Mechanics," or "the inverse problem of the Calculus of Variations." This process cannot, in many cases, be carried out for dimensions higher than one since the forces must satisfy some restrictions.[2] Considering the relation between the Lagrangian and the constant of motion associated with a one-dimensional autonomous system[3–5] (the forces do not depend explicitly on time), the natural question is what restrictions might appear in the constant of motion for higher-dimensional space or even for a relativistic space (hyperbolic flat space). If the system is nonautonomous (the forces depend explicitly on time), it has been shown[6] that a similar simple relation does not exist between the Lagrangian and the constant of motion associated with the system, even for the one-dimensional case. For this reason, this paper restricts itself to the study of autonomous systems.

First, the dynamic system in a flat hyperbolic space is defined, as is its constant of motion. In addition, a relation between this constant of motion and the Lagrangian is obtained. Second, the relation between the Euler-Lagrange equation and the constant of motion is established. The conditions to satisfy the Euler-Lagrange equations are given, bringing about the restrictions on the constant of motion. Next, the relation between the Hamiltonian and the constant of motion is given, as is the relation between the Hamiltonian and the Lagrangian. Finally, the one-dimensional relativistic problem is analyzed using this approach.

II. DYNAMIC SYSTEM, CONSTANT OF MOTION, AND LAGRANGIAN

The equations of motion of a particle moving in a hyperbolic flat space can be written as an n-dimensional dynamical system given by

$$\frac{dx_\nu}{ds} = U_\nu \tag{1a}$$

and

$$\frac{dU_\nu}{ds} = F_\nu, \quad \nu = 1, \ldots, n \ , \tag{1b}$$

where x_ν, U_ν, and F_ν, for $\nu = 1, \ldots n$, are the coordinates, the velocities, and the forces (acting in the particle). The mass has been set equal to unity to simplify the writing. The forces can depend on the coordinates and velocities, but they must not depend on the variable s. This variable is related to the coordinates through the metric of the system,

$$ds^2 = \sum_{\nu=1}^{n} \eta^{\nu\mu} x_\nu x_\mu \ , \tag{2}$$

where the $n \times n$ symmetric metric $(\eta^{\nu\mu})$ is given by

$$(\eta^{\nu\mu}) = \begin{pmatrix} I_r & 0 \\ \hat{0} & -\hat{I}_r \end{pmatrix} \ . \tag{3}$$

I_r and $\hat{I}_r$ are unit matrices of dimensions $r \times r$ and $(n-r) \times (n-r)$, respectively. 0 and $\hat{0}$ are zero matrices of dimensions $r \times (n-r)$ and $(n-r) \times r$. The space defined by the metric (Eq. (3)) is called flat space because gravity is not included in the metric.

A constant of motion associated with the system (Eq. (1)) will be a function $K(x, U)$ satisfying the equation

$$\frac{dK}{ds} = 0 \ , \tag{4}$$

where $x = (x_1, \ldots, x_n)$ and $U = (U_1, \ldots . U_n)$ are vectors, and the operator d/ds is defined as

$$\frac{d}{ds} = \sum_{\mu=1}^{n} \left(U^\mu \frac{\partial}{\partial x_\mu} + F^\mu \frac{\partial}{\partial U_\mu} \right) \ . \tag{5}$$

The components with the upper index are related to the components with the lower indexes in the usual way:

$$x^\mu = \sum_{\nu=1}^{n} \eta^{\nu\mu} x_\nu \ . \tag{6a}$$

The inner product of any two vectors, x and y, in this space is given by

$$x \cdot y = \sum_{\nu\mu} \eta^{\nu\mu} x_\nu y_\mu = \sum_\mu x^\mu y_\mu \ . \tag{6b}$$

The relation between the Lagrangian, L, and the constant of motion is given through the Legendre transformation, which can be written as the following partial differential equation:

$$\sum_{\mu} U^{\mu} \frac{\partial L}{\partial U_{\mu}} - L = K \ . \tag{7}$$

This equation can be solved by the characteristics method[7] yielding the following solution:

$$\begin{aligned} L = & \sum_{\mu} A^{\mu}(x, C_{\lambda\nu}, C_{\nu\lambda})\ U_{\mu}^{s(\mu)} + \\ & + \frac{1}{n} \sum_{\mu} s(\mu) U_{\mu}^{s(\mu)} \int^{U_{\mu}} \frac{K^{\mu}(x,\xi)\ d\xi}{\xi^{1+s(\mu)}} \ , \end{aligned} \tag{8}$$

where the function $s(\mu)$ is defined as

$$s(\mu) = \begin{cases} 1 & \text{if } 1 \le \mu \le r \\ -1 & \text{if } r < \mu \le n \ . \end{cases} \tag{9a}$$

$C_{\nu\mu}$ are the characteristic curves given by

$$C_{\nu\mu} = \begin{cases} 0 & \text{if } \ \nu = \mu \\ U_{\mu} U_{\nu}^{sig(\eta^{\nu\nu}\eta^{\mu\mu})} & \text{otherwise} \ , \end{cases} \tag{9b}$$

where $sig(\eta^{\nu\nu}\eta^{\mu\mu})$ is the function defined as

$$sig(\eta^{\nu\nu}\eta^{\mu\mu}) = \begin{cases} +1 & \text{if } \ \eta^{\nu\nu}\eta^{\mu\mu} = +1 \\ -1 & \text{if } \ \eta^{\nu\nu}\eta^{\mu\mu} = -1 \ . \end{cases} \tag{9c}$$

These characteristic curves have the following symmetric property:

$$C_{\mu\nu} = \begin{cases} 1/C_{\nu\mu} & \text{if } \ \eta^{\nu\nu} = \eta^{\mu\mu} \\ C_{\nu\mu} & \text{otherwise} \ . \end{cases} \tag{9d}$$

A^{μ}, $\mu = 1, \ldots, n$, are arbitrary functions that depend on position and the above characteristic curves, and K_{μ}, $\mu = 1, \ldots, n$, are functions that are defined through the constant of motion as

$$K_{\mu}(x,\xi) = \begin{cases} K\left(x, C_{1,\mu}\xi, \ldots, C_{\mu-1,\mu}, \xi, C_{\mu+1,\mu}\xi, \ldots, C_{r,\mu}\xi, \frac{C_{\mu,r+1}}{\xi}, \ldots, \frac{C_{\mu,n}}{\xi}\right) \\ \text{if } \ \mu \le r \\ K\left(x, \frac{C_{\mu,1}}{\xi}, \ldots, \frac{C_{\mu,r}}{\xi}, C_{r+1,\mu}\xi, \ldots, C_{\mu-1,\mu}\xi, \xi, C_{\mu+1,\mu}\xi, \ldots, C_{n,\mu}\xi\right) \\ \text{if } \ r < \mu \le n \ . \end{cases} \tag{9e}$$

It is not difficult to show that Eq. (8) is in fact the solution of Eq. (7). The first expression on the right-hand side of Eq. (8) corresponds to the homogeneous solution of Eq. (7). The second expression represents the particular solution of this equation. For one-dimensional space ($n = r = 1$), and taking s as the independent variable (not related through Eq. (2)), the solution in Eq. (8) gives the expected Kobussen expression for autonomous systems (References 3 and 5).

III. EULER-LAGRANGE EQUATION AND THE CONSTANT OF MOTION

Applying the operator of Eq. (5) to Eq. (7) and rearranging some terms, the following expression results:

$$\frac{dK}{ds} = \sum_{\mu} U^{\mu} E_{\mu} \; , \tag{10}$$

where E_μ is the component of the Euler-Lagrange vector $E = (E_1, \ldots, E_n)$, defined as

$$E_\mu = \frac{d}{ds}\left(\frac{\partial L}{\partial U_\mu}\right) - \frac{\partial L}{\partial x_\mu} \; . \tag{11}$$

Equation (10) gives us the relation between the function K and the Euler-Lagrange equations that are represented by $E_\mu = 0$. If the Euler-Lagrange equations are satisfied, then K must be a constant of motion of the system. However, if K is a constant of motion of the system, the vectors U and E are orthogonals in this hyperbolic flat space ($U \cdot E = 0$).

Since the Lagrangian can be decomposed in a homogeneous part, L_h, and the particular solution part, L_p, the components of the Euler-Lagrange vector can also be decomposed in a homogeneous part, E^h_λ, and a particular part, E^p_λ, that can be obtained explicitly from Eqs. (8) and (11). These are given by

$$\begin{aligned} E^h_\lambda = & \sum_{\mu} \frac{d}{ds}\left(\frac{\partial A^\mu}{\partial U_\lambda}\right) + \sum_{\mu} s(\mu) \frac{\partial A^\mu}{U_\lambda} U_\mu^{s(\mu)-1} F_\mu \\ & + s(\lambda)[s(\lambda)-1] U_\lambda^{s(\lambda)-2} F_\lambda A^\lambda + s(\lambda) U_\lambda^{s(\lambda)-1} \frac{dA^\lambda}{ds} \\ & - \sum_{\mu} \frac{\partial A^\mu}{\partial x_\lambda} U_\lambda^{s(\mu)} \end{aligned} \tag{12a}$$

and

$$
\begin{aligned}
E^p_\lambda = & \frac{[s(\lambda)-1]}{n} U_\lambda^{s(\lambda)-2} F_\lambda \int^{U_\lambda} \frac{K^\lambda \, d\xi}{\xi^{1+s(\lambda)}} + \frac{[1-s(\lambda)]F_\lambda K^\lambda}{nU_\lambda^2} \\
& + \frac{1}{n} U_\lambda^{s(\lambda)-1} \sum_\mu U^\mu \int^{U_\lambda} \frac{d\xi}{\xi^{1+s(\lambda)}} \frac{\partial K^\lambda}{\partial x_\mu} \\
& - \frac{1}{n} \sum_\mu s(\mu) U_\mu^{s(\mu)} \int^{U_\mu} \frac{d\xi}{\xi^{1+s(\mu)}} \frac{\partial K^\mu}{\partial x_\lambda} . \qquad (12b)
\end{aligned}
$$

For the elliptical space case ($r = n$), the above expressions are simplified in the following way ($s(\nu) = 1, \ \nu = 1, \ldots, n$):

$$
\begin{aligned}
\widetilde{E}^h_\lambda = & \sum_\mu \frac{d}{ds} \left(\frac{\partial A^\mu}{\partial U_\lambda} \right) + \sum_\mu F_\mu \left(\frac{\partial A^\mu}{\partial U_\lambda} + \frac{\partial A^\lambda}{\partial U_\mu} \right) \\
& + \sum_\mu U^\mu \left(\frac{\partial A^\lambda}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\lambda} \right) \qquad (13a)
\end{aligned}
$$

and

$$
\widetilde{E}^p_\lambda = \frac{1}{n} \sum_\mu U^\mu \left\{ \int^{U_\lambda} \frac{d\xi}{\xi^2} \frac{\partial K_\lambda}{\partial x_\mu} - \int^{U_\mu} \frac{d\xi}{\xi^2} \frac{\partial K_\mu}{\partial x_\lambda} \right\} . \qquad (13b)
$$

In addition, if s is the time-independent variable (instead of satisfying Eq. (2)), relations (13a) and (13b) represent the generalization of the Kobussen expression to higher dimensions. These expressions impose strong symmetry conditions for the Euler-Lagrange equations to be satisfied.

IV. GENERALIZED LINEAR MOMENTUM, HAMILTONIAN, LAGRANGIAN, AND CONSTANT OF MOTION

Given the Lagrangian, L, and the constant of motion, K, the generalized linear momentum and Hamiltonian can be given by

$$
p_\mu = \frac{\partial L}{\partial U_\mu} \qquad (14)
$$

and

$$
H(x,p) = K(x, U(x,p)) \ , \qquad (15)
$$

whenever the explicit relation $U_\mu = U_\mu(x,p)$ can be obtained from Eq. (14). However, if the Hamiltonian is given, Eqs. (7) and (15) establish a relationship between the Hamiltonian and Lagrangian through the nonlinear partial differential equation

$$\sum_\mu U^\mu \frac{\partial L}{\partial U_\mu} - L = H\left(x, \frac{\partial L}{\partial U}\right), \tag{16}$$

which brings about the Langrangian and, consequently, the generalized linear momentum in terms of the position and velocity.

V. ONE-DIMENSIONAL RELATIVISTIC MOTION

Assume that a relativistic particle is moving in one-dimensional space where the force acting on the particle depends only on its position. The equation of motion can be written as[8]

$$\frac{dx_1}{ds} = U_1 \tag{17a}$$

and

$$\frac{dU_1}{ds} = F_1\ , \tag{17b}$$

where U_1 and F_1 are expressed in terms of the position, x_1, the velocity, v_1, the mass at rest, m, the speed of light, c, the force, $f(x_1)$, and the relativistic gamma factor, $\gamma = 1/\sqrt{1 - v_1^2/c^2}$, as

$$U_1 = \gamma v_1/c \tag{18a}$$

and

$$F_1 = \gamma f(x_1)/mc^2\ . \tag{18b}$$

The metric of this hyperbolic space is given by

$$ds^2 = (dx_o)^2 - (dx_1)^2\ , \tag{19a}$$

where x_o is written as a function of the time, t, as

$$x_o = ct\ . \tag{19b}$$

From Eq. (19a) one can deduce that

$$dx_o/ds = \gamma \; . \tag{20}$$

Since the rate of energy change is given by

$$dE/dt = f(x_1)v_1 \; , \tag{21}$$

where E is the total free energy, $E = \gamma mc^2$, the following equations result (using also the identification $d/dt = (c/\gamma)d/ds$):

$$\frac{dx_o}{ds} = U_o \tag{22a}$$

and

$$\frac{dU_o}{ds} = F_o \; , \tag{22b}$$

where U_o and F_o are defined as

$$U_o = \gamma \tag{23a}$$

and

$$F_o = U_1 f(x_1)/mc^2 \; . \tag{23b}$$

Equations (17) and (22) form the dynamic equations for the particle moving in the hyperbolic space defined by Eq. (19a) ($n = 2$ and $r = 1$). The constant of motion associated with this system can be obtained from the solution of the equation

$$U^o \frac{\partial K}{\partial x_o} + U^1 \frac{\partial K}{\partial x_1} + F^o \frac{\partial K}{\partial U_o} + F^1 \frac{\partial K}{\partial U_1} = 0 \; . \tag{24}$$

Using Eqs. (18) and (23) in Eq. (24) and solving this equation, the following solution is obtained:

$$K = K(C_1, C_2) \; , \tag{25}$$

where C_1 and C_2 are the characteristic curves of Eq. (24) given by

$$C_1 = mc^2\sqrt{1 + U_1^2} + V(x_1) \tag{26a}$$

and

$$C_2 = U_o - \sqrt{1 + U_1^2} \; . \tag{26b}$$

$V(x_1)$ is the potential function,

$$V(x_1) = -\int f(x_1) dx_1 \; . \tag{26c}$$

However, from Eqs. (21a), (22a), (18a), and the definition of γ, the following relation must be satisfied:

$$C_2 = 0 \; . \tag{27}$$

Therefore, it is possible to select the constant of motion as

$$K = mc^2 \sqrt{1 + U_1^2} + V(x_1) \; . \tag{28}$$

The Lagrangian of Eq. (8) is written for this case as

$$L = A^o U_o + \frac{A^1}{U_1} + \frac{U_o}{2} \int^{U_o} \frac{K^o(x,\xi)}{\xi^2} d\xi - \frac{1}{2U_1} \int^{U_1} K^1(x,\xi) d\xi \; , \tag{29}$$

where A^o and A^1 are arbitrary functions depending on the position, $x = (x_o, x_1)$, and the characteristic curve, $C_{01} = U_o U_1$, associated with the Legendre Eq. (7). The functions K^o and K^1 are given by

$$K^o(x,\xi) = mc^2 \sqrt{1 + C_{01}^2/\xi^2} + V(x_1) \tag{30a}$$

and

$$K^1(x,\xi) = K(x,\xi) \; . \tag{30b}$$

Assuming $A^o = 0$ and $A^1 = 0$, and integrating Eq. (29), it follows that

$$L = -\frac{mc^2}{2} \sqrt{1 + U_1^2} - \frac{mc^2}{2U_1} \log\left(U_1 + \sqrt{1 + U_1^2} \right) - V(x_1) \tag{31}$$

and

$$p_1 = -\frac{mc^2}{2} \frac{\sqrt{1 + U_1^2}}{U_1} + \frac{mc^2}{2U_1^2} \log\left(U_1 + \sqrt{1 + U_1^2} \right) \; . \tag{32}$$

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In this example, E_o^h and E_1^h are zero. The components E_o^p and E_1^p are different from zero, indicating that the Euler-Lagrange equation is not satisfied. However, the relation $U^o E_o^p + U^1 E_1^p = 0$ is satisfied. The constant of motion (Eq. (28)) corresponds to the usual relativistic energy, but the Lagrangian and the generalized linear momentum are quite different from those obtained in a nonhyperbolic space where the time is an independent variable (see Reference 5).

VI. CONCLUSIONS

The constant of motion for a dynamical system defined in a hyperbolic flat space is obtained through the solution of a linear partial-differential (Eq. (4)). This constant of motion is used to obtain the Lagrangian of the system through the Legendre linear partial differential equation (Eq. (7)), which brings about a close expression of the Lagrangian in terms of this constant of motion (Eq. (8)). Although the Euler-Lagrange equations are not always satisfied (Eqs. (12) and (13) must be zero), there is always an orthogonal relation between the Euler-Lagrange and velocity vectors whenever K is a constant of motion (Eq. (10)). Given the Lagrangian, the generalized linear momentum and the Hamiltonian can then be calculated. However, if the Hamiltonian is given, the relation of the generalized linear momentum with the coordinates and velocities is determined by solving a nonlinear partial differential equation (Eq. (16)). The one-dimensional relativistic example is given as an example of the application of the above approach.

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