Modelling three fermion generations with S_3 family symmetry within $\mathbb{C}\ell(8)$

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Abstract. We present a model of three fermion generations with $SU(3) \times U(1)$ gauge symmetry constructed from the complex Clifford algebra $\mathbb{C}\ell(8)$, within which the discrete group S_3 acts as a family symmetry. $\mathbb{C}\ell(8)$ corresponds to the algebra of complex linear maps from the (complexification of the) Cayley-Dickson algebra of sedenions, $\mathbb{S}$, to itself. The automorphism group of $\mathbb{S}$ is $G_2 \times S_3$. We interpret S_3, suitably embedded into $\mathbb{C}\ell(8)$, as a family symmetry. The gauge symmetry $SU(3) \times U(1)$ is invariant under S_3. First-generation states are represented in terms of two even $\mathbb{C}\ell(8)$ semi-spinors, obtained from two minimal left ideals, related to each other via the order-two S_3 symmetry. The remaining two generations are obtained by applying the S_3 symmetry of order-three to the first generation, resulting in three linearly independent generations.

1 Introduction

There have been many endeavours to link division algebras, and their left multiplication algebras, to the mathematical structure of the Standard Model (SM), with a proliferation of recent research papers [1–18]. In many such approaches, the gauge groups, leptons, and quarks are contained within the multiplication algebras, the algebras generated from the actions of division algebras on itself via its endomorphisms.

There are four normed division algebras; $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions), of dimensions one, two, four and eight, respectively. Shortly after the discovery of quarks, Güynadin and Gürsey [19] constructed a model of quark colour symmetry based on the algebra of the split octonions. Subsequently, a series of papers by Barducci et al., and Casalbuoni and Gatto [20–22] explored a unified description of leptons and quarks with internal degrees of freedom in terms of fermionic oscillators (corresponding to a Witt basis of a Clifford algebra, see section 2.3). Three fermionic oscillators are associated with the colour degrees of freedom, whereas the inclusion of electric charge requires a fourth fermionic oscillator.

The early association of the split octonions with quarks in [19] was expanded upon by Dixon [23–25] who revealed that the mathematical characteristics of the SM, encompassing its gauge symmetries and corresponding multiplets to which a single generation of fermions is subject, are inherent in $\mathbb{T}^2$, where $\mathbb{T} = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. Here, $\mathbb{T}^2$ corresponds to a complexified (hyper) spinor in 1+9D spacetime.

In an approach closely related to these earlier works [19–25], Furey encompasses both bosons and fermions within Clifford algebras, arising as the multiplication algebras of compositions of division algebras. Two minimal left ideals of $\mathbb{C}\ell(6)$, the left multiplication algebra of $\mathbb{C} \otimes \mathbb{O}$, transform as a single generation under $SU(3)_C \times U(1)_{em}$, whereas two $\mathbb{C}\ell(4)$ minimal ideals transform as a single generation of chiral fermions under weak $SU(2)$ [26]. These findings integrate into a $\mathbb{C}\ell(10)$ model [6, 7, 27, 28].

Most division-algebraic based constructions to date are limited to describing a single generation of fermions. Despite various attempts [29–32], an algebraic foundation for the existence of three generations within the division algebraic framework remains elusive[1]. Furey endeavours to depict three generations directly from the algebra $\mathbb{C}\ell(6)$ [29, 30]. A similar construction based on $\mathbb{C}\ell(6)$, which includes an $SU(2)$ gauge symmetry, is given by Gording and Schmidt-May [11]. Dixon, on the other hand, characterises three generations using the algebra $\mathbb{T}^6 = \mathbb{C} \otimes \mathbb{H}^2 \otimes \mathbb{O}^3$ [33]. In the unified theories of [22], an additional m fermionic oscillators are included in order to represent 2^m generations (a necessarily even number). However, these additional fermionic oscillators cannot be associated with the division algebras in an obvious way, nor is there any algebraic guidance on what m should be. Other authors have sought to encode three generations within the exceptional Jordan algebra [34–39].

The three-generation model proposed here (based on [1]), which builds on [2, 32], is fundamentally different. We argue that going beyond the division algebras and considering the Cayley-Dickson algebra of sedenions $\mathbb{S}$, with automorphism group $Aut(\mathbb{S}) = Aut(\mathbb{O}) \times S_3$, provides algebraic structure suitable for describing three generations. At the core of our construction is the idea that S_3 serves as the algebraic source for the existence of three generations. The left multiplication algebra that is generated from $\mathbb{C}\otimes\mathbb{S}$ is the algebra $\mathbb{C}\ell(8)$, which will serve a central purpose in our construction.

We initially represent one generation of electrocolour states in terms of a single $\mathbb{C}\ell(8)$ minimal left ideal. We then embed the S_3 automorphisms of $\mathbb{S}$ into $\mathbb{C}\ell(8)$, and subsequently interpret this as a family symmetry. Two additional generations are then obtained by acting on the first with the S_3 symmetry of order three. Although the $SU(3)$ gauge group is invariant under the action of S_3, the group $U(1)$ is not. To include an S_3 invariant $U(1)$ symmetry, each spinor is split into its even and odd-grade semi-spinors. Applying the order-two S_3 symmetry to the even semi-spinor results in a second even semi-spinor whereas the order-three S_3 symmetry generates two additional pairs of semi-spinors. An S_3-invariant $U(1)$ gauge symmetry then arises as the sum of three individual $U(1)$ symmetries, and the three pairs of semi-spinors transform as three generations of fermions under the SM unbroken gauge symmetry $SU(3)_C \times U(1)_{em}$. All three generations are linearly independent.

Coincidentally, several S_3 extensions to the SM have been considered in the literature, particularly as a generation symmetry, and in relation to Higgs, neutrino, and flavour physics [40–49]. This makes us hopeful that our model, once the weak interaction is included, will be able to describe additional SM features such as neutrino oscillations and quark mixing, as well as make phenomenological predictions, something that is mostly lacking in current division algebra based models.

2 $\mathbb{C}\ell(8)$ as the left multiplication algebra of $\mathbb{C}\otimes\mathbb{S}$

2.1 Sedenions, and the left multiplication algebra of $\mathbb{C}\otimes\mathbb{S}$

Starting with $\mathbb{R}$, the Cayley-Dickson (CD) construction produces a sequence of algebras, $\mathbb{A}_n$ (where $\mathbb{A}_0 = \mathbb{R}$), of dimension 2^n, the first three being the remaining three division algebras. Applying the CD construction to $\mathbb{O}$ generates the 16-dimensional algebra of sedenions $\mathbb{S}$. This algebra is non-commutative, non-associative, non-alternative, and contains zero divisors. An orthonormal basis consists of 15 imaginary units s_i $(i = 1, ..., 15)$, as well as the identity s_0. The imaginary units $s_1, ..., s_7$ correspond to the octonion imaginary units $u_1, ..., u_7$. A general sedenion w may then be written as;

$$w = w_0 s_0 + w_1 s_1 + ... + w_{15} s_{15}, \quad w_0, ..., w_{15} \in \mathbb{R}. \tag{1}$$

The product of two sedenions can be determined using the multiplication table in [1]. The involution of a sedenion element w is given by $\overline{w} = w_0 s_0 - w_1 s_1 - ... - w_{15} s_{15}$. The norm $|w|$ is defined by $|w|^2 = w\overline{w} = \overline{w}w$ and the inverse of w (if it exists) is $w^{-1} = \overline{w}/|w|^2$.

It is known that the automorphism group of $\mathbb{A}_4 = \mathbb{S}$ is $Aut(\mathbb{S}) = Aut(\mathbb{O}) \times S_3$ [50]. The automorphisms can be explicitly stated as follows;

$$\phi : A + Bs_8 \to \phi(A) + \phi(B)s_8, \tag{2}$$

$$\epsilon : A + Bs_8 \to A - Bs_8, \tag{3}$$

$$\psi : A + Bs_8 \to \frac{1}{4}[A + 3A^* + \sqrt{3}(B - B^*)] + \frac{1}{4}[B + 3B^* - \sqrt{3}(A - A^*)]s_8, \tag{4}$$

where $A, B \in \mathbb{O}$. A^* is an octonion involution of A such that $(A^*)^* = A$ and $(AB)^* = B^* A^*$, and ϕ is an element of G_2, the automorphism group of $\mathbb{O}$. These result in the following identities; $\epsilon^2 = Id$, $\psi^3 = Id$, $\psi\phi = \phi\psi$, $\epsilon\phi = \phi\epsilon$ and $\epsilon\psi = \psi^2\epsilon$. It follows that ϵ and ψ generate S_3.

[1] The most popular GUTs, such as those based on $SU(5)$, $SO(10)$, and the Pati-Salam model are also inherently single-generation models, lacking a theoretical basis for three generations.

Although $\mathbb{S}$ is both non-associative and non-alternative, it is still possible to define compositions of left or right actions of $\mathbb{S}$ on itself as linear operators, thereby generating an associative algebra [51]. In this paper, we restrict our attention to the left associative multiplication algebra.

Let L_a denote the linear operator corresponding to left multiplication by an element $a \in \mathbb{C} \otimes \mathbb{S}$ onto an element $w \in \mathbb{C} \otimes \mathbb{S}$, defined by;

$$L_a[w] := aw, \quad \forall a, w \in \mathbb{C} \otimes \mathbb{S}. \tag{5}$$

Since L_a corresponds to a linear operator, it can be represented as a 16×16 complex matrix (acting on the vector space $\mathbb{C} \otimes \mathbb{S}$ written as a 16×1 column vector). Due to the non-associativity of $\mathbb{S}$, the left multiplication algebra of $\mathbb{C} \otimes \mathbb{S}$ contains new maps which are not captured by $\mathbb{C} \otimes \mathbb{S}$, because in general;

$$L_a L_b[w] = a(bw) \neq L_{ab}[w] = (ab)w, \quad a, b, w \in \mathbb{C} \otimes \mathbb{S}. \tag{6}$$

There are a total of 256 distinct left-acting complex-linear maps from $\mathbb{C} \otimes \mathbb{S}$ to itself, and these provide a faithful representation of $\mathbb{C}\ell(8)$. It can be shown that $L_{s_i} L_{s_j}[w] = -L_{s_j} L_{s_i}[w]$ and $L_{s_i} L_{s_i}[w] = -w$, $i, j = 1, ..., 8$, $i \neq j$. As a result, the left multiplication actions L_{s_i}, $i = 0, 1, ..., 8$, are a generating basis for $\mathbb{C}\ell(8)$. From now on, the left actions will be denoted simply by $e_i := L_{s_i}$ and assumed to be acting on arbitrary $w \in \mathbb{C} \otimes \mathbb{S}$. That is, instead of writing $L_{s_i} L_{s_j} w$, we simply write $e_i e_j$. The maps e_k, $k = 9, ..., 15$, can then be expressed in terms of these e_i, $i = 1, ..., 8$. For example[2];

$$e_9 = -\frac{1}{2} e_1 e_2 e_3 e_4 e_5 e_8 + \frac{1}{2} e_1 e_2 e_3 e_6 e_7 e_8 + \frac{1}{2} e_1 e_4 e_5 e_6 e_7 e_8 - \frac{1}{2} e_1 e_8. \tag{7}$$

2.2 Constructing a minimal left ideal of $\mathbb{C}\ell(8)$

A general construction for creating spinor spaces as minimal left ideals of Clifford algebras is reviewed in [52]. Define a Witt basis for $\mathbb{C}\ell(8)$ as;

$$a_j := \frac{1}{2}(-e_j + i e_{j+4}), \qquad a_j^\dagger := \frac{1}{2}(e_j + i e_{j+4}), \tag{8}$$

where $j = 1, 2, 3, 4$.These are fermionic oscillators that generate two totally isotropic subspaces, spanned by $\{a_j\}$ and $\{a_j^\dagger\}$, and satisfy the fermionic anticommutation relations;

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0, \quad \{a_i, a_j^\dagger\} = \delta_{ij}. \tag{9}$$

The nilpotents $\Omega_1 = a_1 a_2 a_3 a_4$ and $\Omega_1^\dagger = a_4^\dagger a_3^\dagger a_2^\dagger a_1^\dagger$ can be combined into a primitive idempotent; $v_1 := \Omega_1 \Omega_1^\dagger$, physically representing the vacuum state. A minimal left ideal is then $\mathbb{C}\ell(8) v_1$. Explicitly, we can write the 16 complex-dimensional ideal as follows;

$$T_1 = (r_0 + r_j a_j^\dagger + r_{jk} a_j^\dagger a_k^\dagger + r_{jkl} a_j^\dagger a_k^\dagger a_l^\dagger + r_{1234} a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger) v_1, \tag{10}$$

where $j, k, l \in \{1, 2, 3, 4\}$, $j \neq k \neq l$ and $r_0, r_j, r_{jk}, r_{jkl}, r_{1234}$ are complex coefficients. Notably, T_1 can be expressed as $(\mathbb{C}\ell(6) \oplus \mathbb{C}\ell(6) a_4^\dagger) v_1$, where a_k and $a_k^\dagger$, $k = 1, 2, 3$, constitutes a Witt basis for $\mathbb{C}\ell(6)$.

3 One fermion generation with $SU(3)_C \times U(1)_{em}$ gauge symmetry

We now consider the transformations of the minimal left ideal basis states. Although any transformation of the form

$$e_i \mapsto e^{i\phi_k g_k} e_i e^{-i\phi_k g_k}, \quad \phi_k \in \mathbb{R}, \quad g_k \in \mathbb{C}\ell(8), \tag{11}$$

will preserve the anticommutation relations in eqn. (9), not all such transformations preserve the Witt basis or, equivalently, the isotropic subspaces generated by them. The transformations that preserve the Witt basis generate $U(4) = SU(4) \times U(1)$, which corresponds to a maximal subgroup of $Spin(8)$.

The $SU(4)$ generators $\{\Lambda_1, ..., \Lambda_{15}\}$[3], transform the 16 basis states of T_1 as $1 \oplus 4 \oplus 6 \oplus \bar{4} \oplus 1$ under commutation. $SU(4)$ can be broken to $SU(3) \times U(1)$, where the $SU(3)$ generators are $\{\Lambda_1, ..., \Lambda_8\}$[4]. The

[2]Expressions for all e_k, $k = 9, ..., 15$ as $\mathbb{C}\ell(8)$ elements can be found in [2].

[3]Explicitly defined in [1].

[4]$SU(3) \times U(1)$ corresponds to the subgroup of $SU(4)$ that commutes with the quaternionic structure generated by e_4 and e_8, or equivalently, a_4 and $a_4^\dagger$.

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$SU(3)$ and $U(1)$ generators are the generators that preserve both $\mathbb{C}\ell(6)v_1$ and $\mathbb{C}\ell(6)a_4^\dagger v_1$ individually. Under the action of $SU(3)$, the basis elements of both $\mathbb{C}\ell(6)v_1$ and $\mathbb{C}\ell(6)a_4^\dagger v_1$ transform as $1 \oplus 3 \oplus \overline{3} \oplus 1$, matching that of a single generation of fermions under $SU(3)_C$.

The $U(1)$ generator Λ_{15} is proportional to the electric charge generator defined as;

$$Q_1 := \frac{1}{3}(a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 - 3a_4^\dagger a_4) \quad \in \mathfrak{su}(4), \tag{12}$$

assigning the correct electric charge to each state. The ideal basis states can be identified with the fermions of a single generation based on how they transform under $SU(3)_C$ and $U(1)_{em}$. The $\mathbb{C}\ell(6)v_1$ terms represent the isospin-up states whereas the $\mathbb{C}\ell(6)a_4^\dagger v_1$ terms represent the isospin-down states.

4 S_3 as a generation symmetry

We extend the representation of a single generation of fermions in terms of a single minimal left ideal of $\mathbb{C}\ell(8)$, to three generations by embedding the S_3 automorphisms of $\mathbb{C} \otimes \mathbb{S}$ into $\mathbb{C}\ell(8)$, and subsequently interpreting this discrete group as a generation symmetry.

4.1 Extending the S_3 discrete symmetry to $\mathbb{C}\ell(8)$

The automorphism ϕ defined in eqn. (2) has a natural extension to $\mathbb{C}\ell(8)$; by letting the ϕ automorphisms act on the left actions e_i instead of sedenion elements s_i. The ϵ automorphism likewise has an obvious embedding into $\mathbb{C}\ell(8)$: $e_i \xrightarrow{\epsilon} e_i$ for $i \in \{0, 1, ..., 7\}$ and $e_8 \xrightarrow{\epsilon} -e_8$ within $\mathbb{C}\ell(8)$[5].

We consider the following embedding of $\psi \in Aut(\mathbb{S})$ into $\mathbb{C}\ell(8)$[6];

$$e_i \xrightarrow{\psi} \begin{cases} \frac{1}{4}e_i - \frac{\sqrt{3}}{4}e_i e_8 + \frac{\sqrt{3}}{4}e_{i+8} - \frac{3}{4}e_{i+8}e_8 & i = \{0, ..., 7\}, \\ \frac{1}{4}e_i - \frac{\sqrt{3}}{4}e_i e_8 - \frac{\sqrt{3}}{4}e_{i-8} + \frac{3}{4}e_{i-8}e_8 & i = \{8, ..., 15\}. \end{cases} \tag{13}$$

For example;

$$e_1 \xrightarrow{\psi} \frac{1}{4}e_1 - \frac{\sqrt{3}}{4}e_1 e_8 + \frac{\sqrt{3}}{4}e_9 - \frac{3}{4}e_9 e_8. \tag{14}$$

While the e_{i+8} terms can be rewritten as $\mathbb{C}\ell(8)$ elements as per eqn. (7), it is more convenient to leave them as e_{i+8}. It can then be checked[7] that $\psi^3(e_i) = e_i$, and that both e_0 and e_8 are invariant under ψ. The $\mathbb{C}\ell(8)$ maps ϵ and ψ can then be seen to generate S_3.

One can subsequently check that;

$$\psi(e_i)\psi(e_i) = \psi(e_i^2) = -1, \tag{15}$$

$$\psi(e_i)\psi(e_j) + \psi(e_j)\psi(e_i) = 0, \tag{16}$$

for $i, j \in \{0, 1, ..., 8\}$, $i \neq j$, and likewise for ϵ. These maps therefore extend to $\mathbb{C}\ell(8)$ homomorphisms [53]. Unlike in our previous paper [2], e_i, $\psi(e_i)$ and $\psi^2(e_i)$ are now linearly independent in $\mathbb{C}\ell(8)$.

4.2 Including two additional generations using the order-three symmetry ψ

Applying ψ (and ψ^2) to a_i and $a_i^\dagger$ generates two additional Witt bases;

$$b_i = \psi(a_i), \quad b_i^\dagger = \psi(a_i^\dagger), \quad c_i = \psi^2(a_i), \quad c_i^\dagger = \psi^2(a_i^\dagger), \quad i = \{1, 2, 3, 4\}, \tag{17}$$

satisfying the same fermionic anticommutation relations (eqn. (9)) as our original Witt basis, with $\{a_i, a_i^\dagger\}$ replaced with $\{b_i, b_i^\dagger\}$ or $\{c_i, c_i^\dagger\}$. We can therefore construct two additional minimal left ideals;

$$T_2 = (r_0' + r_j' b_j^\dagger + r_{jk}' b_j^\dagger b_k^\dagger + r_{jkl}' b_j^\dagger b_k^\dagger b_l^\dagger + r_{1234}' b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger) v_2, \tag{18}$$

$$T_3 = (r_0'' + r_j'' C_j^\dagger + r_{jk}'' c_j^\dagger c_k^\dagger + r_{jkl}'' c_j^\dagger c_k^\dagger c_l^\dagger + r_{1234}'' c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger) v_3, \tag{19}$$

where $v_2 = \psi(v_1)$, $v_3 = \psi^2(v_1)$, $j, k, l \in \{1, 2, 3, 4\}$, $j \neq k \neq l$ and $r_0', r_j', r_{jk}', r_{jkl}', r_{1234}', r_0'', r_j'', r_{jk}'', r_{jkl}'', r_{1234}''$ are complex coefficients. Applying ψ to T_3 returns T_1, and so ψ permutes between three minimal ideals.

[5] It then follows (see eqn. (7)) that $e_j \xrightarrow{\epsilon} -e_j$, for $j \in \{9, ..., 15\}$.

[6] This map corresponds to a generalisation of the map we considered in our earlier work [2].

[7] Although many of the calculations that follow are straightforward, they are tedious to carry out by hand and the authors have made extensive use of Mathematica to verify these calculations.

4.3 Generation invariant gauge symmetries

We wish to identify T_2 and T_3 with the second and third generation. However, the $SU(4)$ symmetry does not transform the basis states of T_2 and T_3 as $1 \oplus 4 \oplus 6 \oplus \overline{4} \oplus 1$, as it does for T_1.

In order to identify the gauge symmetries in our model, we require the gauge symmetries to preserve the minimal left ideals under commutation and be invariant under the action of S_3. It can be checked that $\Lambda_i = \psi(\Lambda_i),\ i = \{1, ..., 8\}$, and that T_2 and T_3 transform as one fermion generation under this $SU(3)$. The $SU(3)_C$ identified in the single-generation construction therefore extends to three generations.

On the other hand, the $U(1)$ generator Q_1 turns out not to be invariant under S_3. Nonetheless, it is possible to construct an S_3-invariant $U(1)$ generator as the sum of three individual $U(1)$ generators;

$$Q := \frac{1}{3}(Q_1 + \psi(Q_1) + \psi^2(Q_1)). \tag{20}$$

This new $U(1)$ commutes with $SU(3)$. Although Q preserves the ideal T_1 (and T_2, T_3), it no longer assigns the electric charge corresponding to both isospin-up and isospin-down states. Instead, Q assigns eigenvalues to the basis states of T_1 corresponding to two copies of isospin-down states (and likewise for T_2 and T_3). We will address this issue shortly. The remaining $SU(4)$ generators could likewise be generalised to be S_3-invariant, however, one finds that the minimal ideals are not preserved under commutation with these generators. We therefore exclude them from being part of any viable gauge symmetries.

4.4 Including isospin-up states using the order-two symmetry ϵ

To include the isospin-up states for the first generation, we utilise the order-two symmetry ϵ. Its action on the three Witt bases is as follows;

$$\begin{aligned} &\epsilon(a_i) = a_i, \quad \epsilon(a_4) = -a_4^\dagger, \quad \epsilon(a_i^\dagger) = a_i^\dagger, \quad \epsilon(a_4^\dagger) = -a_4, \\ &\epsilon(b_i) = c_i, \quad \epsilon(b_4) = -c_4^\dagger, \quad \epsilon(b_i^\dagger) = c_i^\dagger, \quad \epsilon(b_4^\dagger) = -c_4, \\ &\epsilon(c_i) = b_i, \quad \epsilon(c_4) = -b_4^\dagger, \quad \epsilon(c_i^\dagger) = b_i^\dagger, \quad \epsilon(c_4^\dagger) = -b_4, \end{aligned} \tag{21}$$

with $i = 1, 2, 3$. Applying ϵ to T_1 produces a second minimal left ideal, defined as $S_1 := \epsilon(T_1)$, built on the primitive idempotent $v_1' := \epsilon(v_1)$. The symmetry generator Q then identifies the basis states of this complementary ideal as (two copies of) isospin-up states.

The fact that each isospin-down state is represented twice in T_1 indicates there is an additional degree of freedom that can be included, most likely chirality. Since this is not the focus of the present paper, we will reduce T_1 (and S_1) to its even semi-spinor via the projector $\rho_+ = \frac{1}{2}(1+e)$, where $e := e_1e_2e_3e_4e_5e_6e_7e_8$ is the $\mathbb{C}\ell(8)$ pseudoscalar. Explicitly, the even semi-spinors $T_1^+ := \rho_+ T_1$ and $S_1^+ := \rho_+ S_1$ are given by;

$$T_1^+ = \begin{pmatrix} \overline{\nu_e} \\ +d^r a_1^\dagger a_2^\dagger + d^g a_1^\dagger a_3^\dagger + d^b a_2^\dagger a_3^\dagger \\ +\overline{u^b} a_1^\dagger a_4^\dagger + \overline{u^g} a_2^\dagger a_4^\dagger + \overline{u^r} a_3^\dagger a_4^\dagger \\ +e^- a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger \end{pmatrix} v_1, \qquad S_1^+ = \begin{pmatrix} \nu_e \\ +u^r a_1^\dagger a_2^\dagger + u^g a_1^\dagger a_3^\dagger + u^b a_2^\dagger a_3^\dagger \\ +\overline{d^b} a_1^\dagger a_4 + \overline{d^g} a_2^\dagger a_4 + \overline{d^r} a_3^\dagger a_4 \\ +e^+ a_1^\dagger a_2^\dagger a_3^\dagger a_4 \end{pmatrix} v_1',$$

where the coefficients indicate how the states transform. Q now assigns the correct electric charge to all states, with T_1^+ (S_1^+) containing the isospin-down (isospin-up) states.

Acting with the order-three symmetry ψ then permutes between the even semi-spinors of the remaining two generations. Three generations with $SU(3)_C \times U(1)_{em}$ gauge symmetry can therefore be represented in terms of three pairs of $\mathbb{C}\ell(8)$ semi-spinors, related via the S_3 order-two symmetry ϵ. Whereas ϵ interchanges isospin-down and isospin-up states, ψ permutes between generations. In contrast to [2], all three generations of states are linearly independent in this construction, a desirable feature.

5 Discussion

In this paper (based on [1]) we have represented three linearly independent generations with $SU(3)_C \times U(1)_{em}$ gauge symmetry within $\mathbb{C}\ell(8)$. Central to our construction is the idea that an S_3 discrete symmetry, arising from the automorphism group of $\mathbb{S}$, $Aut(\mathbb{S}) = G_2 \times S_3$, is the algebraic source for the existence of three generations. $\mathbb{C}\ell(8)$ corresponds to the left multiplication algebra of $\mathbb{C} \otimes \mathbb{S}$. Our generation symmetry then corresponds to an embedding of the S_3 automorphisms of $\mathbb{S}$ into $\mathbb{C}\ell(8)$.

One generation is represented in terms of two semi-spinors, obtained from two minimal left ideals of $\mathbb{C}\ell(8)$, related via the order-two symmetry of S_3. One semi-spinor contains the isospin-down states, whereas the other the isospin-up states. Applying the order-three S_3 symmetry then produces two additional pairs of semi-spinors, linearly independent to the first, to represent the remaining two generations.

The gauge symmetries are identified as the unitary symmetries that both preserve the semi-spinors under commutation and are invariant under S_3. Whereas the $SU(3)_C$ symmetry constructed for the first generation is inherently invariant under S_3, an S_3-invariant $U(1)_{em}$ consists of three separate $U(1)$ symmetries, one associated with each generation, which individually are not S_3-invariant.

The next step to develop this model is to include the symmetry $SU(2)_L \times U(1)_Y$. This will then allow us to relate our model with S_3 family symmetry to S_3 extensions of the SM considered in the literature (see for example [40–49], and references therein), and make phenomenological predictions as a result. In [48, 49] it is found that a S_3 family symmetry strongly constrains the number of free parameters in the fermionic mass matrices and allows one to obtain exact mathematical relations among fermion masses and mixing angles. Furthermore, the extension of family and flavour symmetry to the Higgs sector introduces two additional Higgs doublets, thereby generalising the S_3 invariance of the SM gauge group to include the Yukawa sector. The recent work [54] establishes a connection between S_3 and triality, deriving Yukawa couplings through the trialities associated with division algebras.

In all these proposals however, the appearance of S_3 is put in by hand. In our model this discrete group arises naturally in the context of sedenion automorphisms. This makes us hopeful that our model, once the weak interaction is included, will be able to describe additional features such as neutrino oscillations and quark mixing, as well as make phenomenological predictions, something that is lacking in current division algebra based models. One feature to consider incorporating into the model is the mass hierarchies observed among the three generations of fermions, which currently lack theoretical explanation. Extending our model to include weak interactions may allow for the calculation of mass ratios, perhaps via a similar method to that outlined in [55], where the exceptional Jordan algebra offers insights into fermionic mass ratios.

Other discrete groups (such as A_4, S_4, and D_4) have likewise been proposed as extensions to the SM. Our model based on sedenions is incompatible with such proposed extensions to the SM, as it is uniquely S_3 that arises as a discrete automorphism group of $\mathbb{S}$, and in fact larger CD algebras.

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