Vol. 38 (2007) *ACTA PHYSICA POLONICA B* No 11

DARK ENERGY AND 3-MANIFOLD TOPOLOGY*

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(Received October 15, 2007)

We show that the differential-geometric description of matter by differential structures of spacetime leads to a unifying model of the three types of energy in the cosmos: matter, dark matter and dark energy. Using this model we are able to calculate the ratio of dark energy to the total energy of the cosmos.

PACS numbers: 04.20.Gz, 98.80.JK, 02.40.Vh

1. Introduction

For centuries it has been our firm conviction that matter and energy of the same kind as is surrounding us also constitute the rest of the world. Thorough examinations of supernovae [12,15] and of cosmic background radiation [2,8], however, have replaced this conviction by the insight that the global structure of the cosmos is dominated at 95 % by an energy form that has hitherto been entirely unknown. About two thirds of this energy form consist in “dark energy”, and one third in “dark matter”. This is the most radical revolution in our understanding of the cosmos after Copernicus. In the last years, great effort has been invested to understand these unknown forms of energy [13, 14, 17]. Many explanations of dark energy assume that besides spacetime geometry and baryonic matter, there is an *additional entity* that acts as source of the dark energy. For instance, particle-theoretic models attribute this role to the vacuum energy [3, 20, 24], or introduce additional global scalar fields [16, 21].

* Presented at the XXXI International Conference of Theoretical Physics, “Matter to the Deepest”, Ustroń, Poland, September 5–11, 2007.

A first evaluation of the WMAP data favors a Poincaré sphere as topology of the cosmos [11]. That means that the cosmos is a closed 3-manifold having the same homology as the 3-sphere. Furthermore one knows that this 3-manifold has a positive curvature. In this paper we support this result and show that the dark energy can be explained by the curvature of two Poincaré spheres. Details of the calculation can be found in the expanded version of the paper [1].

2. Basic model

Our model is based on the fact, from general relativity theory, that every form of energy is related to the curvature of the spatial 3-manifold, *i.e.* matter must be interpreted by curved 3-manifolds. Then Einstein's equation is the dynamical equation for the evolution of 3-geometries. Thus we can state our basic assumptions:

Basic assumptions: *The 4-manifold of all possible spacetime events is a compact, closed 4-manifold M which is differentiable and simply connected. The cosmos is an embedded 3-manifold Σ which is compact and closed. The energy density of any kind of matter is described by the curvature of the associated submanifold of Σ.*

Before we study the implications we will motivate these assumptions. Compactness[1] of the 4-manifolds means that *every* series of spacetime events converges to an event belonging to the same 4-manifold. The manifold is closed *i.e.* it has no boundary or in any neighborhood of an arbitrary point there are always inner points of the manifold. The assumptions of compactness and closeness can be interpreted that all points of the manifold are inner points and any spacetime event must be part of the manifold. This assumption seems natural from our knowledge of space and time. The assumption of simple-connectness is more delicate. It means that any closed (time-like) curve is contractable *i.e.* any time circle contradicting causality can be shrunk to a point. In the following we will study the implications via differential topology of the assumptions leading to very strong restrictions on the possible 4- and 3-manifolds.

2.1. Determination of the 4-manifold

The 4-manifold can be determined by the following argumentation. If every kind of energy is described by the curvature of some submanifold of the spatial 3-manifold Σ then there are no source terms in Einstein's equation. Every kind of energy including matter must be given by geometry. Then Einstein's equation $R_{\mu\nu} = 0$ is the statement that the 4-manifold must

[1] Compactness is not contradictory to the possible infiniteness of proper time for world lines. Curves in manifolds can have infinite length like Peano's curve.

be Ricci-flat. But *there is only one Ricci-flat compact, simply-connected 4-manifold, the K3-surface* [23] which is defined by

$$\{(z_1, z_2, z_3, z_4) \in \mathbb{C}P^3 \,|\, z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\}\,.$$

Thus we determine the topology of the 4-manifold by our basic assumptions. The second restriction, which will determine the structure of the 3-manifold, is the choice of a particular differential structure. It is a well-known result [7] that the differential or smooth structure of a compact, simply-connected 4-manifold is determined by a contractable submanifold, the Akbulut cork A. The boundary of the Akbulut cork is a 3-manifold which is a so-called *homology 3-sphere*, *i.e.* a 3-manifold with the same homology as the 3-sphere S^3. Thus we assume *the cosmos Σ is a homology 3-sphere*.

2.2. The 3-manifold and its submanifolds

In case of the K3-surface we know the structure of the Akbulut cork and its boundary $\Sigma = \Sigma(2,5,7)$ [7] the Brieskorn sphere

$$\Sigma(2,5,7) = \{(x,y,z) \in \mathbb{C}^3 \,|\, |x|^2 + |y|^2 + |z|^2 = 1,\ x^2 + y^5 + z^7 = 0\}\,.$$

From the structure theory of 3-manifolds [9,19] we know that there are only *three* kinds of 3-manifolds that can form Σ. In the particular case we obtain

$$\Sigma = K_1 \# K_2 \# K_3 \#_N S^3 \# S^3/I^* \# S^3/I^* \tag{1}$$

with $\#$ as the connected sum between manifolds. The symbol S^3/I^* represents the Poincaré sphere which forms the global structure of the cosmos [11]. Now we identify the pieces with

1. negatively curved pieces K_i (matter, radiation),
2. positively curved 3-spheres S^3 (dark matter),
3. two positively curved Poincaré spheres S^3/I^* (dark energy),

The details can be found in the expanded paper [1]. This remarkable fact motivates the following

Conjecture: *The three types of 3-manifolds that constitute the cosmos as a homology 3-sphere, correspond to the three kinds of matter: baryonic matter, dark matter, and dark energy.*

Thus we obtain a unified approach for all observed kinds of energy densities. The global structure of the cosmos can thus be derived from the differential geometry of spacetime itself, without additional entities, and it is possible to compare the observed energy densities with the curvatures of the three types of Σ. In the next section we will calculate the ratio of the energy densities of the dark energy and the total energy by using a result of Witten [22].

3. Calculation of the dark energy density

The investigation of the global characteristics of the differential structure of spacetime led us to the result that the cosmos is a Brieskorn sphere $\Sigma = \Sigma(2,5,7)$ split up into three types of pieces

$$\Sigma = K_1 \# K_2 \# K_3 \#_N S^3 \# S^3/I^* \# S^3/I^* .$$

Let us now suppose a 4-manifold M with an Akbulut cork bounded by Σ. The metric $g_{\mu\nu}$ of M is given by the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R = \Lambda\, g_{\mu\nu} + \frac{8\pi G}{c^4}\, T_{\mu\nu} \tag{2}$$

with the cosmological constant Λ and the energy-momentum tensor $T_{\mu\nu}$. The cosmological constant

$$\Lambda = \frac{8\pi G}{c^4} \rho_{\mathrm{D}} \tag{3}$$

corresponds to the energy density $\rho_{\mathrm{D}} = E_{\mathrm{D}}/\mathrm{vol}(\Sigma)$ of the dark energy. In the previous section we showed that one can identify the dark energy with the curvature of two Poincaré spheres $\Sigma_{\mathrm{D}} = S^3/I^* \# S^3/I^*$. Let R be the scalar curvature of the cosmos Σ. Inserting the Robertson–Walker-metric

$$ds^2 = c^2 dt^2 - a(t)^2 h_{ik} dx^i dx^j \,, \tag{4}$$

with $\Sigma \times [0,1]$ and the scaling function $a(t)$ in (2), one gets the Friedman equation

$$\left(\frac{\dot{a}(t)}{c\, a(t)}\right)^2 + \frac{k}{a(t)^2} = \frac{8\pi G}{c^4}\frac{\rho}{3} + \frac{\Lambda}{3} = \frac{8\pi G}{c^4}\,\frac{(\rho + \rho_{\mathrm{D}})}{3}$$

with curvature $k = 0, \pm 1$. Then the scalar curvature of the cosmos Σ is $R = 3k/a(t)^2$ and the Hubble constant is given by $\dot{a}(t)/a(t) = H_0$. Thus, the relation of the total density ρ and scalar curvature R of the cosmos is given by

$$\rho = \frac{c^4}{8\pi G} R + \frac{3 H_0^2 c^2}{8\pi G} \,. \tag{5}$$

Using homogeneity and isotropy of the matter distribution we get by integration

$$\rho = \frac{\frac{c^4}{8\pi G} \int\limits_{\Sigma} R \sqrt{h} d^3 x}{\mathrm{vol}(\Sigma)} + \rho_{\mathrm{C}} \,, \qquad \rho_{\mathrm{C}} = \frac{3 H_0^2 c^2}{8\pi G} \,, \tag{6}$$

with the critical density ρ_{C}. Replacing Σ by Σ_{D} we obtain in an analog way the dark energy density

$$\rho_{\mathrm{D}} = \frac{\frac{c^4}{8\pi G}\int\limits_{\Sigma_{\mathrm{D}}}(R_{\mathrm{D}}+R_{\mathrm{C}})\sqrt{h}d^3x}{\mathrm{vol}(\Sigma_{\mathrm{D}})}\,, \qquad R_{\mathrm{C}} = \frac{3H_0^2}{c^2}\,. \tag{7}$$

The main step of the calculation is to solve the integral, *i.e.* the *Einstein–Hilbert action of the dark energy* defined on Σ_{D}

$$S_{\mathrm{EH}}(\Sigma_{\mathrm{D}}) = \int\limits_{\Sigma_{\mathrm{D}}}(R_{\mathrm{D}}+R_{\mathrm{C}})\sqrt{h}d^3x\,.$$

Witten has discussed the 3-dimensional Einstein–Hilbert action in more detail [22]. He was able to derive the important result that S_{EH} is related to a pure topological property — the *Chern–Simons invariant* of the manifold. Then one gets the simple relation between the Chern–Simons invariant of a SU(2) connection A and the Einstein–Hilbert action (see [1] for details)

$$S_{\mathrm{EH}}(\Sigma_{\mathrm{D}}) = \frac{16\pi^2}{(1-R_{\mathrm{C}}/3)^2}\;CS(A,\Sigma_{\mathrm{D}})\,.$$

Using this result we are able to calculate the ratio of the energy density of dark matter (7) and the total density (6) of the cosmos Σ yielding

$$\frac{\rho_{\mathrm{D}}}{\rho} = \frac{CS(A,\Sigma_{\mathrm{D}})}{CS(A,\Sigma)}\,\frac{\mathrm{vol}(\Sigma)}{\mathrm{vol}(\Sigma_{\mathrm{D}})}\,.$$

With the dark energy part $\Sigma_{\mathrm{D}} = S^3/I^*\#S^3/I^*$ by using $CS(A,M_1\#M_2) = CS(A,M_1)+CS(A,M_2)$ we get

$$\frac{\rho_{\mathrm{D}}}{\rho} = \frac{\mathrm{vol}(\Sigma)}{\mathrm{vol}(\Sigma_{\mathrm{D}})}\,\frac{2CS(A,S^3/I^*)}{CS(A,\Sigma)}\,. \tag{8}$$

By homogeneity, the density $\rho|_{\Sigma_{\mathrm{D}}} = E_{\mathrm{D}}/\mathrm{vol}(\Sigma_{\mathrm{D}})$ restricted to the subset Σ_{D} has to be equal to the energy density $\rho = E/\mathrm{vol}(\Sigma)$ on the whole manifold Σ, *i.e.* $E_{\mathrm{D}}/\mathrm{vol}(\Sigma_{\mathrm{D}}) = E/\mathrm{vol}(\Sigma)$. With $E/E_{\mathrm{D}} = \rho/\rho_{\mathrm{D}}$ it yields to $\rho/\rho_{\mathrm{D}} = \mathrm{vol}(\Sigma)/\mathrm{vol}(\Sigma_{\mathrm{D}})$. Inserting in (8) we obtain the *dark energy fraction*

$$\frac{\rho_{\mathrm{D}}}{\rho} = \sqrt{\frac{2CS(A,S^3/I^*)}{CS(A,\Sigma)}}\,, \tag{9}$$

which is a *purely topological invariant.*

To give an explicit expression we need the Chern–Simons invariants of the Poincaré sphere S^3/I^* and the Brieskorn sphere $\Sigma = \Sigma(2,5,7)$. We use the general method of Fintushel and Stern [5,6,10] to calculate both invariants. For an unique determination of the density ratio we need a further constraint: the chosen connection must allow for a Riemann metric, *i.e.* it has to be a Levi–Civita connection. What we need is the so-called *minimal Chern-Simons invariant* $\tau(\Sigma)$ of a homology 3-sphere

$$\tau(S^3/I^*) = \frac{1}{120}\,, \tag{10}$$

$$\tau(\Sigma(2,5,7)) = \frac{9}{280}\,. \tag{11}$$

This invariant corresponds to the self-dual or anti-self-dual solutions of a SU(2) gauge theory on $\Sigma \times \mathbb{R}$. This *minimization principle* permits an unique determination of the Chern–Simons invariants and we obtain for the dark energy fraction

$$\frac{\rho_{\mathrm{D}}}{\rho} = \sqrt{\frac{2\tau(S^3/I^*)}{\tau(\Sigma(2,5,7))}} \tag{12}$$

and thus

$$\frac{\rho_{\mathrm{D}}}{\rho} = \sqrt{\frac{(2\,\frac{1}{120})}{(\frac{9}{280})}} = \sqrt{\frac{14}{27}} \approx 0.720\,. \tag{13}$$

Inserting the observed total energy density $\rho_{\mathrm{obs}} = (1.02 \pm 0.02)\ \rho_{\mathrm{C}}$ from the WMAP data [18] we obtain for the *dark energy density*

$$\Omega_{\mathrm{D}} = \frac{\rho_{\mathrm{D}}}{\rho_{\mathrm{C}}} = 0.734 \pm 0.014\,, \tag{14}$$

with the critical density $\rho_{\mathrm{C}} = 3H_0^2c^2/8\pi G$ and the Hubble-constant H_0. Our calculated value (14) fits very well with the currently observed data. In particular, the fit CMB+2dFGRS+BBN ([4] Table 1) yields $(\Omega_{\mathrm{D}})_{\mathrm{obs}} = 0.73$ and $\Omega_{\mathrm{obs}} = 1.013$, and we obtain $\Omega_{\mathrm{D}} = \sqrt{14/27}\,\Omega_{\mathrm{obs}} = 0.729$. With the observed value of the Hubble constant $(H_0)_{\mathrm{obs}} = (72 \pm 5)\ \frac{\mathrm{km}}{\mathrm{s}}/\mathrm{Mpc}$ we obtain for the cosmological constant

$$\Lambda = \sqrt{\frac{14}{27}}\ \frac{3(H_0)_{\mathrm{obs}}\,\Omega_{\mathrm{obs}}}{c^4}\ (1.4 \pm 0.2) \times 10^{-52}\,\mathrm{m}^{-2}\,.$$

We would like to emphasize that our approach deeply requires a positive curvature of the cosmos, *i.e.* $\Omega > 1$, because our proposed topology of the cosmos — the Brieskorn sphere — is a closed 3-manifold with positive curvature. This provides a strong possibility for falsification and should be determinable by future observations.

4. Discussion

In the paper we show how to calculate the ratio of the dark energy density to the total energy density of the cosmos. Although we do not use a quantum-gravitational argumentation for the calculation, the appearance of the Chern–Simons invariant indicates a possible relation to quantum gravity. The exponential of the Chern–Simons invariant (Kodama state) is a wave function in loop quantum gravity which can be used for cosmology. In a follow-up paper we shall deal with this important battery of questions in more detail.

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