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Evolving Probability Representations of Entangled Cat States in the Potentials of Harmonic and Inverted Oscillators

Matyas Mechler [1,2], Margarita A. Man'ko [3], Vladimir I. Man'ko [3] and Peter Adam [1,2,*]

[1] Institute for Solid State Physics and Optics, HUN-REN Wigner Research Centre for Physics, P.O. Box 49, H-1525 Budapest, Hungary; mechler.matyas@wigner.hun-ren.hu
[2] Institute of Physics, University of Pécs, Ifjúság útja 6, H-7624 Pécs, Hungary
[3] Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia; mankoma@lebedev.ru (M.A.M.); mankovi@lebedev.ru (V.I.M.)
* Correspondence: adam.peter@wigner.hun-ren.hu

Abstract: We determine the evolving probability representation of entangled cat states in the potential of either the harmonic oscillator or the inverted oscillator, assuming that the states are initially prepared in the potential of the harmonic oscillator. Such states have several applications in quantum information processing. The inverted quantum harmonic oscillator, where the potential energy corresponds to imaginary frequencies of the oscillator, can be applied in relation to cosmological problems. We also determine the evolving probability representation of cat states of an oscillating spin-1/2 particle of the inverted oscillator, in which the time evolution of the spin state is described by an arbitrary unitary operator. The properties of the determined entangled probability distributions are discussed.

Keywords: evolving probability representation; cat states; inverted oscillator

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1. Introduction

In quantum mechanics, various representations of quantum states can be used for discussing the properties and dynamics of quantum systems and for performing any calculations. One of them, the probability representation, was proposed in the 1990s, and it has extensively been analyzed in the literature [1–7]. The importance of this representation is that regular non-negative probability distributions defined in the phase space can be used to describe the system states. Knowing the density operator of the quantum state, one can derive the corresponding probability distribution, which contains all information regarding the quantum system. All quantum effects can be explained effectively through the probability representation by the use of the conventional probability theory. We note here that quasiprobability representations such as the Husimi Q-function [8,9], the Glauber–Sudarshan P-function [10,11], or the Wigner function [12] are related to the proposed probability representation through integral transforms [13]. The derivation of probability representations is based on the general formalism of quantizer and dequantizer operators. These operators associate operators defined in a Hilbert space with functions of certain variables, and the general formalism describes all invertible maps between the two sets of quantities [14].

The symplectic tomogram is a type of probability representation of continuous quantum systems widely studied in the literature. An important property of this tomogram is that, for certain sets of the variables, it corresponds to the optical homodyne tomogram [5] that can be determined in quantum optical experiments [15]. Optical tomograms can be used for restoring the density matrix or the Wigner function [16–20]. Symplectic tomograms have already been derived for various states of the harmonic oscillator such as Fock states [7], thermal states [21], coherent states, and even and odd coherent states, also known as Schrödinger cat states [22]. The time evolution of tomograms of these states initially prepared in the potential of the usual harmonic oscillator has also been derived for inverted oscillators [23,24] and for free particle motion [5,7,21,22]. The relevance of the inverted quantum harmonic oscillator model is that it can be used in several physical problems [25–27], e.g., in studying some cosmological problems [28–30], the motion of optically levitated nanoparticles [31–33], or fast frictionless cooling of ultracold atomic mixtures [34–36].

Quantum systems with discrete variables can also be described by probability representations; examples of these were shown for qubit and qudit systems [2,37,38]. Recently, special probability distributions have been presented for one- and two-qubit states where the components of these distributions are actually the probabilities of the spin projections onto opposite directions of the perpendicular x, y, and z axes for the particular qubits [39,40]. Consequently, assuming that a large enough set of identically prepared states are at hand, repeated spin projection measurements can be applied to experimentally measure the components of these probability distributions. As the elements of the density matrix can be expressed by these components [39], any quantum effect and even the time evolution can be addressed by using these probabilities.

Quantum superposition is a basic principle of quantum mechanics resulting in phenomena that cannot be interpreted classically. In harmonic oscillator systems, Schrödinger cat states, also known as even and odd coherent states, are typical examples of such superpositions. These states are superpositions of two macroscopic quasi-classical coherent states [41–43]. Such states can be generated in various quantum optical experiments [44–52], and they can be efficiently applied in quantum information processing schemes, especially in quantum communication [53,54] and optical quantum computation [55–58].

Quantum entanglement is yet another counterintuitive phenomenon that has no classical analog. Widely discussed entangled states are entangled cat states, also known as entangled coherent states, which are the superpositions of macroscopic coherent states for two or more modes [53,54,59–63]. Entangled cat states can be generated in various experiments [59,60,64–72]. Due to entanglement, such states have several applications in quantum information processing, such as quantum teleportation [73,74], quantum key distribution [53], quantum sensing [75], and quantum computation [76]. Recently, cat states of oscillating spin-1/2 particles have also been considered [77,78]. For describing entangled states, entangled probability distributions have been introduced in the literature [39,78]. These distributions differ from classical distributions and, using them, the properties of entanglement can be revealed.

Motivated by these preliminaries, the present paper aims at determining the evolving probability representation of two-mode entangled cat states in the potential of either the harmonic oscillator or the inverted oscillator, assuming that the states are initially prepared in the potential of the harmonic oscillator. We will also determine the evolving probability representation of cat states of an oscillating spin-1/2 particle of both types of oscillators in which the time evolution of the spin state is described by an arbitrary unitary operator. Marginal probability distributions will be derived and the properties of all determined entangled probability distributions will be discussed.

2. Probability Representations and the Formalism of Dequantizers and Quantizers

In this section, we summarize the formalism of dequantizers and quantizers and the theory of probability representations for both continuous and discrete variables.

2.1. Continuous Dimensional Quantum Systems

In Ref. [14], it was shown how the formalism of dequantizer and quantizer operators can be applied for deriving the probability representation of quantum states. The particular operators are labeled by the parameter $\bar{x}$ that can contain either discrete or continuous components $x_1, x_2, \dots, x_n$. Using the dequantizer operators $\hat{U}(\bar{x})$ and quantizer operators $\hat{D}(\bar{x})$, one can create an invertible map between operators $\hat{A}$ acting on the Hilbert space $\mathcal{H}$ and functions $f_A(\bar{x})$ as [14]

$$\begin{aligned} f_A(\bar{x}) &= \mathrm{Tr}\big(\hat{A}\hat{U}(\bar{x})\big), && (1) \\ \hat{A} &= \int f_A(\bar{x})\hat{D}(\bar{x})\,\mathrm{d}\bar{x}\,. && (2) \end{aligned}$$

Since the function $f_A(\bar{x})$ is associated with the operator $\hat{A}$ and it has properties that reflect the properties of the operator, $f_A(\bar{x})$ is termed as the symbol of the operator $\hat{A}$. For discrete variables x_i, the formula above should be modified so that the integral is replaced by a corresponding sum, that is,

$$\begin{aligned} f_A^{(i)} &= \mathrm{Tr}\Big(\hat{A}\hat{U}^{(i)}\Big), \qquad i = 1, \dots, l, && (3) \\ \hat{A} &= \sum_{i=1}^{n} f_A^{(i)}\hat{D}_A^{(i)}. && (4) \end{aligned}$$

The above four formulas are also valid for any density operator. Using the dequantizer operator $\hat{U}(X,\mu,\nu) = \delta\big(X\hat{1} - \mu\hat{q} - \nu\hat{p}\big)$, it becomes possible to map the density operator $\hat{\rho}$ characterizing a continuous-variable quantum system onto the function $w(X \mid \mu,\nu)$, known as a symplectic tomogram, by the formula [1]

$$w(X \mid \mu,\nu) = \mathrm{Tr}\big[\hat{\rho}\,\delta\big(X\hat{1} - \mu\hat{q} - \nu\hat{p}\big)\big]. \tag{5}$$

In this expression, $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively, and the resulting function $w(X \mid \mu,\nu)$ depending on the random position X is a non-negative conditional probability distribution function, which satisfies the normalization condition

$$\int w(X \mid \mu,\nu)\,\mathrm{d}X = 1. \tag{6}$$

The conditions are labeled by the parameters μ and ν in the frame of reference where the position X is measured, that is, the position X can be expressed as $X = \mu q + \nu p$ in the phase space. Then, the inverse transformation can be derived in the form

$$\hat{\rho} = \frac{1}{2\pi}\int w(X \mid \mu,\nu)\hat{D}(X,\mu,\nu)\,\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu\,, \tag{7}$$

where $\hat{D}(X,\mu,\nu) = \exp\big[i\big(X\hat{1} - \mu\hat{q} - \nu\hat{p}\big)\big]$ is the quantizer operator.

In Ref. [7], it was shown that, assuming pure states, that is, $\hat{\rho} = |\psi\rangle\langle\psi|$, Formula (5) can be rewritten as

$$w(X \mid \mu, \nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y) \exp\left(\frac{i\mu}{2\nu}y^2 - \frac{iX}{\nu}y\right) \mathrm{d}y\right|^2 \tag{8}$$

with $\psi(y) = \langle y|\psi\rangle$ being the wave function of the state.

The tomographic symbol of the operators $|\psi\rangle\langle\phi|$ determined by the state vectors $|\phi\rangle$ and $|\psi\rangle$ of a single-mode oscillator is

$$\begin{aligned} \mathrm{Tr}\big(|\psi\rangle\langle\phi|\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})\big) = \frac{1}{2\pi|\nu|}\int \mathrm{d}x\, \phi^*(x) \exp\left(-\frac{i\mu}{2\nu}x^2 + \frac{i}{\nu}Xx\right) \\ \times \int \mathrm{d}x'\, \psi(x') \exp\left(\frac{i\mu}{2\nu}x'^2 - \frac{i}{\nu}Xx'\right). \end{aligned} \tag{9}$$

The Wigner quasiprobability distribution function $W(q,p)$ of a quantum state can be expressed through the density operator as [7]

$$W(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle q - u/2|\hat{\rho}|q + u/2\rangle e^{ipu}\, \mathrm{d}u\,. \tag{10}$$

In this and the following expressions, we assumed that $\hbar = 1$. It can be shown that the symplectic tomogram can be obtained from the Wigner function $W(q,p)$ by the expression

$$w(X \mid \mu, \nu) = \frac{1}{2\pi}\int W(q,p)\delta(X - \mu q - \nu p)\, \mathrm{d}q\, \mathrm{d}p\,, \tag{11}$$

while the Wigner function can be derived from the symplectic tomogram as [7]

$$W(q,p) = \frac{1}{2\pi}\int w(X \mid \mu, \nu) e^{i(X - \mu q - \nu p)}\, \mathrm{d}X\, \mathrm{d}\mu\, \mathrm{d}\nu\,. \tag{12}$$

Note that both the Wigner function $W(q,p)$ and the symplectic tomogram $w(X \mid \mu, \nu)$ can be used to completely characterize the quantum state, as both functions contain all information on the density operator. However, while symplectic tomograms are always non-negative functions and, therefore, are regular probability distributions, in contrast, values of the Wigner functions can be negative. A possible indicator of the nonclassicality of quantum states can be based on the negativity of the Wigner function [79]. In the case of pure states, only Gaussian states have positive Wigner functions [80,81].

The tomogram of a two-mode oscillator can be calculated similarly to Equation (5), that is,

$$w(X_1, X_2 \mid \mu_1, \nu_1, \mu_2, \nu_2) = \mathrm{Tr}\big[\hat{\rho}\delta(X_1\hat{1} - \mu_1\hat{q}_1 - \nu_1\hat{p}_1)\delta(X_2\hat{1} - \mu_2\hat{q}_2 - \nu_2\hat{p}_2)\big]. \tag{13}$$

We note that an alternative definition for the tomogram of two- and multimode oscillators, known as the center-of-mass tomogram, has also been developed [82,83].

In the case of pure states, Equation (13) can be rewritten as

$$w(X, Y \mid \mu_1, \nu_1, \mu_2, \nu_2) = \frac{1}{4\pi^2|\nu_1\nu_2|}\left|\int \Phi(x,y) \exp\left(\frac{i\mu_1}{2\nu_1}x^2 - \frac{iX}{\nu_1}x + \frac{i\mu_2}{2\nu_2}y^2 - \frac{iY}{\nu_2}y\right) \mathrm{d}x\, \mathrm{d}y\right|^2, \tag{14}$$

where $\Phi(x,y)$ is the wave function of the pure two-mode state [39].

Next, we discuss the time evolution of symplectic tomograms. The evolution of the density operator $\hat{\rho}(t)$ of the system can be written as

$$\hat{\rho}(t) = \hat{u}(t)\hat{\rho}(0)\hat{u}^{\dagger}(t), \tag{15}$$

where the unitary operator

$$\hat{u}(t) = \exp\left(-it\hat{H}\right) \tag{16}$$

describes the time evolution governed by the Hamiltonian $\hat{H}$. Then, the tomogram $w(X|\mu,\nu,t)$ corresponding to the density operator $\hat{\rho}(t)$ takes the form

$$w(X|\mu,\nu,t) = \mathrm{Tr}\left(\hat{\rho}(t)\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})\right). \tag{17}$$

Substituting (15) into (17) and taking into account the properties of the trace of product of operators, we obtain

$$w(X|\mu,\nu,t) = \mathrm{Tr}\left(\hat{\rho}(0)\delta(X\hat{1} - \mu\hat{q}_H(t) - \nu\hat{p}_H(t))\right) = w_0(X|\mu_H(t),\nu_H(t)), \tag{18}$$

where $\hat{q}_H(t)$ and $\hat{p}_H(t)$ are the position and momentum operators, respectively, in the Heisenberg representation, that is,

$$\hat{q}_H(t) = \hat{u}^{\dagger}(t)\hat{q}\hat{u}(t), \qquad \hat{p}_H(t) = \hat{u}^{\dagger}(t)\hat{p}\hat{u}(t). \tag{19}$$

The Hamiltonians of the harmonic ($\hat{H}_+$) and the inverted ($\hat{H}_-$) oscillator take the form

$$\hat{H}_{\pm} = \frac{\hat{p}^2}{2} \pm \frac{\hat{q}^2}{2}. \tag{20}$$

Here, the assumptions $m = 1$, $\omega = 1$, and $\hbar = 1$ are taken into account. Applying the unitary operators $\hat{u}_{\pm}(t) = \exp\left(-i\hat{H}_{\pm}t\right)$ in Equation (19), the time-dependent position and momentum operators in the Heisenberg picture for the ordinary oscillator take the forms

$$\hat{q}_H(t) = \hat{q}\cos t + \hat{p}\sin t, \qquad \hat{p}_H(t) = -\hat{q}\sin t + \hat{p}\cos t, \tag{21}$$

while for the inverted oscillator, they read

$$\hat{q}_H(t) = \hat{q}\cosh t + \hat{p}\sinh t, \qquad \hat{p}_H(t) = \hat{q}\sinh t + \hat{p}\cosh t. \tag{22}$$

Using Equations (18), (21) and (22), the coefficients $\mu_H(t)$ and $\nu_H(t)$ appearing in Equation (18) can be determined [23]. In the case of the harmonic oscillator, the time evolution of these coefficients are

$$\mu_H(t) = \mu\cos t - \nu\sin t, \tag{23}$$

$$\nu_H(t) = \mu\sin t + \nu\cos t. \tag{24}$$

For the inverted oscillator, the time evolution of these coefficients can be obtained as

$$\mu_H(t) = \mu\cosh t + \nu\sinh t, \tag{25}$$

$$\nu_H(t) = \mu\sinh t + \nu\cosh t. \tag{26}$$

For two-mode oscillator systems, all these expressions should be applied consequently.

2.2. Finite-Dimensional Quantum Systems

In order to obtain the probability representation of qubit states, one can use the dequantizers [40]

$$\hat{U}(+1/2 \mid 1) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \hat{U}(-1/2 \mid 1) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$
$$\hat{U}(+1/2 \mid 2) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \qquad \hat{U}(-1/2 \mid 2) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \tag{27}$$
$$\hat{U}(+1/2 \mid 3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{U}(-1/2 \mid 3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

These dequantizers are actually density operators corresponding to projectors obtained from the six normalized eigenvectors of the Pauli matrices $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$.

Note that the minimal set of dequantizer operators of a finite d-dimensional quantum system comprises d^2 elements. Accordingly, the above set of dequantizers is not a minimal set. However, it can be shown that four of the above six dequantizers, $\hat{U}(+1/2 \mid 1)$, $\hat{U}(+1/2 \mid 2)$, $\hat{U}(+1/2 \mid 3)$, and $\hat{U}(-1/2 \mid 3)$, form a minimal set.

Based on the dequantizers defined in Equation (27), it is possible to introduce a conditional probability distribution $w(X \mid j)$ characterizing a qubit state described by the density operator $\hat{\rho}$ in the form

$$w(X \mid j) = \mathrm{Tr}\big[\hat{\rho}\hat{U}(X \mid j)\big], \qquad X = \pm\frac{1}{2}, \quad j = 1,2,3. \tag{28}$$

In this formula, the value of the parameter X is chosen from the set $\{+1/2, -1/2\}$, while the value of the parameter j belongs to the set $\{1,2,3\}$. The particular components correspond to the probabilities of the spin projection X onto the three perpendicular directions x $(j = 1)$, y $(j = 2)$, and z $(j = 3)$. Let us introduce the probabilities p_1, p_2 and p_3 as

$$\begin{aligned} &w(+1/2 \mid 1) = p_1, \qquad &&w(+1/2 \mid 2) = p_2, \qquad &&w(+1/2 \mid 3) = p_3, \\ &w(-1/2 \mid 1) = 1 - p_1, &&w(-1/2 \mid 2) = 1 - p_2, &&w(-1/2 \mid 3) = 1 - p_3, \end{aligned} \tag{29}$$

which shows that the conditional probability distribution $w(X \mid j)$ satisfies the condition

$$\sum_X w(X \mid j) = 1, \qquad j = 1,2,3. \tag{30}$$

These probabilities are obviously measurable quantities, and it can be shown that the matrix representation of the density operator of the qubit state can be derived by using these probabilities as

$$\hat{\rho} = \begin{pmatrix} p_3 & (p_1 - 1/2) - i(p_2 - 1/2) \\ (p_1 - 1/2) + i(p_2 - 1/2) & 1 - p_3 \end{pmatrix}, \tag{31}$$

and the non-negativity of the density operator $\hat{\rho}$ poses the constraint

$$\left(p_1 - \frac{1}{2}\right)^2 + \left(p_2 - \frac{1}{2}\right)^2 + \left(p_3 - \frac{1}{2}\right)^2 \leq \frac{1}{4} \tag{32}$$

on the probabilities p_1, p_2, and p_3.

3. Results

In this section, we present our results on evolving probability representations of two-mode entangled cat states and that of even and odd cat states of an oscillating spin-1/2 particle.

First, we determine the evolving probability representation of entangled cat states in the potential of either the harmonic oscillator or the inverted oscillator, assuming that the states are initially prepared in the potential of the harmonic oscillator.

The entangled cat state can be written as

$$|\Phi\rangle = \mathcal{N}(|\alpha\rangle|\alpha\rangle \pm |-\alpha\rangle|-\alpha\rangle). \tag{33}$$

Then, the corresponding density operator takes the form of

$$\begin{aligned} |\Phi\rangle\langle\Phi| \quad = \quad & \mathcal{N}^2(|\alpha\rangle\langle\alpha| \otimes |\alpha\rangle\langle\alpha| \pm |-\alpha\rangle\langle\alpha| \otimes |-\alpha\rangle\langle\alpha| \\ & \pm|\alpha\rangle\langle-\alpha| \otimes |\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle-\alpha| \otimes |-\alpha\rangle\langle-\alpha|). \end{aligned} \tag{34}$$

Applying Equations (8), (9), (14) and (18), the evolving tomogram of this state can be calculated as the sum of four terms containing Gaussian integrals:

$$\begin{aligned} & w(X_1, X_2 \mid \mu_{1,H}(t), \nu_{1,H}(t), \mu_{2,H}(t), \nu_{2,H}(t)) = \frac{\mathcal{N}^2}{4\pi^2|\nu_{1,H}(t)\nu_{2,H}(t)|} \\ & \times\Bigg\{\left|\int \phi_\alpha(x) \exp\left[\frac{i\mu_{1,H}(t)x^2}{2\nu_{1,H}(t)} - \frac{iX_1x}{\nu_{1,H}(t)}\right] \mathrm{d}x\right|^2 \left|\int \phi_\alpha(y) \exp\left[\frac{i\mu_{2,H}(t)y^2}{2\nu_{2,H}(t)} - \frac{iX_2y}{\nu_{2,H}(t)}\right] \mathrm{d}y\right|^2 \\ & +\left|\int \phi_{-\alpha}(x) \exp\left[\frac{i\mu_{1,H}(t)x^2}{2\nu_{1,H}(t)} - \frac{iX_1x}{\nu_{1,H}(t)}\right] \mathrm{d}x\right|^2 \left|\int \phi_{-\alpha}(y) \exp\left[\frac{i\mu_{2,H}(t)y^2}{2\nu_{2,H}(t)} - \frac{iX_2y}{\nu_{2,H}(t)}\right] \mathrm{d}y\right|^2 \\ & \pm \int \phi_\alpha(x') \exp\left[\frac{i\mu_{1,H}(t)x'^2}{2\nu_{1,H}(t)} - \frac{iX_1x'}{\nu_{1,H}(t)}\right] \mathrm{d}x' \int \phi^*_{-\alpha}(x) \exp\left[\frac{-i\mu_{1,H}(t)x^2}{2\nu_{1,H}(t)} + \frac{iX_1x}{\nu_{1,H}(t)}\right] \mathrm{d}x \\ & \times \int \phi_\alpha(y') \exp\left[\frac{i\mu_{2,H}(t)y'^2}{2\nu_{2,H}(t)} - \frac{iX_2y'}{\nu_{2,H}(t)}\right] \mathrm{d}y' \int \phi^*_{-\alpha}(y) \exp\left[\frac{-i\mu_{2,H}(t)y^2}{2\nu_{2,H}(t)} + \frac{iX_2y}{\nu_{2,H}(t)}\right] \mathrm{d}y \\ & \pm \int \phi_{-\alpha}(x') \exp\left[\frac{i\mu_{1,H}(t)x'^2}{2\nu_{1,H}(t)} - \frac{iX_1x'}{\nu_{1,H}(t)}\right] \mathrm{d}x' \int \phi^*_{\alpha}(x) \exp\left[\frac{-i\mu_{1,H}(t)x^2}{2\nu_{1,H}(t)} + \frac{iX_1x}{\nu_{1,H}(t)}\right] \mathrm{d}x \\ & \times \int \phi_{-\alpha}(y') \exp\left[\frac{i\mu_{2,H}(t)y'^2}{2\nu_{2,H}(t)} - \frac{iX_2y'}{\nu_{2,H}(t)}\right] \mathrm{d}y' \int \phi^*_{\alpha}(y) \exp\left[\frac{-i\mu_{2,H}(t)y^2}{2\nu_{2,H}(t)} + \frac{iX_2y}{\nu_{2,H}(t)}\right] \mathrm{d}y\Bigg\}. \end{aligned} \tag{35}$$

Using these formulas and inserting the wave function of coherent states

$$\phi_\alpha(x) = \frac{1}{\pi^{1/4}} \exp\left[-\frac{x^2}{2} + \sqrt{2}\alpha x - \frac{|\alpha|^2}{2} - \frac{\alpha^2}{2}\right] \tag{36}$$

one can obtain

$$
\begin{aligned}
&w(X_1, X_2 \mid \mu_{1,H}(t), \nu_{1,H}(t), \mu_{2,H}(t), \nu_{2,H}(t)) = \\
&= \frac{\mathcal{N}^2}{\pi\sqrt{\mu_{1,H}(t)^2 + \nu_{1,H}(t)^2}\sqrt{\mu_{2,H}(t)^2 + \nu_{2,H}(t)^2}} \\
&\times \exp\left[-\frac{2(\mathrm{Re}(\alpha)\mu_{1,H}(t) + \mathrm{Im}(\alpha)\nu_{1,H}(t))^2 + X_1^2}{\mu_{1,H}(t)^2 + \nu_{1,H}(t)^2} - \frac{2(\mathrm{Re}(\alpha)\mu_{2,H}(t) + \mathrm{Im}(\alpha)\nu_{2,H}(t))^2 + X_2^2}{\mu_{2,H}(t)^2 + \nu_{2,H}(t)^2}\right] \\
&\times \left\{\exp\left[\frac{2^{3/2}X_1(\mathrm{Re}(\alpha)\mu_{1,H}(t) + \mathrm{Im}(\alpha)\nu_{1,H}(t))}{\mu_{1,H}(t)^2 + \nu_{1,H}(t)^2} + \frac{2^{3/2}X_2(\mathrm{Re}(\alpha)\mu_{2,H}(t) + \mathrm{Im}(\alpha)\nu_{2,H}(t))}{\mu_{2,H}(t)^2 + \nu_{2,H}(t)^2}\right]\right. \\
&+ \exp\left[-\frac{2^{3/2}X_1(\mathrm{Re}(\alpha)\mu_{1,H}(t) + \mathrm{Im}(\alpha)\nu_{1,H}(t))}{\mu_{1,H}(t)^2 + \nu_{1,H}(t)^2} - \frac{2^{3/2}X_2(\mathrm{Re}(\alpha)\mu_{2,H}(t) + \mathrm{Im}(\alpha)\nu_{2,H}(t))}{\mu_{2,H}(t)^2 + \nu_{2,H}(t)^2}\right] \\
&\pm \exp\left[\frac{2^{3/2}iX_1(\mathrm{Im}(\alpha)\mu_{1,H}(t) - \mathrm{Re}(\alpha)\nu_{1,H}(t))}{\mu_{1,H}(t)^2 + \nu_{1,H}(t)^2} + \frac{2^{3/2}iX_2(\mathrm{Im}(\alpha)\mu_{2,H}(t) - \mathrm{Re}(\alpha)\nu_{2,H}(t))}{\mu_{2,H}(t)^2 + \nu_{2,H}(t)^2}\right] \\
&\left.\pm \exp\left[-\frac{2^{3/2}iX_1(\mathrm{Im}(\alpha)\mu_{1,H}(t) - \mathrm{Re}(\alpha)\nu_{1,H}(t))}{\mu_{1,H}(t)^2 + \nu_{1,H}(t)^2} - \frac{2^{3/2}iX_2(\mathrm{Im}(\alpha)\mu_{2,H}(t) - \mathrm{Re}(\alpha)\nu_{2,H}(t))}{\mu_{2,H}(t)^2 + \nu_{2,H}(t)^2}\right]\right\} \\
&= \frac{2\mathcal{N}^2}{\pi\sqrt{\mu_{1,H}(t)^2 + \nu_{1,H}(t)^2}\sqrt{\mu_{2,H}(t)^2 + \nu_{2,H}(t)^2}} \\
&\times \exp\left[-\frac{2(\mathrm{Re}(\alpha)\mu_{1,H}(t) + \mathrm{Im}(\alpha)\nu_{1,H}(t))^2 + X_1^2}{\mu_{1,H}(t)^2 + \nu_{1,H}(t)^2} - \frac{2(\mathrm{Re}(\alpha)\mu_{2,H}(t) + \mathrm{Im}(\alpha)\nu_{2,H}(t))^2 + X_2^2}{\mu_{2,H}(t)^2 + \nu_{2,H}(t)^2}\right] \\
&\times \left\{\cosh\left[\frac{2^{3/2}X_1(\mathrm{Re}(\alpha)\mu_{1,H}(t) + \mathrm{Im}(\alpha)\nu_{1,H}(t))}{\mu_{1,H}(t)^2 + \nu_{1,H}(t)^2} + \frac{2^{3/2}X_2(\mathrm{Re}(\alpha)\mu_{2,H}(t) + \mathrm{Im}(\alpha)\nu_{2,H}(t))}{\mu_{2,H}(t)^2 + \nu_{2,H}(t)^2}\right]\right. \\
&\left.\pm \cos\left[\frac{2^{3/2}X_1(\mathrm{Im}(\alpha)\mu_{1,H}(t) - \mathrm{Re}(\alpha)\nu_{1,H}(t))}{\mu_{1,H}(t)^2 + \nu_{1,H}(t)^2} + \frac{2^{3/2}X_2(\mathrm{Im}(\alpha)\mu_{2,H}(t) - \mathrm{Re}(\alpha)\nu_{2,H}(t))}{\mu_{2,H}(t)^2 + \nu_{2,H}(t)^2}\right]\right\}.
\end{aligned} \tag{37}
$$

In this expression, the time-dependent coefficients $\mu_{1,H}(t)$, $\nu_{1,H}(t)$, $\mu_{2,H}(t)$, and $\nu_{2,H}(t)$ can take the forms of Equations (23) and (24) when the states of the two modes evolve in the potential of the harmonic oscillator, and the forms of Equations (25) and (26) when the states of the two modes evolve in the potential of the inverted oscillator. For the case where the states of the two modes evolve in different potentials, the coefficients of the one evolving in the potential of the harmonic oscillator take the form of Equations (23) and (24) and the coefficients of the state of the other mode take the form of Equations (25) and (26). We note that the probability distribution in Equation (37) is an entangled distribution and, in general, it remains entangled throughout the evolution. It means that it cannot be rewritten as the product of two separate distributions concerning the particular modes. This fact can be especially seen from the final form of the expression.

Next, we find the evolving probability representation of even and odd cat states of an oscillating spin-1/2 particle. These states can be defined in the tensor product Hilbert space $\mathcal{H} = \mathcal{H}_{\mathrm{osc}} \otimes \mathcal{H}_{1/2}$ as

$$
\left|\Psi^{\pm}_{\mathrm{cat},1/2}\right\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle \otimes |0\rangle \pm |-\alpha\rangle \otimes |1\rangle). \tag{38}
$$

This state is an entangled state of an oscillating particle and its spin.

The probability representations of even and odd cat states of an oscillating spin-1/2 particle have been derived in Ref. [78]. The density operators of these states read

$$\begin{aligned}\hat{\rho}^{\pm}_{\mathrm{cat},1/2} &= \left|\Psi^{\pm}_{\mathrm{cat},1/2}\right\rangle\left\langle\Psi^{\pm}_{\mathrm{cat},1/2}\right| \\ &= \frac{1}{2}(|\alpha\rangle\langle\alpha|\otimes|0\rangle\langle 0| \pm |\alpha\rangle\langle-\alpha|\otimes|0\rangle\langle 1| \\ &\quad \pm|-\alpha\rangle\langle\alpha|\otimes|1\rangle\langle 0| + |-\alpha\rangle\langle-\alpha|\otimes|1\rangle\langle 1|). \end{aligned} \tag{39}$$

Let us consider the time evolution of this state:

$$\hat{\rho}^{\pm}_{\mathrm{cat},1/2}(t) = \hat{u}_{\mathrm{cat}}(t)\hat{u}_{1/2}(t)\hat{\rho}^{\pm}_{\mathrm{cat},1/2}(0)\hat{u}^{\dagger}_{1/2}(t)\hat{u}^{\dagger}_{\mathrm{cat}}(t), \tag{40}$$

where $\hat{u}_{\mathrm{cat}}(t)$ and $\hat{u}_{1/2}(t)$ describe the time evolution of the oscillator and the spin state, respectively. The operator $\hat{u}_{\mathrm{cat}}(t)$ is defined as

$$\hat{u}_{\mathrm{cat}}(t) = \exp\left(-it\hat{H}_{\pm}\right). \tag{41}$$

The Hamiltonians $\hat{H}_{\pm}$ of the harmonic ($\hat{H}_{+}$) and the inverted ($\hat{H}_{-}$) oscillator are defined in Equation (20). The operator $\hat{u}_{1/2}(t)$ can be represented by a 2-by-2 matrix as

$$\hat{u}_{1/2}(t) = \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix}. \tag{42}$$

The time dependencies of the matrix elements are not indicated in this expression. We will use this simplified notation in the subsequent formulas and tables. Applying Equations (18) and (28), the evolving conditional probability distribution $w^{\pm}_{\mathrm{cat},1/2}(X, Y \mid \mu_H(t), \nu_H(t), j, t)$ of these states can be formulated as

$$\begin{aligned} w^{\pm}_{\mathrm{cat},1/2}(X, Y \mid \mu_H(t), \nu_H(t), j, t) =& \frac{1}{2}\Big[w_{|\alpha\rangle\langle\alpha|}(X \mid \mu_H(t), \nu_H(t))w_{|0\rangle\langle 0|}(Y \mid j, t) \\ &\pm w_{|\alpha\rangle\langle-\alpha|}(X \mid \mu_H(t), \nu_H(t))w_{|0\rangle\langle 1|}(Y \mid j, t) \\ &\pm w_{|-\alpha\rangle\langle\alpha|}(X \mid \mu_H(t), \nu_H(t))w_{|1\rangle\langle 0|}(Y \mid j, t) \\ &+ w_{|-\alpha\rangle\langle-\alpha|}(X \mid \mu_H(t), \nu_H(t))w_{|1\rangle\langle 1|}(Y \mid j, t)\Big]. \end{aligned} \tag{43}$$

This probability distribution is an entangled one, having both a continuous and a discrete variable concerning the oscillator and the spin-1/2 states, respectively. It cannot be rewritten as the product of two separate probability distributions concerning the oscillator and the spin. In this expression, the time-dependent coefficients $\mu_H(t)$ and $\nu_H(t)$ can take the forms of Equations (23) and (24) when the state evolves in the potential of the harmonic oscillator, and the forms of Equations (25) and (26) when the state evolves in the potential of the inverted oscillator. Next, we derive all the factors appearing in Equation (43). Applying Equations (8), (9), (14) and (18) and the corresponding results appearing in Equation (37), the factors $w_{|\pm\alpha\rangle\langle\pm\alpha|}(X \mid \mu_H(t), \nu_H(t))$ in this expression can be obtained as

$$w_{|\alpha\rangle\langle\alpha|}(X \mid \mu_H(t), \nu_H(t)) = \mathcal{N}\exp\left[\frac{2^{3/2}X(\mu_H(t)\,\mathrm{Re}(\alpha) + \nu_H(t)\,\mathrm{Im}(\alpha)) - X^2}{\mu_H(t)^2 + \nu_H(t)^2}\right], \tag{44}$$

$$w_{|\alpha\rangle\langle-\alpha|}(X \mid \mu_H(t), \nu_H(t)) = \mathcal{N}\exp\left[\frac{i2^{3/2}X(\mu_H(t)\,\mathrm{Im}(\alpha) - \nu_H(t)\,\mathrm{Re}(\alpha)) - X^2}{\mu_H(t)^2 + \nu_H(t)^2}\right], \tag{45}$$

$$w_{|-\alpha\rangle\langle\alpha|}(X \mid \mu_H(t), \nu_H(t)) = \mathcal{N} \exp\left[\frac{-i2^{3/2}X(\mu_H(t)\,\mathrm{Im}(\alpha) - \nu_H(t)\,\mathrm{Re}(\alpha)) - X^2}{\mu_H(t)^2 + \nu_H(t)^2}\right], \quad (46)$$

$$w_{|-\alpha\rangle\langle-\alpha|}(X \mid \mu_H(t), \nu_H(t)) = \mathcal{N} \exp\left[\frac{-2^{3/2}X(\mu_H(t)\,\mathrm{Re}(\alpha) + \nu_H(t)\,\mathrm{Im}(\alpha)) - X^2}{\mu_H(t)^2 + \nu_H(t)^2}\right], \quad (47)$$

where

$$\mathcal{N} = \frac{1}{\sqrt{\mu_H(t)^2 + \nu_H(t)^2}} \frac{1}{\sqrt{\pi}\exp(2\,\mathrm{Re}(\alpha)^2))} \times \exp\left[\frac{-4\mu_H(t)\nu_H(t)\,\mathrm{Im}(\alpha)\,\mathrm{Re}(\alpha) + 2\nu_H(t)^2(\mathrm{Re}(\alpha)^2 - \mathrm{Im}(\alpha)^2)}{\mu_H(t)^2 + \nu_H(t)^2}\right]. \quad (48)$$

The other set of factors concerning the spin variable in Equation (43) can be calculated as

$$w_{|k_1\rangle\langle k_2|}(Y \mid j, t) = \mathrm{Tr}\Big(\hat{u}_{1/2}(t)|k_1\rangle\langle k_2|\hat{u}_{1/2}(t)^{\dagger}\hat{U}(Y \mid j)\Big), \qquad k_1, k_2 = 0, 1, \quad (49)$$

and the dequantizer operators $\hat{U}(Y \mid j)$ are the ones defined in Equation (27). The corresponding expressions are shown in Table 1.

Table 1. The evolving factors $w_{|0\rangle\langle 0|}(Y \mid j,t)$, $w_{|0\rangle\langle 1|}(Y \mid j,t)$, $w_{|1\rangle\langle 0|}(Y \mid j,t)$, $w_{|1\rangle\langle 1|}(Y \mid j,t)$ appearing in the conditional probability distribution $w^{\pm}_{\mathrm{cat},1/2}(X, Y \mid \mu_H(t), \nu_H(t), j, t)$ of even and odd cat states of an oscillating spin-1/2 particle displayed in Equation (43) for the pairs of the parameters Y and j.

$Y \mid j$	$w_{\|0\rangle\langle 0\|}(Y \mid j,t)$	$w_{\|0\rangle\langle 1\|}(Y \mid j,t) = w_{\|1\rangle\langle 0\|}(Y \mid j,t)^*$	$w_{\|1\rangle\langle 1\|}(Y \mid j,t)$
$+1/2 \mid 1$	$\frac{\|u_{1,1}+u_{2,1}\|^2}{2}$	$\frac{(u_{1,1}+u_{2,1})(u_{1,2}+u_{2,2})^*}{2}$	$\frac{\|u_{1,2}+u_{2,2}\|^2}{2}$
$+1/2 \mid 2$	$\frac{\|u_{1,1}-iu_{2,1}\|^2}{2}$	$\frac{(u_{1,1}-iu_{2,1})(u_{1,2}-iu_{2,2})^*}{2}$	$\frac{\|u_{1,2}-iu_{2,2}\|^2}{2}$
$+1/2 \mid 3$	$\|u_{1,1}\|^2$	$u_{1,1}u_{1,2}^*$	$\|u_{1,2}\|^2$
$-1/2 \mid 1$	$\frac{\|u_{1,1}-u_{2,1}\|^2}{2}$	$\frac{(u_{1,1}-u_{2,1})(u_{1,2}-u_{2,2})^*}{2}$	$\frac{\|u_{1,2}-u_{2,2}\|^2}{2}$
$-1/2 \mid 2$	$\frac{\|u_{1,1}+iu_{2,1}\|^2}{2}$	$\frac{(u_{1,1}+iu_{2,1})(u_{1,2}+iu_{2,2})^*}{2}$	$\frac{\|u_{1,2}+iu_{2,2}\|^2}{2}$
$-1/2 \mid 3$	$\|u_{2,1}\|^2$	$u_{2,1}u_{2,2}^*$	$\|u_{2,2}\|^2$

Recall that Equation (43) presents the total time-evolving probability representation of even and odd cat states of an oscillating spin-1/2 particle in which the factors of the part describing the oscillator state are given by Equations (44)–(48) and the factors of the part corresponding to the spin-1/2 state are shown in Table 1.

Next, we determine the evolving marginal conditional probability distributions $\tilde{w}_{1/2}(Y \mid j, t)$ and the evolving tomogram $\tilde{w}_{\mathrm{cat}}(X \mid \mu_H(t), \nu_H(t))$ for the evolving states of the spin and the oscillator, respectively. These distributions can be derived as

$$\tilde{w}^{\pm}_{1/2}(Y \mid j, t) = \int w^{\pm}_{\mathrm{cat},1/2}(X, Y \mid \mu_H(t), \nu_H(t), j, t)\mathrm{d}X, \quad (50)$$

$$\tilde{w}^{\pm}_{\mathrm{cat}}(X \mid \mu_H(t), \nu_H(t)) = \sum_Y w^{\pm}_{\mathrm{cat},1/2}(X, Y \mid \mu_H(t), \nu_H(t), j, t). \quad (51)$$

Applying the integral in Equation (50) to the factors $w_{|\pm\alpha\rangle\langle\pm\alpha|}(X \mid \mu_H(t), \nu_H(t))$ appearing in Equations (44)–(48), we obtain

$$\int w_{|\alpha\rangle\langle\alpha|}(X \mid \mu_H(t), \nu_H(t))\mathrm{d}X = \int w_{|-\alpha\rangle\langle-\alpha|}(X \mid \mu_H(t), \nu_H(t))\mathrm{d}X = 1, \tag{52}$$

$$\int w_{|-\alpha\rangle\langle\alpha|}(X \mid \mu_H(t), \nu_H(t))\mathrm{d}X = \int w_{|\alpha\rangle\langle-\alpha|}(X \mid \mu_H(t), \nu_H(t))\mathrm{d}X = \exp\left(-2|\alpha|^2\right). \tag{53}$$

Then, by using the factors presented in Table 1, we eventually obtain the components of the evolving marginal conditional probability distribution $\hat{w}^{\pm}_{1/2}(Y \mid j, t)$ shown in Table 2.

Table 2. The components of the evolving marginal conditional probability distribution $\hat{w}^{\pm}_{1/2}(Y \mid j, t)$ of even and odd cat states of an oscillating spin-1/2 particle shown in Equation (43).

$Y \mid j, t$	$\hat{w}^{\pm}_{1/2}(Y \mid j, t)$
$+1/2 \mid 1$	$\frac{1}{4}\left\{|u_{1,1}+u_{2,1}|^2+|u_{1,2}+u_{2,2}|^2 \pm 2\exp\left(-2|\alpha|^2\right)\mathrm{Re}[(u_{1,1}+u_{2,1})(u_{1,2}+u_{2,2})^*]\right\}$
$+1/2 \mid 2$	$\frac{1}{4}\left\{|u_{1,1}-iu_{2,1}|^2+|u_{1,2}-iu_{2,2}|^2 \pm 2\exp\left(-2|\alpha|^2\right)\mathrm{Re}[(u_{1,1}-iu_{2,1})(u_{1,2}-iu_{2,2})^*]\right\}$
$+1/2 \mid 3$	$\frac{1}{2}\left[|u_{1,1}|^2+|u_{1,2}|^2 \pm 2\exp\left(-2|\alpha|^2\right)\mathrm{Re}(u_{1,1}u^*_{1,2})\right]$
$-1/2 \mid 1$	$\frac{1}{4}\left\{|u_{1,1}-u_{2,1}|^2+|u_{1,2}-u_{2,2}|^2 \pm 2\exp\left(-2|\alpha|^2\right)\mathrm{Re}[(u_{1,1}-u_{2,1})(u_{1,2}-u_{2,2})^*]\right\}$
$-1/2 \mid 2$	$\frac{1}{4}\left\{|u_{1,1}+iu_{2,1}|^2+|u_{1,2}+iu_{2,2}|^2 \pm 2\exp\left(-2|\alpha|^2\right)\mathrm{Re}[(u_{1,1}+iu_{2,1})(u_{1,2}+iu_{2,2})^*]\right\}$
$-1/2 \mid 3$	$\frac{1}{2}\left[|u_{2,1}|^2+|u_{2,2}|^2 \pm 2\exp\left(-2|\alpha|^2\right)\mathrm{Re}(u_{2,1}u^*_{2,2})\right]$

As it can be seen from Table 2, the components of the evolving marginal conditional probability distribution $\hat{w}^{\pm}_{1/2}(Y \mid j, t)$ contain terms in which the parameter α appears. However, in the limit $\alpha \to 0$ where the coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ are practically orthogonal to each other, this dependence vanishes. One can easily check that, in this limit, the evolving marginal conditional probability distribution $\hat{w}^{\pm}_{1/2}(Y \mid j, t)$ corresponds to the conditional probability distribution of the evolving mixed state:

$$\hat{\rho}^{\pm}_{1/2}(t) = \hat{u}^{\dagger}_{1/2}(t)\hat{\rho}^{\pm}_{1/2}\hat{u}_{1/2}(t), \tag{54}$$

where

$$\hat{\rho}^{\pm}_{1/2} = \mathrm{Tr}_{\mathrm{cat}}\left[\hat{\rho}^{\pm}_{\mathrm{cat},1/2}\right] = \frac{1}{2}[|0\rangle\langle 0| + |1\rangle\langle 1|]. \tag{55}$$

Finally, we derive the evolving marginal tomogram $\hat{w}^{\pm}_{\mathrm{cat}}(X \mid \mu_H(t), \nu_H(t))$ by applying the expression presented in Equation (51). Using Equation (43) and the factors presented in Table 1, we obtain

$$\begin{aligned}\hat{w}^{\pm}_{\mathrm{cat}}(X \mid \mu_H(t), \nu_H(t)) =& \frac{1}{2}\Big[w_{|\alpha\rangle\langle\alpha|}(X \mid \mu_H(t), \nu_H(t))(|u_{1,1}|^2 + |u_{2,1}|^2) \\ &\pm w_{|\alpha\rangle\langle-\alpha|}(X \mid \mu_H(t), \nu_H(t))(u_{1,1}u^*_{1,2} + u_{2,1}u^*_{2,2}) \\ &\pm w_{|-\alpha\rangle\langle\alpha|}(X \mid \mu_H(t), \nu_H(t))(u^*_{1,1}u_{1,2} + u^*_{2,1}u_{2,2}) \\ &+ w_{|-\alpha\rangle\langle-\alpha|}(X \mid \mu_H(t), \nu_H(t))(|u_{1,2}|^2 + |u_{2,2}|^2)\Big] \\ =& \frac{1}{2}\Big[w_{|\alpha\rangle\langle\alpha|}(X \mid \mu_H(t), \nu_H(t)) + w_{|-\alpha\rangle\langle-\alpha|}(X \mid \mu_H(t), \nu_H(t))\Big]. \end{aligned} \tag{56}$$

The result in Equation (56) shows that the evolving marginal tomogram $\hat{w}^{\pm}_{\mathrm{cat}}(X \mid \mu_H(t), \nu_H(t))$ corresponds to the tomogram of the evolving mixed state:

$$\hat{\rho}^{\pm}_{\mathrm{cat}}(t) = \hat{u}^{\dagger}_{\mathrm{cat}}(t)\hat{\rho}^{\pm}_{\mathrm{cat}}\hat{u}_{\mathrm{cat}}(t), \tag{57}$$

where

$$\hat{\rho}^{\pm}_{\text{cat}} = \text{Tr}_{1/2}\left[\hat{\rho}^{\pm}_{\text{cat},1/2}\right] = \frac{1}{2}[|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|]. \quad (58)$$

From this result, one can conclude that the entangled probability distribution $w^{\pm}_{\text{cat},1/2}(X, Y \mid \mu_H(t), \nu_H(t), j, t)$ of Equation (43) properly describes the entangled even and odd cat states of an oscillating spin-1/2 particle.

4. Conclusions

We have determined the evolving probability representation of entangled cat states in the potential of either the harmonic oscillator or the inverted oscillator, assuming that the states have been initially prepared in the potential of the harmonic oscillator. In the considered separable time evolution, the evolution of the momentum and position operators of both modes is linear in the Heisenberg picture and the time dependence can be transferred to the parameters of the probability distribution. Determining the evolving probability representation of the discussed entangled states for a general nonseparable time evolution deserves consideration in the future. In this case, the used procedure of the derivation of the evolving probability representations seems to be extendable for systems where the evolution of the position and momentum operators of the two modes in the Heisenberg picture is linear. For describing the nonseparable time evolution of the probability representation, the application of an alternative definition of the representation (13) may also be required.

We have also determined the evolving probability representation of cat states of an oscillating spin-1/2 particle of the inverted oscillator, in which the time evolution of the spin state is described by an arbitrary unitary operator. Finally, we have derived the evolving marginal conditional probability distributions and the evolving tomogram for the evolving states of the spin and the oscillator in the case of the entangled even and odd cat states of an oscillating spin-1/2 particle. The marginal distributions describe the mixed states that can be obtained after tracing out the corresponding system. The determined entangled probability distributions contain all information about the considered entangled states.

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References

1. Mancini, S.; Man'ko, V.I.; Tombesi, P. Symplectic tomography as classical approach to quantum systems. *Phys. Lett. A* **1996**, *213*, 1–6. [CrossRef]
2. Dodonov, V.; Man'ko, V. Positive distribution description for spin states. *Phys. Lett. A* **1997**, *229*, 335–339. [CrossRef]
3. Man'ko, O. Tomography of spin states and classical formulation of quantum mechanics. In *Symmetries in Science X*; Springer: Berlin/Heidelberg, Germany, 1998; pp. 207–222.
4. Ibort, A.; Man'ko, V.I.; Marmo, G.; Simoni, A.; Ventriglia, F. An introduction to the tomographic picture of quantum mechanics. *Phys. Scr.* **2009**, *79*, 65013. [CrossRef]
5. Amosov, G.G.; Korennoy, Y.A.; Man'ko, V.I. Description and measurement of observables in the optical tomographic probability representation of quantum mechanics. *Phys. Rev. A* **2012**, *85*, 52119. [CrossRef]

6. Asorey, M.; Ibort, A.; Marmo, G.; Ventriglia, F. Quantum Tomography twenty years later. *Phys. Scr.* **2015**, *90*, 74031. [CrossRef]
7. Man'ko, O.V.; Man'ko, V.I. Probability Representation of Quantum States. *Entropy* **2021**, *23*, 549. [CrossRef]
8. Husimi, K. Some Formal Properties of the Density Matrix. *Proc. Phys.-Math. Soc. Jpn.* **1940**, *22*, 264–314. [CrossRef]
9. Kano, Y. A New Phase-Space Distribution Function in the Statistical Theory of the Electromagnetic Field. *J. Math. Phys.* **1965**, *6*, 1913–1915. [CrossRef]
10. Glauber, R.J. Photon Correlations. *Phys. Rev. Lett.* **1963**, *10*, 84–86. [CrossRef]
11. Sudarshan, E.C.G. Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams. *Phys. Rev. Lett.* **1963**, *10*, 277–279. [CrossRef]
12. Wigner, E. On the Quantum Correction For Thermodynamic Equilibrium. *Phys. Rev.* **1932**, *40*, 749–759. [CrossRef]
13. Man'ko, V.I.; Markovich, L.A. Integral transforms between tomogram and quasi-probability functions based on quantizer-dequantizer operators formalism. *J. Math. Phys.* **2020**, *61*, 102102. [CrossRef]
14. Man'ko, O.V.; Man'ko, V.I.; Marmo, G. Alternative commutation relations, star products and tomography. *J. Phys. A Math. Gen.* **2002**, *35*, 699. [CrossRef]
15. Smithey, D.T.; Beck, M.; Raymer, M.G.; Faridani, A. Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum. *Phys. Rev. Lett.* **1993**, *70*, 1244–1247. [CrossRef]
16. Bertrand, J.; Bertrand, P. A tomographic approach to Wigner's function. *Found. Phys.* **1987**, *17*, 397–405. [CrossRef]
17. Vogel, K.; Risken, H. Determination of quasiprobability distributions in terms of probability distributions for the rotated quadrature phase. *Phys. Rev. A* **1989**, *40*, 2847–2849. [CrossRef]
18. D'Ariano, G.M.; Leonhardt, U.; Paul, H. Homodyne detection of the density matrix of the radiation field. *Phys. Rev. A* **1995**, *52*, R1801–R1804. [CrossRef]
19. Mauro D'Ariano, G.; Paris, M.G.; Sacchi, M.F. Quantum Tomography. In *Advances in Imaging and Electron Physics*; Elsevier: Amsterdam, The Netherlands, 2003; Volume 128, pp. 205–308. [CrossRef]
20. Lvovsky, A.I.; Raymer, M.G. Continuous-variable optical quantum-state tomography. *Rev. Mod. Phys.* **2009**, *81*, 299–332. [CrossRef]
21. Man'ko, M.A.; Man'ko, V.I. Quantum Oscillator at Temperature T and the Evolution of a Charged-Particle State in the Electric Field in the Probability Representation of Quantum Mechanics. *Entropy* **2023**, *25*, 213. [CrossRef]
22. Ádám, P.; Man'ko, M.A.; Man'ko, V.I. Even and Odd Schrödinger Cat States in the Probability Representation of Quantum Mechanics. *J. Russ. Laser Res.* **2022**, *43*, 1–17. [CrossRef]
23. Man'ko, O.V.; Man'ko, V.I. Inverted Oscillator Quantum States in the Probability Representation. *Entropy* **2023**, *25*, 217. [CrossRef] [PubMed]
24. Chernega, V.N.; Man'ko, O.V. Dynamics of System States in the Probability Representation of Quantum Mechanics. *Entropy* **2023**, *25*, 785. [CrossRef] [PubMed]
25. Sundaram, S.; Burgess, C.P.; O'Dell, D.H.J. Duality between the quantum inverted harmonic oscillator and inverse square potentials. *New J. Phys.* **2024**, *26*, 053023. [CrossRef]
26. Yuce, C. Quantum inverted harmonic potential. *Phys. Scr.* **2021**, *96*, 105006. [CrossRef]
27. Gietka, K.; Busch, T. Inverted harmonic oscillator dynamics of the nonequilibrium phase transition in the Dicke model. *Phys. Rev. E* **2021**, *104*, 034132. [CrossRef]
28. Ullinger, F.; Zimmermann, M.; Schleich, W.P. The logarithmic phase singularity in the inverted harmonic oscillator. *AVS Quantum Sci.* **2022**, *4*, 24402. [CrossRef]
29. Subramanyan, V.; Hegde, S.S.; Vishveshwara, S.; Bradlyn, B. Physics of the Inverted Harmonic Oscillator: From the lowest Landau level to event horizons. *Ann. Phys.* **2021**, *435*, 168470. [CrossRef]
30. Cirilo-Lombardo, D.J.; Sanchez, N.G. Coherent states of quantum spacetimes for black holes and de Sitter spacetime. *Phys. Rev. D* **2023**, *108*, 126001. [CrossRef]
31. Gonzalez-Ballestero, C.; Aspelmeyer, M.; Novotny, L.; Quidant, R.; Romero-Isart, O. Levitodynamics: Levitation and control of microscopic objects in vacuum. *Science* **2021**, *374*. [CrossRef]
32. Duchaň, M.; Šiler, M.; Jákl, P.; Brzobohatý, O.; Rakhubovsky, A.; Filip, R.; Zemánek, P. Experimental amplification and squeezing of a motional state of an optically levitated nanoparticle. *arXiv* **2024**, arXiv:2403.04302.
33. Dago, S.; Rieser, J.; Ciampini, M.A.; Mlynář, V.; Kugi, A.; Aspelmeyer, M.; Deutschmann-Olek, A.; Kiesel, N. Stabilizing nanoparticles in the intensity minimum: Feedback levitation on an inverted potential. *Opt. Express* **2024**, *32*, 45133. [CrossRef]
34. Choi, S.; Onofrio, R.; Sundaram, B. Optimized sympathetic cooling of atomic mixtures via fast adiabatic strategies. *Phys. Rev. A* **2011**, *84*, 051601. [CrossRef]
35. Choi, S.; Onofrio, R.; Sundaram, B. Squeezing and robustness of frictionless cooling strategies. *Phys. Rev. A* **2012**, *86*, 043436. [CrossRef]
36. Choi, S.; Onofrio, R.; Sundaram, B. Ehrenfest dynamics and frictionless cooling methods. *Phys. Rev. A* **2013**, *88*, 053401. [CrossRef]

37. Cunha, M.O.T.; Man'ko, V.I.; Scully, M.O. Quasiprobability and Probability Distributions for Spin-1/2 States. *Found. Phys. Lett.* **2001**, *14*, 103–117. [CrossRef]
38. D'Ariano, G.M.; Maccone, L.; Paini, M. Spin tomography. *J. Opt. B Quantum Semiclass. Opt.* **2003**, *5*, 77. [CrossRef]
39. Chernega, V.N.; Man'ko, O.V.; Man'ko, V.I. Entangled Qubit States and Linear Entropy in the Probability Representation of Quantum Mechanics. *Entropy* **2022**, *24*, 527. [CrossRef]
40. Man'ko, M.A.; Man'ko, V.I. Probability Distributions Describing Qubit-State Superpositions. *Entropy* **2023**, *25*, 1366. [CrossRef]
41. Bužek, V.; Vidiella-Barranco, A.; Knight, P.L. Superpositions of coherent states: Squeezing and dissipation. *Phys. Rev. A* **1992**, *45*, 6570–6585. [CrossRef]
42. Dodonov, V.V.; Man'ko, V.I.; Nikonov, D.E. Even and odd coherent states for multimode parametric systems. *Phys. Rev. A* **1995**, *51*, 3328–3336. [CrossRef]
43. Agarwal, G.; Puri, R.; Singh, R. Atomic Schrödinger cat states. *Phys. Rev. A* **1997**, *56*, 2249–2254. [CrossRef]
44. Brune, M.; Hagley, E.; Dreyer, J.; Maître, X.; Maali, A.; Wunderlich, C.; Raimond, J.M.; Haroche, S. Observing the Progressive Decoherence of the "Meter" in a Quantum Measurement. *Phys. Rev. Lett.* **1996**, *77*, 4887–4890. [CrossRef] [PubMed]
45. Auffeves, A.; Maioli, P.; Meunier, T.; Gleyzes, S.; Nogues, G.; Brune, M.; Raimond, J.M.; Haroche, S. Entanglement of a Mesoscopic Field with an Atom Induced by Photon Graininess in a Cavity. *Phys. Rev. Lett.* **2003**, *91*, 230405. [CrossRef] [PubMed]
46. Neergaard-Nielsen, J.S.; Nielsen, B.M.; Hettich, C.; Mølmer, K.; Polzik, E.S. Generation of a Superposition of Odd Photon Number States for Quantum Information Networks. *Phys. Rev. Lett.* **2006**, *97*, 083604. [CrossRef]
47. Ourjoumtsev, A.; Jeong, H.; Tualle-Brouri, R.; Grangier, P. Generation of optical 'Schrödinger cats' from photon number states. *Nature* **2007**, *448*, 784–786. [CrossRef]
48. Takahashi, H.; Wakui, K.; Suzuki, S.; Takeoka, M.; Hayasaka, K.; Furusawa, A.; Sasaki, M. Generation of Large-Amplitude Coherent-State Superposition via Ancilla-Assisted Photon Subtraction. *Phys. Rev. Lett.* **2008**, *101*, 233605. [CrossRef]
49. He, B.; Nadeem, M.; Bergou, J.A. Scheme for generating coherent-state superpositions with realistic cross-Kerr nonlinearity. *Phys. Rev. A* **2009**, *79*, 035802. [CrossRef]
50. Adam, P.; Darázs, Z.; Kiss, T.; Mechler, M. Double self-Kerr scheme for optical Schrödinger-cat state preparation. *Phys. Scr.* **2011**, *T143*, 014002. [CrossRef]
51. Lee, C.W.; Lee, J.; Nha, H.; Jeong, H. Generating a Schrödinger-cat-like state via a coherent superposition of photonic operations. *Phys. Rev. A* **2012**, *85*, 63815. [CrossRef]
52. Bild, M.; Fadel, M.; Yang, Y.; von Lüpke, U.; Martin, P.; Bruno, A.; Chu, Y. Schrödinger cat states of a 16-microgram mechanical oscillator. *Science* **2023**, *380*, 274–278. [CrossRef]
53. van Enk, S.; Hirota, O. Entangled coherent states: Teleportation and decoherence. *Phys. Rev. A* **2001**, *64*, 022313. [CrossRef]
54. Jeong, H.; Kim, M.S.; Lee, J. Quantum-information processing for a coherent superposition state via a mixedentangled coherent channel. *Phys. Rev. A* **2001**, *64*, 052308. [CrossRef]
55. Cochrane, P.T.; Milburn, G.J.; Munro, W.J. Macroscopically distinct quantum-superposition states as a bosonic code for amplitude damping. *Phys. Rev. A* **1999**, *59*, 2631–2634. [CrossRef]
56. Mirrahimi, M. Cat-qubits for quantum computation. *Comptes Rendus Phys.* **2016**, *17*, 778–787. [CrossRef]
57. Grimm, A.; Frattini, N.E.; Puri, S.; Mundhada, S.O.; Touzard, S.; Mirrahimi, M.; Girvin, S.M.; Shankar, S.; Devoret, M.H. Stabilization and operation of a Kerr-cat qubit. *Nature* **2020**, *584*, 205–209. [CrossRef]
58. Su, Q.P.; Zhang, Y.; Yang, C.P. Single-step implementation of a hybrid controlled-not gate with one superconducting qubit simultaneously controlling multiple target cat-state qubits. *Phys. Rev. A* **2022**, *105*, 062436. [CrossRef]
59. Mogilevtsev, D.; Ya, K.S. The generation of multicomponent entangled Schrödinger cat states via a fully quantized nondegenerate four-wave mixing process. *Opt. Commun.* **1996**, *132*, 452–456. [CrossRef]
60. Vitali, D.; Fortunat, M.; Tombesi, P.; De Martini, F. Generating Entangled Schrödinger Cat States within a Parametric Oscillator. *Fortschr. Phys.* **2000**, *48*, 437–446. [CrossRef]
61. Janszky, J.; Asbóth, J.; Gábris, A.; Vukics, A.; Koniorczyk, M.; Kobayashi, T. Two-mode Schrödinger cats, entanglement and teleportation. *Fortschr. Phys.* **2003**, *51*, 157–171. [CrossRef]
62. Janowicz, M.; Orłowski, A. Decoherence of entangled states in two-mode cavities. *J. Phys. B At. Mol. Opt. Phys.* **2006**, *39*, 1763–1771. [CrossRef]
63. Thenabadu, M.; Reid, M.D. Bipartite Leggett-Garg and macroscopic Bell-inequality violations using cat states: Distinguishing weak and deterministic macroscopic realism. *Phys. Rev. A* **2022**, *105*, 052207. [CrossRef]
64. Zheng, S.B. A scheme for the generation of multi-mode Schrödinger cat states. *Quantum Semiclass. Opt.* **1998**, *10*, 691–694. [CrossRef]
65. Paternostro, M.; Kim, M.S.; Ham, B.S. Generation of entangled coherent states via cross-phase-modulation in a double electro-magnetically induced transparency regime. *Phys. Rev. A* **2003**, *67*, 023811. [CrossRef]
66. Zhou, L.; Xiong, H. A macroscopical entangled coherent state generator in a V configuration atom system. *J. Phys. B At. Mol. Opt. Phys.* **2008**, *41*, 025501. [CrossRef]

67. Liu, T.; Su, Q.P.; Xiong, S.J.; Liu, J.M.; Yang, C.P.; Nori, F. Generation of a macroscopic entangled coherent state using quantum memories in circuit QED. *Sci. Rep.* **2016**, *6*, 32004. [CrossRef]
68. Wang, C.; Gao, Y.Y.; Reinhold, P.; Heeres, R.W.; Ofek, N.; Chou, K.; Axline, C.; Reagor, M.; Blumoff, J.; Sliwa, K.M.; et al. A Schrödinger cat living in two boxes. *Science* **2016**, *352*, 1087–1091. [CrossRef]
69. Guo, R.; Zhou, L.; Gu, S.P.; Wang, X.F.; Sheng, Y.B. Generation of concatenated Greenberger–Horne–Zeilinger-type entangled coherent state based on linear optics. *Quantum Inf. Proc.* **2017**, *16*, 68. [CrossRef]
70. Zhong, Z.R.; Huang, X.J.; Yang, Z.B.; Shen, L.T.; Zheng, S.B. Generation and stabilization of entangled coherent states for the vibrational modes of a trapped ion. *Phys. Rev. A* **2018**, *98*, 032311. [CrossRef]
71. Xiong, B.; Li, X.; Chao, S.L.; Yang, Z.; Zhang, W.Z.; Zhou, L. Generation of entangled Schrödinger cat state of two macroscopic mirrors. *Opt. Express* **2019**, *27*, 13547. [CrossRef]
72. Zhou, Z.Y.; Gneiting, C.; You, J.Q.; Nori, F. Generating and detecting entangled cat states in dissipatively coupled degenerate optical parametric oscillators. *Phys. Rev. A* **2021**, *104*, 013715. [CrossRef]
73. Wang, X. Quantum teleportation of entangled coherent states. *Phys. Rev. A* **2001**, *64*, 022302. [CrossRef]
74. Joo, J.; Ginossar, E. Efficient scheme for hybrid teleportation via entangled coherent states in circuit quantum electrodynamics. *Sci. Rep.* **2016**, *6*, 26338. [CrossRef] [PubMed]
75. Chao-Ping, W.; Hu, X.Y.; Yu, Y.F.; Zhang, Z.M. Phase sensitivity of two nonlinear interferometers with inputting entangled coherent states. *Chin. Phys. B* **2016**, *25*, 040601. [CrossRef]
76. Ralph, T.C.; Gilchrist, A.; Milburn, G.J.; Munro, W.J.; Glancy, S. Quantum computation with optical coherent states. *Phys. Rev. A* **2003**, *68*, 042319. [CrossRef]
77. Liao, J.Q.; Huang, J.F.; Tian, L. Generation of macroscopic Schrödinger-cat states in qubit-oscillator systems. *Phys. Rev. A* **2016**, *93*, 033853. [CrossRef]
78. Mechler, M.; Man'ko, M.A.; Man'ko, V.I.; Adam, P. Even and Odd Cat States of Two and Three Qubits in the Probability Representation of Quantum Mechanics. *Entropy* **2024**, *26*, 485. [CrossRef]
79. Kenfack, A.; Zyczkowski, K. Negativity of the Wigner function as an indicator of non-classicality. *J. Opt. B Quantum Semiclass. Opt*, **2004**, *6*, 396–404. [CrossRef]
80. Hudson, R. When is the wigner quasi-probability density non-negative? *Rep. Math. Phys.* **1974**, *6*, 249–252. [CrossRef]
81. Soto, F.; Claverie, P. When is the Wigner function of multidimensional systems nonnegative? *J. Math. Phys.* **1983**, *24*, 97–100. [CrossRef]
82. Arkhipov, A.S. Tomography for Several Particles with One Random Variable. *J. Russ. Laser Res.* **2003**, *24*, 237–255. [CrossRef]
83. Amosov, G.G.; Man'ko, V.I. A classical limit for the center-of-mass tomogram in view of the central limit theorem. *Phys. Scr.* **2009**, *80*, 025006. [CrossRef]