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Quantization Methods for Fields, from Quantum Hall Effect interactions to Higher order Dual Reducible theories

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“ *A unidade é a variedade, e a variedade na unidade é a lei suprema do universo.*”

Sir Isaac Newton

“ *A pedra que os construtores rejeitaram tornou-se a pedra angular .*”

Mateus 21-42.

“ *Aquilo que se faz por amor está sempre além do bem e do mal .*”

Friedrich Nietzsche

“ *Veni, vidi, vici.*”

Júlio César

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Resumo

Ao longo desta tese pretendemos usar um conjunto de métodos em teoria quântica de campos a fim de investigar determinados modelos pertencentes às classes das teorias planares, eletromagnetismo num calibre não-linear, modelos com derivadas de ordem superior e teorias com simetrias redutíveis. Consideramos o formalismo do campo auxiliary B para quantização covariante de teorias de gauge num espaco de Hilbert de métrica indefinida como o nosso formalismo guia a fim de obter extensões tanto para a sua versão perturbativa quanto para a não perturbativa. Como já mencionado, também consideramos métodos auxiliares alternativos como o uso da representação espectral de Kallen-Lehmann, métodos de espaço de fase como o de Fadeev-Jackiw e a técnica dos grafos de Feynman. Este último método foi empregado para se analisar uma teoria tensorial com simetria local de Weyl, e também de calibre redutível, que é dual a QED_4. Diferentes abordagens para o estudo de integrais de trajetória e sua relação com o formalismo do campo B também são discutidas. Os modelos estudados aqui possuem algumas relações teóricas e conceituais entre si o que motiva o seu estudo num único material completo pois um dado avanço para um deles é, geralmente, o primeiro passo, devido as suas perspectivas e ferramentas desenvolvidas, para o entedimento do próximo modelo a ser investigado.

Palavras-Chave: Método de quantização via campo auxiliar, modelos com simetrias redutíveis.

Área do conhecimento: Teoria quântica de campos.

Abstract

Throughout this thesis we intend to use a set of quantum field theory methods to investigate some specific models belonging to the classes of the planar theories, electromagnetism in a non-linear gauge, higher derivative models and theories with reducible gauge symmetries. We have considered the B field formalism for indefinite metric Hilbert space covariant operator quantization of gauge theories as our guiding formalism in order to provide extensions for its perturbative and also non-perturbative versions. As previously mentioned, we also consider alternative auxiliary methods such as the Kallen-Lehmann spectral representation, phase space methods as the Fadeev-Jackiw formalism and also the Feynman graph technique. This last method was employed to investigate a tensor dual theory of QED_4 with reducible and Weyl gauge symmetry. Different path integral approaches and their relation to the B field formalism are also discussed. The different models studied here have theoretical as well as some conceptual relations between them which motivates their study in a single complete material since a given achievement for one of them is generally the first step, due to its perspectives and developed tools, to understand the next one to be investigated.

Keywords: Quantization method via auxiliary field, models with reducible symmetries.

Area of knowlegde: Quantum field theory.

Contents

Chapter 1

Introduction

The aim of this thesis is to investigate a set of quantum field theory models belonging to the classes of planar gauge theories, QED_4 in a non-linear gauge, vector and tensor field models with reducible gauge symmetries and higher derivative systems. The chapter 2 is devoted to introduce the B field formalism, the chapter 3 is a review about the quantum Hall effect and the remaining ones are new achievements.

In order to accomplish this goal, we also use a set of different but complementary quantum field theory approaches. However, we will adopt the Nakanishi B field formalism [1, 2] as our fundamental approach and use the remaining methods as auxiliary tools that can reveal some physical contents of a given model that are not suitable to be evaluated in the Heisenberg description. Then, before proceed, we first introduce the general idea behind this formalism and then comment about the models to be studied here, their physical significance, and the reason to choose a given specific set of quantization tools to investigate each of these theories. We also call attention to the fact that all of them share some physical content between theirselves.
For example, in chapter 3, we study the general features of the quantum Hall effect and the structure of the graphene, a planar material whose low energy limit excitations lead to a two band system. In the next chapter, we investigate the renormalization of the first quantization properties of the Landau model, effectively described by the Chern-Simons term, due to the interaction with a class of these two band systems. The Fermionic as well as the Bosonic responses are derived and we explicitly obtain the shift in their topological properties.

Then, as promised, after these preliminaries, we introduce the idea of the Nakanishi B-field formalism. It consists of a Heisenberg description quantization which uses an indefinite metric Hilbert space whose positive norm subspace is defined by means of a given subsidiary condition. This condition may be written in terms of the positive frequency part of the auxiliary B field or by using the BRST charge for Abelian and non-Abelian theories, respectively. This formalism furnishes a well-defined system which is free from first class ambiguities [3].

This non-perturbative formalism has a wide variety of applications from Abelian and non-Abelian gauge theories to quantum gravity. It has a perturbative counterpart which gives an exact answer for the quantization of the two dimensional non-Abelian BF theory by means of a well-defined cauchy problem for commutators [4] and can also be used for QED_4 to get its first order radiative corrections [5]. The general structure for the Wightman functions can be inferred by a method which consists of extracting the truncated n point functions from the n point commutators by imposing the requirement of energy positivity. For a review which includes the perturbative B field formalism for QED_4 passing through the string theory and also the two dimensional quantum gravity, see [6].

Now, regarding the explicit structure of this thesis, we intend to extend the B field formalism in order to understand how reducible and higher derivative systems can be incorporated in it.

In order to present the general structure of the thesis, we give some of the fundamental details of each chapter. The second one is devoted to present the B field formalism and how the necessity to adopt an indefinite metric quantization for gauge fields in $D = 3+1$ dimensions arises from the quantum field theory axioms and the structure of the Poincaré group. Moreover, we show how to introduce the auxiliary B field in order to avoid first class ambiguities from the beginning. It also furnish a subsidiary condition to define the physical positive Hilbert space.

The chapter 3 has a conceptual character and presents the history of the quantum Hall effect, from its classical basis and explanation by the Drude model [7] to the quantum field theoretical description [8]. This chapter has a deep relation with the chapter 4, in which the theoretical description of topological two band models coupled to a Chern-Simons field is discussed. The first part of this work presents theoretical arguments in favour of an indefinite metric quantization for gauge theories in the $D = 2 + 1$ dimensional case. The discussion is analogous to the one of the second chapter. We present the whole non-perturbative structure for the Chern-Simons model coupled to the quasi-particle Fermionic excitations in a two band structure and then develop the perturbative version of the B field formalism for it. The polarization tensor is obtained and the resulting potential between static charges is also calculated. The final section is devoted to obtain the self energy tensor contribution and some physical properties related to its renormalized structure as, for example, the emergence of a term that lead to localized boundary states.

In the chapter 5, the Heisenberg description of QED_4 in the non-linear 't Hooft gauge was analyzed. After the non-perturbative general view, we develop the perturbative version in a well-defined way. In the last section, we use a path integral method to show that the physical independence from the unphysical 't Hooft parameter is due to the BRST invariant structure of the model. One reason that motivate its study is the fact that at non-linear gauges, the QED_4 presents an structure analogous to the Yang-Mills model, then it can be understood as a good laboratory to investigate the BRST symmetry, the properties of the ghosts and how a quartet structure emerges in order to avoid the emergence of the auxiliary fields in the physical spectrum. Another motivation is the fact that, for example, in the context of the Abelian dominance in QCD_4, the use of non-linear gauges is welcome and a better theoretical knowledge of them can be important for future research.

Regarding the perturbative aspects, the Nakanishi formalism is not yet totally adequate to investigate non-Abelian systems of the Yang-Mills kind [6]. Therefore, the development of the perturbative version for this laboratory model can be useful to define tools to improve and extend the formalism.

The chapter 6 present the natural evolution of the ideas from the previous one. The same non-linear gauge is considered in a situation in which a matter lagrangian consisting of Fermions and a Higgs field is present. Aspects such as symmetry breaking can also be discussed in the framework of the B field formalism.

The chapter 7 was devoted to extend the B-field formalism for the case in which there is reducible gauge symmetry. Since the Nakanishi formalism is defined in terms of auxiliary fields introduced in such a way to avoid first class ambiguities and correlated pathologies, the new paradigm of this chapter raises interesting questions such as the necessity of adding an extra auxiliary field to correctly quantize the system. We analyze the fact that, at the massless limit, some of these models have a discontinuity in their degrees of freedom [9, 10]. We derive these conclusions by means of Hilbert space analysis instead of phase space methods as in [9]. In the last two sections we give a prescription to build the partition function for the massless phase of these models with all the so-called ghosts of ghosts.

In the chapter 8, a longitudinal, reducible higher derivative dual tensor description of QED_4 is analyzed [11, 12]. We couple the theory to matter and then use path integral methods to derive the Ward identities for both of the local symmetries, the Weyl and the reducible one.

Then, in order to analyze the unitarity constraints, we derive the vertices for the model, the propagators and all the Feynman rules. Then, considering the optical theorem we show that the interacting theory is unitary provided we use a given prescription for the external states. We use dispersion relations to infer that the radiative contributions for the boson field has indeed the right tensor structure and derivative order to be renormalizable as the usual QED_4. Since the propagator has a higher derivative structure while the vertex has also a higher derivative character, the system has the same degree of divergence as the usual QED_4. Due to the tensor structure of the model and its Weyl gauge symmetry, we could find its massive gauge invariant extension and then obtain an integral representation for the self energy tensor by means of dispersion relations. At this point, the natural candidates for a next investigation in the chapter 9 are the higher derivative theories. We try to extend the B field formalism to this kind of model. In this case there are additional phase space variables for the fields due to the Ostrogadskian structure [13, 14] and therefore the gauge fixing sector should take it into account in order to avoid all of these new sources of primary first class structures. The case of a higher derivative longitudinal model and the so-called Podolsky electrodynamics are analyzed. One of our results is the fact that although some longitudinal models have discontinuities in the massless sector, their higher derivative versions may be totally continuous.

The chapter 10 presents the Fadeev-Jackiw method [15, 16] to characterize the phase space and then quantize a given higher order theory. We analyze an extension of the Podolsky model in which an infrared regulator mass is added to the model in a Stuckelberg gauge invariant structure. By analysing the null phase space directions, from which the Hamiltonian are independent, the set of constraints and also the canonical transformations are obtained and, from them, the whole set of gauge symmetry transformations are found. The method allows us to define the reduced phase space in a given gauge. From this structure, the path integral for this model can be calculated.

Finally, in chapter 11, we conclude.

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Chapter 2

Covariant Quantization of Gauge Theories

2.1 The Necessity of an Indefinite Metric Hilbert Space

In this chapter we intend to show that a quantum theory described by a vector field with massless excitations in four dimensions necessarily requires an indefinite metric Hilbert space, [1]. Then, the quantization method should furnish subsidiary conditions to define the physical Hilbert subspace which has positive projections and then allows the well known probabilistic interpretation.

In order to achieve our goal, we divide the discussion into two main parts. First, we introduce the Maxwell action describing the electromagnetism by means of massless photons represented by a vector field. Latter, we infer that the use of a Hilbert space with positive definite metric leads to a problematic situation in which both the electric and magnetic fields must vanish. This conclusion is obtained by means of a theorem valid just for Hilbert spaces with positive norm. Therefore, in order to prevent this problematic situation, the condition of positiveness must be avoided.

The lagrangian describing the electromagnetism is the following

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.1}$$

The Euler-Lagrange equations leads to

$$\Box A_\mu - \partial_\mu \partial^\nu A_\nu = 0 \tag{2.2}$$

In order to provide a quantum analysis, let us suppose, for a while, a Hilbert space with positive definite metric. Then, consider the object below [1]

$$W_{\mu\nu}(x,y) \equiv \langle 0|A_\mu(x)A_\nu(y)|0\rangle \tag{2.3}$$

where $|0\rangle$ denotes the vacuum state of the theory.

This Wightman function can be decomposed in Lorentz covariant structures. The most general one, in $D = 3+1$ dimensions, reads

$$W_{\mu\nu}(x,y) = \eta_{\mu\nu}D_1(x-y) + \partial_\mu\partial_\nu D_2(x-y) \tag{2.4}$$

[1]The field $A_\mu(x)$ must be interpreted as an operator in the Heisenberg picture.

In this expression, D_1 and D_2 denote functions that depends just on the difference of the field arguments. This property can be inferred by the fact that a translation is represented by an unitary transformation $UA_\mu(x)U^\dagger$ acting on the Hermitian operator $A_\mu(x)$ with the form $U = e^{i\hat{P}_\mu . \Delta x^\mu}$, where $\hat{P}_\mu$ denotes the four momentum operator and Δx^μ is the four vector representing the variation of the position due to translation operation. We are considering a Poincaré structure for the space-time which means that $\hat{P}_\mu$ commutes with itself.

These characteristics guarantee the identity

$$W_{\mu\nu}(x+a, y+a) = \langle 0|e^{-i\hat{P}.a}A_\mu(x)e^{i\hat{P}.a}e^{-i\hat{P}.a}A_\nu(y)e^{i\hat{P}.a}|0\rangle = W_{\mu\nu}(x,y) \tag{2.5}$$

if we consider that the vacuum state has no physical properties such as linear and angular momentum. It leads to a translation and Lorentz invariant vacuum.

Applying the differential operator that defines the equation of motion for the field $A_\mu(x)$ on $W_{\mu\nu}(x-y)$, we get

$$\Big(\Box^x \eta_\mu{}^\nu - \partial_\mu \partial^\nu\Big) W_{\nu\beta}(x-y) = \Big(\Box^x \eta_{\nu\beta} - \partial_\mu \partial_\beta\Big) D_1(x-y) = 0 \tag{2.6}$$

The above result must vanish since the operator $A_\mu(x)$ obeys its equation of motion. Considering that the distribution $D_1(x-y)$ must not diverge on the space-time infinity, the condition above implies that it is constant. The distribution $D_2(x-y)$ is not constrained by the previous equation because the photon field differential operator is transverse. Therefore, we have

$$W_{\mu\nu}(x,y) = \eta_{\mu\nu} c + \partial_\mu \partial_\nu D_2(x-y) \tag{2.7}$$

where c is a indefinite c-number constant.

Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, considering the definition of $W_{\mu\nu}(x,y)$ and the previous equation constraining its form, we can derive

$$\langle 0|F_{\mu\nu}(x)F_{\gamma\beta}(y)|0\rangle = 0 \tag{2.8}$$

The fact that we are considering a positive definite Hilbert space implies

$$F_{\gamma\beta}(y)|0\rangle = 0 \tag{2.9}$$

According to the appendix $A2$ of [1], for positive definite spaces, the previous equation leads to

$$F_{\gamma\beta}(y) = 0 \tag{2.10}$$

The previous equation holds due to the so-called vacuum separating property. It is valid just for the case of a local operator, otherwise the physical property related to the vacuum state Poincaré invariance would imply in the vanishing of its generators.

In order to avoid the triviality $F_{\mu\nu}(x) = 0$ of the physical content we conclude that the positivity of the Hilbert space metric must be relaxed. In order to do so, a subsidiary condition is necessary to definite the positive metric subspace where the probabilistic interpretation is possible.

Our next step is to show that besides the previous approach, the necessity of assuming a non-positive definite Hilbert space follows from the structure of the little group of a vector massless particle. The so-called little group refers to a method to classify states by its continuum momentum spectrum and by a subgroup of Lorentz transformations that leaves the four momentum unchanged. This group is the maximum set of the Poincaré group generators compatible with the momentum operator. Then, there are simultaneous eigenstates for this set of operators and they can be used to classify particle states.

For massive particles, we have $P_\mu P^\mu = m^2$ and then we can assume the frame $P_\mu = (m, 0, 0, 0)$ which is invariant under the action of the $SO(3)$ group. This is the little group. In fact, we consider the $SU(2)$ group instead of $SO(3)$ since, despite both of them are compact and the fact that they have the same algebra, the exponentiation of the former leads to a simply connected manifold. This group includes all the elements of $SO(3)$ and also representations classified by semi-integers $j = \frac{n}{2}$, with n belonging to the natural numbers, whose dimension is $2j+1$. These discrete values are used to classify the representations of the little group together with the momentum spectrum [2].

For a massless particle, we have $P_\mu = (p_0, 0, 0, p_0)$. The Lorentz subgroup preserving P_μ is the $E(2)$ which is not compact. It is due to the fact that its algebra is isomorphic to a two dimensional Galileo group with rotations around an axis and $2D$ translations. The sector of translations turn it into a non-compact group. Since there is no observed continuous spectrum for particles besides the momentum one, we fix vanishing eigenvalues for these two dimensional translations and the states are labeled by the $2D$ rotation eigenvalues. It is related to the concept of helicity [2].

In order to explicitly obtain the little group, we must know the Poincaré group algebra

$$\Big[P_\mu, P_\nu\Big] = 0 \ , \ \Big[M_{\mu\nu}, P_\lambda\Big] = i\Big(\eta_{\nu\lambda}P_\mu - \eta_{\mu\lambda}P_\nu\Big) \tag{2.11}$$

$$\Big[M_{\mu\nu}, M_{\lambda\rho}\Big] = i\Big(\eta_{\nu\lambda}M_{\nu\rho} - \eta_{\mu\lambda}M_{\nu\rho} + \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\rho}M_{\mu\lambda}\Big)$$

Defining the following operators $J_i \equiv \epsilon_{ijk}M^{jk}$ and $K_j \equiv M_{0j}$, the Lorentz sector of the algebra assumes the form

$$\Big[J_i, J_j\Big] = i\epsilon_{ijk}J_k \ , \ \Big[K_i, K_j\Big] = -i\epsilon_{ijk}J_k \ , \ \Big[J_i, K_j\Big] = i\epsilon_{ijk}K_k \tag{2.12}$$

We can define a new set of operators $M_i \equiv \frac{1}{2}(J_i + iK_i)$ and $N_i \equiv \frac{1}{2}(J_i - iK_i)$ in order to separate the Lorentz group as a direct product of two independent subgroups

$$\Big[M_i, N_j\Big] = 0 \ , \ \Big[M_i, M_j\Big] = i\epsilon_{ijk}M_k \ , \ \Big[N_i, N_j\Big] = i\epsilon_{ijk}N_k \tag{2.13}$$

The expression above displays two independent $su(2)$ algebras which means that $so(1,3) = su(2) \times su(2)$. Since the $su(2)$ algebra is well-known from the study of spin in quantum mechanics [3], we can use its eigenvalues to classify states. We can define the raising and lowering operators $M_\pm = M_1 \pm M_2$ e $N_\pm = N_1 \pm iN_2$, which, when acting on a eigenstate, raises or lowers the M_3 or N_3 eigenvalues in one unit.

Now, focusing on the obtainment of the little group, we fix $P_\mu = (p_0, 0, 0, p_0)$ and analyze $\Big[M_{\mu\nu}, P_\lambda\Big] = i\Big(\eta_{\nu\lambda}P_\mu - \eta_{\mu\lambda}P_\nu\Big)$. From the definitions of J_i and K_i, we get

$$\Big[J_1, P_\lambda\Big] = -i\eta_{2\lambda}p_0 \ , \ \Big[J_2, P_\lambda\Big] = i\eta_{1\lambda}p_0 \ , \ \Big[J_3, P_\lambda\Big] = 0 \ , \ \Big[K_1, P_\lambda\Big] = i\eta_{1\lambda}p_0 \ , \tag{2.14}$$

$$\Big[K_2, P_\lambda\Big] = i\eta_{2\lambda}p_0 \ , \ \Big[K_3, P_\lambda\Big] = i\Big(\eta_{3\lambda} - \eta_{0\lambda}\Big)p_0$$

In order to show how these results were obtained, we explicitly calculate the first commutator

$$\Big[J_1, P_\lambda\Big] = \epsilon_{123}\Big[M_{23}, P_\lambda\Big] = i\epsilon_{123}\Big(\eta_{\lambda 3}P^2 - \eta_{2\lambda}P^3\Big) = -i\eta_{2\lambda}p_0 \tag{2.15}$$

The information from these commutators allows us to identify the linear combinations $L_1 \equiv J_1 + K_2 = M_- + N_+$, $L_2 \equiv J_2 - K_1 = i(M_- - N_+)$ and J_3 that preserves the four moment of a massless particle and then form the little group. Since $\Big[L_1, L_2\Big] = 0$, $\Big[J_3, L_1\Big] = iL_2$ and $\Big[J_3, L_2\Big] = -iL_1$ we have indeed the $E(2)$ as previously discussed.

Now, in order to characterize the little group eigenstates for a massless particle we use the fact that the Lorentz group can be decomposed in a product $su(2) \times su(2)$. Using the Clebsch-Gordon coefficients, we conclude that a vector representation, which can be understood as the sum of a spin-1 state, represented by the transverse sector of A_μ and a spin-0 state due to its longitudinal contribution, is generated by a product of spin $\frac{1}{2}$ representations. It is easy to see that the dimension of this tensor product is $(2 \times \frac{1}{2} + 1)^2 = 4$, the necessary one to describe a four vector.

The basis elements are characterized by

$$M_3 U(\pm, \sigma) = \pm\frac{1}{2}U(\pm, \sigma) \ , \ N_3 U(\rho, \pm) = \pm\frac{1}{2}U(\rho, \pm) \tag{2.16}$$

It is possible to provide a vector representation for those eigenstates as

$$U^\alpha \equiv U(\rho, \sigma) \to \alpha = 1, ++ \ , \ \alpha = 2, -- \ , \ \alpha = 3, -+ \ , \ \alpha = 4, +- \tag{2.17}$$

This vector representation leads to the definition of its 4×4 operator counterpart for $J_3 = M_3 + N_3$

$$J_3 U^\alpha = \sum_\beta U^\beta (J_3)_\beta{}^\alpha \tag{2.18}$$

The non-vanishing elements are $(J_3)_{11} = -(J_3)_{22} = 1$ which can be inferred by the action of M_3 and N_3 on the eigenstates. In order to provide a matrix representation to the remaining generators of the little group, we note

$$M_\pm U(\pm, \sigma) = 0 \ , \ M_\pm U(\mp, \sigma) = U(\pm, \sigma) \tag{2.19}$$

The same is valid for $N_\pm$.

Since $L_1 = M_- + N_+$, we have $L_1 U^1 = 1U^3$, and then $(L_1)_{14} = (L_1)_{24} = (L_1)_{31} = (L_1)_{32} = 1$.Analogously, using that $L_2 = i(M_- - N_+)$, we obtain the matrix elements $-(L_2)_{14} = (L_2)_{24} = (L_2)_{31} = -(L_2)_{32} = i$. In terms of matrices, the generators of the little group reads

$$J_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ ; \ L_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ ; \ L_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i \\ i & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.20}$$

This is a convenient point to make some comments. First, it is possible to put U^α in a space-time basis $U^\alpha = U^\alpha_\mu e^\mu$. Since we have $(J_3)_{\beta 3} = (L_1)_{\beta 3} = (L_2)_{\beta 3} = 0$, we note that U^3_μ is invariant under the little group. Then, the linear momentum must be proportional to U^3_μ.

Regarding U^4, it is not affected by J_3, L_1, L_2. Therefore, U^1, U^2, U^3 generates an invariant subspace under the action of $E(2)$ since its operations on this space do not leave any of their components out of it.

Now, we analyze the action of the little group on the structure below

$$\langle 0|a_\mu(p)a_\nu^\dagger(p)|0\rangle = \langle 0|a'_\mu(p)a_\nu^{\dagger'}(p)|0\rangle \tag{2.21}$$

where $a_\mu(p)$ is the Fourier transform of $A_\mu(x)$.

This result is due to the fact that the vacuum state is Poincaré invariant and $U^\dagger(\Lambda)a_\mu(p)U(\Lambda) = a'_\mu(\lambda^{-1}p') = \Lambda_\mu{}^\nu a_\nu(p) = a'_\mu(p)$ with $U(\Lambda)$ being the (pseudo) unitary operator representing the aforementioned space time transformation. We are considering that $\Lambda p = p$, if Λ is from the little group.
If we write $a_\mu(p) = \sum_\alpha a_\alpha U_\mu^\alpha$, we can conclude

$$\sum_{\alpha\beta}\langle 0|a_\alpha a_\beta^\dagger|0\rangle U_\mu^\alpha U_\nu^{*\beta} = \sum_{\alpha'\beta'}\langle 0|a_{\alpha'} a_{\beta'}^\dagger|0\rangle U_{\mu'}^{\alpha'} U_{\nu'}^{*\beta'} \Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'} = \sum_{\alpha'\beta'\alpha\beta} M_{\alpha'\beta'}U^\alpha U_\mu^{*\beta} D_{\alpha\alpha'}D_{\beta\beta'}^* \tag{2.22}$$

According to the expression above, $\Lambda U^\alpha = \sum_\beta U^\beta D_{\beta\alpha}$ in which $D_{\beta\alpha}$ is the representation of Λ in the four dimensional internal space generated by the product of spin $\frac{1}{2}$ representations. We call attention to the fact that we are using the definition $M_{\alpha'\beta'} \equiv \langle 0|a_{\alpha'}a_{\beta'}^\dagger|0\rangle$. From the previous result, we infer

$$M_{\alpha\beta} = M_{\alpha'\beta'}D_{\alpha\alpha'}D_{\beta\beta'}^* \equiv DMD^\dagger \tag{2.23}$$

The most general form for the matrix D is $D = I + i(\epsilon_1 J_3 + \epsilon_2 L_1 + \epsilon_3 L_2)$ with all the little group generators and infinitesimal parameters $\epsilon_1.\epsilon_2, \epsilon_3$. From $M = DMD^\dagger$ with just $\epsilon_1 \neq 0$, we have $MJ_3^\dagger = J_3 M$.

Since the little group generators are linearly independent, we conclude that $ML_1^\dagger = L_1 M$ and $ML_2^\dagger = L_2 M$. As the matrix $M_{\alpha'\beta'} \equiv \langle 0|a_{\alpha'}a_{\beta'}^\dagger|0\rangle$ represents projections in the Hilbert space, the previously mentioned conditions are useful to fix a particular form for this matrix. By analysing the sign of its eigenvalues we can investigate the necessity of a indefinite Hilbert space metric. Then, we assume the most general form for the matrix and particularize it using the little group constraints. The first condition $MJ_3^\dagger = J_3 M$ implies that

$$M = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & p & q \end{pmatrix} \tag{2.24}$$

The second condition above leads to

$$M = \begin{pmatrix} h & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & f & h \\ 0 & 0 & h & 0 \end{pmatrix} \tag{2.25}$$

The third condition is automatically fulfilled and cannot provide a further constraint on M. The matrix eigenvalues are $h, h, \frac{f}{2} \pm \sqrt{(\frac{f}{2})^2 + h^2}$. This shows that in general, the quantization of a massless particle described by a vector leads to an indefinite metric Hilbert space.

2.2 The Introduction of the B field

In the previous section we demonstrate that the Maxwell equations leads to a trivial quantum theory for the case of a positive metric Hilbert space. Alternatively, we could have conjectured that the inconsistency was due to the fact that we are considering a classical equation of motion in a quantum context. In fact, both claims are true. Besides the little group analysis regarding the non-positiveness of the Hilbert space, we will show that a covariant quantization requires the introduction of a new term on the equations of motion arising due to the lagrangian term proportional to the so-called B-field. This field also has the function to provide a subsidiary condition to define the physical Hilbert subspace with positive norm.

The B-field content is introduced in the following manner. Suppose a lagrangian depending on n fields in such a way that there is a set of $\alpha < n$ that do not have temporal derivatives. It means that the hamiltonian of the system is constrained [4]. Then, to the original lagrangian $L_0(q,\dot{q})$, we add α extra terms

$$L = L_0(q,\dot{q}) + F_\alpha(q,\dot{q})b^\alpha + G(b) \tag{2.26}$$

The term $F_\alpha(q,\dot{q})$ which depends on the α velocities that do not appear in L_0 must be linear with respect to them. The $G(b)$ is a non-derivative quadratic term in the b field. Moreover, it is important that the condition $det\frac{\partial F_\alpha}{\partial \dot{q}_\beta} \neq 0$ must be fulfilled. The equation of motion for b^α generates a class of conditions over the generalized coordinates which, in the electromagnetic case, for example, generates a set of gauge conditions. It is important to stress that we are not adding new degrees of freedom since b^α, as we are going to see, can be algebraically related to the α canonical momenta of the fields originally without time derivatives. The α new canonical momenta are given below

$$\Pi_\alpha = \frac{\partial L}{\partial \dot{b}^\alpha} = 0 \ , \ p_{n-\alpha} = \frac{\partial L}{\partial \dot{q}^{n-\alpha}} \ , \ p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} = \frac{\partial F_\beta}{\partial \dot{q}^\alpha} b^\beta \tag{2.27}$$

From this expression, we note that the condition $det\frac{\partial F_\alpha}{\partial \dot{q}_\beta} \neq 0$ guarantee that b^β are not new degrees of freedom since it is related to p_α through $b^\beta = (\frac{\partial F_\beta}{\partial \dot{q}^\alpha})^{-1} p^\alpha$.

Since F_β is linear in the velocities, the canonical momenta leads to two constraints

$$\phi_\alpha = p_\alpha - \frac{\partial F_\beta}{\partial \dot{q}^\alpha} b^\beta \approx 0 \ , \ \Pi_\alpha \approx 0 \tag{2.28}$$

In order to proceed, we first define the fundamental Poisson brackets (FPB)

$$\{q_A, p_B\} = \delta_{AB} \ , \ \{b_\alpha, \Pi_\beta\} = \delta_{\alpha\beta} \tag{2.29}$$

We can note that the previous constraints are of second class

$$\{\phi_\alpha, \Pi_\beta\} = -\frac{\partial F_\beta}{\partial \dot{q}_\alpha} \tag{2.30}$$

It means that the first class quantization ambiguities [2] [4] are eliminated in this formalism.

Throughout this thesis, we will use the terminology auxiliary field for an extra operator which changes the equations of motion leading to a well-defined system suitable for quantization by the correspondence principle. Moreover, this field leads to a prescription to define the physical states which turn it into a non-observable quantity. The classical equations of motion are recovered in the average between physical states.

[2] Each first class constraint leads to freedom to perform canonical transformations. Moreover, these quantities do not allow us to build a Dirac bracket describing a well-defined system with no constrainsts.

2.3 B field in the context of the Electromagnetism

In this section we intend to apply the B-field formalism for the case of the electromagnetism. First of all, we note that there is no time derivatives for the $A_0(x)$ field, due to the structure of the lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.31}$$

It means that just one B field must be added to provide a system without first class ambiguities from the beginning. The functional F_α must be a Lorentz scalar containing $\partial_0 A_0(x)$ linearly. Moreover it must fulfil the requirement of invertibility developed in the previous section. We adopt the lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + B\partial_\mu A^\mu + \frac{\alpha}{2}B^2 + eJ_\mu A^\mu \tag{2.32}$$

We included in this expression the B field sector and a source term representing an interaction. Regarding the canonical momenta, we have

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = F^{0\mu} + \eta^{0\mu}B \quad , \quad \Pi = \frac{\partial \mathcal{L}}{\partial \dot{B}} = 0 \tag{2.33}$$

The non-vanishing Poisson brackets are the following

$$\Big\{A_\mu(x), \pi^\nu(y)\Big\} = \delta_\mu^\nu \delta^3(x-y) \quad ; \quad \Big\{B(x), \Pi(y)\Big\} = \delta^3(x-y) \tag{2.34}$$

The definition of canonical momenta leads to two constraints $\pi^0 - B \approx 0$ and $\Pi \approx 0$. They are clearly of second class. Therefore, the introduction of this auxiliary field indeed generates a system free of quantum ambiguities. We can display the constraints in the following way $\Phi^I = (\pi^0 - B, \Pi)$. Since $\{\Phi_I, \Phi_J\} = \epsilon_{IJ}$ we have indeed an invertible constraint matrix, a characteristic of second class systems. It is due to the choice of a functional $\det \frac{\partial F_\alpha}{\partial \dot{q}_\beta} \neq 0$ which, in the present case, has the form $\partial_\mu A^\mu$. It means that $\pi_0 = B$ can be used as a strong equality, and it will help us to quantize the system having in mind the correspondence principle.

Then, the reduced brackets are:

$$\Big\{F(x), G(y)\Big\}_D = \Big\{F(x), G(y)\Big\} - \int d^4w d^4z \Big\{F(x), \Phi^I(w)\Big\}\Big[\Big\{\Phi_I, \Phi_J\Big\}\Big]^{-1}\Big\{\Phi^J(z), G(y)\Big\} \tag{2.35}$$

in which $F(x)$ and $G(x)$ are field operator functionals.

Since the introduction of the B field turn the system into a second class one, we can define brackets in which the second class constraints are automatically taken into account, leading to a well-defined system with no constraints and suitable to consider the correspondence principle. The auxiliary field momentum $\Pi(x)$ disappears from the dynamics. The constraints due to momentum definition can be considered in strong form.

The equations of motion for $A_\mu(x)$ and $B(x)$ reads

$$\partial_\mu F^{\mu\nu} - \partial^\nu B = -eJ^\nu \quad , \quad \partial_\mu A^\mu + \alpha B = 0 \tag{2.36}$$

Considering the system coupled to a conserved source, contracting the gradient with the vector field equation leads to

$$\Box B = 0 \tag{2.37}$$

This property will be useful to define the physical Hilbert subspace.

The B field equations of motion impose a class of gauge fixing labeled by the parameter α.

In order to quantize the system, we use the correspondence principle [3]

$$\Big\{A_i(x), \pi^j(y)\Big\} = \delta_i^j \delta^3(x-y) \to \Big[A_i(x), \pi^j(y)\Big]_0 = i\delta_i^j \delta^3(x-y) \tag{2.38}$$

and

$$\Big\{A_0(x), B(y)\Big\} = \delta^3(x-y) \to \Big[A_0(x), B(y)\Big]_0 = i\delta^3(x-y) \tag{2.39}$$

The index zero denotes quantities taken at equal times.

The next step consists in identifying the temporal derivatives of the fields in terms of their momenta. Then, we have $\dot{A}_k = (\pi_k + \partial_k A_0)$. From the gauge fixing condition, we calculate $\dot{A}_0 = -(\partial_k A^k + \alpha B)$. The temporal derivative of $B(x)$ can be obtained from the time component of the $A_\mu(x)$ equations of motion

$$\partial_\mu F_0^\mu - \dot{B} = -ej^0 \to \dot{B} = eJ_0 - \partial_k F^{0k} \tag{2.40}$$

Considering these results, we have

$$\Big[A_\mu(x), \dot{A}_0(y)\Big]_0 = \Big[A_\mu(x), -\partial_k A^k - \alpha B\Big]_0 = -i\alpha\delta_{\mu 0}\delta^3(x-y)\ ; \tag{2.41}$$

$$\Big[A_\mu(x), \dot{A}_k(y)\Big]_0 = [A_\mu(x), (\pi_k + \partial_k A_0)]_0 = i\delta_{\mu k}\delta^3(x-y)\ ;$$

$$\Big[A_\mu(x), \dot{A}_\nu(y)\Big]_0 = -i\big(\eta_{\mu\nu} - (1-\alpha)\delta_\mu^0\delta_\nu^0\big)\delta^3(x-y)$$

The commutator between A_μ and B is obtained as follows

$$\Big[A_\mu(x), B(y)\Big]_0 = \Big[A_\mu(x), \pi^0(y)\Big]_0 = i\delta_{\mu 0} \quad , \tag{2.42}$$

$$\Big[A_\mu(x), \dot{B}(y)\Big]_0 = \Big[A_\mu(x), eJ_0 - \partial_k F^{0k}(y)\Big]_0 = \Big[A_\mu(x), -\partial_k \dot{A}_k(y)\Big]_0 = -i\partial_k\delta_\mu^k\delta^3(x-y)$$

We also have the commutators

$$\begin{aligned} &\Big[B(x), B(y)\Big]_0 = 0\ , \\ &\Big[B(x), \dot{B}(y)\Big]_0 = \Big[B(x), eJ_0(y) - \partial^k \dot{A}_k(y) + \partial_k\partial^k A_0(y)\Big]_0 \\ &= \Big[B(x), -\partial^k(\pi_k + \partial_k A_0) + \partial_k\partial^k A_0(y)\Big]_0 = 0 \end{aligned} \tag{2.43}$$

[3]Considering the Dirac brackets which leads to a well-defined system with no constraints.

In order to find the complete structure of the commutator between the photon and the auxiliary field at unequal times, it is useful to introduce an integral representation for the $B(x)$ field. Since $B(x)$ obeys the harmonic equation, we can define a conserved non-local current

$$J_\mu(z,y) = \partial_\mu \Delta(y-z)B(z) - \Delta(y-z)\partial_\mu B(z) \tag{2.44}$$

where $\Delta(y-z)$ is the Pauli-Jordan distribution

$$\Box\Delta(x-y) = 0 \quad , \quad \partial_0\Delta(x-y)_{x_0=y_0} = -\delta^3(x-y) \quad , \quad \Delta(x-y)_{x_0=y_0} = 0 \tag{2.45}$$

From this expression, we can find the integral representation

$$B(x) = -\int d^3u\Big[\Delta(x-u)\partial_0^u B(u) - \partial_0^u\Delta(x-u)B(u)\Big] \tag{2.46}$$

This integral representation can be used to obtain the commutators at unequal times. The fact that the expression do not depend on u_0 allow us to use equal-time data to develop the full structure for the commutators.

$$\begin{aligned}\Big[A_\mu(x), B(y)\Big] &= \Big[A_\mu(x), \int_s d^3x'\Big[B(x')\partial^{0'}\Delta(y-x') - \partial^0 B(x')\Delta(y-x')\Big]\Big] \\ &= \int_s d^3x'\Big[\partial_0'\Delta(y-x')i\delta_{\mu 0}\delta^3(x-x') + \Delta(y-x')\partial_k\delta^3(x-x')\delta_\mu^k\Big]\end{aligned} \tag{2.47}$$

where we have fixed $x_0' = x_0$ in order to use the equal-time data.

Then, integrating by parts, we get

$$\Big[A_\mu(x), B(y)\Big] = -i\partial_\mu^x\Delta(x-y) \tag{2.48}$$

The fact that $\Big[B(x), B(y)\Big] = 0$ follows from this same logic. It is interesting to note that the auxiliary field seems to generate the gauge symmetry. We will further analyze it in details.

From the structure of the previous transformations, we find that this auxiliary field commutes with the physical electric and magnetic fields

$$\Big[F_{\mu\nu}(x), B(y)\Big] = 0 \tag{2.49}$$

At this point is important to comment about the case in which the source $J_\mu(x)$ has a quantum microscopic Fermionic structure. The Dirac sector of the lagrangian is

$$\mathcal{L}_D = +i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi - eA_\mu\bar\psi\gamma^\mu\psi \tag{2.50}$$

in which ψ denotes a Dirac spinor.

Considering the fundamental Poisson brackets and the definition of canonical momenta, the correspondence principle furnishes

$$\Big[\psi(x), \bar\psi(y)\Big]_0 = \gamma^0\delta^3(x-y) \tag{2.51}$$

Then, using the relation $\partial_0 B(x) = -\partial^k F_{0k} + e\ J_0$, we find

$$\Big[\psi(x), \partial_0 B(y)\Big]_0 = e\ \psi(x)\ \delta^3(x-y)\ ,\ \Big[\bar{\psi}(x), \partial_0 B(y)\Big]_0 = -e\ \bar{\psi}(x)\ \delta^3(x-y) \tag{2.52}$$

The initial data and the integral representation for the auxiliary field leads to the full commutator structure

$$\Big[\psi(x), B(y)\Big] = e\psi\ \Delta(x-y) \tag{2.53}$$

Then, we conclude that $B(x)$ also acts as a generator of local symmetry transformations for the Fermions. It is also possible to find $\Big[\psi(x), J_\mu(y)\Big] = 0$ with $J_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$. It becomes clear that $B(x)$ commute with the observables of the theory.

Now, we present a brief digression about the role of the auxiliary field in creating a subsidiary condition to define the physical Hilbert space. Since $B(x)$ obeys the harmonic equation, it can be decomposed in its positive and negative frequency parts. The physical states are defined as

$$B^+(x)|\psi\rangle = 0 \tag{2.54}$$

Then, states in the kernel of $B^+(x)$ have positive or null norm and they generate the physical Hilbert subspace denoted as $\mathcal{V}_{phys.}$.

Since the vanishing norm cannot be detected in $\mathcal{V}_{phys.}$ we redefine the positive norm Hilbert space as $\mathcal{H}_{phys.} = \frac{\mathcal{V}_{phys.}}{\mathcal{V}_0}$ in which $\mathcal{V}_0$ designates the subspace of vanishing norm. We also note that the condition is not Hermitian, but it is a desired property since in the opposite case it will lead to ambiguities [1].

The next step is to show that this condition is Poincaré invariant. Using the fact that the temporal evolution is dictated by the temporal component of the momentum operator P_0

$$\Big[iP_0, B^+(x)\Big] = \partial_0 B^+(x) \tag{2.55}$$

We conclude that a state obeying the subsidiary condition will also obey it in a posterior instant

$$\Big[iP_0, B^+(x)\Big]|\psi\rangle = \partial_0 B^+(x)|\psi\rangle = \partial_0(B^+(x)|\psi\rangle) = 0 \tag{2.56}$$

where the last step was possible due to the fact that in the Heisenberg picture the states are time independent.

Since the condition for the kets $B^+(x)|\psi\rangle = 0$ is translated as $0 = \langle\psi|B^-(x)$ for the Bra dual space, the field $B(x) = B^+(x) + B^-(x)$ has vanishing matrix elements between physical states $\langle g|B(x)|f\rangle = 0$.

It means that although the introduction of the B field changes the operator equation of motion, the classical equations are recovered in the average between physical states

$$\langle g|\partial_\mu F^{\mu\nu}|f\rangle = -\langle g|eJ^\nu|f\rangle \tag{2.57}$$

where an analogous procedure allows for finding $\partial_\mu A^\mu(x) = 0$ between physical states.

Although the B-field term breaks part of the local freedom, the system is still invariant with relation to the following residual gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x) \quad , \quad B(x) \to B(x) \quad , \quad \psi(x) \to \psi(x) + ie\Lambda\psi(x) \quad , \quad \Box\Lambda(x) = 0 \tag{2.58}$$

The associate Noether conserved charge due to this symmetry reads

$$Q = \int d^3x \Big\{ B\partial_0\Lambda - \partial_0 B\Lambda \Big\} \tag{2.59}$$

This equation reinforce the idea that the B-field is related to the generation of the local symmetry transformation and then is closely related to these spurious sectors of the model.

2.4 The Free Electromagnetic Field

This section is devoted to the study of the free electromagnetic theory. Then, we analyze again the features of the equations of motion

$$\partial_\mu F^{\mu\nu} - \partial^\nu B = 0 \quad , \quad \partial_\mu A^\mu + \alpha B = 0 \tag{2.60}$$

Those equations can be combined to yield

$$\Box A_\mu - (1-\alpha)\partial_\mu B = 0 \tag{2.61}$$

Considering that $\Box B(x) = 0$, we conclude

$$\Box^2 A_\mu(x) = 0 \tag{2.62}$$

that the photon field obeys a dipole ghost equation.

The Green function for this equation is given in terms of the Pauli-Jordan distribution as

$$E(x) = \frac{1}{(2\pi)^3} \int d^4p \, \frac{\partial}{\partial m^2} \delta(p^2 - m^2)_{m=0} \, e^{-ip_\mu x^\mu} \tag{2.63}$$

It has the following properties

$$\Box E(x-y) = \Delta(x-y) \quad , \quad \Box^2 E(x-y) = 0 \quad , \quad \partial_0^3 E(x-y)|_{x^0=y^0} = -\delta^3(x-y) \tag{2.64}$$

Those distribution properties allow us to find the integral representation for the photon field

$$\begin{aligned} A_\nu(x) = \int_S d^3x' \Big[& A_\nu(x')\partial^{0'}\Delta(x-x') + (1-\alpha)\partial'_\nu B(x')\partial^{0'} E(x-x') \\ & - \partial^{0'} A_\nu(x')\Delta(x-x') - E(x-x')(1-\alpha)\partial^{0'}\partial'_\nu B(x') \Big] \end{aligned} \tag{2.65}$$

where the system is independent of x'_0 and the obtainment of this expression follows the same line as that of the integral representation of the auxiliary field.

In order to define the photon field commutator at unequal times, we use this representation and the initial data, since the system is z_0 independent, to get

$$\Big[A_\mu(x), A_\nu(y)\Big] = \int d^3z \Big\{ i\Delta(y-z)(\eta_{\mu\nu} - (1-\alpha)\delta^0_\mu\delta^0_\nu)\delta^3(x-z)$$
$$- i(1-\alpha)\partial^z_0\partial^z_k E(y-z)\delta_{\mu 0}\delta_{k\nu}\delta^3(x-z) - i\partial^z_0\partial^z_k E(y-z)(1-\alpha)\delta^0_\nu\delta^k_\mu$$
$$+ i(1-\alpha)(\partial^z_k)^2 E(y-z)\delta^0_\nu\delta^0_\mu\delta^3(x-z) - i(1-\alpha)\partial^z_l\partial^z_k E(y-z)\delta^3(x-z)\Big\} \tag{2.66}$$

We used integration by parts.

In order to simplify this expression, we consider $\Box E(x) = \Delta(x)$ which implies $\partial_k\partial^k E(x) = -\Delta(x) + \partial_0^2 E(x)$. Then, we get

$$\Big[A_\mu(x), A_\nu(y)\Big] = \int d^3z \Big\{ i\Delta(y-z)(\eta_{\mu\nu} - (1-\alpha)\delta^0_\mu\delta^0_\nu)\delta^3(x-z) - i(1-\alpha)\partial^z_0\partial^z_k E(y-z)\delta^3(x-z)$$
$$- i(1-\alpha)\partial^z_0\partial^z_k E(y-z)\delta^0_\nu\delta^0_\mu\delta^3(x-z) + i(1-\alpha)(-\Delta(y-z) + \partial^z_0\partial^z_0 E(y-z))\delta^0_\nu\delta^0_\mu\delta^3(x-z)$$
$$- i(1-\alpha)\partial^z_l\partial^z_k E(y-z)\delta^k_\nu\delta^l_\mu\delta^3(x-z)\Big\} \tag{2.67}$$

The final structure for this commutator can be put in the form

$$\Big[A_\mu(x), A_\nu(y)\Big] = -i\eta_{\mu\nu}\Delta(x-y) + i(1-\alpha)\partial_\mu\partial_\nu E(x-y) \tag{2.68}$$

2.5 The Quantum Electrodynamics

In this section, we analyze the best way to proceed in the interacting case. We consider the system coupled to Fermionic matter. The photon field equation of motion from the previous sections coupled to a current density with non-trivial quantum microscopic structure $J_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$ reads

$$\Box A_\mu(x) = -eJ_\mu(x) + (1-\alpha)\partial_\mu B(x) \tag{2.69}$$

It helps us to find an useful differential equation for the commutator

$$\Box^x\Box^y\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle = \langle 0|\Big[\big(-eJ_\mu(x) + (1-\alpha)\partial_\mu B(x)\big), \big(-eJ_\mu(y) + (1-\alpha)\partial_\mu B(y)\big)\Big]|0\rangle$$
$$= e^2\langle 0|\Big[J_\mu(x), J_\nu(y)\Big]|0\rangle$$

We have considered the fact that the auxiliary field commutes with the current operator and has vanishing norm.

The most general solution to the previous equation splits in a homogeneous plus a partial solution sector

$$\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle = \big(a\eta_{\mu\nu} + b\partial_\mu\partial_\nu\big)\Delta(x-y) + \big(c\eta_{\mu\nu} + d\partial_\mu\partial_\nu\big)E(x-y)$$
$$+ \int_{0+}^\infty ds\ s^{-2}(-s\eta_{\mu\nu} - \partial_\mu\partial_\nu)\Pi(s)\Delta(x-y;s) \tag{2.70}$$

Using the initial data and requiring that it must be compatible with the gauge condition, we can particularize this solution. Another point, is the spectral-representation [5] employed to express the current commutator. Since this kind of representation is very important in quantum field theory, we present a short derivation of its properties and then return and obtain the final form for the photon-photon commutator structure.

First of all, we analyze the Wightman function below

$$e^2\langle 0|J_\mu(x)J_\nu(y)|0\rangle = e^2(2\pi)^{-4}\int d^4p\sum_\lambda \langle 0|J_\mu(x)|p,\lambda\rangle\langle\lambda,p|J_\mu(y)|0\rangle \tag{2.71}$$

In this equation we have inserted the identity in terms of the completeness relation of a set of eigenstates of the momentum operator $I = (2\pi)^{-4}\int d^4p \sum_\lambda |p,\lambda\rangle\langle\lambda,p|,$ with $\hat{p}_\mu|p,\lambda\rangle = p_\mu|p,\lambda\rangle$. The discrete sum is over others quantum numbers such as spin.

Considering the fact that the vacuum state is Poincaré invariant and the relation between the source at the origin with the one translated to the space time point x_μ given by an unitary representation of the translation sector of the Poincaré group, $J_\mu(x) = U(x)J_\mu(0)U^\dagger(x),$ with $U(x) = e^{i\hat{P}_\mu x^\mu}$, we get

$$e^2\langle 0|J_\mu(x)J_\nu(y)|0\rangle = e^2(2\pi)^{-4}\int d^4p\sum_\lambda \langle 0|J_\mu(0)|p,\lambda\rangle\langle\lambda,p|J_\nu(0)|0\rangle e^{-ip_\mu(x-y)^\mu} \tag{2.72}$$

Regarding the intermediate states due to the completeness relation, just the ones with positive norm will contribute because $J_\mu(x)$ is an observable, as can be inferred by the fact that it commutes with the auxiliary field defining the physical space.

It is important to define the following structure

$$\Pi_{\mu\nu}(p)\theta(p_0) \equiv \sum_\lambda e^2\langle 0|J_\mu(0)|p,\lambda\rangle\langle\lambda,p|J_\nu(0)|0\rangle \tag{2.73}$$

Due to the current conservation, this object must have the transverse tensor structure [5]

$$\Pi_{\mu\nu}(p) = (-p^2\eta_{\mu\nu} + p_\mu p_\nu)\Pi(p^2) \tag{2.74}$$

Then, the current-current projection can be written as

$$e^2\langle 0|J_\mu(x)J_\nu(y)|0\rangle = (2\pi)^{-4}\int d^4p\int_0^\infty d\mu^2\delta(p^2-\mu^2)e^{-ip_\mu(x-y)^\mu}(-p^2\eta_{\mu\nu}+p_\mu p_\nu)\Pi(\mu^2)\theta(p_0) \tag{2.75}$$

From the definition of the positive frequency part of the Pauli-Jordan distribution $\Delta^+(x-y,m^2) = (2\pi)^{-4}\int d^4p\ \delta(p^2-m^2)\ \theta(p_0)e^{-ip_\mu(x-y)^\mu}$, we have

$$e^2\langle 0|J_\mu(x)J_\nu(y)|0\rangle = \int_0^\infty d\mu^2(-p^2\eta_{\mu\nu} + p_\mu p_\nu)\Pi(\mu^2)\Delta^+(x-y,\mu^2) \tag{2.76}$$

After the presentation of this content, we return to the discussion of equation (2.70). In order to be in agreement with the gauge condition $\partial_\mu A^\mu + \alpha B = 0$, the photon commutator must obey the constraint

$$\langle 0|\partial^\mu A_\mu(x)A_\nu(y)|0\rangle = -\langle 0|\alpha B(x)A_\nu(y)|0\rangle = -\alpha\partial_\nu\Delta(x-y) \tag{2.77}$$

It fixes $c = 0$ and $a + d = -\alpha$. Taking the time derivative with relation to y_0 in (2.70) and considering the expression at equal times leads to

$$\mathrm{i}\eta_{kl}\delta^3(x-y) = -i\Big(a\eta_{kl} + b\partial_k\partial_l\Big)\delta^3(x-y) + i\int_{0+}^{\infty} ds\ s^{-2}\Pi(s)\Big(s\eta_{kl} + \partial_k\partial_l\Big)\delta^3(x-y) \tag{2.78}$$

Then, we determine the constants

$$-a = 1 - \int_{0+}^{\infty} ds\ s^{-1}\Pi(s) \equiv Z_3 \quad , \quad b = \int_{0+}^{\infty} ds\ s^{-2}\Pi(s) \equiv Z_3 K \tag{2.79}$$

Finally, we have

$$\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle = iZ_3\Big\{(-\eta_{\mu\nu} + K\partial_\mu\partial_\nu)\Delta(x-y) + (1 - Z_3^{-1}\alpha)\partial_\mu\partial_\nu E(x-y)\Big\} \tag{2.80}$$

$$+ \int_{0+}^{\infty} ds\ s^{-2}(-s\eta_{\mu\nu} - \partial_\mu\partial_\nu)\Pi(s) iD(x-y;s)$$

The constant Z_3 is called the photon wave function renormalization constant. We postulate that $Z_3 > 0$ since the transverse part of $A_\nu(x)|0\rangle$ belongs to the physical Hilbert space. The term proportional to K can be eliminated from this expression by the redefinition $A'_\mu(x) \to A_\mu(x) - \frac{1}{2}Z_3 K\partial_\mu B(x)$.

2.6 Asymptotic Fields

For the asymptotic photon we assume that it keeps its dispersion relation

$$\langle 0|\Big[A'_\mu(x), A'_\nu(y)\Big]|0\rangle = -iZ_3\eta_{\mu\nu}\Delta(x-y) + iZ_3\Big(1 - Z_3^{-1}\alpha\Big)\partial_\mu\partial_\nu E(x-y) \tag{2.81}$$

The solution can be renormalized as $A^{as}_\mu(x) \equiv Z_3^{-\frac{1}{2}}A'_\mu(x)$. It acquires the form

$$\langle 0|\Big[A^{as}_\mu(x), A^{as}_\nu(y)\Big]|0\rangle = -i\eta_{\mu\nu}\Delta(x-y) + i(1-\alpha_r)\partial_\mu\partial_\nu E(x-y) \tag{2.82}$$

with the renormalization of the non-physical parameter reads $\alpha_r = Z_3^{-1}\alpha$.

The photon field Fourier transform for the positive frequency modes is

$$A^{as}_\mu(x)^+ = (2\pi)^{-\frac{3}{2}}\int d^4p\ a_\mu(p)e^{-ip.x} \tag{2.83}$$

whose norm is

$$\langle 0|\Big[a_\mu(p), a^\dagger_\nu(q)\Big]|0\rangle = \theta(p_0)\big(-\eta_{\mu\nu}\delta(p^2) - (1-\alpha_r)p_\mu p_\nu\delta'(p^2)\big)\delta^4(p-q) \tag{2.84}$$

The Fourier transform of the equations of motion for the asymptotic fields, reads

$$p^2 b(p) = 0, \quad p^2 a_\mu(p) - i(1-\alpha_r)p_\mu b(p) = 0, \quad p_\mu a^\mu(p) - i\alpha_r b(p) = 0 \tag{2.85}$$

In terms of the Fourier modes, the subsidiary condition is $b(p)|f\rangle = 0$. We fix a frame with $p_1 = p_2 = 0$ with $p_0 > 0$. Then, the transverse sector of the photon states, which commute with $b(p)$ is $|p,\ j\rangle_T \equiv a_j^\dagger(p)|0\rangle$ with $j = 1, 2$.

This state has indeed a positive norm since it satisfies the subsidiary condition

$$\langle T, q, j|p, i, T\rangle = \delta_{ij}\delta(p^2)\delta^4(p - q) \tag{2.86}$$

The longitudinal mode $|p\rangle_L \equiv a_3^\dagger(p)|0\rangle$ has the non-positive definite norm

$$\langle L, p|q, L\rangle = \big(\delta(p^2) - (1 - \alpha_r)p_3^2\delta'(p^2)\big)\delta^4(p - q) \tag{2.87}$$

as expected since it do not fulfil the requirement to be a physical state. The so-called scalar photon state $b(p)|0\rangle$ has vanishing norm.

The physical quotient space $\mathcal{H}_{phys.} = \frac{\mathcal{V}_{phys.}}{\mathcal{V}_0}$ is generated by these transverse states. Then, according to the appendix of [1], there is a theorem stating that if the Hamiltonian is Hermitian, and the total Hilbert space has a unitary S matrix, its restriction to the physical quotient space $S_{phys.}$ is also unitary.

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Chapter 3

An introduction to Quantum Hall Effect, From the Drude Conductivity Model to the Chern-Simons Field Description

3.1 Experimental Origin of the Classical Hall Effect

Here, we intend to discuss the Hall effect from its classical origins to the quantum structure effectively described by field theory methods [1].

The classical Hall effect was discovered in 1879 by Edwin H. Hall, from J. Hopkins university [2]. Contrary to the Maxwell's Treatise on Electricity and Magnetism, which claimed that the effect of a magnetic field on a fixed conductor is just to create a fast transient induction currents due to the variation of magnetic flux due to turning on the external magnetic field, Hall demonstrated that by applying an electric field to this system a transverse current arises and the resistance to its establishment is

$$R_H = \frac{\mathcal{B}}{nq} \tag{3.1}$$

where $\mathcal{B}$ denotes the magnetic field, n denotes the charge density and q denotes its charge.

It is interesting to mention that this effect can be used as a method to infer the charge or the density of charge carriers of the sample. The qualitative behaviour of the Hall resistivity with the increase of a magnetic field is given in figure 3.1.

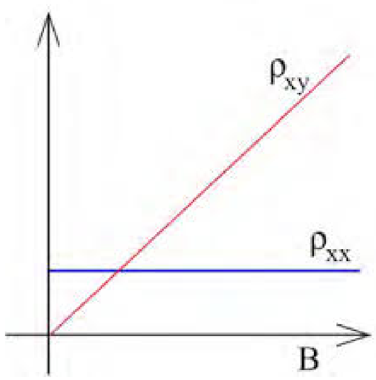

Figure 3.1: Transverse and longitudinal resistivity dependence on the external magnetic field. [3]

3.2 The Theoretical Description using the Drude Model

In order to have a realistic description, the properties of the medium in which the charged particle propagates should be taken into account. Adding to the Lorentz force a term representing the effect of the material medium on the charge carriers, we get the equation of motion [3]

$$m\frac{d\vec{V}}{dt} = -e\vec{E} - e\vec{V} \times \vec{B} - m\frac{\vec{V}}{\tau} \tag{3.2}$$

where the τ denotes the average time between two successive charge collisions with the material medium. This is the Drude model. We are considering a magnetic field vector perpendicular to the plane of the sample.

The equilibrium solution (steady current) $\frac{d\vec{V}}{dt} = 0$ gives

$$J_i = \sigma_{ij} E_j \quad , \quad \sigma = \frac{ne^2\tau}{m(1+\omega_B^2\tau^2)} \begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix} \tag{3.3}$$

where $\omega_B = \frac{eB}{m}$ is the cyclotron frequency due to the interaction with the magnetic field. The current density is written in terms of the variables of the model as $J_i = -neV_i$.

The resistivity ρ (σ^{-1}) is then

$$\rho = \frac{m}{ne^2\tau} \begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix} \tag{3.4}$$

For $\tau \to \infty$ (no influence from medium) $\sigma_{xx} = \rho_{xx} = 0$!. And $\rho_{xy} = \frac{m\omega_B}{ne^2} = R_H = \frac{B}{ne}$.

It means that the transverse conducting properties are independent of the material medium content which leads to its universality and robustness. Namely, this is why this effect is observed in a wide variety of materials and furnishes a clear experimental evidence of its existence.

3.3 The Integer Quantum Hall Effect

The integer quantum Hall effect (IQHE) was first observed in 1980 by v. Klitizing [4] in a MOSFET (field effect semiconductor transistor.) in an environment of low temperatures. For both IQHE and fractional quantum Hall effect (FQHE), the sample has between 10^{11} and 10^{12} electrons per cm^2, [4, 5].

The transverse resistance versus external magnetic field graphic has plateaus at some fixed values

$$\rho_{xy} = \frac{h}{e^2\nu} \tag{3.5}$$

with ν belonging to the integer numbers.

Its dependence on the external magnetic field is given in figure 3.2, where we note plateaus at some fixed quantized levels and peaks in the longitudinal resistance in the transition to the next plateau.

Since 1990, the IQHE quantum $R_K = \frac{h}{e^2} = 25812.807\Omega$ is regarded as the resistance standard with the subscript K being in honour of v.Klitizing [6].

3.3.1 The Integer Quantum Hall Effect; Landau Levels

The lagrangian of a charged particle under external magnetic field perpendicular to the plane is given by

$$L = \frac{m\dot{x}^2}{2} - e\dot{x}_i A^i \tag{3.6}$$

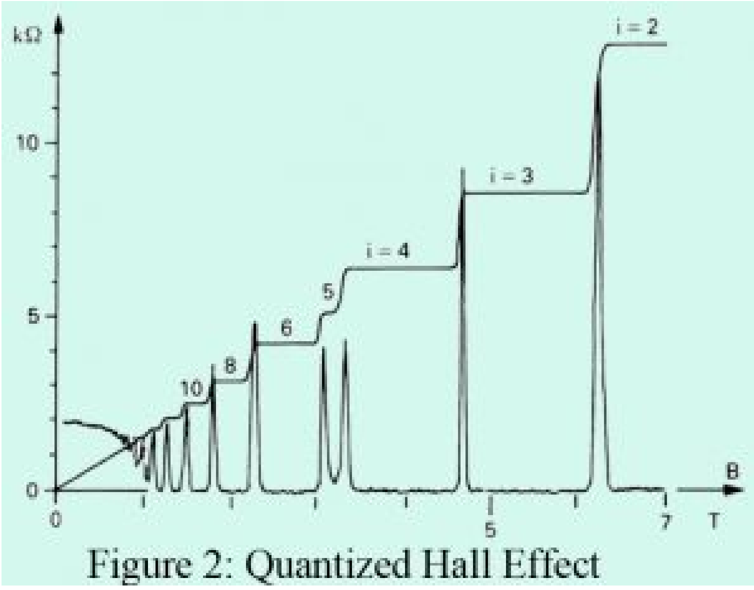

Figure 3.2: The graph relating the transverse resistance and the magnetic field reveals plateaus with allowed quantized levels.[6]

The $A^i(x)$ field denotes the vector potential. We choose the Landau gauge $\vec{A} = xBj$ where j denote the unitary vector in the y direction. It means that there are translation symmetry just in the y axis.

The canonical momenta and the Hamiltonian are, respectively,

$$p_i = m\dot{x}_i - eA_i \ , \ H = \frac{(p_i + eA_i)^2}{2m} \tag{3.7}$$

From the well known fundamental Poisson brackets and the correspondence principle [1] [7], we get

$$\Big[\pi_x, \pi_y\Big] = i\hbar B \quad , \quad \Big[a, a^\dagger\Big] = 1 \tag{3.8}$$

with $\pi_i = m\dot{x}_i$ and $a = \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y)$.

The problem reduces to the quantum harmonic oscillator algebra. The Hamiltonian can be rewritten as

$$H = \frac{1}{2m}\pi_i\pi^i = \hbar\omega_B(a^\dagger a + \frac{1}{2}) \tag{3.9}$$

where ω_B is the cyclotron frequency.

The spin degree of freedom is not taken into account since, although, in principle, the energy displacement due to the Zeeman effect has the same value than the Landau level quanta, in practice, the cyclotron frequency must be computed with regard to an effective mass due to interaction with environment. For Ga AS system, the Landau quanta is 70× bigger than the Zeeman effect one [3]. The magnetic field is considered to be intense and therefore we consider that the spins are all polarized to minimize this interaction energy [6].

According to the chosen gauge, the y translation invariance motivates us to consider the following ansatz for the wave function $\psi_k(x, y) = e^{iky} f_k(x)$.

The solution has two labels defining the eigenstates of the Hamiltonian

$$\psi_{n,k}(x, y) \sim e^{iky} H_n(x + kl_B^2) e^{-\frac{(x+kl_B^2)^2}{2l_B^2}} \tag{3.10}$$

[1] $i\hbar\{,\} \rightarrow [,]$

n denotes the Landau level, k is a real number and H_n are the Hermite Polynomials.

The degeneracy of the system is given by the formula

$$\mathcal{N} = eBA/h \tag{3.11}$$

where A is the area of the planar sample.

The quantum Hall effect phenomenology is achieved by introducing an electric field in the x direction leading to a change in the Hamiltonian as $H \to H + eEx$. The new eigenstates are

$$\psi(x,y) = \psi_{n,k}(x + \frac{mE}{eB^2}, y) \tag{3.12}$$

The energy now depends on k

$$E_{n,k} = \hbar\omega_B(a^\dagger a + \frac{1}{2}) - eE(kl_B^2 + \frac{eE}{m\omega_B^2}) + \frac{mE^2}{2B^2} \tag{3.13}$$

A group velocity in the y direction, related to the transverse current, arises

$$V_y = \frac{\partial E_{n,k}}{\partial k} = -\frac{E}{B} \tag{3.14}$$

It will be useful for us to also consider the so-called symmetric gauge (which is convenient for FQHE)

$$2\vec{A} = \vec{r} \times \vec{B} = -yBi + xBj \tag{3.15}$$

It breaks translation invariance but there is rotational invariance yet. Defining the variable $z = x - iy$, we have the wave function

$$\psi_m(x,y) \sim (\frac{z}{l_B})^m e^{-\frac{z^* z}{4l_B^2}} \quad , \quad J_z\psi_m = \hbar m\psi_m \tag{3.16}$$

which is also an eigenstate of angular momentum.

3.4 Quantized Conductivity

Since the mechanical momenta can be written as $m\dot{x}_i = p_i + eA_i$ and the current is defined as $I_i = -e\dot{x}_i$, we have (using Landau gauge)

$$I = -\frac{e}{m}\sum_n^{\nu}\sum_k \langle\psi_{n,k}|(-i\vec{\nabla} + e\vec{A})\psi_{n,k}\rangle \tag{3.17}$$

$$I_x = 0 \tag{3.18}$$

where the sum over k is related to the degeneracy of the system.
The last expression resembles the expectation value of the momentum in a harmonic oscillator since $A_x = 0$. We also have

$$I_y = Ae^2\nu\frac{E}{h} \quad , \quad \rho_{xy} = \frac{h}{e^2\nu} \tag{3.19}$$

ν is the filling fraction, the ratio of the number of electrons and the degeneracy of the Landau levels. A is the area of the sample.

In a sample with boundary, the bulk electrons are confined in an area delimited by its cyclotron movement. At the boundary, there are non-vanishing velocity in the y direction. Each side has a different direction of motion for the conducting states, see figure 3.3.

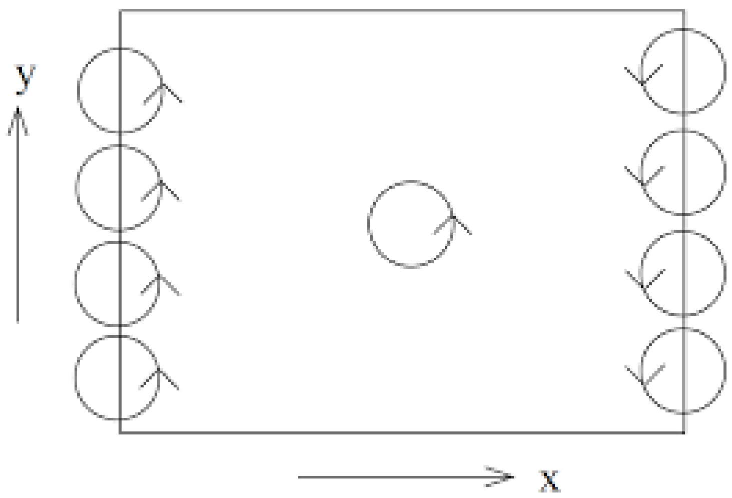

Figure 3.3: Due to the cyclotron movement, the conducting states appear just at the boundary and each one with opposite direction.

3.5 Adding Impurity to the Sample

Consider a very clean sample, in which the impurity may be modeled by a small perturbation. We also model it as a localized potential $|\vec{\nabla}V| << \frac{\hbar\omega_B}{l_B}$. The energy graph change as in Figure 3.4.

Since the magnetic field is intense, we focus on a situation with just an external B field (perpendicular to the sample.) and the Lorentz force

$$x(t) = X - Rsen(\omega_B t + \phi) \quad , \quad y(t) = Y + Rcos(\omega_B t + \phi) \tag{3.20}$$

The radius R and the phase ϕ and the center (X,Y) are arbitrary.

In a situation with impurity the Hamiltonian change as $H \to H + V$. Since $X = x - \frac{\pi_y}{m\omega_B}$ and $Y = y + \frac{\pi_x}{m\omega_B}$, we have

$$i\hbar\dot{X} = [X, H + V] = il_B^2\frac{\partial V}{\partial Y} \quad , \quad i\hbar\dot{Y} = [Y, H + V] = -il_B^2\frac{\partial V}{\partial X} \tag{3.21}$$

The guiding center drift along equipotentials. $\dot{\vec{R}}$ is perpendicular to $\vec{\nabla}V$.

3.5.1 Localized and Extended States

The localized states are the bulk ones and the extended states lie on the boundary. According to figure 3.4, since the Landau Levels (LL) are gaped, by increasing the energy, the electrons starts to populate the bulk energy levels and, due to the Fermionic nature, they cannot occupy the same state. Therefore, when the energy reaches the threshold value the electrons start to fill the next (LL) and the transverse current increases [6, 8]. It explains the formation of plateaus as due to the invariance of the transport properties under small variations of the parameters of the system.

A similar idea explains the behaviour of figure 3.2. with increasing magnetic field the degeneracy as well as the distance between two consecutive Landau levels increase and when a given previously filled level becomes bigger than the Fermi energy the electrons start to scatter to fill the next lower energy level which is possible due to the increase of the degeneracy. The filling factor decreases and the transverse resistivity goes to the next plateau. In this transition both the longitudinal current and the longitudinal resistance have a peak since the latter is related

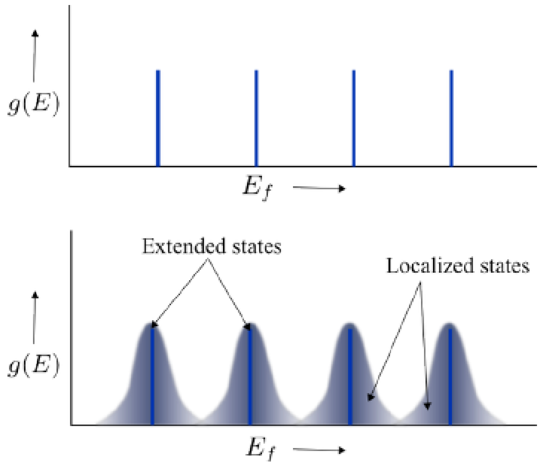

Figure 3.4: The inclusion of the impurity defines a continuous density of states distribution as a function of energy, instead of just the discrete Landau levels [6].

to the former as $\rho_{xx} = \sigma_{xx}/\left(\sigma_{xx}^2 + \sigma_{xy}^2\right)$. After the lower energy level is filled, the electron have nowhere to scatter and the longitudinal resistance falls back to zero again. We also note that the bigger the magnetic field is, the bigger the amplitude of the peak becomes since the degeneracy increases and there are more states for the electrons to scatter into.

In a crystal structure, the momenta lies on a torus T^2 with $-\pi/a < k_x < \pi/a$ and $-\pi/b < k_y < \pi/b$ with a being the distance between two consecutive sites in the x direction and b being the distance between two sites in the y direction. From the eigenstates one get the Berry connection $A_i(k) = -i\langle U(k)\frac{\partial}{\partial k^i}|U(k)\rangle$. The curvature $F_{xy} = \frac{\partial}{\partial k^x}A_y - \frac{\partial}{\partial k^y}A_x$ is used to construct the T.K.N.N topological invariant, which is related to the conductance as

$$C = -\frac{1}{2\pi}\int_{T^2} d^2k F_{xy} \quad , \quad \sigma_{xy} = \frac{e^2}{2\pi\hbar}\sum_\alpha C_\alpha \tag{3.22}$$

The C_α denotes the invariant in each filled band. It is valued on the natural numbers. It is robust against small variation of the system's parameters showing that the QHE phenomenology has indeed a topological nature.

3.6 Fractional Quantum Hall Effect

In 1983 it was discovered quantum Hall systems [5] (GaAs) with fractional filling fractions ν with $\nu = 1/3$ and latter with $\nu = 2/5, 7/3, ...$. So, they are semi-filled Landau levels.

In order to explain it we must take into account the interactions of electron inside a given LL. It is an interacting system with an extremely high number of particles. An exact answer is too much difficult, then one must find an approximation method. We have the hierarchy of scales

$$\hbar\omega_B >> E_{Coulomb} >> V_{disorder} \tag{3.23}$$

The Coulomb interaction is of the form $V(|\vec{r}_1 - \vec{r}_2|)$, so the angular momentum is conserved. We work on the symmetric gauge on a wave function for many particles which is also eigenstates of angular momentum, the Laughlin wave function

$$\psi(z_1, ..., z_n) = \Pi_{i<j}(z_i - z_j)^m \exp\left(-\sum_{i=1}^{N}|z_i|^2/4l_B^2\right) \tag{3.24}$$

m is odd in order to respect the Fermionic nature under exchange of z_i and z_j.

3.6.1 The Filling Factor

We are going to show that we have indeed the right filling fraction. Focusing on the particle 1 the Laughlin wave function gives $m(N-1)$ as the maximum exponent of the particle at z_1. So, $J_{max.} = m(N-1)\hbar$.

Treating the particle 1 approximately as a free one, we see that the its possible to show that its wave function has support on an area of radius $R = \sqrt{2mN}l_B$. So, the area $A \sim 2\pi mNl_B^2$. The degeneracy is $\mathcal{N} = eBA/h = mN$, so the number of particles over the degeneracy gives the filling factor

$$\nu = \frac{1}{m} \tag{3.25}$$

3.6.2 Charged Excitations: Quasi m-Holes

A quasi M-hole at position η (complex coordinate) is represented by the wave function [3]

$$\psi(z_1, ..., z_n) = \Pi_{i=1}^{N}\Pi_{j=1}^{M}(z_i - \eta_j)\Pi_{k<l}(z_k - z_l)^m \exp{(-\sum_{n=1}^{N}|z_n|^2/4l_B^2)} \tag{3.26}$$

The electron density vanishes in the M hole positions η_j. That's the reason to the name M-hole.

In order to heuristically infer its charge, we consider the case in which all the M-holes are in the same position η and $M = m$.

$$\psi(z_1, ..., z_n) = \Pi_{i=1}^{N}(z_i - \eta)^m\Pi_{k<l}(z_k - z_l)^m \exp{(-\sum_{n=1}^{N}|z_n|^2/4l_B^2)} \tag{3.27}$$

If it was not a parameter but a coordinate it would be the wave function for $N + 1$ electrons, the m- hole would be equivalent to an extra electron. But since it is not, a m-hole represents the deficit of one electron. Therefore, one single hole has charge $q = +\frac{e}{m}$.

3.7 A Primer in Graphene Description

The graphene is a two-dimensional material composed by carbon atoms. Namely, the valence electrons are disposed in a sp^2 hibridization and the remaining one is in a p_z orbital. The former electrons form sigma bounds with the sp^2 ones of the neighboring atoms. The electrons in the p_z orbitals do not enter in these bonds. The sp^2 orbitals are in the same plane and form angles of 120^0 between them [10, 11]. Novoselov and Geim won the Nobel prize in 2010 due to previous experimental works related to the graphene two dimensional material. It has the lattice structure of the figure 3.5. Remarkably there is two different kinds of elements in this structure (which is called honeycomb lattice.). They have a different crystallographic classification since their lattice neighborhoods are different. This lattice is defined by the vector basis $\delta_3 = \frac{a}{\sqrt{3}}e_y$, $\delta_1 = -\frac{a}{\sqrt{3}}(\frac{\sqrt{3}}{2}e_x + \frac{1}{2}e_y)$ and $\delta_2 = \frac{a}{\sqrt{3}}(\frac{\sqrt{3}}{2}e_x - \frac{1}{2}e_y)$, with $\frac{a}{\sqrt{3}} = 0,142nm$, which connect one basis element to its first neighbors.

The Hamiltonian first obtained by Philip R. Wallace to model the band structure of mono-layer graphite (graphene.) [12] presents a tight binding structure in which the p_z electrons are governed by the Hamiltonian [2]

$$H = -t \sum_{R;i=1,2,3;\sigma=\pm} \left(C_B^\dagger(R + \delta_i, \sigma)C_A(R, \sigma) + H.C.\right) \tag{3.28}$$

[2]The coupling between electrons of the same kind of site, which are beyond the nearest neighbors, is one order of magnitude smaller than t.

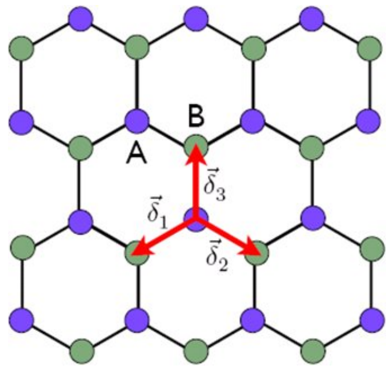

Figure 3.5: Lattice structure of graphene [10].

where the coupling t is of order 3 eV , σ denotes the spin orientations and $C_B^\dagger(R+\delta_i,\sigma)$ is a creation operator of a p_z electron with spin σ.

Using the Fourier transform

$$C_{A,B}(r,\sigma) = \frac{1}{N}\sum_k C_{A,B}(k,\sigma)e^{ik.r} \tag{3.29}$$

The Hamiltonian can be rewritten as

$$H = \sum_{k,\sigma}\big(-tC_A(k,\sigma)C_B^\dagger(k,\sigma)\sum_i e^{ik.\delta_i} + H.C.\big) = \sum_{k,\sigma}\psi^\dagger(k,\sigma)\begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix}\psi(k,\sigma) \tag{3.30}$$

with $\phi(k) = -t\sum_l e^{ik.\delta_l}$ and $\psi^\dagger(k,\sigma) \equiv \big(C_A^\dagger(k,\sigma), C_B^\dagger(k,\sigma)\big)$.

The energy eigenvalues define the two band structure $E(k) = \pm|\phi(k)| = \pm\sqrt{\sum_{l,j} e^{ik.(\delta_l-\delta_i)}}$. There are two points in the reciprocal lattice in which the valence and the conduction bands have a zero energy gap between them [3]. They are at the Brillouin zone corners $K_\pm = \pm\frac{4\pi e_x}{3a}$ leading to $\phi(K_+) = \phi(K_-) = 1 + e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}} = 0$.

A low energy Hamiltonian can be obtained considering the expansion near the Dirac points $K = K_\pm + p$ which, after the canonical transformation $\psi_{K_-}(k,\sigma) \to e^{i\frac{\sigma_z\pi}{2}}\sigma_x\psi_{K_-}(k,\sigma)$, assumes the form

$$H_{K_+} = \psi_{K_+}^\dagger(k,\sigma)h_{K_+}\psi_{K_+}(k,\sigma) \tag{3.31}$$

$$H_{K_-} = \psi_{K_-}^\dagger(k,\sigma)h_{K_-}\psi_{K_-}(k,\sigma) \tag{3.32}$$

with $h_{K_+} = h_{K_-} = v\big(p_x\sigma_x + p_y\sigma_y\big) + \mathcal{O}(p^2)$ and the effective velocity $v = \frac{\sqrt{3}ta}{2}$ and depends on the microscopic details of the lattice. The energy eigenvalues are $E(\vec{k}) = \pm v|\vec{k}|$. It is a relativistic velocity since it is comparable to the light velocity $v \sim \frac{c}{300}$. The two solutions for the low energy Hamiltonian are the so-called Dirac valleys.

The graphene is a kind of material with promising applications in technology. The electrons in this environment must be treated as relativistic ones with a vanishing effective mass [9]. The effective Hamiltonian of the system under an external magnetic field is obtained by the use of the minimal coupling procedure (in the Landau Gauge). Let's analyze one valley [6]

$$H = v\Big(\vec{p} + e\vec{A}\Big).\vec{\sigma} = v\begin{pmatrix} 0 & \pi_x - i\pi_y \\ \pi_x + i\pi_y & 0 \end{pmatrix} = \sqrt{2}\frac{\hbar v}{l_B}\begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix} \tag{3.33}$$

[3] That is why it is called a semi-metal.

where v is the drift velocity in graphene and a and $a^\dagger$ are the creation and annihilation operators previously defined in the Landau problem analysis. The relativistic Landau levels are $E_n = \epsilon\sqrt{2n}\frac{\hbar v}{l_B}$ with $\epsilon = \pm$, where the $+$ is related to the conductance band and $-$ to the valence band.

Regarding the experimental observation of relativistic Landau levels, they have been experimentally observed in transmission spectroscopy by shinning light on the sample and measuring the intensity of the transmitted light. The experiments have been performed on the epitaxial graphene 11 [13] and later on exfoliated graphene [14]. When the monochromatic light is in resonance with a dipole-allowed transition from the (partially) filled LL ($\pm n$) to the (partially) unoccupied LL ($\pm, n \pm 1$), the light is absorbed due to an electronic excitation between the two levels. The allowed transitions are listed below

$$\Delta_{n,\epsilon} = \frac{\hbar v}{l_B}\left[\sqrt{2n+1} - \epsilon\sqrt{2n}\right] \tag{3.34}$$

3.8 Effective Field Description: FQHE for $\nu = \frac{1}{2k+1}$

The Hamiltonian for a two dimensional sample with a macroscopic number of electrons under a perpendicular magnetic field and a scalar potential reads [15]

$$H = \frac{1}{2m}\sum_i \left[\vec{p}_i - \frac{e}{c}\vec{A}(x_i)\right]^2 + \sum_i eA_0(x_i) + \sum_{i<j} V(x_i - x_j) \tag{3.35}$$

We use the symmetric gauge $2A_i = B\epsilon_{ij}x_j$. The Coulomb interaction between electrons is $V(x_i - x_j) = -e^2/4\pi\epsilon_0|x_i - x_j|$.

The eigenvalue problem is $H\psi = E\psi$, where the wave function $\psi(x_1, ..., x_n)$ is antisymmetric under exchange of x_i and x_j, due to the Fermionic nature. We do not take into account the spin since we consider a magnetic field strong enough to polarize the electrons.

We are going to show that this Hamiltonian can be related by an unitary transformation to

$$H = \frac{1}{2m}\sum_i \left[\vec{p}_i - \frac{e}{c}(\vec{A}(x_i) + \vec{a}(x_i))\right]^2 + \sum_i eA_0(x_i) + \sum_{i<j} V(x_i - x_j) \tag{3.36}$$

where $\vec{a}(x_i) = \frac{\hbar c\theta}{e\pi}\sum_{j\neq i}\vec{\nabla}\alpha_{ij}$ and $H\phi = E\phi$. The system has the same eigenvalues as before, but $\phi(x_1, ..., x_n)$ is now symmetric under particle exchange, it is Bosonic!. The α_{ij} is the angle between the i and the j particle with relation to the origin.

It is possible to show that just for the values $\theta = (2k+1)\pi$, with k being a natural number, there is an unitary transformation $U = exp[-i(\frac{\theta}{\pi}\sum_{j>i}\alpha_{ij})]$, with $\psi = U\phi$

$$U\left[\vec{p}_i - \frac{e}{c}(\vec{A}(x_i) + \vec{a}(x_i))\right]U^{-1} = \vec{p}_i - \frac{e}{c}\vec{A}(x_i) \tag{3.37}$$

We work with the Bosonic system quantized as $[\phi(x), \phi(y)] = \delta^2(\vec{x} - \vec{y})$. In a second quantized notation, we have

$$H_1 = \int d^2x\phi^\dagger\Big(\frac{1}{2m}\left[\vec{\nabla} - \frac{e}{c}(\vec{A}(x) + \vec{a}(x))\right]^2 + eA_0(x)\Big)\phi + \frac{1}{2}\int d^2xd^2y\delta\rho(x)V(x-y)\delta\rho(y) \tag{3.38}$$

where the fluctuation $\delta\rho(x) = \rho(x) - p$. with $\rho(x) = \phi^\dagger(x)\phi(x)$ and p is the average particle density.

In a continuous notation

$$a^i(x) = -\frac{\hbar c\theta}{e\pi}\epsilon^{ij}\int d^2y\frac{x^j - y^j}{|x^j - y^j|}\rho(y) \tag{3.39}$$

The lagrangian that may lead to the new vector field equations of motion is

$$\mathcal{L}_2 = \frac{e^2\pi}{2\theta hc}\epsilon^{\nu\alpha\beta}a_\nu\partial_\alpha a_\beta - a_\mu J^\mu \tag{3.40}$$

It is invariant under $a_\mu \to a_\mu + \partial_\mu\Lambda$ up to a boundary term. The a_0 field can be understood as a Lagrange multiplier.

The equations of motion, in the case of the mean field solution $\vec{a}(x) = -\vec{A}(x)$, the so-called uniform solution, expresses the quantized Hall conductivity and the fact that a magnetic flux is attached to the charge carriers

$$e\rho(x) = \mathcal{K}B(x) \tag{3.41}$$

$$J^i(x) = \mathcal{K}\epsilon^{ij}E_j(x) \tag{3.42}$$

with $\mathcal{K} = \frac{e^2}{hc(2k+1)}$.

We can see that the expressions above represent the expected phenomenology of the Hall effect for a class of fractionally quantized conductivities.

The circulation of the statistical field is (Using $\int d^2y\rho(y) = 1$)

$$\oint \vec{a}.d\vec{l} = (2k+1)\frac{hc}{e} \tag{3.43}$$

Therefore, since the Chern-Simons field attaches a magnetic flux on the matter particles, they receive an Aharonov-Bohm phase due their exchange which is equal to $exp(ie/hc\int_0^\pi \vec{a}.d\vec{l}) = -1$. This extra phase can be interpreted as statistics changing, from Fermions to bosons and vice versa. It is another confirmation of the fact that the mentioned unitary transformation can relate field descriptions with different associated statistics for the matter sector.

The scalar field lagrangian (non-relativistic) is

$$\mathcal{L}_1 = \phi^\dagger\Big(i\hbar\partial_t - e(A_0+a_0)\Big)\phi - \phi^\dagger\Big(\frac{1}{2m}\Big[\vec{\nabla} - \frac{e}{c}(\vec{A}(x)+\vec{a}(x))\Big]^2 + eA_0(x)\Big)\phi - \frac{1}{2}\int d^2y\delta\rho(x)V(x-y)\delta\rho(y) \tag{3.44}$$

3.8.1 Vortex Solution

Now, we consider another solution for the Bosonic scalar field, namely, a vortex one. The vortex solution is found by imposing non-trivial conditions on these fields at the spatial infinite. Although a neutral vortex need infinite energy, a charged one requires just a finite amount of energy. The asymptotic configuration at $|\vec{x}| \to \infty$ is

$$a_0 = 0, \quad \phi = \sqrt{p}e^{\pm i\alpha(x)}, \quad \delta\vec{a} = \pm\frac{\hbar c}{e}\vec{\nabla}\alpha(x) = \pm\frac{\hbar c}{e}\frac{\vec{z}}{|\vec{z}|}\times\frac{\vec{x}}{|\vec{x}|^2} \tag{3.45}$$

with $\delta\vec{a} \equiv \vec{A} + \vec{a}$ denoting a perturbation in the mean field solution.
In this kind of solution there is functional of the exponent of the asymptotic scalar field configuration called winding number, which is a positive/negative integer, related to the number of times the system gives a complete rotation in the internal field space when a complete rotation is performed in the two dimensional space. This number is a topological invariant and separates the space of solutions into disjoint classes.

We have finite energy since for $|\vec{x}| \to \infty$ we have $|(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\delta\vec{a})\phi| \to 0$ with the gauge field approaching something analogous to a pure gauge configuration [4] at the spatial infinite. The charged vortex solution is conceived with a particular structure to ensure a finite energy. Then, the asymptotic gauge field is dependent from the exponent of the asymptotic scalar field.

Therefore, if we consider a class of solutions with the property that the phase of the scalar field gives a $\pm 2\pi$ twist for each time a complete rotation in the space is done, the following condition must be considered $\oint \delta\vec{a}.d\vec{l} = \pm\frac{hc}{e}$ with the integral being performed in the line that delimits the region where the vortex solution occur.

The charge density related to a given gauge field configuration is given by the following equation of motion

$$\tilde{\rho}(x) = -\frac{\partial S}{\partial a_0} = \frac{e^2}{hc}\nu\epsilon^{ij}\partial_i a_j \tag{3.46}$$

with $\tilde{\rho}(x)$ denoting the charge density. Therefore, the vortex charge can be calculated considering the specific gauge field solution for this case [15] $Q = \int d^2x\tilde{\rho}(x) = \pm\frac{e}{2k+1}$.

3.9 On the Relation between the lagrangian of a planar Charged Particle in a Constant Magnetic Field and the Maxwell-Chern-Simons Model

In the previous section we have found a deep relationship between the description of the quantum Hall effect and the Maxwell-Chern-Simons (MCS) theory. Here, we intend to point out some additional analogies between these systems [8]. Namely, we focus on their canonical analysis having in mind the correspondence principle. The MCS lagrangian reads

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{k}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{3.47}$$

The canonical momenta is defined as

$$\Pi_i = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_i)} = \frac{\partial_0 A_i}{e^2} + \frac{k}{2}\epsilon^{ij}A_j \tag{3.48}$$

From the definition of the Poisson brackets, we obtain the following result for the electric fields [8]

$$\Big\{E_i(\vec{x}), E_j(\vec{y})\Big\} = ke^4\epsilon_{ij}\delta^2(\vec{x}-\vec{y}) \tag{3.49}$$

with $E_i(x) = \partial_0 A_i(x) - \partial_i A_0(x)$.

By the other hand, the lagrangian for a charged particle on a plane under the effect of a perpendicular constant magnetic field is

$$L = \frac{1}{2}m\dot{x}^2 + \frac{b}{2}\epsilon^{ij}\dot{x}_i x_j \tag{3.50}$$

The canonical momenta reads

$$p_i = m\dot{x}_i + b\epsilon^{ij}x_j \tag{3.51}$$

[4] It has just an analogous form but is not equal since α behaves as a polar angle and then the derivatives applied on this function do not commute since it is not well-defined when the radial coordinate is zero. Therefore it is not a symmetry of the model, the Chern-Simons strength tensor do not vanish in this configuration.

It leads to

$$\Big\{\dot{x}_i,\dot{x}_j\Big\} = \frac{b}{m^2}\epsilon_{ij} \tag{3.52}$$

Then, regarding the non-relativistic case considering the long wavelength limit in which all the spatial derivatives vanishes, we note that x_i plays the role of A_i and we conclude that e^2 from MCS theory plays the role of $\frac{1}{m^2}$ in the Landau model as well as k and the MCS mass ke^2 plays the role of the Landau system variables b and $\frac{b}{m}$, respectively. Regarding $\frac{b}{m}$ it is equal to the cyclotron frequency of the Landau system. If $\frac{b}{m}\to\infty$ the gap between the two successive Landau levels becomes infinite and the system describes just low energy excitations. Looking at the expression of MCS lagrangian and the map between the parameters of both models mentioned here we conclude that, in this limit, the pure Chern-Simons term becomes dominant.

3.10 How to project a Four Dimensional Theory on a Plane?

Here we discuss a topic related to the effective description of a four dimensional field theory, specifically the Maxwell one, for the case in which there is just non-vanishing current density on a plane [16]. This situation is obviously compatible with the quantum Hall effect phenomenology.

Considering, for simplicity, the model coupled to a c-number external source constrained to lie on a plane, we have

$$\mathcal{L}_{(4D)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eJ_\mu A^\mu + \frac{\epsilon}{2}\big(\partial_\mu A^\mu\big)^2 \tag{3.53}$$

We are considering the Euclidean theory.

Now, consider that the external source is of the form $J^\mu(x) = J^\mu(x^4,x^1,x^2)\ \delta(x^3)$ for $\mu\neq 3$ and $J^\mu(x)=0$ for $\mu=3$. Then, having in mind the form of the Euclidean propagator

$$P^{\mu\nu} = \Big(-\Box_E\delta^{\mu\nu} + \big(1-\frac{1}{\epsilon}\big)\partial^\mu\partial^\nu\Big)\frac{1}{\Box_E^2} \tag{3.54}$$

The functional generator becomes

$$Z = \int DA_\mu \exp\Big(-\int d^4z_E\mathcal{L}_{4D}\Big) = e^{-S_{eff.}(J_\mu)} \tag{3.55}$$

The effective action on the plane reads

$$S_{eff.} = -\frac{e^2}{2}\int d^3z_E d^3\tilde{z}_E J^\mu K_E(z-\tilde{z}|z^3=\tilde{z}^3=0)\ J_\mu \tag{3.56}$$

with

$$K_E = \frac{1}{8\pi|z-\tilde{z}|^2_{3D}} \tag{3.57}$$

It means that if one compute the potential energy for static charges, the correct $\frac{1}{r}$ behaviour is obtained instead of the unphysical $\log(r)$ potential that would arise if we have considered the usual model for QED in $2+1$ dimensions.

In order to find an effective description for the theory on the plane, we first consider the fact that K_E can be written in a Fourier transform structure just in terms of the planar coordinates

$$K_E = \frac{1}{8\pi|z-\tilde{z}|^2_{3D}} = \frac{1}{4}\int \frac{d^3k_{3D}}{(2\pi)^3}\frac{e^{i\left[k.(z-\tilde{z})\right]_{3D}}}{\sqrt{k^2_{3D}}} = \frac{1}{4\sqrt{\Box^{3D}_E}} \tag{3.58}$$

Another way to justify this pseudo differential structure is to consider the static potential at a given plane

$$\frac{1}{4\pi r_{||}} = \left[\int \frac{d^3p}{(2\pi)^3}\frac{e^{-i\vec{p}.\vec{r}}}{(\vec{p})^2}\right]_{r_z=0} = \frac{1}{2}\int \frac{d^2p}{(2\pi)^2}e^{-i\vec{p}_{||}.\vec{r}_{||}}\frac{1}{\sqrt{(\vec{p})^2_{||}}} \tag{3.59}$$

with $r_{||}$ denoting the coordinates parallel to plane.

Then, it is possible to show that the three dimensional action that indeed gives the correct potential is the following effective one

$$\mathcal{L}_{3D} = -\frac{1}{4}F_{\mu\nu}\frac{1}{\sqrt{\Box}}F^{\mu\nu} - eJ_\mu A^\mu + \frac{\epsilon}{2}\frac{\left(\partial_\mu A^\mu\right)^2}{\sqrt{\Box}} \tag{3.60}$$

Although it may look strange due to the non-local pseudo differential structure, it is totally well behaved. Such approach may provide a more accurate description for the planar phenomena such as the quantum Hall effect [17].

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Chapter 4

The Features of the Chern-Simons system coupled to Topological Two Band models, a Heisenberg approach

4.1 Introduction

The quantum Hall [1] effect (QHE) is an important condensed matter phenomena which has a good experimental [2, 3] and theoretical basis [4, 5, 6]. Roughly speaking it is related to the appearance of a quantized transverse conductivity in a class of materials possessing a high scattering time with the material crystalline lattice, such that their transport properties depends just on the intense external magnetic field of order 1 Tesla. The samples are two-dimensional layers of semiconductors like $GaAS$ or the MOSFET transistors.

The phenomena is observed at very low temperatures and high external magnetic field [6] in order to avoid temperature fluctuation noise. Then, a natural non-relativistic description for the integer case is given by the Landau model which gives gaps in the energy levels. The fractional case, the one concerning a class of fractional values for conductance, are described by the so-called Laughlin wave function [4].

Regarding the Chern-Simons model, it can be related to (QHE) by means of the seminal work of [7] which is based on a unitary transformation that relates the Hamiltonian describing polarized Fermions interacting with an external photon field to a system of bosons interacting with external photons and a quantum field responsible for changing the statistical nature of the charge carriers. These transformations exists just for some lagrangian parameter values that leads exactly to a class of observed fractional conductivities. This models also predicts the existence of fractionally charged vortices.

The Chern-Simons model also arises in other contexts. For example, if one couple an external electromagnetic field to Fermions and integrate them out, the effective action obtained contains a Chern-Simons term in the low energy regime [5].

Another convenient way to effectively describe the low energy excitations of the quantum Hall system is given by means of the action [4, 5]

$$S = \int d^3x \Big(\frac{\mathcal{K}}{2} \epsilon^{\mu\nu\beta} A_\mu \partial_\nu A_\beta + A_\mu J^\mu \Big) \tag{4.1}$$

where $J^\mu(x)$ denotes a source describing the charged particles and $\mathcal{K} = -\frac{e^2 n}{2\pi} \equiv -\sigma_H$ with n being a positive integer. We use $c = \hbar = 1$ throughout.

From the equations of motion, it follows that

$$\rho(x) = \sigma_H B(x) \tag{4.2}$$

$$J_i(x) = \sigma_H \epsilon_{ij} E_j(x) \tag{4.3}$$

where $\rho(x)$ denotes the charge density, $B(x)$ the magnetic field in the z-direction and $E_j(x)$, the electric field. The attachment of magnetic flux at the position of the charged particles leads to an anyonic statistics for them due to the Aharonov-Bohm effect.

From the above results we find a class of fractional quantized transverse conductivities given by $\sigma^{ij} = \epsilon^{ij} \frac{e^2 n}{2\pi}$.

There is an interesting article [8] that analyzes the $2+1$ dimensional QED which isolates what they called as the statistical interaction responsible to generate an anyonic nature for the matter interacting with the Chern-Simons field. It is given by the potential

$$U = \frac{e}{\mathcal{K}} \int d^3x \epsilon^{\mu\nu\rho} J_\mu \partial_\alpha \Big(\frac{1}{-\Box_E} \Big) J_\rho \tag{4.4}$$

where $\Box_E$ denotes the Euclidean version of the d'Alembert operator.

It is also interesting to mention that the Landau system can be mapped into the non-relativistic Maxwell-Chern-Simons (CS) model [5] and, in the limit of high magnetic field, the topological becomes dominant.

Our intention here is to describe corrections to the quantum Hall effect due the microscopic structure of the charge carriers. We consider the gauge sector as the Chern-Simons system with a suitable coefficient in order to take into account the Landau level structure due to the first quantization and a specific Fermionic content for the matter sector. In order to calculate these corrections, we use the B field formalism which is based on an indefinite metric Hilbert space in the Heisenberg picture. We also use the Lorentz group to prove that even in $2+1$ dimensions, the quantization of a massless vector field indeed demands a Hilbert space with no definite metric as in the four dimensional case [9]. with these tools, we can effectively describe the action of an external magnetic field over a range of Fermionic topological two band systems arising from the low energy regime of some condensed matter lattice models. They include the case of graphene [10, 11], some topological insulators [12, 13], and the Haldane model [14]. Recently, a realization of the Haldane model in Fe based materials was experimentally conceived [15]. There is interest in the independent behaviour of the valley excitations of these materials, in particular, for the increasing field of valleytronics [16]. We will analyze the specific contribution from each of these models for the renormalization of the topological properties of the gauge sector related to the Hall conductivity which, as we are going to prove, is shifted by the Chern numbers of the models considered here. It is an expected result since, according to our previous spectral representation analysis [17], there is a shift in the (CS) coefficient that is due to the topological sector of the polarization tensor associated to the interaction with a discrete symmetry breaking matter lagrangian. We also discuss the fact that the drift velocity of the quasi-particles are different from c, the speed of light, leading to a breaking of the symmetry between space and time.

In order to perform such description, the sources are considered to have a Fermionic quantum structure described by its faithful two dimensional representation (which naturally arises from the low energy regime of a given valley of those two band models.). We furnish the system's well-defined quantization and then discuss which are the implications of the model's radiative corrections and then generalize for the case of a system with two valleys with different associated masses. Both Bosonic and Fermionic responses are calculated and the latter reveals that the interaction with the (CS) field leads to a low energy propagator with non-trivial topological properties. Interestingly, the interaction of the gauge field with a time reversal invariant matter leads to a renormalized effective Fermionic response with non trivial topology characterized by a non-vanishing Chern number, which is related to the emergence of a transverse conductivity associated to localized states at the boundary. The investigations done here can provide a useful framework in the context of valley filters, see [18].

Regarding the method employed here, the Kugo-Ojima-Nakanishi (KON) B field formalism [19] consists of a Heisenberg description quantization which uses an indefinite metric Hilbert space whose positive norm subspace is defined by means of a subsidiary condition. This condition may be written in terms of the positive frequency part of the auxiliary field or by using the BRST charge for Abelian and non-Abelian theories, respectively. This formalism furnishes a well-defined system that is free from first class ambiguities [20].

This non-perturbative formalism has a wide variety of applications from Abelian and non-Abelian gauge theories to quantum gravity. It has a perturbative counterpart which, for example, gives an exact quantization for the two dimensional BF theory by means of a well-defined Cauchy problem for the commutators [21] and can be also used for QED_4 to get its first order radiative corrections [22]. Additionally, the general structure for the Wightman functions can be inferred by a method which consists of extracting the truncated n point functions from the n point commutators by imposing the requirement of positive energy. For a review which includes perturbative B field QED_4 passing through an academic one loop model, string theory and the two dimensional quantum gravity, see [23].

Using this formalism, we obtain the system's vacuum average commutators for two photons and two Fermions field operators. In order to obtain the effective action and then compute the desired low energy system's transport properties we should have the retarded version of these commutator distributions.

In possession of the radiative structure we are able to discuss its effects on the conductivity, the radiative correction for the Fermionic sector and some other aspects. Regarding the theoretical description in $2+1$ dimensions, besides the (KON) formalism [17], we use some results from previous works in $2+1$ dimensional QED such as [24, 25, 26] and we can also cite [27] with uses an super symmetric Chern-Simons structure.

The work is organized as follows. In section 2, we display explicitly the two band models to be analyzed here, its particularities, such as their topological properties and symmetries, and comment about its origin as the low energy quasi-particle description of honeycomb lattices with two sites. We also comment about the introduction of the interaction with the Chern-Simons field.

After this digression, in section 3 we present the arguments in support of the use of an indefinite Hilbert space metric, based on the $2+1$ dimensional Poincaré stability group of a massless particle described by a gauge field.

In the section 4 the Nakanishi B-field non perturbative quantization of the Chern-Simons model coupled to Fermions in its two dimensional faithful representation [25] is contemplated. We also how that all the class of models studied here can be contemplated by this approach with a simple generalization. It is then found that the (CS) model has observables with positive projections just in the interacting case, otherwise their norm vanish. The section 5 is devoted to present the lowest order explicit development of the complete non-perturbative expressions previously found. Although the radiative corrections related to the commutators are finite, its time ordered versions may not have well-defined Fourier transform due to multiplication by Heaviside functions [28]. We explicitly obtain the first contribution to the polarization tensor which leads to a shift to the conductivity properties which, in the low energy regime, is proportional to the Chern number of the matter field. Then, we calculate how it flows with the energy. Latter, since the quasi-particles from the medium have a drift velocity different from that of the light, we address this issue of Lorentz breaking with a careful construction of the complete propagator for the gauge field. We show that a massless pole has its residue renormalized by the topological properties of the matter field and a massive pole with a unitary sign arises since, according to section 4, the interacting case generates a positive norm for the gauge field observables. We also calculate the electric potential and the magnetic field for the case of a static point test charge.

In section 6 we present the construction of the self energy anti-commutator tensor, and show that the low energy effective Fermionic response for the case of interaction with the (CS) model has a topological structure defined by the sum of the Chern number of its bare version and a contribution from the mass correction due to the interaction with the gauge field. We show that it implies in localized boundary states with an associated group velocity. Finally, in section 7 we conclude and point out new perspectives.

4.2 Topological two Band models

We begin by the description of the graphene [10] and then generalize for others two band models. The graphene is a two-dimensional material composed by carbon atoms. Namely, the valence electrons are disposed in a sp^2 hibridization and the remaining one is in a p_z orbital. The former electrons form sigma bounds with the sp^2 ones of the neighboring atoms. The electrons in the p_z orbitals do not enter these bonds. The sp^2 orbitals are in the same plane and form angles of 120^0 between them.

Remarkably there are two different kinds of elements in this structure (which is called honeycomb lattice.). They have a different crystallographic classification since their lattice neighborhoods are different. This lattice is defined by the vector basis $\delta_3 = \frac{a}{\sqrt{3}} e_y$, $\delta_1 = -\frac{a}{\sqrt{3}}(\frac{\sqrt{3}}{2} e_x + \frac{1}{2} e_y)$ and $\delta_2 = \frac{a}{\sqrt{3}}(\frac{\sqrt{3}}{2} e_x - \frac{1}{2} e_y)$, with $\frac{a}{\sqrt{3}} = 0,142nm$, which connect one basis element to its first neighbors.

The Hamiltonian presents a tight binding structure in which the p_z electrons are governed by the Hamiltonian

$$H = -t \sum_{R;i=1,2,3;\sigma=\pm} \big(C_B^\dagger (R + \delta_i, \sigma) C_A(R, \sigma) + H.C. \big) \tag{4.5}$$

where the coupling t is of order 3 e.V. , σ denotes the spin orientations and $C_B^\dagger(R + \delta_i, \sigma)$ is a creation operator of a p_z electron with spin σ.

Using the Fourier transform

$$H = \sum_{k,\sigma} \big(- t C_A(k, \sigma) C_B^\dagger(k, \sigma) \sum_i e^{ik.\delta_i} + H.C. \big) = \sum_{k,\sigma} \psi^\dagger(k, \sigma) \begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix} \psi(k, \sigma) \tag{4.6}$$

with $\phi(k) = -t \sum_l e^{ik.\delta_l}$ and $\psi^\dagger(k, \sigma) \equiv \big(C_A^\dagger(k, \sigma), C_B^\dagger(k, \sigma) \big)$.

The energy eigenvalues define the two band structure $E(k) = \pm|\phi(k)| = \pm\sqrt{\sum_{l,j} e^{ik.(\delta_l - \delta_i)}}$. There are two points in the reciprocal lattice in which the valence and the conduction bands have a zero energy gap between them [1]. They are at the Brillouin zone corners $K_\pm = \pm \frac{4\pi e_x}{3a}$ leading to $\phi(k_+) = \phi(k_-) = 1 + e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}} = 0$.

A low energy Hamiltonian can be obtained considering the expansion near the Dirac points $K = K_\pm + p$ which, after the canonical transformation $\psi_{K_-}(k, \sigma) \to e^{i\frac{\sigma_z \pi}{2}} \sigma_x \psi_{K_-}(k, \sigma)$, assumes the form

$$H_{K_+} = \psi_{K_+}^\dagger(k, \sigma) h_{K_+} \psi_{K_+}(k, \sigma) \tag{4.7}$$

[1]That is why it is called a semi-metal.

$$H_{K_-} = \psi^\dagger_{K_-}(k,\sigma) h_{K_-} \psi_{K_-}(k,\sigma) \tag{4.8}$$

with $h_{K_+} = h_{K_-} = v\big(p_x\sigma_x + p_y\sigma_y\big) + \mathcal{O}(p^2)$ and the effective velocity $v = \frac{\sqrt{3}ta}{2}$ and depends on the microscopic details of the lattice. The σ_i denote the Pauli matrices. It is a relativistic velocity since it is comparable to the light velocity $v \sim \frac{c}{300}$. The two solutions for the low energy Hamiltonian are the so-called Dirac valleys. It is worth mention that this model preserves the so-called pseudo-helicity $\vec{p}.\vec{\sigma}$. Then, the different valleys have electrons with different pseudo-helicity eigenvalues in its upper and lower cones [29]. This model, although formally equal to the massless Dirac one, breaks the Lorentz symmetry since $v \neq c$ being c the velocity of the light. In the following sections we are going to discuss this point.

The Haldane model consists in a honeycomb lattice structure in which the couplings between the C_A and C_B with theirselves are also considered. Its low energy behaviour leads to a two valley system, analogous to the graphene one, but with mass terms proportional to the σ_3 Pauli matrix and given by $M_k = M - 3\sqrt{3}t_2 sen\phi$ and $M_{k'} = -M - 3\sqrt{3}t_2 sen\phi$ where t_2 is the coupling between equal sites and ϕ is a hypothetical intrinsic magnetic flux through the plaquettes of the second nearest neighbors. A version of this model which takes spin into account is the Kane-Mele one [30].

An isolated single valley would break discrete symmetries and lead to an intrinsic Hall effect. This is due to to non-trivial overall topological properties of the Hamiltonian associated to a non-vanishing Chern number. A time reversal invariant topological insulator can be provided by the Haldane theory with $\phi = 0$. This situation demands two valleys whose mass terms have different signs but equal intensity. The reciprocal lattice points, with minimum gap, are related by time reversal operation. It can be regarded as a superposition of two opposite intrinsic anomalous Hall effects, leading to an overall insulating phase. Recent atomic engineering methods are capable of isolating one of these valleys, something useful in valleytronics [16].

The Hamiltonian for a two band system has the general form

$$H = n_0(k)I + \vec{n}(k).\vec{\sigma} \tag{4.9}$$

with energies $E_\pm = n_0 \pm \sqrt{\vec{n}.\vec{n}}$.

The topological classes of these models are defined by the Chern number

$$n_{chern} = -\frac{1}{4\pi}\int d^3k\big[\vec{n}.(\partial_{k_x}\vec{n}\times\partial_{k_y}\vec{n})\big]/|n|^3 \tag{4.10}$$

For the models considered $n_{chern} = \frac{1}{2}\big(sign(m_k) + sign(m_{k'})\big)$. It vanishes for a time reversal invariant system since the masses are opposite and the two valleys are linked by this operation. We also mention that a single valley, if massive, breaks time reversal invariance since there is no representation for this operator for $D = 2+1$ dimensional Fermions in the two-dimensional representation [31].

In order to study planar systems we adopt the units of mass for the fields in order to have a dimensionless electric charge as it should be. Moreover, we should consider a system as

$$S = \int d^3x\Big(\frac{\mathcal{K}}{2}\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha - A_\mu J_\mu\Big) \tag{4.11}$$

with $\mu = 0, 1, 2$.

The bare Chern-Simons term, according to our approach, have quantized coefficient [2] $\mathcal{K} = -\frac{e^2 n}{4\pi}$, with n being an integer. This is also the condition to ensure gauge invariance of the partition function for the specific case in which a finite temperature is considered [4]. These considerations are given in order to take into account the overall phenomenology arising from

[2] We are considering the natural units with $c = \hbar = 1$ that we adopt from now on.

the Landau level structure due to the first quantization of the system composed of a charged particle in an external magnetic field orthogonal to the planar sample. Therefore, throughout this work, we intend to analyze this system coupled to a matter lagrangian whose microscopic content is described by Fermionic low energy quasi-particle excitations that belongs to the class of two band models that arise in several contexts of condensed matter.

Due to the form of the coefficient of the Chern-Simons term, a non-perturbative method seems to be suitable to describe the system. For the case of high conductivity with $n \gg 1$, also perturbative approaches can be provided and may furnish useful insights. This feature motivates the general structure of this work.

4.3 Indefinite Hilbert Space Metric in $D = 2+1$ dimensions

We are going to show that as well as happens in the four dimensional case [19], for $D = 2+1$ dimensions, the quantization of a massless gauge vector fields requires an indefinite Hilbert space metric.

In order to demonstrate it, we consider the three-dimensional Poincaré group generator algebra in order to find the maximum set of compatible operators in the case of a massless particle of momentum $P_\mu = (p_0, 0, p_0)$

$$\Big[J_\mu, J_\nu\Big] = \epsilon_{\mu\nu\gamma} J^\gamma \quad , \quad \Big[M_{\mu\nu}, P_\lambda\Big] = i\Big(\eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu\Big) \quad , \quad \Big[P_\mu, P_\lambda\Big] = 0 \tag{4.12}$$

where the $SO(1,2)$ generators J_μ are defined as $\epsilon^{\mu\nu\beta} M_{\nu\beta}$ with no implicit index sum. The maximum compatible set of generators are given by $L \equiv J_0 + J_1$ and P_μ.

Now, it is useful to consider an explicit 2+1 space-time dimensional representation for these generators given in [5]

$$\Big(J^\mu\Big)_{\alpha\beta} = \epsilon^\mu{}_{\gamma\sigma} P^\gamma \frac{\delta}{\delta P_\sigma} \delta_{\alpha\beta} + i\epsilon^\mu{}_{\alpha\beta} \tag{4.13}$$

with regard to the operator L,the orbital part do not contribute.

The explicit form of the $(L)_{\alpha\beta}$ operator is

$$(L)^{AB} = i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{4.14}$$

Then, considering Lorentz transformations that preserve the light like momentum $\Lambda p = p$ and due to the Poincaré invariance of the vacuum state, such that for $\Lambda_\mu{}^\nu a_\nu(\Lambda^{-1}p) = U^\dagger a_\mu(p) U$, we have $U|0\rangle = |0\rangle$, where U denotes the (pseudo) unitary representation of the Lorentz group. Than, we conclude that the projection [3] $M_{\mu\nu} = \langle 0| a_\mu(p) a^\dagger_\nu(p) |0\rangle$ is invariant under the transformations generated by $(L)_{\alpha\beta}$ with $\Lambda^{-1}p = p$. In matrix language, we have

$$M = \Lambda M \Lambda^t \tag{4.15}$$

where t denote transposition.

Since the infinitesimal form of $\Lambda_{\mu\nu} = \delta_{\mu\nu} + i(L)_{\mu\nu}\varepsilon$, with the parameter ε being small, we have $LM = ML$, where we have considered the matrix anti symmetry. This condition allows us to obtain the general form

[3] where $a_\mu(p)$ denotes the Fourier transform of the asymptotic gauge field.

$$M^{AB} = \begin{pmatrix} a & 0 & c \\ 0 & (a-c) & 0 \\ c & 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b & 0 \\ -b & 0 & b \\ 0 & -b & 0 \end{pmatrix} \tag{4.16}$$

Due to the hermiticity of the projection matrix, the elements from the symmetric part must be real, and the ones from the antisymmetric one must be purely imaginary. the emergence of this antisymmetric part is due to the possibility of adding a Chern-Simons term in $2+1$ dimensions when the system described allows discrete symmetry violation.

Regarding the secular equation, the eigenvalues of the projection matrix are the following $\lambda = a+c$ and $\lambda = a-c \pm |b|$ with $|b|$ being the modulus of the imaginary b number. This result means that the sign of the matrix projections is indeed indefinite. In order to know more about irreducible representations of the $2+1$ dimensional Poincaré group, we indicate the work [34].

4.4 The B field Quantization

The theory describing Fermions in its two-dimensional representation interacting with the Chern-Simons field is given by

$$\mathcal{L} = \frac{\mathcal{K}}{2}\epsilon^{\mu\nu\beta}A_\mu\partial_\nu A_\beta + B\partial_\mu A^\mu - \frac{\alpha}{2}B^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + eA_\mu\bar{\psi}\gamma^\mu\psi \tag{4.17}$$

with $\mathcal{K} = -\frac{e^2 n}{2\pi}$.

We will consider, for a while, the system with such a structure that is analogous to just one valley (see the definition of the gamma matrices in appendix B.) and no space-time breaking due to the drift velocity of the quasi-particles. These issues are going to be discussed in convenient points throughout the work.

The equations of motion are

$$\mathcal{K}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho(x) = -e\bar{\psi}(x)\gamma^\nu\psi(x) + \partial_\mu B(x) \tag{4.18}$$

$$\partial_\mu A^\mu(x) = \alpha B \tag{4.19}$$

$$\Big[i\gamma^\mu\Big(\partial_\mu + ieA_\mu(x)\Big) - m\Big]\psi(x) = 0 \tag{4.20}$$

$$\bar{\psi}(x)\Big[i\gamma^\mu\Big(\overleftarrow{\partial}_\mu - ieA_\mu(x)\Big) + m\Big] = 0 \tag{4.21}$$

From them, we can derive the Fermion current conservation

$$\partial_\mu\Big(\bar{\psi}(x)\gamma^\mu\psi(x)\Big) = 0 \tag{4.22}$$

Then, taking the divergence of the photon field equation of motion, we find

$$\Box B(x) = 0 \tag{4.23}$$

The positive metric Hilbert subspace $\mathcal{H}_{phys}$ is defined by means of the positive frequency part of the auxiliary field as [19]

$$B^+(x)|phys\rangle = 0 \quad , \quad \forall|phys\rangle \ \in \mathcal{H}_{phys} \tag{4.24}$$

This definition is Poincaré invariant.

In order to derive the quantum field commutators, we must find their initial conditions by means of the correspondence principle [19]. The canonical momenta are the following:

$$\pi^i(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i(x))} = \frac{\mathcal{K}}{2}\epsilon^{ij}A_j, \tag{4.25}$$

$$\pi^0(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0(x))} = B(x), \tag{4.26}$$

$$\pi_B(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 B(x))} = 0, \tag{4.27}$$

$$\pi_\psi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi(x))} = i\bar{\psi}(x)\gamma^0 \tag{4.28}$$

$$\pi_{\bar{\psi}}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi}(x))} = 0 \tag{4.29}$$

Although all the above momenta define constraint relations, the addition of the $B(x)$ field turn the system into a second class one [20] and then the quantum ambiguities are avoided from the beginning. Therefore, we can build a well-defined reduced phase space and then consider the correspondence principle. This is one of the main reasons to introduce $B(x)$.

The Poisson brackets have the following structure [4]

$$\left\{A_i(x), \pi^j(y)\right\} = \frac{1}{2}\delta_i^j\delta^2(x-y) \tag{4.30}$$

$$\left\{A_0(x), B(y)\right\} = \delta^2(x-y) \tag{4.31}$$

$$\left\{\psi(x), \pi_\psi(y)\right\} = \delta^2(x-y) \tag{4.32}$$

$$\tag{4.33}$$

with the moment definition in terms of the fields in strong form.

Owing to the correspondence principle, and considering the equations of motion, the non-vanishing initial conditions for the commutators are

$$\Big[\psi(x), \bar{\psi}(y)\Big]_0 = \gamma^0\delta^2(x-y) \tag{4.34}$$

$$\Big[A_i(x), A_j(y)\Big]_0 = \frac{i}{\mathcal{K}}\epsilon_{ij}\delta^2(x-y) \tag{4.35}$$

$$\Big[A_0(x), B(y)\Big]_0 = i\delta^3(x-y) \tag{4.36}$$

$$\Big[A_0(x), \partial_0 A_0(y)\Big]_0 = i\alpha\delta^2(x-y) \tag{4.37}$$

$$\Big[B(x), \partial_0 B(y)\Big]_0 = 0 \tag{4.38}$$

$$\Big[A_\mu(x), \partial_0 B(y)\Big]_0 = -i\partial_k^y\delta_\mu^k\delta^2(x-y) \tag{4.39}$$

$$\Big[\partial_0 A_i(x), A_0\Big]_0 = i\frac{\epsilon_{ij}}{\mathcal{K}}\partial^j\delta^2(x-y) \tag{4.40}$$

$$\Big[\partial_0 A_0(x), A_i\Big]_0 = -i\frac{\epsilon_{ji}}{\mathcal{K}}\partial^j\delta^2(x-y) \tag{4.41}$$

$$\Big[\psi(x), \partial_0 B(y)\Big]_0 = e\psi(x)\delta^2(x-y) \tag{4.42}$$

$$\Big[\bar{\psi}(x), \partial_0 B(y)\Big]_0 = -e\bar{\psi}(x)\delta^2(x-y) \tag{4.43}$$

[4] Since the system is of second class from the beginning, the unusual factor $\frac{1}{2}$ appears when one considers the Dirac brackets, see appendix *A*.

where the subscript 0 denotes quantities evaluated at equal times.

For an operator $F(x-y)$ obeying $\hat{O}^x F(x,y) = G(x,y)$, where $\hat{O}^x$ is a given second order differential operator acting in coordinate x, there is the following integral representation [21, 23]

$$\begin{aligned} F(x,y) = & \int d^3u \; \varepsilon(x,y,u)\tau(x-u)G(u,y) \\ & - \int d^2u \Big[\tau(x-u)\partial_0^u F(u,y) - \partial_0^u \tau(x-u)F(u,y)\Big]_{u^0=y^0} \end{aligned} \tag{4.44}$$

where $\tau(x-y)$ is an operator valued distribution such that $\hat{O}^x\tau(x-y) = 0$. The symbol $\varepsilon(x,y,u)$ is defined in terms of Heaviside functions as $\Theta(x_0-u_0) - \Theta(y_0-u_0)$.

The integral representation for the case of a first order differential operator $\mathcal{O}^x$ associated to the Dirac equation with $\mathcal{O}^x f(x,y) = g(x,y)$ reads

$$f(x,y) = \int d^3u \; \varepsilon(x,y,u)S(x-u)g(u,y) + i\int d^2u \Big[S(x-u)\gamma^0 f(u,y)\Big]_{u^0=y^0} \tag{4.45}$$

where $S(x-u)$, which is going to be further defined, obeys $\mathcal{O}^x S(x-u) = 0$.

The B-field commutators then reads

$$\Big[B(x), B(y)\Big] = 0; \tag{4.46}$$

$$\Big[A_\mu(x), B(y)\Big] = i\partial_\mu D(x-y); \tag{4.47}$$

$$\Big[\psi(x), B(y)\Big] = e\psi(x)D(x-y); \tag{4.48}$$

$$\Big[\bar{\psi}(x), B(y)\Big] = -e\bar{\psi}(x)D(x-y) \tag{4.49}$$

where we have the definitions of the Pauli-Jordan distributions in terms of its negative and positive frequency parts

$$\begin{aligned} iD(x-y;s) = & D^+(x-y;s) + D^-(x-y;s) \\ \Box D(x-y,s) = -sD(x-y,s) \quad , \quad & D(x-y,s)|_{x_0=y_0} = 0 \quad , \quad \partial_0 D(x-y,s)|_{x_0=y_0} = -\delta^2(x-y) \\ D^\pm(x-y;s) = \pm & \frac{1}{(2\pi)^2}\int d^3p \; \delta(p^2-s)\Theta(\pm p_0)e^{-ip.(x-y)} \end{aligned} \tag{4.50}$$

Let's analyze, for a moment, the free case, the model with no interaction terms. Then, contracting the operator $\epsilon_{\mu\nu\beta}\partial^\beta$ with the photon field equation and considering the zero norm character of the B field, we get

$$\Box_x \Box_y \Big[A_\mu(x), A_\nu(y)\Big] = 0 \tag{4.51}$$

The initial conditions and the constraints from the equations of motion allied to the integral representation gives

$$\Big[A_\rho(x), A_\beta(y)\Big] = -\frac{i}{\mathcal{K}}\epsilon_{\rho\beta\mu}\partial^\mu D(x-y,0) + i\alpha\partial_\rho\partial_\beta E(x-y) \tag{4.52}$$

We note that the gauge fields have all null norm and non-observable projections between then. where the new distribution $E(x-y,s)$ above has the properties

$$\begin{aligned} (\Box + s)E(x-y,s) = D(x-y,s) \; , \; \partial_0^3 E(x-y,s)|_{x_0=y_0} = -\delta^2(x-y) \; , \; E(x-y,s)|_{x_0=y_0} = 0, \\ E(x) \equiv -\int d^3u \; \varepsilon(x,0,u)D(x-u)D(u) \end{aligned} \tag{4.53}$$

It has positive and negative frequency parts $iE(x-y,s) = E^{+}(x-y,s)+E^{-}(x-y,s)$, with

$$E^{\pm}(x-y,s) = \pm(2\pi)^{-2}\int d^3p\theta(\pm p_0)\frac{d\delta(p^2-s^2)}{dp^2}e^{-ipx} \tag{4.54}$$

For completeness, if we consider the free Dirac field case, we have

$$(i\partial\!\!\!/ - m)_x\Big\{\psi(x),\bar{\psi}(y)\Big\} = 0 \tag{4.55}$$

and taking into account the initial conditions, the integral representation gives

$$\Big\{\psi(x),\bar{\psi}(y)\Big\} = i(i\partial\!\!\!/ + m)D(x-y,m^2) \equiv iS(x-y) \tag{4.56}$$

It is worth mention that the form of this propagator is the same for a system with isotropically broken Lorentz symmetry as happens for the quasi-particles or low energy excitations of graphene, for example. Its Fourier transform would be the same as the Lorentzian one but with the momentum variable replaced by $\tilde{K}_\mu$ defined as $\tilde{K}_\mu = (k_0, v\vec{k})$ in which v is the upper limit on the drift velocity of the quasi-particles. It is a function of the parameters of the lattice model. In order to have a gauge invariant coupling, we must also have $A_\mu\psi\gamma^\mu\psi \to A^\mu\big(\bar{\psi}\gamma_0\psi\delta^0_\mu + v\bar{\psi}\gamma_j\psi\delta^j_\mu\big)$.

A massless pole field which obeys the subsidiary condition, is given by $\mathcal{A}_\mu(x) = \mathcal{K}\epsilon_{\mu\nu\alpha}\partial^\nu A^\alpha(x)$. Its norm, in the free case, vanishes

$$\langle 0|\Big[\mathcal{A}^\mu(x),\mathcal{A}^\nu(y)\Big]|0\rangle = 0 \tag{4.57}$$

The pole field $\mathcal{A}^\mu(x)$ has its temporal component proportional to the magnetic field and the spatial ones are related to the electric field. From the structure of the gauge field commutator, we note that it also do not have positive observable projections.

Therefore, the Chern-Simons model, which is responsible for the effective description of the quantum Hall effect, has no observable projections in the free case. This is in accordance with the fact that this model does not have any local degree of freedom. When the interaction is turned on, this observable present a positive norm due to the current-current commutator structure. In order to understand it, let us first note that it is possible, from the previous developments, to prove that

$$\Big[\bar{\psi}(x)\gamma^\mu\psi(x), B(y)\Big] = 0 \tag{4.58}$$

This result reveals that the current is an observable, as it should be, and then has positive projections.

Regarding the $\mathcal{A}^\mu(x)$ field, we now conclude that it also has positive norm, since

$$\langle 0|\Big[\mathcal{A}^\mu(x),\mathcal{A}^\nu(y)\Big]|0\rangle = e^2\langle 0|\Big[J^\mu(x),J^\nu(y)\Big]0\rangle \tag{4.59}$$

with $J_\mu(x) \equiv \bar{\psi}(x)\gamma_\mu\psi(x)$.

This result reveals the emergence of vector field positive projections due to the presence of the interaction. It indicates that the transverse conducting properties possibly receive radiative corrections and then shift from their bare values. The analysis of the renormalized propagator will reveal that this interaction indeed generates a massive pole with a positive norm besides the massless one contained in the topological sector.

Regarding the commutator between the vector fields, we can show that

$$\Box_x\Box_x\Big[A_\mu(x),A_\nu(y)\Big] = \frac{e^2}{\mathcal{K}^2}\epsilon_{\mu\sigma\beta}\epsilon_{\nu\omega\gamma}\partial^\sigma_x\partial^\omega_x\Big[J^\beta(x),J^\gamma(y)\Big] \tag{4.60}$$

Then, using the integral representation formula twice, and considering the initial conditions, we can define the general solution

$$\begin{aligned}\Big[A_\mu(x),A_\nu(y)\Big] &= -\frac{i}{\mathcal{K}}\epsilon_{\mu\nu\alpha}\partial^\alpha D(x-y,0)+i\alpha\partial_\mu\partial_\nu E(x-y)\\ &-\frac{e^2}{\mathcal{K}^2}\epsilon_{\mu\sigma\beta}\epsilon_{\nu\omega\gamma}\partial^\sigma_x\partial^\omega_x\int d^3\omega d^3u\ \epsilon(y,x;u)\epsilon(x,u;\omega)D(x-\omega,0)D(y-u,0)\Big[J^\beta(\omega),J^\gamma(u)\Big]\end{aligned} \tag{4.61}$$

The vacuum average reads

$$\begin{aligned}\langle 0|\Big[A_\mu(x),A_\nu(y)\Big]|0\rangle &= -\frac{i}{\mathcal{K}}\epsilon_{\mu\nu\alpha}\partial^\alpha D(x-y,0)+i\alpha\partial_\mu\partial_\nu E(x-y)\\ &-\frac{e^2}{\mathcal{K}^2}\epsilon_{\mu\sigma\beta}\epsilon_{\nu\omega\gamma}\partial^\sigma_x\partial^\omega_x\int d^3\omega d^3u\ \epsilon(y,x;u)\epsilon(x,u;\omega)D(x-\omega,0)D(y-u,0)\langle 0|\Big[J^\beta(\omega),J^\gamma(u)\Big]|0\rangle\end{aligned} \tag{4.62}$$

Considering the operator equations of motion, we can also obtain

$$\begin{aligned}\Big[A_\mu(x),\psi(y)\Big] &= \frac{1}{\mathcal{K}}\epsilon_{\mu\sigma\nu}\partial^\sigma_x\int d^3u\ \varepsilon(x,y,u)D(x-u)\Big[eJ^\nu(u),\psi(y)\Big]\\ &-\frac{\alpha e}{\mathcal{K}}\partial^x_\mu\int d^3u\varepsilon(x,y,u)D(x-u)D(u-y)\psi(y)\end{aligned} \tag{4.63}$$

The equations of motion for the Fermion fields furnishes

$$\Big(i\partial\!\!\!/^{x}-m\Big)\Big\{\psi(x),\bar\psi(y)\Big\}\Big(i\overleftarrow{\partial\!\!\!/}^{y}+m\Big) = -e^2\Big\{\gamma^\mu A_\mu(x)\psi(x),\bar\psi(y)A_\nu(y)\gamma^\nu\Big\} \tag{4.64}$$

where the left side denotes the electron self energy.

The general solution is then obtained by means of the initial data and the integral representation for the anti-commutator equation above. It leads to

$$\begin{aligned}\Big\{\psi(x),\bar\psi(y)\Big\} =& iS(x-y)-\int d^3\omega d^3u\ \epsilon(y,x;u)\epsilon(x,u;\omega)S(x-\omega)\hat\Sigma(\omega,u)S(u-y)\\ &-i\int d^3\omega\epsilon(y,x,\omega)S(x-\omega)eA\!\!\!/(\omega)S(w-y)\end{aligned} \tag{4.65}$$

At this point is important to mention that although the (CS) model is been employed to effectively describe the action of an external magnetic field on a planar sample, the mean value of the gauge field vanishes in this effective model since to simulate these effects of true QED it is enough to consider the topological model with no one point functions.

Therefore, the vacuum average reads

$$\langle 0|\Big\{\psi(x),\bar\psi(y)\Big\}|0\rangle = iS(x-y)-\int d^3\omega d^3u\ \epsilon(y,x;u)\epsilon(x,u;\omega)S(x-\omega)\Sigma(\omega-u)S(u-y) \tag{4.66}$$

with $\Sigma(x-y)\equiv e^2\langle 0|\Big\{\gamma^\mu A_\mu(x)\psi(x),\bar\psi(y)A_\nu(y)\gamma^\nu\Big\}|0\rangle$.

If one wants to include the Fermionic content from the other valley to generalize the result for a wider class of models, we must consider the other quasi-particle excitation with the same anti-commutator structure with its respective mass m_2. The interaction term becomes $eA_\mu \sum_{i=1}^{2} \bar{\psi}_i \gamma^\mu \psi_i$. Then, their anti-commutator reads

$$\langle 0| \Big\{ \psi_1(x), \bar{\psi}_2(y) \Big\} |0\rangle = -\int d^3\omega d^3u \; \epsilon(y,x;u)\epsilon(x,u;\omega) S(x-\omega, m_1)\Sigma(\omega - u) S(u-y, m_2) \tag{4.67}$$

with $\Sigma(x-y) \equiv e^2 \langle 0| \Big\{ \gamma^\mu A_\mu(x)\psi_1(x), \bar{\psi}_2(y) A_\nu(y)\gamma^\nu \Big\} |0\rangle$.

These expressions are the complete formal ones, they give a complete non-perturbative characterization of the system. We must find a way to extract explicit numerical information from that general expressions and, in order to do so, an approximation method must be considered.

Regarding the first order perturbative analysis of the next section, due to this last result, we can consider $\langle 0|[J^1_\mu(x) + J^2_\mu(x), J^1_\mu(y) + J^2_\mu(y)]|0\rangle = \langle 0|[J^1_\mu(x), J^1_\mu(y)]|0\rangle + \langle 0|[J^2_\mu(x), J^2_\mu(y)]|0\rangle$. It implies that, in this order, the polarization and the self energy tensor receives two independent contributions from each valley.

4.5 Lowest Order Corrections

In order to evaluate the photon commutator at order e^2 we must compute the source's commutator at the lowest perturbation order. This distribution has an analogy to the self energy of the topological photon, which at the lowest order is equal to the vacuum polarization tensor [35] and, at this order, has the following form [5] (considering just the free commutators in order to find the first correction.)

$$\begin{aligned} e^2 \langle 0| \Big[J_\mu(x), J_\nu(y) \Big] |0\rangle = \\ e^2 \langle 0| \bar{\psi}_a(x) \gamma^{ab}_\mu \Big\{ \psi_b(x), \bar{\psi}_c(y) \Big\} \gamma^{cd}_\nu \psi_d(y) |0\rangle - e^2 \langle 0| \bar{\psi}_c(y) \gamma^{ab}_\mu \Big\{ \psi_d(y), \bar{\psi}_a(x) \Big\} \gamma^{cd}_\nu \psi_b(x) |0\rangle \\ = e^2 tr\Big(\gamma_\mu S^+(x-y)\gamma_\nu S^-(y-x) - \gamma_\nu S^-(x-y)\gamma_\mu S^+(y-x) \Big) \end{aligned} \tag{4.68}$$

We have used the definition of the Pauli-Jordan distribution in terms of its positive and negative frequency parts, the cyclicality of the trace and the information from the previous section regarding the free solution, from which we can deduce $\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = S^+(x-y)$, $\langle 0|\bar{\psi}(y)\psi(x)|0\rangle = S^-(x-y)$ and $\pm$, denotes the positive and negative frequency parts of the distribution.

The next step is to calculate the Fourier transform of the following term $P_{\mu\nu}(x-y) = tr\Big(\gamma_\mu S^+(x-y)\gamma_\nu S^-(y-x) \Big)$ and then infer the whole expression of the commutator $\langle 0| \Big[J_\mu(x), J_\nu(y) \Big] |0\rangle = P_{\mu\nu}(x-y) - P_{\nu\mu}(y-x)$ in momentum space using the expressions for the Pauli-Jordan distributions from the previous section[6]

$$\begin{aligned} P_{\mu\nu}(k) = \int d^3z P_{\mu\nu}(z) e^{ikz} = \\ -\int \frac{d^3p}{(2\pi)} \, tr\Big(\gamma_\mu (\not{p} + m)\gamma_\nu (\not{k} - \not{p} - m) \Big) \delta(p^2 - m^2)\theta(p_0)\delta((k-p)^2 - m^2)\theta((k-p)_0) \end{aligned} \tag{4.69}$$

[5]For the case of a drift velocity with upper limit $v \neq 1$ we must have $J_\mu \to (\bar{\psi}\gamma_0\psi\delta^0_\mu + v\bar{\psi}\gamma_j\psi\delta^j_\mu)$.

[6]The expression of the Fermionic anti commutator in terms of its Fourier transform can be put in the following form $S^\pm(z) = \frac{i}{(2\pi)^2} \int d^3p (\not{p} \pm m) e^{-i(\pm)pz}\theta(p_0)$

Evaluating the trace, we get for the tensor part in the numerator (see the appendix B)

$$U^{\mu\nu} = -2\Big[(m^2 - p^2)\eta^{\mu\nu} + 2p^\mu p^\nu - (p^\mu k^\nu + p^\nu k^\mu) + \eta^{\mu\nu} pk + im\epsilon^{\mu\nu\alpha}k_\alpha\Big] \tag{4.70}$$

We can contract k_μ with $P_{\mu\nu}(k)$ and using the relations implied by the delta functions conclude that it is transverse. Since the antisymmetric part is obviously transverse, it means that the symmetric part has the form $(k_\mu k_\nu - \eta^{\mu\nu}k^2)B(k)$ with $P^{\mu\nu}\eta_{\mu\nu} = -2k^2 B(k)$. Therefore, we conclude

$$B(k) = -\frac{\left(4m^2 + k^2\right)}{2k^2}\int \frac{d^3p}{(2\pi)}\delta(p^2 - m^2)\theta(p_0)\delta((k-p)^2 - m^2)\theta((k-p)_0) \tag{4.71}$$

The antisymmetric part is also proportional to the integral

$$\int \frac{d^3p}{(2\pi)}\delta(p^2 - m^2)\theta(p_0)\delta((k-p)^2 - m^2)\theta((k-p)_0) = \frac{\theta(k^2 - 4m^2)\theta(k_0)}{4\sqrt{k^2}} \tag{4.72}$$

where we use the frame $k_\mu = (k_0, \vec{0})$.

In order to deduce this expression we considered the identity $\delta(p^2 - m^2)\theta(p_0) = \frac{\delta(p_0 - E_p)}{2E_p}$, the relativistic relation $EdE = |\vec{p}|d|\vec{p}|$ and the fact that going back to the general frame we have to consider $|k_0| \to \sqrt{k^2}$. It is also taken into account the fact that due to the relations implied by the delta functions, we have $|\vec{p}| = \sqrt{\frac{k^2}{4} - m^2}$.

Owing to the form of $P^{\mu\nu}(k)$ the Fourier transform of the current commutator is

$$\Pi_{\mu\nu}(k) = e^2\langle 0|\Big[J_\mu(k), J_\nu(-k)\Big]|0\rangle = -\frac{e^2}{8}\Big(1 + \frac{4m^2}{k^2}\Big)\theta(k^2 - 4m^2)\frac{sign(k_0)}{\sqrt{k^2}}\Big(k_\mu k_\nu - k^2\eta_{\mu\nu}\Big)$$
$$+ i\epsilon_{\mu\nu\alpha}k^\alpha\frac{e^2 m}{2\sqrt{k^2}}\theta(k^2 - 4m^2)sign(k_0) \tag{4.73}$$

Therefore, we obtained an explicit expression for the commutator version of the vacuum polarization tensor. However, in order to calculate the complete propagator structure and then the effect of the quantum corrections on the transport properties, the knowledge of the time ordered version of the polarization tensor is necessary. The complete propagator has the structure $\mathcal{D}^{-1F}_{\mu\nu}(k) = D^{-1F}_{(0)\mu\nu}(k) - i\Pi^F_{\mu\nu}(k)$, from which the two photon sector of the effective action can be achieved.

Fortunately, there is a link between the commutator function and its time ordered version. Due to the possibility of a spectral representation for the former, it is possible to infer that

$$\Pi_{\mu\nu}(k) = 2\pi sign(k_0)\big[(\eta_{\mu\nu} - k_\mu k_\nu/k^2)\rho^1(k^2) + i\epsilon_{\mu\nu\gamma}k^\gamma\rho^2(k^2)\big] \tag{4.74}$$

in which $\rho^{1,2}(k^2)$ represents the spectral density for the tensor and pseudo-tensor sectors. Therefore, it is possible to obtain an integral representation for the Feynman distributions in terms of $\rho^{1,2}(k^2)$ which can be obtained from the $\Pi_{\mu\nu}(k)$.

However, there is also an alternative convenient and instructive way to derive the Feynman distributions from its (anti) commutator versions. First of all, we should obtain the truncated Wightman function related to the current-current commutator

$$\langle 0|\Big[J_\mu(x), J_\nu(y)\Big]|0\rangle = \langle 0|J_\mu(x)J_\nu(y)|0\rangle_T - \langle 0|J_\nu(y)J_\mu(x)|0\rangle_T \tag{4.75}$$
$$= tr\Big(\gamma_\mu S^+(x-y)\gamma_\nu S^-(y-x) - \gamma_\mu S^-(x-y)\gamma_\nu S^+(y-x)\Big)$$

Considering the positive energy condition [23], it is possible to find the truncated structure

$$\langle 0|J_\mu(x)J_\nu(y)|0\rangle_T = tr\Big(\gamma_\mu S^+(x-y)\gamma_\nu S^-(y-x)\Big) \tag{4.76}$$

Therefore, the time ordered product is defined by

$$\langle 0|\mathcal{T}J_\mu(x)J_\nu(y)|0\rangle = \theta(x_0-y_0)tr\Big(\gamma_\mu S^+(x-y)\gamma_\nu S^-(y-x)\Big)+\theta(y_0-x_0)tr\Big(\gamma_\mu S^-(x-y)\gamma_\nu S^+(y-x)\Big) \tag{4.77}$$

Owing to the definition of the free Fermion anti-commutator $S(x-y)$ and the Feynman distribution as

$$D_F(x-y) \equiv \theta(x_0-y_0)D^+(x-y)+\theta(y_0-x_0)D^-(y-x) = \frac{i}{(2\pi)^3}\int d^3p\frac{e^{-ip.(x-y)}}{(p^2-m^2+i\sigma)} \tag{4.78}$$

with $\sigma \to 0$.
the time ordered distribution can be expressed as

$$\langle 0|\mathcal{T}J_\mu(x)J_\nu(y)|0\rangle = tr\Big(\gamma_\mu S_F(x-y)\gamma_\nu S_F(y-x)\Big) \tag{4.79}$$

with $S^F(x-y) \equiv \big(i\partial\!\!\!/ + m\big)D^F(x-y)$.

The expression above has, in general, no well-defined Fourier transform since the multiplication by a Heaviside function turn it more singular [28]. Therefore, in order to evaluate this expression, a regularizing method must be employed. We use the result of [26]. We adopt the definition of the photon self energy which, in this first approximation, is equal to the polarization tensor

$$i\Pi^F_{\mu\nu}(k^2) \equiv e^2\langle 0|\mathcal{T}J_\mu(k)J_\nu(-k)|0\rangle = -\frac{e^2}{(2\pi)^3}Tr\int d^3p\frac{\gamma_\mu(m+p\!\!\!/)\gamma_\nu(m+p\!\!\!/-k\!\!\!/)}{(p^2-m^2+i\sigma)((k-p)^2-m^2+i\sigma)} \tag{4.80}$$

The integration yield the result

$$\begin{aligned}\Pi^F_{\mu\nu}(k^2) =&\Big(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\Big)\frac{e^2k^2}{16\pi}\Big[\frac{1}{\sqrt{k^2}}(1+\frac{4m^2}{k^2})\,\log\frac{1-\sqrt{k^2/4m^2}}{1+\sqrt{k^2/4m^2}}+\frac{4m}{k^2}\Big] \\ &-im\epsilon_{\mu\nu\alpha}k^\alpha\Big(\frac{e^2}{4\pi\sqrt{k^2}}\log\frac{1-\sqrt{k^2/4m^2}}{1+\sqrt{k^2/4m^2}}\Big)\end{aligned} \tag{4.81}$$

We are interested in the explicit k^2 dependent expression for the topological sector and not just the $\Pi^{top}_F(0)$ since we want to study how the transverse conductivity flows with the energy.

In order to build the two topological photon sector of the effective action, it is required to know its free Feynman propagator

$$\begin{aligned}\langle 0|\Big[A_\rho(x),A_\beta(y)\Big]|0\rangle =& -\frac{i}{\mathcal{K}}\epsilon_{\rho\beta\mu}\partial^\mu D(x-y,0)+i\alpha\partial_\rho\partial_\beta E(x-y) \\ =&\langle 0|A_\rho(x)A_\beta(y)|0\rangle_T - \langle 0|A_\beta(y)A_\rho(x)|0\rangle_T\end{aligned} \tag{4.82}$$

From the positive energy requirement, we obtain the Wightman function

$$\langle 0|A_\rho(x)A_\beta(y)|0\rangle_T = -\frac{1}{\mathcal{K}}\epsilon_{\rho\beta\mu}\partial^\mu D^+(x-y,0)+\alpha\partial_\rho\partial_\beta E^+(x-y) \tag{4.83}$$

Then, its time ordered version is given below

$$\langle 0|\mathcal{T}A_\rho(x)A_\beta(y)|0\rangle = -\frac{1}{\mathcal{K}}\epsilon_{\rho\beta\mu}\partial^\mu D_F(x-y,0) + \alpha\partial_\rho\partial_\beta E_F(x-y) \tag{4.84}$$

where the initial conditions of $D^\pm(x-y,0)$ and $E^\pm(x-y,0)$ were considered.

Its Fourier transform reads [36, 37]

$$D^F(k)_{\mu\nu}(k) = -\epsilon_{\mu\nu\alpha}\frac{k^\alpha}{\mathcal{K}(k^2+i\sigma)} + i\alpha\frac{k_\mu k_\nu}{(k^4+i\sigma)} \tag{4.85}$$

Applying the spin projector algebra presented in the next subsection, it is easy to obtain its inverse

$$D^F(k)_{\mu\nu}{}^{-1} = \mathcal{K}\epsilon_{\mu\nu\alpha}k^\alpha + \frac{i}{\alpha}k_\mu k_\nu \tag{4.86}$$

The knowledge of the time ordered polarization tensor and the free Feynman propagator allows the obtainment of the quantum action

$$\Gamma^{(2)} = -i\int d^3k\ A^\mu(k)\Big[\Big(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)\frac{e^2k^2}{32\pi}\Big[\frac{1}{\sqrt{k^2}}(1+\frac{4m^2}{k^2})\ \log\frac{1-\sqrt{k^2/4m^2}}{1+\sqrt{k^2/4m^2}} + \frac{4m}{k^2}\Big] \tag{4.87}$$
$$+ i\epsilon_{\mu\nu\alpha}k^\alpha\Big(-m\frac{e^2}{8\pi\sqrt{k^2}}\log\frac{1-\sqrt{k^2/4m^2}}{1+\sqrt{k^2/4m^2}} + \frac{\mathcal{K}}{2}\Big) - \frac{1}{2\alpha}k_\mu k_\nu\Big]A^\nu(-k)$$

where the gauge parameter α do not affect the physical transverse conductivity, as it should be.

From the low energy limit of the previous expression and considering two interacting valleys, one note that the quantized transverse conductance receives a radiative correction of order e^2

$$\sigma_{xy} \to \sigma_{xy} + m_1\frac{e^2}{4\pi\sqrt{k^2}}\log\frac{1-\sqrt{k^2/4m_1^2}}{1+\sqrt{k^2/4m_1^2}} + m_2\frac{e^2}{4\pi\sqrt{k^2}}\log\frac{1-\sqrt{k^2/4m_2^2}}{1+\sqrt{k^2/4m_2^2}} \tag{4.88}$$

The transport properties are defined in the low energy limit [7]

$$\lim_{k\to 0}\sigma_{xy}(k) = \frac{e^2}{2\pi}\Big(n - \frac{1}{2}(sign(m_1) + sign(m_2))\Big) \tag{4.89}$$

We obtained a microscopic description for the quantum Hall effect over topological materials. We noticed that its renormalized version depends on the sum of the topological numbers [8] of the the interacting valleys. For the case of a time reversal invariant topological insulator there is no shift in the conductance [9]. However, for the Haldane-like material [15], depending on its lattice parameters, and also for the case of a single valley [16, 18], the conductivity indeed receives a shift. The Coleman-Hill theorem, that is still valid for this kind of model (even with Lorentz symmetry breaking [11].) guarantee that this structure for the quantized conductivity is indeed the exact one since any shift must be originated by the first radiative correction. It shows that the quantized conductivity arising from the Landau model is indeed robust, as the experiments reveal, even when considering the corrections due to the microscopic structure of the charge carriers.

The transverse conductivity flows with the energy as

[7] As we are going to see, the low energy limit of the topological sector of the radiative corrections is the same for both the relativistic and also for the case with quasi-particles whose velocity is different than the light one.

[8] which happens to be related to the material's intrinsic conductivity

[9] the time reversal symmetry implies in opposite masses for each valley.

$$\frac{d\sigma_{xy}}{d\sqrt{k^2}} = -\sum_i \frac{e^2}{8\pi}\Big(\log(\frac{1-\sqrt{k^2/4m_i^2}}{1+\sqrt{k^2/4m_i^2}}) + \frac{2}{\sqrt{k^2/4m_i^2}(1-k^2/4m_i^2)}\Big) \tag{4.90}$$

where the sum is over the masses of quasi-particles of each valley.
It is worth mention that this expression has no divergence for $k^2 \to 0$.

4.5.1 Static Potential Computation

In possession of the renormalized structure, physical aspects such as the potential and magnetic field contributions due to a charge distribution can be investigated.

Although the interaction potential between two static charged particles vanishes for the bare Chern-Simons model, in an interacting scenario, the radiative corrections generate a non trivial low energy scalar potential. In order to calculate it, we need to find the propagator by inverting the gauge field two point structure. To this end, we decompose the polarization tensor in terms of the spin projectors as

$$\Pi^T_{\mu\nu}(k^2) = \Pi^1(k^2)\Big(P^1_{\mu\nu}(k^2) + P^2_{\mu\nu}(k^2)\Big) + m\Big(P^1_{\mu\nu}(k^2) - P^2_{\mu\nu}(k^2)\Big)\sqrt{k^2}\Pi^2(k^2) \tag{4.91}$$

where $\Pi_1(k^2)$ and $\Pi_2(k^2)$ are the coefficients of the transverse and the topological sector, respectively.

Regarding the projectors used above, we first define the transverse and longitudinal operators as follows

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \quad , \quad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} \quad , \quad E_{\mu\nu} = \frac{i}{\sqrt{k^2}}\epsilon_{\mu\nu\alpha}k^\alpha \tag{4.92}$$

We use the same projection definitions as in [25]

$$P^1_{\mu\nu} = \frac{1}{2}\Big(\theta_{\mu\nu} + E_{\mu\nu}\Big) \quad , \quad P^2_{\mu\nu} = \frac{1}{2}\Big(\theta_{\mu\nu} - E_{\mu\nu}\Big) \quad , \quad P^3_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} \tag{4.93}$$

with $P^1_{\mu\nu} + P^2_{\mu\nu} + P^2_{\mu\nu} = \eta_{\mu\nu}$ and $P^3_{\mu\alpha}P^{(i)\alpha\beta} = 0$ for $i = 1, 2$.

The complete inverse propagator obeys the relation $\mathcal{D}^{-1}_{F\ \mu\nu}(k) = D^{-1}_{F(0)\mu\nu}(k) - i\Pi^F_{\mu\nu}$, where its free version has the form $D^{-1}_{F\ (0)\mu\nu}(k) = -i\mathcal{K}\sqrt{k^2}\Big(P^1_{\mu\nu} - P^2_{\mu\nu}\Big) - i\alpha P^3_{\mu\nu}$. Using the projectors algebra, the complete propagator reads

$$\mathcal{D}^F_{\mu\nu}(k) = i\frac{\Pi_1\theta_{\mu\nu}}{\Big[\Pi_1^2 - k^2(\mathcal{K}+m\Pi_2)^2\Big]} - i\frac{E_{\mu\nu}\sqrt{k^2}\Big(\mathcal{K}+m\Pi_2\Big)}{\Big[\Pi_1^2 - k^2(\mathcal{K}+m\Pi_2)^2\Big]} - i\frac{\alpha P_{(3)\mu\nu}}{k^2} \tag{4.94}$$

Considering the low energy limit, the coefficients tends to $\Pi_2(k^2) \to c'$ and $\Pi_1(k^2) \to \tilde{n}k^2$. If one adds the content of the two valleys, we must consider $m\Pi_2(k^2, m) \to m_1\Pi_2(k^2, m_1) + m_2\Pi_2(k^2, m_2)$ with $c'_1 = \frac{e^2}{4\pi}\frac{1}{|m_1|}$, $c'_2 = \frac{e^2}{4\pi}\frac{1}{|m_2|}$ and also $\tilde{n} = -\frac{e^2}{16\pi}\big(\frac{1}{|m_1|} + \frac{1}{|m_2|}\big)$.

The non-topological sector of the propagator presents a massive pole of positive norm in accordance with our previous non-perturbative analysis for the interacting case. However, it do not imply in the existence of a free external state since, due to the topological character of the Chern-Simons model, the norm of the projections of its observables is algebraically related to the one of the currents.

Regarding the topological sector, at the low energy limit, it can be written as

$$\Pi^{top}_{\mu\nu}(k^2) = \frac{1}{\tilde{\mathcal{K}}}\epsilon_{\mu\nu\alpha}k^\alpha\Big(-\frac{1}{k^2}+\frac{\tilde{n}^2}{\tilde{n}^2k^2-\tilde{\mathcal{K}}^2}\Big) \tag{4.95}$$

the massless pole has its residue renormalized by the interaction as $\tilde{\mathcal{K}} = \mathcal{K} + \frac{e^2}{4\pi}\sum_i n^i_{chern}$ with $\sum_i n^i_{chern}$ meaning the sum of the chern numbers of each valley.

Now, we consider the fact that the upper limit on the drift velocity of the quasi-particle is different than the light velocity c generating a different behaviour for space and time. We follow the discussion of [11], but here the objective is to explicitly derive the complete propagator for this case. First of all, we should consider the discussion of the previous section regarding the structure of the Fermionic propagator for the case of isotropically broken Lorentz symmetry. Then, from the evaluation of the time and space components of the time ordered version of the polarization tensor for this specific case by the use of an appropriate redefinition of the loop integration variable and considering the following projectors [38]

$$P^{ij}_T = \delta^{ij} - \frac{k^ik^j}{\vec{k}^2} \quad ; \quad P^{\mu\nu}_L = -\eta^{\mu\nu} + \frac{k^\mu k^\nu}{k^2} - P^{\mu\nu}_T \tag{4.96}$$

with $P^{00}_T = P^{0i}_T = 0$, $P_T.P_L = 0$ and the algebra $P^2_L = -P_L$, $P^2_T = -P_T$, $P_L + P_T = \theta$ it is possible to show that the polarization tensor can be written in terms of them as

$$\Pi_{\mu\nu}(\vec{k}^2) = FP_{L\ \mu\nu} + GP_{T\ \mu\nu} + mE_{\mu\nu}(\vec{k}^2)\sqrt{\vec{k}^2}\Pi^2(v\vec{k}^2) \tag{4.97}$$

We consider the static case in order to derive the potentials created due to a point charge.

Then, we have $G = -\Pi_1(v\vec{k},0) = \sum_i \frac{e^2v^2\vec{k}^2}{16\pi|m_i|}$ and $F = -\frac{\Pi_1}{v^2}(v\vec{k},0) = \sum_i \frac{e^2\vec{k}^2}{16\pi|m_i|}$ with v being the upper limit on the velocity of the quasi particle. In the low energy limit, the topological sector remains the same as for the case with $v = 1$. It means that the features of this sector, at long distances, do not depend on the specific microscopic structure of the material. It can be associated to the universality property of the quantum Hall effect [4].

The static propagator with $v \neq 1$ reads

$$D^F_{\mu\nu} = \theta_{\mu\nu}\frac{i\Pi_1}{v^2\big(\tilde{\mathcal{K}}^2\vec{k}^2+G^2\big)} - P_{T\ \mu\nu}\frac{i(G-F)}{\big(\tilde{\mathcal{K}}^2\vec{k}^2+G^2\big)} - iE_{\mu\nu}\frac{\sqrt{\vec{k}^2}\tilde{\mathcal{K}}}{\big(G^2+\vec{k}^2\tilde{\mathcal{K}}^2\big)} - i\frac{\alpha P_{(3)\mu\nu}}{k^2} \tag{4.98}$$

We can use the renormalized propagator to to calculate the non-relativistic scalar potential energy due to the presence of a static charge q, whose current $J_\mu(k) = (2\pi)\Big(q\delta_{\mu 0}\delta(k_0)e^{-i\vec{k}.\vec{r}_1}\Big)$, is located at $\vec{r}_1$. It reads [10] (we are considering the low energy limit.)

$$A_0 = \int\frac{d^2\vec{k}}{(2\pi)}\mathcal{D}^R_{00}(\vec{k})e^{i\vec{k}(\vec{x}_1-\vec{x}_2)}q = \int\frac{d^2\vec{k}}{(2\pi)\tilde{n}v^4}\frac{q}{\big(|\vec{k}|^2+m^{2\prime}\big)}e^{i\vec{k}(\vec{x}_1-\vec{x}_2)} \tag{4.99}$$

where the parameter m' has the value $\tilde{\mathcal{K}}/|\tilde{n}|v^2$.

The scalar potential is given by

$$A_0 = q\frac{K_0(m'.r)}{\tilde{n}v^4} = \frac{q}{\tilde{n}v^4}\frac{e^{-m'r}\pi^{1/2}}{\sqrt{2m'r}}\Big[1-\frac{1}{8m'r}\Big(1-\frac{9}{16m'r}(1-\frac{25}{24m'r})\Big)\Big] \tag{4.100}$$

[10]The A_0 field is a c-number object representing the average value of the gauge field due to the presence of the external current.

where K_0 denotes the modified Bessel function of the second kind of order 0. The radiative corrections are responsible for the existence of a non-vanishing scalar potential due to a static charge. In accordance with the fact that for the case of free Chern-Simons system an static charge cannot give rise to a non-vanishing potential, the expression above tends to zero for $e \to 0$.

We have used the retarded propagator which is related to the Feynman one by $D^R(x) = iD_F(x)$ associated to a different prescription for the contour adding a piece of the form $i0p_0$ at the denominator.

The vector potential reads

$$A_i(r) = q\int \frac{d^2p}{(2\pi)}\, e^{ip.r}\frac{\sqrt{\vec{p}^2}\tilde{\mathcal{K}}E_{i0}}{G^2+\vec{p}^2\tilde{\mathcal{K}}^2} = -q\frac{i}{\tilde{\mathcal{K}}}\int \frac{d^2p}{(2\pi)}\, e^{ip.r}\epsilon_{i0j}p^j\Big(-\frac{1}{\vec{p}^2}+\frac{\tilde{c}^2}{\tilde{c}^2\vec{p}^2+\tilde{\mathcal{K}}^2}\Big) \quad (4.101)$$

with $\tilde{c} = \tilde{n}v^2$.

The magnetic field $\mathcal{B}(r) = \epsilon^{ij}\partial_i A_j(r)$ has the form

$$\mathcal{B}(r) = -\frac{q}{\tilde{\mathcal{K}}}\int \frac{d^2p}{(2\pi)}\, e^{ip.r}p_ip^i\Big(-\frac{1}{\vec{p}^2}+\frac{\tilde{c}^2}{\tilde{c}^2\vec{p}^2+\tilde{\mathcal{K}}^2}\Big) = \frac{q}{\tilde{\mathcal{K}}}\big[2\pi\delta^2(\vec{r})+m'^2K_0(m'.r)\big] \quad (4.102)$$

Interestingly, besides the traditional delta function, now renormalized, we have one extra contribution to the magnetic field, with a bigger range, due to the radiative corrections.

4.6 The Electron Self Energy

Regarding the quantum corrections due to the anti-commutator version of the electron self energy tensor, from the non-perturbative solution of section 4 evaluated at its lowest order[11], using the free (anti) commutators, we have

$$\begin{aligned}\Sigma(x-y) &= e^2\langle 0|\Big\{\gamma^\mu A_\mu(x)\psi(x), \bar{\psi}(y)A_\nu(y)\gamma^\nu\Big\}|0\rangle \\ &= e^2\langle 0|\gamma^\mu A_\mu(x)\Big\{\psi(x),\bar{\psi}(y)\Big\}A_\nu(y)\gamma^\nu|0\rangle - e^2\langle 0|\gamma^\mu\bar{\psi}(y)\Big[A_\mu(x),A_\nu(y)\Big]\psi(x)\gamma^\nu|0\rangle \\ &= \frac{e^2}{\mathcal{K}}\gamma^\mu\Big(\epsilon_{\mu\nu\alpha}\partial^\alpha D^+(x-y,0)S^+(x-y) - \epsilon_{\mu\nu\alpha}\partial^\alpha D^-(x-y,0)S^-(x-y)\Big)\gamma^\nu\end{aligned} \quad (4.103)$$

where, for simplicity, we consider the Landau gauge with $\alpha = 0$. The quantity above is gauge dependent although the observable formed by its spinor indices contracted with the spinor structure of the external particles is independent, since any longitudinal contribution coming from the gauge field propagator vanishes due to the Gordon identity. We also do not consider the Lorentz breaking effects due to $v \neq 1$, with v being the drift velocity of the quasi-particle, since the main phenomenological features of the system can be discussed by our approach. Therefore, we leave this point for a forthcoming work. The analysis done here is enough to contemplate the key aspects of the quantum Hall effect phenomenology.

Although the Chern-Simons term is topological, we are going to show that it indeed contribute to a nontrivial anti-commutator version of the self energy.

The Fourier transform of the positive frequency part of the self energy commutator reads

[11] We must mention again that due to the structure of the coefficient of the Chern-Simons term in order to use perturbation theory the $n \gg 1$ limit of high transverse conductivity must be considered.

$$\Sigma^+(p) = \int d^3z\, \Sigma^+(z)e^{ip.z} = \frac{ie^2}{(2\pi)\mathcal{K}} \int d^3q \gamma^\mu \epsilon_{\mu\nu\rho}(p^\rho - q^\rho) D^+(p-q,0)(\not{q} + m) D^+(q,m^2)\gamma^\nu \tag{4.104}$$

In order to reduce this expression, we use the gamma matrices identities $[\gamma^\mu, \gamma^\nu] = -2i\epsilon^{\mu\nu\rho}\gamma_\rho$ and $\not{q}\gamma^\nu = -\gamma^\nu \not{q} + 2q^\nu$. It then becomes

$$\begin{aligned}\Sigma^+(p) = \frac{e^2}{\pi\mathcal{K}} \int \Big[\Big(\not{p} - \not{q}\Big) D^+(p-q,0)D^+(q,m^2)m - \Big(\not{p} - \not{q}\Big) D^+(p-q,0)D^+(q,m^2)\not{q} \\ - i\epsilon_{\mu\nu\rho}\gamma^\mu p^\rho q^\nu D^+(p-q,0)D^+(q,m^2)\Big] d^3q\end{aligned} \tag{4.105}$$

This expression can be even more simplified. In fact, considering the rest frame $p_\mu = (p_0, \vec{0})$, due to the integration symmetry under $q^i \to -q^i$, it is possible to show that the term proportional to the Levi-Civitta tensor vanishes

$$\Sigma^+(p) = \frac{e^2}{\pi\mathcal{K}} \int d^3q \Big(-(\not{p} + m)\not{q} + \not{p}m + q^2 \Big) D^+(p-q,0)D^+(q,m^2) \tag{4.106}$$

Regarding the first term, we obtain

$$\frac{e^2}{\pi\mathcal{K}}(\not{p} + m) \int d^2|\vec{q}|\gamma^0 E_q \frac{\delta\Big(E_q - \frac{p_0^2+m^2}{2p_0}\Big)}{4|p_0|E_q}\theta(p_0 - E_q) \tag{4.107}$$

We considered $\not{q} = \gamma_0 q_0$ since this is the only term that contributes to the integral.

Considering the relativistic relation $EdE = |\vec{q}|d|\vec{q}|$, and also that $p_0 \geq 0$, $E_q = \frac{(p_0^2+m^2)}{2p_0}$ and $|\vec{q}| = \frac{p_0^2-m^2}{2p_0}$, it implies in $p^2 - m^2 \geq 0$. Therefore, this first term can be managed to yield

$$\frac{e^2}{4\mathcal{K}} \frac{(\not{p} + m)}{\sqrt{p^2}} \Big(1 + \frac{m^2}{p^2}\Big)\not{p}\; \theta(p^2 - m^2)\theta(p_0) \tag{4.108}$$

We considered the fact that going back to a general frame $|p_0| \to \sqrt{p^2}$.

Following analogous steps, we find for the second term

$$-\frac{e^2}{2\mathcal{K}} \frac{\big(\not{p}m + m^2\big)}{\sqrt{p^2}}\theta(p^2 - m^2)\theta(p_0) \tag{4.109}$$

Therefore, the anti commutator version of the self energy tensor has the form

$$\Sigma(p) = \frac{e^2}{4\mathcal{K}\sqrt{p^2}} \Big((\not{p} + m)\Big(1 + \frac{m^2}{p^2}\Big)\not{p} - 2m^2 - 2\not{p}m \Big)\theta(p^2 - m^2) sign(p_0) \tag{4.110}$$

It is worth mention that its positive frequency part is associated to the imaginary part of the time ordered version. They can be related through an integral representation with a given number of subtractions according to the $\Sigma^+(p)$ divergence degree.

Analogously to the previous section, the positive energy condition leads to the one loop time ordered self energy tensor defined as $-i\Sigma_F(x-y) = e^2\gamma^\mu\epsilon_{\mu\alpha\nu}\partial^\alpha D^F(x-y,0)S^F(x-y)\gamma^\nu$. From this expression, we can find the self energy definition, whose Fourier transform reads [36]

$$-i\Sigma_F(p,m) = -\frac{2e^2}{\mathcal{K}}\int\frac{d^3q}{(2\pi)^3}\Big[\frac{q^2+(\not p-m)\not q}{q^2\big((q+p)^2-m^2\big)}\Big]$$
$$= \frac{ie^2}{4\pi\mathcal{K}}\Big(2m-(\not p-m)\frac{\not p}{m}\Big[\frac{m^2}{p^2}+\frac{m^3}{2p^2\sqrt{p^2}}(1-\frac{p^2}{m^2})\log(\frac{|1-\sqrt{p^2/m^2}|}{|1+\sqrt{p^2/m^2}|})\Big]\Big) \quad (4.111)$$

At low energies, the self energy tensor becomes $\Sigma_F(p,m) = -\frac{e^2}{4\pi\mathcal{K}}\big(2m-\frac{p^2}{|m|}+sign(m)\not p\big)$. In order to guarantee the physical requirement of a pole in the effective mass with residue equal one, we must redefine the fields and also the mass as $m = m_R+\delta m$, leading to a self energy of the form

$$\Sigma^R_F(p,m_R) = (Z-1)(\not p-m_R)-\delta m+\Sigma_F(p,m_R) \quad (4.112)$$

The requirement to have a pole in m_R is equivalent to the condition $\Sigma^R_F(m_R) = 0$ which implies in $\delta m = -\frac{e^2 m}{2\pi\mathcal{K}}$. The unitary residue can be achieved if

$$\frac{\partial\Sigma^R_F(p,m_R)}{\partial\not p}|_{\not p=m_R} = 0 \quad (4.113)$$

which furnishes $Z-1 = \frac{e^2 sign(m_R)}{4\pi\mathcal{K}}$.

Therefore, the inverse of the complete Fermion propagator, at this order, becomes

$$\mathcal{S}^{-1}(p) = -i\Big[\not p-m_R-\Sigma^R_F(p,m_R)\Big] =$$
$$-i\Big[(\not p-m_R)\big(1-\frac{e^2}{4\pi\mathcal{K}}sign(m_R)\big)-p^2\frac{e^2}{4\pi\mathcal{K}|m_R|}+\frac{\not p\; e^2 sign(m_R)}{4\pi\mathcal{K}}\Big] \quad (4.114)$$

Owing to the emergence of the p^2 term, a non trivial topology for the Fermionic response may occur if the Chern number, which is a function [39] $F(\mathcal{S}^{-1},\mathcal{S},\partial_{k_i}\mathcal{S},\partial_{k_i}\mathcal{S}^{-1})|_{E=0}$ of the Green function and its variations with relation to the spatial momentum components at $E=0$, associated to this effective low energy model, which is given by [33]

$$n_{chern} = \frac{1}{2}\Big[sign\big[m_R\big(1-\frac{e^2}{4\pi\mathcal{K}}sign(m_R)\big)\big]+sign(\frac{e^2}{4\pi\mathcal{K}|m_R|})\Big] \quad (4.115)$$

do not vanish.

It means that if the valley mass is negative, the Chern number do not vanish and the system possesses a non trivial topological content leading to an intrinsic Hall conductivity (The Hall conductivity is proportional to this number.). Therefore, for a material whose low energy excitations can be described by just one massive valley, changing the sign of $\mathcal{K}$, leads to a new phase belonging to a different topological class. There are several ways to isolate a single valley from a given material. A combined effect of an external magnetic field with a strain field can be used as a valley filter for graphene-like materials [18].

Since the topological properties are calculated in the $E=0$ frame, our analysis leads to the same qualitative behaviour as for the case with $v \neq 1$ because the symmetry under spatial rotations is kept in both cases.

Regarding the fermionic response, we analyze a system that generalize our result, show that it contains localized boundary states and then particularize for our case [33]. We consider a semi-infinite planar sample with a boundary at $y = 0$. Therefore, p_x is a good quantum number and we replace p_y by $-i\partial_y$

$$AE\psi(p) = \Big(\tilde{M}\sigma_z - C\big(p_x^2 - \partial_y^2\big)^2\sigma_z + A\big(\sigma_x p_x - i\sigma_y\partial_y\big)\Big)\psi(p) \tag{4.116}$$

with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denoting the Pauli matrices. We are assuming the definition

$$\tilde{M} \equiv m_R\big(1 - \frac{e^2}{4\pi\mathcal{K}} sign(m_R)\big) + \frac{e^2E^2}{4\pi\mathcal{K}|m_R|} \tag{4.117}$$

considering the following ansatz for a localized boundary state

$$\psi(p) = \begin{bmatrix} c(p_x) \\ d(p_x) \end{bmatrix} e^{-y\Lambda} \tag{4.118}$$

we have that $\Lambda_{1,2} = p_x^2 + F \pm \sqrt{F^2 - \frac{\tilde{M}^2 - E^2}{C^2}}$ with $F \equiv \big(A^2 - 2\tilde{M}C\big)/2C^2$. y denotes the coordinate orthogonal to the boundary of the sample.

A relation for the energy can be found

$$AE = \tilde{M} - C\Lambda_1\Lambda_2 - C(\Lambda_1 + \Lambda_2)p_x - Cp_x^2 \tag{4.119}$$

A solution that vanishes in the boundary but is concentrated near it, being normalizable, must be of the form

$$\psi(p) = \begin{bmatrix} c(p_x) \\ d(p_x) \end{bmatrix} \Big(e^{-y\Lambda_1} - e^{-y\Lambda_2}\Big) \tag{4.120}$$

Therefore, for the limit $p_x = 0$, we have $E = 0$, and

$$\Lambda_1\Lambda_2 = m_R\big(1 - \frac{e^2}{4\pi\mathcal{K}} sign(m_R)\big)/\frac{e^2}{4\pi\mathcal{K}|m_R|} \tag{4.121}$$

$\Lambda_1\Lambda_2 > 0$ is the condition for the existence of the localized boundary state. In this limit, we also have $\Lambda_1 + \Lambda_2 = |A|/|C|$. Interestingly, it is the condition to ensure a non-trivial Chern number for the system. In accordance with our previous discussions, if $n_{chern} \neq 0$, the valley may present boundary states with a group velocity

$$\mathcal{V} \equiv \frac{dE}{dp_x}|_{p_x=0} = -signAsignC \tag{4.122}$$

for our specific case, it yields

$$\mathcal{V} = -sign\big(\frac{e^2}{4\pi\mathcal{K}|m_R|}\big) \tag{4.123}$$

In order to encompass the two band models mentioned here, we consider the above situation now with two valleys. Then, as mentioned in the section 4, there are two independent complete propagators written in terms of their respective masses. For a time reversal invariant bare matter lagrangian (which is a good model for the QHE samples.) the masses are opposite and then, due to the interaction with the gauge field, the Fermionic response associated to a given valley acquires a non-vanishing Chern number while the other not, leading to an overall group velocity at the boundary. It is a consequence of the asymmetry between the valleys resulting from the interaction with the gauge field.

4.7 Conclusion

This work had two main objectives. First, a purely theoretical one related to extend the B-field formalism by giving it a well-defined perturbative counterpart for the specific case of a topological gauge theory coupled to two band models. Regarding the B field formalism, it helped us to obtain the non-perturbative complete solution and also furnished a well-defined road through the positive energy condition to derive the time ordered objects in a well-defined lowest order solution. We also analyzed the $2+1$ Poincaré stability group for a massless gauge field and then inferred the that the Hilbert space has indefinite metric. The radiative correction for the Fermionic sector was obtained and we demonstrated that it is in agreement with the unitarity requirements.

The second objective was to relate these formal achievements with a relevant physical situation. Since the Chern-Simons model can provide a useful effective representation for the transverse conductivity we refine the model by adding a quantum structure for both the gauge and matter fields and analyzed the radiative contributions. The shift on the transverse conductivities appears as an emergent low energy quantity and is proportional to the topological numbers of the matter field. The radiative contributions generated a renormalized propagator from which we have shown that even a static point charge can generate not just the delta function contribution to the magnetic field, as in the bare case, but also a new one with a bigger range due to renormalization effects.

We also considered for the gauge field response the physical situation that the quasi-particles arising from the low energy description of the materials have a velocity different than c and it generates a breaking of symmetry between space and time. This fact motivated us to use a new set of projectors and derive the propagators and the physical quantities for this case, in order to evaluate the influence of the velocity parameter in the observables.

Regarding the Fermionic response, we have used the Nakanishi B field formalism for the anti-commutators and from them we derived the time ordered objects. Their low energy behaviour leads to a system that has the possibility of localized boundary states in agreement with the phenomenology that we were modeling, the quantum Hall effect. We did not considered the velocity of the quasi-particles for the self energy discussion since it would become cumbersome and the main insights could be given by the approach followed by us. But, of course, this discussion is indeed one of our future perspectives.

For future research, we intend to give a similar analysis based on indefinite metric and on the correspondence principle for the DKP scalar electrodynamics [40] and also analyze the case of $2+1$ dimensional triad gravity [37, 41]as a toy model for understanding the four dimensional case [42] . After analysing the t'Hooft gauge electrodynamics [9] , we are writing a work in which a similar approach regarding the B field formalism is used but now in the presence of a Higgs and a Fermionic sector. The Kallen-Lehmann representation uses asymptotic field operators and is compatible to our approach [43] . We also intend to link these techniques to describe interesting situations such as [44] which dealt with non-commutative QFT which was an inspirational content to model soft condensed matter [45].

4.8 Appendix A: Dirac brackets

Here we intend to derive the Dirac brackets which give us the equal time commutators via the correspondence principle. The B field formalism fixes the action in such a way that there is no first class constraints ambiguity and the Dirac brackets are used since the beginning. In order to see it let's build a matrix with all the Poisson brackets of the Bosonic sector of the Legendre transform constraints:

$$M^{IJ}(x,y) = \{\Phi^I(x), \Phi^J(y)\} = \begin{pmatrix} \epsilon^{nk}\mathcal{K} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \delta^2(x-y) \tag{4.124}$$

where

$$\Phi^I(x) = \begin{pmatrix} \pi_i(x) + \frac{\epsilon_{ij}A^j(x)\mathcal{K}}{2} \\ \pi_B(x) \\ \pi_0(x) - B(x) \end{pmatrix} \tag{4.125}$$

So, the inverse matrix is given by:

$$\tilde{M^{IJ}}(x,y) = \{\Phi^I(x), \Phi^J(y)\} = \begin{pmatrix} \frac{\epsilon^{nk}}{\mathcal{K}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \delta^2(x-y) \tag{4.126}$$

Then, the reduced vector brackets are:

$$\{A_i(x), \pi_j(y)\}_D = \{A_i(x), \pi_j(y)\} - \int d^3w d^3z \{A_i(x), \Phi^1_n(w)\} \frac{\epsilon_{mn}}{\mathcal{K}} \{\Phi^1_m(z), \pi_j(y)\} = \frac{\delta_{ij}}{2}\delta^2(x-y) \tag{4.127}$$

Instead of $\delta_{ij}\delta^2(x-y)$. It permit us to derive the correct factor for the commutator between the spatial vector fields. We also have that π_B desappears from the dynamics and $\pi_0 = B$ in the strong form. Therefore, under this bracket, the system has no constraints, is well-defined and the correspondence principle can be applied. An analogous procedure for the Fermionic sector ensures that we can cosider its momentum constraint in strong form with $\pi_{\bar{\psi}} = 0$.

4.9 Appendix B: Gamma Matrices Algebra

The two dimensional faithful representation for Fermions is given in terms of Pauli matrices with $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_1$ and $\gamma^2 = i\sigma_2$. They obey

$$\Big\{\gamma^\mu, \gamma^\nu\Big\} = 2I\eta^{\mu\nu} \tag{4.128}$$

where $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\epsilon^{\mu\nu\alpha}\gamma_\alpha$ and I denotes the 2×2 unity matrix. For the traces, we have

$$\begin{aligned} tr(\gamma^\mu) =& 0\ ,\ tr(\gamma^\mu\gamma^\nu) = 2\eta^{\mu\nu} \\ tr(\gamma^\mu\gamma^\nu\gamma^\rho) =& -2i\epsilon^{\mu\nu\rho} \\ tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) =& 2\Big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\Big) \end{aligned} \tag{4.129}$$

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Chapter 5

Electrodynamics in the 't Hooft gauge, a covariant operator approach

5.1 Motivation [1]

The 't Hooft-Veltman gauge in four-dimensional quantum electrodynamics (QED_4) was proposed in the seventies to model Yang-Mills theories with complicated gauge structures [1]. The aim of the approach was to define good lagrangians, that is, non-singular lagrangians with well-defined propagators in theories with gauge symmetries, and to show that those definitions do not modify the physical content of the original singular lagrangian. Since the investigation of Yang-Mills theories is a harder problem, QED_4 in this gauge can be considered as a testing ground for four-dimensional quantum chromodynamics (QCD_4). In fact, the manipulation of this gauge by means of path integral methods implies the addition of ghost fields that will be coupled to the electromagnetic field as well as self-interaction, in analogy to what happens in QCD_4 in which the gluons are also coupled to the ghost fields of the theory.

In general, gauge-ghost couplings may turn the quantization of the system trickier since, besides the appearance of new vertices, there are more subtle problems as the possibility of Gribov copies [2], that is, that even fixing the gauge condition the path integral may take into account more than one representative element of the gauge orbits [3]. Nevertheless, the version of QED_4 in the 't Hooft-Veltman gauge is simple enough to give us control over this issue [4].

It is possible to show that the use of this gauge has no phenomenological consequences, as it is expected from its attainability and well-definiteness. 't Hooft and Veltman established this by an explicit computation of the contributions to the photon-photon scattering at tree level which must vanish in ordinary electromagnetism. They showed that the required cancellation indeed occurs. For the general case of loops, they showed this by using diagrammatic Slavnov-Taylor identities [5, 6]. However, many of works such as [7, 8, 9] explore the gauge structure of the theory by means of perturbation theory.

Although the use of linear gauges are, in general, the most useful ones in practical loop calculations, non-linear gauges has received recent interest since they are adequate to study some aspects of the infrared dynamics of non-Abelian gauge theories [10]. In this context, we can cite the Curci-Ferrari [11] gauge and the Maximal Abelian gauge (MAG)[12]. They are used to analyze the Abelian dominance hypothesis, which basically refers to the idea that diagonal gluons receive a smaller dynamically generated mass than the off-diagonal ones. In view of this Abelian dominance possibility, an investigation of QED_4 in a non-linear gauge may be welcome to develop insight and a well-defined consistent path. Since the 't Hooft parameter is unphysical, there is no bound in its value, so it can be big and then just non-perturbative general

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methods can be employed to explicitly verify that it really renormalize just non-physical sectors of the Hilbert space. In this work, we are concerned with the development of a completely non-perturbative study of the two-point function of the gauge field. The authors had this particular interest since in references [13, 14] there is an apparent longitudinal contribution to the vacuum polarization tensor which is controlled by Ward identities. However, the fictitious g parameter (see equation (5.1) below) just renormalize the longitudinal non-physical sectors of the photon propagator. Since we want to compare our results with [13, 14] , then besides the non-perturbative approach, which is the most adequate in this case, we also extend the formalism to a perturbative Heisenberg quantization in the situation of small g. In theorectical grounds, this is an important achievement since, as previously mentioned, this theory can be used as an insightful toy model for QCD_4, and there is not yet a complete understanding of a Heisenberg perturbation for quantum chromodynamics [15].

In order to proceed we will use the Kujo-Ojima-Nakanishi formalism [16, 17]. It consists of an indefinite metric quantization in which an auxiliary B-field is suitably introduced to provide a second class system in the sense of the Dirac-Bergman Hamiltonian analysis [18]. The resulting theory is then free of quantum ordering ambiguities and it is described by Dirac brackets which may be turned into equal-time (anti)commutators via correspondence principle. Since we are dealing with indefinite metric, we must find a way to well-define the positive norm subspace or physical space. In the case of QED_4 in linear gauges, the positive frequency part of the B-field does this job by annihilating the physical states. On the other hand, however, the use of the 't Hooft-Veltman gauge does not allow the B-field to obey a free-field equation and, thus, the previous condition can no longer be used. Yet, we can still use the BRST charge to define physical states as expected.

We also show that the indefinite metric formalism have a tool to avoid the presence of the B-field, ghost and longitudinal gauge fields in the physical subspace by the so-called quartet mechanism. The canonical quantization is employed and the propagator of the gauge fields is obtained. In order to do so, we had to use the quantum Cauchy problem for the propagator of ghosts [19, 15] and use the BRST symmetry to relate it to the B field and the longitudinal gauge field projection. Latter, we also show that in a general case, all the amplitudes are independent from the g 't Hooft parameter [20].

This work is organized as follows. In section 2, we construct the lagrangian of QED_4 in the 't Hooft-Veltman gauge, we deduce the corresponding equations of motion and the BRST charge, and we define the physical states. In section 3, we perform the covariant quantization of the theory. We derive the canonical structure and infer the propagator of the photon field. In section 4, we present a discussion about ghost fields and the relation between the BRST charge and the ghost number operator. We also present a discussion about the quartet structure. Section 5 is devoted to show that all the amplitudes are independent from the g 't Hooft gauge parameter. Finally, we conclude in section 6.

5.2 The lagrangian and its residual symmetry

The 't Hooft-Veltman gauge fixing condition is given by the following relation

$$\partial_\mu A^\mu(x) + gA_\mu(x)A^\mu(x) = 0, \tag{5.1}$$

where g is an arbitrary parameter. It is important to mention that this is a good gauge condition since it is attainable, that is, by suitable gauge transformations it is always possible to write down relation (5.1), and since it satisfies the Dirac criteria.

Let us start by generalizing equation (5.1) to its α-gauge condition using the following lagrangian density [17]

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + B\left(\partial_\mu A^\mu + gA_\mu A^\mu\right) - \frac{\alpha}{2}B^2, \tag{5.2}$$

where α is the gauge parameter. The equations of motion for the Lagrange multiplier or B-field provides an operator equation

$$\partial_\mu A^\mu(x) + gA^\mu(x)A_\mu(x) - \alpha B(x) = 0, \tag{5.3}$$

which depends on α and controls the gauge fixing condition. In particular, for $\alpha = 0$ we reproduce the original 't Hooft-Veltman choice.

Before proceeding with the quantization of this theory, we must identify which is the residual gauge invariance of the theory. Since the non-gauge sector of the lagrangian is invariant under the local gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)$, where $\Lambda(x)$ is an *a priori* c-number field, it is enough to vary the equation of motion (5.3) with respect to the previous transformation in order to find a condition that defines the residual symmetry. Explicitly, we obtain

$$\delta\left[\partial_\mu A^\mu(x) + gA^\mu(x)A_\mu(x) - \alpha B(x)\right] = 0 \quad \to \quad \left(\Box + 2gA_\mu(x)\partial^\mu\right)\Lambda(x) = 0. \tag{5.4}$$

These residual gauge transformations are defined by a class of $\Lambda(x)$ "functions" satisfying equation (5.4). However, we must be careful. Unlike in linear gauges, the residual symmetry (5.4) is defined by an operator equation, thus, the set of $\Lambda(x)$ must be operators as well instead of ordinary functions. Therefore, equation (5.4) plays the role of a constraint and should be incorporated in the lagrangian density (5.2) by means of a Lagrange multiplier operator. In other words, we end up with the following modified lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + B\left(\partial_\mu A^\mu + gA^\mu A_\mu\right) - \frac{\alpha}{2}B^2 - i\bar{c}\left(\Box + 2gA^\mu\partial_\mu\right)c. \tag{5.5}$$

with regard to $\bar{c}(x)$, it is the aforementioned Lagrange multiplier field whose equation of motion furnishes the residual symmetry (5.4) and has a Grassmann character. Moreover, in order to reproduce equation (5.4) we must have $c(x) = \varepsilon\Lambda(x)$ where ε is a Grassmann number and $\Lambda(x)$ has acquired an operator field status. This Grassmannian nature is necessary if one wants a conserved residual-symmetry generating charge. It also has an important relation with unitarity where non-physical fields can be arranged in a non-observable structure called quartets. We will call $\bar{c}(x)$ and $c(x)$ as ghost fields. Needless to say, the imaginary unit has been added to recover the Hermitian characteristic of the lagrangian.

After these observations, it is straightforward to show that the system presents the following operatorial global symmetry tranformations, known as BRST transformations,

$$Q_B A_\mu(x) = \partial_\mu c(x), \qquad Q_B c(x) = 0, \qquad Q_B\bar{c}(x) = iB(x), \qquad Q_B B(x) = 0, \tag{5.6}$$

where Q_B denotes the BRST generating charge. From (5.6), we note that the BRST charge is nilpotent, that is, $Q_B^2 = 0$. This nilpotency property allows us to write the lagrangian (5.5) in an explicit BRST invariant way

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + Q_B\left[-i\bar{c}\left(\partial_\mu A^\mu + gA^\mu A_\mu\right) + i\frac{\alpha}{2}\bar{c}B\right] + A_\mu J^\mu, \tag{5.7}$$

where we have added to the system a Fermionic matter current denoted by J_μ.

From (5.7) it follows the equation of motion

$$\partial_\mu F^{\mu\nu}(x) + Q_B\tilde{J}^\nu(x) + J^\nu(x) = 0, \tag{5.8}$$

wherein

$$Q_B\tilde{J}^\nu(x) = -\partial^\nu B(x) + 2gB(x)A^\nu(x) - 2ig\bar{c}(x)\partial^\nu c(x). \tag{5.9}$$

with $\tilde{J}^\nu(x) = i\partial^\nu \bar{c}(x) - 2ig\bar{c}(x)A^\nu(x)$.
Since the first term on the left-hand side of (5.8) and the matter current are transverse operators, we conclude that $Q_B\tilde{J}_\mu(x)$ must also be transverse.

It is important to comment that with covariant linear gauges, the B-field satisfies a free-field equation, thus, the physical Hilbert space, $\mathcal{H}_{\text{phys}}$, is constructed by imposing the following subsidiary condition

$$B^+(x)|\text{phys}\rangle = 0, \quad \forall|\text{phys}\rangle \in \mathcal{H}_{\text{phys}}. \tag{5.10}$$

This expression is Poincaré invariant since it is defined in terms of the positive frequency part of the B-field. However, in the present case, the use of the 't Hooft-Veltman gauge does not permit B to obey a free equation as we can see by acting with the differential operator ∂_μ on equation (5.8). The generalization of (5.10) is given by

$$Q_B|\text{phys}\rangle = 0, \quad \forall|\text{phys}\rangle \in \mathcal{H}_{\text{phys}} \equiv \frac{\mathcal{V}}{\mathcal{V}_0}. \tag{5.11}$$

This definition is also Poincaré invariant.

It means that physical states must be invariant by residual gauge transformations. This is a consistent definition due to its nilpotency.

In fact, the quotient space $\mathcal{V}/\mathcal{V}_0$ above tells us that the physical states are defined by $Q_B|\text{phys}\rangle = 0$ and also by the requirement that it should not be of the form $Q_B|\Psi\rangle$ for any state Ψ since the nilpotency of Q_B would make it unobservable.

5.3 Covariant quantization

In order to quantize the theory we compute the corresponding canonical momenta variables and write down the equal-time commutators by using the correspondence principle. We start from lagrangian (5.5) to obtain that

$$\pi^i(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i(x))} = \partial_0 A^i(x) - \partial^i A^0(x), \tag{5.12}$$

$$\pi^0(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0(x))} = B(x), \tag{5.13}$$

are the momenta for the gauge field, that

$$\pi_{\bar{c}}(x) = \frac{\partial_L \mathcal{L}}{\partial_L(\partial_0 \bar{c}(x))} = i\partial_0 c(x), \tag{5.14}$$

$$\pi_c(x) = \frac{\partial_L \mathcal{L}}{\partial_L(\partial_0 c(x))} = i\partial_0 \bar{c}(x) - i2g\bar{c}(x)A^0(x), \tag{5.15}$$

are those for the ghost fields, and that

$$\pi_B(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 B(x))} = 0, \tag{5.16}$$

is the one for the the auxiliary B-field.
The expressions above prove that also in the ‘t Hooft gauge QED_4 case, the careful introduction of the B field structure furnishes already a well-defined system without any first class ambiguities [18] .

Since the system is of second class, we can defined a bracket in which the primary constrainst hold in the strong form. Therefore, we have

$$\big\{A_j(x), \pi^i(y)\big\} = \delta_i^j \delta^3(x-y) \tag{5.17}$$
$$\big\{A_0(x), B(y)\big\} = \delta^3(x-y) \tag{5.18}$$
$$\big\{c(x), \pi_c(y)\big\} = \delta^3(x-y) \tag{5.19}$$
$$\big\{\bar{c}(x), \pi_{\bar{c}}(y)\big\} = \delta^3(x-y), \tag{5.20}$$

give the fundamental (anti)commutation relations that will serve as starting points to derive initial conditions in our analysis below. They are

$$\Big[A_i(x), \partial_0 A^l(y)\Big]_0 = i\delta_i^l \delta^3(x-y) \tag{5.21}$$
$$\Big[A_0(x), B(y)\Big]_0 = i\delta^3(x-y) \tag{5.22}$$
$$\Big\{\bar{c}(x), \partial_0 c(y)\Big\}_0 = \delta^3(x-y) \tag{5.23}$$
$$\Big\{c(x), \partial_0 \bar{c}(y)\Big\}_0 = \delta^3(x-y) \tag{5.24}$$
$$\Big[B(x), B(y)\Big]_0 = 0, \tag{5.25}$$

where the last commutator has been added for completeness and the subscript 0 means equal-time, that is, $x_0 = y_0$.

From the gauge condition (5.3), we obtain

$$\Big[A_\mu(x), \partial_0 A_0(y)\Big]_0 = i\alpha\delta_\mu^0 \delta^3(x-y). \tag{5.26}$$

Using the $\nu = 0$ component of the equation of motion (5.8) to obtain the time derivative of the B-field in terms of the other fields, we get

$$\Big[A_\mu(x), \partial_0 B(y)\Big]_0 = -i\partial_k^y \delta_\mu^k \delta^3(x-y) + 2igA_0(x)\delta^3(x-y) \tag{5.27}$$
$$\Big[B(x), \partial_0 B(y)\Big]_0 = -2igB(x)\delta^3(x-y). \tag{5.28}$$

In order to find the general structure for the vacuum expectation value of the photon field we first manipulate the equation of motion to write it in the following form

$$\Box A_\mu(x) + g\partial_\mu\left(A_\beta(x)A^\beta(x)\right) = \alpha\partial_\mu B(x) - Q_B\tilde{J}_\mu(x) - J_\mu(x). \tag{5.29}$$

Thus, we have

$$\begin{aligned}
&\Box^x\Box^y\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle = \\
&\quad\langle 0|\Big[-g\partial_\mu\left(A_\beta(x)A^\beta(x)\right) + \alpha\partial_\mu B(x) - Q_B\tilde{J}_\mu(x) - J_\mu(x), -g\partial_\nu\left(A_\gamma(y)A^\gamma(y)\right) \\
&\quad + \alpha\partial_\nu B(y) - Q_B\tilde{J}_\nu(y) - J_\nu(y)\Big]|0\rangle.
\end{aligned} \tag{5.30}$$

This seemingly complicated expression can be simplified if one considers that B has a vanishing norm and is a Q_B-boundary term, i.e., $Q_B B(x) = 0$, that the matter current $J_\mu(x)$ is physical in the sense of $Q_B J_\mu(x)|\text{phys}\rangle = 0$, and that both currents $J_\mu(x)$ and $Q_B\tilde{J}_\mu(x)$ commute with

$\partial_\nu A^\nu(x)$. [2] In fact, we have

$$\Box^x\Box^y\langle 0|\Big[A_\mu(x),A_\nu(y)\Big]|0\rangle = g^2\partial^x_\mu\partial^y_\nu\langle 0|\Big[A_\beta(x)A^\beta(x),A_\gamma(y)A^\gamma(y)\Big]|0\rangle + \langle 0|\Big[J_\mu(x),J_\nu(y)\Big]|0\rangle$$
$$-\alpha g\partial^x_\mu\partial^y_\nu\langle 0|\Big[A_\beta(x)A^\beta(x),B(y)\Big]|0\rangle - \alpha g\partial^x_\mu\partial^y_\nu\langle 0|\Big[B(x),A_\beta(y)A^\beta(y)\Big]|0\rangle. \tag{5.31}$$

Using, again, the gauge condition (5.3) on the second line of this expression in order to relate it with $\partial_\mu A^\mu(x)$, we obtain

$$\Box^x\Box^y\langle 0|\Big[A_\mu(x),A_\nu(y)\Big]|0\rangle = g^2\partial^x_\mu\partial^y_\nu\langle 0|\Big[A_\beta(x)A^\beta(x),A_\gamma(y)A^\gamma(y)\Big]|0\rangle + \langle 0|\Big[J_\mu(x),J_\nu(y)\Big]|0\rangle$$
$$+\alpha\partial^x_\mu\partial^y_\nu\langle 0|\Big[\partial^x_\beta A^\beta(x),B(y)\Big]|0\rangle + \alpha\partial^x_\mu\partial^y_\nu\langle 0|\Big[B(x),\partial^y_\beta A^\beta(y)\Big]|0\rangle. \tag{5.32}$$

From this we can immediately conclude that the non-linear self-interaction part contributes to the renormalization of the longitudinal non-physical sector. Therefore, we expect that the physical sector is independent of the fictitious parameter g.

To find the general expression for the two-point function it is mandatory to compute the commutator between the photon field A_μ and the B-field, as we can see in (5.32). Since the use of the 't Hooft-Veltman gauge forbids the harmonic character of the B-field, the desired commutator cannot be easily found by means of the initial data. An alternative approach is to use BRST symmetry. In fact,

$$0 = \langle 0|Q_B\Big[A_\mu(x),\bar{c}(y)\Big]|0\rangle = \partial^x_\mu\langle 0|\Big\{c(x),\bar{c}(y)\Big\}|0\rangle + i\langle 0|\Big[A_\mu(x),B(y)\Big]|0\rangle. \tag{5.33}$$

This expression shows us that in order to find the vacuum expectation value of the commutator between A_μ and B we must first compute the two-point function for ghosts which is defined by the following Cauchy problem

$$\Box^x\langle 0|\Big\{c(x),\bar{c}(y)\Big\}|0\rangle = -2g\langle 0|\Big\{A^\mu(x)\partial^x_\mu c(x),\bar{c}(y)\Big\}|0\rangle \tag{5.34}$$
$$\langle 0|\Big\{c(x),\bar{c}(y)\Big\}_0|0\rangle = 0 \tag{5.35}$$
$$\partial^x_0\langle 0|\Big\{c(x),\bar{c}(y)\Big\}_0|0\rangle = -\delta^3(x-y). \tag{5.36}$$

We shall define

$$\mathcal{D}(x-y;A) \equiv \langle 0|\Big\{c(x),\bar{c}(y)\Big\}|0\rangle, \tag{5.37}$$

for convenience. It only depends on the difference of the coordinates due to Poincaré invariance [22]. Hence, the formal solution of (5.34) is the following integro-differential equation [19]

$$\mathcal{D}(x-y;A) = -2g\int d^4u\ \varepsilon(x,y,u)\Delta(x-u)\langle 0|\Big\{A^\beta(u)\partial^u_\beta c(u),\bar{c}(y)\Big\}|0\rangle$$
$$-\int d^3u\Big[\Delta(x-u)\partial^u_0\mathcal{D}(u-y;A) - \partial^u_0\Delta(x-u)\mathcal{D}(u-y;A)\Big]_{u^0=y^0}$$
$$= -2g\int d^4u\ \varepsilon(x,y,u)\Delta(x-u)\langle 0|\Big\{A^\beta(u)\partial^u_\beta c(u),\bar{c}(y)\Big\}|0\rangle - \Delta(x-y) \tag{5.38}$$

[2] Explicitly, we have $[\partial^\mu A_\mu(x),J^\nu(y)] = [\partial^\mu A_\mu(x),Q_B\tilde{J}^\nu(y)] = 0$. This result is obtained by commuting the longitudinal part of $A_\mu(x)$ with $J^\nu(y)$ and $Q_B\tilde{J}^\nu(y)$, respectively. The trasnverse nature of the ν index implies that $[Q_B\tilde{J}_\nu(x),A^L_\beta(y)] = C\partial_\nu\partial_\beta\Delta(x-y,0)$ and $[J_\nu(x),A^L_\beta(y)] = D\partial_\nu\partial_\beta\Delta(x-y,0)$ where $\Delta(x-y,0)$ is defined in (5.39). C and D are indefinite constants.

where $\Delta(x-y;s)$ is defined by the following Cauchy data

$$\Box\Delta(x-y;s) = -s\Delta(x-y;s), \quad \Delta(x-y;s)|_0 = 0, \quad \partial_0^x\Delta(x-y;s)|_0 = -\delta^3(x-y), \quad (5.39)$$

and we have defined $\varepsilon(x,y,u)$ in terms of the Heaviside function as

$$\varepsilon(x,y,u) = \Theta(x_0-u_0) - \Theta(y_0-u_0). \quad (5.40)$$

Summing up, we have from (5.33) that

$$\langle 0|\Big[A_\mu(x), B(y)\Big]|0\rangle = i\partial_\mu^x\mathcal{D}(x-y;A), \quad (5.41)$$

with $\mathcal{D}$ defined in (5.38).

Having these results in mind, it follows from (5.32) that

$$\begin{aligned}\Box^x\Box^y\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle &= g^2\partial_\mu^x\partial_\nu^y\langle 0|\Big[A_\beta(x)A^\beta(x), A_\gamma(y)A^\gamma(y)\Big]|0\rangle + \langle 0|\Big[J_\mu(x), J_\nu(y)\Big]|0\rangle \\ &+ i\alpha\partial_\mu^x\partial_\nu^y\Box^x\mathcal{D}(x-y;A) - i\alpha\partial_\mu^x\partial_\nu^y\Box^y\mathcal{D}(y-x;A), \quad (5.42)\end{aligned}$$

is the differential equation we must solve in order to compute the vacuum expectation value of the gauge field commutator. In view of this, we shall define the following spectral representation [17]

$$\frac{\langle 0|[J_\mu(x), J_\nu(y)]|0\rangle}{\Box^x\Box^y} \equiv -i\int ds\ \rho(s)\big(s\eta_{\mu\nu} + \partial_\mu\partial_\nu\big)\Delta(x-y;s) \quad (5.43)$$

and use (5.38) to obtain that the general solution of equation (5.42) is given by

$$\begin{aligned}\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle &= a\eta_{\mu\nu}\Delta(x-y) + b\partial_\mu\partial_\nu E(x-y) + c\partial_\mu\partial_\nu\Delta(x-y) + d\eta_{\mu\nu}E(x-y) \\ &- i\int ds\ \rho(s)\big(s\eta_{\mu\nu} + \partial_\mu\partial_\nu\big)\Delta(x-y;s) \\ &- g^2\partial_\mu^x\partial_\nu^x\int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega)\Delta(y-u)\langle 0|\Big[A_\mu(\omega)A^\mu(\omega), A_\beta(u)A^\beta(u)\Big]|0\rangle \\ &- i\alpha\partial_\mu^x\partial_\nu^x\int d^4u\ \epsilon(x,y,u)\Delta(y-u)\mathcal{D}(x-u;A) - i\alpha\partial_\mu^x\partial_\nu^x\int d^4u\ \epsilon(x,y,u)\Delta(x-u)\mathcal{D}(y-u;A),\end{aligned} \quad (5.44)$$

where $E(x-y)$ is defined as

$$\big(\Box + s\big)E(x-y;s) = \Delta(x-y;s), \quad E(x-y;s)|_0 = 0, \quad (\partial_0^x)^3E(x-y;s)|_0 = -\delta^3(x-y). \quad (5.45)$$

The unknown coefficients a, b and c are determined using the initial conditions as follows. Take the spatial components $\mu = k, \nu = l$ of (5.44) and act to it with ∂_0^y, set $x_0 = y_0$, and finally use (5.21) to obtain that

$$a = -i\left(1 - \int ds\ s\rho(s)\right) \equiv -iZ \quad (5.46)$$

$$c = i\int ds\ \Big(\rho(s)\Big). \quad (5.47)$$

Moreover, let us act with ∂_y^ν to (5.44), set $x_0 = y_0$, and use the $\mu = 0$ component of (5.26) to find that

$$a + b = -i\alpha \quad (5.48)$$

or, more precisely,

$$b = i\,(Z - \alpha)\,. \tag{5.49}$$

Thus, the vacuum expectation value of the gauge field commutator reads as

$$\begin{aligned}
\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle &= a\bigg(\eta_{\mu\nu}\Delta(x-y) - \partial_\mu\partial_\nu E(x-y)\bigg) - i\alpha\partial_\mu\partial_\nu E(x-y) + d\eta_{\mu\nu}E(x-y)\\
&+ c\partial_\mu\partial_\nu\Delta(x-y) - i\int ds\ \rho(s)\big(s\eta_{\mu\nu} + \partial_\mu\partial_\nu\big)\Delta(x-y;s)\\
&-g^2\partial^x_\mu\partial^x_\nu\int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega)\Delta(y-u)\langle 0|\Big[A_\mu(\omega)A^\mu(\omega), A_\beta(u)A^\beta(u)\Big]|0\rangle\\
&- i\alpha\partial^x_\mu\partial^x_\nu\int d^4u\ \epsilon(x,y,u)\Delta(y-u)\mathcal{D}(x-u;A)\\
&- i\alpha\partial^x_\mu\partial^x_\nu\int d^4u\ \epsilon(x,y,u)\Delta(x-u)\mathcal{D}(y-u;A).
\end{aligned} \tag{5.50}$$

This is a good point to stop developing and to introduce two consistency checks to test that our solution (5.50) is correct. Let us first consider the absence of matter current, namely, $\rho(s) \to 0$, and the limit $g^2 \to 0$. We should expect to end up in the theory of a free electromagnetic field with Lorenz gauge condition. In other words, we expect that (5.50) becomes

$$\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle_{\text{Free}} = -i\eta_{\mu\nu}\Delta(x-y) + i\,(1+\alpha)\,\partial_\nu\partial_\mu E(x-y) \tag{5.51}$$

From (8.1) and (8.4) we have that $a \to -i$ and $c \to 0$, respectively. Thus, from (5.50) we obtain

$$\begin{aligned}
\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle &\longrightarrow -i\bigg(\eta_{\mu\nu}\Delta(x-y) - \partial_\mu\partial_\nu E(x-y)\bigg) - i\alpha\partial_\mu\partial_\nu E(x-y) + d\eta_{\mu\nu}E(x-y)\\
&- i\alpha\partial^x_\mu\partial^x_\nu\lim_{g^2\to 0}\int d^4u\ \epsilon(x,y,u)\bigg(\Delta(y-u)\mathcal{D}(x-u;A) + \Delta(x-u)\mathcal{D}(y-u;A)\bigg).
\end{aligned} \tag{5.52}$$

For the last term we use the fact that

$$E(x-y) = -\int d^4u\ \epsilon(x,y,u)\Delta(x-u)\Delta(u-y), \tag{5.53}$$

and that in the $g^2 \to 0$ limit the function $\mathcal{D}(x-y;A)$ goes to $-\Delta(x-y)$ to obtain that

$$-i\alpha\partial^x_\mu\partial^x_\nu\lim_{g^2\to 0}\int d^4u\ \epsilon(x,y,u)\bigg(\Delta(y-u)\mathcal{D}(x-u;A) + \Delta(x-u)\mathcal{D}(y-u;A)\bigg) = 2i\alpha\partial_\mu\partial_\nu E(x-y). \tag{5.54}$$

Plugging this result back in equation (5.52) we realize that it becomes the expected free solution for QED_4 if $d = 0$, a constant that was undetermined until this point. We are going to see that this condition can be obtained also by making our solution compatible with the gauge condition.

Our second consistency check is to consider again the limit $g^2 \to 0$ but in the presence of the Fermionic current. This time we should expect the theory of an interacting electromagnetic field with Lorenz gauge condition, namely,

$$\begin{aligned}
\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle_{\text{Int}} &= -i\left(Z\eta_{\mu\nu} - \int ds\rho(s)\partial_\mu\partial_\nu\right)\Delta(x-y) + i\,(Z+\alpha)\,\partial_\nu\partial_\mu E(x-y)\\
&- i\int ds\ \rho(s)\big(s\eta_{\mu\nu} + \partial_\mu\partial_\nu\big)\Delta(x-y;s).
\end{aligned} \tag{5.55}$$

Similarly, from (5.50) we obtain

$$
\begin{aligned}
\langle 0|\Big[A_\mu(x),A_\nu(y)\Big]|0\rangle \longrightarrow & -iZ\Big(\eta_{\mu\nu}\Delta(x-y)-\partial_\mu\partial_\nu E(x-y)\Big)-i\alpha\partial_\mu\partial_\nu E(x-y)\\
&+i\int ds\rho(s)\partial_\mu\partial_\nu\Delta(x-y)-i\int ds\ \rho(s)\big(s\eta_{\mu\nu}+\partial_\mu\partial_\nu\big)\Delta(x-y;s)\\
&-i\alpha\partial^x_\mu\partial^x_\nu\lim_{g^2\to 0}\int d^4u\ \epsilon(x,y,u)\Big(\Delta(y-u)\mathcal{D}(x-u;A)+\Delta(x-u)\mathcal{D}(y-u;A)\Big). \quad (5.56)
\end{aligned}
$$

Using again (5.54) we end up with the expected solution (5.55). These results show that (5.50) is the non-perturbative two-point function of an interacting electromagnetic field in the 't Hooft-Veltman gauge.

It is also noteworthy to mention that these results are compatible with the gauge condition. In fact, on the one hand we have [3]

$$
\begin{aligned}
\langle 0|\Big[\partial^\mu_x A_\mu(x),A_\nu(y)\Big]|0\rangle =& d\partial_\nu E(x-y)+g^2\partial^x_\nu\int d^4\omega\ \epsilon(x,y;\omega)\Delta(x-\omega)\langle 0|\Big[A_\mu(\omega)A^\mu(\omega),A_\beta(y)A^\beta(y)\Big]|0\rangle\\
&-i\alpha\partial^y_\nu\int d^4u\ \epsilon(x,y,u)\Delta(y-u)\Box^x\mathcal{D}(x-u;A)-i\alpha\partial^y_\nu\mathcal{D}(y-x)\\
&, \quad (5.57)
\end{aligned}
$$

but on the other hand, by using the gauge condition, we obtain

$$
\begin{aligned}
\langle 0|\Big[\partial^\mu_x A_\mu(x),A_\nu(y)\Big]|0\rangle &= \langle 0|\Big[-gA^\mu(x)A_\mu(x)+\alpha B(x),A_\nu(y)\Big]|0\rangle\\
&=-g\langle 0|\Big[A^\mu(x)A_\mu(x),A_\nu(y)\Big]|0\rangle-i\alpha\partial^y_\nu\mathcal{D}(y-x;A). \quad (5.58)
\end{aligned}
$$

For the first term we use the equations of motion in the following way

$$
\begin{aligned}
-g&\langle 0|\Big[A^\mu(x)A_\mu(x),A_\nu(y)\Big]|0\rangle\\
&=-g\langle 0|\Big[A^\mu(x)A_\mu(x),-g\frac{\partial^y_\nu(A^\beta(y)A_\beta(y))}{\Box^y}-\frac{Q_B\tilde{J}_\nu(y)}{\Box^y}-\frac{J_\nu(y)}{\Box^y}+\alpha\frac{\partial^y_\nu B(y)}{\Box^y}\Big]|0\rangle\\
&=g^2\partial^y_\nu\frac{\langle 0|\Big[A^\mu(x)A_\mu(x),A_\beta(y)A^\beta(y)\Big]|0\rangle}{\Box^y}-\alpha g\partial^y_\nu\frac{\langle 0|\Big[A^\mu(x)A_\mu(x),B(y)\Big]|0\rangle}{\Box^y}\\
&=g^2\partial^x_\nu\int d^4\omega\ \epsilon(x,y;\omega)\Delta(x-\omega)\langle 0|\Big[A_\mu(\omega)A^\mu(\omega),A_\beta(y)A^\beta(y)\Big]|0\rangle\\
&\quad -i\alpha\partial^y_\nu\int d^4u\ \epsilon(x,y,u)\Delta(y-u)\Box^x\mathcal{D}(x-u;A). \quad (5.59)
\end{aligned}
$$

The $J_\mu(x)$ and $Q_B\tilde{J}_\mu(x)$ did not contribute to the previous expressions [4] due to the same reasons as in (5.31).

Therefore, comparing with equation (5.57) we obtain again that $d=0$.

5.4 Perturbative analysis in powers of g^2

Note that our well-established solution (5.50) is only formal since it depends on the photon field commutator. In order to have information about the change in the structure of the two-point function due solely to this part, we set $\alpha=0$ since in this case equation (5.3) reduces

[3] Use had been made of $-i\alpha\Box_x\int d^4u\epsilon(x,y,u)\Delta(y-u)\mathcal{D}(x-u)=+i\alpha\Delta(x-y)-i\alpha\int d^4u\epsilon(x,y,u)\Box_x\mathcal{D}(x-u)\Delta(y-u)$ and $-i\alpha\Box_x\int d^4u\epsilon(x,y,u)\Delta(x-u)\mathcal{D}(y-u)=i\alpha\mathcal{D}(y-x)$

[4] It can be understood by using the gauge condition to relate $gA_\mu A^\mu$ with $\partial_\mu A^\mu$ and the B field.

to the original 't Hooft-Veltman choice[5], and we will make use of the Heisenberg perturbation theory [15]. We also consider the system without interaction with the Fermions. We must call attention to the fact that g is an unphysical parameter and it may have any value and then non-perturbative methods may be used. But since we want to compare our results to the diagrammatic approach [13], we consider the limit of small g in this section. Thus, from our formal solution we expect that for $\alpha = 0$ we must have

$$\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle = -i\Big(\eta_{\mu\nu}\Delta(x-y) - \partial_\mu\partial_\nu E(x-y)\Big)$$
$$-g^2\partial^x_\mu\partial^x_\nu \int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega)\Delta(y-u)\langle 0|\Big[A_\mu(\omega)A^\mu(\omega), A_\beta(u)A^\beta(u)\Big]|0\rangle \tag{5.60}$$

as the solution of the following differential equation

$$\Box^x\Box^y\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle = -g^2\partial^x_\mu\partial^x_\nu\langle 0|\Big[A_\beta(x)A^\beta(x), A_\gamma(y)A^\gamma(y)\Big]|0\rangle, \tag{5.61}$$

which will serve us as a guiding block in the perturbative expansion of (5.61).

Let us consider g^2 as an infinitesimal parameter and expand the gauge field commutator as follows [23]

$$\Big[A_\mu(x), A_\nu(y)\Big] = \sum_{n=0}^{\infty}\left(g^2\right)^n \mathcal{O}^{(n)}_{\mu\nu}(x,y), \tag{5.62}$$

where $\mathcal{O}^{(n)}_{\mu\nu}(x,y)$ are operators [6] whose vacuum expectation values must be computed by solving a chain of coupled differential equations arising from (5.61). Hence, the initial conditions (5.21) and (5.26) with $\alpha = 0$ translate to

$$\langle 0|\partial^y_0\mathcal{O}^{(n)l}_k(x,y)|0\rangle_0 = i\delta^l_k\delta^3(x-y) \tag{5.63}$$

and

$$\langle 0|\partial^y_0\mathcal{O}^{(n)}_{\nu 0}(x,y)|0\rangle_0 = 0 \quad \text{for any } n, \tag{5.64}$$

respectively. Using identities for the self-interaction commutator we have that

$$\Big[A_\beta(x)A^\beta(x), A_\gamma(y)A^\gamma(y)\Big] = \sum_{n=0}^{\infty}\left(g^2\right)^n \Big[\mathcal{O}^{(n)}_{\beta\gamma}(x,y)A^\beta(x)A^\gamma(y) + A^\gamma(y)\mathcal{O}^{(n)}_{\beta\gamma}(x,y)A^\beta(x)$$
$$+ A^\beta(x)\mathcal{O}^{(n)}_{\beta\gamma}(x,y)A^\gamma(y) + A^\gamma(y)A^\beta(x)\mathcal{O}^{(n)}_{\beta\gamma}(x,y)\Big]. \tag{5.65}$$

Thus, plugging these results back in (5.61) we obtain

$$\sum_{n=0}^{\infty}\left(g^2\right)^n\Box^x\Box^y\langle 0|\mathcal{O}^{(n)}_{\mu\nu}(x,y)|0\rangle = -\sum_{k=0}^{\infty}\left(g^2\right)^{k+1}\partial^x_\mu\partial^x_\nu\langle 0|\Big(\mathcal{O}^{(k)}_{\beta\gamma}(x,y)A^\beta(x)A^\gamma(y)$$
$$+ A^\gamma(y)\mathcal{O}^{(k)}_{\beta\gamma}(x,y)A^\beta(x) + A^\beta(x)\mathcal{O}^{(k)}_{\beta\gamma}(x,y)A^\gamma(y) + A^\gamma(y)A^\beta(x)\mathcal{O}^{(k)}_{\beta\gamma}(x,y)\Big)|0\rangle, \tag{5.66}$$

[5]Moreover, we chose to work with $\alpha = 0$ for maximum simplicity since the term proportional to α in (5.50) may also have corrections in powers of g.

[6]We cannot specify a given coordinate dependence for this operator, although it can be shown that the first term of the approximation depends on its difference. However, their vacuum expectation values must depend on the mentioned difference due to Poincaré invariance.

from which we obtain the following differential equations

$$\Box^x\Box^y\langle 0|\mathcal{O}^{(0)}_{\mu\nu}(x,y)|0\rangle = 0 \tag{5.67}$$

$$g^2\Box^x\Box^y\langle 0|\mathcal{O}^{(1)}_{\mu\nu}(x,y)|0\rangle = -g^2\partial^x_\mu\partial^x_\nu\langle 0|\Big(\mathcal{O}^{(0)}_{\beta\gamma}(x,y)A^\beta(x)A^\gamma(y) + A^\gamma(y)\mathcal{O}^{(0)}_{\beta\gamma}(x,y)A^\beta(x) + A^\beta(x)\mathcal{O}^{(0)}_{\beta\gamma}(x,y)A^\gamma(y) + A^\gamma(y)A^\beta(x)\mathcal{O}^{(0)}_{\beta\gamma}(x,y)\Big)|0\rangle, \tag{5.68}$$

for the zeroth order and $n=1$ contributions, respectively.

The former differential equation is easily solved by taking the $g^2\to 0$ limit in the non-perturbative solution (5.60). In fact, we have

$$\langle 0|\mathcal{O}^{(0)}_{\mu\nu}(x,y)|0\rangle = -i\Big(\eta_{\mu\nu}\Delta(x-y) - \partial_\mu\partial_\nu E(x-y)\Big) \tag{5.69}$$

which satisfies the requirements (5.63) and (5.64). Since this is the solution of a free theory we conclude that $\mathcal{O}^{(0)}_{\mu\nu}(x,y)$ is actually a c-number (Which depends on the coordinate difference.) which we will denote it as $D^{(0)}_{\mu\nu}(x-y)$ as it is usually found in the literature. Hence, the latter differential equation is reduced to

$$g^2\Box^x\Box^y\langle 0|\mathcal{O}^{(1)}_{\mu\nu}(x,y)|0\rangle = -2g^2\partial^x_\mu\partial^x_\nu\Big(D^{(0)}_{\beta\gamma}(x,y)\langle 0|\Big(A^\beta(x)A^\gamma(y) + A^\gamma(y)A^\beta(x)\Big)|0\rangle\Big). \tag{5.70}$$

For the right-hand side we introduce the following expansion[7]

$$A_\beta(x)A_\gamma(y) = \sum_{n=0}^{\infty}\left(g^2\right)^n\mathcal{O}^{+(n)}_{\beta\gamma}(x,y), \tag{5.71}$$

to obtain that the first order contribution is given by

$$\Box^x\Box^y\langle 0|\mathcal{O}^{(1)}_{\mu\nu}(x,y)|0\rangle = -2\partial^x_\mu\partial^x_\nu\Big(D^{(0)\beta\gamma}(x,y)\langle 0|\Big(\mathcal{O}^{+(0)}_{\beta\gamma}(x,y) + \mathcal{O}^{+(0)}_{\beta\gamma}(y,x)\Big)|0\rangle\Big). \tag{5.72}$$

Since we have concluded that the zeroth contribution is a c-number we finally have[8]

$$\Box^x\Box^y\langle 0|\mathcal{O}^{(1)}_{\mu\nu}(x,y)|0\rangle = -2\partial^x_\mu\partial^x_\nu\Big(D^{(0)\beta\gamma}(x,y)\left(D^{+(0)}_{\beta\gamma}(x,y) + D^{+(0)}_{\beta\gamma}(y,x)\right)\Big), \tag{5.73}$$

from which it follows that

$$\langle 0|\mathcal{O}^{(1)}_{\mu\nu}(x,y)|0\rangle = \tilde{a}\eta_{\mu\nu}\Delta(x-y) + \tilde{b}\partial_\mu\partial_\nu\Delta(x-y) + \tilde{c}\eta_{\mu\nu}E(x-y) + \tilde{d}\partial_\mu\partial_\nu E(x-y) - 2\partial^x_\mu\partial^x_\nu\int d^4\omega d^4u\;\epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega)\Delta(y-u)M^{(0)}(\omega-u), \tag{5.74}$$

where we have defined

$$M^{(0)}(x-y) = D^{(0)\beta\gamma}(x-y)\left(D^{+(0)}_{\beta\gamma}(x-y) + D^{+(0)}_{\beta\gamma}(y-x)\right). \tag{5.75}$$

[7]The use of the notation + is to emphasize that its vacuum expectation value gives the positive frequency part of the associated distributions.

[8]The distribution $D^{+(0)}_{\beta\gamma}(x-y)$ denote the positive frequency part of $D^{(0)}_{\beta\gamma}(x-y)$ which is defined as $iD^{(0)}_{\beta\gamma}(x-y) = D^{+(0)}_{\beta\gamma}(x-y) + D^{+(0)}_{\beta\gamma}(y-x)$.

Now we set $\tilde{a} = \tilde{b} = \tilde{d} = 0$ since it can be verified that the loop term do not contribute to the initial conditions. We also set $\tilde{c} = 0$ in order to be compatible with the ‘t Hooft gauge condition. Therefore, we have that

$$\langle 0|\mathcal{O}^{(1)}_{\mu\nu}(x,y)|0\rangle = -2\partial^x_\mu\partial^x_\nu \int d^4\omega d^4u \; \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega)\Delta(y-u)M^{(0)}(\omega-u) \quad (5.76)$$

Then, the propagator evaluated until order g^2 has the form

$$\begin{aligned}\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle &= -i\Big(\eta_{\mu\nu}\Delta(x-y) - \partial_\mu\partial_\nu E(x-y)\Big) \\ &- 2g^2\partial^x_\mu\partial^x_\nu \int d^4\omega d^4u \; \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega)\Delta(y-u)M^{(0)}(\omega-u)\end{aligned} \quad (5.77)$$

Again, now in the Heisenberg perturbative framework, we conclude that the ‘t Hooft parameter affects just non-physical longitudinal contributions in accordance with the diagrammatic approach [13, 14].

5.5 A digression about the ghost fields

This section has the purpose of exposing some important theoretical content about the auxiliary Grassmann fields usually called ghosts. Firstly, the need of an odd parity for those fields is inferred demanding that the BRST symmetry current must be conserved. It is given below

$$J^B_\mu(x) = -F_{\mu\nu}(x)\partial^\nu c(x) + B(x)\partial_\mu c(x) \quad (5.78)$$

Its divergence is evaluated as

$$\partial^\mu J^B_\mu(x) = -\partial^\mu F_{\mu\nu}(x)\partial^\nu c(x) - F_{\mu\nu}(x)\partial^\mu\partial^\nu c(x) + \partial^\mu B(x)\partial_\mu c(x) + B(x)\Box c(x) \quad (5.79)$$

Using the operator equations of motion we have:

$$\partial^\mu J^B_\mu(x) = B(x)\Big(\Box + 2gA_\nu(x)\partial^\nu\Big)c(x) + i2g\bar{c}(x)\partial_\nu c(x)\partial^\nu c(x) = 0 \quad (5.80)$$

The first term vanishes in all Hilbert space due to the ghost field equation. The second one do not contribute if $c(x)$ has a Grassmann character. That is the reason to consider this kind of parity.

Regarding the ghost field and its lagrangian one may find a global ghost number symmetry expressed by the invariance under the transformations

$$c(x) \to c(x)e^\theta \quad ; \quad \bar{c}(x) \to \bar{c}(x)e^\theta \quad (5.81)$$

The θ parameter that appears above must be real in order to preserve the field’s Hermitian nature. Its associate current reads

$$J^c_\mu(x) = i\Big(\partial_\mu\bar{c}(x) - 2g\bar{c}(x)A_\mu(x)\Big)\theta c(x) - i\partial_\mu c(x)\;\theta\;\bar{c}(x) = J^\dagger_\mu(x) \quad (5.82)$$

We can show that the charge $Q_c = \int d^3x J_0^c$ indeed generates this symmetry

$$\left\{ iQ_c, c(x) \right\} = c(x) \quad ; \quad \left\{ iQ_c, \bar{c}(x) \right\} = -\bar{c}(x) \tag{5.83}$$

It is possible to show that there is a relation between Q_c and the BRST charge

$$Q_B J_0^c(x) = -\partial_0 B(x)c(x) + B(x)\partial_0 c(x) + 2gB(x)A_0(x)c(x) \tag{5.84}$$

Indeed, the BRST charge can be rewritten as

$$J_0^B(x) = B(x)\partial_0 c(x) - \partial_0 B(x)c(x) + 2gB(x)A_0(x)c(x) \tag{5.85}$$

So we have the relation

$$Q_B Q_c = Q_B \tag{5.86}$$

It shows that the application of the BSRT charge raises the ghost number eigenvalue in one unit.

5.5.1 Quartet structure

An interesting property shared by Q_c eigenstates with a given N eigenvalue is the fact that it obeys orthogonality relations of the kind [17]

$$\langle \phi_N | \psi_M \rangle \sim \delta_{M-N} \tag{5.87}$$

This result can be inferred by the Hermitian nature of Q_c. Having this result in mind we can introduce the concept of quartet configuration. It is nothing more than a way to dispose the non-physical fields in an non-observable structure. Since the physical Hilbert space is defined as

$$\mathcal{H}_{phys.} = \frac{\mathcal{V}}{\mathcal{V}_0} \tag{5.88}$$

We must confine the auxiliary fields outside $\mathcal{H}_{phys.}$, in other words, in the negative or null norm sectors.

So, if non-physical fields such as $B(x)$ and the longitudinal $A_\mu(x)$ part are in the quartet configuration they cannot be detected.

In order to define such a structure, suppose four states defined in the following manner

$$|\pi_k^N\rangle, \quad |\delta_k^{N+1}\rangle = Q_B|\pi_k^N\rangle, \quad |\pi_k^{-N-1}\rangle, \quad |\delta_k^{-N}\rangle = Q_B|\pi_k^{-N-1}\rangle \tag{5.89}$$

where N denotes a given eigenvalue of the ghost number symmetry charge.

Evidently, two of those states are non-physical while the other two has vanishing norm. From the previously introduced orthogonality relations the first state is not orthogonal to the last one while the second is not orthogonal just to the third. We can write those states as asymptotic creation operator acting on the vacuum state

$$|\pi_k^N\rangle = \chi_k^\dagger|0\rangle, \quad |\delta_k^{N+1}\rangle = -i\gamma_k^\dagger|0\rangle, \quad |\pi_k^{-N-1}\rangle = -\bar{\gamma}_k^\dagger|0\rangle, \quad |\delta_k^{-N}\rangle = -\beta_k^\dagger|0\rangle \tag{5.90}$$

Regarding the symmetry transformations, according to the quartet definition, considering $N = 0$, we have

$$\Big[Q_B, \chi_k\Big] = \gamma_k, \quad \Big\{Q_B, \bar{\gamma}_k\Big\} = i\beta_k, \quad \Big[Q_B, \beta_k\Big] = \Big\{Q_B, \gamma_k\Big\} = 0 \tag{5.91}$$

Those previous relations are the same as considering $A_\mu(x) = \partial_\mu \chi(x)$, $B(x) = \beta(x)$, $c(x) = \gamma(x)$ and $\bar{c}(x) = \bar{\gamma}(x)$. It allow us to conclude that the non-physical auxiliary fields in 't Hooft gauge electrodynamics are harmless since they are confined in a quartet non-observable structure.

5.6 Parameter independence

In this final section we intend to use the BRST symmetry to show that all physical amplitudes are independent of the g parameter that appears in the expression defining the 't Hooft gauge.

In order to do so we define the generator functional of connected Green function [17, 21]:

$$\exp iW(J) \equiv \langle 0|\mathcal{T} \exp iS(J)|0\rangle \tag{5.92}$$

The vacuum above is still the Heisenberg one, $\mathcal{T}$ denotes the time ordering operator and $S(J)$ is given by:

$$S(J) = \int d^4x \Big(\Sigma^\mu(x) A_\mu(x) + \beta(x)\bar{c}(x) + \bar{\beta}c(x) + \sigma(x)B(x)\Big) \tag{5.93}$$

where $\Sigma^\mu(x)$, $\beta(x)$, $\bar{\beta}(x)$ and $\sigma(x)$ are c-number external sources with the appropriate Grassmann parity.

The c-number fields are defined collectively as:

$$\phi_I(x) \equiv \frac{\delta W(J)}{\delta J_I(x)} = \langle 0|\mathcal{T}\hat{\phi}_I(x) \exp iS(J)|0\rangle \exp\left(-iW(J)\right) \tag{5.94}$$

where we emphasize the operator nature of the fields by using the hat.

Since the $\mathcal{T} \exp iS(J)$ is a gauge invariant object, we use this fact explicitly to derive the generator of the Ward identities:

$$0 = \langle 0|\Big[Q_B, \mathcal{T} \exp iS(J)\Big]|0\rangle = \langle 0|\mathcal{T} \int d^4x \Big(\Sigma^\mu \partial_\mu c + i\beta B\Big) \exp iS(J)|0\rangle \tag{5.95}$$

It is possible to show that the generator of connected Green functions can be written in terms of a path integral [21]

$$e^{iW(J)} \equiv \mathcal{Z} = \int DA_\mu(x) DB(x) Dc(x) D\bar{c}(x) \exp i(\mathcal{S}_t + iS(J)) \tag{5.96}$$

where $\mathcal{S}_t = \int d^4x \mathcal{L}(x)$ with $\mathcal{L}(x)$ defined in equation (7.5).

If we impose that this expression is invariant with relation to the g parameter the following condition arises

$$\langle 0| \int d^4x \Big(A^\mu(x) A_\mu(x) B(x) - 2i\bar{c}(x) A_\mu(x) \partial^\mu c(x)\Big)|0\rangle = 0 \tag{5.97}$$

The relation above is the B field version of [20].

Now we show that this expression indeed follows from BRST invariance. Then, we vary the generator of connected Green functions with relation to $\frac{\delta^3}{i\delta\Sigma_\mu(x)\Sigma^\mu(x)\delta\beta(x)}$ and taking the limit of vanishing sources in the end of the process

$$\langle 0| \int d^4x \Big(A^\mu(x)A_\mu(x)B(x) - 2i\bar{c}(x)A_\mu(x)\partial^\mu c(x) \Big) |0\rangle = 0 \tag{5.98}$$

The same expression than the one required for amplitudes to be independent from the parameter g is recovered. So, the BRST symmetry indeed furnishes a sufficient condition to it.

5.7 Conclusion

Throughout this chapter we have shown that the non-linear 't Hooft gauge do not has any influence on the physical output of the theory, as it should be. In order to do so we have employed the Nakanishi B field formalism, in its non-perturbative and perturbative versions [23]. Then, we concluded that just longitudinal sectors of the two point function are dependent of the g parameter, a result in agreement with [13, 14].

To generalize the result to an alpha gauge it was required to obtain the vacuum projection between the B and photon field . It was achieved by relating this projection with the ghost fields two point function by means of BRST symmetry.

This discussion had a constructive approach in which the ghost fields and the BRST and ghost number charges where introduced in a didactical way. We also show that the Maxwell equations are recovered between physical states and that the auxiliary fields are harmless since they are confined in the quartet non-observable configuration. The prove of g parameter independence was obtained in its B field version and also in a Heisenberg point of view which was latter related to the conventional path integral formalism. We also employed Heisenberg perturbation theory to show that the 't Hooft gauge self interaction just generates longitudinal contributions which do not has any physical influence as one can infer by the action of the BRST symmetry and the definition of the positive Hilbert space sector.

For future perspectives we intend to extend the B field formalism to also contemplate Ostrogadskian systems [24, 25] in order to study, for example, the Podolsky electrodynamics [26, 27, 28, 29]. We also want to use this non perturbative approach to clearly point out some quantum field theory phenomena, such as the elimination of the negative norm Ostrogadisky ghost in the free and interacting cases. Then, we intend to use a well-defined perturbation theory to give an explicit numerical representation to our efforts.

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Chapter 6

Electrodynamics in the 't Hooft gauge II, the inclusion of the Higgs and the Matter Sector

6.1 Motivation

The importance of mass generation mechanisms in particle physics can be traced back to the case of the Tamm-Yukawa description [1, 2] of nuclear forces in terms of massive pions exchange leading to a localized potential that falls off exponentially with the distance according to the observed phenomenology. Also, the addition of mass for quantum fields namely, the electromagnetic one, appeared in the context of the London description of the superconductivity phenomenon in which the fact that the magnetic field is expelled from the bulk of the material under a given critical temperature, the Meissner effect [3], could be explained by postulating a current density proportional to the vector potential [4] leading to a exponential decay of such a field and also the correlated electric and magnetic fields inside the material. It can be rephrased in terms of adding mass to the electromagnetic potentials.

Still regarding superconductivity models, Landau and Ginsburg [5] described this phase transition by means of a self interacting scalar field. It was an inspiration for a latter understanding of symmetry breaking in terms of a relativistic quartic potential for a complex scalar field with a negative mass parameter leading to an infinite number of different vacua. This mechanism can generate a mass for a gauge field interacting with the self interacting scalar field. The mass term appears in a gauge invariant Stueckelberg [6, 7] combination with the so-called Goldstone boson (a massless field that appears due to the breaking of the symmetry.) and it is responsible for good renormalization properties. This is the basis of the Higgs mechanism developed in its non-relativistic version by Anderson while Englert, Brout, Higgs, Guralnik, Hagen, and Kibble developed it for relativistic fields [8, 9, 10, 11, 12] .

with relation to applications in particle physics, the 4 Fermion interaction describing the weak interaction could be explained as the low energy limit of a theory with a QED_4-like interaction term with a massive intermediate boson. The apparent negative mass dimension coupling constant was the combination of a dimensionless one divided by the mass squared appearing in the low energy propagator. This idea was first proposed by [13] . Latter, Schwinger, Glashow, Salam, Ward and Weinberg [14, 15, 16, 17] worked independently in order to combine this idea with a careful analysis of this model coupled to a Higgs field in order to generate mass in a well-defined manner in the context of a $SU(2) \times U(1)$ internal symmetry.

with regard to our intentions in this present work, we recall that the 't Hooft-Veltman gauge in four-dimensional quantum electrodynamics (QED_4) was proposed in the seventies to model Yang-Mills theories with complicated gauge structures [18] as the non-Abelian ones describing

Weak interactions and QCD_4. The aim of the approach was to define good lagrangians, that is, non-singular lagrangians with well-defined propagators in theories with gauge symmetries, and to show that those definitions do not modify the physical content of the original singular lagrangian. Since the investigation of Yang-Mills theories is a harder problem, QED_4 in this gauge can be considered as a testing ground for four-dimensional quantum chromodynamics (QCD_4). In fact, the manipulation of this gauge by means of path integral methods implies the addition of ghost fields that will be coupled to the electromagnetic field as well as self-interaction, in analogy to what happens in QCD_4 in which the gluons are also coupled to the ghost fields of the theory.

We intend to give a well-defined Heisenberg non-perturbative description for such a system as a laboratory to use these techniques in the field of QCD_4 in which non-perturbative methods are welcome to describe phenomena such as the confinement. We also know that in QCD_4 the coupling constant can become non-perturbative in the low energy regime.

We consider the electromagnetic field in such a non-linear gauge as a good testing ground to improve the understanding of QCD_4 Heisenberg quantization and also as a purely theoretical achievement for the quantum field theory since this is the most fundamental description. Also, we are performing a logical continuation of [19] now with the addition of the Higgs and the Fermionic matter sector which is a next step in complexity [20].

Although the calculations becomes harder, it furnishes a more general and realistic treatment since the quantum gauge field interacts with matter. The addition of the Higgs field generates a good laboratory for Heisenberg quantization of the weak interaction or can be a effective simplified description for dynamical mass generation in QCD_4 [21] in which alternative symmetry breaking processes also occur, as the one effectively described by the Nambu and Jona-Lasinio model [22], and a recent related version that can be rewritten in terms of Higgs-Like fields [23].

Despite the use of linear gauges are, in general, the most useful ones in practical loop calculations, non-linear gauges has received recent interest since they are adequate to study some aspects of the infrared dynamics of non-Abelian gauge theories [24]. In this context, we can cite the Curci-Ferrari [25] gauge and the Maximal Abelian gauge (MAG) [26]. They are used to analyze the Abelian dominance hypothesis, which basically refers to the idea that diagonal gluons receive a smaller dynamically generated mass than the off-diagonal ones. In view of this Abelian dominance possibility, an investigation of QED_4 in a non-linear gauge may be welcome to develop insight and a well-defined consistent path through the Heisenberg description. Regarding this latter issue, we have published [19, 27, 28] in order to give new contributions to this field.

with regard to specific theoretical problems, gauge-ghost couplings may turn the quantization of the system trickier since, besides the appearance of new vertices, there are more subtle problems as the possibility of Gribov copies [29], that is, that even fixing the gauge condition the path integral may take into account more than one representative element of the gauge orbits [30]. Nevertheless, the version of QED_4 in the 't Hooft-Veltman gauge is simple enough to give us control over this issue [31].

It is possible to show that the use of this gauge has no phenomenological consequences, as it is expected from its attainability and well-definiteness. 't Hooft and Veltman established this by an explicit computation of the contributions to the photon-photon scattering at tree level which must vanish in ordinary electromagnetism. They showed that the required cancellation indeed occurs. For the general case of loops, they showed this by using diagrammatic Slavnov-Taylor identities [32, 33]. However, many of works such as [34, 35, 36] explore the gauge structure of the theory by means of perturbation theory. In this work, we are concerned with the development of a completely non-perturbative study of the two-point function of the gauge field. The authors had this particular interest since in references [37, 38] there is an apparent longitudinal contribution to the vacuum polarization tensor which is controlled by Ward identities. However, the fictitious g parameter just renormalize the longitudinal non-physical sectors of the photon propagator.

In order to proceed we will use the Kujo-Ojima-Nakanishi formalism [39, 40]. It consists of an indefinite metric quantization in which an auxiliary B-field is suitably introduced to provide a second class system in the sense of the Dirac-Bergman Hamiltonian analysis [41]. The resulting theory is then free of quantum ordering ambiguities and it is described by Dirac brackets which may be turned into equal-time (anti)commutators via correspondence principle. Since we are dealing with indefinite metric, we must find a way to well-define the positive norm subspace or physical space. In the case of QED_4 in linear gauges, the positive frequency part of the B-field does this job by annihilating the physical states. On the other hand, however, the use of the 't Hooft-Veltman gauge does not allow the B-field to obey a free-field equation and, thus, the previous condition can no longer be used. Yet, we can still use the BRST charge to define physical states as expected.

We also show that the indefinite metric formalism have a tool to avoid the presence of the B-field, ghost and longitudinal gauge fields in the physical subspace by the so-called quartet mechanism. The canonical quantization is employed and the propagator of the gauge fields is obtained. In order to do so, we had to use the quantum Cauchy problem for the propagator of ghosts [42, 43] and use the BRST symmetry to relate it to the B field and the longitudinal gauge field projection. Latter, we also show that in a general case, all the amplitudes are independent from the g 't Hooft parameter [44].

This work is organized as follows. In section 2, we construct the lagrangian of QED_4 in the 't Hooft-Veltman gauge coupled with the Higgs boson and Fermionic matter, we deduce the corresponding equations of motion and the BRST charge, and we define the physical states. In section 3, we perform the covariant quantization of the theory. We derive the canonical structure and infer the propagator of the photon field. The Section 5 is devoted to show that all the amplitudes are independent from the g 't Hooft gauge parameter and to obtain the contributions from the matter sector with a careful analysis of the current-current contribution for the photon commutator. Finally, we conclude in section 6.

6.2 The lagrangian and its Residual Symmetry

The lagrangian describing the interaction of the gauge field responsible for the electromagnetic interaction coupled with a complex self interacting scalar field, which can give it mass by a symmetry breaking mechanism, and also naturally coupled to a Fermionic field representing the matter content is [40]

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \left(\partial_\mu + ieA_\mu\right)\phi^\dagger\left(\partial^\mu - ieA^\mu\right)\phi - u\phi^\dagger\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - eA_\mu\bar{\psi}\gamma^\mu\psi \tag{6.1}$$

where $u < 0$, $\lambda > 0$ and the phase invariance is spontaneously broken by the non-vanishing vacuum expectation value $\sqrt{2}\langle 0|\phi(x)|0\rangle = \sqrt{\frac{-4u}{\lambda}} \equiv v$.

As already mentioned in the introduction, this model can be a laboratory to describe the Abelian sector of the weak interaction in a scenario of mass generation in which a more realistic treatment should be taken in account. Since the use of a non-linear gauge can be useful in such a case, we introduce, by means of an auxiliary B-field, the t'Hooft-Veltman gauge condition generalized to an α gauge. Expressing the lagrangian in terms of the fluctuations of the scalar field components around the vacuum $\sqrt{2}\phi = v + \varphi + i\chi$, we get [40]

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{M^2}{2}\big(A_\mu - \frac{1}{M}\partial_\mu\chi\big)^2 + \frac{1}{2}\big(\partial_\mu\varphi\partial^\mu\varphi + \mu^2\varphi\varphi\big) + eA^\mu\big(\chi\partial_\mu\varphi - \varphi\partial_\mu\chi\big)\\ &+ eMA_\mu A^\mu\varphi + \frac{1}{2}e^2A_\mu A^\mu\big(\varphi^2+\chi^2\big) - \frac{1}{2}\gamma\mu\varphi\big(\varphi^2+\chi^2\big) - \frac{1}{8}\gamma^2\big(\varphi^2+\chi^2\big)^2\\ &+ B\left(\partial_\mu A^\mu + gA_\mu A^\mu\right) - \frac{\alpha}{2}B^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - eA_\mu\bar{\psi}\gamma^\mu\psi \end{aligned} \tag{6.2}$$

where α is the gauge parameter, the mass parameter is given by $M = ev$, the coupling constant γ has the form $\gamma = \sqrt{\frac{\lambda}{2}}$, the scalar field components $\chi(x)$ and $\varphi(x)$ are real fields and the mass parameter for this last field is written as $\mu^2 = -2u$. The Higgs field $\chi(x)$ appearing above is the so-called Goldstone boson and we will see that in the asymptotic limit it is a massless pole expected to appear due to the symmetry breaking process [45] .

We can see that the mass term appears in a field combination which we are going to show that can be cast in a Stueckelberg structure, responsible for a Higgs mass generation that avoids gauge symmetry violation.

The equations of motion for the B-field provides an operator equation

$$\partial_\mu A^\mu(x) + gA^\mu(x)A_\mu(x) - \alpha B(x) = 0, \tag{6.3}$$

which depends on α and controls the gauge fixing condition. In particular, for $\alpha = 0$ we reproduce the original 't Hooft-Veltman choice with an extra term responsible for decoupling the photon and the Goldstone boson equation of motion.

Before proceeding with the quantization of this theory, we must identify which is the residual gauge invariance of the theory. Since the non-gauge sector of the lagrangian is invariant under the local gauge transformations which expresses, the Stueckelberg mechanism responsible for giving mass to the photon without breaking the gauge symmetry $A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)$ and $\chi(x) \to \chi(x) + M\Lambda(x)$, where $\Lambda(x)$ is an *a priori* c-number field, it is enough to vary the equation of motion (6.3) with respect to the previous transformation in order to find a condition that defines the residual symmetry. Explicitly, we obtain

$$\delta\big[\partial_\mu A^\mu(x) + gA^\mu(x)A_\mu(x) - \alpha B(x)\big] = 0 \quad \to \quad \left(\Box + 2gA_\mu(x)\partial^\mu\right)\Lambda(x) = 0. \tag{6.4}$$

Since this condition can be fulfilled just by an operator valued symmetry parameter field, a set of extra fields, the ghosts, must be added to the system

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{M^2}{2}\big(A_\mu - \frac{1}{M}\partial_\mu\chi\big)^2 + \frac{1}{2}\big(\partial_\mu\varphi\partial^\mu\varphi + \mu^2\varphi\varphi\big) + eA^\mu\big(\chi\partial_\mu\varphi - \varphi\partial_\mu\chi\big) + eMA_\mu A^\mu\varphi\\ &\frac{1}{2}e^2A_\mu A^\mu\big(\varphi^2+\chi^2\big) - \frac{1}{2}\gamma\mu\varphi\big(\varphi^2+\chi^2\big) - \frac{1}{8}\gamma^2\big(\varphi^2+\chi^2\big)^2 + B\left(\partial_\mu A^\mu + gA_\mu A^\mu\right) - \frac{\alpha}{2}B^2\\ &i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - i\bar{c}\left(\Box + 2gA^\mu\partial_\mu\right)c \end{aligned} \tag{6.5}$$

with regard to $\bar{c}(x)$, it is the aforementioned Lagrange multiplier field whose equation of motion furnishes the residual symmetry (6.4) and has a Grassmann character. Moreover, in order to reproduce equation (6.4) we must have $c(x) = \varepsilon\Lambda(x)$ where ε is a Grassmann number and $\Lambda(x)$ has acquired an operator field status. This Grassmannian nature is necessary if one wants a conserved residual-symmetry generating charge [19]. It also has an important relation with unitarity where non-physical fields can be arranged in a non-observable structure called quartets. We will call $\bar{c}(x)$ and $c(x)$ as ghost fields. Needless to say, the imaginary unit has been added to recover the Hermitian characteristic of the lagrangian. Finally, besides the complex scalar field fluctuations around the vacuum, we also added Fermions as our matter lagrangian.

After these observations, it is straightforward to show that the system presents the following operatorial global symmetry transformations, known as BRST transformations,

$$
\begin{aligned}
&Q_B A_\mu(x) = \partial_\mu c(x), \quad Q_B c(x) = 0, \quad Q_B \bar{c}(x) = iB(x), \quad Q_B B(x) = 0, \quad Q_B \chi(x) = Mc(x) \\
&Q_B \psi(x) = ic(x)\psi(x), \quad Q_B \bar{\psi}(x) = -ic(x)\bar{\psi}(x)
\end{aligned}
\tag{6.6}
$$

where Q_B denotes the BRST generating charge. From (6.6), we note that the BRST charge is nilpotent, that is, $Q_B^2 = 0$. This nilpotency property allows us to write the lagrangian (6.5) in an explicit BRST invariant way

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{M^2}{2}\big(A_\mu - \frac{1}{M}\partial_\mu\chi\big)^2 + \frac{1}{2}\big(\partial_\mu\varphi\partial^\mu\varphi + \mu^2\varphi\varphi\big) + eA^\mu\big(\chi\partial_\mu\varphi - \varphi\partial_\mu\chi\big) + eMA_\mu A^\mu\varphi \\
&\frac{1}{2}A_\mu A^\mu\big(\varphi^2+\chi^2\big) - \frac{1}{2}\gamma\mu\varphi\big(\varphi^2+\chi^2\big) - \frac{1}{8}\gamma^2\big(\varphi^2+\chi^2\big)^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - eA_\mu\bar{\psi}\gamma^\mu\psi + \\
&\qquad Q_B\left[-i\bar{c}\left(\partial_\mu A^\mu + gA^\mu A_\mu\right) + i\frac{\alpha}{2}\bar{c}B\right]
\end{aligned}
\tag{6.7}
$$

where we can see that the mass term combination for the gauge field is BRST invariant, expressing the Stueckelberg trick of adding mass to the gauge field without breaking local symmetries, which also contribute to good renormalization properties [6].

From (6.5) it follows the equation of motion

$$
\partial_\mu F^{\mu\nu}(x) + Q_B\tilde{J}^\nu(x) + J^\nu(x) = 0, \tag{6.8}
$$

wherein

$$
Q_B\tilde{J}^\nu(x) = -\partial^\nu B(x) + 2gB(x)A^\nu(x) - 2ig\bar{c}(x)\partial^\nu c(x). \tag{6.9}
$$

with $\tilde{J}^\mu(x) = i\partial^\mu\bar{c}(x) - 2ig\bar{c}(x)A^\mu$.
Since the first term on the left-hand side of (6.8) and the matter current are transverse operators, we conclude that $Q_B\tilde{J}_\mu(x)$ must also be transverse. The transverse matter current $J_\mu(x)$ that appears above has the form

$$
\begin{aligned}
J_\mu(x) = &e\Big(\chi(x)\partial_\mu\varphi(x) - \varphi(x)\partial_\mu\chi(x)\Big) + M^2\Big(A_\mu(x) - \frac{1}{M}\partial_\mu\chi(x)\Big) + 2eMA_\mu(x)\varphi(x) \\
&+ e^2M^2A_\mu(x)\Big(\varphi^2(x) + \chi^2(x)\Big) + e\bar{\psi}(x)\gamma_\mu\psi(x)
\end{aligned}
\tag{6.10}
$$

It is important to comment that with covariant linear gauges, the B-field satisfies a free-field equation, thus, the physical Hilbert space, $\mathcal{H}_{\text{phys}}$, is constructed by imposing the following subsidiary condition

$$
B^+(x)|\text{phys}\rangle = 0, \quad \forall|\text{phys}\rangle \in \mathcal{H}_{\text{phys}}. \tag{6.11}
$$

This expression is Poincaré invariant since it is defined in terms of the positive frequency part of the B-field. However, in the present case, the use of the 't Hooft-Veltman gauge does not permit B to obey a free equation as we can see by acting with the differential operator ∂_μ on equation (6.8). The generalization of (6.11) for the case of the non-linear t'Hooft gauge, is given below

$$
Q_B|\text{phys}\rangle = 0, \quad \forall|\text{phys}\rangle \in \mathcal{H}_{\text{phys}} \equiv \frac{\mathcal{V}}{\mathcal{V}_0}. \tag{6.12}
$$

This definition is also Poincaré invariant. It means that physical states must be invariant by residual gauge transformations. This is a consistent definition due to its nilpotency. In fact, the quotient space $\mathcal{V}/\mathcal{V}_0$ above tells us that the physical states are defined by $Q_B|\text{phys}\rangle = 0$ and also by the requirement that it should not be of the form $Q_B|\Psi\rangle$ for any state Ψ since the nilpotency of Q_B would make it unobservable.

For now on, in order to simplify the calculations, we can consider $\mu \to \infty$ to eliminate the Higgs from the model. Since the Higgs has indeed a huge mass, we can use our Abelian laboratory spontaneous symmetry breaking model to contemplate this situation. Due to the existence of gauge symmetry, in order to consistently eliminate $\varphi(x)$ from the theory we must also eliminate non-linear terms with $\chi(x)$ from the lagrangian. Otherwise, it would lead to an operator equation of motion with non-conserved sources coupled to the photon. Thus, our model turn out to be the gauged Proca model, with a Stueckelberg particle, coupled to Fermionic matter.

6.3 Covariant Quantization

In order to quantize the theory we compute the corresponding canonical momenta variables and write down the equal-time commutators by using the correspondence principle. We start from lagrangian (6.5) to obtain that

$$\pi^i(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i(x))} = \partial_0 A^i(x) - \partial^i A^0(x), \tag{6.13}$$

$$\pi^0(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0(x))} = B(x), \tag{6.14}$$

are the momenta for the gauge field, that

$$\pi_{\bar{c}}(x) = \frac{\partial_L \mathcal{L}}{\partial_L(\partial_0 \bar{c}(x))} = i\partial_0 c(x), \tag{6.15}$$

$$\pi_c(x) = \frac{\partial_L \mathcal{L}}{\partial_L(\partial_0 c(x))} = i\partial_0 \bar{c}(x) - i2g\bar{c}(x)A^0(x), \tag{6.16}$$

are those for the ghost fields in which L denotes left derivative, and that

$$\pi_B(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 B(x))} = 0, \tag{6.17}$$

$$\pi_\chi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \chi(x))} = \big(\partial_0 \chi - MA_0\big), \tag{6.18}$$

$$\pi_\psi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi(x))} = i\bar{\psi}(x)\gamma^0, \tag{6.19}$$

$$\pi_{\bar{\psi}}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi}(x))} = 0 \tag{6.20}$$

are the ones for the the auxiliary B-field, the Fermionic matter sector and the components of the Higgs field.
The expressions above prove that also in the ‘t Hooft gauge QED_4 case, the careful introduction of the B field structure furnishes already a well-defined system without any first class ambiguities [41] . The system is of second class from the beginning.

Therefore, we can build a reduced bracket in which all the constraints can be taken in the strong form. Then, we have

$$\begin{aligned}
\{A_i(x), \pi^j(y)\} &= \delta_i^j \delta^3(x-y) && (6.21)\\
\{A_0(x), B(y)\} &= \delta^3(x-y) && (6.22)\\
\{c(x), \pi_c(y)\} &= \delta^3(x-y) && (6.23)\\
\{\bar{c}(x), \pi_{\bar{c}}(y)\} &= \delta^3(x-y) && (6.24)\\
\{\psi(x), \pi_\psi(y)\} &= \delta^3(x-y) && (6.25)\\
\{\chi(x), \pi_\chi(y)\} &= \delta^3(x-y) && (6.26)\\
& && (6.27)
\end{aligned}$$

which by means of the correspondence principle [40] give the fundamental (anti)commutation relations that will serve as starting points to derive initial conditions in our analysis below. They are

$$\begin{aligned}
\Big[A_i(x), \partial_0 A^l(y)\Big]_0 &= i\delta_i^l \delta^3(x-y) && (6.28)\\
\Big[A_0(x), B(y)\Big]_0 &= i\delta^3(x-y) && (6.29)\\
\Big\{\bar{c}(x), \partial_0 c(y)\Big\}_0 &= \delta^3(x-y) && (6.30)\\
\Big\{c(x), \partial_0 \bar{c}(y)\Big\}_0 &= \delta^3(x-y) && (6.31)\\
\Big[B(x), B(y)\Big]_0 &= 0 && (6.32)\\
\Big[\psi(x), \bar{\psi}(y)\Big]_0 &= \gamma^0 \delta^3(x-y) && (6.33)\\
\Big[\chi(x), \partial_0 \chi(y)\Big]_0 &= i\delta^3(x-y) && (6.34)\\
& && (6.35)
\end{aligned}$$

where the subscript 0 means quantities taken at equal-times, that is, $x_0 = y_0$.

From the gauge condition (6.3), we obtain

$$\Big[A_\mu(x), \partial_0 A_0(y)\Big]_0 = i\alpha\delta_\mu^0 \delta^3(x-y). \quad (6.36)$$

Using the $\nu = 0$ component of the equation of motion (6.8) to obtain the time derivative of the B-field in terms of the other fields, we get

$$\begin{aligned}
\Big[A_\mu(x), \partial_0 B(y)\Big]_0 &= -i\partial_k^y \delta_\mu^k \delta^3(x-y) + 2igA_0(x)\delta^3(x-y) && (6.37)\\
\Big[B(x), \partial_0 B(y)\Big]_0 &= -2igB(x)\delta^3(x-y). && (6.38)
\end{aligned}$$

Commuting the zeroth component of (6.8) with $\chi(x)$, $\varphi(x)$ and $\psi(x)$ gives, respectively

$$\begin{aligned}
\Big[\chi(x), \partial_0 B(y)\Big]_0 &= -\, iM\delta^3(x-y) && (6.39)\\
\Big[\psi(x), \partial_0 B(y)\Big]_0 &= -\, e\psi(x)\delta^3(x-y) && (6.40)\\
\Big[\bar{\psi}(x), \partial_0 B(y)\Big]_0 &= e\bar{\psi}(x)\delta^3(x-y), && (6.41)
\end{aligned}$$

The photon equation of motion can be cast in a form that reveals the mass generation due to the spontaneous symmetry breaking

$$\Box A_\mu(x) - \partial_\mu\big(\partial_\nu A^\nu(x)\big) + M^2\Big(A_\mu(x) - \frac{1}{M}\partial_\mu\chi(x)\Big) = Q_B\tilde{J}_\mu(x) + e\bar{\psi}(x)\gamma_\mu\psi(x) \tag{6.42}$$

In order to find the general structure for the vacuum expectation value of the photon field we first manipulate the equation of motion to write it in the following form

$$\begin{aligned}&\Big(\Box + M^2\Big)A_\mu(x) + g\partial_\mu\left(A_\beta(x)A^\beta(x)\right) = +\alpha\partial_\mu B(x) - Q_B\tilde{J}_\mu(x)\\ &-\Big(J_\mu(x) - M^2\big(A_\mu(x) - \frac{1}{M}\partial_\mu\chi(x)\big)\Big) + M\partial_\mu\chi(x)\end{aligned} \tag{6.43}$$

The reason to proceed like this is the fact that on the right hand side there are conserved currents and a term proportional to the mass that is BRST invariant and these properties will be very useful for us in the next steps.

In order to find the structure of the commutator, and show that even in the case where different kinds of interaction are present, the g t'Hooft parameter does not affect the observable sector of the theory, and also to understand the implications of such an interaction, we need to obtain the differential equations obeyed by it which allows us to find its explicit structure using the initial conditions. We have

$$\begin{aligned}&\Big(\Box + M^2\Big)^x\Big(\Box + M^2\Big)^y\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle =\\ &\langle 0|\Big[-g\partial_\mu\left(A_\beta(x)A^\beta(x)\right) + \alpha\partial_\mu B(x) - Q_B\tilde{J}_\mu(x) - \Big(J_\mu(x) - M^2\big(A_\mu(x) - \frac{1}{M}\partial_\mu\chi(x)\big)\Big)\\ &+ M\partial_\mu\chi(x) \quad , \quad -g\partial_\nu\left(A_\gamma(y)A^\gamma(y)\right) + \alpha\partial_\nu B(y) - Q_B\tilde{J}_\nu(y)\\ &-\Big(J_\nu(y) - M^2\big(A_\nu(y) - \frac{1}{M}\partial_\nu\chi(y)\big)\Big) + M\partial_\nu\chi(y)\Big]|0\rangle.\end{aligned} \tag{6.44}$$

If one considers that the B field has a vanishing norm and is a Q_B-boundary term, i.e., $Q_B B(x) = 0$, that the matter current $J_\mu(x)$ is physical in the sense of $Q_B J_\mu(x)|\text{phys}\rangle = 0$, and that both currents $J_\mu(x)$ and $Q_B\tilde{J}_\mu(x)$ commute with $\partial_\nu A^\nu(x)$. [1] We also consider that $\langle 0|[\Big(J_\nu(x) + Q_B\tilde{J}_\nu(x)\Big), \chi(y)]|0\rangle = 0$ [2] and $Q_B\big(A_\mu(x) - \frac{1}{M}\partial_\mu\chi(x)\big)\Big) = 0$. The simplified expression reads

$$\begin{aligned}&\Big(\Box + M^2\Big)^x\Big(\Box + M^2\Big)^y\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle = g^2\partial^x_\mu\partial^y_\nu\langle 0|\Big[A_\beta(x)A^\beta(x), A_\gamma(y)A^\gamma(y)\Big]|0\rangle\\ &+\langle 0|\Big[J^\psi_\mu(x), \bar{J}^\psi_\nu(y)\Big]|0\rangle - \alpha g\partial^x_\mu\partial^y_\nu\langle 0|\Big[A_\beta(x)A^\beta(x), B(y)\Big]|0\rangle - \alpha g\partial^x_\mu\partial^y_\nu\langle 0|\Big[B(x), A_\beta(y)A^\beta(y)\Big]|0\rangle\\ &+ M^2\partial^x_\mu\partial^y_\nu\Big(\mathcal{G}(x-y) - \mathcal{G}(y-x)\Big) - M^2\partial^x_\mu\partial^y_\nu\langle 0|\Big[\chi(x), \chi(y)\Big]|0\rangle\\ &+ M^3\Big(\langle 0|\Big[\partial_\mu\chi(x), A_\nu(y)\Big]|0\rangle - \langle 0|\Big[A_\mu(x), \partial_\nu\chi(y)\Big]|0\rangle\Big)\end{aligned} \tag{6.45}$$

[1] Explicitly, we have $[\partial^\mu A_\mu(x), J^\nu(y)] = [\partial^\mu A_\mu(x), Q_B\tilde{J}^\nu(y)] = 0$. This result is obtained by commuting the longitudinal part of $A_\mu(x)$ with $J^\nu(y)$ and $Q_B\tilde{J}^\nu(y)$, respectively. The transverse nature of the ν index implies that $[Q_B\tilde{J}_\nu(x), A^L_\beta(y)] = C\partial_\nu\partial_\beta\Delta(x-y,0)$ and $[J_\nu(x), A^L_\beta(y)] = D\partial_\nu\partial_\beta\Delta(x-y,0)$ where $\Delta(x-y,0)$ is defined in (6.53). C and D are indefinite constants.

[2] Since both currents are conserved, we must have $\langle 0|[J_\nu(x), \chi(y)]|0\rangle = C\partial_\nu D(x-y,0)$, $\langle 0|[Q_B\tilde{J}_\nu(x), \chi(y)]|0\rangle = D\partial_\nu D(x-y,0)$. Then, owing to the definition of these currents, the fact that all the fields have vanishing vacuum averages and the initial conditions for the commutators we conclude that $C = -iM = -D$.

We have considered $J^{\psi}_{\nu}(x) = J_{\nu}(x) - M^2\big(A_{\nu}(x) - \frac{1}{M}\partial_{\nu}\chi(x)\big)$ and due to Poincaré invariance $\partial_{\nu}\mathcal{G}(x-y) \equiv \langle 0|\Big[\partial^{\mu}A_{\mu}(x), A_{\nu}(y)\Big]|0\rangle$.
Using, again, the gauge condition (6.3) on the second line of this expression in order to relate it with $\partial_{\mu}A^{\mu}(x)$, we obtain the following equivalent version

$$\begin{aligned}
\Big(\Box + M^2\Big)^x \Big(\Box + M^2\Big)^y \langle 0|\Big[A_{\mu}(x), A_{\nu}(y)\Big]|0\rangle &= g^2 \partial^x_{\mu}\partial^y_{\nu}\langle 0|\Big[A_{\beta}(x)A^{\beta}(x), A_{\gamma}(y)A^{\gamma}(y)\Big]|0\rangle \\
&+ \langle 0|\Big[J^{\psi}_{\mu}(x), J^{\psi}_{\nu}(y)\Big]|0\rangle + \alpha\partial^x_{\mu}\partial^y_{\nu}\langle 0|\Big[\partial^x_{\beta}A^{\beta}(x), B(y)\Big]|0\rangle + \alpha\partial^x_{\mu}\partial^y_{\nu}\langle 0|\Big[B(x), \partial^y_{\beta}A^{\beta}(y)\Big]|0\rangle \\
&- M^2\partial^x_{\mu}\partial^y_{\nu}\langle 0|\Big[\chi(x), \chi(y)\Big]|0\rangle + M^2\partial^x_{\mu}\partial^y_{\nu}\Big(\mathcal{G}(x-y) - \mathcal{G}(y-x)\Big) \\
&+ M^3\Big(\langle 0|\Big[\partial_{\mu}\chi(x), A_{\nu}(y)\Big]|0\rangle - \langle 0|\Big[A_{\mu}(x), \partial_{\nu}\chi(y)\Big]|0\rangle\Big) \qquad (6.46)
\end{aligned}$$

From this we can immediately conclude that the non-linear self-interaction part contributes to the renormalization of the longitudinal non-physical sector. Therefore, we expect that the physical sector is independent of the fictitious parameter g. Then, we need to know the structure of all the commutator appearing above.

First of all, to find the general expression for the two-point function it is mandatory to compute the commutator between the photon field A_{μ} and the B-field, as we can see in (6.46). Since the use of the 't Hooft-Veltman gauge forbids the harmonic character of the B-field, the desired commutator cannot be easily found by means of the initial data. An alternative approach is to use BRST symmetry. In fact,

$$0 = \langle 0|Q_B\Big[A_{\mu}(x), \bar{c}(y)\Big]|0\rangle = \partial^x_{\mu}\langle 0|\Big\{c(x), \bar{c}(y)\Big\}|0\rangle + i\langle 0|\Big[A_{\mu}(x), B(y)\Big]|0\rangle. \qquad (6.47)$$

This expression shows us that in order to find the vacuum expectation value of the commutator between A_{μ} and B we must first compute the two-point function for ghosts which is defined by the following Cauchy problem

$$\Box^x\langle 0|\Big\{c(x), \bar{c}(y)\Big\}|0\rangle = -2g\langle 0|\Big\{A^{\mu}(x)\partial^x_{\mu}c(x), \bar{c}(y)\Big\}|0\rangle \qquad (6.48)$$

$$\langle 0|\Big\{c(x), \bar{c}(y)\Big\}_0|0\rangle = 0 \qquad (6.49)$$

$$\partial^x_0\langle 0|\Big\{c(x), \bar{c}(y)\Big\}_0|0\rangle = -\delta^3(x-y). \qquad (6.50)$$

We shall define

$$\mathcal{D}(x-y; A) \equiv \langle 0|\Big\{c(x), \bar{c}(y)\Big\}|0\rangle, \qquad (6.51)$$

for convenience. It only depends on the difference of the coordinates due to Poincaré invariance [46]. Hence, the formal solution of (6.48) is the following integral representation [42, 43]

$$\begin{aligned}
\mathcal{D}(x-y; A) &= -2g\int d^4u\ \varepsilon(x,y,u)\Delta(x-u,0)\langle 0|\Big\{A^{\beta}(u)\partial^u_{\beta}c(u), \bar{c}(y)\Big\}|0\rangle \\
&\quad - \int d^3u\Big[\Delta(x-u,0)\partial^u_0\mathcal{D}(u-y;A) - \partial^u_0\Delta(x-u,0)\mathcal{D}(u-y;A)\Big]_{u^0=y^0} \\
&= -2g\int d^4u\ \varepsilon(x,y,u)\Delta(x-u,0)\langle 0|\Big\{A^{\beta}(u)\partial^u_{\beta}c(u), \bar{c}(y)\Big\}|0\rangle - \Delta(x-y,0)
\end{aligned} \qquad (6.52)$$

where Pauli-Jordan distribution $\Delta(x-y;s)$ is defined by the following Cauchy data

$$\Box\Delta(x-y;s) = -s\Delta(x-y;s), \quad \Delta(x-y;s)|_0 = 0, \quad \partial^x_0\Delta(x-y;s)|_0 = -\delta^3(x-y), \qquad (6.53)$$

and we have defined $\varepsilon(x,y,u)$ in terms of the Heaviside function as

$$\varepsilon(x,y,u) = \Theta(x_0 - u_0) - \Theta(y_0 - u_0). \tag{6.54}$$

We have used the fact that for a given operator $F(x,y)$ obeying $\hat{O}^x F(x,y) = G(x,y)$, where $\hat{O}^x$ is a given differential operator acting in coordinate x, there is the following integral representation [43]

$$\begin{aligned} F(x,y) = &\int d^4u \; \varepsilon(x,y,u)\tau(x-u)G(u,y) \\ &- \int d^3u \Big[\tau(x-u)\partial_0^u F(u,y) - \partial_0^u \tau(x-u)F(u,y)\Big]_{u^0=y^0} \end{aligned} \tag{6.55}$$

where $\tau(x-y)$ is an operator valued distribution such that $\hat{O}^x\tau(x-y) = 0$.

The integral representation for the case of a first order differential operator $\mathcal{O}^x$ associated to the Dirac equation with $\mathcal{O}^x f(x,y) = g(x,y)$ reads

$$f(x,y) = \int d^4u \; \varepsilon(x,y,u)S(x-u)g(u,y) + i\int d^3u \Big[S(x-u)\gamma^0 f(u,y)\Big]_{u^0=y^0} \tag{6.56}$$

where $S(x-u)$, which is going to be further defined, obeys $\mathcal{O}^x S(x-u) = 0$.

Summing up, we have from (6.3) that

$$\langle 0|\Big[A_\mu(x), B(y)\Big]|0\rangle = i\partial_\mu^x \mathcal{D}(x-y;A), \tag{6.57}$$

with $\mathcal{D}$ defined in (6.55).

The BRST symmetry can be also used to find a relation for the commutator between the Goldstone boson and the auxiliary B field

$$0 = \langle 0|Q_B\Big[\chi(x), \bar{c}(y)\Big]|0\rangle = M\langle 0|\Big\{c(x), \bar{c}(y)\Big\}|0\rangle + i\langle 0|\Big[\chi(x), B(y)\Big]|0\rangle. \tag{6.58}$$

So, we have

$$-iM\mathcal{D}(x-y;A) = \langle 0|\Big[\chi(x), B(y)\Big]|0\rangle \tag{6.59}$$

In the case of a linear gauge, this expression would mean that the Goldstone boson, commonly considered to be eaten up by the gauge field, is made unobservable in the present formalism. Since we are using non-linear gauges (which can be used as an inspiration to investigate the Abelian dominance in QCD_4 or other non-Abelian gauge theories, now in the physical context of the standard model.) the non-physical character of $\chi(x)$ is due to the fact that it is not annihilated by the BRST charge, being out from the positive metric Hilbert subspace.

We also need more two commutators, the ones involving $\chi(x)$ and the photon and with itself. Considering its equation of motion

$$\Box\chi(x) = +\alpha MB(x) + MgA^\mu(x)A_\mu(x) \tag{6.60}$$

The above equation, together with the initial conditions, can be used to give an integral representation for the following commutators [19, 43]

$$\mathcal{D}_{\chi\chi}(x-y) = \langle 0|\Big[\chi(x), \chi(y)\Big]|0\rangle = \int d^4u\, \varepsilon(x,y,u)\Delta(x-u,0)\Big(\langle 0|\Big[+MgA^{\mu}(u)A_{\mu}(u), \chi(y)\Big]|0\rangle - i\alpha M^2\mathcal{D}(u-y;A)\Big) + i\Delta(x-y,0) \quad (6.61)$$

Regarding the commutator with the photon field, we have

$$\mathcal{D}_{\chi\ A_\nu}(x-y) = \langle 0|\Big[\chi(x), A_{\nu}(y)\Big]|0\rangle = \int d^4u\, \varepsilon(x,y,u)\Delta(x-u,0)\Big(\langle 0|\Big[MgA^{\mu}(u)A_{\mu}(u), A_{\nu}(y)\Big]|0\rangle - iM\alpha\partial_{\nu}\mathcal{D}(u-y;A)\Big)$$

The Fermionic sector is also relevant in our discussion. Therefore, we display

$$\Big(i\gamma^{\mu}\partial^{x}_{\mu} - m\Big)\langle 0|\Big\{\psi(x), \bar{\psi}(y)\Big\}|0\rangle\Big(i\gamma^{\mu}\overleftarrow{\partial}^{y}_{\mu} + m\Big) = -e^2\langle 0|\Big\{\gamma^{\mu}A_{\mu}(x)\psi(x), \bar{\psi}(y)A_{\nu}(y)\gamma^{\nu}\Big\}|0\rangle \quad (6.62)$$

where the left side denotes the anti commutator version of the electron self energy.

The general solution then gives

$$\langle 0|\Big\{\psi(x), \bar{\psi}(y)\Big\}|0\rangle = iS(x-y) - \int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)S(x-\omega)\Sigma(\omega-u)S(u-y) \quad (6.63)$$

with $\Sigma(x-y) \equiv e^2\langle 0|\Big\{\gamma^{\mu}A_{\mu}(x)\psi(x), \bar{\psi}(y)A_{\nu}(y)\gamma^{\nu}\Big\}|0\rangle$. The distribution above is defined as $\Big(i\gamma^{\mu}\partial_{\mu} + m\Big)\Delta(x-y,m) \equiv S(x-y)$.

Having these results in mind, it follows from (6.46) that

$$\Big(\Box + M^2\Big)^x\Big(\Box + M^2\Big)^y\langle 0|\Big[A_{\mu}(x), A_{\nu}(y)\Big]|0\rangle = g^2\partial^x_{\mu}\partial^y_{\nu}\langle 0|\Big[A_{\beta}(x)A^{\beta}(x), A_{\gamma}(y)A^{\gamma}(y)\Big]|0\rangle + \langle 0|\Big[J^{\psi}_{\mu}(x), J^{\psi}_{\nu}(y)\Big]|0\rangle + i\alpha\partial^x_{\mu}\partial^y_{\nu}\Box^x\mathcal{D}(x-y;A) - i\alpha\partial^x_{\mu}\partial^y_{\nu}\Box^y\mathcal{D}(y-x;A) - M^2\partial^x_{\mu}\partial^y_{\nu}\mathcal{D}_{\chi\chi}(x-y) + M^2\partial^x_{\mu}\partial^y_{\nu}\Big(\mathcal{G}(x-y) - \mathcal{G}(y-x)\Big) - M^3\Big(\partial^y_{\mu}\mathcal{D}_{\chi A_\nu}(x-y) - \partial^x_{\nu}\mathcal{D}_{\chi A_\mu}(y-x)\Big) \quad (6.64)$$

is the differential equation we must solve in order to compute the vacuum expectation value of the gauge field commutator. Owing to Lorentz covariance, every vacuum average of a commutator between a scalar and a vector field is proportional to a gradient of a scalar function. So, we define $\mathcal{D}_{\chi A_\nu}(x-y) \equiv \partial^x_{\nu}\mathcal{T}(x-y)$.

The general solution that must be fixed by initial conditions is

$$
\begin{aligned}
\langle 0|\Big[A_\mu(x),A_\nu(y)\Big]|0\rangle &= b\partial_\mu\partial_\nu E(x-y,M)+c\partial_\mu\partial_\nu\Delta(x-y,M)\\
&+a\Big(\eta_{\mu\nu}\Delta(x-y,M)+\frac{\partial_\mu\partial_\nu}{M^2}\Delta(x-y,M)\Big)\\
-g^2\partial^x_\mu\partial^x_\nu &\int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)\langle 0|\Big[A_\mu(\omega)A^\mu(\omega),A_\beta(u)A^\beta(u)\Big]|0\rangle\\
&-i\alpha\partial^x_\mu\partial^x_\nu\int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)\Big(\Box_u\mathcal{D}(\omega-u;A)-\Box_u\mathcal{D}(u-\omega;A)\Big)\\
&-\partial^x_\mu\partial^x_\nu\int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)\Big(-M^2\mathcal{D}_{\chi\chi}(x-y)\\
&-M^2\Big(\mathcal{G}(x-y)-\mathcal{G}(y-x)\Big)-M^3\big(\mathcal{T}(x-y)-\mathcal{T}(y-x)\big)\Big)\\
&+\int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)\langle 0|\Big[J^\psi_\mu(\omega),J^\psi_\nu(u)\Big]|0\rangle \qquad (6.65)
\end{aligned}
$$

The complete non perturbative solution above shows that the unphysical parameter g do not have any influence on the physics since it appears just in longitudinal sectors of the photon commutator. This is in agreement with the diagrammatic approach of [44, 47], when just the ‘t Hooft gauge self interaction was taken into account. We can also mention [48] in which the DeWitt background field method revealed that the physical quantities, such as the charge renormalization, is independent from the ‘t Hooft gauge parameter.
We also mention that since we do not know the parity of the above complete distributions under $(x-y)\to-(x-y)$ the expression of the commutator cannot be more simplified.
The distribution $E(x-y)$ appearing above is defined as

$$
\big(\Box+s\big)E(x-y;s)=\Delta(x-y;s),\quad E(x-y;s)|_0=0,\quad (\partial^x_0)^3E(x-y;s)|_0=-\delta^3(x-y). \qquad (6.66)
$$

with the following definition in terms of $\Delta(x-u,s)$

$$
E(x-y,s)=-\int d^4u\ \epsilon(x,y,u)\Delta(x-u,s)\Delta(u-y,s), \qquad (6.67)
$$

In order to specify our general solution, we consider the initial conditions for the photon commutator and the fact that just the discrete pole part of the complete distributions previously defined, that are being integrated in the expression of the photon commutator, contributes at equal times $x_0=y_0$, as we did in [19].

An important step is to consider that owing to the definition of the complete distributions, their discrete pole part goes as $\mathcal{D}_{\chi\chi}(x-y)\to i\Delta(x-y,0)-i\alpha E(x-y,0)$, $\mathcal{D}_{\chi A_\nu}(x-y)\to -i\alpha\partial_\nu E(x-y,0)$ and the use of the gauge condition gives $\mathcal{G}(x-y)\to -i\alpha\Delta(x-y,0)$. Now, we must call attention to the fact that differently from [19] , we are using a double application of a massive Pauli-Jordan distribution to express the photon commutator. Then, to achieve our goal, the following formal equations are of fundamental importance

$$
\Big(\Box+M^2\Big)^{-2}_x\Delta(x-y,0)=\frac{\Delta(x-y,0)}{M^4} \qquad (6.68)
$$

$$
\Big(\Box+M^2\Big)^{-2}_x E(x-y,0)=\frac{E(x-y,0)}{M^4}-2\frac{\Delta(x-y,0)}{M^6} \qquad (6.69)
$$

Considering these relations and also the simple pole expressions for the complete distributions we find that $b=c=0$ and $a=-i$.

This is a good point to stop developing the result and to introduce a consistency check to test that our solution (6.65) is correct. Let us first consider the asymptotic limit with $e \to 0$.

It is possible to rewrite the photon operator equation of motion to obtain

$$(\Box + M^2)U^{as}_\mu = Q_B \tilde{J}_\mu - (\Box + M^2\alpha)B \tag{6.70}$$

with $U^{as}_\mu(x) \equiv A^{as}_\mu(x) - \frac{1}{M}\partial_\mu \chi^{as}(x) - \frac{1}{M^2}\partial_\mu B^{as}(x)$.

Considering that $B(x)$ is a Q_B-boundary term and the vacuum state is annihilated by Q_B with $Q_B^2 = 0$, we have

$$(\Box + M^2)^x(\Box + M^2)^y \langle 0|\Big[U^{as}_\mu(x), U^{as}_\nu(y)\Big]|0\rangle = 0 \quad ; \quad \langle 0|\Big[\partial^\mu U^{as}_\mu(x), U^{as}_\nu(y)\Big]|0\rangle = 0 \tag{6.71}$$

Using the integral representation [3] and the initial conditions leads to

$$\langle 0|\Big[U^{as}_\mu(x), U^{as}_\nu(y)\Big]|0\rangle = -i\Big(\eta_{\mu\nu}\Delta(x-y,M) + \frac{\partial_\mu\partial_\nu}{M^2}\Delta(x-y,M)\Big) \tag{6.72}$$

$$Q_B U^{as}_\mu(x) = 0 \tag{6.73}$$

The fact that the $\chi^{as}(x)$ appears in the definition of the massive field is the so-called process of eating the Goldstone boson by the gauge field. We proved that is valid even for the case with $g \neq 0$. This is expected since the norm of a gauge invariant field combination must be independent from gauge fixing parameters.

Regarding the photon commutator, we can check that the $g \to 0$ result is recovered by our theory

$$\langle 0|\Big[A_\mu(x), A_\nu(y)\Big]|0\rangle_{\text{Free}} = -i\Big(\eta_{\mu\nu}\Delta(x-y,M) + \frac{\partial_\mu\partial_\nu}{M^2}\Delta(x-y,M)\Big) + i\frac{1}{M^2}\partial_\mu\partial_\nu\Delta(x-y,0) + i\alpha\partial_\nu\partial_\mu E(x-y,0) \tag{6.74}$$

6.4 Perturbative Analysis

Note that our well-established solution (6.65) is only formal since it depends on the photon field as well as the others field commutators. In order to have information about the change in the structure of the two-point function due solely to this part, we set $\alpha = 0$ since in this case equation (6.3) reduces to the original 't Hooft-Veltman choice[4] , and we will make use of the Heisenberg perturbation theory [43].

Although the gauge fixing parameters are not physical and do not need to be small, since we have found the general non-perturbative solution in the previous section, we found it interesting to consider the perturbative limit case in order to give a new contribution to the development of the perturbative version of the Heisenberg picture quantization formalism and also have a more intuitive view of the theory.

We are interested just in the first quadratic contribution in g and the electric charge e.

[3] we also consider that all the fields have zero vacuum expectation value.

[4] Moreover, we chose to work with $\alpha = 0$ for maximum simplicity since the term proportional to α in (6.65) may also have corrections in powers of g.

Thus, from our formal solution we obtain the complete photon commutator

$$\langle 0|\Big[A_\mu(x),A_\nu(y)\Big]|0\rangle = -i\Big(\eta_{\mu\nu}\Delta(x-y,M)+\frac{\partial_\mu\partial_\nu}{M^2}\Delta(x-y,M)\Big)+i\frac{1}{M^2}\partial_\mu\partial_\nu\Delta(x-y,0)$$
$$-g^2\partial^x_\mu\partial^x_\nu\int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)\langle 0|\Big[A_\mu(\omega)A^\mu(\omega),A_\beta(u)A^\beta(u)\Big]|0\rangle$$
$$+g^2M^4\partial^x_\mu\partial^x_\nu\int d^4\omega d^4u d^4\tilde{\omega}d^4\tilde{u}\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)\epsilon(\omega,u;\tilde{u})$$
$$\times\,\theta(u_0-\tilde{u}_0)\Delta(\omega-\tilde{\omega},0)\Delta(u-\tilde{u},0)\langle 0|\Big[A_\mu(\tilde{\omega})A^\mu(\tilde{\omega}),A_\beta(\tilde{u})A^\beta(\tilde{u})\Big]|0\rangle$$
$$+\,2M^2g^2\partial^x_\mu\partial^x_\nu\int d^4\omega d^4u d^4\Omega\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)$$
$$\times\Delta(u-\Omega,0)\theta(u_0-\Omega_0)\langle 0|\Big[A_\mu(\omega)A^\mu(\omega),A_\beta(\Omega)A^\beta(\Omega)\Big]|0\rangle$$
$$+\int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)\langle 0|[J^\psi_\mu(\omega),J^\psi_\nu(u)]|0\rangle \tag{6.75}$$

We have used the simple pole approximation given in the previous section and also the relations (6.69). Besides that, to obtain the corrections proportional to the g parameter, we considered the photon equation of motion rewritten in the following convenient way

$$\Box A_\mu(x)=-Q_B\tilde{J}_\mu(x)-J_\mu(x)-\partial_\mu\Big(gA_\beta(x)A^\beta(x)\Big) \tag{6.76}$$

Which gave us an useful expression for this field operator in terms of the integral representation given in the previous section evaluated for the specific case of just one space-time argument

$$A_\mu(x)=-\int d^4v\theta(x_0-v_0)\Delta(x-v,0)\Big(-Q_B\tilde{J}_\mu(v)-J_\mu(v)-\partial_\mu\big(gA_\beta(v)A^\beta(v)\big)\Big) \tag{6.77}$$

We did not considered the harmonic solution since our spectrum is massive as we have shown.

For the cases of $\mathcal{D}_{\chi A_\nu}(x-y)$ and $\mathcal{G}(x-y)$ we considered this equation and used the gauge condition to relate the divergence of the photon field to $gA_\beta(x)A^\beta(x)$. An important simplification comes from the fact that the conserved currents commute with this divergence. We also considered that $\Big[\chi(x),Q_B\tilde{J}_\mu(x)+J_\mu(x)\Big]=0$ which was proved in the second footnote of the previous section. This same approach was also used to investigate the content from the $\mathcal{G}(x-y)$ distribution.

In order to analyze the $\mathcal{D}_{\chi\chi}(x-y)$ contribution for the photon propagator, we consider the integral representation for the Goldstone boson operator

$$\chi(x)=\chi^0(x)-\int d^4v\theta(x_0-v_0)\Delta(x-v,0)\Big(MgA_\beta(x)A^\beta(x)\Big) \tag{6.78}$$

The contribution coming from this distribution has the same form as the one from $\mathcal{D}_{\chi A_\nu}(x-y)$, but with a different sign, so the term proportional to M^4 in the photon commutator is their algebraic sum.

The fact that this expression for the propagator is in agreement with the gauge condition can be verified considering conservation of the Fermionic sources. Then, considering the formal expression for the inverse of differential operators, we conclude

$$\langle 0|\Big[\partial_\mu A^\mu(x),A_\nu(y)\Big]|0\rangle = g^2\Box\frac{1}{\big(\Box+M^2\big)^2}\big(1+\frac{M^4}{\Box^2}+\frac{2M^2}{\Box}\big)\langle 0|\Big[gA^\mu(x)A^\mu(x),\partial^y_\nu\Big(gA^\mu(y)A^\mu(y)\Big)\Big]|0\rangle$$
$$=\frac{1}{\Box}\langle 0|\Big[gA^\mu(x)A^\mu(x),\partial^y_\nu\Big(gA^\mu(y)A^\mu(y)\Big)\Big]|0\rangle = -\langle 0|\Big[gA^\mu(x)A^\mu(x),A^\nu(y)\Big]|0\rangle \tag{6.79}$$

where in the last step we considered the photon equation of motion rewritten as $\Box A_\mu(x) = -Q_B\tilde{J}_\mu(x) - J_\mu(x) - g\partial_\mu\Big(A_\beta(x)A^\beta(x)\Big)$ and the fact that the transverse currents commute with the divergence of the photon field, which is related to the $gA_\beta(x)A^\beta(x)$ due to the gauge condition.

Now, developing the commutator regarding the self interaction for the photon, in our first approximation, we get [19]

$$\langle 0|\Big[A_\beta(x)A^\beta(x), A_\gamma(y)A^\gamma(y)\Big]|0\rangle = D^{(0)\beta\gamma}(x-y)\left(D^{+(0)}_{\beta\gamma}(x-y) + D^{+(0)}_{\beta\gamma}(y-x)\right) \tag{6.80}$$

where $D^{+(0)}_{\beta\gamma}(x-y)$ denotes the positive frequency part of the distribution. Since these distributions are related to the Pauli-Jordan one, we display its explicit definition

$$\Delta^+(x-y;s) = -\frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-s)\Theta(p_0)e^{-ip.x}\;,\;\Delta^-(x-y;s) = \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-s)\,\Theta(-p_0)e^{-ip.x} \tag{6.81}$$

with $i\Delta(x-y,s) = \Delta^+(x-y,s) + \Delta^-(x-y,s)$.

From now on we use the shorthand notation

$$M^{(0)}(x-y) = D^{(0)\beta\gamma}(x-y)\left(D^{+(0)}_{\beta\gamma}(x-y) + D^{+(0)}_{\beta\gamma}(y-x)\right). \tag{6.82}$$

where

$$D^{(0)}_{\mu\nu}(x-y) = -i\Big(\eta_{\mu\nu}\Delta(x-y,M_r) + \frac{\partial_\mu\partial_\nu}{M^2}\Delta(x-y,M)\Big) + i\frac{1}{M^2}\partial_\mu\partial_\nu\Delta(x-y,0) \tag{6.83}$$

.

Considering the Fermion contribution for current-current commutator, at order e^2, we have the following expression (using explicit Fermion indices.)

$$\begin{aligned}\langle 0|\Big[J^\psi_\mu,(x),J^\psi_\nu(y)\Big]|0\rangle = \\ e^2\langle 0|\bar\psi_a(x)\gamma^{ab}_\mu\Big\{\psi_b(x),\bar\psi_c(y)\Big\}\gamma^{cd}_\nu\psi_d(y)|0\rangle - e^2\langle 0|\bar\psi_c(y)\gamma^{ab}_\mu\Big\{\psi_d(y),\bar\psi_a(x)\Big\}\gamma^{cd}_\nu\psi_b(x)|0\rangle \\ = e^2tr\Big(\gamma_\mu S^+(x-y)\gamma_\nu S^-(y-x) - \gamma_\nu S^-(x-y)\gamma_\mu S^+(y-x)\Big)\end{aligned} \tag{6.84}$$

Again, now in the Heisenberg perturbative framework, we conclude that the ‘t Hooft parameter affects just non-physical longitudinal contributions in accordance with the diagrammatic approach [37, 38]. Besides that, we have found the generalized conclusion that just the coupling proportional to e^2 gives contributions in the transverse sector, while the gauge parameter g appears just in longitudinal modes. Now, it is the point to present the quadratic approximation explicitly

$$\langle 0|\Big[A_\mu(x),A_\nu(y)\Big]|0\rangle = -i\Big(\eta_{\mu\nu}\Delta(x-y,M)+\frac{\partial_\mu\partial_\nu}{M^2}\Delta(x-y,M)\Big)+i\frac{1}{M^2}\partial_\mu\partial_\nu\Delta(x-y,0)$$
$$-g^2\partial^x_\mu\partial^x_\nu\int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)M^{(0)}(\omega-u)$$
$$+g^2M^4\partial^x_\mu\partial^x_\nu\int d^4\omega d^4u d^4\tilde{\omega}d^4\tilde{u}\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)\epsilon(\omega,u;\tilde{u})$$
$$\times\,\theta(u_0-\tilde{u}_0)\Delta(\omega-\tilde{\omega},0)\Delta(u-\tilde{u},0)M^{(0)}(\tilde{\omega}-\tilde{u})$$
$$+\,2M^2g^2\partial^x_\mu\partial^x_\nu\int d^4\omega d^4u d^4\Omega\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)$$
$$\times\Delta(u-\Omega,0)\theta(u_0-\Omega_0)M^{(0)}(\omega-\Omega)$$
$$+\int d^4\omega d^4u\ \epsilon(y,x;u)\epsilon(x,u;\omega)\Delta(x-\omega,M)\Delta(y-u,M)$$
$$\times\Big\{e^2tr\Big(\gamma_\mu S^+(x-y)\gamma_\nu S^-(y-x)-\gamma_\nu S^-(x-y)\gamma_\mu S^+(y-x)\Big)\Big\} \tag{6.85}$$

6.5 Parameter Independence

In this final section we intend to use the BRST symmetry to show that all physical amplitudes are independent of the g parameter that appears in the expression defining the 't Hooft gauge even in the case where there is Higgs and matter interaction.

In order to do so we define the generator functional of connected Green function [40, 49]:

$$\exp iW(J)\equiv\langle 0|\mathcal{T}\exp iS(J)|0\rangle \tag{6.86}$$

The vacuum above is still the Heisenberg one, $\mathcal{T}$ denotes the time ordering operator and $S(J)$ is given by:

$$S(J)=\int d^4x\Big(\Sigma^\mu(x)A_\mu(x)+\beta(x)\bar{c}(x)+\bar{\beta}c(x)+\gamma(x)B(x)+\Omega(x)\chi(x)+\psi(x)\bar{\eta}(x)+\bar{\psi}(x)\eta(x)\Big) \tag{6.87}$$

where $\Sigma^\mu(x)$, $\beta(x)$, $\bar{\beta}(x)$, $\Omega(x)$, $\eta(x)$, $\bar{\eta}(x)$ and $\gamma(x)$ are c-number external sources with the appropriate Grassmann parity.

The c-number fields are defined collectively as:

$$\phi_I(x)\equiv\frac{\delta W(J)}{\delta J_I(x)}=\langle 0|\mathcal{T}\hat{\phi}_I(x)\exp iS(J)|0\rangle\exp\left(-iW(J)\right) \tag{6.88}$$

where we emphasize the operator nature of the fields by using the hat.

Since the $\mathcal{T}\exp iS(J)$ is a gauge invariant object, we use this fact explicitly to derive the generator of the Ward identities:

$$0=\langle 0|\Big[Q_B,\mathcal{T}\exp iS(J)\Big]|0\rangle=\langle 0|\mathcal{T}\int d^4x\Big(\Sigma^\mu\partial_\mu c+i\beta B+\Omega Mc-ic\bar{\psi}\eta+ic\psi\bar{\eta}\Big)\exp iS(J)|0\rangle \tag{6.89}$$

It is possible to show that the generator of connected Green functions can be written in terms of a path integral [49]

$$e^{iW(J)} \equiv \mathcal{Z} = \int DA_\mu(x)D\psi(x)D\bar{\psi}(x)D\chi(x)D\varphi(x)DB(x)Dc(x)D\bar{c}(x)\exp i(\mathcal{S}_t + iS(J)) \tag{6.90}$$

where $\mathcal{S}_t = \int d^4x\mathcal{L}(x)$ with $\mathcal{L}(x)$ defined in equation (8.5).

If we impose that this expression is invariant with relation to the g parameter the following condition arises

$$\langle 0| \int d^4x \Big(A^\mu(x)A_\mu(x)B(x) - 2i\bar{c}(x)A_\mu(x)\partial^\mu c(x) \Big) |0\rangle = 0 \tag{6.91}$$

The relation above is the B field version of [44].

Now we show that this expression indeed follows from BRST invariance. Then, we vary the generator of Ward identities with relation to $\frac{\delta^3}{i\delta\Sigma_\mu(x)\Sigma^\mu(x)\delta\beta(x)}$ and taking the limit of vanishing sources in the end of the process

$$\langle 0| \int d^4x \Big(A^\mu(x)A_\mu(x)B(x) - 2i\bar{c}(x)A_\mu(x)\partial^\mu c(x) \Big) |0\rangle = 0 \tag{6.92}$$

The same expression than the one required for amplitudes to be independent from the parameter g is recovered. So, the BRST symmetry indeed furnishes a sufficient condition to it.

6.6 Conclusion

Throughout this work we have shown that the non-linear 't Hooft gauge coupled to a Higgs scalar field and also with Fermionic matter do not has any influence on the physical output of the theory, as it should be. In order to do so we have employed the Nakanishi B field formalism, in its non-perturbative and perturbative versions [50]. Then, we concluded that just longitudinal sectors of the two point function are dependent of the g parameter, a result in agreement with [37, 38] regarding the case with no matter coupling.

To generalize the result to an alpha gauge it was required to obtain the vacuum projection between the B and photon field . It was achieved by relating this projection with the ghost fields two point function by means of BRST symmetry. We also used the BRST symmetry to obtain the commutator between the Goldstone boson field operator and the B field. Also, by means of a well-defined integral representation, we found the non-perturbative expression for the Fermion anti commutator, the commutator between the Goldstone boson and with this field and the photon field. They were required to obtain the complete photon commutator.

The prove of g parameter independence was obtained in its B field version and also in a Heisenberg point of view which was latter related to the conventional path integral formalism. We also employed Heisenberg perturbation theory to show that the 't Hooft gauge self interaction just generates longitudinal contributions which do not has any physical influence as one can infer by the action of the BRST symmetry and the definition of the positive Hilbert space sector. We needed to give a careful analysis of the complete distributions related to the commutator of the Goldstone boson and the one with this field and the photon operator to obtain the first approximation and then verify the gauge condition in this case where the photon field is massive and then, compared to the 't Hooft gauge free QED_4 case, 2 more contributions proportional to g^2 appeared. The matter sector was also analyzed and the Fermion current-current commutator was obtained using the explicit spinor indices to find the traced structure.

The remaining contributions due to the scalar field coupling were also calculated.

For future perspectives we intend to extend the B field formalism to also contemplate Ostrogadskian systems [51, 52, 53] in order to study, for example, the Podolsky electrodynamics [54, 55, 56, 57] and also higher derivative gravity models [58]. The case of three dimensional gravity written in terms of tetrad and spin connection [59] can be a interesting first approach to study real non-Abelian theories in the B-field formalism after the experience from this work and [19]. We also intend to investigate the alternative correlated Heisenberg formalism in which [60] not the (anti)-commutators, but the fields itselves are expanded in the coupling constant. A recent interesting application in the DKP model was given in [61], in which a physical renormalization process was used for the vacuum polarization tensor.

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Chapter 7

Indefinite metric quantization for models with reducible constraint structure

7.1 Motivation

The present work consists in the B field indefinite metric Hilbert space Heisenberg picture formulation [1, 2] for dual theories describing spin 0 and spin 1 particles by means of vector and antisymmetric tensors, respectively. We also present an alternative partition function formulation for their massless limits following the principles of this formalism. This kind of higher rank tensor description can be found in the context of effective theories for low energy excitations of bound states with integer spin in QCD, [3, 4, 5]. The use of these models can provide a wider quantity of couplings which are of particular interest in a more complete low energy description, more precisely, for the matching with the underlying high energy theory.

The Hamiltonian analysis [6] shows that these models, in general, presents discontinuities in the massless limit, for example, jumping from a massive spin 1 canonical structure to a massless spin 0 one [7]. This is a well known characteristic of the Kalb-Ramond model [8, 9], which describes one of the particles of the string spectrum [10]. At least for antisymmetric fields, this was expected since massless p forms are dual to $D-p-2$ forms while massive ones are dual to $D-p-1$ forms [11]. In a general view, this is due to the fact that their massless versions present reducible constraints [12], namely, there is a linear dependence between them. Therefore, one of our objectives is to understand what happens in this limit to the positive Hilbert subspace, where the physical spectrum of the theory lies.

The Kugo-Ojima-Nakanishi B field formulation[1, 13] basically consists in adding auxiliary fields that impose some conditions over the gauge field operator that are consistent with their equations of motion and whose corresponding generating lagrangian term turn the theory into a second class one from the beginning, avoiding every quantum ambiguity [6]. We are going to see that in the case of the models considered here, in order to achieve this goal, one more auxiliary field must be added to consistently quantize the system. Then, they must be confined in quartet configurations to avoid their presence in the physical subspace. These fields are related to the generation of the gauge symmetries in the massless case. They also provide subsidiary conditions to define the positive Hilbert subspace of the system. It is worth mention that this method, besides the non-perturbative analysis, has also a well-defined perturbative version [14, 15], see [16, 17, 18] for more recent achievements.

Since the issue of reducible models was not analyzed in the previous articles related to the B field formulation, we find it relevant to analyze them under this perspective. Another important feature that we must point out is the fact that, in this case, for the quantum ambiguities to be avoided a new set of extra auxiliary fields must be included.

Besides the previous remarks, we are going to provide a complete gauge fixing lagrangian for the massless models with all the required ghosts. Although it is not necessary in Abelian gauge theories since they are decoupled, it is necessary to include them in the partition function if one wants to find the correct thermodynamic degrees of freedom for the system. We are going to perform the functional integration to show that the system described by this extended lagrangian indeed leads to the correct degrees of freedom.

Regarding the construction of the partition function, this is also one of our achievements since our procedure to derive the ghost sector can be an alternative to the Fradkin-Vilkovisky method [19]. Here, we just use the requirement of invariance under ghost number symmetry and the prescription to avoid any quantum ambiguities by introducing suitable auxiliary fields in the lagrangian leading to two sets of quartet configurations [16, 20, 21] which confine these extra fields in a non-observable configuration. It leads to the so-called ghost of ghost structure, characteristic of these reducible systems [22].

The work is organized as follows. Since the Hamiltonian analysis that reveals the reducible constraint structure of the theory described by the antisymmetric tensor can be found in [7], we present the one for the vector reducible case [23] as a conceptual introduction in section 2. In the section 3, the B field quantization for the vector model is presented and we learn some important features for the case of a spin 1 model described by an antisymmetric field [5] and study the behaviour of the positive metric Hilbert subspace in the massless limit in order to understand the degree of freedom discontinuity. The section 5 presents the path integral representation for these fields and the computation of the partition function, with all the required ghost of ghost structure built by means of the B-field quantization principles and it can be understood as an alternative approach to elucidate the features of the previous Heisenberg analysis. The section 6 is devoted to our conclusions and new perspectives.

7.2 Hamiltonian Analysis

A massive spin 0 particle can be described by a vector field by means of the following action [23]

$$S = \frac{1}{2}\int d^Dx \Bigg\{ (\partial^\mu A_\mu(x))^2 + m^2 A_\nu(x)A^\nu(x) \Bigg\} \tag{7.1}$$

The equation of motion are

$$-\partial^\nu\Big(\partial^\gamma A_\gamma(x)\Big) + m^2 A^\nu(x) = 0 \tag{7.2}$$

From which a massive pole can be found

$$\Big(\Box - m^2\Big)\partial^\nu A_\nu(x) = 0 \tag{7.3}$$

where $\partial^\nu A_\nu(x)$ represents a scalar massive degree of freedom.

The Dirac-Bergmann Hamiltonian analysis [6] can be used as a powerful method to analyze the system's degrees of freedom. So, we define the canonical momenta

$$\pi^0 = \frac{\partial\mathcal{L}}{\partial(\dot{A}_0)} = \dot{A}^0 + \partial^i A_i \quad , \quad \pi^i = 0 = \frac{\partial\mathcal{L}}{\partial(\dot{A}_i)} \tag{7.4}$$

The fundamental Poisson brackets are

$$\Big\{A_\mu(x), \pi^\nu(y)\Big\} = \delta^\nu_\mu \delta^{D-1}(x-y) \tag{7.5}$$

The canonical Hamiltonian reads

$$H_c = \int d^{D-1}x \Big\{ \frac{1}{2}(\pi_0)^2 - \pi^0(\partial^i A_i(x)) - \frac{1}{2}m^2 A_\mu(x)A^\mu(x) \Big\} \tag{7.6}$$

Then, the primary Hamiltonian takes the form, where λ^i are Lagrange multipliers for the primary fields.

$$H_p = H_c + \int d^{D-1}x \ \lambda^i \ \pi_i \tag{7.7}$$

The consistency condition for the constraints are

$$\dot{\pi}^i(x) = \Big\{ \pi^i(x), H_p \Big\} = -\partial^i \pi^0(x) - m^2 A^i(x) \approx 0 \tag{7.8}$$

The above condition leads to the secondary constraint $\phi^i(x) = -\partial^i \pi^0(x) - m^2 A^i(x)$. The vanishing of its time evolution determines the lagrangian multipliers

$$\dot{\phi}^i(x) = \{\phi^i(x), H_p\} = -m^2 \partial^i A_0(x) - m^2 \lambda^i(x) \approx 0 \tag{7.9}$$

Then, all Lagrange multipliers are determined which is characteristic of a system without gauge symmetry. We also had $2D$ phase space degrees of freedom and found $2(D-1)$ second class constraints, yielding just 2 remaining degrees of freedom. The configuration space is therefore of a scalar particle.

Let's see what happens in the massless case and how the existence of reducibility relations between the constraints turn the system trivial in the massless limit. The system now has a gauge symmetry of the form

$$\delta A_\mu(x) \to \partial^\nu T_{[\nu\mu]}(x) \tag{7.10}$$

where $T_{[\nu\mu]}(x)$ denotes a general antisymmetric tensor.

The canonical momenta and the primary constraints are the still the same.

The primary Hamiltonian lies below

$$H_p = \int d^{D-1}x \Big\{ \frac{1}{2}(\pi_0)^2 - \pi^0(\partial^i A_i) \Big\} + \int d^{D-1}x \ \lambda^i \ \pi_i \tag{7.11}$$

The consistency condition for the primary constraints is

$$\dot{\pi}^i(x) = \Big\{ \pi^i(x), H_p \Big\} = -\partial^i \pi^0(x) \approx 0 \tag{7.12}$$

It furnishes us a secondary constraint

$$\gamma^i(x) = \partial^i \pi^0(x) \approx 0 \tag{7.13}$$

In this point we find the source of the aforementioned discontinuity. In the massive case, the constraint above was second class and then removes $D-1$ degrees of freedom. In most cases, the massless limit of the theory becomes first class and the double of these degrees of freedom were suppose to be removed. But the fact is that this constraint becomes just a scalar one since it is the derivative of a scalar function. It lies in the zero mode of a transverse spatial operator. This is our particular example of reducibility relations, this constraint is projected in just one linear independent spin projector.

The constraint temporal evolution is automatically fulfilled leaving the Lagrange multipliers undetermined, which is characteristic feature of gauge systems

$$\dot{\gamma}^i(x) = \Big\{\gamma^i(x), H_p\Big\} \approx 0 \tag{7.14}$$

Regarding the degree of freedom counting, we have $D-1$ constraints plus a scalar one. Since the system is of first class, each of them takes off 2 phase space elements. So, in the end of the day, we are left with zero degrees of freedom and not a massless scalar one, as expected. This kind of discontinuity was analyzed for an antisymmetric tensor model [7] in which a Hamiltonian and unitarity analysis were done. There are considerable examples of models described by longitudinal modes of higher order tensor that may present these discontinuities. They appear in the context of effective descriptions of QCD [5], and also in the string theory one as happens for the Kalb-Ramond model [10].

7.3 The B field formalism for a vector reducible model

Our intention is to quantize the spin-0 system represented by the vector field $A_\mu(x)$. Since its dynamics lies in longitudinal modes it would be interesting to use an auxiliary lagrange multiplier field to fix a condition of the form $F_{\mu\nu}(A)(x) = 0$ [1]. Although the massive field do not have gauge symmetry, fixing such a condition do not affect the dynamics (since it is compatible with the field equations.) and is a way to avoid first class ambiguities by turning the system into a second class one by the beginning. This same principle was used to develop a well-defined quantization for the Proca model [1].

Our first guess for the system's lagrangian would be

$$\mathcal{L} = \frac{1}{2}\Big((\partial_\mu A^\mu)^2 - m^2 A_\mu A^\mu + B^{\mu\nu}F_{\mu\nu}(A)\Big) \tag{7.15}$$

First of all, we should calculate the canonical momenta and verify that indeed no ambiguity remain. They are given by

$$\pi_A^0 = \partial_0 A^0 + \partial^i A_i \quad , \quad \pi_A^k = B^{0k} \quad , \quad \pi_B^{ij} = \pi_B^{0i} = 0 \tag{7.16}$$

We note that almost all constraints are second class ones but $\pi_B^{ij} = 0$ is not. Therefore, in order to obtain a second class system suitable for the use of the corresponding principle, we must consider an extra set of auxiliary fields. In D dimensions, we have

$$\mathcal{L}_{aux.} = \epsilon^{\mu\nu\alpha_1 \ldots \alpha_{D-2}} F_{\mu\nu}(A)\Gamma_{\alpha_1 \ldots \alpha_{D-2}} + \partial^{\alpha_1}\Gamma_{\alpha_1 \ldots \alpha_{D-2}}\bar{\Gamma}^{\alpha_2 \ldots \alpha_{D-2}} + \ldots + \partial^{\alpha_{D-3}}\Omega_{\alpha_{D-3}\alpha_{D-2}}\phi^{\alpha_{D-2}} + b\partial_\mu\phi^\mu \tag{7.17}$$

in which the fields have appropriated parity under discrete symmetry transformations to ensure a scalar nature to the lagrangian. The auxiliary tensor fields with rank ≥ 2 are totally antisymmetric.

For simplicity, we consider the $D = 2+1$ dimensions since, in this case, just two auxiliary fields are necessary. Therefore, we have

$$\mathcal{L} = \frac{1}{2}\Big((\partial_\mu A^\mu)^2 - m^2 A_\mu A^\mu\Big) + \epsilon^{\rho\mu\nu}\phi_\rho\partial_\nu A_\mu + b\partial^\mu\phi_\mu \tag{7.18}$$

[1] We have the definition $F_{\mu\nu}(A)(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$

where the non-gauge fixed massless limit is invariant under the local transformation $A_\mu(x) \to A_\mu(x) + \epsilon_{\mu\nu\beta}\partial^\nu \Lambda^\beta(x)$.
In order to have a parity invariant lagrangian, $\phi_\mu(x)$ and $b(x)$ should be a pseudo vector and a pseudo scalar, respectively [2].

The canonical momenta reads

$$\pi_A^0 = \partial_0 A^0 + \partial^i A_i \quad , \quad \pi_A^k = \epsilon^{k0i}\phi_i \quad , \quad \pi_k^\phi = 0 \quad , \quad \pi^b = 0 \quad ; \quad \pi_0^\phi = b \tag{7.19}$$

The theory is of second class from the beginning and we learn that quantizing theories whose dynamical poles lie in longitudinal modes may be possibly trickier than the conventional ones due to the necessity of adding extra auxiliary fields. The introduction of new auxiliary fields must be followed by a procedure to eliminate them from the observable positive Hilbert subspace. Further we are going to introduce a subsidiary condition that can fulfil these requirements.

The equations of motion are the following

$$\begin{aligned} \epsilon^{\mu\nu\rho}\partial_\nu A_\rho &= \partial_\mu b \\ \partial^\mu \phi_\mu &= 0 \\ \partial_\mu(\partial^\beta A_\beta) + m^2 A_\mu &= \epsilon_{\mu\nu\rho}\partial^\nu \phi^\rho \end{aligned} \tag{7.20}$$

Applying the following differential operator $\epsilon^{\beta\omega\mu}\partial_\omega$ on the last equation above, we find

$$m^2 \epsilon^{\beta\omega\mu}\partial_\omega A_\mu = -\Box \phi^\beta \tag{7.21}$$

This result can be written, by means of the vector field equation of motion, in the form $-\Box\phi^\beta = m^2 \partial^\beta b$. If we take the divergence of the first equation we obtain a harmonic equation for the auxiliary field $b(x)$

$$\Box b(x) = 0 \tag{7.22}$$

This results implies that

$$\Box^2 \phi_\beta(x) = 0 \tag{7.23}$$

It means that the $\phi_\beta(x)$ can be decomposed in positive and negative frequency parts and can possibly provide a good subsidiary condition for defining the positive metric Hilbert subspace.

The divergence of the last equation gives the expected result defining a massive spin 0 excitation

$$(\Box + m^2)\partial^\mu A_\mu = 0 \tag{7.24}$$

This second class system leads to well-defined brackets suitable for the use of the correspondence principle. The primary constraints can be used in the strong form. The non-vanishing brackets are

$$\left\{ A_0(x), \pi_A^0(y) \right\} = \delta^2(x-y) \tag{7.25}$$

$$\left\{ A_i(x), \epsilon^{k0j}\phi_j(y) \right\} = \delta_i^k \delta^2(x-y) \tag{7.26}$$

$$\left\{ \phi_i(x), \pi_\phi^j(y) \right\} = \delta_i^j \delta^2(x-y) \tag{7.27}$$

$$\left\{ \phi_0(x), b(y) \right\} = \delta^2(x-y) \tag{7.28}$$

$$\tag{7.29}$$

[2] In order to have a lagrangian that transforms as a scalar under discrete symmetries.

the correspondence principle allows us to obtain the equal time commutators

$$\Big[A_0(x), \partial_0 A_0(y)\Big]_0 = i\delta^2(x-y) \tag{7.30}$$

$$\Big[A_\mu(x), \epsilon^{k0i}\phi_k(y)\Big]_0 = i\delta^i_\mu \delta^2(x-y) \tag{7.31}$$

$$\Big[\phi_0(x), b(y)\Big]_0 = i\delta^2(x-y) \tag{7.32}$$

The knowledge of the i spatial component of the last equation of motion above allied to the information from the equal time commutators, yields

$$\Big[A_\mu(x), \epsilon^{i0k}\partial_0\phi_k(y)\Big]_0 = -i\delta^0_\mu \partial^i \delta^2(x-y) \tag{7.33}$$

The equal time commutators between $A_\mu(x)$ and $\phi_\mu(x)$ can be put in the following compact forms $\Big[A_m(x), \phi_z(y)\Big]_0 = i\epsilon_{mz}\delta^2(x-y)$ and $\Big[A_\mu(x), \partial_0\phi_z(y)\Big]_0 = -i\delta^0_\mu \epsilon_{zi}\partial^i\delta^2(x-y)$.

The transverse condition on $\phi_\mu(x)$ helps us to show

$$\Big[A_\mu(x), \partial_0\phi_0(y)\Big]_0 = -i\epsilon_{mz}\partial^z\delta^2(x-y)\delta^m_\mu \tag{7.34}$$

. Now, since $\Big[A_\mu(x), b(y)\Big] = 0$ (as we are going to prove.), the equations of motion implies

$$\Box\Big[A_\mu(x), \phi_\nu(y)\Big] = 0 \tag{7.35}$$

Then, using the integral representation

$$\begin{aligned} F(x-y) = &\int d^3u\ \varepsilon(x,y,u)\tau(x-u)G(u-y) \\ &- \int d^2u \Big[\tau(x-u)\partial^u_0 F(u-y) - \partial^u_0\tau(x-u)F(u-y)\Big]_{u^0=y^0} \end{aligned} \tag{7.36}$$

where $\varepsilon(x,y,u)$ is defined as

$$\varepsilon(x,y,u) = \Theta(x_0-u_0) - \Theta(y_0-u_0). \tag{7.37}$$

for an operator obeying the relation $\hat{O}^x F(x-y) = G(x-y)$ such that $\hat{O}^x\tau(x-y) = 0$, with $\hat{O}^x$ being a differential operator, and considering the distribution properties below

$$\begin{aligned} &\Box\Delta(x-y;s) = -s\Delta(x-y;s), \quad \Delta(x-y;s)|_0 = 0, \quad \partial^x_0\Delta(x-y;s)|_0 = \delta^2(x-y), \\ &\big(\Box+s\big)E(x-y;s) = \Delta(x-y;s), \quad E(x-y;s)|_{x_0=y_0} = 0, \quad (\partial^x_0)^3 E(x-y;s)|_{x_0=y_0} = \delta^2(x-y). \end{aligned} \tag{7.38}$$

we can provide, given the previously calculated initial data, the following commutator at unequal times

$$\Big[A_\mu(x), \phi_\Omega(y)\Big] = -i\epsilon_{\mu\Omega\beta}\partial^\beta\Delta(x-y) \tag{7.39}$$

It is interesting to note that the auxiliary field is the generator of the residual gauge symmetry that occurs in the massless limit in analogy to the mentioned Proca theory case [1]. Then, it can provide subsidiary conditions to define the physical Hilbert space.

Regarding the $b(x)$ field, the integral representation of the harmonic operator $b(x)$ and its initial conditions furnishes

$$\Big[b(x),\ b(y)\Big] = 0 \tag{7.40}$$

. This field also commute with $A_\mu(x)$ since, due to the correspondence principle, its time derivative commutes with the vector field at equal times. The integral representation for this commutator would then imply that it is zero also for $x_0 \neq y_0$.

The equation of motion relating non-physical transverse sectors of $A_\mu(x)$ and $\dot{b}(x)$ and the initial conditions for commutators, gives

$$\Big[\partial_0 b(x), \phi_\nu(y)\Big]_0 = -i\delta^i_\nu \partial_i \delta^2(x-y) \tag{7.41}$$

The initial conditions for these fields allied to the equation of motion $\Box\Big[b(x), \phi_\mu(y)\Big] = 0$ leads, by means of the integral representation, to

$$\Big[b(x), \phi_\nu(y)\Big] = i\partial_\nu \Delta(x-y) \tag{7.42}$$

where the second auxiliary field do not commute with the longitudinal sector of the first one. It can provide a generator for the $\phi_\nu(y)$ field residual local longitudinal freedom.

Using the initial commutator conditions for $\phi_\mu(x)$ and $A_\mu(x)$, we consider the i spatial component of the vector spin 0 field equation of motion to derive

$$\Big[\partial_0 \phi^j(x), \phi_m(y)\Big]_0 = -m^2 \delta^j_m \delta^2(x-y) \tag{7.43}$$

we also consider that the divergence of $A_\mu(x)$ is proportional to $\pi^0_A(x)$.
Using the transverse condition $\partial^\mu \phi_\mu = 0$ in the above equation, we find

$$\Big[\phi_\mu(x), \phi_\beta(y)\Big] = im^2\Big(\eta_{\mu\beta}\Delta(x-y) - \partial_\mu\partial_\beta E(x-y)\Big) \tag{7.44}$$

This result was obtained considering a general distribution annihilated by $\Box^2$, transverse and in agreement with the initial conditions.

The previous results shows that $\phi_\mu(x)$ has non-vanishing commutators just with the non-physical gauge sectors of the vector field. So, a good subsidiary condition, in analogy to the QED_4 case [16], is

$$\phi^+_\mu(x)|\text{phys}\rangle = 0, \quad \forall|\text{phys}\rangle \in \mathcal{H}_{\text{phys}}. \tag{7.45}$$

where + denotes the positive frequency part of the field. That this condition is Poincaré invariant, it follows from the action of this group on the functional form of the vector, that can be written in terms of its components, and the orbital part composed of derivatives acting on it. Since the states are not space time dependent in the Heisenberg description, the mentioned invariance is achieved as a consequence of the definition of $\mathcal{H}_{\text{phys}}$.

Besides the pure gauge sectors of the vector field $A_\mu(x)$ this condition also eliminates $b(x)$ from the observable spectrum.

Regarding the $A_\mu(x)$ field commutator, a structure that respects the constraints from the equations of motion is

$$\Big[A_\mu(x), A_\nu(y)\Big] = \partial_\mu\partial_\nu\Big(a\Delta(x-y,m^2) + b\Delta(x-y,0)\Big) \tag{7.46}$$

in order to be compatible with the initial conditions we must have $a = -b = \frac{i}{m^2}$.

Another way to understand the behaviour of the Hilbert space at this limit it is to verify the commutator between the massive pole functions $\partial_\mu A^\mu(x)$. It has the following form

$$\Big[\partial^\mu A_\mu(x), \partial^\nu A_\nu(y)\Big] = im^2\Delta(x-y, m^2) \tag{7.47}$$

which present a positive norm that vanishes in the massless limit.

7.4 Spin 1 field via an antisymmetric tensor, a B field quantization

The model that going to be analyzed has some analogies with the one of the previous section. Namely, it massless limit also has reducible constraints and, in this case, taking this limit lead us from a spin 1 theory to a spin 0 one. In [7] we did a Dirac-Bergman analysis followed by an unitarity inspection in D space-time dimensions which revealed the decrease in the degrees of freedom and poles that moved from spin 1 sectors to the spin 0 in the massless limit.

Here, we provide the indefinite metric quantization, find the right number of necessary auxiliary fields and how to get rid of them from the observable spectrum.

The full lagrangian with the addition of auxiliary pseudo vector and pseudo scalar fields, in $D = 3+1$ dimensions, reads

$$\mathcal{L} = -\Big(\partial^\mu B_{\mu\nu}(x)\Big)^2 + \frac{m^2}{2}B_{\mu\nu}(x)B^{\mu\nu}(x) + \epsilon^{\mu\nu\alpha\beta}\phi_\mu(x)\partial_\nu B_{\alpha\beta}(x) + \partial^\mu\phi_\mu(x)\Omega(x) \tag{7.48}$$

the non-gauge fixed massless lagrangian is invariant under $B_{\mu\nu}(x) \to B_{\mu\nu}(x) + \epsilon_{\mu\nu\beta\gamma}\partial^\beta\Lambda^\gamma(x)$.

The canonical momenta are the following

$$\pi_{0i}(x) = -2\Big(\partial_0 B_{0i} + \partial^j B_{ji}(x)\Big) \tag{7.49}$$

$$\pi_{ij}(x) = \epsilon_{0ijk}\phi^k(x) \tag{7.50}$$

$$\pi_\mu^{(\phi)}(x) = \delta_0^\mu\Omega(x) \tag{7.51}$$

$$\pi^{(\Omega)}(x) = 0 \tag{7.52}$$

We can see that this lagrangian really ensures that the system is a second class one from the beginning, avoiding any quantum ambiguity.

The system's equations of motion are

$$0 = -\epsilon^{\mu\nu\alpha\beta}\partial_\nu\phi_\mu + \big(\partial^\alpha(\partial_\nu B^{\nu\beta}) - \partial^\beta(\partial_\nu B^{\nu\alpha})\big) + m^2 B^{\alpha\beta} \tag{7.53}$$

$$0 = \partial^\mu\phi_\mu \tag{7.54}$$

$$0 = \epsilon^{\mu\nu\alpha\beta}\partial_\nu B_{\alpha\beta} - \partial^\mu\Omega \tag{7.55}$$

The divergence of the first equation leads to the propagating $s = 1$ massive mode

$$\Big(\Box + m^2\Big)\partial^\nu B_{\nu\mu}(x) = 0 \tag{7.56}$$

The divergence of the last equation of (8.1) furnishes a d'Alembert equation for the scalar auxiliary field

$$\Box\Omega(x) = 0 \tag{7.57}$$

In order to find other useful relations, we apply the following differential operator $\epsilon_{\alpha\beta\sigma\omega}\partial^{\omega}$ in the first and use also the transverse nature of $\phi_\mu(x)$. It then reads

$$\Box^2\phi_\mu(x) = 0 \tag{7.58}$$

The second class nature of the system leads to well-defined brackets in which the constraints are valid in the strong form. The non-vanishing ones are

$$\Big\{B_{ij}(x), \epsilon_{0mnk}\phi^k(y)\Big\} = \frac{\Big(\delta_i^m\delta_j^n - \delta_i^n\delta_j^m\Big)}{2}\delta^3(x-y) \tag{7.59}$$

$$\Big\{B_{0i}(x), \pi^{0k}(y)\Big\} = \delta_i^k\delta^3(x-y) \tag{7.60}$$

$$\Big\{\phi^i(x), \pi_j^{(\phi)}(y)\Big\} = \delta_j^i\delta^3(x-y) \tag{7.61}$$

$$\Big\{\phi_0(x), \Omega(y)\Big\} = \delta^3(x-y) \tag{7.62}$$

$$\tag{7.63}$$

From, the correspondence principle, we have

$$\Big[B_{0l}(x), \partial_0 B^{0i}(y)\Big]_0 = -\frac{i}{2}\delta_l^i\delta^3(x-y) \tag{7.64}$$

$$\Big[B^{kl}(x), \epsilon_{m0ij}\phi^m(y)]_0 = i\frac{\Big(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k\Big)}{2}\delta^3(x-y) \tag{7.65}$$

$$\Big[B^{kl}(x), \phi^m(y)\Big]_0 = -\frac{i}{2}\epsilon^{klm}\delta^3(x-y) \tag{7.66}$$

$$\Big[\phi_0(x), \Omega(y)\Big]_0 = i\delta^3(x-y) \tag{7.67}$$

The equations of motion can be used to relate $-\epsilon^{k0ij}\partial_0\phi_k(x)$ with $m^2B^{ij}(x)$ such that the initial conditions furnishes

$$\Big[\partial_0\phi_l(x), \phi^k(y)\Big]_0 = -\frac{i}{2}m^2\delta_k^l\delta^3(x-y) \tag{7.68}$$

we also considered that the divergence $\partial_\mu B^{\mu i}(x)$ is proportional to $\pi_{0i}(x)$.

From the correspondence principle, considering the operator equation due to the auxiliary vector field leads to $\Big[\partial_0\Omega(x), \Omega(y)\Big]_0 = 0$. Then, this initial condition implies in $\Big[\Omega(x), \Omega(y)\Big] = 0$ as we can deduce from the integral representation for $\Box\Big[\Omega(x), \Omega(y)\Big] = 0$.

Using the gauge condition to relate $\partial_0\Omega$ with spatial derivatives of the tensor field leads to a vanishing equal time commutator between this field and the scalar one. Since this commutator must be in the kernel of $\Box^x$, the integral representation and the initial condition implies in zero commutator also for $x_0 \neq y_0$.

The equation for the commutator $\Box^2\Big[\phi_\mu(x), \phi^\nu(y)\Big] = 0$ is solved by the most general linear combination of functions annihilated by the differential operator constrained by the fact that it must be transverse and obey the initial conditions. It is then given by

$$\begin{aligned}\Big[\phi_\mu(x), \phi_\nu(y)\Big] =& a(\eta_{\mu\nu}\Delta(x-y,0) - \partial_\mu\partial_\nu E(x-y,0)) + b\partial_\mu\partial_\nu\Delta(x-y,0) \\ &+ d\eta_{\mu\nu}E(x-y,0) + e\partial_\mu\partial_\nu E(x-y)\end{aligned} \tag{7.69}$$

The transverse nature of ϕ_μ leads to the initial condition $\Big[\phi_0(x), \partial_0\phi_0(y)\Big]_0 = 0$. It implies that $b = e = 0$. By the other hand

$$\Big[\partial_0\phi_l(x), \phi^k(y)\Big]_0 = -\frac{i}{2}m^2\delta^l_k\delta^3(x-y) \tag{7.70}$$

fixes $a = \frac{i}{2}m^2$. So, owing to the transverse constraint, we must have

$$\Big[\phi_\mu(x), \phi_\nu(y)\Big] = \frac{i}{2}m^2\Big(\eta_{\mu\nu}\Delta(x-y,0) - \partial_\mu\partial_\nu E(x-y,0)\Big) \tag{7.71}$$

Therefore, the auxiliary field projections cannot be detected in the physical spectrum. as it should be.

The commutator $\Big[\phi_\mu(x), \Omega(y)\Big] = -i\partial_\mu\Delta(x-y,0)$ is obtained by means of the $\Omega(x)$ integral representation in a complete analogy to the case of the previous section.

In order to find the commutator between the $\phi_\mu(x)$ and $B_{\mu\nu}(x)$ fields, we must take into account the fact that acting with $\epsilon_{\alpha\beta\sigma\omega}\partial^\sigma$ at the $B_{\mu\nu}(x)$ equation of motion would give us the following operator relation

$$\Box\phi_\sigma(x) + m^2\epsilon_{\sigma\omega\alpha\beta}\partial^\omega B^{\alpha\beta}(x) = 0 \tag{7.72}$$

. Then, using the auxiliary condition for the antisymmetric field yields

$$\Box\phi_\mu(x) + m^2\partial_\mu\Omega(x) = 0 \tag{7.73}$$

Therefore, we can find the commutator equations $\Box^y\Big[B_{\mu\nu}(x), \phi_\alpha(y)\Big] = 0$. The integral representation developed in the last section furnishes

$$\Big[B_{\mu\nu}(x), \phi_\alpha(y)\Big] = \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\beta\Delta(x-y,0) \tag{7.74}$$

The analogy with the previous section is remarkable. Since the auxiliary field indeed generates the residual symmetry for the massless case, it can provide an useful subsidiary condition for the physical subspace even in the massive case.

The condition below, avoids observing non-physical sector projections of the antisymmetric field and also avoids the emergence of the auxiliary $\Omega(x)$ field in the physical space

$$\phi^+_\mu(x)|\text{phys}\rangle = 0, \quad \forall|\text{phys}\rangle \in \mathcal{H}_{\text{phys}}. \tag{7.75}$$

Then, it easily follows that $\partial_\mu B^{\mu\nu}(x)$ lies in the physical subspace.

In order to find the commutator $\Big[B_{\mu\beta}(x), B_{\nu\gamma}(y)\Big]$ we must take into account the condition from the equations of motion $\partial_\mu B_{\nu\alpha}(x) + \partial_\alpha B_{\mu\nu}(x) + \partial_\nu B_{\alpha\mu}(x) = 0$, so that the answer must be proportional to the tensor given below [3]

$$\begin{aligned}\Big[B_{\mu\nu}(x), B^{\alpha\lambda}(y)\Big] =& \Big(\eta^\alpha_\mu\partial_\nu\partial^\lambda + \eta^\lambda_\nu\partial_\mu\partial^\alpha - \eta^\lambda_\mu\partial_\nu\partial^\alpha - \eta^\alpha_\nu\partial_\mu\partial^\lambda\Big)\Big(a\Delta(x-y,m^2) + b\Delta(x-y,0)\\ &+ cE(x-y,0) + dE(x-y,m^2)\Big)\end{aligned} \tag{7.76}$$

[3]The operator condition involves the $\Omega(x)$ field but since it commute with the tensor field, the condition to be taken for the $B_{\nu\gamma}(x)$ commutator is indeed this one.

For the system to fulfil the massive pole equation $\Big(\Box + m^2\Big)\Big[\partial^\nu B_{\nu\mu}, B_{\alpha\gamma}(y)\Big] = 0$ we must have $c = d = 0$. The agreement with the initial condition $\Big[B_{0l}, \partial_0 B^{0i}\Big]_0 = -i/2\delta^i_l \delta^3(x-y)$, fixes $a = -b$ and $a = -\frac{i}{2m^2}$. So, we are left with

$$\Big[B_{\mu\nu}(x), B^{\alpha\lambda}(y)\Big] = -\frac{i}{2m^2}\Big(\eta^\alpha_\mu \partial_\nu \partial^\lambda + \eta^\lambda_\nu \partial_\mu \partial^\alpha - \eta^\lambda_\mu \partial_\nu \partial^\alpha - \eta^\alpha_\nu \partial_\mu \partial^\lambda\Big)\Big(\Delta(x-y, m^2) - \Delta(x-y, 0)\Big) \tag{7.77}$$

In the massless limit, considering the expansion $\Delta(x-y, m^2) = \Delta(x-y, 0) - m^2 E(x-y, 0) + ...$, the commutator is given in terms of the double pole distributions $E(x-y, 0)$. We are going to see that, in this case, some physical content remains, although different from the massive case.

It is instructive to analyze the commutator between the massive pole fields

$$\Big[\partial^\mu B_{\mu\nu}, \partial^\alpha B_{\alpha\lambda}\Big] = -\frac{i}{2m^2}\Big(m^4 \eta_{\nu\lambda} + m^2 \partial_\nu \partial_\lambda\Big)\Delta(x-y, m^2) \tag{7.78}$$

The commutator indicates a spin 1 content. We can also notice that the massless case has a longitudinal scalar nature

$$\Big[\partial^\mu B_{\mu\nu}, \partial^\alpha B_{\alpha\lambda}\Big] = -\frac{i}{2}\partial_\nu \partial_\lambda \Delta(x-y, 0) \tag{7.79}$$

This is in agreement with the analysis of [7] where the transition from the spin 1 to the spin 0 regime at the massless limit was investigated by phase space degree of freedom counting and also the analysis of the residue of the saturated amplitude of poles that move from the spin 1 to the spin 0 sector at the mentioned limit.

Regarding the nature of the projections of the vector pole in the massive/massless phase it is useful to consider explicitly their Fourier component algebra since the analysis is not straightforward as in the previous section. For the massive case, we have

$$\Big[\tilde{a}_\mu(p), \tilde{a}^\dagger_\nu(q)\Big] = \frac{1}{2}\theta(p_0)\big(-m^2\eta_{\mu\nu} + p_\mu p_\nu\big)\delta(p^2 - m^2)\delta^4(p-q) \tag{7.80}$$

Assuming the frame $p_\mu = \big(m, 0, 0, 0\big)$, we find that the three spatial components have positive norm and the temporal one has null projection. Since the vector pole is transverse by construction we also have, in accordance with this last claim, that $\tilde{a}_0(p) = 0$. This is compatible with the spin 1 content of this phase. Its commutator structure is in agreement with the fact that there is a master action correspondence between the divergence of the tensor field and the Proca one as $A_\mu \longleftrightarrow \frac{\sqrt{2}}{m}\partial^\lambda B_{\lambda\mu}$, see [7]. Since this pole field commutes with the auxiliary one defining the subsidiary condition, it indeed should have no negative projections.

For the massless limit, we have

$$\Big[\tilde{a}_\mu(p), \tilde{a}^\dagger_\nu(q)\Big] = \frac{1}{2}\theta(p_0)p_\mu p_\nu \delta(p^2)\delta^4(p-q) \tag{7.81}$$

Considering the frame $p_\mu = \big(p_3, 0, 0, p_3\big)$, we note that both the zeroth and the third component has positive projections while the others vanish. Due to the transverse constraint on this field these components are linearly dependent and therefore there is just one degree of freedom in accordance with the spin 0 nature of the massless phase.

7.5 Path integral representation

In this section we intend to evaluate the massless limits of the previously analyzed models. Although in an Abelian theory the contribution from the ghosts can be factorized, here we want to obtain a gauge fixed action with all the required ghosts to count degrees of freedom from the partition function and compare with the conclusions of the Heisenberg analysis. Since in the present case there is the necessity of implementing ghosts of ghosts due to the constraint reducibility, this procedure may be instructive. Also, we show that using the B-field formalism prescription of avoiding any first class ambiguity by the beginning, we can obtain the whole structure of ghosts of ghost by simply adding suitable auxiliary fields and imposing the equations defining the residual local symmetries as operator ones.

Regarding our intentions, the procedure of evaluating the complete ghost of ghost structure and its path integral[4] is to show the usefulness of B field formalism in this context and also that the expected degrees of freedom from constraint and the Heisenberg analysis can also be obtained in this alternative approach.

The massless vector field case is the first one to be analyzed. Varying its gauge condition with respect to the symmetry parameter [5] we find that it must respect the following equation

$$\Big[\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu\Big]\Lambda^\nu(x) = 0 \tag{7.82}$$

Although the system is Abelian, it is useful to provide a BRST structure to the system in order to define the positive physical Hilbert space. Then, we promote the c number parameter $\Lambda^\nu(x)$ to a Grassmann field $C^\nu(x)$ in order to impose the above equation as an operator equation of motion. To this end we must add another auxiliary field. The necessity for a grasmannian nature is to obtain a conserved BRST current [16] that is going to define the physical subspace. We will show that this definition allows us to get rid of the extra auxiliary fields from the physical spectrum.

The lagrangian is given below

$$\mathcal{L} = \frac{1}{2}\Big(\partial_\mu A^\mu\Big)^2 + \epsilon^{\rho\mu\nu}\phi_\rho\partial_\nu A_\mu + b\partial^\mu\phi_\mu - i\bar{C}^\mu\Big(\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu\Big)C^\nu - i\bar{\gamma}\partial_\mu C^\mu \tag{7.83}$$

The extra Grassmann field $\bar{\gamma}(x)$, which should have ghost number $+1$, is necessary to provide a system without any first class ambiguities since the ghost field differential operator is not invertible and leads to first class primary constraints. The $\bar{C}(x)$ is a Lagrange multiplier with appropriate ghost number. Additionally, the system is also invariant by the ghost number symmetry transformation

$$C_\mu(x) \to e^\theta C_\mu(x),\ \bar{C}_\mu(x) \to e^{-\theta}\bar{C}_\mu(x),\ \bar{\gamma} \to e^{-\theta}\bar{\gamma} \tag{7.84}$$

where θ is a constant c number parameter. So, the auxiliary field is also Grassmannian and transform accordingly under this symmetry.

The above structure can be put on a BRST language. The auxiliary sector can be partially written as

$$Q_B\Big(-i\bar{C}^\mu\epsilon_{\mu\nu\alpha}\partial^\nu A^\alpha - ib\partial_\mu\bar{C}^\mu\Big) \tag{7.85}$$

where Q_B generates the following symmetry transformation

$$Q_B\bar{C}_\nu(x) = i\phi_\nu(x) \quad , \quad Q_B A_\mu(x) = \epsilon_{\mu\nu\alpha}\partial^\nu C^\alpha(x) \tag{7.86}$$

[4] The path integral formalism is compatible with the approach of this article as we can see in [1] and [16].

[5] The massless limit of the model is invariant under variations of the form $\delta A_\mu = \epsilon_{\mu\nu\rho}\partial^\nu\Lambda^\rho$

It is nilpotent and will help us to define a physical subspace. The i factor in front of Grassmann ghost term is to ensure an Hermitian nature to the lagrangian. Although just part of the action can be written as a BRST coboundary term, since $Q_B^2 = 0$ and the remaining fields are invariant under the action of this charge, the system is indeed BRST invariant.

Another important point is that the BRST charge raises the ghost number eigenvalue in one unit. It is possible to show that a projection between any two fields of ghost number eigenvalues M and N is proportional to δ_{M-N}. The physical subspace is defined as

$$Q_B|\text{phys}\rangle = 0, \quad \forall|\text{phys}\rangle \in \mathcal{H}_{\text{phys}} \equiv \frac{\mathcal{V}}{\mathcal{V}_0}. \tag{7.87}$$

where the null norm subspace has the form $\mathcal{V}_0 \equiv Q_B|a\rangle$ with $|a\rangle$ being a non-physical state.

These properties are responsible for confining the auxiliary fields ($A^t_\mu(x)$, C^t_μ, $\bar{C}_\mu(x)$ and $\phi_\mu(x)$, where t denotes the transverse part.) in a non-observable quartet-like configuration [1, 16].

We can group the remaining fields that do not transform under BRST symmetry, with the inclusion of two extra Bosonic fields, as

$$Q\Big(\bar{\beta}\partial_\mu C^\mu\Big) \tag{7.88}$$

where Q is a nilpotent charge that generates the transformation

$$Q\bar{\beta}(x) = -i\bar{\gamma}(x) \quad , \quad QC_\mu(x) = \partial_\mu\beta(x) \tag{7.89}$$

We are promoting the equation expressing the residual ghost local symmetry to an operator one, analogously to the usual procedure of adding ghosts.

Since it acts on the non-physical subspace, a consistency condition is $Q|\text{phys}\rangle = 0$. It anti-commutes with the BRST charge which opens the possibility to define a physical subspace that is indeed invariant by both symmetries. As an observation, it is worth mention that a BRST-like transformation involving the longitudinal sector of $\phi^\nu(x)$ necessarily have a generating charge that would not anti commute with the BRST one as revealed by its action on the $\bar{C}_\mu(x)$ ghost. This is the reason to do not consider this possibility here.

This new charge also has the property of raising the ghost number eigenvalue. This is an useful property related to the avoidance of the auxiliary fields in the physical spectrum due to the emergence of a quartet configuration. It is easy to see that it acts on the longitudinal sector of the vector ghost field, which is in the kernel of the BRST charge transformation, a characteristic of the reducible systems [12]. In order to have a consistent non-ambiguous quantization, this null sector must be carefully investigated. Since the action of Q raises the ghost number in one unit, these extra ghosts are commuting quantities, they are the so-called ghost of ghosts. Moreover, these characteristics are responsible to confine the auxiliary fields in quartet configurations.

The total lagrangian then reads

$$\mathcal{L} \;=\; \frac{1}{2}\Big(\partial_\mu A^\mu\Big)^2 + \epsilon^{\rho\mu\nu}\phi_\rho\partial_\nu A_\mu + b\partial^\mu\phi_\mu - i\bar{C}^\mu\Big(\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu\Big)C^\nu - i\bar{\gamma}\partial_\mu C^\mu + \bar{\beta}\Box\beta \tag{7.90}$$

Since the objective of our approach is to define a system with no first class ambiguities, we should point out the new constraints that appear in the model. They are the following

$$\pi_{\bar{\gamma}} = 0 \quad , \quad \pi^c_0 = -i\partial_i\bar{C}^i - i\bar{\gamma} \quad , \quad \pi^{\bar{c}}_0 = -i\partial_i C^i \tag{7.91}$$

where we have used integration by parts to express the ghost lagrangian as a functional depending just on the field's first derivatives. Adding a surface term may change the momentum definition but it do not change the first/second class nature of the associated primary constraints.

If we consider the constraint imposed by the $\bar{\gamma}(x)$ equation of motion on all Hilbert space, namely $\partial_\mu C^\mu(x) = 0$, the last momentum definition is no more a constraint and becomes proportional to $\partial_0 C_0(x)$ while the remaining terms are second class constraints. Therefore, this system is free of first class structures.

Now we are going to perform the path integral over the fields and show that the expected behaviour for this massless limit (no degrees of freedom.) is recovered in this alternative approach, as it should be.

Taking into account the nature of the fields, the path integral after the integration of the ghost fields, up to normalization factors ,reads

$$\int \mathcal{D}\mathcal{A}\mathcal{D}\phi\mathcal{D}b(\det\Box)^3(\det\Box)^{-1}\exp i\int d^3x\Big(\frac{1}{2}A_\mu\Box A^\mu + A^\alpha\epsilon_{\alpha\beta\nu}F_\phi^{\beta\nu} - \partial_\mu b\phi^\mu\Big) \tag{7.92}$$

where $F_\phi^{\beta\nu} = \partial_\beta\phi_\nu - \partial_\nu\phi_\beta$.

The first term of the action is obtained when we consider that [6] $\partial_\mu\partial_\nu = \Box\eta_{\mu\nu} - \Box\Theta_{\mu\nu}$, and the constraint imposed by the auxiliary vector field which can be expressed as a functional delta function restricting all the quantum configurations.

The action can be rewritten as [8]

$$\int d^3x\Big[\frac{1}{2}\Big(A_\mu + \frac{\epsilon_{\alpha\beta\nu}F_\phi^{\beta\nu}}{\Box}\Big)\Box\Big(A_\mu + \frac{\epsilon_{\alpha\beta\nu}F_\phi^{\beta\nu}}{\Box}\Big) - \frac{F_\phi^{\beta\nu}F_{\beta\nu\ \ \phi}}{\Box} - \partial_\mu b\phi^\mu\Big] \tag{7.93}$$

Defining $X_\mu \equiv \Big(A_\mu + \frac{\epsilon_{\alpha\beta\nu}F_\phi^{\beta\nu}}{\Box}\Big)$ we get a $(\det\Box)^{-3/2}$ factor after integration, and considering the transverse nature of the $\phi_\mu(x)$ the remaining action term is

$$\int d^3x\Big(\phi_\mu\phi^\mu - \partial_\mu b\phi^\mu\Big) = \frac{1}{4}\int d^3x b\Box b \tag{7.94}$$

The relation above is due to Gaussian integration in $\phi_\mu(x)$. Then, after integrating this last term, we have the following product of determinants $(\det\Box)^3(\det\Box)^{-1}(\det\Box)^{-3/2}(\det\Box)^{-1/2}$, which give $\det\Box^0$ which is the correct result since constraint and Heisenberg picture analysis reveals that this massless limit has no degrees of freedom.

7.5.1 Massless antisymmetric tensor field model

The tools to perform the path integral and also the BRST quantization of the massless reducible models were introduced in the previous subsection. Therefore, our intention here is to apply such a procedure for the case of the massless limit of the spin 1 model via antisymmetric tensor which we know that describes a scalar degree of freedom due to the spin jumping in the massless limit [7].

Varying the gauge fixing condition with respect to the local gauge symmetry [7] furnishes

$$\Big[\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu\Big]\tilde{\Lambda}^\nu(x) = 0 \tag{7.95}$$

The situation is analogous to the previous section case since the vector symmetry parameter is required to fulfil the same equation which contains zero modes. Promoting this parameter to

[6] The $\Theta_{\mu\nu}$ is the transverse projection operator, defined in the appendix. The constraint due to the auxiliary vector field, which can be written as a path integral delta function, implies in $\Box\theta_{\mu\nu}A^\nu = 0$.

[7] The massless system is invariant under $\delta B_{\mu\nu} = \epsilon_{\mu\nu\gamma\rho}\partial^\gamma\Lambda^\rho$

a Grassmann ghost field $C^\nu(x)$, using a Lagrange multiplier field to get the former condition as an operator equation of motion, adding an auxiliary Grassmann field coupled to the divergence of these ghosts to avoid first class ambiguities leads to a theory whose auxiliary sector can be partially written as the sum of the following terms below

$$Q_B\Big(- i\bar{C}^\nu \epsilon_{\mu\nu\alpha\gamma}\partial^\nu B^{\alpha\gamma} - ib\partial_\mu \bar{C}^\mu\Big) \tag{7.96}$$

where Q_B generates the following symmetry transformation

$$Q_B\bar{C}^\nu(x) = i\phi_\mu(x) \quad , \quad Q_B B_{\mu\gamma}(x) = \epsilon_{\mu\gamma\nu\alpha}\partial^\nu C^\alpha(x) \tag{7.97}$$

It is nilpotent and will help us to define a physical subspace as

$$Q_B|\text{phys}\rangle = 0, \quad \forall|\text{phys}\rangle \in \mathcal{H}_{\text{phys}} \equiv \frac{\mathcal{V}}{\mathcal{V}_0}. \tag{7.98}$$

The remaining terms can be grouped as given below with the cost of adding a new set of ghosts of ghosts

$$Q\Big(\bar{\beta}\partial_\mu C^\mu\Big) \tag{7.99}$$

where Q generates the following transformation

$$Q\bar{\beta}(x) = -i\bar{\gamma}(x) \quad , \quad QC_\mu(x) = \partial_\mu\beta(x) \tag{7.100}$$

Both charges have the same properties as explained in the last subsection and the physical Hilbert space has the also the same definition. Regarding the quartet-like structure and the decomposition in transverse and longitudinal sectors of the fields follows the same logic as before. The only difference is considering $B^t_{\mu\nu}(x)$, the transverse sector of the anti-symmetric field, as one of the fields of the quartet structure related to the BRST charge.

The extended lagrangian has the form

$$\mathcal{L} = -\Big(\partial_\mu B^{\mu\gamma}\Big)^2 + \epsilon^{\rho\nu\mu\sigma}\phi_\rho\partial_\nu B_{\mu\sigma} + b\partial^\mu\phi_\mu - i\bar{C}^\mu\Big(\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu\Big)C^\nu - i\bar{\gamma}\partial_\mu C^\mu + \bar{\beta}\Box\beta \tag{7.101}$$

The fact that this lagrangian avoids any kind of quantum ambiguity has exactly the same explanation as given in the previously section.

The path integral, taking into account the nature of the auxiliary fields, up to normalization factors, reads

$$Z = \int \mathcal{D}B\mathcal{D}\phi\mathcal{D}b(\det\Box)^4(\det\Box)^{-1}\exp i\int d^4x\Big(B_{\mu\nu}\Box B^{\mu\nu} + B^{\alpha\gamma}\epsilon_{\alpha\gamma\beta\nu}F^{\beta\nu}_\phi - \partial_\mu b\phi^\mu\Big) \tag{7.102}$$

The fact that $B_{\mu\nu}\Box B^{\mu\nu}$ appears as the kinetic term is due to the auxiliary vector field that implement $\epsilon_{\alpha\beta\gamma\sigma}\partial^\beta B^{\gamma\sigma} - \partial_\alpha b = 0$. Since $\Big(\partial_\mu B^{\mu\gamma}\Big)^2$ can be written as [8]

$$B^{\mu\gamma}\Big(\Box\Delta^{\sigma\beta}{}_{\mu\gamma} - \Box P^{\sigma\beta[1b]}_{\mu\gamma}\Big)B_{\sigma\beta} \tag{7.103}$$

and the condition enforced by the auxiliary fields is equivalent to the vanishing of the second term in parenthesis, we arrive at the kinetic term above. We have defined $\Delta^{\sigma\beta}{}_{\mu\gamma} \equiv \frac{1}{2}\big(\delta^\sigma_\mu\delta^\beta_\gamma -$

[8]where the projection operator $P^{\sigma\beta[1b]}_{\mu\gamma}$ is defined in the appendix.

$\delta_\mu^\beta \delta_\gamma^\sigma$).

The action can be rewritten in the form

$$\int d^4x \Big[\frac{1}{2}\Big(B_{\mu\gamma} + \frac{\epsilon_{\mu\gamma\beta\nu} F_\phi^{\beta\nu}}{\Box}\Big)\Box\Big(B^{\mu\gamma} + \frac{\epsilon^{\mu\gamma\beta\nu} F_\phi^{\beta\nu}}{\Box}\Big) - \frac{F_\phi^{\beta\nu} F_{\beta\nu \ \ \phi}}{\Box} - \partial_\mu b \phi^\mu\Big] \tag{7.104}$$

Defining the field $X_{\mu\gamma} \equiv \Big(B_{\mu\gamma} + \frac{\epsilon_{\mu\gamma\beta\nu} F_\phi^{\beta\nu}}{\Box}\Big)$ and performing the Gaussian integration gives a factor $(\det\Box)^{-3}$ to the path integral. Owing to the transverse nature of $\phi_\mu(x)$, the action becomes

$$\int d^4x \Big(\phi_\mu\phi^\mu - \partial_\mu b \phi^\mu\Big) = \frac{1}{4}\int d^4x b\Box b \tag{7.105}$$

The equality above is due to the Gaussian integration. Now, integrating in b, we get the expression, up to normalization constants, for the partition function

$$(\det\Box)^4 (\det\Box)^{-1} (\det\Box)^{-1/2} (\det\Box)^{-3} \tag{7.106}$$

This partition function is indeed compatible with a scalar degree of freedom $(\det\Box)^{-1/2}$. This is in accordance with the previous Hilbert space analysis and also with the Hamiltonian and master action approach of [7].

7.6 Conclusions and Perspectives

Throughout this work we have discussed the quantization of models describing dual versions of spin 0 and spin 1 theories by means of vector and antisymmetric tensors, respectively. In the section 2 we analyzed the Hamiltonian aspects of the reducible symmetry structure of the spin 0 vector model. The next section presented some unusual features with relation to the standart B field quantization procedure, for example, the necessity of adding new auxiliary fields and the fact that its massless limit has no particle content.

The structure for the model described by the antisymmetric field in section 4 have a remarkable analogy to the vector one. The form of the auxiliary field commutators with the physical fields implies that they are related to the non-physical sectors which generates the redundant local symmetries in the massless case. Although the massive case has no gauge symmetry, it allows a suitable introduction of auxiliary fields in a well-defined manner as in the Proca theory model [1]. The model presents a spin 1 particle content in the massive phase and a spin 0 in the massless one. We carefully analyzed the change in the Hilbert space content at this limit.

The derivation of the partition function for the massless limit was done in section 5. The ghost of ghost structure arised from the requirement of avoiding first class ambiguities by adding new auxiliary fields. They are confined in a non-observable quartet structure due to the BRST symmetry and the remaining set of auxiliary fields are also contained in another quartet structure due to the existence of a BRST-like structure acting on the reducible sectors which lies in the kernel of the transformations of the BRST charge.

The results from this last section confirmed some of the previous conclusions and is, by itself, an interesting achievement since it is an alternative for constructing the path integral for these reducible models.

For future perspectives, we are interested in extending the Nakanishi B field formalism for Ostrogadskian systems [26, 27] such as the Podolsky model, see [28]. with this knowledge, we intend to construct the same approach developed here for the psin 1 symmetric tensor models of [24, 25]. It would be a next step to test our achievements since these models contain a more complex reducible structure and some of them are of higher derivative nature

7.7 Appendix: Spin projectors

Using the spin-0 and spin-1 projection operators acting on vector fields, respectively,

$$\omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\Box} \ , \ \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \quad , \tag{7.107}$$

as building blocks, one can define the projection operators in D dimensions acting on antisymmetric rank-2 tensors. First, we define the transverse and longitudinal operators as follows

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \qquad \omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\Box}. \tag{7.108}$$

The above set of operator satisfies

$$\theta^2 = \theta, \quad \omega^2 = \omega \quad \text{and} \quad \theta\omega = \omega\theta = 0. \tag{7.109}$$

On the other hand, the set of the antisymmetric four-dimensional Barnes-Rivers operator are given by

$$P^{[1b]}_{\mu\nu,\alpha\lambda} = \frac{1}{2}(\theta_{\mu\alpha}\theta_{\nu\lambda} - \theta_{\mu\lambda}\theta_{\nu\alpha}), \tag{7.110}$$

$$P^{[1e]}_{\mu\nu,\alpha\lambda} = \frac{1}{2}(\theta_{\mu\alpha}\omega_{\nu\lambda} + \theta_{\nu\lambda}\omega_{\mu\alpha} - \theta_{\mu\lambda}\omega_{\nu\alpha} - \theta_{\nu\alpha}\omega_{\mu\lambda}). \tag{7.111}$$

They satisfy the very simple algebra

$$(P^{[1b]})^2 = P^{[1b]}, \quad (P^{[1e]})^2 = P^{[1e]}, \tag{7.112}$$

$$P^{[1b]}P^{[1e]} = P^{[1e]}P^{[1b]} = 0. \tag{7.113}$$

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Chapter 8

Master Action Duality, The Case of a Generalized Higher Derivative Tensor Electrodynamics

8.1 Motivation

The issue of duality has important implications in physics. We can cite the particle-vortex web of dualities in $2+1$ dimensions [1, 2] with applications in the context of quantum Hall effect [3], the case of three dimensional QCD [4], the boson-Fermion duality [5] and the well-established weak dualities in condensed matter [6, 7].

In the case of quantum electrodynamics in $2+1$ dimensions there is the possibility of adding a topological charge which can be used to classify different vortex solutions [8]. But the point is, who is charged under this symmetry? Again, in order to answer it, duality can play an useful role relating the vector gauge field to a scalar field by means of a master action with a given structure that interpolates a scalar field, the so-called dual photon, and the electromagnetic potential by means of different integrations in the path integral measure. The invariance of the dual theory under the shift symmetry is then related to the mentioned topological charge [9].

with regard to the origin of the master action duality approach, see [10, 11, 12], we mention the relation between the so-called self dual model and the Maxwell-Chern-Simons one. It is known that QED in lower dimensions have severe infrared (IR) singularities. Then, in the eighties, a topological mass term was introduced to provide an infrared cut-off for these theories in $2+1$ space-time dimensions without neither violating gauge symmetry nor adding new degrees of freedom [13].

The resulting theory is the so-called Maxwell-Chern-Simons (MCS) theory which describes a massive excitation. A nice review of these ideas is given in reference [14]. After that, a self-dual spin 1 model was introduced by Townsend and collaborators [15]. Even though this new theory describes a massive vector field without imposing gauge symmetry, it was realized that the model is related to the above topological massive model through a dual correspondence derived by the previously mentioned master action approach [16].

with relation to the power of duality, we should also cite the promising field of ADS-CFT correspondence [17], from which interesting perspectives for condensed matter physics are obtained [18].

In this present work we intend to develop a dual electrodynamics by means of a higher derivative spin 1 model described by a symmetric rank 2 tensor field proposed in [19], whose massless limit present a Weyl symmetry that under a dual correspondence perspective can be related to the $U(1)$ symmetry of the QED_4. This higher order theory was obtained from a master action that interpolates between it and a similar tensor model of second order in derivatives whose unitarity constraints and Hamiltonian properties were discussed in [20]. The aspects regarding a necessary condition for the unitarity of the massive version of the fourth order model analyzed here are found in [19] but they are restricted to a semiclassical analysis. Moreover, as an example of a duality map in the context of a non-liner interacting scalar model, we can cite [21].

Our objective here is to extend the analysis for the case in which the system is coupled to microscopic matter with a non-trivial quantum structure described by a Fermionic field. We intend to derive Ward-like identities ensuring that no new degrees of freedom are radiatively generated for gauge redundant sectors avoiding unitarity violation. We explicitly verify these assumptions by a careful analysis of the one loop contributions for the vacuum polarization and self energy tensors. For this last structure, we show that the conventional result from the usual quantum electrodynamics is recovered, proving that the dual description leads to the same phenomenology, as it should be.

In order to verify if the structures obtained are in accordance with the unitarity requirement, we use the cutting rules in order to obtain the imaginary part of the diagrams constructed by means of the specific Feynman rules for this dual electrodynamics and then consider the optical theorem. This procedure requires a prescription for the external states but since they are on-shell, the knowledge of the Gaussian master action dual map is enough to suggest us the necessary structure which we are going to take as a limiting procedure in the massless case. We also recall the fact that, although the model has derivative couplings, it exhibits the same superficial degree of divergence structure as the conventional QED_4, since the gauge boson is described by a higher derivative kinetic term.

The use of higher rank tensor fields to provide dual descriptions of Bosonic theories, regarding the case of anti-symmetric tensor models, can be traced back to the Kalb-Ramond model [22] case and the effective theory for low energy excitations in QCD [23, 24]. These models, and also the ones described by symmetric tensors, often have degree of freedom discontinuities in its massless limit [25] which can be understood as an implication of its reducible Hamiltonian constraint structure [26, 27]. It is interesting to mention that the model studied here do not present this feature as explicitly demonstrated in the second section.

The importance of deriving such dual descriptions, besides the purely theoretical attempt of generating well-defined equivalent structures, is the possibility of a wider variety of couplings for the case of the higher rank tensor fields which can be used as extensions of the present theories that may, for example, describe some phenomenological content [28, 29] as discussed in the end of the section 6.

We consider the present work as a laboratory for a future discussion of new extensions of spin 2 theories, particularly, gravity models. Using the approach of [19, 20] and also the correlated work [30], devoted to the study of spin one models in terms of symmetric tensor fields, which uses similar techniques as in [31, 32], and also having in mind its interesting non-symmetric tensor field counterpart [33, 34], we believe that it is possible to investigate and provide an analogous one loop analysis (after adding non-linear interactions.), as the one employed in this present work, for the case of gravity described by a higher order rank three tensor model [19], and also for the case of spin 3 theories [35, 36], in order to directly verify the optical theorem. Therefore, due to the higher rank dual structure, we can search for extensions such as new couplings and also the possibility of constructing a gauge invariant mass terms which can be interesting for the research field of massive gravity, see the review [37] and also [38]. Regarding the reference [36], we believe that a dual model for spin 2 can be built by means of a three indices tensor as $\partial^\nu h_{\nu\mu\beta}$ in a suitable action that gives dynamics just for the spin 2 degrees of freedom, in strict analogy to the model studied here. It is interesting to mention that in [19], they did something similar to this idea for the specific case of the new massive gravity.

Regarding the extensions for the model studied here, we analyze, in the section 6, possible new couplings for the present model and, in this specific case, conclude that they are not unitary. However, we derive a well-defined unitary gauge invariant mass term without the need of compensating Stueckelberg fields [39] as it would be required for a vector field description. We also provide an integral representation for the self energy tensor to analyze the effect of adding a massive gauge boson internal line for this structure.

The work is organized as follows. In the section 2, the basic aspects of the previously mentioned higher derivative tensor model as well as its obtainment are briefly discussed, using also non-relativistic potential tools to improve the arguments. We also choose a convenient gauge condition for the massless case and derive a prescription for on-shell external states. The section 3 is devoted to show that the massless limit of the higher derivative tensor model can reproduce the surface potential interaction associated to the electromagnetism in the presence of a conducting surface if we use the map between this model and the Maxwell theory suggested in section 2.

Throughout the section 4 , Ward-Like identities are derived for the case of the model interacting with spinors. These identities are explicitly verified in the section 5 regarding the tensor structure of some radiative corrections for the spinor dual electrodynamics. We also present its unitarity analysis by means of the optical theorem.

In the section 6 we look for extensions of quantum electrodynamics exploring the higher rank structure of the field. Although we did not found a new consistent interaction, we managed to derive a gauge invariant mass term without the need of compensating Stueckelberg fields. We also give an integral representation for the self energy tensor emerging from the interaction of an electron with the massive gauge boson and then discuss some possible phenomenological motivations for this model. The section 7 is devoted for conclusions and new perspectives. The metric signature $(+,-,-,-)$ is used throughout.

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8.2 The Obtainment of the Model, Behaviour of its Potential and the Prescription for External States

In order to derive the higher derivative model from which a renormalizable dual tensor electrodynamics can be generated, we give a brief overview on the master action procedure related to its obtainment, for more details, see [19]. We first analyze an action interpolating a second order tensor model with the Proca theory where c-number sources are added in order to obtain a dual map between these fields. We also investigate the physical content of its massless limit by evaluating the behaviour of its inter-particle non-relativistic potential. Latter, another master action is presented relating this model which describes spin 1 particles by means of a symmetric tensor with a similar model which is of fourth order in derivatives. The unitarity and the inter-particle potential analysis reveals that this higher derivative model has a well-defined massless limit in which, besides the appearance of a reducible gauge symmetry which is also present in the second derivative order model, a Weyl local symmetry emerges.

We consider a gauge fixing condition of the form $\partial^\mu\partial^\nu N_{\mu\nu}(x) = 0$. The reason is that besides leading to a fourth derivative order gauge fixing lagrangian as well as the physical sector of the model, it allows a straightforward association of the divergence of such a field with a Maxwell four-vector potential in the Lorenz gauge and also provide a radiative structure that recovers the Feynman gauge electrodynamics as we are going to see in the next section in which the system are coupled to a spinor field. Regarding this association with a spin 1 vector field, the equations of motion and the previously mentioned dual map between tensor and vector fields suggest a prescription for the on-shell gauge boson external states employed in the next section in the explicit verification of unitarity in the case of a dual electrodynamics.

The action relating the second order tensor model with the Maxwell-Proca theory [19, 20] reads

$$S_M\big[W,A,J,T\big] = \int d^4x\Big(W_{\mu\nu}W^{\mu\nu} - \frac{W^2}{3} - 2W^{\mu\nu}\partial_{(\mu}A_{\nu)} + \frac{m^2}{2}A_\mu A^\mu + J_\mu A^\mu + W_{\mu\nu}T^{\mu\nu}\Big) \tag{8.1}$$

where this action must be understood in a path integral context in which the c-number sources are added to generate relations between n-point Green function from both kind of fields.

Redefining the tensor field, in such a way that leaves invariant the path integral measure, according to the expression below

$$\tilde{W}_{\mu\nu}(x) \to W_{\mu\nu}(x) + \partial_{(\mu}A_{\nu)}(x) - \eta_{\mu\nu}\partial_\mu A^\mu(x) + \frac{1}{2}T^{\mu\nu}(x) - \frac{1}{2}\eta_{\mu\nu}T(x) \tag{8.2}$$

leads to the Maxwell-Proca theory plus source and non-dynamical tensor terms that can be integrated out in the path integral and contributes just to an overall normalization factor

$$\begin{aligned} S_M\big[W,A,J,T\big] = \int d^4x\Big(\tilde{W}_{\mu\nu}\tilde{W}^{\mu\nu} - \frac{\tilde{W}^2}{3} - T^{\mu\nu}\big[\frac{1}{2}\eta_{\mu\nu}\partial_\beta A^\beta - \partial_{(\mu}A_{\nu)}\big] + \frac{T^2 - T^{\mu\nu}T^{\mu\nu}}{4} \\ - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu + J_\mu A^\mu\Big) \end{aligned} \tag{8.3}$$

Now, performing the redefinition $\tilde{A}_\mu(x) \to A_\mu(x) - \frac{\sqrt{2}}{m}\partial^\nu W_{\mu\nu}(x) - \frac{1}{m^2}J_\mu(x)$ and for the tensor field $W^{\mu\nu}(x) \to \frac{\sqrt{2}}{m}W^{\mu\nu}(x)$ on (8.1), we get an action for a spin-1 model described by a tensor field plus source and non-dynamical vector terms

$$S_M = \int d^4x\Big(- \big(\partial^\nu W_{\mu\nu}\big)^2 + \frac{m^2}{2}\big(W_{\mu\nu}W^{\mu\nu} - \frac{W^2}{3}\big) + \frac{J_\mu J^\mu}{2m^2} + \frac{\sqrt{2}}{m}J^\mu\partial^\nu W_{\mu\nu} + \frac{m^2}{2}\tilde{A}_\mu\tilde{A}^\mu\Big) \tag{8.4}$$

Varying the path integral equivalently written in terms of (8.3) and c with relation to the sources gives, up to contact terms , the following dual map between both theories

$$W_{\mu\nu}(x) \longleftrightarrow -\frac{\sqrt{2}}{m}\Big[\frac{1}{2}\eta_{\mu\nu}\partial_\beta A^\beta(x) - \partial_{(\mu}A_{\nu)}(x)\Big] \quad , \quad A_\mu(x) \longleftrightarrow \frac{\sqrt{2}}{m}\partial^\nu W_{\nu\mu}(x) \tag{8.5}$$

The fact that this model is unitary and has well-defined Hamiltonian properties is verified in [20]. There, it is demonstrated that the massless limit of the theory has no particle content.

Here, as an alternative approach, we intend to obtain its non-relativistic inter-particle potential and show that it vanishes in this limit. For the cases in which the physical poles of a given model are projected in longitudinal irreducible spin sectors, we should consider a source term capable of being projected in this sector to extract the physical content. As we are going to see, the spin 1 content of a tensor model couples with a source of the form

$$J^{\mu\nu}_{TL}(x) = \frac{1}{\sqrt{2}}\eta^{\mu 0}\partial^\nu\Big(Q_1\delta^3(\vec{x}-\vec{a}_1) + Q_2\delta^3(\vec{x}-\vec{a}_2)\Big) + \frac{1}{\sqrt{2}}\eta^{\nu 0}\partial^\mu\Big(Q_1\delta^3(\vec{x}-\vec{a}_1) + Q_2\delta^3(\vec{x}-\vec{a}_2)\Big) \tag{8.6}$$

See [40] where a general source decomposition for vector, symmetric and anti-symmetric tensor models are derived and tested in order to obtain the low energy inter-particle potentials for a variety of models, including the dual ones studied here and the so-called Kalb-Ramond model which is a kind of dual anti-symmetric tensor description of spin 1 and spin 0 theories in its massive and massless phases [25], respectively.

The expression for the potential, up to contact terms, is given below

$$E_4 = \frac{1}{2\tau}\int\int d^4x d^4y J^{\mu\nu}_{TL}(x)\mathcal{P}_{\mu\nu\alpha\beta}(x-y)J^{\alpha\beta}_{TL}(y) = \frac{m^2 Q_1 Q_2\ e^{-mr}}{4\pi r} \tag{8.7}$$

where $r = |\vec{a}_1 - \vec{a}_1|$, τ stands for a time interval and $\mathcal{P}_{\mu\nu\alpha\beta}(x-y)$ denotes the propagator of the model [20, 40].
The expression is obtained up to self interaction terms. As expected, it clearly vanishes in the massless limit as well as its previously analyzed particle content. It makes sense since physics relies on interactions. This discontinuity in the massless limit is commonly related to models in which reducible symmetries and constraints appears [26]. The Kalb-Ramond model [22] has similar characteristics and there is also an alternative anti-symmetric tensor model [27] for which this discontinuity is analyzed in a Hamiltonian perspective, in the context of its saturated two point amplitude structure and also by means of the master action technique.

In order to derive the fourth order derivative theory, we first add a non-physical term to this model [1]

$$S_M = \int d^4x\Big(- \big(\partial^\nu W_{\mu\nu}\big)^2 + \frac{m^2}{2}\big(W_{\mu\nu}W^{\mu\nu} - \frac{W^2}{3}\big) + \big(\partial^\nu N_{\mu\nu}\big)^2\Big) \tag{8.8}$$

Similar manipulations as the ones employed in (8.1) leads to an unitary fourth order model in its massive phase [19] plus non-dynamical terms

$$S_M = \int d^4x\Big(\tilde{W}_{\mu\nu}\tilde{W}^{\mu\nu} - \frac{\tilde{W}^2}{3} + \frac{1}{2}\partial^\alpha N_{\mu\alpha}\big(\eta^{\mu\nu}(\Box + m^2) - \partial^\mu\partial^\nu\big)\partial^\alpha N_{\nu\alpha}\Big) \tag{8.9}$$

with regard to this model, we have also calculated its potential [40] and it is indeed in accordance with the previous analysis of [19] ensuring that it has a well-defined massless limit

$$E(r) = \frac{Q_1 Q_2\ e^{-mr}}{4\pi r} \tag{8.10}$$

The equations of motion are the following

$$\partial_\alpha\Big(\big(\Box + m^2\big)\partial^\beta N_{\nu\beta}(x) - \partial_\nu\big(\partial_\mu\partial_\sigma N^{\mu\sigma}(x)\big)\Big) + \partial_\nu\Big(\big(\Box + m^2\big)\partial^\beta N_{\alpha\beta}(x) - \partial_\alpha\big(\partial_\mu\partial_\sigma N^{\mu\sigma}(x)\big)\Big) = 0 \tag{8.11}$$

These equations of motion, considering that the fields vanishes at the space-time infinity, recover the Maxwell-Proca ones if we adopt the map $\partial^\beta N_{\nu\beta}(x) \leftrightarrow A_\beta(x)$

$$\big(\Box + m^2\big)\partial^\beta N_{\nu\beta}(x) - \partial_\nu\big(\partial_\mu\partial_\sigma N^{\mu\sigma}(x)\big) = 0 \tag{8.12}$$

[1] The fact that it is non-dynamical is due to the fact that it is the massless limit of the second derivative order model studied here, which has no particle content at this limit.

Even the massive version of this model is invariant under the following reducible local symmetry transformation

$$N^{\mu\nu}(x) \to N^{\mu\nu}(x) - \epsilon^{\alpha\beta\gamma(\mu}\epsilon^{\nu)\sigma\omega\varepsilon}\partial_\alpha\ \partial_\sigma\Lambda_{[\omega\varepsilon][\beta\gamma]}(x) \tag{8.13}$$

where $\Lambda_{[\omega\varepsilon][\beta\gamma]}(x)$ is an arbitrary field with antisymmetric behaviour under the exchange of the indices inside the square brackets.

The massless limit of such a model is of interest to us since it presents a Weyl symmetry that under the previously suggested identification of $N^{\mu\nu}(x)$ with a vector field can be an inspiration to derive a dual electrodynamics. In this case, beyond the reducible symmetry, the action is also invariant under the mentioned local transformation

$$N_{\nu\beta}(x) \to N_{\nu\beta}(x) + \eta_{\nu\beta}\ \phi(x) \tag{8.14}$$

with $\phi(x)$ being a given scalar field.

For the case in which there are local symmetries, to obtain the propagator of the theory it is required to consider symmetry breaking conditions. Regarding the reducible symmetry, we assume the following one

$$\Box^2 N^{\nu\alpha}(x) + \partial^\nu\partial^\alpha\big(\partial_\mu\partial_\sigma N^{\mu\sigma}(x)\big) - 2\Box\partial_\mu\partial^{(\nu}N^{\alpha)\mu}(x) = 0 \tag{8.15}$$

In order to fix the Weyl symmetry, we find it more convenient to consider

$$\partial_\mu\partial_\nu N^{\mu\nu}(x) = 0 \tag{8.16}$$

It allows the obtainment of a Klein-Gordon equation for the field $\partial_\nu N^{\mu\nu}(x)$.
It is compatible with the association of $\partial^\beta N_{\nu\beta}(x)$ and a massless spin 1 vector field in the Lorenz gauge. Moreover, the gauge fixing lagrangian would be of the same order as the kinetic terms and, as we are going to see, we can recover the Feynman gauge radiative corrections considering for the parameter $\tilde{\lambda} = 1$ appearing below.

The gauge fixed action written in terms of the spin projectors has the form [2]

$$S_M = \frac{1}{2}\int d^4x N^{\mu\nu}\Big(-\frac{1}{2}\Box^2 P^{(1)}_{ss} + \tilde{\lambda}\Box^2 P^{(0)}_{\omega\omega} + \lambda\big(\Box^4 P^{(2)}_{ss} + \Box^4 P^{(0)}_{ss}\big)\Big)_{\mu\nu\alpha\beta} N^{\alpha\beta} \tag{8.17}$$

Using the spin projector algebra and completeness relation we can obtain the propagator in the momentum space

$$\mathcal{G}_{\mu\nu\alpha\beta} = \Big(\frac{1}{\lambda k^8}P^{(2)}_{ss} - \frac{2}{k^4}P^{(1)}_{ss} + \frac{1}{\tilde{\lambda}k^4}P^{(0)}_{\omega\omega} + \frac{1}{\lambda k^8}P^{(0)}_{ss}\Big)_{\mu\nu\alpha\beta} \tag{8.18}$$

Now, we add a source term that must be in accordance with both the gauge symmetries of the system. It must have the form $T^{\alpha\beta}(k) = k^\alpha J^\beta_T(k) + k^\beta J^\alpha_T(k)$ where $J^\alpha_T(k)$ denotes a transverse current. It has the same structure as the current employed to obtain the non-relativistic potential for this model. It is worth mention that the reducible local symmetry implies the structure in terms of a vector current and the Weyl one imposes the transverse nature to this current.

In order to derive a necessary condition for the model to be unitary, we consider the saturated two point amplitude

[2] See appendix A

$$\mathcal{A}(k) = iT^{*\mu\nu}(k)\ \mathcal{G}_{\mu\nu\alpha\beta}T^{\alpha\beta}(k) = -4i\frac{J^{*}_{T\mu}J^{\mu}_{T}}{k^2} \tag{8.19}$$

We must evaluate the sign of the imaginary part of the residue of this expression. Using the constraint due to transversality $k_\mu J^\mu_T(k) = 0$ and assuming the frame compatible to a light-like momenta $k^\mu = \big(k, k, 0, 0\big)$, we obtain the desired necessary condition for unitarity

$$\mathcal{R}\big[\mathcal{A}(k)\big] = 4i\sum_{n=2}^{3}|J^n_T|^2 \tag{8.20}$$

since the imaginary part of the residue $\Im(\mathcal{R}\mathcal{A}(k)) > 0$ is positive.

Although the previous analysis were made in terms of Gaussian actions, these observations and suggestions can be useful even in the interacting case. In order to explicitly verify unitarity constraints in the case of dual electrodynamics we use the following prescription for the external gauge boson states

$$\epsilon^r_{\mu\nu}(k) = \frac{\Big(k_\mu\epsilon^r_\nu(k) + k_\nu\epsilon^r_\mu(k)\Big)}{M^2} \tag{8.21}$$

The polarization vectors given above are transverse and obey the completeness relation $\sum_r \epsilon^r_\mu(k)\epsilon^{*r}_\nu(k) = -\Big(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\Big)$. We also have $k^2\epsilon_\mu(k) = M^2\epsilon_\mu(k)$. The prescription is in accordance with the relation $\partial^\nu N_{\nu\beta}(x) \leftrightarrow A_\beta(x)$, with both sides being transverse relative to the β index, suggested by the the equation (8.11) regarding the massive case when the mass is set equal to M. Since we are interested in the massless limit, for the further calculations we consider the poles shifted by a mass term M^2, and a well-defined limit is taken at the end of the calculations. This addition of a regularizing mass term is not a novelty and is applied, for example, in the calculation of the one loop contribution to the renormalization of the electric charge of QED$_4$ [41, 42]. We will see that this procedure is enough to prove the required unitarity relations. The reason to use this limiting procedure is to avoid a non-local map that may be not well-defined when the momenta has a light-like character.

8.3 A first semiclassical test: The theory in the presence of a conducting surface

As a first semiclassical test for our dual Maxwell model, the massless limit of the higher derivative tensor description of a spin 1 particle, we consider the case in which the system is in the presence of a conducting surface. The formalism [43] briefly reviewed here is particularly useful in the study of the Casimir effect or just to consider the system in the presence of a conducting plate. The condition to define a conductor is the absence of a Lorentz force parallel to it. In vector QED_4 it is expressed as $\frac{1}{2}\eta_\mu\epsilon^{\mu\nu\rho\beta}F_{\rho\beta}|_{\partial S} = 0$ in which $\eta_\mu \equiv \delta^3_\mu$ represent the unit normal vector of the surface, pointing to the z axis, and $F_{\mu\nu}(x)$ denotes the field strength tensor. Then, we should translate this situation for our tensor model and verify if it recovers or not the surface physics that occur in the vector QED_4 model. Using the suggested map from section 2, the condition can be written by means of the operator $\eta_\mu\epsilon^{\mu\nu\gamma\alpha}\partial_\gamma\partial^\sigma\Delta_{\sigma\alpha}{}^{\chi\tau}N_{\chi\tau} \equiv \chi^\nu{}_{\chi\tau}N^{\chi\tau} = 0$ with $\Delta_{\sigma\alpha}{}^{\chi\tau}$ being the symmetric identity. In order to implement it on the functional generator, we write the condition in a functional Dirac delta structure by means of an auxiliary field $B_\nu(x)$

$$Z[j_\mu] = \int DN_{\mu\beta}DB_\mu \exp\{i\int d^4xN_{\mu\nu}\frac{1}{2}\mathcal{O}^{\mu\nu\alpha\beta}N_{\alpha\beta} + j_\mu\partial_\nu N^{\mu\nu}\}\exp\Big\{i\int dSB_\nu(x_{||})\chi^\nu{}_{\chi\tau}N^{\chi\tau}\Big\}$$
$$\times\exp\Big\{\frac{-i}{2\Lambda}\int dS_xdS_yB_\mu(x_{||})\partial^\mu\partial^\nu Q(x_{||}-y_{||})B_\nu(y_{||})\Big\} \tag{8.22}$$

we added an point like static external source $j_\mu(x)$ for the gauge field and the last term is due to the fact that $\chi^\nu{}_{\chi\tau}$ has zero modes and therefore the auxiliary field has the local freedom $B_\mu(x_{||}) \to B_\mu(x_{||}) + \partial_\mu f(x_{||})$ which allow us to fix the gauge $\partial_\nu B^\nu = 0$ leading to this term with an arbitrary functional $Q(x_{||} - y_{||})$ which is going to be fixed by a convenient choice. Λ is an arbitrary gauge parameter, $dS = d^4x\delta(x_\perp - a)$ denotes the boundary integration measure and $x^\mu_{||}$ are the coordinates parallel to our plane boundary at $x_\perp = a$. The $\mathcal{O}^{\mu\nu\alpha\beta}$ is the differential operator defining the dynamics of the gauge fixed model presented in equation 8.17.

In order to decouple the fields, we perform the redefinition[3]

$$N_{\alpha\beta}(x) \to N_{\alpha\beta}(x) + \int d^4y\ \chi_{\nu\omega\lambda}\big(B^\nu(y_{||})\delta(y_\perp - a)\big)\mathcal{G}^{\omega\lambda}{}_{\alpha\beta}(y,x) \tag{8.23}$$

Now, defining [4] $I_\lambda(y) \equiv \chi_\lambda{}^{\omega\rho}(y)\int d^4x\mathcal{G}_{\omega\rho\alpha\beta}(y,x)\partial^{(\alpha}j^{\beta)}(x) = \epsilon_{3\lambda\omega\rho}\partial^\omega_y\int d^4xD^{\rho\mu}(y,x)j_\mu(x)$, we get the decoupled functional generator

$$Z[j_\mu] = \int DN_{\mu\beta}DB_\mu \exp\{i\int d^4xN_{\mu\nu}\frac{1}{2}\mathcal{O}^{\mu\nu\alpha\beta}N_{\alpha\beta} + j_\mu\partial_\nu N^{\mu\nu}\}\exp\Big\{i\int dS_xB_\nu(x_{||})I^\nu(x)\Big\}$$
$$\times\exp\Big\{\frac{-i}{2}\int dS_xdS_yB_\mu(x_{||})\{\big(\chi^\nu{}_{\alpha\beta}(y)\chi^\mu{}_{\omega\tau}(x)\mathcal{G}^{\alpha\beta\omega\tau}(y,x)\big)_{x=x_{||},y=y_{||}} + \frac{1}{\Lambda}\partial^\mu\partial^\nu Q(x_{||}-y_{||})\}B_\nu(y_{||})\Big\} \tag{8.24}$$

Considering the structure of this propagator found in the previous section, the identity $\eta^\mu\epsilon_{\mu\alpha\gamma\chi}p^\gamma\eta^\nu\epsilon_{\nu\beta\tau\omega}p^\tau g^{\chi\omega} = g^{\lambda\alpha}p^2_{||} - p^\lambda p^\alpha$ and the fact that, at the conducting surface, we have

$$\int\frac{d^4p}{(2\pi)^4p^2}e^{-ip(x-y)} = -\frac{i}{2}\int\frac{d^3p_{||}}{(2\pi)^3\sqrt{p^2_{||}}}e^{-ip_{||}(x_{||}-y_{||})} \tag{8.25}$$

It is easy to note that, after integrating the fields, it yields the same surface potential contribution [5] as in the vector QED_4 if we choose $Q(x_{||} - y_{||}) = \frac{i}{2}\int\frac{d^3p_{||}}{(2\pi)^3\sqrt{p^2_{||}}}e^{-ip_{||}(x_{||}-y_{||})}$ with $\Lambda = 1$. This claim can be verified by noticing that the $I_\mu(x)$ generalized current and the differential operator for the auxiliary field $B_\mu(x)$ given below

$$\big(\chi^\nu{}_{\alpha\beta}(y)\chi^\mu{}_{\omega\tau}(x)\mathcal{G}^{\alpha\beta\omega\tau}(y,x)\big)_{x=x_{||},y=y_{||}} = \frac{i}{2}\int\frac{d^3p_{||}}{(2\pi)^3}\frac{e^{-i(y-x)_{||}}}{\sqrt{p^2_{||}}}\big(\eta^{\mu\nu}p^2_{||} - p^\mu p^\nu\big) \tag{8.26}$$

are the same as in the conventional QED_4 case. Alternatively, it can be verified considering the work [43], by taking the infinite mass limit for the Lie-Wick model surface potential interaction

[3]We consider the following definition and the fundamental property of the propagator $\mathcal{G}_{\alpha\beta\omega\lambda}(x,y) = \int\frac{d^4p}{(2\pi)^4}\mathcal{G}_{\alpha\beta\omega\lambda}(p)e^{-ip(x-y)}$ and $\mathcal{O}^{\mu\nu\alpha\beta}\mathcal{G}_{\alpha\beta\omega\lambda}(x,y) = \delta^4(x-y)\Delta^{\mu\nu}_{\omega\lambda}$. Regarding the specific form of the redefinition, it is due to the fact that, differently from the usual QED_4 case, the differential boundary operator acts on $y_\perp$.

[4]where $D^{\rho\mu}(y,x)$ denotes the QED_4 propagator.

[5]This can be obtained by considering $Z = \exp\big[-iET\big]$, with T being the time interval and E the non-relativistic potential energy that splits in a bulk and a boundary term.

. It is important to stress that the choice for $Q(x_{||}-y_{||})$ do not have any influence on physics since it is related to the non-physical longitudinal sector. As a future perspective, the detailed analysis of the surface structure and the Casimir interaction between different surfaces for the case in which the field to be considered is governed by the second derivative order massive tensor model presented in the previous section seems to be a good objective since it leads to a saturated propagator with no analog in the conventional quantum field theories which may lead to new interaction terms at the boundary. Also, the issue of degree of freedom discontinuity in the massless limit may have an interesting counterpart in the framework of the behaviour of such surface interaction potentials.

8.4 Ward-like Identities for Generalized Spinor Electrodynamics

The goal of this section is to derive Ward-like identities for the dual electrodynamics described by the action below

$$S_{eff.}=\int d^4x\Big(+i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi-m\bar{\Psi}\Psi-e\partial^{\nu}N_{\nu\mu}\bar{\Psi}\gamma^{\mu}\Psi+\frac{1}{2}\partial^{\alpha}N_{\mu\alpha}\big(\eta^{\mu\nu}\Box-\partial^{\mu}\partial^{\nu}\big)\partial^{\alpha}N_{\nu\alpha}$$
$$+\frac{\tilde{\lambda}}{2}(\partial^{\nu}\partial^{\alpha}N_{\nu\alpha})^2+\frac{\lambda}{2}(\Box^2N^{\nu\alpha}+\partial^{\nu}\partial^{\alpha}\big(\partial_{\mu}\partial_{\sigma}N^{\mu\sigma}\big)-2\Box\partial_{\mu}\partial^{(\nu}N^{\alpha)\mu})^2\Big) \tag{8.27}$$

where $\Psi(x)$ denote a Fermionic field.

The action is invariant under the Weyl and the reducible symmetry transformations, respectively

$$\Psi(x)\to\Psi(x)+i\alpha(x)\Psi(x)\quad;\quad N_{\mu\nu}(x)\to N_{\mu\nu}(x)+\eta_{\mu\nu}\alpha(x) \tag{8.28}$$

$$N^{\mu\nu}(x)\to N^{\mu\nu}(x)-\epsilon^{\alpha\beta\gamma(\mu}\epsilon^{\nu)\sigma\omega\varepsilon}\partial_{\alpha}\,\partial_{\sigma}\Lambda_{[\omega\varepsilon][\beta\gamma]}(x) \tag{8.29}$$

The transformation parameterized by the field $\alpha(x)$ recovers the $U(1)$ transformation of QED_4 since the action is written in terms of the divergence of the tensor field. This last feature is the origin of the additional reducible symmetry under transformations parametrized by $\Lambda_{[\omega\varepsilon][\beta\gamma]}(x)$ [6] which has no analogous in the usual vector theory of QED_4.

The importance of deriving the Ward identities in the context of unitarity is to due to the fact that if they are not violated, as we are going to prove when analysing the one loop structures, it means that the radiative corrections do not generate dynamics for unphysical gauge redundant degrees of freedom that are absent when just the free theory is considered. the emergence of these extra degrees of freedom may generally violate unitarity.

The expression for the functional generator is given below

$$Z\big[\eta,\bar{\eta},T\big]=\int DN_{\mu\gamma}D\Psi D\bar{\Psi}\exp\Big[i\int d^4x\Big(+i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi-m\bar{\Psi}\Psi-e\partial^{\nu}N_{\nu\mu}\bar{\Psi}\gamma^{\mu}\Psi$$
$$+\frac{1}{2}\partial^{\alpha}N_{\mu\alpha}\big(\eta^{\mu\nu}\Box-\partial^{\mu}\partial^{\nu}\big)\partial^{\alpha}N_{\nu\alpha}+T_{\nu\alpha}N^{\nu\alpha}+\bar{\eta}\Psi+\bar{\Psi}\eta$$
$$+\frac{\tilde{\lambda}}{2}(\partial^{\nu}\partial^{\alpha}N_{\nu\alpha})^2+\frac{\lambda}{2}(\Box^2N^{\nu\alpha}+\partial^{\nu}\partial^{\alpha}\big(\partial_{\mu}\partial_{\sigma}N^{\mu\sigma}\big)-2\Box\partial_{\mu}\partial^{(\nu}N^{\alpha)\mu})^2\Big)\Big] \tag{8.30}$$

[6] The square brackets denotes anti-symmetrization.

where c-number sources are added to the fields in order to derive n-point Green functions for the fields.

Varying the functional generator with relation to a infinitesimal Weyl symmetry transformation gives

$$\delta Z\big[\eta,\bar{\eta},T\big] = \int DN_{\mu\gamma}D\Psi D\bar{\Psi}\int d^4x\Big(\tilde{\lambda}\Box\partial_\mu\partial_\sigma N^{\mu\sigma} + \eta^{\gamma\omega}T_{\gamma\omega} + i\bar{\eta}\Psi - i\bar{\Psi}\eta$$
$$+\lambda\Box^2\theta_{\alpha\nu}\big(\Box^2 N^{\nu\alpha} + \partial^\nu\partial^\alpha\big(\partial_\mu\partial_\sigma N^{\mu\sigma}\big) - 2\Box\partial_\mu\partial^{(\nu}N^{\alpha)\mu}\big)\Big)\alpha(x)e^{iS_{eff.}} \tag{8.31}$$

where the $\theta_{\mu\nu}$ transverse projector is defined in appendix A.

We did not considered the variation of the measure since the Jacobian due to this infinitesimal gauge transformation is unitary up to second order terms

$$J = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-i\alpha(x) & 0 \\ 0 & 0 & 1+i\alpha(x) \end{pmatrix} = 1+\alpha^2(x) \tag{8.32}$$

Expressing the gauge invariance of the functional generator in terms of the connected Green function generator defined by $Z = e^{iW}$, we get the following expression

$$\int d^4x\Big(\tilde{\lambda}\partial_\mu\partial_\sigma\Box\frac{\delta W}{\delta T_{\mu\sigma}} + \eta^{\gamma\omega}T_{\gamma\omega} + i\bar{\eta}\frac{\delta W}{\delta\bar{\eta}} - i\frac{\delta W}{\delta\eta}\eta$$
$$+\,\lambda\,\Box^2\theta_{\alpha\nu}\big(\Box^2\frac{\delta W}{\delta T_{\nu\alpha}} + \partial^\nu\partial^\alpha\big(\partial_\mu\partial_\sigma\frac{\delta W}{\delta T_{\mu\sigma}}\big) - 2\Box\partial_\mu\partial^{(\nu}\frac{\delta W}{\delta T_{\alpha)\mu}}\big)\Big)Z = 0 \tag{8.33}$$

Considering the expression for the effective action [41] $\Gamma = W - \int d^4x\Big(\bar{\eta}\Psi + \bar{\Psi}\eta + T_{\mu\nu}N^{\mu\nu}\Big)$, we can derive a relation which depends just on the fields and the Γ variations with relation to them

$$\Big(\tilde{\lambda}\Box\partial_\mu\partial_\sigma N^{\mu\sigma} - \eta^{\gamma\omega}\frac{\delta\Gamma}{\delta N^{\gamma\omega}} - i\frac{\delta\Gamma}{\delta\bar{\Psi}}\Psi + i\bar{\Psi}\frac{\delta\Gamma}{\delta\Psi} +$$
$$\lambda\Box^2\theta_{\alpha\nu}\big(\Box^2N^{\nu\alpha} + \partial^\nu\partial^\alpha\big(\partial_\mu\partial_\sigma N^{\mu\sigma}\big) - 2\Box\partial_\mu\partial^{(\nu}N^{\alpha)\mu}\big)\Big) = 0 \tag{8.34}$$

Now,varying this expression with relation to the gauge field and taking the limit of all field going to zero at the end of the calculations, we can obtain an expression whose momentum space version is

$$\lambda k^8\theta_{\mu\nu} - \eta^{\gamma\omega}\frac{\delta^2\Gamma}{\delta N^{\gamma\omega}(k)\delta N^{\mu\nu}(-k)} + \tilde{\lambda}k^2k_\mu k_\nu = 0 \tag{8.35}$$

Since $\frac{\delta^2\Gamma}{\delta N^{\gamma\omega}(k)\delta N^{\mu\nu}(-k)}$ is the inverse of the complete propagator, it is easy to see that the trace above has contributions just from its free part, as we can see from (8.17) where it is displayed multiplied by a factor $\frac{1}{2}$ integrated and contracted with the gauge fields. It means that the quantum corrections due to the physical interactions do not act on the gauge redundant sector. The inverse of the complete propagator can be written as $\mathcal{G}^{-1(c)}_{\mu\nu\beta\gamma} = \mathcal{G}^{-1(0)}_{\mu\nu\beta\gamma} - i\pi_{\mu\nu\beta\gamma}$, where the first and the second terms on the right hand side of this equality are the inverse of the free propagator and the vacuum polarization tensor, respectively. It implies that the Ward identity results in a traceless condition for the radiative correction $\pi_{\mu\nu\beta\gamma}(p)$. We are going to confirm this observation by explicitly analysing it in the next section. As a consequence of this constraint and the next one due to the reducible symmetry, the radiative corrections are all proportional to the $P_{ss}^{(1)}$ projector.

Varying equation (8.34) with relation to $\Psi(y)$ and $\bar{\Psi}(z)$, taking the limit of vanishing fields at the end of the calculations furnishes an expression whose momentum space version has the form

$$\mathcal{S}^{-1}(\tilde{p}+k)-\mathcal{S}^{-1}(\tilde{p})=\eta^{\mu\nu}\frac{\delta^3\Gamma(p,\tilde{p},k=p-\tilde{p})}{\delta\bar{\psi}(p)\delta\psi(\tilde{p})\delta N_{\mu\nu}(k)}$$
$$=\eta^{\mu\nu}\int d^4l\frac{\delta^3\Gamma(p,\tilde{p},k=p-\tilde{p})}{\delta\bar{\psi}(p)\delta\psi(\tilde{p})\delta(l^\beta N_{\beta\alpha}(l))}\frac{\delta(l^\beta N_{\beta\alpha}(l))}{\delta N_{\mu\nu}(k)}=k_\alpha\frac{\delta^3\Gamma(p,\tilde{p},k=p-\tilde{p})}{\delta\bar{\psi}(p)\delta\psi(\tilde{p})\delta(k^\beta N_{\beta\alpha}(k))} \tag{8.36}$$

This relation expresses the fact that the renormalization constant of the trilinear interaction term is equal to the one from the Fermion field which means that the gauge invariance is kept in the radiative corrections.

Varying the action with relation to the reducible symmetry and imposing its invariance leads to

$$\delta Z\big[\eta,\bar{\eta},T\big]=\int DN_{\mu\gamma}D\Psi D\bar{\Psi}\int d^4x\Big(\partial^\alpha\partial^\gamma T^{\mu\nu}\epsilon_{\mu\sigma\alpha\beta}\epsilon_{\nu\Omega\gamma\Gamma}$$
$$+\lambda\partial^\omega\partial^\gamma\Box^2\big(\Box^2N^{\mu\alpha}+\partial^\mu\partial^\alpha\big(\partial_\nu\partial_\sigma N^{\nu\sigma}\big)-2\Box\partial_\nu\partial^{(\mu}N^{\alpha)\nu}\big)\epsilon_{\mu\sigma\omega\beta}\epsilon_{\alpha\Omega\gamma\Gamma}\Big)\Lambda^{[\sigma\beta][\Omega\Gamma]}(x)e^{iS_{eff.}}=0 \tag{8.37}$$

where the integration measure is obviously invariant in this case.

Expressing the above relation in terms of the effective action, varying it with relation to the gauge field and going to the momentum space gives the following constraint on the inverse of the complete propagator

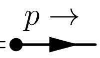
$$\Big(-k^\alpha k^\gamma\frac{\delta^2\Gamma}{\delta N_{\mu\nu}(k)\delta N_{\chi\varepsilon}(-k)}\epsilon_{\mu\sigma\alpha\beta}\epsilon_{\nu\Omega\gamma\Gamma}-\lambda k^\omega k^\gamma k^8\epsilon_{\sigma\omega\beta(\chi}\epsilon_{\varepsilon)\Omega\gamma\Gamma}\Big)=0 \tag{8.38}$$

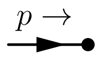
This is also the expected result from the inverse of the free propagator, if one sums up the contributions from the spin projectors $\big(\Box^4P_{ss}^{(2)}+\Box^4P_{ss}^{(0)}\big)$ and consider their definitions. Since we are working with the complete structure, it means that the quantum corrections to the propagator are invariant under the action of the reducible symmetry operator, as it should be, since the coupling involves just the $\partial^\mu N_{\mu\nu}(x)$ structure which is invariant under it.

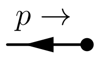
8.5 Feynman Rules, Radiative Corrections and Unitarity

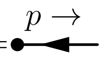
After presenting the general constraints due to the Ward identities, we provide the Feynman rules for the dual spinor electrodynamics in order to obtain the explicit one loop radiative corrections for the vacuum polarization and the electron self energy.

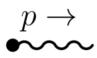
Regarding the rules for the external lines, we have [42]

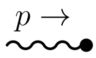
$\bar{u}_s(p) =$ 

$u_s(p) =$ 

$\bar{v}_s(p) =$ 

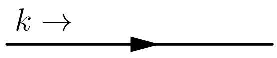
$v_s(p) =$ 

$\epsilon^{*r}_{\mu\nu}(p) =$ 

$\epsilon^{r}_{\mu\nu}(p) =$ 

(8.40)

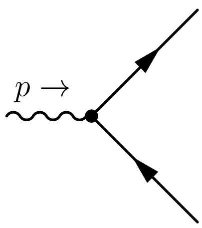
For the propagators, we have



$$= \frac{i\left(\gamma^\beta k_\beta + m\right)}{\left(k^2 - m^2 + i\epsilon\right)} \tag{8.41}$$

$$= i\mathcal{G}_{\mu\nu\alpha\beta}(k) \tag{8.42}$$

with $\lim \epsilon \to 0$.
The interaction vertex has the form



$$= e p_{(\mu}\gamma_{\nu)} \tag{8.43}$$

where the orientation of the gauge boson momentum must be taken into account. Since in this case there is no imaginary unit, the minus sign due to i^2 is replaced by the one appearing due to the orientation of the momenta in the one loop calculations. The gamma matrices algebra is displayed in appendix B.

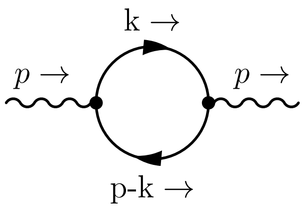
Since the model describing the gauge boson is of fourth order in derivatives and the extra momentum term from the vertex contributes to raise the degree of divergence of a graph just in the case when it is contracted to an internal gauge field line, we can conclude, analysing the topology of the graphs, that this model has the same superficial divergence structure as the usual QED_4. It can be understood as follows: Since the diagrams have the same topology as the vector QED_4, the relations [7] $V = 2P_\gamma + N_\gamma = P_e + \frac{1}{2}N_e$ and $L = P_e + P_\gamma - V + 1$ are kept. The expression for the superficial degree of divergence becomes $D = 4L + \left(V - N_\gamma\right) - P_e - 4P_\gamma$, the coefficient 4 for the boson internal line is due to the higher order structure of the propagator. The term $\left(V - N_\gamma\right)$ expresses the fact that since the momentum is attached to the photon line,

[7] V denotes the number of vertices, P_γ is the number of boson propagators, P_e the number of Fermionic propagators, with N_γ and N_e being the Bosonic and Fermionic external lines, respectively. L denotes the number of loops.

when it is internal, the vertex leads to an increase in the divergence order of the graph but in the case in which it is external there is no colaboration to the divergence at all. Therefore, these constraints lead to $D = 4 - N_\gamma - \frac{3}{2}N_e$, which is the same expression for the conventional formulation of QED_4.

The expression for the vacuum polarization tensor is the following

$$i\pi_{\mu\nu\alpha\beta}(p) = -e^2 tr \int \frac{d^4k}{(2\pi)^4} \frac{p_{(\mu}\gamma_{\nu)}(\gamma^\sigma(k_\sigma - p_\sigma) + m)p_{(\alpha}\gamma_{\beta)}(\gamma^\rho k_\rho + m)}{\big(k^2 - m^2 + i\epsilon\big)\big((p-k)^2 - m^2 + i\epsilon\big)} = \tag{8.44}$$



The polarization tensor can be written as

$$\pi_{\mu\nu\alpha\beta}(p) = p_{(\mu}\Big[\Pi_{\nu)(\alpha}(p)\Big]p_{\beta)} \tag{8.45}$$

with $\Pi_{\mu\nu}(p)$ being the polarization tensor of the vector QED_4.

This structure is clearly in accordance with the Ward identity due to the reducible symmetry.

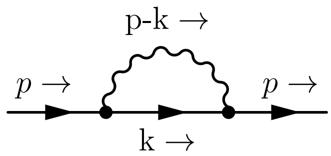
Regarding the Weyl symmetry, we can show that the fulfilment of its associated Ward identity is a heritage from the transverse nature of the usual QED_4 radiative corrections

$$\eta^{\mu\nu}\pi_{\mu\nu\alpha\beta}(p) = p^\mu\Big[\Pi_{\mu(\alpha}(p)\Big]p_{\beta)} = 0 \tag{8.46}$$

This result, which respects both Weyl and the reducible symmetry, imply in radiative corrections that act just on the $P^{(1)\mu\nu}_{ss\ \ \ \rho\sigma}$ sector in accordance with the spin 1 content of the free part of the model. Also, the power counting analysis [8] shows that all the divergences from the polarization tensor can be absorbed in the counter-terms arising from the gauge field renormalization. This is due to the fact that the boson field kinetic operator is proportional to $p_{(\mu}\Big[p^2\theta_{\nu)(\alpha}(p)\Big]p_{\beta)}$, with $\theta_{\mu\nu}$ being the transverse projector (see appendix A).

Using the Feynman rules for the dual electrodynamics, we can obtain the self energy tensor

$$i\Sigma(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{(p-k)^{(\mu}\gamma^{\nu)} i(\gamma^\sigma k_\sigma + m)(i)\mathcal{G}_{\mu\nu\alpha\beta}\big[(p-k)\big](p-k)^{(\alpha}\gamma^{\beta)}}{\big(k^2 - m^2 + i\epsilon\big)} = \tag{8.47}$$



Working out this expression, we find [9]

$$\begin{aligned} i\Sigma(p) =& -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{(p-k)^{(\mu}\gamma^{\nu)} i(\gamma^\sigma k_\sigma + m)(i)\mathcal{G}_{\mu\nu\alpha\beta}\big[(p-k)\big](p-k)^{(\alpha}\gamma^{\beta)}}{\big(k^2 - m^2 + i\epsilon\big)} \\ =& -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\gamma^\sigma k_\sigma + m)}{\big(k^2 - m^2 + i\epsilon\big)} \frac{\big(\eta_{\mu\beta} - \omega_{\mu\beta} + \omega_{\mu\beta}/\tilde{\lambda}\big)\gamma^\beta}{\big((p-k)^2 + i\epsilon\big)} \end{aligned} \tag{8.48}$$

[8] Which is the same as the vector QED_4 one due to the compensation of the addition of a derivative vertex by the presence of a higher order pole in the propagator.

[9] where we have defined $\omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}$

We see that the same expression as the one obtained from the usual QED_4 in the Feynman gauge is recovered if we fix $\tilde{\lambda} = 1$.

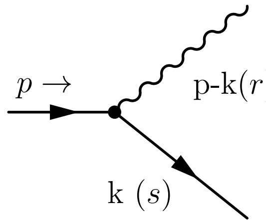
In order to prove unitarity by means of the optical theorem, we must use the expression for the self energy and calculate the imaginary part of the amplitude [42, 44] associated to the process of an incoming electron with a given momentum that emits a photon in a given point and then absorb it in another and then emerges with the same momentum

$$\bar{u}(p) 2\Im\big[\Sigma(p)(p^2)\big] u(p) = \\ -e^2 \int \tfrac{d^4k}{(2\pi)^2} \theta(k_0)\theta(p_0 - k_0)\bar{u}(p)\gamma^\mu(\gamma^\sigma k_\sigma + m)\gamma_\mu u(p)\delta\big((p-k)^2\big)\delta\big(k^2 - m^2\big) \tag{8.50}$$

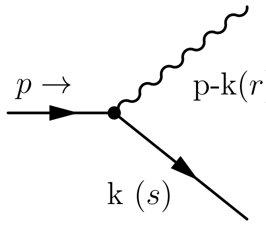
we also need to calculate the decay rate expression in which our prescription for the external gauge boson states have to be tested in order to verify if the map suggested by the master action technique is strong enough to be in agreement with the unitarity requirement even in the interacting case.

As previously mentioned, we consider a mass M to the boson field according to the prescription for the external states and then take the massless limit at the end of the calculations. The decay rate reads

$$\lim_{M\to 0} \sum_{r,s} \int \frac{d^4k}{(2\pi)^2} \theta(k_0)\theta(p_0 - k_0)\delta\Big(k^2 - m^2\Big)\delta\Big((p-k)^2 - M^2\Big) \left| \quad \right|^2 = 2m\Gamma \tag{8.51}$$



The amplitude appearing above has the following value,



$$\sum_{r,s} \left| \quad \right|^2 = e^2 \sum_{r,s} \bar{u}(p)(p-k)^{(\mu}\gamma^{\nu)} u_s(k)\bar{u}_s(k)(p-k)^{(\alpha}\gamma^{\beta)} u(p)\epsilon^r_{\mu\nu}[p-k]\epsilon^{r*}_{\alpha\beta}[p-k] \\ = -e^2\bar{u}(p)\gamma^\nu(\gamma^\beta k_\beta + m)\gamma_\nu u(p)\frac{\big((p-k)^2\big)^2}{M^4} \tag{8.52}$$

where the longitudinal terms coming from the sum over the vector polarizations do not contribute due to the Gordon identity [42].

Therefore, since the optical theorem implies that

$$\bar{u}(p)\Im\big[\Sigma(p)(p^2)\big]u(p) = m\Gamma \tag{8.53}$$

we conclude that this process is indeed unitary at one loop.

Regarding the amplitudes for processes with Bosonic external lines, the vertex has a structure in which the momentum attached to these lines is contracted with the external states in such a way that the vanishing mass limit is totally well-defined. Moreover, considering the power counting analysis and these mentioned features, all the vector QED_4 amplitudes are recovered by the tensor model.

8.6 New Couplings, other Models and Gauge Invariant Mass term

The use of a dual description by means of higher rank tensor fields may be useful to derive new couplings and extensions from a given theory. A higher rank tensor has bigger possibilities of couplings then a lower rank one, but they must be carefully investigated to avoid inconsistencies with the principles of the quantum field theory. Here we analyze possible extensions due to a tensor description of QED_4.

with regard to the present model, due to the tensor structure of the boson field it is possible to add a term relating one gauge boson and two current operators as $N_{\mu\nu}(x)\bar{\Psi}(x)\gamma^{\mu}\Psi(x)\bar{\Psi}(x)\gamma^{\nu}\Psi(x)$ but this term generates a contribution for the gauge field equations of motion that couples not just with the spin 1 sector but also with others irreducible components. The loops formed from this graph, besides receiving contributions from the spin 1 sector of the gauge field internal lines, also receives contributions from its other auxiliary irreducible sectors. Moreover, we note that this interaction clearly violates gauge invariance which means that there is no Ward identities that guarantee radiative corrections with just the spin 1 tensor structure. It also breaks the charge conjugation symmetry. We recall that unitarity cannot be taken as granted and owing to our prescription for external states, that involves just spin 1 particles, it is possible to note that this coupling violates the optical theorem when we analyze the imaginary part of the self energy tensor and consider our prescription for the external states. This interaction term would also be non-renormalizable since, due to the dimension of the fields, it would need to be multiplied by a negative mass dimension constant when written in the action.

Alternatively, we can also add a local Weyl invariant quartic interaction term for this Abelian model, something that is not possible when considering a vector field. It violates unitarity since it breaks part of the reducible symmetry that also contributes to project the dynamical content of the theory into the spin 1 sector

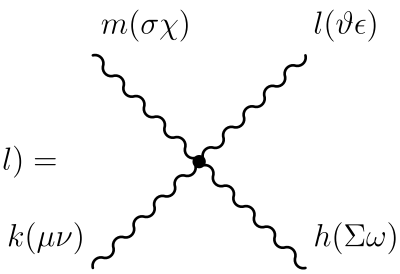
$$\mathcal{L}_I = \frac{\Omega}{4!}\Big(\big(\partial^{\nu}N_{\nu\alpha} - \frac{1}{4}\partial_{\alpha}N\big)\big(\partial_{\beta}N^{\beta\alpha} - \frac{1}{4}\partial^{\alpha}N\big)\Big)^2 \tag{8.54}$$

The interaction vertex is the following

$$\begin{aligned} 3\mathcal{V}^{\vartheta\epsilon\sigma\chi}_{\mu\nu\Sigma\omega}(k,m,h,l) = \Omega\Big\{ & \Big(\Delta^{\alpha\beta}_{\mu\nu}k_{\alpha} - k^{\beta}\frac{\eta_{\mu\nu}}{4}\Big)\Big(\Delta^{\sigma\chi}_{\tau\beta}m^{\tau} - m_{\beta}\frac{\eta^{\sigma\chi}}{4}\Big)\Big(\Delta^{\alpha\gamma}_{\Sigma\omega}h_{\alpha} - h^{\gamma}\frac{\eta_{\Sigma\omega}}{4}\Big)\Big(\Delta^{\vartheta\epsilon}_{\rho\gamma}l^{\rho} - l_{\gamma}\frac{\eta^{\vartheta\epsilon}}{4}\Big) \\ & + \Big(\Delta^{\alpha\gamma}_{\mu\nu}k_{\alpha} - k^{\gamma}\frac{\eta_{\mu\nu}}{4}\Big)\Big(\Delta^{\sigma\chi}_{\tau\beta}m^{\tau} - m_{\beta}\frac{\eta^{\sigma\chi}}{4}\Big)\Big(\Delta^{\alpha\beta}_{\Sigma\omega}h_{\alpha} - h^{\beta}\frac{\eta_{\Sigma\omega}}{4}\Big)\Big(\Delta^{\vartheta\epsilon}_{\rho\gamma}l^{\rho} - l_{\gamma}\frac{\eta^{\vartheta\epsilon}}{4}\Big) \\ & + \Big(\Delta_{\mu\nu\alpha\beta}k^{\alpha} - k_{\beta}\frac{\eta_{\mu\nu}}{4}\Big)\Big(\Delta^{\tau\gamma\sigma\chi}m_{\tau} - m^{\gamma}\frac{\eta^{\sigma\chi}}{4}\Big)\Big(\Delta^{\alpha\beta}_{\Sigma\omega}h_{\alpha} - h^{\beta}\frac{\eta_{\Sigma\omega}}{4}\Big)\Big(\Delta^{\vartheta\epsilon}_{\rho\gamma}l^{\rho} - l_{\gamma}\frac{\eta^{\vartheta\epsilon}}{4}\Big)\Big\} \end{aligned} \tag{8.55}$$

The symbol $\Delta^{\alpha\beta}_{\mu\nu} = \frac{1}{2}\big(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}\big)$ denotes the symmetric identity.

The Feynman graph representing this vertex is given below

$\mathcal{V}^{\vartheta\epsilon\sigma\chi}_{\mu\nu\Sigma\omega}(k,m,h,l) =$ 

(8.56)

Regarding the contribution for the polarization tensor due to this interaction, it is traceless by construction as a kind of Ward identity due to the Weyl symmetry. However, it is not invariant under the action of the reducible symmetry operator. Then, according to the Wilsonian perspective, it is associated to the emergence of radiative terms in the quantum action that are not present in the bare lagrangian. They generate new poles associated to the spin 0 sector violating the unitarity of the theory.

Interestingly, this interaction is renormalizable. The associated coefficient is dimensionless and the superficial degree of divergence of the diagrams composed by just this kind of vertex do not increase with its number. It is given by $D = 4L - 4P_\gamma + \big(4V - N_\gamma\big)$. The last term is due to the presence of 4 derivatives in each vertex distributed into the four boson lines. Using the topological relation $L = P_\gamma - V + 1$, we get $D = 4 - N_\gamma$.

Now we provide a gauge invariant massive extension of this theory with good unitarity and renormalization properties. We could derive a model with similar properties, also given in terms of a symmetric tensor field, in $D = 1+1$ dimensions from the Kaluza-Klein dimensional reduction of the linearized K-term of the new massive gravity [45]. For the present case, our proposal for a gauge invariant massive electrodynamics without the need of Stueckelberg compensating fields, is the following

$$S_M = \frac{1}{2}\int d^4x\Big(\partial^\alpha N_{\mu\alpha}\big(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\big)\partial^\alpha N_{\nu\alpha} + M^2\big(\partial^\nu N_{\nu\alpha} - \frac{1}{4}\partial_\alpha N\big)^2\Big) \tag{8.57}$$

Although the full reducible symmetry is now restricted to ones whose local parameter obeys

$$\epsilon^{\gamma\sigma\beta\rho}\epsilon_{\gamma\mu\nu\alpha}\partial_\sigma\partial^\mu\Lambda^{\nu\alpha}{}_{\beta\rho} = 0 \tag{8.58}$$

the radiative corrections still respects its full version, ensuring a spin 1 character for the model, if we consider interaction terms with the same structure as the ones of the previously analyzed models and, in this case, the model is perfectly unitary as we are going to see.

The restricted reducible symmetry implies in a different structure for the field operator functional representing the source that couples to the gauge field $T_{\mu\nu}(x) = \eta_{\mu\nu}R(x) + \partial_{(\nu}J_{\alpha)}(x)$ where $J_\alpha(x)$ is not necessarily conserved.

The Weyl symmetry imposes

$$4R(x) + \partial^\mu J_\mu(x) = 0 \tag{8.59}$$

It is possible to find a Ward-like identity by applying the full reducible symmetry operator on the gauge field inverse propagator. In order to illustrate it, let's consider a specific source term with $R(x) = -\frac{\Box}{4}(\bar\psi\psi)$ (as an example for the case of Fermionic matter with $J_\mu = \bar\psi\gamma_\mu\psi + \partial_\mu(\bar\psi\psi)$) which leads to

$$\partial^\alpha\partial^\gamma\frac{\delta^2\Gamma}{\delta N_{\mu\nu}(x)\delta N_{\chi\varepsilon}(y)}\epsilon_{\mu\sigma\alpha\beta}\epsilon_{\nu\Omega\gamma\psi} - i\frac{\Box}{4}\epsilon^{\gamma}{}_{\sigma\omega\beta}\epsilon_{\gamma\Omega\Gamma\psi}\partial^\omega\partial^\Gamma\int d^4u d^4\omega\,\mathcal{S}_{ac}(u,x)\frac{\delta^3\Gamma}{\delta N_{\chi\epsilon}(y)\delta\psi_c(u)\delta\bar\psi_b(\omega)}\mathcal{S}_{ba}(u,x) + \quad \dots \quad = 0 \tag{8.60}$$

where the terms in ellipsis denote gauge fixing and massive breaking terms from the quadratic part, the derivatives acts on the coordinate x and $\mathcal{S}_{ab}(x,y)$ denotes the complete Fermion propagator with explicit spinor indices.

It clearly shows that even in the massive case, if $R(x) = 0$, the violation of the residual symmetry is soft (it is due to the Gaussian sector.). This, together with the Weyl symmetry invariance, implies that the interaction terms, in this case, won't generate radiative contributions to the two point structure which are not proportional to $P_{ss}^{(1)}$, see the previous section. For $R(x) \neq 0$, the renormalized quantum action give dynamics for spin 0 zero sectors avoiding the possibility of a purely spin 1 theory. Further, we will analyze the saturated amplitude and make a connection with this result.

Now, we make a digression to investigate the physical content of the theory from a semi-classical analysis. It is possible to show that the equations of motion below, for the specific case of a conserved current $J_\mu(x)$ (with $R = 0.$), describe spin one degrees of freedom

$$-\frac{1}{2}\partial_\alpha\Big(\big(\Box + M^2\big)\partial^\beta N_{\nu\beta}(x) - \partial_\nu\big(\partial_\mu\partial_\sigma N^{\mu\sigma}(x)\big)\Big) - \frac{1}{2}\partial_\nu\Big(\big(\Box + M^2\big)\partial^\beta N_{\alpha\beta}(x) - \partial_\alpha\big(\partial_\mu\partial_\sigma N^{\mu\sigma}(x)\big)\Big)$$
$$- \frac{M^2\Box}{16}N\eta_{\alpha\nu} + \frac{1}{4}\Big(M^2\eta_{\alpha\nu}\partial^\gamma\partial^\chi N_{\gamma\chi} + M^2\partial_\alpha\partial_\nu N\Big) + \partial_{(\nu}J_{\alpha)}(x) = 0 \tag{8.61}$$

where a source term $T_{\nu\alpha}(x) = \partial_{(\nu}J_{\alpha)}(x)$ is added to the system.

This equation is traceless as expected by its Weyl symmetry (we are considering the constraint on the sources.). Then, we are allowed to fix the gauge condition $\partial^\gamma\partial^\chi N_{\gamma\chi} = 0$.

Applying the $P_{ss}^{(1)}(k)$ spin 1 projector on the Fourier transform of this equation, we get

$$k_\alpha\Big(\big(-k^2 + M^2\big)k^\beta N^T_{\nu\beta}(k)\Big) + k_\nu\Big(\big(-k^2 + M^2\big)k^\beta N^T_{\alpha\beta}(k)\Big) = ik_{(\nu}J_{\alpha)}(k) \tag{8.62}$$

where the superscript T denotes the transverse part of a vector.

Now, using this equation ad the gauge condition, contracting ∂^α with the equation (8.61), we find

$$k^2 N(k) = 0 \tag{8.63}$$

The model has still a Weyl residual symmetry with harmonic parameters. Then, the trace can be gauged away by such transformations. There is also the residual symmetry due to the gauge fixing $k^4 {P_{ss}^{(2)}}_{\mu\nu\alpha\beta}N^{\alpha\beta} = 0$ [10] of the reducible transformations. They can be used to eliminate the transverse and traceless part of the tensor field. Using these results in the original equation of motion, we get

$$\Big(\big(-k^2 + M^2\big)k^\beta N_{\nu\beta}(k)\Big) = iJ_\nu(k) \tag{8.64}$$

Therefore, we have a vector defined as $k^\beta N_{\nu\beta}(k) \equiv V_\nu(k)$ with $k_\mu V^\mu(k) = 0$. Considering the constraints, we get the spin 1 structure for the tensor field. For the case of a free field, we have

$$N_{\nu\beta}(k) = \frac{k_\nu V_\beta(k) + k_\beta V_\nu(k)}{M^2} \tag{8.65}$$

The possibility of deriving such a mass term is due to the use of a higher rank tensor in this dual description. Regarding theories with a spin 3-like tensor structure [35, 36], the idea of describing linearized gravity by means of longitudinal excitations, in analogy to what is done here, opens a bigger variety of possible couplings due to the three indices of the field. The possibility of a massive gravity, which is a topic of recent discussions [37], with a gauge invariant structure would be welcome since this invariance is related to the principle of relativity which is the epistemological basis of the theory.

Regarding the massive model, we can write it conveniently in terms of spin projectors

[10] Which is a suitable condition to fix these transverse and traceless transformations.

$$S_M = \frac{1}{2}\int d^4x N^{\mu\nu}\Big(-\frac{1}{2}\Box(\Box + M^2)P^{(1)}_{ss} + \big(\tilde{\lambda}\Box^2 - \frac{9M^2\Box}{16}\big)P^{(0)}_{\omega\omega} + \lambda\big(\Box^4 P^{(2)}_{ss}\big) - \frac{3M^2\Box}{16}P^{(0)}_{ss} + \frac{3\sqrt{3}\Box M^2}{16}\big(P^{(0)}_{\omega s} + P^{(0)}_{s\omega}\big)\Big)_{\mu\nu\alpha\beta} N^{\alpha\beta} \tag{8.66}$$

where the gauge fixing sector related to the reducible local freedom is proportional to the square of the condition $\Box^2 P^{(2)}_{ss\ \mu\nu\alpha\beta}N^{\alpha\beta} = 0$, a convenient gauge fixation for this case in which the reducible gauge symmetry is restricted to transverse and traceless transformations which are proportional to the spin 2 projector. The Weyl symmetry is fixed as in the previous sections.

From the completeness and the specific structure of the algebra of the spin projectors, we can calculate the propagator

$$\mathcal{G}_{\mu\nu\alpha\beta} = \Big(\frac{1}{\lambda k^8}P^{(2)}_{ss} - \frac{2}{k^2(k^2 - M^2)}P^{(1)}_{ss} + \frac{3(1 + \frac{16k^2\tilde{\lambda}}{9M^2})}{\tilde{\lambda}k^4}P^{(0)}_{ss} + \frac{1}{\tilde{\lambda}k^4}P^{(0)}_{\omega\omega} + \frac{\sqrt{3}}{\tilde{\lambda}k^4}\big(P^{(0)}_{\omega s} + P^{(0)}_{s\omega}\big)\Big)_{\mu\nu\alpha\beta} \tag{8.67}$$

The saturated amplitude from which we derive the necessary condition for tree level unitarity, with the sources restricted to the conventional spin one structure ($R(x) = 0$, see (8.59)) reads

$$A(k) = iT^{*\mu\nu}(k)\ \mathcal{G}_{\mu\nu\alpha\beta}T^{\alpha\beta}(k) = -i\frac{J^*_{T\mu}J^\mu_T}{\big(k^2 - M^2\big)} \tag{8.68}$$

Since the system is gauge invariant, the source must be of the form $T^{\mu\nu}(k) = k^{(\mu}J^{\nu)}_T$ for the case of vanishing $R(x)$. Then, using the transverse condition $k_\mu J^\mu_T(k) = 0$ and the frame $k^\mu = \big(m, 0, 0, 0\big)$, we get the residue

$$\mathcal{RA}(k) = i\sum_{n=1}^{3}|J^n_T|^2 \tag{8.69}$$

whose imaginary part $\Im(\mathcal{RA}(k)) > 0$ is positive.

Therefore, this theory fulfils the necessary condition for unitarity and it is also in agreement with our Ward identity analysis. Moreover, all the steps regarding the verification of the optical theorem derived in the previous sections can be applied to this model. It is indeed an unitary gauge invariant massive extension of the dual electrodynamics. It is worth mention that when $R(x) \neq 0$, using (8.59) leads to a saturated amplitude independent of the gauge parameters, as it should be, but with massive/massless poles in the $P^{(1)}_{ss}$ and $P^{(0)}_{ss}$ sectors whose residues have indefinite sign leading to unitarity violation.

with regard to the Ward identities, the Weyl symmetry invariance implies in

$$-\eta^{\gamma\omega}\frac{\delta^2\Gamma}{\delta N^{\gamma\omega}(k)\delta N^{\mu\nu}(-k)} + \tilde{\lambda}k^2 k_\mu k_\nu = 0 \tag{8.70}$$

For the case with $R(x) = 0$, we also have a condition on the gauge field self energy

$$-k^\alpha k^\gamma \pi^{\mu\nu\chi\varepsilon}(k)\epsilon_{\mu\sigma\alpha\beta}\epsilon_{\nu\Omega\gamma\psi} = 0 \tag{8.71}$$

It means that the trace of the inverse of the complete propagator has no radiative corrections and it can be easily verified by a careful analysis of (8.66) and the use of the definition of the spin projectors. For the case $R(x) = 0$, the radiative corrections for the two point part of the action are also in the kernel of the full reducible symmetry operator even though there is a break at the Gaussian sector.

Regarding the loops of this theory, if we use the same derivative vertex as the one employed in dual spinor QED_4 the radiative contributions would be the same (up to longitudinal terms that do not contribute to the gauge invariant amplitudes due to the Gordon identity.) as in this massless case but with a massive pole. Regarding the previous section developments, we note that our gauge invariant massive QED_4 also fulfill the unitarity requirements due to the optical theorem with the difference being the fact that in this specific case we do not take the massless limit at the end of the calculations.

These conclusions are due to the fact that considering a derivative vertex ($R(x) = 0$) contracted to the propagators appearing in the internal gauge field lines, the contribution of the $P_{ss}^{(0)}$ sector is eliminated in all loops in accordance with our discussion from the classical and quantum aspects of the model.

Although the one loop vacuum polarization tensor for this new model has the same form as the one obtained in the previous sections, the self energy tensor has now an internal line with a massive gauge boson structure. We also should mention that the model is renormalizable and the renormalization factor of the mass parameter is related to the one of the Bosonic gauge field as $Z_m - 1 = 1 - Z_{field}$ because the one loop radiative corrections for $R(x) = 0$, according to the Ward-like identity analysis and power counting, are proportional to $p^2 P_{ss}^{(1)}(p^2)\big(div.{+}\log[f(p^2)]\big)$, with the first term being a logarithm divergent one and the second a logarithm of a given function of p^2 (The preservation of the full reducible symmetry in the radiative corrections imply in a pure spin 1 content for them.). A similar discussion can be found in [46].

Differently from the Proca theory, our model has gauge symmetry and considering the same interaction vertex as in the dual spinor QED_4 case, the internal gauge field line insertions on the loops leads to factors of the form $\gamma^\mu\gamma_\mu/(p^2 - M^2) + \lambda p_\mu\gamma^\mu p_\nu\gamma^\nu/p^4 + p_\mu\gamma^\mu p_\nu\gamma^\nu/p^2(p^2 - M^2)$ which indeed goes as $1/p^2$ at the ultraviolet regime which means that the model has the same power counting structure as the QED_4. This is due to the fact that the $P_{ss}^{(0)}$ sector of the propagator is orthogonal to the vertex tensor structure and therefore do not contribute to the radiative corrections. It guarantee the general form for the polarization tensor outlined in the previous paragraph. However, it is important to mention that although the Proca theory propagator have a bad high energy behaviour, it is due to longitudinal sectors that do not contribute if the sources are conserved or the interaction has the minimal coupling structure. Therefore, in this case, it is also renormalizable [47].

Since this is not just a dual description of quantum electrodynamics but its massive extension, it is interesting to derive an explicit integral formula for the self energy tensor. We consider the case of the spinor electrodynamics.

First of all, we develop the expression of its imaginary part [48], up to longitudinal terms that do not contribute when the external Fermions are taken into account in order to build the gauge invariant amplitude [11]

[11] This part is proportional to the integration of $\gamma^\mu\gamma^\nu(p-k)_\mu(p-k)_\nu\big(\frac{1}{(p-k)^2((p-k)^2-M^2)} + \frac{\lambda}{(p-k)^4}\big)$

$$
\begin{aligned}
-2\Im\big[\Sigma(p_0)\big] =& e^2\int\frac{d^4k}{(2\pi)^2}\theta(k_0)\theta(p_0-k_0)\gamma^\mu(\gamma^0k_0+m)\gamma_\mu\delta\big((p-k)^2-M^2\big)\delta\big(k^2-m^2\big)\\
=&e^2\int\frac{d^3kdk_0}{(2\pi)^2}\theta(p_0-k_0)\gamma^\mu(\gamma^0k_0+m)\gamma_\mu\delta\big((p-k)^2-M^2\big)\frac{\delta\big(k_0-E_k\big)}{2E_k}\\
=&e^2\int\frac{d^3k}{(2\pi)^2 2E_k}\theta(p_0-E_k)\gamma^\mu(\gamma^0E_k+m)\gamma_\mu\delta\big(p_0^2-2p_0E_k+\Delta m^2\big)\\
=&e^2\int\frac{d|\vec{k}||\vec{k}|^2}{4\pi E_k|p_0|}\theta\big(p^2-(M+m)^2\big)\theta(p_0-E_k)\gamma^\mu(\gamma^0E_k+m)\gamma_\mu\delta\Big(E_k-\frac{p_0^2+\Delta m^2}{2p_0}\Big)\\
=&e^2\int\frac{dE_k|\vec{k}|}{4\pi|p_0|}\theta\big(p^2-(M+m)^2\big)\theta(p_0-E_k)\gamma^\mu(\gamma^0E_k+m)\gamma_\mu\delta\Big(E_k-\frac{p_0^2+\Delta m^2}{2p_0}\Big)\\
=&\theta\big(p^2-(M+m)^2\big)\frac{e^2}{8\pi}\theta(p_0)\gamma^\mu\sqrt{\big(1+\frac{2(\Delta m^2-2m^2)}{p_0^2}+\frac{\Delta m^4}{p_0^4}\big)}\Big(m+\gamma^0p_0\frac{(p_0^2+\Delta m^2)}{2p_0^2}\Big)\gamma_\mu
\end{aligned}
\tag{8.72}
$$

where we have considered the definition $\Delta m^2 = m^2 - M^2$ and the expression

$$
|\vec{k}| = \frac{|p_0|}{2}\sqrt{\big(1+\frac{2(\Delta m^2-2m^2)}{p_0^2}+\frac{\Delta m^4}{p_0^4}\big)}
\tag{8.73}
$$

for the modulus of the momentum which can be derived from the delta function constraint on the energy. We adopted the frame $p_\mu = (p_0, 0, 0, 0)$ in order to simplify the calculations. The Heaviside function is added in order to express a relation that can be easily derived from the momentum conservation in the vertices and is also in accordance with the expression of $|\vec{k}|$. We did not considered the term $\gamma^i k_i$ inside the integral since it vanishes due to the symmetry of the integral under $k_i \to -k_i$.

Now, going to a general frame and using the gamma matrices algebra, we get

$$
\Im\big[\Sigma(p)\big] = -\theta\big(p^2-(M+m)^2\big)\frac{e^2}{8\pi}\theta(p_0)\sqrt{\big(1+\frac{2(\Delta m^2-2m^2)}{p^2}+\frac{\Delta m^4}{p^4}\big)}\Big(2m-\gamma^\mu p_\mu\frac{(p^2+\Delta m^2)}{2p^2}\Big)
\tag{8.74}
$$

Since this model has the same degree of divergence as the usual QED_4, we can use dispersion relations and just one subtraction to obtain an integral representation [44] for the self energy tensor

$$
\begin{aligned}
\Sigma(p)-\Sigma(p_1) = & -\frac{e^2}{8\pi^2}\int_{(M+m)^2}^{\infty}ds\sqrt{\big(1+\frac{2(\Delta m^2-2m^2)}{s}+\frac{\Delta m^4}{s^2}\big)}\Big[\frac{2m(p^2-p_1^2)}{\big(s-p_1^2\big)\big[s-p^2-i\epsilon\big]}\\
&-\frac{\big(\gamma^\mu p_\mu(s-p_1^2)-\gamma^\mu p^1_\mu(s-p^2)\big)}{\big(s-p^2-i\epsilon\big)\big(s-p_1^2\big)}\frac{(s+\Delta m^2)}{2s}\Big]
\end{aligned}
\tag{8.75}
$$

where p_1^μ is a given arbitrary momentum four vector.

In the case in which the gauge boson mass is lower than $2m$ the same kind of phenomenology obtained from the usual quantum electrodynamics is recovered. However, in the opposite case, it can be used as an effective description for the phenomenon observed in He^4 in which the explanation of the experimental output could be modeled by an unknown massive gauge boson which decays creating an electron-positron pair [28]. This same process was observed previously in Be^8 samples [29]. Although there is an attempt to explain this situation by means of a well-defined axionic extension of the standard model, there is still a considerable possibility for this process to be an overall description of a collective phenomena instead of a new fundamental interaction. In this case, a gauge invariant massive electrodynamics as the one derived here can be a good effective description for this experimental fact. A massive spin 1 model with gauge symmetry can also play an important role in the effective description of the mediation of the interaction between charged pions.

8.7 Conclusion

Throughout this work, we obtained a dual generalized electrodynamics written in terms of a higher derivative tensor model. Although there is a similar model, related by master action approach to this one, it cannot provide a dual QED_4 since its derivative order is such that no radiative structure of QED_4 is recovered. Also, it do not have a well-defined massless limit differently than the model studied here.

In the section 3 we have shown that the physics in a conducting boundary for the dual electrodynamics indeed recover the result from the conventional QED_4 if we use a prescription inferred from the master action approach revised in section 2.

In the section 4, Ward-like identities were derived for the case of the dual spinor electrodynamics. It receives contributions from both the reducible and the Weyl gauge symmetry. One of the achievements of the remaining sections was to explicitly verify these identities.

Regarding the section 5, the Feynman rules for the dual electrodynamics were derived and the expected form for the self energy tensor was recovered. Also the Ward identities were verified as a heritage of the usual transverse structure of quantum electrodynamics. The fact that it is compatible with the optical theorem was verified due to the use of the prescription for the external states suggested by the master action technique .

The section 6 was devoted to analyze some new interaction terms for the system, exploring its higher rank field structure. We analyzed both gauge-Fermion couplings and a gauge invariant self interaction. Although they have some interesting features, some important content such as unitarity are missing due to the violation of part of the full reducible symmetry. Then, we focused on the search for new massive terms with a structure that is invariant under local Weyl symmetry transformations. We could manage to find it by the cost of restricting the possible interacting vertex structure of the model, something that is not possible for the usual vector description if one wants to keep the same superficial divergence structure as in the usual QED_4. We also give an integral representation for the spinor self energy tensor in the presence of the massive gauge boson internal line.

As a future perspective, we intend to provide an analogous treatment as the one given here for the case of spin 2 theories described by a rank 3 tensor field with just the spin 2 content being active, something as a theory described by $\partial^\nu h_{\nu\mu\gamma}$. with the knowledge of the spin 3 projector algebra [35, 36] and writing the linearized Einstein-Hilbert model in terms of $\partial^\nu h_{\nu\mu\gamma}$ in analogy to our present model, which can be put in a similar form but in terms of the Maxwell action [19, 40], we can find a higher derivative dual description of gravity with derivative coupling with matter energy-momentum tensor ensuring that the same divergence structure is recovered. An interesting idea is to provide an even higher derivative extension in order to obtain a renormalizable model. with three indices, even more possibilities of new interactions are available and there is also the possibility of a gauge invariant massive gravity [37], such that the $\partial^\nu h_{\nu\mu\gamma}$ transformations may recover the required linearized diffeomorphism structure.

Another possibility is to verify if the self dual spin 1 model and the Maxwell-Chern-Simons one, besides being related by a master action duality relation, also exhibits the same radiative corrections when non-linear interactions with matter fields are added to the system.

8.8 Appendix A

The spin 1 projectors, from which the spin 2 are built, are listed below

$$\theta_{\mu\nu}(k) = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \quad , \quad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} \tag{8.76}$$

with the properties $\theta_{\mu\nu}(k)\theta^{\mu\alpha}(k) = \theta^{\alpha}_{\nu}(k)$, $\omega_{\mu\nu}(k)\omega^{\mu\alpha}(k) = \omega^{\alpha}_{\nu}(k)$, $\theta_{\mu\nu}(k)\omega^{\mu\alpha}(k) = 0$ and $\theta_{\mu\nu}(k) + \omega_{\mu\nu}(k) = \eta_{\mu\nu}$.

The spin 2 projectors are of the form

$$P^{(2)\mu\nu}_{ss\ \ \rho\sigma} = \frac{1}{2}\Big(\theta^{\mu}{}_{\rho}\theta^{\nu}{}_{\sigma} + \theta^{\mu}{}_{\sigma}\theta^{\nu}{}_{\rho}\Big) - \frac{\theta^{\mu\nu}\theta_{\rho\sigma}}{(D-1)} \tag{8.77}$$

$$P^{(1)\mu\nu}_{ss\ \ \rho\sigma} = \frac{1}{2}\Big(\theta^{\mu}{}_{\rho}\omega^{\nu}{}_{\sigma} + \theta^{\mu}{}_{\sigma}\omega^{\nu}{}_{\rho} + \theta^{\nu}{}_{\rho}\omega^{\mu}{}_{\sigma} + \theta^{\nu}{}_{\sigma}\omega^{\mu}{}_{\rho}\Big) \tag{8.78}$$

$$P^{(0)\mu\nu}_{ss\ \ \rho\sigma} = \frac{\theta^{\mu\nu}\theta_{\rho\sigma}}{(D-1)} \tag{8.79}$$

$$P^{(0)\mu\nu}_{\omega\omega\ \ \rho\sigma} = \omega^{\mu\nu}\omega_{\rho\sigma} \tag{8.80}$$

$$P^{(0)\mu\nu}_{\omega s\ \ \rho\sigma} = \frac{\omega^{\mu\nu}\theta_{\rho\sigma}}{\sqrt{(D-1)}} \tag{8.81}$$

$$P^{(0)\mu\nu}_{s\omega\ \ \rho\sigma} = \frac{\theta^{\mu\nu}\omega_{\rho\sigma}}{\sqrt{(D-1)}} \tag{8.82}$$

The completeness relations is the following

$$P^{(2)\mu\nu}_{ss\ \ \rho\sigma} + P^{(1)\mu\nu}_{ss\ \ \rho\sigma} + P^{(0)\mu\nu}_{ss\ \ \rho\sigma} + P^{(0)\mu\nu}_{\omega\omega\ \ \rho\sigma} = \frac{1}{2}\big(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \delta^{\nu}_{\rho}\delta^{\mu}_{\sigma}\big) \tag{8.83}$$

The projector algebra reads $P^{(s)}_{IL}P^{(\tilde{s}}_{JK} = \delta^{s\tilde{s}}P^{(s)}_{IK}$

8.9 Appendix B

The gamma matrices obey the relations

$$\Big\{\gamma^{\mu}, \gamma^{\nu}\Big\} = 2I_{4\times 4}\eta^{\mu\nu} \tag{8.84}$$

For the traces, we have

$$\begin{aligned} tr(\gamma^{\mu}) &= 0 = tr(\gamma^{\mu\ 1}...\gamma^{\mu n+1}) \ , \ tr(\gamma^{\mu}\gamma^{\nu}) = 4\eta^{\mu\nu} \\ tr(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) &= 4\Big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\Big) \end{aligned} \tag{8.85}$$

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Chapter 9

Higher Derivative Models in the Covariant Operator Formalism

9.1 Motivation

Higher derivative field theories can hardly be overestimated due to a wide range of their applications in different areas of physics. Interestingly, the addition of higher derivative operators can also be understood as quantum corrections [1]. Accordingly, as in [2], the Lee-Wick model, a higher derivative system, can be regarded as a model obtained by elevating the status of the Pauli-Villars regulators to dynamical degrees of freedom. It is then possible to cite the correlated Podolsky model [3, 4, 5] which is also of higher derivative order, a model that provides a well behaved potential energy for static charges even at the origin [6]. Then, it naturally leads to a finite self energy tensor for the electron.

This mentioned property is generally related to an improvement of the power counting renormalizability of a field theory and emerge in some specific contexts as the necessary addition of extra counter-terms in order to quantize the gravitational theory. It can be easily verified from the functional renormalization group analysis [7, 8, 9] for the gravitational theory. We can also cite the framework of the string theory in which a generalized gravitational theory with the addition of higher derivative curvature terms emerges in the low energy limit [10].

Another interesting feature of higher derivative systems is related to the dual description of field theories. It is possible to show that a particle with a given spin can be described by the longitudinal modes of tensors of higher rank than the usual ones generally employed to describe them. For example, a spin 0 particle can be described by a vector field instead of a scalar operator [11, 12], or even by an antisymmetric field as in the case of the massless Kalb-Ramond model [13, 14, 15]. Interestingly, the massive version of the latter describes spin 1 particles which means that,in the massless limit, a degree of freedom discontinuity may occur in some of these higher rank dual descriptions. A spin 1 particle also admits a symmetric tensor field description by means of a suitable lagrangian [11]. This model also has the previously mentioned degree of freedom jump in the massless limit as we can see by the behaviour of the potential between charged particles interacting by means of the exchange of a spin 1 boson represented by this tensor field. Therefore, we have founda that as well as the particle content disappears in the massless limit, the inter particle potential also vanishes [16]. Then, regarding the subject of our discussion, it is interesting to note that it is possible to provide higher derivative extensions of those dual models [11] with non-vanishing potentials in the massless limit [16] avoiding this degree of freedom discontinuity. Namely, the analysis of the case of a higher derivative vector model for a spin 0 particle is one of the subjects of this work.

Here, we intend to extend the Nakanishi covariant operator formalism [17] to quantize two Ostrogadskian systems [18, 19], more specifically, the one describing spinless particles by a vector field [11] and the Podolsky generalized electrodynamics[3]. We discuss features related to the reducible gauge symmetry structure [20] and a well-defined massless limit for the first and issues related to non-positive norm sector for the latter model.

The Nakanishi B field formalism consists of a Heisenberg description quantization which uses an indefinite metric Hilbert space whose positive norm subspace is defined by means of a subsidiary condition. It may be written in terms of the positive frequency part of the auxiliary B field or by using the BRST charge for Abelian and non-Abelian theories, respectively. This formalism furnishes a well-defined system that is free from first class ambiguities [21]. It means that, in the present case, we must have in mind the Ostrogadskian phase space structure, evaluate the content of the Poisson brackets for this specific situation and then obtain the initial data for the system by the correspondence principle. The complete structure for the commutators of the theory is obtained from this data and from the equations of motion of the system. We also call attention to the fact that in order to correctly quantize these models, without ambiguities, a higher derivative structure for the B field sector, the gauge fixing lagrangian, must be provided [22] and this is a point in which some researchers are confused.

This non-perturbative formalism has an important variety of applications from Abelian and non-Abelian gauge theories to quantum gravity. It has also a perturbative counterpart which gives an exact answer for the two dimensional BF theory by means of a well-defined Cauchy problem for commutators [23] and can be also used for QED_4 to get its first order radiative corrections [24]. Moreover, the general structure for the Wightman functions can be inferred by a method which consists of extracting the truncated n point functions from the n point commutators by imposing the requirement of energy positivity. For a review which includes perturbative B field quantization for QED_4 passing through an academic one loop model and string theory to the two dimensional quantum gravity, see [25].

The work is organized as follows. The second section is devoted to the vector field model describing spin 0 particles. There, we present the Ostrogadskian canonical momenta and infer the necessity to add extra auxiliary fields to quantize the system. In the next section, the operator equations of motion for this model are analyzed. In the fourth section, the correspondence principle is evoked to determine the initial data and, from the dynamics of the system, obtain the complete commutator structure. From the lessons of the previous sections, the fifth one is regarded to the B field quantization of the Podolsky electrodynamics in which the same previous steps are followed for this specific case. We comment about the subsidiary condition and others features of this electrodynamics. We also show that the commutator structure for the usual QED_4 is obtained in the infinite Podolsky mass limit, as it should be. The metric (+,-,-,-) is used throughout this work.

9.2 Canonical momenta and constraint analysis for the spin 0 vector model

The lagrangian that describes a spin 0 particle by means of a forth order derivative theory in terms of a vector field is given below [11]

$$\mathcal{L} = \frac{1}{2}\Big[\partial_\beta\Big(\partial_\mu B^\mu\Big)\partial^\beta\Big(\partial_\mu B^\mu\Big) - m^2\Big(\partial_\mu B^\mu\Big)^2\Big] + \epsilon^{\mu\nu\rho}\partial_\nu\phi_\rho\Big(\Box + m^2\Big)B_\mu + \partial^\mu\Omega\Big(\Box + m^2\Big)\phi_\mu \tag{9.1}$$

we choose to work in $D = 2 + 1$ dimensions since in this case just two auxiliary fields are necessary to provide a second class system from the beggining.

The gauge field presents the following local symmetry transformation $\delta B_\mu(x) \to \epsilon_{\mu\nu\sigma}\partial^\nu\Lambda_\sigma(x)$

and $\Lambda_\sigma(x)$ is a c-number vector symmetry parameter. This transformation has a clear reducible structure, obeying the constraint of transversality.

Now we are going to show that the introduction of two auxiliary terms written by means of a pseudo scalar and pseudo vector [1] fields turn the model into a second class one and avoids any quantum ambiguity. To this end, we must introduce the definition of momenta in a Ostrogadskian system [18], namely, for the case in which the lagrangian depends on the velocities and accelerations. We are going to see that the auxiliary vector field imposes a constraint[2] $(\Box + m^2)\Big(\epsilon^{\mu\nu\beta}F^B_{\mu\nu}(x) - 2\partial^\beta\Omega\Big) = 0$ which is compatible with the equations of motion and guarantee the system's consistency. The necessity to introduce the pseudo scalar field is to eliminate the first class structures that comes from the definition of the auxiliary vector field momenta [17].

After this digression, we return to the generalized canonical momenta discussion which, in this case, turn out to be

$$
\begin{aligned}
p_\Phi(x) \equiv & \Big[\frac{\partial\mathcal{L}}{\partial(\partial_0\Phi(x))} - 2\partial_i\Big(\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\Phi(x))}\Big) - \partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\Phi(x))}\Big]\\
\pi_\Phi(x) \equiv & \frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\Phi(x))}
\end{aligned}
\tag{9.2}
$$

where $\Phi(x)$ is a generic notation for the fields occurring in the lagrangian.

Then, according to this definition, we have

$$
\begin{aligned}
&\pi^B_\mu(x) = \Big(\partial_0\partial_0 B^0(x) + \partial_i\partial_0 B^i(x)\Big)\delta^0_\mu + \epsilon_{\mu\nu\rho}\partial_\nu\phi_\rho(x),\\
&p^B_\mu(x) = -m^2\Big(\partial_0 B^0(x) + \partial_i B^i(x)\Big)\delta^0_\mu + 2\partial^k\partial_k\Big(\partial_0 B^0(x) + \partial_i B^i(x)\Big)\delta^0_\mu\\
&\ - 2\partial^k\Big(\partial_0\partial_0 B^0(x) + \partial_i\partial_0 B^i(x)\Big)\delta^\mu_k, - \Big(\partial_0\partial_0\partial_0 B^0(x) + \partial_i\partial_0\partial_0 B^i(x)\Big)\delta^0_\mu - \epsilon_{\mu\nu\rho}\partial^\nu\partial_0\phi_\rho(x),\\
&p^\phi_\mu(x) = \epsilon^{k0\mu}\Big(\Box + m^2\Big)B_k(x) - \partial_0\partial^\mu\Omega(x),\\
&\pi^\phi_\mu(x) = \partial_\mu\Omega(x),\\
&p^\Omega(x) = \Big(\Box + m^2\Big)\phi_0(x),\\
&\pi^\Omega(x) = 0
\end{aligned}
\tag{9.3}
$$

In order to define the nature of the primary constraints, we first introduce the fundamental Poisson brackets in this extended phase space $\Big(\Phi(x), \partial_0\Phi(x), p^\Phi(x), \pi^\Phi(x)\Big)$, with $\Phi(x)$ being a general notation for the fields of the theory.

Following this general structure, the non-vanishing brackets reads

$$
\begin{aligned}
\Big\{\Phi(x), p^\Phi(y)\Big\} &= \mathcal{I}\delta^2(x-y)\\
\Big\{\partial_0\Phi(x), \pi^\Phi(y)\Big\} &= \mathcal{I}\delta^2(x-y)
\end{aligned}
\tag{9.4}
$$

where $\mathcal{I}$ denotes the identity written in terms of the specific field tensor structure.

The primary constraints are the ones which generate non-dynamical relations between the phase space degrees of freedom. They are the following

[1] The pseudo vector and pseudo scalar structure is to ensure a lagrangian that changes as a scalar under discrete symmetry transformations.

[2] where $F^B_{\mu\nu}(x) \equiv \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x)$

$$\pi_j^B(x) = \epsilon^{j0k}\partial_0\phi_k(x) + \epsilon^{jl0}\partial_l\phi_0(x)$$
$$\pi_\mu^\phi(x) = \partial^\mu\Omega(x)$$
$$\pi^\Omega(x) = 0 \tag{9.5}$$

Considering the Poisson brackets structure we easily note that the system is of second class and all the quantum ambiguities are consistently avoided.

9.3 Equations of motion

The equations of motion for the model reads

$$-\partial^\mu\Big(\Box + m^2\Big)\Big(\partial_\nu B^\nu(x)\Big) + \Big(\Box + m^2\Big)\epsilon^{\mu\nu\rho}\partial_\nu\phi_\rho(x) = 0\ ,$$
$$\Big(\Box + m^2\Big)\partial_\mu\phi^\mu = 0\ ,$$
$$-\epsilon^{\mu\nu\rho}\partial_\nu\Big(\Box + m^2\Big)B_\mu(x) + \partial^\rho\Big(\Box + m^2\Big)\Omega(x) = 0 \tag{9.6}$$

The $B_\nu(x)$ equation of motion is the first one and we easily see that its divergence gives

$$\Box\Big(\Box + m^2\Big)\partial_\mu B^\mu(x) = 0 \tag{9.7}$$

. This equation has a deep analogy with the two pole structure of the Podolsky generalized electrodynamics, the difference being the facts that there is no negative residue and the pole field is a scalar combination $\partial_\mu B^\mu(x)$.

The last equation, which gives the gauge condition on the fields can be used to derive a massive pole equation to the auxiliary field $\Omega(x)$

$$\Box(\Box + m^2)\Omega(x) = 0 \tag{9.8}$$

Applying the differential operator $\epsilon_{\mu\nu\alpha}\partial^\nu$ on the first equation and using the auxiliary vector field transverse nature, gives

$$\Box\Big(\Box + m^2\Big)\phi^\mu(x) = 0 \tag{9.9}$$

9.4 From the commutator initial conditions to its final form at unequal times

The system is of second class. Therefore, a well-defined bracket in which the constraints are incorporated can be obtained. This is suitable for the use of the correspondence principle. Therefore, considering the Ostrogadskian phase space variables and the specific structure for the generalized momenta, we have

$$\Big[B_\mu(x), p_B^\nu(y)\Big]_0 = i\delta_\mu^\nu\delta^2(x-y)$$
$$\Big[\partial_0 B_\mu(x), \pi_B^\nu(y)\Big]_0 = i\delta_\mu^\nu\delta^2(x-y)$$
$$\Big[\Omega(x), p^\Omega(y)\Big]_0 = i\delta^2(x-y)$$
$$\Big[\phi_\mu(x), p_\phi^\nu(y)\Big]_0 = i\delta_\mu^\nu\delta^2(x-y)$$
$$\Big[\partial_0\phi_\mu(x), \pi_\phi^\nu(y)\Big]_0 = i\delta_\mu^\nu\delta^2(x-y)$$

which leads to

$$
\begin{aligned}
\Big[\partial_0 B_0(x), \partial_0\partial_0 B_0(y)\Big]_0 &= i\delta^2(x-y), \\
\Big[\partial_0 B_j(x), \epsilon^{i0k}\partial_0\phi_k(y)\Big]_0 &= i\delta^i_j\delta^2(x-y), \\
\Big[B_0(x), -\partial_0\partial_0\partial_0 B_0(y)\Big]_0 &= i\delta^2(x-y), \\
\Big[\phi_0(x), -\partial_0\partial_0\Omega(y)\Big]_0 &= i\delta^2(x-y), \\
\Big[\partial_0\phi_0(x), \partial_0\Omega(y)\Big]_0 &= i\delta^2(x-y), \\
\Big[\phi_i(x), -\epsilon^{j0k}\partial_0\partial_0 B_k(x)\Big]_0 &= i\delta^j_i\delta^2(x-y)
\end{aligned}
\tag{9.10}
$$

The operator equation of motion furnishes $\Box\Big(\Box+m^2\Big)^x\Big[\Omega(x),\Omega(y)\Big]=0$. Then, using the properties of the distributions below [17]

$$
\begin{aligned}
&\Box\Delta(x-y;s) = -s\Delta(x-y;s), \quad \Delta(x-y;s)|_0 = 0, \quad \partial^x_0\Delta(x-y;s)|_0 = \delta^2(x-y), \\
&\big(\Box+s\big)E(x-y;s) = \Delta(x-y;s), \quad E(x-y;s)|_{x_0=y_0} = 0, \quad (\partial^x_0)^3 E(x-y;s)|_{x_0=y_0} = \delta^2(x-y).
\end{aligned}
\tag{9.11}
$$

the general solution for this commutator is obtained

$$
\Big[\Omega(x),\Omega(y)\Big] = a\Delta(x-y,m) + b\Delta(x-y,0) \tag{9.12}
$$

Owing to the canonical momenta definition (9.3), we can obtain $\partial_0\Omega(x)$ in terms of $\pi^\phi_0(x)$ and then the general Ostrogadskian phase space structure (9.4), due to the correspondence principle, leads to

$$
\Big[\partial_0\Omega(x),\Omega(y)\Big]_0 = 0 \tag{9.13}
$$

It implies that $a+b=0$. We can also obtain $\partial^2_0\Omega(x)$ by means of $p^\phi_0(x)$ in (9.3). Since $p^\phi_0(x)$ and $\pi^\phi_0(x)$ are not phase space conjugate variables, we have

$$
\Big[\partial_0\Omega(x),\partial^2_0\Omega(y)\Big]_0 = 0 \tag{9.14}
$$

Then, we conclude that $a=0$ and therefore $\Big[\Omega(x),\Omega(y)\Big]=0$.

Since the equations of motion yields $\Box\Big(\Box+m^2\Big)^x\Big[\Omega(x),B_\nu(y)\Big]=0$, proceeding as in the previous lines, the initial data furnishes

$$
\Big[\Omega(x),B_\nu(y)\Big] = 0 \tag{9.15}
$$

The next step is to find the commutator structure for the gauge field. The easiest approach is to find the most general distribution that is in the kernel of its equation of motion differential operator

$$
\begin{aligned}
\Big[B_\mu(x),B^\nu(y)\Big] =& a\partial_\mu\partial_\nu\Delta(x-y,m^2) + d\eta_{\mu\nu}\Delta(x-y,m^2) + c\partial_\mu\partial_\nu E(x-y,0) + \\
& e\partial_\mu\partial_\nu\Delta(x-y,0) + g\eta_{\mu\nu}\Delta(x-y,0)
\end{aligned}
\tag{9.16}
$$

It must also be in accordance with the subsidiary condition imposed on the $B_\mu(x)$ field due to the auxiliary pseudo vector field equation of motion. It fixes $g = 0$.

The distributions properties and the initial conditions can be used to specify the arbitrary constants

$$\Big[\partial_0 B_0, \partial_0\partial_0 B_0\Big]_0 = i\delta^2(x-y) = \Big(a\nabla^4 - 2am^2\nabla^2 + am^4 + e\nabla^4 + d\nabla^2 - dm^2 + 2c\nabla^2\Big) \quad (9.17)$$

In order to cancel the ∇^4 term we must have $a = -e$. To eliminate the ∇^2 the constraint $2c - 2am^2 + d = 0$ must be imposed. Also, to recover the initial data, it is also required that $am^4 - dm^2 = i$.

To obtain further relations between the undetermined constants, we use the following initial conditions

$$0 = \Big[B_i(x), \partial_0 B^j(y)\Big]_0 = a\partial_i\partial_j\delta^2(x-y) + e\partial_i\partial_j\delta^2(x-y) + d\eta_{ij}\delta^2(x-y) \quad (9.18)$$

This definition is due to the correspondence principle and the Ostrogadskian phase space structure.

Then, we are left with $d = 0$ and $a = -e$. Using the previous constraints between the constants, we finally obtain

$$\Big[B_\mu(x), B_\nu(y)\Big] = \frac{i}{m^4}\partial_\mu\partial_\nu\Big(\Delta(x-y,m^2) - \Delta(x-y,0)\Big) + \frac{i}{m^2}\partial_\mu\partial_\nu E(x-y,0) \quad (9.19)$$

It leaves us with the exact scalar commutator for the pole operators

$$\Big[\partial^\mu B_\mu(x), \partial^\nu B_\nu(y)\Big] = i\Delta(x-y,m^2) \quad (9.20)$$

It is worth mention that this structure defines a spin 0 particle even at the massless limit. An alternative check to this conclusion is the fact that the inter particle potential derived for this model in [16] is nontrivial at this limit. This model is a higher derivative version of the second order vector spin 0 one [11]. Since it loses its particle content at $m \to 0$, it indicates that the use of a higher derivative structure may avoid discontinuities.

The next step is to calculate the commutator between the auxiliary pseudo vector and pseudo scalar fields. Latter, we calculate the commutator between the gauge field and this mentioned pseudo vector field.

These two results are useful for deriving a suitable subsidiary condition to avoid the emergence of the auxiliary fields in the positive Hilbert subspace.

The general form for the mentioned commutator is given below

$$\Big[\phi_\rho(x), \Omega(y)\Big] = \partial_\rho\Big(a\Delta(x-y,m^2) + b\Delta(x-y,0)\Big) \quad (9.21)$$

This is due to the fact that both field operators are in the kernel of $\Box\Big(\Box + m^2\Big)$ and the scalar one is coupled to the longitudinal part of $\phi_\mu(x)$.

Using the initial condition $\Big[\phi_0(x), \partial_0\partial_0\Omega(y)\Big]_0 = -i\delta^2(x-y)$ and the distribution properties, we find $a = -b$ and $a = -\frac{i}{m^2}$. Then, we have

$$\Big[\phi_\rho(x), \Omega(y)\Big] = -\frac{i}{m^2}\partial_\rho\Big(\Delta(x-y,m^2) - \Delta(x-y,0)\Big) \quad (9.22)$$

Now we derive the commutator between the gauge and vector auxiliary field. As already mentioned, this and the previous commutator are the key ones to understand how a consistent definition of the positive Hilbert space must be achieved. The general form is given below

$$\begin{aligned}\Big[B_\mu(x),\phi_\rho(y)\Big] &= i\epsilon_{\mu\rho\nu}\partial^\nu\Big(a\Delta(x-y,m^2)+b\Delta(x-y,0)\Big)\\ &+c\,\partial_\mu\partial_\rho\Delta(x-y,0)+d\Big(\eta_{\mu\rho}+\frac{\partial_\mu\partial_\rho}{m^2}\Big)\Delta(x-y,m^2)\end{aligned}\tag{9.23}$$

This commutator is in accordance with the fact that $\phi_\rho(x)$ has a pole in $\Box\Big(\Box+m^2\Big)$ and the constraint $\partial_\mu\phi^\mu(x)=0$. Using the initial conditions, we have

$$\Big[B_j,\partial_0\partial_0\phi_m\Big]_0=\epsilon_{jm0}\partial^0\Big(a\partial_0\partial_0\Delta(x-y,m^2)|_0+b\partial_0\partial_0\Delta(x-y,0)|_0\Big)=i\epsilon_{mj}\delta^2(x-y)\tag{9.24}$$

In order to fulfil the requirement above, we must set $a=-b$ and $a=\frac{i}{m^2}$. From the phase space structure, the correspondence principle gives $\Big[B_i(x),\phi_0(y)\Big]_0=0$, it then generate a relation between the constants $c=-\frac{d}{m^2}$. At this point, to simplify the expression, we demand that the commutator structure must be in accordance with the subsidiary condition $\epsilon^{\mu\nu\rho}\partial_\nu\Big(\Box+m^2\Big)B_\mu(x)=\partial^\rho\Big(\Box+m^2\Big)\Omega(x)$ and the previously obtained commutator structure can obviously help us in this procedure. This requirement fixes $c=d=0$ which allow us to obtain the final form of the commutator

$$\Big[B_\mu(x),\phi_\rho(y)\Big]=\frac{i}{m^2}\epsilon_{\mu\rho\nu}\partial^\nu\Big(\Delta(x-y,m^2)-\Delta(x-y,0)\Big)\tag{9.25}$$

This shows that the auxiliary field interact with just non-physical (the transverse sector.) projections of the gauge field, as it should be. One easily note that, at the massless limit, the auxiliary field acts in the redundant gauge sector. The fact that this structure has a well-defined massless limit can inferred by the following Taylor expansion $\Delta(x-y,m^2)=\Delta(x-y,0)-m^2E(x-y,0)+...$ for the Pauli Jordan distribution. Therefore, a good subsidiary condition to define the physical subspace is

$$\phi^+_\mu(x)|\text{phys}\rangle=0,\quad\forall|\text{phys}\rangle\in\mathcal{H}_{\text{phys}}.\tag{9.26}$$

where $\phi^+_\mu(x)=\phi^+_{\mu\ m}(x)+\phi^+_{\mu\ 0}(x)$ denotes the sum of the positive frequency parts of the massive and massless solutions of the $\phi(x)$ field equations of motion. It avoids the spurious gauge field projections as well as both the auxiliary fields to appear on the measurable positive metric Hilbert subspace. We imeadiatelly notice that $\partial_\mu B^\mu(x)$ fulfils the condition to be in the positive Hilbert subspace.

The remaining commutator that we should investigate is $\Big[\phi_\mu(x),\phi^\nu(y)\Big]$. The most general form that takes into account the operator equation of motion and the transverse condition is

$$\begin{aligned}\Big[\phi_\mu(x),\phi_\nu(y)\Big] &= a\Big(\eta_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{m^2}\Big)\Delta(x-y,m^2)+b\partial_\mu\partial_\nu\Delta(x-y,0)\\ &+\epsilon_{\mu\nu\rho}\partial^\rho\Big(c\Delta(x-y,m^2)-d\Delta(x-y,0)\Big)\end{aligned}\tag{9.27}$$

In higher derivative theories, $\partial_0\phi_\mu(x)$ and $\phi_\mu(x)$ are not conjugated phase space variables and then, by the correspondence principle, must have the following vanishing equal time commutators.

$$0=\Big[\phi_k(x),\partial_0\phi_l(x)\Big]_0=a\Big(\eta_{kl}+\frac{\partial_k\partial_l}{m^2}\Big)\delta^2(x-y)+b\partial_k\partial_l\,\delta^2(x-y)\tag{9.28}$$

The result above implies that $a = b = 0$. If we use the transverse nature of $\phi_\mu(x)$, we obtain that $\partial_0\phi_0(x) = -\partial^i\phi_i(x)$. Then, since $\Big[\phi_k(x), \phi_l(y)\Big]_0 = 0$ by definition of the phase space variables, we have

$$\Big[\phi_i(x), \partial_0\phi_0(y)\Big]_0 = 0 = \epsilon_{i0k}\partial^k\delta^2(x-y)\Big(c-d\Big) \tag{9.29}$$

It implies that $c = d$. Again, by definition of phase space variables, we have

$$\Big[\partial_0\phi_i(x), \partial_0\phi_j(y)\Big]_0 = 0 \tag{9.30}$$

which imposes $c = 0$. Then, it means that $\phi_\mu(x)$ is a zero norm field.

The previous calculations show that differently from the case of a spin 0 particle described by a second order vector model, the higher derivative case, which is related to the former one by a master action technique [11], and therefore share the same particle content, is totally continuous in the massless limit. Namely, it continues to describes a spin 0 particle in that limit, as the commutator structure clearly reveals. It is in opposition to the case of a spin 1 particle described by a symmetric tensor model of second order in derivatives which loses its particle content in that limit [11, 12].

9.5 The Podolsky Higher Derivative Electrodynamics

The lagrangian for the Podolsky higher order electrodynamic theory reads [3, 4, 5]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2m^2}\partial_\lambda F^{\alpha\lambda}\partial^\rho F_{\alpha\rho} + \partial^\mu B(\frac{\Box}{m^2}+1)A_\mu - \frac{1}{2}\alpha B^2 \tag{9.31}$$

We add a higher derivative term for the gauge sector for two reasons. First, as we are going to see, it is the most general condition compatible to a pole equation for the vector field in the physical subspace [22]. The other is that, considering the Ostrogadskian phase space structure previously introduced, this higher order term will contribute to generate a set of non-vanishing generalized momenta responsible to turn the system into a second class one from the beginning since it is the most fundamental principle of the covariant operator formalism [17, 25].

The equations of motion are the following

$$\begin{aligned}
&\Big(\Box + m^2\Big)\partial^\mu A_\mu = +m^2\alpha B \\
&\Big(\Box + m^2\Big)\Big(\partial^\nu F_{\nu\mu} + \partial_\mu B\Big) = 0 \\
&\Box\Big(\Box + m^2\Big)B = 0
\end{aligned} \tag{9.32}$$

As previously mentioned, the gauge condition allows for finding $\Box\Big(\Box + m^2\Big)A_\mu(x) = 0$ up to B field terms that do not contribute in transition between physical states, as we are going to prove.

The phase space variables in this Ostrogadskian [18, 19] system reads

$$\begin{aligned}
p_\alpha(x) &= -F_{0\alpha}(x) - \frac{1}{m^2}\Big(\partial_k\partial_\lambda F^{0\lambda}(x)\delta^k_\alpha - \partial_0\partial_\lambda F_\alpha{}^\lambda - \partial_\alpha\partial_0 B\Big) \\
\pi_\alpha(x) &= \frac{1}{m^2}\Big(\partial_\lambda F^{0\lambda}(x)\delta^0_\alpha - \partial_\lambda F_\alpha{}^\lambda(x)\Big) + \frac{1}{m^2}\partial_\alpha B(x) \\
p_B(x) &= (\frac{\Box}{m^2}+1)A_0 \\
\pi_B(x) &= 0
\end{aligned} \tag{9.33}$$

The momenta definition furnishes two constraints which are of second class

$$\pi_0(x) = \frac{1}{m^2}\partial_0 B(x) \quad , \quad \pi_B(x) = 0 \tag{9.34}$$

Therefore, the careful introduction of the gauge fixing sector was successful in generating a system without any quantum ambiguities [21], see [22] for more information about it. Then, it is possible to obtain a well-defined bracket for which the costraints are valid in the strong form. Thus, considering the correspondence principle, we have

$$\begin{aligned}
\Big[\partial_0 A_i(x), \pi^j(y)\Big]_0 &= \Big[\partial_0 A_i(x), -\frac{1}{m^2}\partial_\lambda F^{j\lambda}\Big]_0 \to \Big[\partial_0 A_i(x), \partial_0\partial_0 A^j\Big]_0 = +im^2\delta_i^j\delta^3(x-y), \\
\Big[A_0(x), p_0(y)\Big]_0 &= \Big[A_0(x), \frac{1}{m^2}\partial_0\partial_0 B(y)\Big]_0 \to \Big[A_0(x), \partial_0\partial_0 B(y)\Big]_0 = im^2\delta^3(x-y), \\
\Big[A_i(x), p^j\Big]_0 &= \Big[A_i(x), -\partial_0 A^j(y) - \frac{1}{m^2}\partial^j\partial_k\big(\partial^0 A^k(y) - \partial^k A^0(y)\big) \\
&\quad - \frac{1}{m^2}\partial_0\Big(\partial_k(\partial_i A^k - \partial^k A_i) - \partial_0(\partial_i A^0 - \partial^0 A_i)\Big)\Big]_0 = i\delta_i^j\delta^3(x-y)
\end{aligned} \tag{9.35}$$

We also have

$$\Big[A_\mu(x), \partial_0 A^\nu(y)\Big]_0 = 0 \tag{9.36}$$

as it was expected due to the definition of the ostrogaskian phase space variables since in this case $\Gamma_\mu \equiv \partial_0 A_\mu$ is the canonical pair of π_ν. This information and the definition of p_B leads to

$$\Big[A_i(x), p^j\Big]_0 = i\delta_i^j\delta^3(x-y) \to \Big[A_i(x), \partial_0^3 A^j\Big]_0 = -im^2\delta_i^j\delta^3(x-y) \tag{9.37}$$

In order to infer the general structure of the commutator involving the gauge and the auxiliary one, we use its equations of motion to find the unequal time constraint

$$\Box\Big(\Box + m^2\Big)^y \Big[A_\mu(x), B(y)\Big] = 0 \tag{9.38}$$

Then, owing to Lorentz covariance, we can build the most general commutator

$$\Big[A_\mu(x), B(y)\Big] = a\partial_\mu\Delta(x-y;0) + b\partial_\mu\Delta(x-y;m^2) \tag{9.39}$$

If we use the initial condition below

$$\Big[A_0(x), \partial_0\partial_0 B(y)\Big]_0 = im^2\delta^3(x-y) \tag{9.40}$$

and the distribution properties

$$\partial_0^3\Delta(x-y;m^2) = \Big(\nabla^2 + m^2\Big)\partial_0\Delta(x-y;m^2) \tag{9.41}$$

and

$$\partial_0^x\Delta(x-y;m^2) = -\delta^3(x-y) \tag{9.42}$$

we find that $a = -b$ and $a = i$

$$\Big[A_\mu(x), B(y)\Big] = -i\partial_\mu\Big(\Delta(x-y;0) - \Delta(x-y;m^2)\Big) \tag{9.43}$$

Again, we conclude that the auxiliary field do not commute with the spurious longitudinal projection of the gauge field.

Commuting the B field with the operator equation of motion expressing the α gauge condition, and using the information from the commutator between the photon field and the $B(x)$, we find

$$\Big[B(x), B(y)\Big] = 0 \tag{9.44}$$

Which means that this is a zero norm field.

Regarding the commutator of the vector fields, we can obtain a equation to relate it with the ones just calculated

$$\Big(\Box + m^2\Big)\Box\Big[A_\mu(x), A_\nu(y)\Big] = -\alpha\partial_\mu\Big[B(x), A_\nu(y)\Big] - \frac{1}{m^2}\Big(\Box + m^2\Big)\Big[\partial_\mu B(x), A_\nu(y)\Big] \tag{9.45}$$

Therefore, we must have

$$\Big(\Box + m^2\Big)^x\Box^x\Big(\Box + m^2\Big)^y\Box^y\Big[A_\mu(x), A_\nu(y)\Big] = 0 \tag{9.46}$$

The general solution is[3]

$$\Big[A_\mu(x), A_\nu(y)\Big] = a\Big(\eta_{\mu\nu}\Delta(x-y;0) - \partial_\mu\partial_\nu E(x-y;0)\Big) + b\Big(\eta_{\mu\nu}\Delta(x-y,m^2) + \frac{1}{m^2}\partial_\mu\partial_\nu\Delta(x-y,m^2)\Big)$$
$$+ d\partial_\mu\partial_\nu E(x-y,0) + e\partial_\mu\partial_\nu E(x-y,m^2) + f\partial_\mu\partial_\nu\Delta(x-y,m^2) + c\partial_\mu\partial_\nu\Delta(x-y,0)$$

In order to obey the gauge condition, the system must fulfil the requirement

$$\Big(\Box + m^2\Big)\partial^\mu\Big[A_\mu(x), A_\nu(y)\Big] = +m^2\alpha\Big[B(x), A_\nu(y)\Big] = im^2\alpha\partial_\nu\Big(\Delta(x-y;0) - \Delta(x-y;m^2)\Big) \tag{9.47}$$

It helps us to specify the constants $d = i\alpha$ and $e = i\alpha$.

From the initial condition $\Big[A_i(x), \partial_0 A^j(y)\Big]_0 = 0$ we obtain $a + b = 0$ and $\frac{b}{m^2} + c + f = 0$.

In order to the commutator be in agreement with $\Big[\partial_0 A_i(x), \partial_0^2 A^j(y)\Big]_0 = im^2\delta_i^j\delta^3(x-y)$, we must equalize both members below

$$im^2\delta_i^j\delta^3(x-y) = -\delta_i^j\Big(a\nabla^2 + b(\nabla^2 - m^2)\Big)\delta^3(x-y)$$
$$+ \partial_i\partial_j\Big(-\frac{1}{m^2}(\nabla^2 - m^2)b + 2i\alpha - a - (\nabla^2 - m^2)f - c\nabla^2\Big)\delta^3(x-y) \tag{9.48}$$

It leads to the following conclusions $b = -a = i$, $\Big(\frac{i}{m^2} + f + c\Big) = 0$ and

$$\Big(2i\alpha - i - a + m^2 f\Big) = 0 \tag{9.49}$$

It furnishes $c = 2i\frac{\alpha}{m^2} - \frac{i}{m^2}$ and $f = -2i\frac{\alpha}{m^2}$. Therefore, the gauge field propagator reads

$$\Big[A_\mu(x), A_\nu(y)\Big] = -i\Big(\eta_{\mu\nu}\Delta(x-y,0) - \partial_\mu\partial_\nu E(x-y,0)\Big) + i\alpha\partial_\mu\partial_\nu E(x-y,0)$$
$$+ i\Big(\eta_{\mu\nu}\Delta(x-y,m^2) + \frac{1}{m^2}\partial_\mu\partial_\nu\Delta(x-y,m^2)\Big) + i\alpha\partial_\mu\partial_\nu E(x-y,m^2)$$
$$i\frac{(2\alpha-1)}{m^2}\partial_\mu\partial_\nu\Delta(x-y,0) - i\frac{2\alpha}{m^2}\partial_\mu\partial_\nu\Delta(x-y,m^2) \tag{9.50}$$

[3]In fact, this is not the most general one, since we are avoiding terms like $\eta_{\mu\nu}E(x-y,m^2)$ since the photon equation of motion clearly reveals that there is no poles like this in the transverse sector.

Using the fact that the simple and double pole massive Pauli-Jordan distributions tend to zero at $m \to \infty$, the maxwellian form for the commutator is recovered at this limit

$$\Big[A_\mu(x), A_\nu(y)\Big] = -i\Big(\eta_{\mu\nu}\Delta(x-y;0) - \partial_\mu\partial_\nu E(x-y;0)\Big) + i\alpha\partial_\mu\partial_\nu E(x-y;0) \tag{9.51}$$

The propagator for the photon field has a massive and a massless sector as its equation of motion suggests. The point is that the massless sector describes positive norm particles while the massive one describes negative norm states. In general we can use the auxiliary field to derive a consistent condition to define the positive Hilbert subspace. But in this case the situation is much different. Suppose we define this mentioned subspace as

$$B^+(x)|\text{phys}\rangle = 0, \quad \forall|\text{phys}\rangle \in \mathcal{H}_{\text{phys}}. \tag{9.52}$$

where $B^+(x) = B^+_{m=0}(x) + B^+_{m\neq 0}(x)$ is the sum of massive and massless positive frequency parts of the auxiliary operator.

Although this condition avoids spurious gauge projections to appear in the physical subspace, it do not eliminate transverse massive negative norm states.

This feature is not a failure of the B field approach, this model indeed present this kind of ghost particle as well as, for example, some higher derivative gravity models [26]. However, considering a huge mass for these negative norm particles to avoid their detection, this model, in the interacting case, can be used as an interesting effective description with an ultraviolet behaviour improvement when interactions are taken into account. This last property is reflected in the fact that the inter particle potential for this model is finite at the origin which is related to the finiteness of the electron self energy [1] for the case of the generalized QED.

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Chapter 10

Fadeev-Jackiw quantization of the Podolsky-Stueckelberg theory

10.1 Motivation

In this study, we demonstrate that the Fadeev-Jackiw method for the phase space constraint analysis [1, 2] for field theories can provide a suitable framework to obtain the canonical transformations, gauge symmetry and gauge fixing structures and, finally, the construction of the path integral for the so-called Podolsky-Stueckelberg theory [3, 4]. This model was recently proposed as a generalization of the Podolsky theory, [5]. This latter theory describes massive and massless photons in a Ostrogadskian structure [6, 7] of fourth order in derivatives which leads to an ultraviolet improvement for the interacting theory with Fermions. Although it provide the mentioned improvement, it still have infrared divergences. Then, the intent of this extension is to provide an infrared regularization for the Podolsky theory.

It consists on the usual Podolsky plus a mass term written in a Stueckelberg combination [8] responsible for keeping the gauge invariance. the emergence of both mass terms can be traced back to a generalization of the Higgs mechanism [4] and has an equivalent form to the thermodynamic mass generation responsible for the Debye screening [3]. The lagrangian has the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2m^2}\partial_\lambda F^{\alpha\lambda}\partial^\rho F_{\alpha\rho} + \frac{M^2}{2}\Big(A_\mu + \frac{1}{m_s}\partial_\mu B\Big)^2 \tag{10.1}$$

Which is invariant under the local transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda$, $B(x) \to B(x) - m_s\Lambda$.

Fixing the Lorenz gauge and using the residual invariance to eliminate the harmonic Stueckelberg field, we get the equation of motion

$$\Big[(\frac{\Box}{m^2} + 1)\Box + M^2\Big]A_\mu = J_\mu \tag{10.2}$$

where $J_\mu(x)$ represents an external source.

For the case of a static charge at the origin $J_\mu(x) = Q\delta_{\mu 0}\delta^3(\vec{x})$, the inter particle potential energy is of the form

$$A^0(\vec{x}) = \frac{Qm^2\big(e^{-m_- r} - e^{-m_+ r}\big)}{4\pi r\big(m_+^2 - m_-^2\big)} \tag{10.3}$$

with r representing the modulus of $\vec{x}$ and the pole masses defined as

$$m_\pm = \sqrt{\frac{m^2 \pm m\sqrt{m^2 - 4M^2}}{2}} \tag{10.4}$$

It is interesting to mention that depending on the values of m and M a complex pole can be reached leading to a confining phase. It resembles what happens in the Zwanziger-Gribov action [9] for Yang-Mills theory in which the presence of localizing ghosts responsible to restrict the physics to the Gribov region leads to an enhanced confining boson propagator without real poles. For the present model, the addition of interaction terms may lead to a possible dynamical phase transition between confining and non-confining configurations due to radiative mass corrections.

Regarding the method employed to analyze such a model, the Fadeev-Jackiw formalism, we write the lagrangian in a Legendre transform structure as

$$\mathcal{L} = a_i(\epsilon)\partial_0\epsilon^i - \mathcal{H}(\epsilon) \tag{10.5}$$

where ϵ^i denotes the phase space variables and $a_i(\epsilon)$ is a one form that depends on them. The Euler-Lagrange equations leads to [10, 11, 12]

$$\mathcal{F}_{ij}\partial_0\epsilon^j = \frac{\partial\mathcal{H}(\epsilon)}{\partial\epsilon^i} \tag{10.6}$$

with $\mathcal{F}_{ij} \equiv \frac{\partial a_j}{\partial\epsilon^i} - \frac{\partial a_i}{\partial\epsilon^j}$.

Then, the method consists in finding null eingenvectors V_i^A, labelled by A, of the $\mathcal{F}$ matrix because they represent phase space directions from which the Hamiltonian is independent. Obtaining these null directions leads to constraints $\Omega^A = V_i^A\frac{\partial\mathcal{H}(\epsilon)}{\partial\epsilon_i} = 0$ which are after grouped in a generalized extended phase space one form a_i' and are associated to a corresponding Lagrange multiplier field. The lagrangian becomes

$$\mathcal{L}^1 = \mathcal{L} + \lambda_A\Omega^A \tag{10.7}$$

Then, since the constraints are time independent and the dynamics is invariant under the addition of total time derivative terms, the lagrangian can be written in the following equivalent form

$$\mathcal{L} = a_i(\epsilon)\partial_0\epsilon^i + \Omega^A\partial_0\lambda_A - \mathcal{H}(\epsilon)|_\Omega \tag{10.8}$$

where $\mathcal{H}|_\Omega(\epsilon)$ denotes the Hamiltonian evaluated at the constraint surface.

If in the end of the process of finding all the constraints, there is yet some null eigenvectors that do not generate any new constraint, we can state that the system has gauge symmetry and new constraints must be imposed by hand to avoid this degree of freedom redundancy and then obtain a uniquely defined time evolution for the phase space variables in terms of an invertible generalized $\mathcal{F}$ matrix. In the case that the phase space do not present this degeneracy, all the null eigenvectors are related to constraints, the system do not have any local symmetry and the process stops when all of them are found and implemented in an extended phase space. In both cases, the final phase space $\mathcal{F}_{ij}$ matrix is found to be invertible, allowing to obtain the mentioned well-defined expression for the time evolution of the phase space variables ϵ_i.

Since it is possible to prove that the final extended $\mathcal{F}_{ij}$ matrix furnishes the inverse of the Dirac-brackets [12], a conclusion that easily follows when comparing our Lagrange equations with the equivalent expression given in terms of Poisson brackets $\partial_0\epsilon_i = \left\{\epsilon_i, \epsilon_j\right\}_D \frac{\partial\mathcal{H}(\epsilon)}{\partial\epsilon_j}$, it is then possible to link the Fadeev-Jackiw formalism to the theory's path integral formulation. By the use of the procedure of [13], the knowledge of the model phase space structure leads to a well-defined measure built by means of the $\mathcal{F}_{ij}$ matrix determinant. The construction of the path integral is based on the Darboux theorem, see the discussion of [10].

The section 2 is devoted to perform the Fadeev-Jackiw phase space procedure for the Podolsky-Stueckelberg model, finding all the constraints, canonical transformations, and the gauge fixing conditions for the model. Latter, in section 3, the path integral for the model is built by means of its phase space constraint structure.

10.2 Constraints, symmetries and gauge fixing

In order to proceed with the Fadeev-Jackiw analysis of the Podolsky-Stueckelberg model phase space content, we need to rewrite its lagrangian in a Legendre transform structure. Since this theory has higher order derivatives, we must employ the Ostrogadsky approach [6, 7] which means that an enlarged phase space must be considered to include the higher order cotangent variables into the game.

Owing to the lagrangian structure presented in the introduction, the generalized momentum for this higher derivative system reads

$$\begin{aligned}
\Pi^\mu \equiv & \Big[\frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu(x))} - 2\partial_i\Big(\frac{\partial \mathcal{L}}{\partial(\partial_i\partial_0 A_\mu(x))}\Big) - \partial_0\frac{\partial \mathcal{L}}{\partial(\partial_0\partial_0 A_\mu(x))}\Big] = F^{\mu 0} + \frac{1}{m^2}\left[\eta^{\mu j}\partial_j\partial_l F^{l0} - \partial_0\partial_\nu F^{\nu\mu}\right] \\
\phi^\mu(x) \equiv & \frac{\partial \mathcal{L}}{\partial(\partial_0\partial_0 A_\mu(x))} = \frac{1}{m^2}\Big(\partial_\nu F^{\nu\mu} - \eta^{\mu 0}\partial_j F^{j0}\Big) \\
\pi \equiv & \frac{\partial \mathcal{L}}{\partial(\partial_0\ B(x))} = \frac{M^2}{m_s}\big(A_0 + \frac{1}{m_s}\partial_0 B\big)
\end{aligned} \tag{10.9}$$

The Hamiltonian, considering the phase space Ostrogadski momenta structure of [4, 5], is given by

$$\begin{aligned}
\mathcal{H} = \Pi_0\Gamma^0 + \Pi_k\Gamma^k + \phi^k(\partial_k\Gamma_0 - \partial^l F_{lk} + \frac{\phi_k}{2a^2}) + \frac{1}{4}F_{kl}^2 + \frac{1}{2}(\Gamma_j - \partial_j A_0)(\Gamma^j - \partial^j A_0) \\
+ \frac{m_s^2}{2M^2}\pi^2 - m_s A_0\pi + \frac{M^2}{2}(A_k + \frac{\partial_k B}{m_s})^2 - \frac{1}{2m^2}(\partial^j\Gamma_j - \nabla^2 A_0)^2
\end{aligned} \tag{10.10}$$

where the new phase space elements are ϕ_k, the conjugate momentum for $\Gamma_k = \partial_0 A_k$, see [4], and we also have the definition $a^2 \equiv \frac{1}{m^2}$.

Its lagrangian has the following Legendre transform structure

$$\mathcal{L} = \Pi_\mu \dot{A}^\mu + \phi_k\dot{\Gamma}^k + \pi\dot{B} - \mathcal{H} \tag{10.11}$$

The phase space variables are defined as $\epsilon^I = \Big(A_\nu, \Pi_\nu, \Gamma_k, \phi_k, \Gamma_0, B, \pi\Big)$ in terms of which the lagrangian becomes $\mathcal{L} = a_I\dot{\epsilon}^I - \mathcal{H}(\epsilon)$. The non-vanishing one form coefficient elements a_I are given below:

$$a_\nu^A = \Pi_\nu \quad , \quad a_k^\Gamma = \phi_k \quad , \quad a^B = \pi \tag{10.12}$$

This result allow us to build the matrix $\mathcal{F}_{IJ}(x,y) = \frac{\delta}{\delta\epsilon_I(x)}a_J(y) - \frac{\delta}{\delta\epsilon_J(y)}a_I(x)$ which can be used to rewrite the Hamilton's equations as

$$\int d^3y \mathcal{F}_{IJ}(x,y)\dot{\epsilon}^J(y) = \int d^3z \frac{\delta\mathcal{H}(\epsilon(z))}{\delta\epsilon^I(x)} \tag{10.13}$$

The initial form of the matrix $\mathcal{F}_{IJ}$ reads

$$\mathcal{F}^0_{IJ} = \begin{pmatrix} 0 & -\eta_{\mu\nu} & 0 & 0 & 0 & 0 & 0 \\ \eta_{\mu\nu} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_{kl} & 0 & 0 & 0 \\ 0 & 0 & \delta_{kl} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \delta^3(x-y) \tag{10.14}$$

From this first expression we can easily recognize the null eigenvector $\mathcal{V} = \Big(0,0,0,0,1,0,0\Big)$. Since the Hamiltonian must be invariant under translations in the direction of this phase space vector, we have

$$\int d^3x d^3y \mathcal{V}^i \frac{\partial \mathcal{H}(y)}{\partial \epsilon_i(x)} = 0 = \int d^3x \big(\Pi_0 - \partial^k \phi_k\big) \equiv \Omega^\eta \tag{10.15}$$

In order to implement the above constraint into our analysis we define the extra phase space variable η and the one form coefficient $a^\eta \equiv \Big(\Pi_0 - \partial^k \phi_k\Big)$. Before providing the next steps, we must impose this constraint in strong form in the Hamiltonian and then Γ_0 disappears completely from the entire dynamics [5], that is, do not contribute for the Hamiltonian and the one form coefficients. The enlarged phase space is $\tilde{\epsilon}^I = \Big(A_\nu, \Pi_\nu, \Gamma_k, \phi_k, B, \pi, \eta\Big)$. The hamiltonian in this surface constraint becomes

$$\begin{aligned} \mathcal{H}^1 &= \Pi_k \Gamma^k + \phi^k(-\partial^l F_{lk} + \frac{\phi_k}{2a^2}) + \frac{1}{4} F^2_{kl} + \frac{1}{2}(\Gamma_j - \partial_j A_0)(\Gamma^j - \partial^j A_0) \\ &+ \frac{m_s^2}{2M^2}\pi^2 - m_s A_0 \pi + \frac{M^2}{2}(A_k + \frac{\partial_k B}{m_s})^2 - \frac{1}{2m^2}(\partial^j \Gamma_j - \nabla^2 A_0)^2 \end{aligned} \tag{10.16}$$

The second iterated matrix has the form

$$\mathcal{F}^1_{IJ} = \begin{pmatrix} 0 & -\eta_{\mu\nu} & 0 & 0 & 0 & 0 & 0 \\ \eta_{\mu\nu} & 0 & 0 & 0 & 0 & 0 & \delta_{0\mu} \\ 0 & 0 & 0 & -\delta_{kl} & 0 & 0 & 0 \\ 0 & 0 & \delta_{kl} & 0 & 0 & 0 & -\nabla_l \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\delta_{0\nu} & 0 & -\nabla_k & 0 & 0 & 0 \end{pmatrix} \delta^3(x-y) \tag{10.17}$$

The $\mathcal{F}^1$ matrix has the following null eigenvector $\mathcal{V}_2 = \Big(\beta\delta_{\nu 0}, 0, -\nabla_k\beta, 0, 0, 0, -\beta\Big)$. Now, considering the Hamiltonian computed in the first constraint surface, its invariance with relation to $\mathcal{V}_2$ phase space direction furnishes an addition constraint and then we must consider an associated new phase space variable λ whose one form coefficient is

$$\Omega^\lambda \equiv 0 = \nabla.\Pi - m_s\pi \rightarrow a^\lambda = \nabla.\Pi - m_s\pi \tag{10.18}$$

must be added to the system. The phase admit an additional degree of freedom $\epsilon'^I = \Big(A_\nu, \Pi_\nu, \Gamma_k, \phi_k, B, \pi, \eta, \lambda\Big)$.

Therefore, we are left with the new matrix $\mathcal{F}^2_{IJ}$ which is computed below

$$\mathcal{F}^2_{IJ} = \begin{pmatrix} 0 & -\eta_{\mu\nu} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_{\mu\nu} & 0 & 0 & 0 & 0 & 0 & \delta_{0\mu} & \delta^i_\mu\nabla_i \\ 0 & 0 & 0 & -\delta_{kl} & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_{kl} & 0 & 0 & 0 & -\nabla_l & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -m_s \\ 0 & -\delta_{0\nu} & 0 & -\nabla_k & 0 & 0 & 0 & 0 \\ 0 & \delta^i_\nu\nabla_i & 0 & 0 & 0 & m_s & 0 & 0 \end{pmatrix} \delta^3(x-y) \tag{10.19}$$

where we have eliminated the line and column corresponding to Γ_0 since it disappeared from the dynamics.

This matrix has the null eigenvector $\mathcal{V}_3 = \Big(-\delta^j_\nu\nabla_j\frac{F}{m_s}+D\delta_{0\nu}, 0, -\nabla_k D, 0, F, 0, -D, \frac{F}{m_s}\Big)$. The hamiltonian independence of this phase space direction do not generate any new phase space constraint. It then implies that the theory is invariant with relation to these transformations

$$\delta A_i = \nabla_i\sigma, \quad \delta B = -m_s\sigma, \quad \delta\eta = -D, \quad \delta A_0 = D, \quad \delta\lambda = -\sigma, \quad \delta\Gamma_k = -\nabla_k D \tag{10.20}$$

We are defining $-F \equiv m_s\sigma$.

This local phase space invariance is nothing more than a freedom to perform canonical transformations. If one impose the Lorentz symmetry, that is, considering $D = -\partial_0\sigma$, the well known lagrangian gauge transformations are recovered for the tangent variables.

In order to have a well-defined time evolution to the phase space fields we need to fix an extra constraints to eliminate the freedom due to the canonical transformations. Then, we choose the generalized Coulomb condition [1]

$$\Omega^\rho \equiv \Big(1 + \frac{\Box}{m^2}\Big)\vec{\nabla}.\vec{A} - \frac{M^2}{m_s}B = 0 \tag{10.21}$$

[1] This gauge condition leads to $\Gamma_0 = 0$, but since it already disappeared from the dynamics, we do not need to impose it.

with the intention to define an invertible $\mathcal{F}_{IJ}$ that leads to a well-defined dynamics for the system. It is worth mention that the last constraint (and the fact that it should be time invariant.) together with the equations of motion imply in $A_0 = 0$ [4]. Therefore, we also consider this additional constraint Ω^γ in our former set.

The hamiltonian computed at the constraint surface can be written as

$$\mathcal{H}|_\Omega = \Pi_k\Gamma^k + \phi_k\Big(\frac{\phi^k}{2a^2} - \partial_l F^{lk}\Big) + \frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}\Gamma_j^2 - \frac{1}{2m^2}\Big(\partial_k\Gamma^k\Big)^2 + \frac{m_s^2}{2m^2}\pi^2 + \frac{M^2}{2}\Big(A_k - \frac{\partial_k B}{m_s}\Big)^2 \tag{10.22}$$

Such extra conditions are implemented by means of an extra phase space multiplier field and it associated one forms $a^\rho = \Big(1 + \frac{\Box}{m^2}\Big)\vec{\nabla}.\vec{A} - \frac{M^2}{m_s}B$ and $a^\gamma = A_0$.
The complete phase space structure, owing to the final form of the constrained hamiltonian, is[2] $\epsilon''^I = \Big(A_\nu, \Pi_\nu, \Gamma_k, \phi_k, B, \pi, \eta, \lambda, \rho, \gamma\Big)$. The final form of the $\mathcal{F}_{IJ}$ matrix is:

$$\mathcal{F}^3_{IJ} =$$

$$\begin{pmatrix}
0 & -\eta_{\mu\nu} & 0 & 0 & 0 & 0 & 0 & 0 & (1+\frac{\Box^2}{m^2})\nabla_n\delta^n_\mu & \delta_{0\mu} \\
\eta_{\mu\nu} & 0 & 0 & 0 & 0 & 0 & \delta_{0\mu} & \delta^i_\mu\nabla_i & 0 & 0 \\
0 & 0 & 0 & -\delta_{kl} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta_{kl} & 0 & 0 & 0 & -\nabla_l & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\frac{M^2}{m_s} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -m_s & 0 & 0 \\
0 & -\delta_{0\nu} & 0 & -\nabla_k & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta^n_\nu\nabla_n & 0 & 0 & 0 & m_s & 0 & 0 & 0 & 0 \\
(1+\frac{\Box^2}{m^2})\nabla_j\delta^j_\nu & 0 & 0 & 0 & \frac{M^2}{m_s} & 0 & 0 & 0 & 0 & 0 \\
-\delta_{0\nu} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \times \delta^3(x-y) \tag{10.23}$$

The determinant is proportional to det $\Big[\Big(1+\frac{\Box}{m^2}\Big)\nabla^2 - M^2\Big]^2$ which is indeed the same value obtained by the Dirac-Bergmann algorithm [14] in the approach of [4] . It can be used to build

[2] Although we should consider this entire set in order to obtain the Legendre transformed lagrangian, from the structure of the Hamiltonian in terms in the physical surface, we note that there is 14 physical phase space variables and therefore 7 configuration space degrees of freedom, six from the gauge field (two sets of massive spin one fields), and one from the Stueckelberg particle.

the theory's path integral measure. This determinant was obtained by using the identity below

$$\mathcal{M}_{n\times n} = \begin{pmatrix} A_{m,m} & B_{m,n-m} \\ \\ C_{n-m,m} & D_{n-m,n-m} \end{pmatrix} \quad , \quad \det \mathcal{M} = \det D\Big(\det(A - BD^{-1}C)\Big) \tag{10.24}$$

then, in order to find a convenient form for the matrix $\mathcal{F}^3_{IJ}$ to calculate this determinant, we have moved its last 4 lines to occupy the first four lines and moved the last 4 columns to the left to occupy the 4 initial ones.

10.3 The Path Integral Formulation

In order to build the path integral for the Podolsky-Stueckelberg theory, we use the approach of [13], which is compatible to the Fadeev-Jackiw formalism employed here.

Considering the Schwinger variational principle of quantum action and the Darboux theorem [10], the transition amplitude can be obtained from the phase space as

$$Z = \int D\epsilon_I \sqrt{|det\mathcal{F}^3|}\exp i\int \big(a^I(\epsilon)\dot{\epsilon}_I - \mathcal{H}|_\Omega\big)dt \tag{10.25}$$

Then, it reads

$$Z = \int DA_\mu D\rho D\lambda D\eta D\Pi_\mu D\Gamma_k D\phi^k D\pi DB\sqrt{|det\mathcal{F}^3|}\exp i\int d^4x\Big(\pi\partial_0 B + \Pi^\mu\partial_0 A_\mu + \phi_k\partial_0\Gamma^k + \partial_0\eta\ \Omega^\eta + \partial_0\lambda\ \Omega^\lambda + \partial_0\rho\ \Omega^\rho + \partial_0\gamma\ \Omega^\gamma - \mathcal{H}|_\Omega\Big) \tag{10.26}$$

where Ω denotes the constraints and $\mathcal{H}|_\Omega$ denotes the Hamiltonian evaluated at the constraint surface. It is given by

$$\mathcal{H}|_\Omega = \Pi_k\Gamma^k + \phi_k\big(\frac{\phi^k}{2a^2} - \partial_l F^{lk}\big) + \frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}\Gamma_j^2 - \frac{1}{2m^2}\Big(\partial_k\Gamma^k\Big)^2 + \frac{m_s^2}{2m^2}\pi^2 + \frac{M^2}{2}\big(A_k + \frac{\partial_k B}{m_s}\big)^2 \tag{10.27}$$

As in the motivation section, we can use the invariance of the dynamics under the addition of total time derivative terms to rewrite the lagrangian in its equivalent form and then integrate in the extra phase space variables to obtain

$$Z = \int DA_\mu D\Pi_\mu D\Gamma_k D\phi^k D\pi DB\sqrt{|det\mathcal{F}^3|}\delta(A_0)\delta(\Pi_0 - \partial_k\phi^k)\delta(m_s\pi + \partial_k\Pi^k)\delta\Big(\Big(1 + \frac{\Box}{m^2}\Big)\vec{\nabla}.\vec{A} - \frac{M^2}{m_s}B\Big)$$
$$\times \exp i\int d^4x\Big(\pi\partial_0 B + \Pi^\nu\partial_0 A_\nu + \phi_k\partial_0\Gamma^k - \Pi_k\Gamma^k - \phi_k\big(\frac{\phi^k}{2a^2} - \partial_l F^{lk}\big) - \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}\Gamma_j^2 - \frac{m_s^2}{2M^2}\pi^2 + \frac{1}{2m^2}\big(\partial^j\Gamma_j\big)^2 - \frac{M^2}{2}\big(A_k + \frac{\partial_k B}{m_s}\big)^2\Big) \tag{10.28}$$

Integrating in Π_0 and A_0 and then giving a functional representation for $\delta(m_s\pi + \partial_k\Pi^k)$, we get

$$Z=\int D\theta DA_kD\Pi_kD\Gamma_kD\phi^kD\pi DB\sqrt{|det\mathcal{F}^3|}\delta\Big(\Big(1+\frac{\Box}{m^2}\Big)\vec{\nabla}.\vec{A}-\frac{M^2}{m_s}B\Big)$$
$$\times\exp i\int d^4x\Big(\pi\partial_0B+\Pi^k\partial_0A_k+\phi_k\partial_0\Gamma^k+\Pi_k\Gamma^k-\phi_k\big(\frac{\phi^k}{2a^2}-\partial_lF^{lk}\big)-\frac{1}{4}F_{ij}F^{ij}$$
$$+\frac{1}{2}\Gamma_j^2-\frac{m_s^2}{2M^2}\pi^2+\frac{1}{2m^2}\big(\partial^j\Gamma_j\big)^2-\frac{M^2}{2}\big(A_k+\frac{\partial_kB}{m_s}\big)^2+\theta\big(m_s\pi+\partial_k\Pi^k\big)\Big)\quad(10.29)$$

Now, we integrate the Π_k variable leading to $\delta\big(\partial_0A_k-\partial_k\theta-\Gamma_k\big)$ in the path integral measure. Then, the next most convenient variables to be integrated are Γ_l and π (in order to Gaussian integrate and obtain the Stueckelberg mass term.) which gives

$$Z=\int DA_\mu D\phi^kDB\sqrt{|det\mathcal{F}^3|}\delta\Big(\Big(1+\frac{\Box}{m^2}\Big)\vec{\nabla}.\vec{A}-\frac{M^2}{m_s}B\Big)$$
$$\times\exp i\int d^4x\Big(-\frac{F_{ij}F^{ij}}{4}+\frac{F_{i0}F_{i0}}{2}+\frac{1}{2m^2}\big(\partial_lF^{l0}\big)^2-\frac{\phi_k\phi^k}{2a^2}-\phi_k\big(\partial_0F^{0k}+\partial_lF^{lk}\big)+\frac{M^2}{2}\big(A_\mu+\frac{\partial_\mu B}{m_s}\big)^2\Big)\quad(10.30)$$

where we have renamed $\theta=A_0$.

The Gaussian integration of ϕ_k leads to the coordinate space path integral, up to normalization factors,

$$Z=\int DA_\mu DB\sqrt{|det\mathcal{F}^3|}\delta\Big(\Big(1+\frac{\Box}{m^2}\Big)\vec{\nabla}.\vec{A}-\frac{M^2}{m_s}B\Big)$$
$$\times\exp i\int d^4x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2m^2}\partial_\lambda F^{\alpha\lambda}\partial^\rho F_{\alpha\rho}+\frac{M^2}{2}\big(A_\mu+\frac{\partial_\mu B}{m_s}\big)^2\Big)\quad(10.31)$$

This transition amplitude is correct but not Lorentz covariant. In order to turn the system into a covariant one, we should consider the following identity with a covariant gauge condition [4]

$$I=\det\Big(\big(1+\frac{\Box}{m^2}\big)\Box+M^2\Big)\int D\alpha(x)\delta\Big(\big(1+\frac{\Box}{m^2}\big)\partial^\mu A_\mu^\alpha-\frac{M^2}{m_s}B^\alpha+f(x)\Big)\quad(10.32)$$

with A_μ^α and B^α being the gauge transformed fields

$$A_\mu(x)\to A_\mu(x)+\partial_\mu\alpha(x)\ ,\ B(x)\to B(x)-m_s\alpha(x)\quad(10.33)$$

and $f(x)$ being a scalar function.

We can insert this identity into the path integral, perform the inverse gauge transformation, considering the fact that the system is gauge invariant, and then integrate in α to eliminate the presence of $\sqrt{|det\mathcal{F}^3|}\delta\Big(\Big(1+\frac{\Box}{m^2}\Big)\vec{\nabla}.\vec{A}-\frac{M^2}{m_s}B\Big)$ from the integration measure. After this procedure we can insert $\int df\mathcal{M}(f)=1$ with $\mathcal{M}(f)=e^{-i\int d^4xf^2(x)/2}$ leading, up to normalization factors, to

$$Z=\int DA_\mu DB\exp i\int d^4x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2m^2}\partial_\lambda F^{\alpha\lambda}\partial^\rho F_{\alpha\rho}+\frac{M^2}{2}\big(A_\mu+\frac{\partial_\mu B}{m_s}\big)^2$$
$$-\frac{1}{2}\Big[\Big(1+\frac{\Box}{m^2}\Big)\partial_\mu A^\mu-\frac{M^2}{m_s}B\Big]^2\Big)\quad(10.34)$$

The result is equivalent to the one of [4] which has used the alternative approach of the Fadeev-Senjanovic method [15, 16] allied to the Dirac-Bergman algorithm [14] for Ostrogadskian systems.

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Chapter 11

Conclusion

This thesis was focused on new applications and extensions of the so-called B-field formalism. We also have used some additional quantization methods depending on the specific model to be investigated.

The structure of the thesis is the following, an introductory second chapter for the covariant operator formalism, next, a chapter that was devoted to conceptual and phenomenological discussions about the quantum Hall effect. It was a motivation to the chapter 4 which describes the action of an external magnetic field, modeled by the Chern-Simons model, coupled to topological two band systems representing quasi-particles excitations of the low energy limit of planar materials with honeycomb lattice structure with two different kind of sites. Then, as a future perspective we also intend to effectively describe other phenomena as the three dimensional topological insulators, for example.

In the chapter 5 we studied the electrodynamics in a non-linear gauge leading to a discussion about BRST symmetry and the quartet mechanism even for the case of an Abelian theory. It provided a laboratory to the study of the so-called Abelian dominance in QCD_4. The chapter 6 was a natural extension of the previous one in order to include a Higgs and a matter sector. As a new perspective, the perturbative study of the Yang-Mills theory by means of the Heisenberg description is an interesting goal, as mentioned in the introduction.

The discussion about models whose dynamical poles are contained in longitudinal sectors leading to possible reducible structure was done in chapter 7 and 8. In this latter chapter, we analyzed a dual description of a spin 1 theory described by a higher derivative and higher rank tensor model. We used the diagrammatic approach to investigate the unitarity aspects of the model coupled to a matter sector. We also analyzed the Ward identities for the reducible and Weyl gauge symmetries. As a new perspective, trying to mimic this same approach for linearized gravity described by a rank 3 tensor field seems to be an interesting perspective since we can use this higher rank structure to look for new couplings and a possible gauge invariant mass term (Which, at least for the case of the dual electrodynamics, constrains the classes of possible couplings in order to keep unitarity.).

The idea of studying higher derivative models motivated the extension of the covariant operator formalism to these kind of theories in chapter 9. The achievements due to the previous investigations were an inspiration to the chapter 10 that was devoted to present the Fadeev-Jackiw quantization for an extension of the higher derivative Podolsky model in which an infrared regulator mass is added in a gauge invariant Stueckelberg structure. Then, we had to deal with a massive gauge theory with an extra Stueckelberg field and perform a Fadeev-Jackiw analysis for an ostrogadiskian phase space. We also its obtained the path integral.