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String Theory and the Size of Hadrons

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Abstract. We begin by outlining the ancient puzzle of off shell currents and the infinite size particles in a string theory of hadrons. We then consider the problem from the modern AdS/CFT perspective. We argue that although hadrons should be thought of as ideal thin strings from the 5-dimensional bulk point of view, the 4-dimensional strings are a superposition of "fat" strings of different thickness.

We also find that the warped nature of the target geometry provides a mechanism for taming the infinite zero point fluctuations which apparently produce a divergent result for hadronic radii.

1 Meeting Holger

When I was a school kid during the early 1950's we used to have to read a magazine called "The Reader's Digest". It was full of corny articles about patriotic platitudes which were very boring but it always had an interesting section called "My Most Unforgettable Character ". It was usually about a somewhat eccentric but admirable character that the writer had especially fond memories of. Well, for me (and I suspect anyone else who knows him), Holger will always be one of the most unforgettable characters I've ever met.

Holger and I first met through the mail in 1970. He had seen a paper that I wrote claiming that the Veneziano amplitude described the scattering of some kind of elastic strings. Unfortunately I no longer have the hand written letter but I can still see his distinctive curly handwriting and the signature - Holger Bech Nielsen. Most of all I remember his almost child-like enthusiam and simplicity. He too had been working on a similar idea [1]. Unlike so many messages that I've received over the years, this one had nothing to do with staking a claim or as we say, pissing on territory. The letter straightforwardly expressed his excitement and joyously shared his own ideas. It was completely clear to me that I had met a larger than life, most unforgettable character.

That year I invited Holger to spend a month visiting me in New York and what a month it was. We ate too much, drank too much and yelled too loud but the physics excitement was palpable. I have never had more fun doing physics than during that time. At the time, string theory was of course a theory of hadrons. Mesons were strings with quarks at their ends. Both of us were disturbed by

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[1] Nambu had also been working on the same ideas but I don't believe that Holger or I were aware of it.

something that we thought was a very serious shortcoming of the theory. At about that time the electromagnetic properties of hadrons were under intensive investigation at SLAC and other places. Electromagnetic form factors of nucleons were already well measured. SLAC had measured deep inelastic electroproduction and Feynman and Bjorken had explained the data with their parton ideas. The problem that puzzled Holger and me was that we could see no way to define the local electromagnetic current of a hadron using string theory. Every time we tried we got nonsense. Hadrons came out infinitely big and in a sense, infinitely soft. Holger and I had a wonderful time thinking about the problem. I think it is fair to say that many of the themes of my future work trace back to that brief month and to Holger's profound influence on me .

What was our solution to the problem? Both of us were inclined to think of the string as an idealized limit of a discrete system. In my case I viewed it as a system of partons in the light cone frame. Holger had a more covariant view which he had been discussing with Aage Kraemmer in Copenhagen. According to this view, the Koba Nielsen disc [1] was really the continuum limit of an infinitely dense planar Feynman diagram or more precisely, a sum over such diagrams. At that time we had no idea why planar diagrams should dominate. That had to wait for 't Hooft [4].

One of the ideas in the paper was that the geometry of a large planar diagram defined a kind of metric which could be gauge fixed to what would now be called the conformal gauge. We realized that a proper treatment should include a sum over diagrams which could be represented as a path integral over a diagram density. Holger wanted to treat this degree of freedom as an additional dimension which a decade later, following the work of Polyakov, became the Liouville field. I thought it was a bad idea since I could not see how an additional infinite direction could fit into hadron physics. For this reason we decided that the integration should be dominated by some specific density that didn't fluctuate too much.

In fact the form factor problem forced us to conclude that the continuum limit was just too extreme. Hadrons might be described by fairly dense systems of partons but not a continuum. There had to be a cutoff which limited the zero-point fluctuations that blew the string up to infinite size [2,3] and also removed the hard effects of discrete partons. Together with Kraemmer we wrote a paper [2] formulating what we called the Dual Parton Model which tried to keep the good features of strings without passing to the extreme limit. It is a great pleasure to come back to this problem which so occupied our thoughts during that month thirty one years ago and to contribute some new thoughts for Holger's Festschrift.

2 The Puzzle of Infinite Size

The obvious difficulties with hadronic string theory involved the spectrum which invariably included massless vectors, scalars and tensor particles. There were also the subtle problems of local currents that Holger and I had wrestled with. Technically speaking, there was no possibility of continuing string theory off the mass shell to construct the matrix elements needed to describe the interaction of hadrons with electromagnetism and gravitation [2]. The natural candidates, vertex operators like $\exp ikX$ can not be sensibly continued away from specific discrete "on shell" values of k^2. Closely connected with this was the divergence encountered in attempting to compute the hadronic electromagnetic or gravitational radius [2] [3]. Thus string theory was abandoned as a theory of hadrons and replaced by QCD. The success of string theory in understanding Regge Trajectories and quark confinement was understood in terms of an approximate string-like behavior of chromo-electric flux tubes. According to this view, hadronic strings are not the infinitely thin idealized objects of mathematical string theory but are thick tubes similar to the quantized flux lines in superconductors[5]. The ideal string theory was relegated to the world of quantum gravity.

However more recent developments have strongly suggested that an idealized form of string theory may exactly describe certain gauge theories which are quite similar to QCD [6] [7]. We have returned full circle to the suspicion that hadrons may be precisely described by an idealized string theory, especially in the 't Hooft limit [4]. The new string theories are certainly more complicated than the original versions and it seems very plausible that the problems with the massless spectrum of particles will be overcome. Less however has been studied about the problems connected with local currents. In this contribution I will show that the new insights from the AdS/CFT correspondence provide a solution to the form factor problem.

I begin by reviewing the problem. For definiteness we work in the light cone frame in which string theory has the form of a conventional Galilean-invariant Hamiltonian quantum mechanics. The degrees of freedom of the first-quantized string include $D-2$ transverse coordinates $X^m(\sigma)$ and the Lagrangian for these variables is

$$L = \frac{1}{4\pi}\int_0^{2\pi P_-} d\sigma(\dot{X}\dot{X} - (\alpha')^{-2}X'X') \tag{1}$$

where $\dot{X}$ and X' mean derivative with respect to light-cone time τ and string parameter σ. The light-cone momentum P_- is conjugate to the light like coordinate x^-. All irrelevant constants have been set to unity.

An important feature of the light-cone theory involves the local distribution of P_- on the string. The rule is that the distribution of P_- is uniform with respect to σ. In other words the longitudinal momentum dP_- carried on a segment of string $d\sigma$ is exactly $d\sigma/2\pi$.

Let us now consider the transverse density of P_-. In a space-time field theory this would be given by

$$\rho(X) = \int dx^- T_{--}(X, x^-) \tag{2}$$

where T is the energy momentum tensor of the field theory. Matrix elements of ρ between strings of equal P_- define form factors for gravitational interactions of the string and are entirely analogous to electromagnetic form factors.

The natural object in string theory to identify with $\rho(X)$ is

$$\frac{1}{2\pi}\int d\sigma[\delta(X - X(\sigma))] \tag{3}$$

In other words $\rho(X)$ receives contributions from every element of string localized at X. The Fourier transform of $\rho(X)$

$$\tilde{\rho}(k) = \int d\sigma \exp ikX(\sigma) \tag{4}$$

defines a system of form factors by its matrix elements between string states.

The mean square radius of the distribution function is given by

$$\bar{R}^2 = \langle \int X^2 \rho(X) \rangle \tag{5}$$

and can be rewritten in terms of $\tilde{\rho}$.

$$\bar{R}^2 = -\partial_k \partial_k \langle \tilde{\rho} \rangle|_{k=0}. \tag{6}$$

Eq.(6) is the standard definition of the mean-square radius in terms of the momentum space form factor.

The squared radius is also given by

$$\langle X(\sigma)^2 \rangle. \tag{7}$$

where the value of σ is arbitrary.

For a field theory with a mass gap, such as pure QCD it is possible to prove that $\bar{R}^2$ is finite. This follows from the standard analytic properties of form factors. The problem arises when we attempt to apply the world sheet field theory to compute $\langle X(\sigma)^2 \rangle$. An elementary calculation based on the oscillator representation of X gives a sum over modes

$$\langle X^2 \rangle \sim \alpha' \sum_0^\infty \frac{1}{n} = \alpha' \log \infty. \tag{8}$$

A related disaster occurs when we compute the form factor which is easily seen to have the form

$$\langle \tilde{\rho}(k) \rangle = \exp -k^2 \langle X^2 \rangle. \tag{9}$$

Evidently it is only non-zero at $k^2 = 0$.

In a covariant description of string theory the problem has its roots in the fact that the graviton vertex operator is only well defined on the mass shell of the graviton, $k^2 = 0$. Vertex operators to be well defined must correspond to perturbations with vanishing world sheet β function. This implies that they should correspond to on shell solutions of the appropriate space-time gravitational theory. For the kinematical situation in which the graviton carries vanishing $k_\pm$ the transverse momentum must vanish. Thus no well defined off shell continuation of the form factor exists.

One might wonder if the divergence of X^2 is special to the case of a free world sheet field theory. The answer is that the divergence can only be made worse by interactions. The 2-point function of a unitary quantum field theory is at least as divergent as the corresponding free field theory. This follows from the spectral representation for the two point function and the positivity of the spectral function. Thus it is hard to see how an ideal string theory can ever describe hadrons.

3 Light Cone Strings in AdS

There are good reasons to believe that certain confining deformations of maximally supersymmetric Yang Mills theory are string theories albeit in higher dimensions. The strings move in a 5 dimensional space [2] that is asymptotically AdS. In the ′t Hooft limit these theories are believed to be free string theories. Evidently if this is so there must exist a well defined string prescription for form factors in the 4-D theory.

What we will see is that although the theory in bulk of AdS is an ideal thin-string theory the 4-D boundary field theory is not described by thin strings. That may seem surprising. Suppose that in the light-cone frame the thin 5-D string has the form

$$X(\sigma), Y(\sigma) \tag{1}$$

where X are the transverse coordinates of 4-D Minkowski space and Y is the additional coordinate perpendicular to the boundary of AdS. Then it would seem natural to consider the projection of the string onto the X plane to define a thin string. According to this view the mean-squared radius would again be $\langle X^2 \rangle$ and we would be no better off than before. Before discussing the resolution of this problem let us work out the bosonic part of the light-cone string Lagrangian in AdS. I will make no attempt to derive the full supersymmetric form of the theory in this paper. I believe the resolution of the form factor problem does not require this. On the geometric side I will also ignore the 5-sphere component of the geometry implied by the usual R-symmetry of the $N = 4$ supersymmetry.

The metric of AdS is given by

$$ds^2 = R^2 \frac{dx^+ dx^- - dX^2 - dY^2}{Y^2} \tag{2}$$

I have defined the overall scale of the AdS (radius of curvature) to be R.

In order to pass to the light cone frame we must also introduce the world sheet metric h_{ij}. In the usual flat space theory it is possible to fix the world sheet metric to be in both the light cone gauge $\sigma_0 = \tau = x^+$ and also the conformal gauge $h_{00} = -h_{11}, \;\; h_{01} = 0$. However this is not generally possible since it entails 3 gauge conditions which is one too many. The special feature of flat space which permit the over-fixing of the gauge is not shared by AdS. Thus we must give up the conformal gauge if we wish to work in light-cone gauge.

Let us fix the gauge by choosing 2 conditions

$$\begin{aligned} \sigma_0 &= x^+ \\ h_{01} &= 0. \end{aligned} \tag{3}$$

Let us further define

$$\sqrt{\frac{-h_{11}}{h_{00}}} = E. \tag{4}$$

[2] Strictly speaking the target space is 10 dimensional with the form AdS_5 times a compact space such as S_5. In this paper the compact factor plays no role.

Setting $\alpha' = 1$, an elementary calculation gives the Hamiltonian

$$H = \int d\sigma \left(P_X P_X + P_Y P_Y + \frac{R^4}{Y^4} (\partial_\sigma X \partial_\sigma X + \partial_\sigma Y \partial_\sigma Y) \right) . \tag{5}$$

The precise version of the supersymmetric Hamiltonian was given in [8]. This is a more or less conventional string action with the unusual feature that the effective string tension scales like $1/Y^4$. Thus the tension blows up at the AdS boundary $Y = 0$ and tends to zero at the horizon $Y = \infty$. This of course is a manifestation of the usual UV/IR connection .

The Hamiltonian could be obtained from an action

$$S = \int d\sigma d\tau \left(\dot{X}\dot{X} + \dot{Y}\dot{Y} - \frac{R^4}{Y^4} (\partial_\sigma X \partial_\sigma X + \partial_\sigma Y \partial_\sigma Y) \right) . \tag{6}$$

This action thought of as a $1+1$ dimensional field theory is not Lorentz invariant in the world sheet sense. However it is classically scale invariant if we assume X and Y are dimension zero. The Hamiltonian has dimension 1 and therefore scales as the inverse length of the σ circle which is $\sim P_-$. We recognize this scale symmetry as space-time longitudinal boost invariance under which H and P_- scale oppositely and X, Y are invariant. No doubt the actual Lagrangian when properly super-symmetrized retains this symmetry when quantized.

Let us consider the equal time correlation function $\langle X(0)X(\sigma)\rangle$ in the field theory defined by (6) or more precisely in its

supersymmetrized version. By inserting a complete set of eigenstates of the energy and (world sheet) momentum we obtain

$$\begin{aligned} \langle X(0)X(\sigma)\rangle &= \sum \int e^{ip\sigma} |\langle X(0)|E,p\rangle|^2 \frac{1}{p^2} dE dp \\ &= \int F(E,p) e^{ip\sigma} \frac{1}{p^2} dE dp \end{aligned} \tag{7}$$

with $F \geq 0$

The measure of integration $dEdp/p^2$ follows from the fact that X has "engineering" dimension zero under the longitudinal boost rescaling. Furthermore the assumption that the scale invariance is preserved in the quantum theory requires $F(E,p) = F(E/p)$ for large p, E. It follows that as long as F does not go to zero in this limit that the correlation function diverges as $\sigma \to 0$. This would imply $X^2 = \infty$

4 Dressing the Vertex with Y Dependence

Let us consider the problem from the point of view of the vertex operator exp ikX. One problem that I have emphasized is that it is not a solution of the on-shell condition. We can try to fix this by replacing it with a solution of the wave equation for a graviton in AdS space. The relevant equation is

$$\left(\partial_\mu \partial^\mu + Y^3 \partial_Y Y^{-3} \partial_Y\right) \Phi = 0 \tag{1}$$

where μ runs over the four dimensions of flat Minkowski space.

The particular solutions we are looking for are independent of the $x^{\pm}$ and have the form

$$\Phi = \exp ikXF(k, Y) \tag{2}$$

where F satisfies

$$Y^3 \partial_Y Y^{-3} \partial_Y F(k, Y) = k^2 F \tag{3}$$

Thus the on shell vertex operator has the form

$$\int d\sigma \exp ikX(\sigma) F(k, Y(\sigma)) \tag{4}$$

The factor F is a dressing of the vertex, necessary to make its matrix elements well defined for $k \neq 0$.

Let us consider the mean square radius of the hadron defined by eq.(6).

$$\bar{R}^2 = -\partial_k \partial_k \langle F(k, Y) \exp ikX \rangle|_{k=0} \tag{5}$$

or

$$\bar{R}^2 = \langle X^2 F(0, Y) - 2iX \cdot F'(0, Y) - F''(0, Y) \rangle \tag{6}$$

where $F'' \equiv \partial_k \partial_k F$.

For a state of zero angular momentum in the X plane the term linear in X vanishes and we have

$$\bar{R}^2 = \langle X^2 F(0, Y) - F''(0, Y) \rangle. \tag{7}$$

One possibility for resolving the infinite radius problem is a cancellation of the two terms in eq.(7). To compute F and F'' we Taylor expand $F(k, Y)$ in powers of k and substitute into eq.(3). There are two linearly independent solutions.

$$F(k, Y) = Y^4 + \frac{1}{12} k^2 Y^6 + \cdots \tag{8}$$

and

$$F(k, Y) = 1 - \frac{1}{4} k^2 Y^2 + \cdots \tag{9}$$

Only the second of these is relevant to the problem of vertex operators. To see this we need only note that the vertex at $k = 0$ is just the operator that measures P_-. For states with $P_- = 1$ this operator in just the identity. This implies that $F(0, Y) = 1$.

Thus we find

$$\bar{R}^2 = \langle [X^2 + Y^2] \rangle \tag{10}$$

and $\bar{R}^2$ is the sum of a divergent term and a positive term. The mean radius continues to be divergent. Evidently cancellation is not the answer.

The dressing of the vertex by the factor $F(k, Y)$ obviously modifies the expression (3) for the transverse density ρ. If we define the Fourier transform of F with respect to k to be $\tilde{F}(X, Y)$ eq.(3) is replaced by

$$\rho \sim \int d\sigma \tilde{F}(X - X(\sigma), Y). \tag{11}$$

This means that an ideal thin string in the AdS bulk space is smeared out by the holographic projection onto the boundary. This is of course the familiar UV/IR correspondence at work. Bulk strings near the boundary are projected as very thin strings in the 4-D theory but those far from the boundary are fat. The extra term $\langle Y^2 \rangle$ in eq.(10) represents this fattening. Evidently I have only made things worse by including the dressing.

Before discussing the solution to the problem let us make some remarks about confining deformations in the context of AdS/CFT. Bulk descriptions of confining deformations of super Yang Mills theory have an effective infrared "wall" at a value of Y which represents the confinement scale. In these cases the metric (2) is modified in the infrared region.

$$ds^2 = h(y)\left(dx^+ dx^- - dX^2 - dY^2\right) \tag{12}$$

where, as in the conformal case, $h \sim 1/Y^2$ for $Y \to 0$. Assume that h has a minimum at the confinement scale, $Y = Y^*$.

The light-cone hamiltonian is easily worked out,

$$H = \int d\sigma \left(P_X P_X + P_Y P_Y + h(Y)^2 (\partial_\sigma X \partial_\sigma X + \partial_\sigma Y \partial_\sigma Y)\right) \tag{13}$$

Consider a string stretched along the X direction and choose σ so that $\partial_\sigma X = 1$. The potential energy of the string is then given by

$$V(Y) = h(Y)^2 \tag{14}$$

which has a minimum at $Y = Y^*$. Thus a classical long straight string will be in equilibrium at this value of Y. This classical bulk string corresponds to a field theory configuration which, according to the UV/IR connection, is thickened to a size $\sim Y^*$, that is, the QCD scale.

Quantum fluctuations will cause the wave function of the string to fluctuate away from Y^*. The implication is that the QCD string is a superposition of different thickness values extending from infinitely thin to QCD scale. Indeed different parts of the string can fluctuate in thickness over this range. The portions of the string near $Y = 0$ will be very thin and will determine the large momentum behavior of the form factor.

5 Finiteness of $\langle X^2 \rangle$

I believe that despite the argument given at the end of Section 3 the value of $\langle X^2 \rangle$ is finite. This can only be if the function $F = \sum |\langle X(0)|E, p\rangle|^2$ vanishes in the scaling limit of large E, p. I will first give an intuitive argument and follow it with a more technical renormalization group analysis that is due to Joe Polchinski.

First suppose the string is "stuck" at some value of Y. In that case the action for X in eq.(6) is a conventional string action except that the string tension is replaced by $1/Y^4$. The divergence in X^2 would then be given by

$$\langle X^2 \rangle = Y^2 |\log \epsilon|. \tag{1}$$

If we ignore quantum fluctuations of Y we could replace Y by Y^*. But Y fluctuates as well as X and can be expected to fluctuate toward the boundary as ϵ tends to zero. This is just the usual UV/IR connection in AdS. Therefore as we remove the cutoff the fluctuations of X are diminished because the string moves into a region of increasing effective stiffness. If for example the average value of Y^2 tends to zero as $|1/\log\epsilon|$ or faster then the fluctuations of X would remain bounded. To see that this happens we consider the renormalization running of the operator X^2.

Begin with the bare theory defined with a cutoff length ϵ on the world sheet. We can then ask how a given operator in this bare theory is described in a renormalized version of the theory with a cutoff at some longer distance l. A general operator $\phi(X,Y)$ runs to lower momentum scales according to the renormalization group equation

$$(l\partial/\partial l)\phi(X,Y,l) = (\alpha'/2)\nabla^2\phi(X,Y,l). \quad (2)$$

For example, consider flat space and the operator X^2. We look for a solution of eq.(2) with

$$\phi(X,\epsilon) = X^2. \quad (3)$$

The solution is

$$\phi(X,l) = X^2 + \alpha'\log l/\epsilon. \quad (4)$$

Thus if we regulate the theory at some fixed scale, for example $l \sim 1$, the matrix elements of X^2 blow up as send $\epsilon \to 0$.

By contrast, consider the the case of AdS space where

$$\nabla^2 = R^{-2}(Y^2\partial_X^2 + Y^5\partial_Y Y^{-3}\partial_Y)\ . \quad (5)$$

For a solution of the form $X^2 + f(l)Y^2$ this becomes

$$(l\partial/\partial l)f = (2\alpha'/R^2)(1-f)\ . \quad (6)$$

With $f(\epsilon) = 0$ the solution is

$$f(l) = 1 - (\epsilon/l)^{2\alpha'/R^2}\ . \quad (7)$$

So if we fix the scale l and take the cutoff length ϵ to zero the matrix elements tend to finite limits and the problem of infinite radii is resolved . If, however, we expand in powers of α' there are logarithmic divergences.

Note that the operator X^2 runs to a fixed point $X^2 + Y^2$ which is just the operator in eq.(10) which represents the mean squared radius $\bar{R}^2$.

The reader may wonder how the finiteness of X^2 can be explained in covariant gauges such as the conformal gauge in which the world sheet theory has the form of a relativistic field theory. A standard argument insures that the singularity in a two point function can not be less singular than a free field; in this case logarithmic. The argument is based on the positivity of spectral functions which in turn assumes the metric in the space of states is positive. In general this is not the case in covariant gauges.

6 Discussion

The original attempt to describe hadrons as idealized strings was frustrated by the infinite zero point oscillations in the size of strings. Early ideas for modifying string theory such as replacing the idealized strings by fat flux tubes or as collections of partons which approximate strings fit well with QCD but seemed to preclude an idealized mathematical string description.

More recent evidence from AdS/CFT type dualities suggest that idealized string theory in higher dimensions may provide an exact description of the 't Hooft limit of QCD-like theories. I have argued that an ideal bulk string theory in five dimensions is fully compatible with a fat non-ideal string in four dimensions.

The fifth dimension can be divided into two regions. The "wall" region near $Y = Y^*$ corresponds to the confinement scale Λ. If we ignore high frequency fluctuations, the string spends most of its time in this region. The usual UV/IR spreading gives the string a thickness of order Λ. High frequency fluctuations of small sections of string can occur which cause it to fluctuate toward $Y = 0$, the region corresponding to short distance behavior in space-time. These fluctuations will control the large momentum behavior of form factors as well as deep inelastic matrix elements. Such fluctuations give the string a parton-like makeup. We have also seen that these fluctuations stiffen the effective string tension so much that the infinite zero point size that Holger and I worried about so long ago is now eliminated.

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