

# **Orbits and Invariants in Quantum Information Theory**

by

Ian Tan

A dissertation submitted to the Graduate Faculty of  
Auburn University  
in partial fulfillment of the  
requirements for the Degree of  
Doctor of Philosophy

Auburn, Alabama  
May 10, 2025

Keywords: tensor decomposition, HOSVD, entanglement, quantum information

Copyright 2025 by Ian Tan

Approved by

Luke Oeding, Associate Professor  
Michael Brown, Assistant Professor  
Huajun Huang, Associate Professor  
Evangelos Miliordos, James E. Land Associate Professor  
Henry Schenck, Professor and Rosemary Kopel Brown Chair

## Abstract

In the field of quantum information theory, one studies—among other things—the information processing tasks that can be achieved by taking advantage of a quantum phenomenon known as entanglement. There is a mathematical formalism that captures the notion of entanglement by elements of a vector space called state vectors. The local unitary and SLOCC (stochastic local operations with classical communication) groups act on this space, producing natural equivalence classes of state vectors. In this work, we consider group actions, their invariants, and how these can be used to classify and distinguish state vectors.

In Chapter 1, we give an introduction to quantum information theory. In Chapter 2, we show how results from Lie theory can be used to help find stationary points of invariant polynomials; these points correspond to highly entangled states. In Chapter 3, we discuss the problem of classifying orbits in these state spaces.

## Acknowledgments

I dedicate this to the most amazing woman. While here, some called her Lee Yuen (麗云) and others called her Lydia. To me, she was always Mama.

## Table of Contents

Abstract . . . . .	ii
Acknowledgments . . . . .	iii
List of Tables . . . . .	viii
List of Figures . . . . .	ix
1 Introduction to Quantum Information Theory . . . . .	1
1.1 Quantum mechanics . . . . .	2
1.1.1 Hilbert spaces . . . . .	2
1.1.2 The qubit . . . . .	3
1.1.3 Separable pair . . . . .	3
1.1.4 Bell state . . . . .	4
1.1.5 Unitary operations . . . . .	4
1.1.6 Measurements . . . . .	5
1.1.7 Postulates of quantum mechanics . . . . .	8
1.1.8 Projective space . . . . .	9
1.2 Quantum information processing protocols . . . . .	9
1.2.1 Pauli matrices . . . . .	10
1.2.2 Superdense coding . . . . .	10
1.2.3 The Mermin-Peres magic square . . . . .	11
1.3 Group actions and invariants . . . . .	14

1.3.1	The LU and SLOCC groups . . . . .	14
1.3.2	Density matrices . . . . .	15
1.3.3	Von Neumann entropy . . . . .	18
2	Four-qubit critical states . . . . .	20
	Abstract . . . . .	21
2.1	Introduction . . . . .	22
2.1.1	A selection of 4-qubit critical states . . . . .	23
2.1.2	Organization . . . . .	26
2.1.3	Notation and conventions . . . . .	27
2.2	Stationary points and critical points . . . . .	28
2.2.1	Stationary points . . . . .	28
2.2.2	Complex and real Lie groups . . . . .	29
2.2.3	The critical set . . . . .	30
2.3	Vinberg theory . . . . .	32
2.3.1	A graded Lie algebra . . . . .	32
2.3.2	The Cartan subspace . . . . .	33
2.3.3	The Weyl group . . . . .	34
2.3.4	Critical states on the Cartan . . . . .	35
2.3.5	SLOCC invariants . . . . .	37
2.3.6	Symmetric invariants . . . . .	37
2.4	Stationary points of symmetric invariants . . . . .	38
2.4.1	Systems of equations . . . . .	38

2.4.2	Four-qubit stationary points . . . . .	40
2.5	Acknowledgements . . . . .	48
3	Tensor decompositions . . . . .	49
Abstract	. . . . .	50
3.1	Introduction . . . . .	51
3.1.1	Historical highlights . . . . .	52
3.1.2	Our approach . . . . .	53
3.1.3	Wider context and applications . . . . .	54
3.1.4	Organization . . . . .	55
3.1.5	Notation and conventions . . . . .	55
3.2	HOSVD and its complex orthogonal counterpart . . . . .	56
3.2.1	Reduction maps . . . . .	56
3.2.2	Core elements . . . . .	60
3.2.3	HOSVD . . . . .	64
3.2.4	Orthogonal HOSVD . . . . .	65
3.2.5	Stabilizers of Jordan blocks . . . . .	67
3.3	Normal forms for almost all qubit states . . . . .	69
3.3.1	General points and generic properties . . . . .	70
3.3.2	LU group . . . . .	70
3.3.3	SLOCC group, even case . . . . .	73
3.3.4	SLOCC group, odd case . . . . .	79

References . . . . . 89

## List of Tables

2.1	Four-qubit stationary points . . . . .	26
2.2	More four-qubit stationary points (see Section 2.4.2.6) . . . . .	43
3.1	Reduction maps of Examples 3.3 to 3.6. . . . .	59
3.2	Ranks of the reduction maps for 3 qubits. . . . .	87

## List of Figures

1.1	Classical and quantum strategies for the magic square game . . . . .	12
-----	--	----

## Chapter 1

### Introduction to Quantum Information Theory

## 1.1 Quantum mechanics

In this chapter, we introduce the ideas that make up quantum mechanics, summarized as four postulates in Section 1.1.7. We will only introduce concepts necessary to understand quantum information theory, leaving aside deeper questions about physics. Although counterintuitive, there exist physical systems that behave as described in this chapter. Our goal is to give some intuition about these systems and, in doing so, to develop a well-motivated formalism which provides a setting to ask mathematical questions.

### 1.1.1 Hilbert spaces

Suppose  $\mathcal{H}$  be a  $d$ -dimensional complex Hilbert space. Every Hilbert space that we will encounter is finite-dimensional. In quantum information theory, the standard basis vectors of  $\mathcal{H}$  are traditionally written as  $|0\rangle, |1\rangle, \dots, |d-1\rangle$ . Let

$$|\varphi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle \quad \text{and} \quad |\psi\rangle = \sum_{i=0}^{d-1} \beta_i |i\rangle$$

be elements of  $\mathcal{H}$ , thought of as column vectors. Then  $\langle\varphi|$  denotes the Hermitian transpose of  $|\varphi\rangle$  and the inner product  $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  on the pair  $(|\varphi\rangle, |\psi\rangle)$  is written as

$$\langle\varphi|\psi\rangle = \sum_{i=0}^{d-1} \bar{\alpha}_i \beta_i.$$

Given a collection  $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$  of Hilbert spaces, we write

$$|i_1 i_2 \dots i_n\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_n\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n$$

for the basis vectors in the tensor product, where each  $|i_j\rangle$  is a basis vector of  $\mathcal{H}_j$ . The following equations are easy to verify and will be used frequently in this chapter without reference:

1.  $\langle\varphi|\varphi\rangle = \text{tr}(|\varphi\rangle\langle\varphi|)$ , and

$$2. (\langle \varphi | \otimes \langle \psi |)(|\varphi\rangle \otimes |\psi\rangle) = \langle \varphi | \varphi \rangle \cdot \langle \psi | \psi \rangle$$

for all  $|\varphi\rangle, |\psi\rangle \in \mathcal{H}$ .

### 1.1.2 The qubit

The qubit is a fundamental concept in the field of quantum information theory. As its name implies, it is a quantum analog of the classical bit. The word “bit” is itself a portmanteau of binary digit: one of the two values 0 and 1. Similarly, a qubit can be in the 0 state or in the 1 state. These states correspond to the standard basis vectors  $|0\rangle$  and  $|1\rangle$  of  $\mathbb{C}^2$ . A qubit may also be in a so-called “superposition” of the 0 state and the 1 state; in this case, the state of a qubit is described by a unit vector  $|\varphi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$  in  $\mathbb{C}^2$ . When a party measures such a qubit, rather than seeing a complex linear combination of states, they observe either the 0 state or the 1 state. Mathematically, a measurement “collapses” the state of the qubit to one of two possibilities:  $|0\rangle$  or  $|1\rangle$ . The probability of observing each outcome is the squared norm of the corresponding coefficient. That is, the probability is  $|\alpha_0|^2$  for the  $|0\rangle$  state and  $|\alpha_1|^2$  for the  $|1\rangle$  state. This naturally leads to the condition that  $|\varphi\rangle$  is a unit vector or that  $|\alpha_0|^2 + |\alpha_1|^2 = 1$ .

Similarly, a qutrit has a state described by a unit vector in  $\mathbb{C}^3$  and, in general, a qudit has a state described by a unit vector in  $\mathbb{C}^d$ . In this chapter we will focus on qubits, but the mathematical formalisms we develop can be straightforwardly generalized to qudits. For example, when a qudit is measured, its state collapses to one of the basis vectors  $|i\rangle \in \mathbb{C}^d$  and the probability of each outcome is again the squared magnitude of the corresponding coefficient.

### 1.1.3 Separable pair

Next, we describe how to compose systems to make larger systems. Consider a system composed of a pair of qubits that are not entangled. In this case, the first qubit has the state described by  $|\varphi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$  and the second qubit has the state  $|\psi\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$ . The state of the

system of qubits is described by the tensor product

$$|\varphi\rangle \otimes |\psi\rangle = \alpha_0\beta_0 |00\rangle + \alpha_0\beta_1 |01\rangle + \alpha_1\beta_0 |10\rangle + \alpha_1\beta_1 |11\rangle. \quad (1.1)$$

The induced basis vectors  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$  of  $\mathbb{C}^2 \otimes \mathbb{C}^2$  correspond to the four possible outcomes when both qubits are measured. As before, the probability of each outcome is the squared norm of the corresponding coefficient. The lack of entanglement means that the observed states of the first and second qubits are independent events. That is, the probability  $|\alpha_i\beta_j|^2$  that the first qubit is in the  $i$  state and the second qubit is in the  $j$  state is the product of the probability  $|\alpha_i|^2$  that the first qubit is in the  $i$  state and the probability  $|\beta_j|^2$  that the second qubit is in the  $j$  state.

#### 1.1.4 Bell state

In general, the state of a closed system of  $n$  qubits is described by a unit vector in the  $n$ -fold tensor product  $(\mathbb{C}^2)^{\otimes n}$ . States that correspond to rank-one tensors in  $(\mathbb{C}^2)^{\otimes n}$ , such as (1.1), are called **separable** or **unentangled**. A system of qubits may also be **entangled** or in a state not described by a rank-one tensor. A prototypical example of entanglement is given by the Bell state

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (1.2)$$

If a party measures the first qubit of the pair and observes the  $|0\rangle$  state, they may infer that the second qubit is also in the  $|0\rangle$  state, and similarly if they observe the  $|1\rangle$  state. This type of inference is possible since the pair is entangled, which implies that the observed states of the first and second qubits are not independent events.

#### 1.1.5 Unitary operations

If a party is in possession of a system of  $n$  qubits in the state  $|\varphi\rangle \in (\mathbb{C}^2)^{\otimes n}$ , we assume that they can apply any unitary operation to  $|\varphi\rangle$ . This updates the state of the system to  $U|\varphi\rangle$ , where  $U \in U(2^n)$ . If the collection of  $n$  qubits is part of a larger closed system of  $n + m$  qubits in the

state  $|\varphi\rangle \in (\mathbb{C}^2)^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes m}$ , then the same operation updates the state of the larger system to  $|\varphi\rangle \mapsto (U \otimes I) |\varphi\rangle$ , where  $I$  is the identity on  $(\mathbb{C}^2)^{\otimes m}$ .

### 1.1.6 Measurements

We now discuss in depth the concept of measurement in quantum mechanics. A measurement may be thought of as an experiment on a system of qubits that produces one outcome from a set of possible outcomes. For a given measurement, the probability of each outcome depends on the state of the system.

First, we introduce what is known as a **projective measurement**. Consider an experiment where a party measures a system of  $n$  qubits. Mathematically, such a measurement is described by an orthogonal basis  $\{u_i\}$  of the state space  $(\mathbb{C}^2)^{\otimes n}$  which corresponds to projection operators  $P_i = |u_i\rangle\langle u_i|$ . The indices  $i$  correspond to the outcomes that may occur in the measurement. If a system is in the state  $|\varphi\rangle \in (\mathbb{C}^2)^{\otimes n}$  before a measurement then the probability that outcome  $i$  occurs is

$$p(i) = \langle\varphi| P_i |\varphi\rangle = |\langle u_i|\varphi\rangle|^2 \quad (1.3)$$

and in this case the state of the system becomes

$$\frac{P_i |\varphi\rangle}{\sqrt{\langle\varphi| P_i |\varphi\rangle}}$$

after the measurement. Since  $\{u_i\}$  is an orthogonal basis, the projectors satisfy the equation  $\sum_i P_i = I$  and the probabilities sum to one:

$$\sum_{i=1}^{2^n} \langle\varphi| P_i |\varphi\rangle = \langle\varphi| \left(\sum_{i=1}^{2^n} P_i\right) |\varphi\rangle = \langle\varphi|\varphi\rangle = 1.$$

In the case where  $\{u_i\}$  is the standard basis, the probabilities  $p(i) = |\langle i|\varphi\rangle|^2$  are the squared magnitudes of the coefficients of  $|\varphi\rangle$ . This recovers the description of measurement given in Section

1.1.2. Allowing  $\{u_i\}$  to be an arbitrary orthogonal basis accounts for the fact that a party may apply arbitrary unitary operations before and after measuring the system.

If the collection of  $n$  qubits is part of a larger closed system of  $n + m$  qubits in the state  $|\varphi\rangle \in (\mathbb{C}^2)^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes m}$ , then the same measurement is described by the set of projection operators  $\{P_i \otimes I\}$ , where  $I$  is the identity on  $(\mathbb{C}^2)^{\otimes m}$ . The probability that outcome  $i$  occurs is

$$p(i) = \langle \varphi | (P_i \otimes I) | \varphi \rangle \quad (1.4)$$

and in this case the state of the system becomes

$$\frac{(P_i \otimes I) |\varphi\rangle}{\sqrt{\langle \varphi | (P_i \otimes I) | \varphi \rangle}} \quad (1.5)$$

after the measurement.

Given any state  $|\varphi\rangle \in (\mathbb{C}^2)^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes m}$ , the orthogonal basis  $\{u_i\}$  of  $(\mathbb{C}^2)^{\otimes n}$  uniquely determines vectors  $|\varphi_i\rangle$  such that

$$|\varphi\rangle = \sum_{i=1}^{2^n} |u_i\rangle \otimes |\varphi_i\rangle.$$

This allows us to re-express (1.4) and (1.5) as

$$p(i) = \langle \psi_i | \psi_i \rangle \quad \text{and} \quad \frac{(P_i \otimes I_m) |\varphi\rangle}{\sqrt{\langle \varphi | (P_i \otimes I_m) | \varphi \rangle}} = \frac{|u_i\rangle \otimes |\varphi_i\rangle}{\sqrt{\langle \varphi_i | \varphi_i \rangle}}.$$

**Example 1.1.** Consider an entangled pair of qubits in the state

$$|\varphi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2.$$

Suppose the first qubit is measured with the standard basis  $\{|0\rangle, |1\rangle\}$ . We calculate

$$(P_0 \otimes I_2) |\varphi\rangle = \frac{1}{\sqrt{3}}(P_0 |0\rangle \otimes |0\rangle + P_0 |0\rangle \otimes |1\rangle + P_0 |1\rangle \otimes |1\rangle) = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle),$$

$$(P_1 \otimes I_2) |\varphi\rangle = \frac{1}{\sqrt{3}}(P_1 |0\rangle \otimes |0\rangle + P_1 |0\rangle \otimes |1\rangle + P_1 |1\rangle \otimes |1\rangle) = \frac{1}{\sqrt{3}} |11\rangle.$$

The probabilities of outcomes 0 and 1 are the squared norms of the vectors above:

$$p(0) = \langle \varphi | (P_0 \otimes I_2) |\varphi\rangle = \frac{2}{3}, \quad \text{and} \quad p(1) = \langle \varphi | (P_1 \otimes I_2) |\varphi\rangle = \frac{1}{3}.$$

In the case that outcome 0 occurs, the state of the system becomes

$$\frac{(P_0 \otimes I_2) |\varphi\rangle}{\sqrt{\langle \varphi | (P_0 \otimes I_2) |\varphi\rangle}} = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle).$$

In the case that outcome 1 occurs, the state of the system becomes

$$\frac{(P_1 \otimes I_2) |\varphi\rangle}{\sqrt{\langle \varphi | (P_1 \otimes I_2) |\varphi\rangle}} = |11\rangle.$$

◇

A more general type of measurement that may be applied to a quantum system, called a Positive Operator-Valued Measure or **POVM**. Given a system of  $n$  qubits in the state  $|\varphi\rangle \in (\mathbb{C}^2)^{\otimes n}$ , we may prepare a *qudit* in state  $|e\rangle \in \mathbb{C}^d$  and apply a unitary operation  $U \in \text{GL}((\mathbb{C}^2)^{\otimes n} \otimes \mathbb{C}^d)$  to the composite system. The state of the composite system then takes the form

$$U(|\varphi\rangle \otimes |e\rangle) = \sum_{i=0}^{d-1} |\varphi_i\rangle \otimes |i\rangle \tag{1.6}$$

for some  $|\varphi_i\rangle \in (\mathbb{C}^2)^{\otimes n}$ . By the linearity of  $U$ , we must have  $|\varphi_i\rangle = A_i |\varphi\rangle$  for some operators  $A_i$  not depending on  $|\varphi\rangle$ . Taking the inner product of  $U(|\varphi\rangle \otimes |e\rangle)$  and  $U(|\psi\rangle \otimes |e\rangle)$ , we have

$$\langle \varphi | \psi \rangle = \sum_{i=0}^{k-1} \langle \varphi | A_i^* A_i | \psi \rangle.$$

The above holds for all  $|\varphi\rangle, |\psi\rangle \in \mathbb{C}^2$ , so the operators satisfy the completeness equation

$$\sum_{i=0}^{k-1} A_i^* A_i = I. \quad (1.7)$$

When the ancillary qudit is measured, the projection  $I \otimes P_i$  is applied to the state  $|\tilde{\varphi}\rangle = |\varphi\rangle \otimes |e\rangle$  with probability

$$\langle \tilde{\varphi} | I \otimes P_i | \tilde{\varphi} \rangle = \langle \varphi | A_i^* A_i | \varphi \rangle,$$

where  $P_i = |i\rangle \langle i|$ . In the other direction, given a set  $\{A_i\}$  of operators satisfying (1.7), we can define a unitary map  $U : (\mathbb{C}^2)^{\otimes n} \otimes |e\rangle \rightarrow (\mathbb{C}^2)^{\otimes n} \otimes \mathbb{C}^d$  by (1.6), where  $|\varphi_i\rangle = A_i |\varphi\rangle$ . This map can be extended to a unitary operator on  $(\mathbb{C}^2)^{\otimes n} \otimes \mathbb{C}^d$ . This brings us to the description of a general measurement given by the fourth postulate of Section 1.1.7.

We note an important difference between a projective measurement and a POVM. Suppose a party has in its possession a system in some unknown state  $|\varphi\rangle$ . If they perform a projective measurement, the state of the system becomes known. However, if they apply a POVM, they will only know the relationship between the state before measurement and the state after measurement.

### 1.1.7 Postulates of quantum mechanics

To summarize our discussion thus far, let us present a list of postulates for quantum mechanics. These are the postulates that appear in Section 2.2 of [46], although they are numbered differently. The term ‘‘closed system’’ in the first postulate can be understood to mean a system safe from unwanted outside influence.

**Postulate 1.** Associated to any closed physical system is a Hilbert space  $\mathcal{H}$  known as the **state space** of the system. The system is described by a unit vector in  $\mathcal{H}$ , the system's **state vector**.

**Postulate 2.** The union of  $n$  closed systems is a composite system whose state space is the tensor product  $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$  of the state spaces  $\mathcal{H}_i$  of the component systems. Moreover, if the  $i$ th component system is prepared in the state  $|\varphi_i\rangle$ , then the joint state of the total system is  $|\varphi_1\rangle \otimes \cdots \otimes |\varphi_n\rangle$ .

**Postulate 3.** The state  $|\varphi\rangle$  of a system at time  $t_1$  is related to the state  $U|\varphi\rangle$  of a system at time  $t_2$  by a unitary operator  $U$  that depends only on the times  $t_1$  and  $t_2$ .

**Postulate 4.** Quantum measurements are described by a collection  $\{A_i\}$  of operators acting on the state space. The index  $i$  refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is  $|\varphi\rangle$  immediately before the measurement then the probability that result  $i$  occurs is given by  $p(i) = \langle\varphi|A_i^*A_i|\varphi\rangle$ , and the state of the system after the measurement is  $A_i|\varphi\rangle/\sqrt{p(i)}$ . The operators satisfy the completeness equation  $\sum_i A_i^*A_i = I$ .

### 1.1.8 Projective space

Consider two states  $|\varphi\rangle \in (\mathbb{C}^2)^{\otimes n}$  and  $|\varphi'\rangle = e^{it}|\varphi\rangle$ , where  $t \in \mathbb{R}$ . We say that  $|\varphi\rangle$  and  $|\varphi'\rangle$  are the same up to a **global phase change**, i.e., multiplication by a unit complex number. Notice that for any operator  $A$  on the state space, we have  $\langle\varphi|A^*A|\varphi\rangle = \langle\varphi'|A^*A|\varphi'\rangle$ . This means that no measurement can distinguish between the states  $|\varphi\rangle$  and  $|\varphi'\rangle$ . It is therefore appropriate to think of quantum states as equivalence classes or orbits of unit vectors under the action of the circle group  $\{e^{it} : t \in \mathbb{R}\}$ . These orbits are in correspondence with points of the projective space  $\mathbb{P}(\mathbb{C}^2)^{\otimes n}$ .

## 1.2 Quantum information processing protocols

In order to gain the proper intuition, we argue that it is essential to see examples of information-theoretical tasks that can be achieved with quantum mechanics. We present two such examples, known as **quantum protocols**.

### 1.2.1 Pauli matrices

The three matrices below, known as the **Pauli matrices**, show up frequently in the field of quantum information theory.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Each Pauli matrix is unitary, Hermitian, and has two eigenvectors with eigenvalues 1 and -1. Furthermore, they pairwise anticommute. The **Pauli group** on  $n$  qubits is the set of operators having the form  $\lambda A_1 \otimes \cdots \otimes A_n$ , where  $A_i \in \{I, X, Y, Z\}$  and  $\lambda^4 = 1$ . If a subset of the Pauli group consists of pairwise commuting elements, then the elements of the subset are simultaneously diagonalizable [34].

### 1.2.2 Superdense coding

For a first example, let us introduce a quantum protocol known as superdense coding [46]. This protocol allows two parties, Alice and Bob, to communicate securely.

Two qubits are prepared in advance to be in the Bell state (1.2). The first qubit of the system is sent to Alice and the other to Bob; the two parties are assumed to be a large distance apart from each other. Suppose Alice wants to send the message  $(i, j)$  to Bob, where  $i, j \in \{0, 1\}$ . She applies the unitary operation  $U_{ij} \otimes I_2$  to the system, where

$$U_{00} = I_2, \quad U_{01} = X, \quad U_{10} = iY, \quad U_{11} = Z.$$

Then the state updates by the rule

$$|\varphi\rangle \mapsto \begin{cases} |\varphi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & \text{if } (i, j) = (0, 0), \\ |\varphi_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) & \text{if } (i, j) = (0, 1), \\ |\varphi_{10}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) & \text{if } (i, j) = (1, 0), \\ |\varphi_{11}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) & \text{if } (i, j) = (1, 1). \end{cases}$$

After this, Alice sends her qubit to Bob. Notice that the vectors  $|\varphi_{00}\rangle, |\varphi_{01}\rangle, |\varphi_{10}\rangle, |\varphi_{11}\rangle$ , form an orthogonal basis of  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . Therefore, now in possession of both qubits, Bob can apply the projective measurement corresponding to this basis. The outcome of Bob's measurement is the message  $(i, j)$  with probability 1.

There are two noteworthy features of superdense coding. First, by taking advantage of shared entanglement, Alice sends 2 classical bits of information by sending only one quantum qubit. Second, Alice's message is safe from eavesdropping: a bad actor needs both qubits to decode the message and not just the qubit Alice sends to Bob.

### 1.2.3 The Mermin-Peres magic square

Consider the following game with players Alice and Bob, who must coordinate in order to win, and referee Charlie. The game presupposes the existence of a  $3 \times 3$  grid consisting of 9 cells. Charlie assigns a random row of the grid to Alice and a random column of the grid to Bob. Alice and Bob must assign numbers from  $\{\pm 1\}$  to each cell of their row or column such that

1. the product of all of Alice's row numbers is 1,
2. the product of all of Bob's column numbers is  $-1$ , and
3. Alice and Bob pick the same number for the intersection of their row and column.

Alice and Bob may devise a strategy beforehand, but cannot communicate once the game has started. One idea for a strategy is to pre-populate the  $3 \times 3$  grid with numbers from  $\{\pm 1\}$  such that

1	1	1
1	-1	-1
-1	1	?

$I_2 \otimes Z$	$Z \otimes I_2$	$Z \otimes Z$
$X \otimes I_2$	$I_2 \otimes X$	$X \otimes X$
$-X \otimes Z$	$-Z \otimes X$	$Y \otimes Y$

Figure 1.1: Classical and quantum strategies for the magic square game

each row product is 1 and each column product is  $-1$ . However, we find that we can fill 8 out of the 9 empty cells of the grid before running into a problem (see the left side of Figure 1.1). It is in fact impossible—and this not hard to show—to fill out the entire grid subject to these constraints. This means that Alice and Bob cannot guarantee that they will win the game if they know which numbers they will choose for each possible row or column they are assigned.

However, we may devise a quantum strategy by filling the cells of the grid with operators. In particular, we use the arrangement of operators given in the right side of Figure 1.1, known as the **Mermin-Peres magic square** [52]. This magic square has the following properties:

1. the operators in any single column or row mutually commute,
2. the product of operators in each row is  $I_4$ , and
3. the product of operators in each column is  $-I_4$ .

A consequence of the first item is that the operators in any row or column are simultaneously unitarily diagonalizable. For example, the matrices of the top row are already diagonal and can be written out as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Mermin-Peres magic square gives a quantum strategy with which Alice and Bob can win the game with probability 1. First, Alice and Bob prepare the 4-qubit entangled state

$$|\varphi\rangle = \frac{1}{4}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle). \quad (1.8)$$

Before the game starts, the first two qubits are kept with Alice and the last two qubits are kept with Bob. It can be checked that  $(U \otimes U)|\varphi\rangle = |\varphi\rangle$  where  $U$  is any of the 9 operators in the magic square. That is,  $|\varphi\rangle$  is an eigenvalue of  $U \otimes U$  with eigenvalue 1.

After Charlie assigns a row and a column, Alice and Bob each perform a projective measurement on their pair of qubits with the orthogonal basis of  $\mathbb{C}^2 \otimes \mathbb{C}^2$  consisting of simultaneous eigenvectors determined by their row or column of the magic square. They fill out their row and column with the eigenvalues corresponding to their outcome. Due to items 2 and 3 in the second list of this section, items 1 and 2 of the first list immediately hold. It remains to show that Alice and Bob pick the same number for the intersection of their row and column.

Suppose Alice measures her pair of qubits with the orthogonal basis of simultaneous eigenvectors  $\{|a_1\rangle, \dots, |a_4\rangle\}$  and Bob measures his pair of qubits with the orthogonal basis of simultaneous eigenvectors  $\{|b_1\rangle, \dots, |b_4\rangle\}$ . Acting as a combined party, Alice and Bob measure the state of the 4-qubit system with the orthogonal basis of  $(\mathbb{C}^2)^{\otimes 4}$  consisting of eigenvectors  $\{|a_i\rangle \otimes |b_j\rangle : 1 \leq i, j \leq 4\}$  of  $U \otimes U$ , where  $U$  is the operator in the intersection of Alice's row and Bob's column. If  $\{u_i\}$  is any orthogonal basis of eigenvectors of  $U \otimes U$ , then we may write  $|\varphi\rangle = \sum_i \alpha_i |u_i\rangle$  for some  $\alpha_i \in \mathbb{C}$ . Suppose  $|u_j\rangle$  has eigenvalue  $-1$ . Since  $|\varphi\rangle$  is an eigenvector of  $U \otimes U$  with eigenvalue 1, we have  $\alpha_j = 0$ . By (1.3), the probability that the outcome associated with  $|u_j\rangle$  occurs is

$$|\langle u_j | \varphi \rangle|^2 = \left| \sum_{i=1}^{16} \alpha_i \langle u_j | u_i \rangle \right|^2 = |\alpha_j \langle u_j | u_j \rangle|^2 = 0.$$

Therefore the outcome of this joint measurement must be associated with the eigenvalue 1. It follows that the outcomes of Alice's and Bob's measurements both give the eigenvalue 1 or both give the eigenvalue -1, since the product of these eigenvalues must equal 1.

### 1.3 Group actions and invariants

Entanglement is a resource for carrying out information-processing tasks. We see this in the quantum protocols introduced previously: superdense coding utilizes the entangled Bell state (1.2) while the four-qubit entangled state (1.8) is used to win the Mermin-Peres magic square game. In light of this view, it is natural to try to find ways of classifying and measuring entanglement. We shall achieve this by considering groups that act on state spaces, inducing natural equivalence classes of quantum states.

#### 1.3.1 The LU and SLOCC groups

Suppose a system of  $n$ -qubits is prepared in an entangled state  $|\varphi\rangle \in (\mathbb{C}^2)^{\otimes n}$ . Each of the  $n$  qubits is then sent to different parties in laboratories physically separated from one another. Each party can apply a unitary operation to only their qubit, so a unitary operation on the system takes the form

$$|\varphi\rangle \mapsto (U_1 \otimes \cdots \otimes U_n) |\varphi\rangle, \quad U_i \in \text{U}(2).$$

This is called a local unitary operation and the set of all such operations is the Local Unitary or **LU group**. This gives a natural notion of equivalence between quantum states since all of the states in a given local unitary orbit may be used to achieve the same information-processing tasks.

It is also possible for each party to apply a POVM to their qubit. Let us first show that for every  $A \in \text{GL}_2$  we can find  $B \in \text{GL}_2$  and  $\lambda > 0$  such that  $A^*A + B^*B = \lambda I$ . So any invertible operation can be applied to a qubit with some nonzero probability. Given  $A \in \text{GL}_2$  we want  $B$  and  $\lambda$  such that  $B^*B = \lambda I - A^*A$ . The right side of this equation is a Hermitian matrix, so has real eigenvalues; if  $\lambda > 0$  is large these eigenvalues are positive. Thus, by the spectral theorem, there

exists a unitary matrix  $U$  and a positive-definite diagonal matrix  $D$  such that  $UDU^* = \lambda I - A^*A$ . Then one can define  $B^* = U\sqrt{D}V$  for some unitary  $V$ .

Now, suppose the  $i$ th party applies to their qubit the POVM defined by invertible operators  $A_i, B_i \in \text{GL}_2$ , where the operation  $A_i$  is applied with probability  $p_i$  and  $B_i$  is applied with probability  $1 - p_i$ . After this, each party tells every other party through classical communication channels the measurement outcome they observed. With probability  $\prod_i p_i$  the state of the system updates by

$$|\varphi\rangle \mapsto \frac{(A_1 \otimes \cdots \otimes A_n) |\varphi\rangle}{\sqrt{\langle \varphi | (A_1^* A_1 \otimes \cdots \otimes A_n^* A_n) | \varphi \rangle}}, \quad A_i \in \text{GL}_2. \quad (1.9)$$

The set of all operations of the form (1.9) is known as the group of Stochastic Local Operations with Classical Communication or the **SLOCC group**. The word “stochastic” in the name refers to the fact that there is only a nonzero probability of converting a given state to a different state in the same equivalence class. This group gives a coarser classification than the LU group, but with a similar interpretation: any information-processing task that can be achieved with one state can also be achieved *with enough supply* of an equivalent state.

We note the distinction, or lack thereof, between the groups  $\text{SL}_N$  and  $\text{GL}_N$ . Any  $\tilde{A} \in \text{GL}_N$  has the form  $\lambda A$  where  $\lambda \in \mathbb{C}$  and  $A \in \text{SL}_N$ . For any  $|\varphi\rangle \in \mathbb{C}^N$  we have

$$\frac{\tilde{A} |\varphi\rangle}{\sqrt{\langle \varphi | \tilde{A}^* \tilde{A} | \varphi \rangle}} = \frac{\lambda}{|\lambda|} \cdot \frac{A |\varphi\rangle}{\sqrt{\langle \varphi | A^* A | \varphi \rangle}}$$

so the action of  $\text{GL}_N$  on states agrees with the action of  $\text{SL}_N$  up to a global phase change. Another way of saying this is that  $\text{GL}_N$  and  $\text{SL}_N$  both act as  $\text{PGL}_N$  on the projectivization  $\mathbb{P}(\mathbb{C}^N)$ .

### 1.3.2 Density matrices

A **density matrix**  $\rho$  is a positive semidefinite (having nonnegative eigenvalues) Hermitian matrix satisfying  $\text{tr}(\rho) = 1$ . Density matrices are an alternate way of representing quantum states and allow for more general states called “mixed states,” a term which is not necessary for us to define. These are in contrast to “pure states,” or what we have hitherto simply called quantum states.

### 1.3.2.1 Pure state

Define a map  $\rho : \mathbb{C}^N \rightarrow \text{End}(\mathbb{C}^N)$  by the rule  $\rho(|\varphi\rangle) = |\varphi\rangle \langle \varphi|$ . The operator  $|\varphi\rangle \langle \varphi|$  is called the **density matrix of the pure state**  $|\varphi\rangle$ . This map satisfies

$$\rho(U|\varphi\rangle) = U|\varphi\rangle \langle \varphi|U^* = U\rho(|\varphi\rangle)U^*, \quad \text{for all } U \in U(2^N)$$

so the action of  $U$  on  $\mathbb{C}^N$  corresponds via  $\rho$  to the conjugation action of  $U$  on  $\text{End}(\mathbb{C}^N)$ . It follows that any unitary invariant  $f : \text{End}(\mathbb{C}^N) \rightarrow \mathbb{C}$  pulls back to an unitary invariant  $f \circ \rho : \mathbb{C}^N \rightarrow \mathbb{C}$ . Note that  $|\varphi\rangle \langle \varphi|$  is Hermitian and has exactly one nonzero eigenvalue which equals 1, and from this it is easy to check to other properties of a density matrix.

### 1.3.2.2 Reduced density matrix

We may also use the reduced density matrix to find invariants, which is defined using the partial trace. Given two finite-dimensional Hilbert spaces  $V$  and  $W$ , the **partial trace**  $\text{Tr}_W$  is the unique linear map  $\text{End}(V \otimes W) \rightarrow \text{End}(V)$  such that  $\text{Tr}_W(A \otimes B) = \text{tr}(B)A$  for all  $A \in \text{End}(V)$  and  $B \in \text{End}(W)$ . Define a map  $\rho_W : V \otimes W \rightarrow \text{End}(V)$  by the rule

$$\rho_W(|\varphi\rangle) = \text{Tr}_W(|\varphi\rangle \langle \varphi|).$$

The operator  $\rho_W(|\varphi\rangle)$  is called the **reduced density matrix** of  $|\varphi\rangle$ .

Any state vector  $|\varphi\rangle \in V \otimes W$  uniquely determines vectors  $|\varphi_i\rangle \in V$  such that

$$|\varphi\rangle = \sum_{i=1}^d |\varphi_i\rangle \otimes |i\rangle.$$

Let  $|V|$  and  $|W|$  denote the dimensions of  $V$  and  $W$  respectively. Calculating the reduced density matrix of this expression, we find

$$\mathrm{Tr}_W(|\varphi\rangle\langle\varphi|) = \mathrm{Tr}_W\left(\sum_{i=1}^{|W|}\sum_{j=1}^{|W|}|\varphi_i\rangle\langle\varphi_j|\otimes|i\rangle\langle j|\right) = \sum_{i=1}^{|W|}|\varphi_i\rangle\langle\varphi_i|. \quad (1.10)$$

From Equation (1.10), we see that  $\rho_W(|\varphi\rangle)$  has eigenvectors  $|\varphi_i\rangle$  with eigenvalues  $\langle\varphi_i|\varphi_i\rangle$ . Therefore the image of  $\rho_W$  consists of positive semidefinite Hermitian matrices. The condition that the trace of  $\rho_W(|\varphi\rangle)$  equals 1 comes from the fact that  $|\varphi\rangle$  is a unit vector:

$$\mathrm{tr}(\rho_W(|\varphi\rangle)) = \sum_{i=1}^{|W|}\mathrm{tr}(|\varphi_i\rangle\langle\varphi_i|) = \sum_{i=1}^{|W|}\langle\varphi_i|\varphi_i\rangle = \langle\varphi|\varphi\rangle = 1.$$

We also observe from Equation (1.10) that

$$\rho_W((U \otimes I)|\varphi\rangle) = \sum_{i=1}^{|W|}U|\varphi_i\rangle\langle\varphi_i|U^* = U\rho_W(|\varphi\rangle)U^* \quad (1.11)$$

where  $U \in \mathrm{GL}(V)$  is unitary and  $I$  is the identity on  $W$ . On the other hand,  $|\varphi\rangle$  uniquely determines vectors  $|\psi_i\rangle \in W$  such that  $|\varphi\rangle = \sum_{i=1}^{|V|}|i\rangle \otimes |\psi_i\rangle$ . Then the reduced density matrix can also be expressed as

$$\mathrm{Tr}_W(|\varphi\rangle\langle\varphi|) = \mathrm{Tr}_W\left(\sum_{i=1}^{|V|}\sum_{j=1}^{|V|}|i\rangle\langle j|\otimes|\psi_i\rangle\langle\psi_j|\right) = \sum_{i=1}^{|V|}\sum_{j=1}^{|V|}\langle\psi_i|\psi_j\rangle E_{ij}, \quad (1.12)$$

where  $E_{ij} = |i\rangle\langle j|$ . It follows that

$$\rho_W((I \otimes U)|\varphi\rangle) = \sum_{i,j=1}^{|V|}\langle\psi_i|U^*U|\psi_j\rangle E_{ij} = \sum_{i,j=1}^{|V|}\langle\psi_i|\psi_j\rangle E_{ij} = \rho_W(|\varphi\rangle) \quad (1.13)$$

where  $U \in \text{GL}(W)$  is unitary and  $I$  is the identity on  $V$ . Putting Equations (1.11) and (1.13) together, we conclude that

$$\rho_W((U_1 \otimes U_2) |\varphi\rangle) = U_1 \rho_W(|\varphi\rangle) U_1^* \quad (1.14)$$

for any unitaries  $U_1 \in \text{GL}(V)$  and  $U_2 \in \text{GL}(W)$ .

A system of  $N$  qubits can be partitioned into two subsystems in many ways. Each bipartition determines reduced density matrices for its state, as follows. Choose an integer partition  $n = r + s$  and an isomorphism  $(\mathbb{C}^2)^{\otimes n} \cong V \otimes W$ , where  $V = (\mathbb{C}^2)^{\otimes r}$  and  $W = (\mathbb{C}^2)^{\otimes s}$ . Any such grouping of tensor factors determines a function  $\rho_W : (\mathbb{C}^2)^{\otimes n} \rightarrow \text{End}(V)$ . By Equation (1.14), any function  $f : \text{End}(V) \rightarrow \mathbb{C}$  that is invariant under conjugation by unitary matrices pulls back to an invariant  $f \circ \rho_W$  of the LU group.

### 1.3.3 Von Neumann entropy

Let  $\lambda_1, \dots, \lambda_N$  denote the real nonnegative eigenvalues of an  $N \times N$  density matrix  $\rho$ . The eigenvalues of a matrix are fixed by conjugation, therefore any function of  $\lambda_1, \dots, \lambda_N$  can be pulled back to an invariant for the LU group. One example often used in the field is the **Von Neumann entropy**

$$E(\rho) = - \sum_{i=1}^N \lambda_i \ln \lambda_i$$

which can be used to define a measure of entanglement on quantum states  $|\varphi\rangle \in (\mathbb{C}^2)^{\otimes N}$ . Note that  $E(\rho)$  is nonnegative.

To find the density matrices that maximize  $E$ , one may maximize  $E$  over the subspace of Hermitian matrices  $\rho$  subject to the constraint

$$1 = \text{tr}(\rho) = \lambda_1 + \dots + \lambda_N. \quad (1.15)$$

By the method of Lagrange multipliers, the maximum occurs when  $\lambda_1 = \dots = \lambda_N$ . It follows that  $E$  is maximized when  $\rho = \frac{1}{N}I$ .

On the other hand, a density matrix  $\rho$  satisfies  $E(\rho) = 0$  if and only if it is the density matrix of some pure state  $|\varphi\rangle \in \mathbb{C}^N$ . It is clear that  $E(|\varphi\rangle\langle\varphi|) = 0$ . Conversely, suppose  $E(\rho) = 0$ . Then we have  $\lambda_i \ln \lambda_i = 0$  or  $\lambda_i \in \{0, 1\}$  for each  $1 \leq i \leq N$ . By Equation (1.15),  $\lambda_i = 1$  for exactly one index  $i$  and the other eigenvalues all equal 0. Therefore  $\rho$  is a Hermitian matrix with exactly one nonzero eigenvalue  $\lambda = 1$ . By the spectral theorem,  $\rho = |\varphi\rangle\langle\varphi|$  for some unit vector  $|\varphi\rangle \in \mathbb{C}^N$ .

**Example 1.2.** Let  $V = W = \mathbb{C}^2$  and let  $|\varphi\rangle \in V \otimes W$  be the Bell state (1.2). If  $E_{ij} = |i\rangle\langle j|$ , then the reduced density matrix of  $|\varphi\rangle$  is

$$\begin{aligned} \text{Tr}_W(|\varphi\rangle\langle\varphi|) &= \frac{1}{2}\text{Tr}_W(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \\ &= \frac{1}{2}\text{Tr}_W(E_{00} \otimes E_{00} + E_{01} \otimes E_{01} + E_{10} \otimes E_{10} + E_{11} \otimes E_{11}) \\ &= \frac{1}{2}(E_{00} + E_{11}) \end{aligned}$$

which maximizes the Von Neumann entropy. On the other hand, if  $|\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \in V \otimes W$  is any separable state, then the reduced density matrix

$$\text{Tr}_W(|\varphi\rangle\langle\varphi|) = \text{Tr}_W(|\varphi_1\rangle\langle\varphi_1| \otimes |\varphi_2\rangle\langle\varphi_2|) = |\varphi_1\rangle\langle\varphi_1|$$

is the density matrix of the pure state  $|\varphi_1\rangle$ , hence has zero Von Neumann entropy. ◇

## Chapter 2

### Four-qubit critical states

Note: This work was written with the candidate's Ph. D. advisor, has already been submitted for publication and appears on a preprint server.

Oeding, L. and Tan, I., *Four-qubit critical states*, (2024), arXiv:2410.08317.

## Abstract

Verstraete, Dehaene, and De Moor (2003) showed that SLOCC invariants provide entanglement monotones. We observe that many highly entangled or useful four-qubit states that appear in prior literature are stationary points of such entanglement measures. This motivates the search for more stationary points. We use the notion of critical points (in the sense of the Kempf-Ness theorem) together with Vinberg theory to reduce the complexity of the problem significantly. We solve the corresponding systems utilizing modern numerical nonlinear algebra methods and reduce the solutions by natural symmetries. This method produces an extended list of four-qubit stationary points, which includes all the critical states in the survey by Enríquez et al (2016).

## 2.1 Introduction

In quantum information theory one studies the information-theoretical tasks that can be accomplished by utilizing the quantum phenomenon of entanglement. Naturally, one might ask for ways to quantify entanglement. This brings us to the notion of an entanglement measure, defined below as a function with certain desirable properties.

Let  $\mathcal{H}_n := (\mathbb{C}^2)^{\otimes n}$  denote the Hilbert space of unnormalized  $n$ -qubit state vectors. The group  $\text{GL}_2(\mathbb{C})^{\times n}$  has a natural representation  $\rho : \text{GL}_2(\mathbb{C})^{\times n} \rightarrow \text{GL}(\mathcal{H}_n)$  defined by

$$\rho(g_1, \dots, g_n)v_1 \otimes \cdots \otimes v_n = g_1v_1 \otimes \cdots \otimes g_nv_n \quad (2.1)$$

where  $g_i \in \text{GL}_2(\mathbb{C})$  and  $v_i \in \mathbb{C}^2$ . Two important groups are the local unitary group  $\text{U}_2^{\times n}$  and the SLOCC group  $\text{SL}_2(\mathbb{C})^{\times n}$  (the group of stochastic local operations with classical communication). Both act on  $\mathcal{H}_n$  by restricting  $\rho$ . Furthermore, the symmetric group  $\mathfrak{S}_n$  acts on  $\mathcal{H}_n$  by permuting tensor factors:

$$\sigma.v_1 \otimes \cdots \otimes v_n = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad \sigma \in \mathfrak{S}_n.$$

A polynomial  $f : \mathcal{H}_n \rightarrow \mathbb{C}$  is **symmetric** if it is  $\mathfrak{S}_n$ -invariant. Notions of measurements of entanglement vary in the literature [46]. For our purposes, we define an **entanglement measure** to be a function  $E : \mathcal{H}_n \rightarrow [0, \infty)$  such that

1.  $E(\varphi) = 0$  whenever  $\varphi \in \mathcal{H}_n$  is separable (i.e. is a rank-one tensor),
2.  $E$  is invariant under the action of the local unitary group,
3.  $E$  is invariant under the action of the symmetric group, and
4.  $E$  is an entanglement monotone [57], i.e. it does not increase, on average, under local operations.

Let  $f \in \mathbb{C}[\mathcal{H}_n]$  be a homogeneous symmetric SLOCC invariant polynomial of degree  $m > 0$ . We claim that the following function is an entanglement measure.

$$E : \mathcal{H}_n \rightarrow [0, \infty), \quad E(\varphi) = |f(\varphi)|^{1/m}$$

Since  $f$  vanishes on rank-one tensors (this is true for any nonconstant SLOCC invariant), so does  $E$ . Note that any  $A \in \mathbb{U}_2$  has the form  $e^{it}A'$  for some  $t \in \mathbb{R}$  and  $A' \in \mathbb{S}\mathbb{U}_2$ . Then the second and third items hold because  $E$  is invariant under the SLOCC group action, multiplication by unit complex numbers, and the symmetric group action. The fourth item holds by [56, Theorem 2]. Examples of entanglement measures with this form include the Christensen-Wong  $n$ -tangle [61], and those arising from the invariant-comb approach of Osterloh and Siewert [51].

In past works [33, 29, 24, 50], researchers have been interested in finding highly entangled states by maximizing an entanglement measure  $E$  over the real manifold  $M \subset \mathcal{H}_n$  of unit-norm vectors. If  $\varphi \in M$  is such a maximizer, then  $\varphi$  is a stationary point of  $E|_M$  in the sense that the gradient vanishes. In this paper, we argue that it is worthwhile to understand the complete set of stationary points. We focus on the four-qubit case  $n = 4$ . The important observation is that many known highly entangled states turn out to be stationary points of  $E|_M$  for natural choices of  $f$ .

### 2.1.1 A selection of 4-qubit critical states

We now present some interesting highly-entangled states  $\varphi \in \mathcal{H}_4$ . All of these states are equivalent to ones that appear in the survey of Enríquez et al. [16]; see also [17]. Note that these states have the property that every single-qubit reduced density matrix of  $\varphi$  maximizes the von Neumann entropy. Such states are called “critical” (we define this term in Section 2.2.3). We will write the basis vectors of  $\mathcal{H}_n$  as “ket” vectors, so that  $|i_1 i_2 \dots i_n\rangle := e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}$  where each  $i_j \in \{0, 1\}$  and  $\{e_0, e_1\}$  is the basis for  $\mathbb{C}^2$ .

The first state of interest is the well-known GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle).$$

Next, the state

$$|MP\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$$

can be used to win pseudo-telepathy games such as the Mermin-Peres magic square game [35].

The Yeo-Chua state

$$|YC\rangle = \frac{1}{\sqrt{8}}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle),$$

can be used to teleport an arbitrary 2-qubit entangled state [62]. The Higuchi-Sudberry state

$$|HS\rangle = \frac{1}{\sqrt{6}}[|0011\rangle + |1100\rangle + \omega(|0101\rangle + |1010\rangle) + \omega^2(|0110\rangle + |1001\rangle)],$$

where  $\omega = e^{2\pi i/3}$ , was found to maximize the average von Neumann entropy across all splittings into bipartite subsystems [29]. The state

$$|BSSB\rangle = \frac{1}{\sqrt{12}}[|0110\rangle + |1011\rangle + i(|0010\rangle + |1111\rangle) + (1+i)(|0101\rangle + |1000\rangle)],$$

found via numerical search by Brown et al., is highly entangled considering the negative eigenvalues of all partial transposes [8]. The cluster states

$$\begin{aligned} |C_1\rangle &= \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle), \\ |C_2\rangle &= \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle), \\ |C_3\rangle &= \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle - |1111\rangle), \end{aligned}$$

can be used in one-way quantum computing; this has in fact been experimentally realized [60]. The cluster states were also shown to be the only states that maximize the Renyi  $\alpha$ -entropy for all  $\alpha \geq 2$  [24]. Note that  $|C_1\rangle$  and  $|C_2\rangle$  are related by the transposition  $(1, 3) \in \mathfrak{S}_4$  swapping the first and third qubits, while  $|C_1\rangle$  and  $|C_3\rangle$  are related by the transposition  $(2, 3) \in \mathfrak{S}_4$  swapping the second and third qubits. Hence the cluster states have the same value on  $|f\rangle$  for any symmetric  $f : \mathcal{H}_4 \rightarrow \mathbb{R}$ . The hyperdeterminant state

$$|HD\rangle = \frac{1}{\sqrt{6}}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle + \sqrt{2}|1111\rangle),$$

was claimed by Osterloh and Siewert [50] to maximize the 4-qubit hyperdeterminant. Also found in [50] is the state

$$|OS\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |1100\rangle + |1111\rangle)$$

which has nonzero expectation for the so-called four-qubit sixth-order filter. Finally, Gour and Wallach [24] show that the state

$$|L\rangle = \frac{1}{4}[(1 + \omega)(|0000\rangle + |1111\rangle) + (1 - \omega)(|0011\rangle + |1100\rangle) + \omega^2(|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle)]$$

maximizes the  $\alpha$ -Tsallis entropy for all  $\alpha > 2$ , while the state

$$|M\rangle = \frac{1}{\sqrt{18}}[(1 + \sqrt{3}i)(|0000\rangle + |1111\rangle) + (-1 + \sqrt{3}i)(|0011\rangle + |1100\rangle) + |0101\rangle + |1010\rangle]$$

maximizes the  $\alpha$ -Tsallis entropy for all  $0 < \alpha < 2$ . In Section 2.3.4 we introduce an algorithm for computing normal forms of critical states. Applying this algorithm, we find that, up to the joint action of the local unitary and symmetric groups,  $|YC\rangle = |C_1\rangle = |OS\rangle$ ,  $|HS\rangle = |M\rangle$ , and  $|HD\rangle = |L\rangle$ .

	$ GHZ\rangle$	$ MP\rangle$	$ C_1\rangle$	$ HS\rangle$	$ HD\rangle$	$ BSSB\rangle$
$ \mathcal{F}_1 $	6	6	0	0	0	0
$ \mathcal{F}_3 $	9	6	0	$2\bar{6}$	$2\bar{6}$	0
$ \mathcal{F}_4 $	16.5	6	2.5	0	0	1.875
$ \mathcal{F}_6 $	64.125	6	1.875	$\sim 0.790$	$\sim 3.01$	0
$ \text{Hdet} $	0	0	0	0	$\sim 5.08\text{E-}5$	$\sim 1.53\text{E-}5$

Table 2.1: Four-qubit stationary points

The algebra of symmetric SLOCC invariant polynomials  $f \in \mathbb{C}[\mathcal{H}_4]$  is known to be generated by invariants  $\mathcal{F}_1$ ,  $\mathcal{F}_3$ ,  $\mathcal{F}_4$ , and  $\mathcal{F}_6$  of degrees 2, 6, 8, and 12 respectively. The generating set is not unique; we choose generators from [26] which are written out explicitly in Section 2.3.6. Let  $M$  be the set of unit-norm vectors in  $\mathcal{H}_4$ . Remarkably, we find that for each of the states  $\varphi \in M$  mentioned above, either  $f(\varphi) = 0$  or  $\varphi$  is a nonvanishing stationary point of  $|f|_M|^{1/m}$  where  $f = \mathcal{F}_k$  for some  $k = \{1, 3, 4, 6\}$  or  $f = \text{HDet}$  is the hyperdeterminant. Table 2.1 lists the values attained on  $|f|$  for each distinct state up to the joint action of the local unitary group and the symmetric group  $\mathfrak{S}_4$ . A tilde in front of a number indicates that the number is rounded to three significant figures.

### 2.1.2 Organization

In Section 2.2 we define the notions of stationary and critical points. An important fact is that, in our setting, nonvanishing stationary points are critical points. In Section 2.3 we bring in results from Vinberg's theory of  $\theta$ -groups. We define the Cartan subspace and Weyl group for  $\mathcal{H}_4$ , give a simple algorithm that finds equivalent representatives in the Cartan subspace for critical points, then give expressions for the invariants in Table 2.1 as well as their restrictions to the Cartan subspace. In Section 2.4 we compute nonvanishing stationary points. First we explain how to set up of systems of polynomial equations for the task. In the easiest case, we are able to describe, with a mathematical proof, all of the stationary points of  $|\mathcal{F}_1|$ . We then find stationary points of  $|\mathcal{F}_3|$  and  $|\mathcal{F}_4|$  using techniques from numerical algebraic geometry. The states in Table 2.1 are among the solutions found.

### 2.1.3 Notation and conventions

Let  $V$  be an inner product space over a field  $\mathbb{K}$ . We denote the inner product with angle brackets:

$$V \times V \rightarrow \mathbb{R}, \quad (v, w) \mapsto \langle v, w \rangle.$$

If  $\mathbb{K} = \mathbb{C}$ , then  $\langle v, w \rangle$  is the standard Hermitian inner product. If  $\mathbb{K} = \mathbb{R}$ , then  $\langle v, w \rangle$  is the standard Euclidean inner product. Since the Hermitian inner product restricts to the Euclidean inner product on real vectors, there is no conflict of notation. The inner product endows  $V$  with a norm  $\|v\| = \sqrt{\langle v, v \rangle}$ .

We shall frequently identify a complex vector space  $\mathbb{C}^N$  with the real vector space  $\mathbb{R}^{2N}$  via

$$\varphi = (z_1, \dots, z_N) \in \mathbb{C}^N \mapsto \varphi' = (x_1, y_1, \dots, x_N, y_N) \in \mathbb{R}^{2N} \quad (2.2)$$

where  $z_i = x_i + iy_i$  for each  $1 \leq i \leq N$ . We denote by  $\mathbb{S}^{k-1}$  the real smooth manifold consisting of unit-norm vectors in a  $k$ -dimensional real vector space. The norm on  $\mathbb{R}^{2N}$  induced by the map  $\varphi \mapsto \varphi'$  agrees with the usual norm. Hence we understand that  $\mathbb{S}^{2N-1} \subset (\mathbb{C}^d)^{\otimes n}$ , where  $N = d^n$ , is the set of unit-norm vectors in  $(\mathbb{C}^d)^{\otimes n}$  and that this is equivalent to the set of unit-norm vectors in  $\mathbb{R}^{2N}$  via the identifications (as real normed spaces)  $\mathcal{H}_n = \mathbb{C}^N = \mathbb{R}^{2N}$ .

We write  $\mathrm{GL}_d(\mathbb{K})$ ,  $\mathrm{SL}_d(\mathbb{K})$ , and  $\mathrm{SO}_d(\mathbb{K})$  for the general linear group, special linear group, and special orthogonal linear group respectively, each consisting of linear maps  $\mathbb{K}^d \rightarrow \mathbb{K}^d$ . It is assumed that  $\mathbb{K} = \mathbb{C}$  when the field is unspecified so that, for example,  $\mathrm{GL}_d = \mathrm{GL}_d(\mathbb{C})$ . Subgroups of  $\mathrm{GL}_d^{\times n}$  act on the space of tensors  $(\mathbb{C}^d)^{\otimes n}$  by restricting the obvious generalization of the representation  $\rho : \mathrm{GL}_2(\mathbb{C})^{\times n} \rightarrow \mathrm{GL}(\mathcal{H}_n)$  given in Equation (2.1).

## 2.2 Stationary points and critical points

### 2.2.1 Stationary points

We recall the definition of a stationary point of a function on a manifold, then discuss some basic facts about such points.

**Definition 2.1.** Let  $M$  be a real smooth manifold of dimension  $k$  and  $g : M \rightarrow \mathbb{R}$  a smooth map. We say that  $x \in M$  is a **stationary point** of  $g$  if the gradient of  $g \circ \alpha^{-1}$  vanishes at  $x$ , where  $\alpha : U \rightarrow \mathbb{R}^k$  is a chart on an open subset  $U \subset M$  containing  $x$ . ■

Note that  $x \in M$  is a stationary point of  $g : M \rightarrow \mathbb{R}$  if and only if  $\frac{d}{dt}\big|_{t=0} g \circ \gamma(t) = 0$  for any smooth curve  $\gamma : \mathbb{R} \rightarrow M$  such that  $\gamma(0) = x$ . Supposing that  $\alpha$  is centered at  $x$ , this follows from the equation

$$\frac{d}{dt}\bigg|_{t=0} g \circ \gamma(t) = \frac{d}{dt}\bigg|_{t=0} g \circ \alpha^{-1} \circ \alpha \circ \gamma(t) = \langle \nabla(g \circ \alpha^{-1})(0), (\alpha \circ \gamma)'(0) \rangle. \quad (2.3)$$

If the gradient of  $g \circ \alpha^{-1}$  vanishes, so does the whole expression. Conversely, if (2.3) vanishes for every choice of  $\gamma$ , then  $\nabla(g \circ \alpha^{-1}) = 0$  by the nondegeneracy of the inner product.

Suppose  $f : M \rightarrow \mathbb{C}$  is complex-valued. Proposition 2.2 tells us that the nonvanishing stationary points of  $|f|$  are the same as those of  $|f|^p$  for any nonzero  $p \in \mathbb{R}$ . It will be useful for us to take  $p = 2$ , since  $|f|^2 = f_1^2 + f_2^2$  where  $f_1$  and  $f_2$  are real-valued functions such that  $f = f_1 + if_2$ .

**Proposition 2.2.** Let  $M$  and  $g : M \rightarrow \mathbb{R}$  be as in Definition 2.1. Suppose  $x \in M$  such that  $g(x) \neq 0$ . Then  $x$  is a critical point of  $g$  if and only if  $x$  is a critical point of  $g^p$ , where  $0 \neq p \in \mathbb{R}$ .

*Proof.* Let  $\alpha : U \rightarrow \mathbb{R}^k$  be a chart centered at  $x$  and let  $F = g \circ \alpha^{-1}$ . Then  $\nabla(F^p) = pF^{p-1}\nabla F$ . If  $g(x) \neq 0$ , then  $\nabla(F^p)(0) = 0$  if and only if  $\nabla F(0) = 0$ . □

The set of stationary points is partitioned into equivalence classes by a natural group action, given in Proposition 2.3. Since we can act on representatives, it suffices to find stationary points up to equivalence.

**Proposition 2.3.** *Let  $M$  and  $g : M \rightarrow \mathbb{R}$  be as in Definition 2.1. Suppose  $\Phi : M \rightarrow M$  is a smooth bijection such that  $g = g \circ \Phi$ . Then  $x \in M$  is a stationary point of  $g$  if and only if  $\Phi(x)$  is a stationary point of  $g$ .*

*Proof.* Let  $x \in M$ . If  $\alpha : U \rightarrow \mathbb{R}^k$  is a chart centered at  $x \in M$  then  $\alpha \circ \Phi^{-1} : \Phi(U) \rightarrow \mathbb{R}^k$  is a chart centered at  $\Phi(x)$ . Since  $g = g \circ \Phi$ , we have  $\nabla(g \circ \alpha^{-1})(0) = \nabla(g \circ \Phi \circ \alpha^{-1})(0)$ . Hence  $x = \alpha^{-1}(0)$  is stationary if and only if  $\Phi(x) = \Phi \circ \alpha^{-1}(0)$  stationary.  $\square$

## 2.2.2 Complex and real Lie groups

We explain how to interpret  $\mathrm{SL}_N(\mathbb{C})$  as a real Lie group; the facts presented here are standard.

Recall the identification  $\mathbb{C}^N = \mathbb{R}^{2N}$  via the map  $\varphi \mapsto \varphi'$  given in (2.2). While it is true that  $\langle \varphi, \varphi \rangle = \langle \varphi', \varphi' \rangle$  (so the induced norm is the usual norm), it is not generally true that  $\langle \varphi, \psi \rangle = \langle \varphi', \psi' \rangle$  for  $\varphi, \psi \in \mathbb{C}^N$ . The correct formula reads

$$\langle \varphi, \psi \rangle = \langle \varphi', \psi' \rangle + i \langle J^{\oplus N} \varphi', \psi' \rangle, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.4)$$

Similarly, matrices in  $\mathbb{C}^{N \times N}$  embed into  $\mathbb{R}^{2N \times 2N}$  by the map

$$M = \begin{pmatrix} z_{11} & \dots & z_{1N} \\ \vdots & & \vdots \\ z_{N1} & \dots & z_{NN} \end{pmatrix} \mapsto M' = \begin{pmatrix} U(z_{11}) & \dots & U(z_{1N}) \\ \vdots & & \vdots \\ U(z_{N1}) & \dots & U(z_{NN}) \end{pmatrix},$$

where  $U(z_{ij}) = \begin{pmatrix} x_{ij} & -y_{ij} \\ y_{ij} & x_{ij} \end{pmatrix}$  and  $x_{ij}, y_{ij}$  are the real and imaginary parts of  $z_{ij} = x_{ij} + iy_{ij}$ .

This embedding respects our identification of  $\mathbb{C}^N$  with  $\mathbb{R}^{2N}$ , i.e.  $(M\varphi)' = M'\varphi'$ . Furthermore, the

embedding respects the matrix exponential, i.e. it makes the following diagram commute.

$$\begin{array}{ccc}
 \mathfrak{sl}_N^{\mathbb{C}} & \xrightarrow{\quad} & \mathfrak{sl}_{2N}^{\mathbb{R}} \\
 \exp \downarrow & & \downarrow \exp \\
 \mathrm{SL}_N(\mathbb{C}) & \xrightarrow{\quad} & \mathrm{SL}_{2N}(\mathbb{R})
 \end{array}$$

### 2.2.3 The critical set

As in the previous section, make the identification  $(\mathbb{C}^d)^{\otimes n} = \mathbb{C}^N = \mathbb{R}^{2N}$ , where  $N = d^n$ , via the map  $\varphi \mapsto \varphi'$  given in (2.2). The embedding  $\mathrm{SL}_N(\mathbb{C}) \mapsto \mathrm{SL}_{2N}(\mathbb{R})$  restricts to an embedding  $\rho: \mathrm{SL}_d(\mathbb{C})^{\times n} \rightarrow \mathrm{SL}_{2N}(\mathbb{R})$ . Let  $G_{\mathbb{C}} = \mathrm{SL}_d(\mathbb{C})^{\otimes n}$  and let  $G_{\mathbb{R}} \subset \mathrm{GL}(\mathbb{R}^N)$  be the corresponding real Lie group. We say that  $\varphi \in (\mathbb{C}^d)^{\otimes n}$  is **critical** if  $\langle X\varphi, \varphi \rangle = 0$  for all  $X \in \mathrm{Lie}(G_{\mathbb{C}})$ . It is equivalent to say that  $\langle X'\varphi', \varphi' \rangle = 0$  for all  $X' \in \mathrm{Lie}(G_{\mathbb{R}})$ . To see this, specialize equation (2.4):

$$\langle X\varphi, \varphi \rangle = \langle X'\varphi', \varphi' \rangle + i\langle J^{\oplus N} X'\varphi', \varphi' \rangle.$$

Observing that  $J^{\oplus N} X' = (iX)'$  if and only if  $X \in \mathrm{Lie}(G_{\mathbb{C}})$ , the claim follows.

**Proposition 2.4** (Kempf-Ness). *The point  $\varphi \in \mathcal{H}_n$  is critical if and only if  $\|g.\varphi\| \geq \|\varphi\|$  for all  $g \in \mathrm{SL}_2^{\times n}$ . If  $\varphi$  is critical and  $g \in \mathrm{SL}_2^{\times n}$  is such that  $\|g.\varphi\| = \|\varphi\|$ , then there exists  $h \in \mathrm{SU}_2^{\times n}$  such that  $g.\varphi = h.\varphi$ .*

*Proof.* See [25, Theorem 2]. □

In other words, Proposition 2.4 states that  $\varphi$  is critical if and only if it minimizes the norm on its  $\mathrm{SL}_d^{\times n}$ -orbit. If  $\psi$  is another minimizer in the  $\mathrm{SL}_d^{\times n}$ -orbit, then  $\psi$  is in fact in the  $\mathrm{SU}_d^{\times n}$ -orbit of  $\varphi$ .

**Proposition 2.5.** *The point  $\varphi \in \mathcal{H}_n$  is critical if and only if all single-qubit reduced density matrices of  $\varphi$  are proportional to the identity.*

*Proof.* See [25, Theorem 3]. □

Note that the von Neumann entropy of a reduced density matrix [46] is maximized if and only if the matrix is proportional to the identity. Together with Proposition 2.5, this justifies a claim made at the beginning of Section 2.1.1.

Our primary interest in critical points is due to Proposition 2.6. A consequence is that, to find the nonvanishing stationary points of  $|f|$ , it suffices to search over the set of critical points.

**Proposition 2.6.** *Let  $f : (\mathbb{C}^d)^{\otimes n} \rightarrow (\mathbb{C}^d)^{\otimes n}$  be a homogeneous  $\mathrm{SL}_d^{\times n}$ -invariant of degree  $m > 0$  and let  $N = d^n$ . Suppose  $\varphi \in \mathbb{S}^{2N-1}$  is a nonvanishing stationary point of  $|f(x)|$  restricted to  $\mathbb{S}^{2N-1}$ . Then  $\varphi$  is critical.*

*Proof.* Consider  $\varphi$  to be an element of the real vector space  $\mathbb{R}^{2N}$ . We prove the contrapositive. Suppose  $\varphi$  is not critical. Then there exists  $X \in \mathrm{Lie}(G_{\mathbb{R}})$  such that  $\langle X\varphi, \varphi \rangle \neq 0$ . Set  $\gamma(t) = \exp(tX)\varphi$ . We have

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \|\gamma(t)\|^2 &= \left. \frac{d}{dt} \right|_{t=0} \langle \exp(tX)\varphi, \exp(tX)\varphi \rangle \\ &= \langle \exp(tX)X\varphi, \exp(tX)\varphi \rangle + \langle \exp(tX)\varphi, \exp(tX)X\varphi \rangle \Big|_{t=0} \\ &= 2\langle \exp(tX)X\varphi, \exp(tX)\varphi \rangle \Big|_{t=0} \\ &= 2\langle X\varphi, \varphi \rangle. \end{aligned}$$

If  $f(\varphi) \neq 0$ , then

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} |f(\gamma(t)/\|\gamma(t)\||) &= \left. \frac{d}{dt} \right|_{t=0} \|\gamma(t)\|^{-m} |f(\varphi)| \\ &= |f(\varphi)| \left. \frac{d}{dt} \right|_{t=0} (\|\gamma(t)\|^2)^{-m/2} \\ &= |f(\varphi)| (-m\|\varphi\|^{-m-2} \langle X\varphi, \varphi \rangle) \neq 0. \end{aligned}$$

Therefore  $\varphi$  is not stationary. □

## 2.3 Vinberg theory

### 2.3.1 A graded Lie algebra

The results we need from Vinberg's theory rely on the fact that the SLOCC module  $\mathcal{H} := \mathcal{H}_4$  embeds into the  $\mathbb{Z}_2$ -graded Lie algebra  $\mathfrak{so}_8 \cong \mathfrak{so}_4^{\times 2} \oplus \mathcal{H}$ . We refer the reader to [59] for details. We now describe a construction of this embedding following Chterental and Djokovic [10].

Each tensor  $\varphi \in \mathcal{H}$  corresponds to the matrix  $\tilde{\varphi} \in \mathbb{C}^{4 \times 4}$  of a linear map  $(\mathbb{C}^2 \otimes \mathbb{C}^2)^* \rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$  associated to  $\varphi$ . Explicitly, the correspondence is

$$\varphi = \sum_{i,j,k,l=0}^1 a_{ijkl} |ijkl\rangle \mapsto \tilde{\varphi} = \begin{pmatrix} a_{0000} & a_{0001} & a_{0010} & a_{0011} \\ a_{0100} & a_{0101} & a_{0110} & a_{0111} \\ a_{1000} & a_{1001} & a_{1010} & a_{1011} \\ a_{1100} & a_{1101} & a_{1110} & a_{1111} \end{pmatrix}. \quad (2.5)$$

Let  $\kappa : \mathrm{SL}_4 \times \mathrm{SL}_4 \rightarrow \mathrm{GL}(\mathbb{C}^{4 \times 4})$  denote the representation defined by  $\kappa(A, B)(\tilde{\varphi}) = A\tilde{\varphi}B$ . Then the induced  $\mathrm{SL}_2^{\times 4}$ -action on  $\mathbb{C}^{4 \times 4}$  factors through  $\mathrm{SL}_4 \times \mathrm{SL}_4$ , having the form

$$\mathrm{SL}_2^{\times 4} \longrightarrow \mathrm{SL}_4 \times \mathrm{SL}_4 \xrightarrow{\kappa} \mathrm{GL}(\mathbb{C}^{4 \times 4})$$

where the first map is  $(A_1, A_2, A_3, A_4) \mapsto (A_1 \otimes A_2, (A_3 \otimes A_4)^\top)$ . There exists a double cover  $\mathrm{SL}_2 \times \mathrm{SL}_2 \rightarrow \mathrm{SO}_4$  defined using conjugation  $(A_1, A_2) \mapsto T(A_1 \otimes A_2)T^*$  by the unitary matrix

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \end{pmatrix}.$$

Applying the change of coordinates  $\tilde{\varphi} \mapsto R_\varphi := T\tilde{\varphi}T^*$ , the induced  $SL_2^{\times 4}$ -action on  $\mathbb{C}^{4 \times 4}$  now factors through  $SO_4 \times SO_4$ , having the form

$$SL_2^{\times 4} \longrightarrow SO_4 \times SO_4 \xrightarrow{\kappa} GL(\mathbb{C}^{4 \times 4})$$

where the first map is  $(A_1, A_2, A_3, A_4) \mapsto (T(A_1 \otimes A_2)T^*, T(A_3 \otimes A_4)^\top T^*)$ . Conjugation by the diagonal matrix  $I_4 \oplus (-I_4)$  decomposes the Lie algebra  $\mathfrak{so}_8$  of skew-symmetric matrices into a direct sum of eigenspaces consisting of matrices with the following block formats:

$$\mathfrak{so}_8 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in \mathfrak{so}_4 \right\} \oplus \left\{ \begin{pmatrix} 0 & R_\varphi \\ -R_\varphi^\top & 0 \end{pmatrix} : \varphi \in \mathcal{H} \right\}.$$

This establishes the decomposition  $\mathfrak{so}_8 \cong \mathfrak{so}_4^{\times 2} \oplus \mathcal{H}$ . A calculation shows

$$\left[ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} 0 & R_\varphi \\ -R_\varphi^\top & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & AR_\varphi + R_\varphi B^\top \\ -(AR_\varphi + R_\varphi B^\top)^\top & 0 \end{pmatrix}$$

so the Lie algebra representation  $\mathfrak{so}_4^{\times 2} \rightarrow \text{Aut}(\mathcal{H})$  obtained from the adjoint action corresponds to the natural Lie group representation  $SO_4^{\times 2} \rightarrow GL(\mathcal{H})$ .

### 2.3.2 The Cartan subspace

The semisimple elements in  $\mathbb{C}^{4 \times 4} \subset \mathfrak{so}_8$  are those that are diagonalizable by the action of  $SO_4 \times SO_4$ . We choose a so-called **Cartan subspace** to be the 4-dimensional subspace of diagonal matrices in the new coordinates where the  $SL_2^{\times 4}$ -action factors through  $SO_4^{\times 2}$ . Translating back to the original coordinates, the Cartan  $\mathfrak{a} \subset \mathcal{H}$  is the complex span of

$$\begin{aligned} u_1 &= \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle), & u_2 &= \frac{1}{2}(|0000\rangle - |0011\rangle - |1100\rangle + |1111\rangle), \\ u_3 &= \frac{1}{2}(|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle), & u_4 &= \frac{1}{2}(|0101\rangle - |0110\rangle - |1001\rangle + |1010\rangle). \end{aligned}$$

We will identify  $\mathfrak{a}$  with the set of vectors  $z \in \mathbb{C}^4$ , writing

$$(z_1, z_2, z_3, z_4) := z_1 u_1 + z_2 u_2 + z_3 u_3 + z_4 u_4.$$

A calculation shows that

$$\|z_1 u_1 + z_2 u_2 + z_3 u_3 + z_4 u_4\|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2,$$

so the induced norm on  $\mathbb{C}^4$  is the usual Hermitian norm.

The following result, together with Propositions 2.3 and 2.6, allows us to further restrict the search for stationary points to the linear subspace  $\mathfrak{a}$ .

**Proposition 2.7.** *The set of critical points in  $\mathcal{H}$  is the  $\mathrm{SU}_2^{\times 4}$ -orbit of  $\mathfrak{a}$ .*

*Proof.* This is a special case of [59, Proposition 3.75]. □

### 2.3.3 The Weyl group

The normalizer and centralizer of  $\mathfrak{a}$  are defined as the subgroups

$$\begin{aligned} N(\mathfrak{a}) &:= \{g \in \mathrm{SL}_2^{\times 4} : g \cdot \mathfrak{a} = \mathfrak{a}\} \quad \text{and} \\ C(\mathfrak{a}) &:= \{g \in \mathrm{SL}_2^{\times 4} : g \cdot x = x \text{ for all } x \in \mathfrak{a}\} \end{aligned}$$

respectively. The **Weyl group** of  $\mathfrak{a}$  is the quotient  $W := N(\mathfrak{a})/C(\mathfrak{a})$ . The group  $W$  acts on the Cartan subspace  $\mathfrak{a}$  like the set of linear maps

$$(z_1, z_2, z_3, z_4) \mapsto (\varepsilon_1 z_{\pi(1)}, \varepsilon_2 z_{\pi(2)}, \varepsilon_3 z_{\pi(3)}, \varepsilon_4 z_{\pi(4)}), \tag{2.6}$$

such that  $\pi \in \mathfrak{S}_4$ ,  $\varepsilon_i = \pm 1$ , and  $\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 = 1$  (see [59, Exercise 3, p. 179]).

**Proposition 2.8.** *The restriction map  $f \mapsto f|_{\mathfrak{a}}$  for  $f \in \mathbb{C}[\mathcal{H}]$  induces an isomorphism of algebras  $\mathbb{C}[\mathcal{H}]^{\mathrm{SL}_2^{\times 4}} \cong \mathbb{C}[z_1, z_2, z_3, z_4]^W$ .*

*Proof.* This is a special case of [59, Theorem 3.62]. □

### 2.3.4 Critical states on the Cartan

Given a critical point  $\varphi \in \mathcal{H}$ , we explain how to find a point  $\varphi' \in \mathfrak{a}$  in the Cartan subspace equivalent to  $\varphi$  under the joint action of the local unitary group  $U_2^{\times 4}$  and the symmetric group  $\mathfrak{S}_4$ .

As discussed in Section 2.3.1, there is an equivariant isomorphism

$$\mathcal{H} \rightarrow \mathbb{C}^{4 \times 4}, \quad \varphi \mapsto R_\varphi$$

where the representation  $SL_2^{\times 4} \rightarrow GL(\mathbb{C}^{4 \times 4})$  factors through  $SO_4^{\times 2}$ . The group  $SO_4^{\times 2}$  acts by left and right multiplication on the space of matrices  $\mathbb{C}^{4 \times 4}$ . Let  $\{e_1, e_2, e_3, e_4\}$  be the standard basis for  $\mathbb{C}^4$  and  $\{E_{ij} : 1 \leq i, j \leq 4\}$  be the basis of operators  $E_{ij}e_k = \delta_{jk}e_i$  for  $\mathbb{C}^{4 \times 4}$ . The basis vectors  $u_1, \dots, u_4$  for  $\mathfrak{a}$  correspond to the basis operators  $E_{11}, \dots, E_{44}$  which span the subspace of diagonal matrices. Thus, by Proposition 2.7, every critical  $\varphi \in \mathcal{H}$  corresponds to a matrix  $R_\varphi$  that is  $SO_4^{\times 2}$ -equivalent to a diagonal matrix. Define  $\tau : \mathbb{C}^{4 \times 4} \rightarrow \mathbb{C}^{4 \times 4}$  by  $\tau(R_\varphi) = R_\varphi R_\varphi^\top$ . The set of eigenvalues of  $\tau(R_\varphi)$  is invariant under  $SO_2^{\times 4}$  since

$$\tau(AR_\varphi B^\top) = AR_\varphi R_\varphi^\top A^\top = A\tau(R_\varphi)A^\top$$

for all  $(A, B) \in SO_4^{\times 2}$ . Observing that

$$\tau(\lambda_1 E_{11} + \lambda_2 E_{22} + \lambda_3 E_{33} + \lambda_4 E_{44}) = \lambda_1^2 E_{11} + \lambda_2^2 E_{22} + \lambda_3^2 E_{33} + \lambda_4^2 E_{44},$$

we obtain an algorithm for computing  $\varphi'$  in two steps.

1. Compute the eigenvalues  $\mu_1, \mu_2, \mu_3, \mu_4$  of  $\tau(R_\varphi)$ .
2. Set  $\varphi' = \sqrt{\mu_1}u_1 + \sqrt{\mu_2}u_2 + \sqrt{\mu_3}u_3 + \sqrt{\mu_4}u_4$ .

The  $\mathfrak{S}_4$ -representation  $\mathfrak{S}_4 \rightarrow \text{GL}(\mathcal{H})$  restricts to a subrepresentation  $\pi : \mathfrak{S}_4 \rightarrow \text{GL}(\mathfrak{a})$  on  $\mathfrak{a} \cong \mathbb{C}^4$ . Let  $\sigma_i \in \mathfrak{S}_4$  denote the transposition  $(i, i + 1)$ . The symmetric group  $\mathfrak{S}_4$  is generated by  $\sigma_1, \sigma_2$ , and  $\sigma_3$ . Written as matrices, these have the form (also noted in [26])

$$\pi(\sigma_1) = \pi(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \pi(\sigma_2) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}. \quad (2.7)$$

Using the actions of  $\pi(\sigma_1)$  (2.7) and the Weyl group (2.6) we see that neither the order of the eigenvalues  $\mu_1, \dots, \mu_4$  nor the choice of square roots are important in choosing  $\varphi'$ .

Using the algorithm described above and further simplifying by the actions of the Weyl group, the symmetric group, and the circle group

$$\mathbb{T} := \{e^{it} : t \in \mathbb{R}\}$$

(which acts by scalar multiplication), we find that the states introduced in Section 2.1.1 have the following representatives in  $\mathfrak{a}$ . For ease of notation, we do not normalize the vectors.

$$\begin{aligned} |GHZ\rangle &\cong (1, 1, 0, 0), & |MP\rangle &\cong (1, 0, 0, 0), \\ |YC\rangle &\cong |C_1\rangle \cong |OS\rangle \cong (1, i, 0, 0), & |HS\rangle &\cong |M\rangle \cong (\sqrt{3}, i, i, i), \\ |HD\rangle &\cong |L\rangle \cong (1, e^{\pi i/3}, e^{2\pi i/3}, 0), & |BSSB\rangle &\cong (1, i, e^{-\pi i/4}, e^{\pi i/4}). \end{aligned}$$

It was previously observed [16] that  $|HS\rangle$  and  $|M\rangle$  are equivalent. The fact that  $|HD\rangle$  and  $|L\rangle$  are equivalent is no surprise either, since it was claimed (likely based on numerical computations) in [50] that  $|HD\rangle$  is the unique global maximizer of the norm of the four-qubit hyperdeterminant, while the same thing was conjectured about  $|L\rangle$  in [26]. This conjecture was later proved by Chen and Đoković [9].

### 2.3.5 SLOCC invariants

The algebra  $\mathbb{C}[z_1, z_2, z_3, z_4]^W$  is freely generated by the invariants

$$\mathcal{E}_0(z) = z_1 z_2 z_3 z_4 \quad \text{and} \quad \mathcal{E}_i(z) = z_1^{2i} + z_2^{2i} + z_3^{2i} + z_4^{2i} \text{ for } 1 \leq i \leq 3.$$

Identifying  $\mathcal{H}$  with the space of matrices  $\tilde{\varphi} \in \mathbb{C}^{4 \times 4}$  via the map (2.5), we define the following polynomials (which appear in [24]):

$$\mathcal{G}_0(\tilde{\varphi}) = \det(\tilde{\varphi}) \quad \text{and} \quad \mathcal{G}_i(\tilde{\varphi}) = \text{tr}((\tilde{\varphi}(J \otimes J)\tilde{\varphi}^\top(J \otimes J))^i) \text{ for } 1 \leq i \leq 3$$

with  $J$  as in (2.4). The polynomials  $\mathcal{G}_i$  are  $\text{SL}_2^{\times 4}$ -invariant due to the fact that  $\text{SL}_2$  preserves the symplectic bilinear form associated to  $J$ . These  $\text{SL}_2^{\times 4}$ -invariants correspond to the  $W$ -invariants via the restriction map, i.e.  $\mathcal{G}_i|_{\mathfrak{a}} = \mathcal{E}_i$  for all  $0 \leq i \leq 4$ . Note that  $\mathcal{G}_1$  is the unique  $\text{SL}_2^{\times 4}$ -invariant in degree 2 and  $|\mathcal{G}_1|^2$  coincides with the Christensen-Wong 4-tangle [61].

### 2.3.6 Symmetric invariants

We are particularly interested in those SLOCC invariants which are also symmetric. The algebra consisting of such polynomials is known [26] to be generated by  $\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_6$  whose restrictions have the form

$$\mathcal{F}_k|_{\mathfrak{a}}(z) = \sum_{i < j} (z_i - z_j)^{2k} + \sum_{i < j} (z_i + z_j)^{2k}, \text{ for } k \in \{1, 3, 4, 6\}.$$

Written in terms of the polynomials  $\mathcal{E}_i$ , these read

$$\mathcal{F}_1|_{\mathfrak{a}} = 6\mathcal{E}_1,$$

$$\mathcal{F}_3|_{\mathfrak{a}} = 30\mathcal{E}_1\mathcal{E}_2 - 24\mathcal{E}_3,$$

$$\mathcal{F}_4|_{\mathfrak{a}} = -20\mathcal{E}_1^4 + 120\mathcal{E}_1^2\mathcal{E}_2 + 480\mathcal{E}_0^2 + 10\mathcal{E}_2^2 - 104\mathcal{E}_1\mathcal{E}_3,$$

and

$$\begin{aligned}\mathcal{F}_6|_{\mathfrak{a}} = & -148\mathcal{E}_1^6 + 565\mathcal{E}_1^4\mathcal{E}_2 + 5460\mathcal{E}_0^2\mathcal{E}_1^2 + 540\mathcal{E}_1^2\mathcal{E}_2^2 \\ & - 570\mathcal{E}_1^3\mathcal{E}_3 + 2160\mathcal{E}_0^2\mathcal{E}_2 - 15\mathcal{E}_2^3 - 610\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3 + 244\mathcal{E}_3^2.\end{aligned}$$

Since the restriction map is an isomorphism (Proposition 2.8), one obtains corresponding expressions for the generators  $\mathcal{F}_k$  by substituting  $\mathcal{E}_i = \mathcal{G}_i$  for  $0 \leq i \leq 3$ .

### 2.3.6.1 Hyperdeterminant

Miyake [42] gives an expression for the four-qubit hyperdeterminant restricted to  $\mathfrak{a}$ . In our basis, it reads

$$\text{Hdet}|_{\mathfrak{a}}(z) = \prod_{i < j} (z_i^2 - z_j^2)^2.$$

In terms of the symmetric generators  $\mathcal{G}_1 = \frac{1}{6}\mathcal{F}_1$ ,  $\mathcal{F}_3$ ,  $\mathcal{F}_4$ , and  $\mathcal{F}_6$ , the hyperdeterminant is

$$\begin{aligned}\text{Hdet} = & 314928000^{-1}(-23794560\mathcal{G}_1^{12} + 14450400\mathcal{G}_1^9\mathcal{F}_3 - 6828300\mathcal{G}_1^8\mathcal{F}_4 - 2211120\mathcal{G}_1^6\mathcal{F}_3^2 \\ & + 2043360\mathcal{G}_1^5\mathcal{F}_3\mathcal{F}_4 + 563760\mathcal{G}_1^6\mathcal{F}_6 + 5376\mathcal{G}_1^3\mathcal{F}_3^3 - 484380\mathcal{G}_1^4\mathcal{F}_4^2 + 6552\mathcal{G}_1^2\mathcal{F}_3^2\mathcal{F}_4 \\ & - 172800\mathcal{G}_1^3\mathcal{F}_3\mathcal{F}_6 - 40\mathcal{F}_3^4 - 5832\mathcal{G}_1\mathcal{F}_3\mathcal{F}_4^2 + 81000\mathcal{G}_1^2\mathcal{F}_4\mathcal{F}_6 + 729\mathcal{F}_4^3 + 720\mathcal{F}_3^2\mathcal{F}_6 - 3240\mathcal{F}_6^2).\end{aligned}$$

Holweck and Oeding give another expression for this hyperdeterminant using a different set of generators [30].

## 2.4 Stationary points of symmetric invariants

### 2.4.1 Systems of equations

Let  $g : \mathbb{R}^k \rightarrow \mathbb{R}$  be a homogeneous polynomial. Consider the problem of finding nonvanishing stationary points  $\varphi \in \mathbb{S}^{k-1}$  of the restriction of  $g$  to  $\mathbb{S}^{k-1}$ . By the method of Lagrange multipliers,

$\varphi$  is a stationary point of  $g$  if and only if

$$\nabla g(\varphi) = \lambda(x_1, x_2, \dots, x_k) \Big|_{x=\varphi} \quad (2.8)$$

for some  $\lambda \in \mathbb{R}$ . Indeed, the condition above says that  $\nabla g(\varphi) \in (T_\varphi \mathbb{S}^{k-1})^\perp$ . This is equivalent to  $\langle \nabla g(\varphi), u \rangle = 0$  for every  $u \in T_\varphi \mathbb{S}^{k-1}$ , which is equivalent to  $\frac{d}{dt} \Big|_{t=0} g(\gamma(t)) = 0$  for every smooth  $\gamma : \mathbb{R} \rightarrow \mathbb{S}^{k-1}$  such that  $\gamma(0) = \varphi$ .

We can turn (2.8) into a system of equations by taking all  $2 \times 2$  minors of the  $2 \times k$  matrix obtained from the vectors that appear in the equation, since the rows of the matrix are proportional if and only if the matrix is rank 1. It is enough to take  $k - 1$  of these matrix minors, reasoning as follows. For  $j = 1, \dots, k$  let  $U_j \subset \mathbb{S}^{k-1}$  be the dense open set consisting of  $\varphi \in \mathbb{S}^{k-1}$  such that  $x_j(\varphi) \neq 0$ . Then  $\varphi \in U_j$  is stationary if and only if it solves

$$\frac{\partial g}{\partial x_i} x_j - \frac{\partial g}{\partial x_j} x_i = 0, \quad i \neq j. \quad (2.9)$$

Since  $g$  is a homogeneous polynomial, the the system (2.9) consists of homogeneous polynomials and its solution set in the ambient space  $\mathbb{R}^k$  is a union of lines. Any point on a line can easily be normalized to get a corresponding point on  $\mathbb{S}^{k-1}$ . Therefore, to find nonvanishing stationary points  $\varphi \in U_j$ , we may set  $x_j = 1$  and solve the dehomogenized system

$$\left( \frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial x_j} x_i \right) \Big|_{x_j=1} = 0, \quad i \neq j. \quad (2.10)$$

We are particularly interested in the case where  $g : \mathbb{C}^r \rightarrow \mathbb{R}$  has the form  $g(z) = |f(z)|^2$ , where  $f \in \mathbb{C}[z_1, \dots, z_r]$  is a complex homogeneous polynomial of degree  $m > 0$ . Note that  $f(e^{it}z) = e^{imt}f(z)$ , hence  $g$  is invariant under multiplication by elements of the circle group  $\mathbb{T}$ . By Proposition 2.3, the solutions of (2.9) in  $\mathbb{S}^{2r-1} \subset \mathbb{C}^r$  is a union of  $\mathbb{T}$ -orbits, hence the variety has no 0-dimensional components. To fix this issue, we add the constraint  $f(\varphi) \in \mathbb{R}$ . In this

case, it suffices to solve the system (2.9) or (2.10) substituting  $g$  with the real part of the complex polynomial  $f$ . The following proposition explains.

**Proposition 2.9.** *Let  $f = f_1 + if_2$  be a polynomial in  $\mathbb{C}[z_1, \dots, z_r]$  with real and imaginary parts  $f_1$  and  $f_2$  respectively. Suppose  $\varphi \in \mathbb{S}^{2r-1} \subset \mathbb{C}^r$  such that  $f(\varphi)$  is real. Then  $\varphi$  is a nonvanishing stationary point of  $|f(z)|^2$  restricted to  $\mathbb{S}^{2r-1}$  if and only if  $\varphi$  is a nonvanishing stationary point of  $f_1$  restricted to  $\mathbb{S}^{2r-1}$ .*

*Proof.* Choose a chart  $\alpha : U \rightarrow \mathbb{R}^{2r-1}$  on an open set  $U \subset \mathbb{S}^{2r-1}$  centered at  $\varphi$ . Let  $F : \mathbb{R}^{2r-1} \rightarrow \mathbb{R}$  be the composition

$$F(x) = |f(\alpha^{-1}(x))|^2 = f_1(\alpha^{-1}(x))^2 + f_2(\alpha^{-1}(x))^2.$$

Since  $f(\varphi)$  is real,  $f_2(\varphi) = 0$ . Hence

$$\begin{aligned} \frac{\partial F}{\partial x_i}(0) &= 2f_1(\varphi) \frac{\partial}{\partial x_i}(f_1 \circ \alpha^{-1})(0) + 2f_2(\varphi) \frac{\partial}{\partial x_i}(f_2 \circ \alpha^{-1})(0) \\ &= 2f_1(\varphi) \frac{\partial}{\partial x_i}(f_1 \circ \alpha^{-1})(0). \end{aligned}$$

Then the gradient at  $\varphi$  is  $\nabla F(0) = 2f_1(\varphi) \nabla(f_1 \circ \alpha^{-1})(0)$ . If  $f_1(\varphi) \neq 0$ , then  $\nabla F(0) = 0$  if and only if  $\nabla(f_1 \circ \alpha^{-1})(0) = 0$ .  $\square$

#### 2.4.2 Four-qubit stationary points

We return to the four-qubit SLOCC module  $\mathcal{H} = \mathcal{H}_4$  and continue using the notation and definitions introduced in Section 2.3. We also define  $\mathbb{S}^{15} \subset \mathcal{H}$  to be the set of unit vectors in  $\mathcal{H}$ ,  $\mathbb{S}^7 := \mathbb{S}^{15} \cap \mathfrak{a}$  to be the set of unit vectors in the Cartan subspace, and  $\mathbb{T}$  to be the circle group. Let  $f \in \mathbb{C}[\mathcal{H}]$  be a homogeneous symmetric SLOCC invariant polynomial of degree  $m > 0$ , and define  $F : \mathcal{H} \rightarrow \mathbb{R}$  by  $F(z) = |f(z)|^2$ . Our goal is to find the nonvanishing stationary points of  $F|_{\mathbb{S}^{15}}$ .

### 2.4.2.1 Reducing to the Cartan subspace

Let  $S \subset \mathbb{C}[\mathcal{H}]$  denote the set of homogeneous symmetric SLOCC invariant polynomials. Define

$$\tilde{G} := \{\text{smooth } \Phi : \mathbb{S}^{15} \rightarrow \mathbb{S}^{15} \mid |f \circ \Phi| = |f| \text{ for all } f \in S\}. \quad (2.11)$$

Recall, by Proposition 2.3, that  $\tilde{G}$  preserves the stationary points of  $F|_{\mathbb{S}^{15}}$ . By Proposition 2.6, every nonvanishing stationary point  $\varphi \in \mathbb{S}^{15}$  is critical. By Proposition 2.7, the set of critical points in  $\mathbb{S}^{15}$  is the  $\text{SU}_2^{\times 4}$ -orbit of  $\mathbb{S}^7$ . Since  $\rho(\text{SU}_2^{\times 4}) \subset \tilde{G}$ , where  $\rho : \text{GL}_2^{\times 4} \rightarrow \text{GL}(\mathcal{H})$  is the natural representation defined in (2.1), we can find every nonvanishing critical point of  $F$  on  $\mathbb{S}^{15}$  up to  $\tilde{G}$ -equivalence by searching over points in  $\mathbb{S}^7$ .

### 2.4.2.2 Symmetries of stationary points

Now define

$$\tilde{W} := \{\text{smooth } \Phi : \mathbb{S}^7 \rightarrow \mathbb{S}^7 \mid |f \circ \Phi| = |f| \text{ for all } f \in S\}.$$

By Proposition 2.3,  $\tilde{W}$  preserves the stationary points of  $F|_{\mathbb{S}^7}$ . Since the Weyl group arises from restricting the  $\text{SL}_2^{\times 4}$ -action, we have  $\rho'(W) \subset \tilde{W}$ , where  $\rho'$  is the homomorphism induced by  $\rho$ . Since  $f$  is symmetric,  $\pi(\mathfrak{S}_4) \subset \tilde{W}$ , where  $\pi : \mathfrak{S}_4 \rightarrow \text{GL}(\mathfrak{a})$  is the representation given explicitly in (2.7). Finally, we note that entrywise complex conjugation  $\varphi \mapsto \varphi^*$  is also an element of  $\tilde{W}$ .

### 2.4.2.3 Stationary points of $|\mathcal{F}_1|$

The invariant  $\mathcal{F}_1 = 6\mathcal{G}_1$  is simple enough that we can find all of its stationary points of  $|\mathcal{F}_1|$  analytically. The following proposition summarizes the situation.

**Proposition 2.10.** *The state  $\varphi \in \mathbb{S}^{15} \subset \mathcal{H}$  is a global maximizer of the 4-tangle  $|\mathcal{F}_1|^2$  restricted to  $\mathbb{S}^{15}$  if and only if  $\varphi$  is in the local unitary orbit of the set  $\mathbb{S}_{\mathbb{R}}^7 \subset \mathbb{S}^7$  of real unit vectors in the Cartan subspace. There are no other nonvanishing stationary points.*

*Proof.* Let  $F : \mathcal{H} \rightarrow \mathbb{R}$  be the map  $z \mapsto |\mathcal{F}_1(z)|$  and define  $\tilde{G}$  as in (2.11). Make the identification  $\mathfrak{a} = \mathbb{C}^4 = \mathbb{R}^8$  with the embedding (2.2). The real part of  $\mathcal{F}_1|_{\mathfrak{a}}$  is

$$\operatorname{Re}(\mathcal{F}_1|_{\mathfrak{a}}(z)) = 6(x_1^2 - y_1^2 + x_2^2 - y_2^2 + x_3^2 - y_3^2 + x_4^2 - y_4^2). \quad (2.12)$$

The ideal generated by the dehomogenized polynomials (2.10) with  $g$  set equal to the right side of (2.12) is  $\langle y_1, y_2, y_3, y_4 \rangle$ . By Proposition 2.9, if  $\varphi \in \mathbb{S}^7$  is a nonvanishing stationary point of  $F|_{\mathbb{S}^7}$  such that  $\mathcal{F}_1(\varphi)$  is real and  $x_1(\varphi) \neq 0$ , then  $\varphi$  is real. Repeating the calculation with different orderings on the variables, we conclude that if  $\varphi \in \mathbb{S}^7$  is a nonvanishing stationary point of  $F|_{\mathbb{S}^7}$  such that  $\mathcal{F}_1(\varphi)$  is real, then  $\varphi$  is purely real or purely imaginary. Given any  $\psi \in \mathbb{S}^7$  there exists  $\varphi$  in the  $\mathbb{T}$ -orbit of  $\psi$  such that  $\mathcal{F}_1(\varphi)$  is real. Since  $\psi \mapsto e^{it}\psi$  is an element of  $\tilde{G}$ , we showed that

$$X \cap \mathbb{S}^7 \subset \mathbb{T}(\mathbb{S}_{\mathbb{R}}^7 \cup i\mathbb{S}_{\mathbb{R}}^7) = \mathbb{TS}_{\mathbb{R}}^7, \quad (2.13)$$

where  $X \subset \mathbb{S}^{15}$  is the set of nonvanishing stationary points of  $F|_{\mathbb{S}^{15}}$ . By Propositions 2.6 and 2.7,

$$X \subset (\mathrm{SU}_2^{\times 4} \mathfrak{a}) \cap \mathbb{S}^{15} = \mathrm{SU}_2^{\times 4} \mathbb{S}^7. \quad (2.14)$$

Since  $\mathrm{SU}_2^{\times 4}$  acts on  $\mathbb{S}^{15}$  like elements of  $\tilde{G}$ ,  $X$  is a union of  $\mathrm{SU}_2^{\times 4}$ -orbits. By (2.14), each  $\mathrm{SU}_2^{\times 4}$ -orbit intersects  $\mathbb{S}^7$ ; by (2.13), the intersection consists of elements in  $\mathbb{TS}_{\mathbb{R}}^7$ . It follows that

$$X \subset \mathrm{SU}_2^{\times 4}(\mathbb{TS}_{\mathbb{R}}^7) = \mathrm{U}_2^{\times 4} \mathbb{S}_{\mathbb{R}}^7.$$

Since  $\mathcal{F}_1(\varphi) = 6\|\varphi\|^2 = 6$  whenever  $\varphi \in \mathbb{S}_{\mathbb{R}}^7$ , we have  $F(\varphi) = 6$  whenever  $\varphi \in \mathrm{U}_2^{\times 4} \mathbb{S}_{\mathbb{R}}^7$ . By compactness,  $F|_{\mathbb{S}^{15}}$  has a global maximizer, which is a nonvanishing stationary point. Therefore the global maximum is 6 and  $X = \mathrm{U}_2^{\times 4} \mathbb{S}_{\mathbb{R}}^7$  is the set of global maximizers.  $\square$

	$ \mathcal{F}_1 $	$ \mathcal{F}_3 $	$ \mathcal{F}_4 $	$ \mathcal{F}_6 $	$\mathbf{H}( \mathcal{F}_3 )$	$\mathbf{H}( \mathcal{F}_4 )$	$\mathbf{H}( \mathcal{F}_6 )$
$\varphi_1^*$	6	6	6	6	(3,1,3)	(3,1,3)	(3,1,3)
$\varphi_2^*$	6	9	16.5	64.125	(6,1,0)	(6,1,0)	(6,1,0)
$\varphi_3$	6	$7.\bar{3}$	$9.\bar{5}$	$\sim 16.9$	(4,1,2)	(4,1,2)	(4,1,2)
$\varphi_4$	6	$7.\bar{6}$	$10.7\bar{2}$	$\sim 23.0$	(5,1,1)	(5,1,1)	(5,1,1)
$\varphi_5$	2	$0.\bar{6}$	$1.2\bar{5}9$	$1.2\bar{5}9$	(3,1,3)	n/a	n/a
$\varphi_6$	0	2.25	1.40625	$\sim 1.35$	(4,1,2)	n/a	n/a
$\varphi_7$	1.2	2.64	1.392	0.912768	(5,1,1)	n/a	n/a
$\varphi_8^*$	0	$2.\bar{6}$	0	$\sim 0.790$	(6,1,0)	n/a	n/a
$\varphi_9^*$	0	$2.\bar{6}$	0	$\sim 3.01$	(6,1,0)	n/a	n/a
$\varphi_{10}$	0	$\sim 2.45$	0	$1.6041\bar{6}$	(5,1,1)	n/a	n/a
$\varphi_{11}$	0.4	$2.24\bar{8}$	$\sim 1.93$	$\sim 1.55$	(3,1,3)	n/a	n/a
$\varphi_{12}$	0.96	2.6496	1.8223104	$\sim 3.23$	(5,1,1)	n/a	n/a
$\varphi_{13}$	$\sim 1.13$	$\sim 2.23$	$\sim 1.20$	$\sim 1.23$	(3,1,3)	n/a	n/a
$\varphi_{14}$	$\sim 0.963$	$\sim 2.43$	$\sim 1.41$	$\sim 1.74$	(4,1,2)	n/a	n/a
$\psi_5^*$	0	0	2.5	1.875	n/a	(5,1,1)	(5,1,1)
$\psi_6^*$	0	0	1.875	0	n/a	(3,1,3)	n/a
$\psi_7$	$\sim 3.77$	$\sim 0.342$	$0.\bar{0}6$	$\sim 2.36$	n/a	(3,1,3)	n/a
$\psi_8$	$\sim 3.17$	$\sim 0.932$	$\sim 1.66$	$\sim 0.00699$	n/a	(3,1,3)	n/a
$\psi_9$	$\sim 2.88$	$\sim 1.47$	$\sim 2.41$	$\sim 1.30$	n/a	(3,1,3)	n/a
$\psi_{10}$	$\sim 2.84$	$\sim 1.34$	$\sim 3.87$	$\sim 4.65$	n/a	(5,1,1)	n/a
$\psi_{11}$	$\sim 0.758$	$\sim 0.440$	$\sim 2.27$	$\sim 0.834$	n/a	(4,1,2)	n/a
$\psi_{12}$	$\sim 1.13$	$\sim 0.624$	$\sim 2.17$	$\sim 0.311$	n/a	(3,1,3)	n/a
$\psi_{13}$	$\sim 2.69$	$\sim 1.33$	$\sim 3.01$	$\sim 3.46$	n/a	(4,1,2)	n/a

Table 2.2: More four-qubit stationary points (see Section 2.4.2.6)

#### 2.4.2.4 Stationary points of $|\mathcal{F}_3|$

To find stationary points of  $|\mathcal{F}_3|$ , we set up a system of dehomogenized equations as in (2.10) with  $g$  equal to the real part of  $\mathcal{F}_3$ . This system consists of seven polynomial equations (inhomogeneous) of degree 6. We solved the system using the Homotopy Continuation library in Julia [7], which tracks 131,505 paths (the mixed volume) and finds (after about 15 minutes of computation) 9,471 nonsingular real points, which are highly accurate approximations of solutions. From these approximations, we hypothesized a real affine subspace in which the exact solutions lie. Then we performed symbolic computations with Macaulay2 [27] to find expressions or polynomial relations for these solutions, verifying that they are stationary on  $\mathbb{S}^7$ . This method has an advantage

over just attempting to decompose the original ideal since the additional constraints simplify the polynomials and make symbolic methods tractable.

After reducing by the action of the group  $\tilde{W}$  (see Section 2.4.2.2), we find 14 distinct equivalence classes with representatives  $\varphi_i$  listed below, where  $1 \leq i \leq 14$ . For ease of notation, we do not normalize the vectors. We choose representatives which have a real first coordinate; this can always be done by the action of  $\mathbb{T}$ . The  $\varphi_i$  are in distinct  $\tilde{G}$ -orbits because they have distinct images under the map

$$\varphi_i \mapsto (|\mathcal{F}_1(\varphi_i)|, |\mathcal{F}_3(\varphi_i)|, |\mathcal{F}_4(\varphi_i)|, |\mathcal{F}_6(\varphi_i)|).$$

Using the polynomials in Section 2.3.6, we check numerically that these points are stationary for  $\|\mathcal{F}_3\|$  restricted to  $\mathbb{S}^{15}$  and not just restricted to  $\mathbb{S}^7$ . Note that  $\varphi_1, \varphi_2, \varphi_8$ , and  $\varphi_9$  are  $\tilde{G}$ -equivalent to states that appear in Section 2.1.1.

- $\varphi_1 = (1, 0, 0, 0) \cong |MP\rangle$
- $\varphi_2 = (1, 1, 0, 0) \cong |GHZ\rangle$
- $\varphi_3 = (1, 1, 1, 0)$
- $\varphi_4 = (2, 1, 1, 0)$
- $\varphi_5 = (\sqrt{2}, i, 0, 0)$
- $\varphi_6 = (\sqrt{2}, i, i, 0)$
- $\varphi_7 = (\sqrt{2}, i, i, i)$
- $\varphi_8 = (\sqrt{3}, i, i, i) \cong |HS\rangle$
- $\varphi_9 = (1, e^{\pi i/3}, e^{2\pi i/3}, 0) \cong |HD\rangle$
- $\varphi_{10} = (1, 1 - \frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{\sqrt{3}}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}}i, (\sqrt{3} - \sqrt{2})i)$

- $\varphi_{11} = (7, 2\sqrt{7}i, 2\sqrt{7}i, 0)$
- $\varphi_{12} = (18, 11 - \sqrt{203}i, 7 + \sqrt{203}i, 0)$
- $\varphi_{13} = (1, a + bi, -a - bi, 0)$ , where  $a \approx 0.0933$ ,  $b \approx 0.622$  satisfy

$$1000a^2b^2 - 872b^4 + 85a^2 + 345b^2 - 7 = 0,$$

$$100a^4 - 244b^4 + 45a^2 + 65b^2 + 11 = 0.$$

- $\varphi_{14} = (1, a - bi, a + bi, ci)$ , where  $a \approx 0.217$ ,  $b \approx 0.830$ ,  $c \approx 0.366$  satisfy

$$5a^4c - 30a^2b^2c + 5b^4c - 5a^2c^3 + 5b^2c^3 + 2c^5 + 5a^2c - 5b^2c - 5c^3 + 2c = 0,$$

$$10a^4b - 40a^2b^3 + 14b^5 - 15a^2bc^2 + 5b^3c^2 + 5bc^4 + 25a^2b - 15b^3 - 5bc^2 + 4b = 0,$$

$$4a^5 + 20a^3b^2 - 5a^3c^2 + 15ab^2c^2 - 5a^3 - 5ab^2 + 5ac^2 + a = 0.$$

#### 2.4.2.5 Stationary points of $|\mathcal{F}_4|$

Repeating the process of Section 2.4.2.4, we obtain a system of 7 polynomial equations (inhomogeneous) of degree 8 in 7 variables. We experimented with a variety of methods to solve this system, such as primary decomposition in symbolic algebra software [27], Newton's method for many initial points, and numerical irreducible decomposition or solving with regeneration in Bertini [3]. Ultimately the method that worked was to do a standard solve command with HomotopyContinuation.jl [7] on a 2020 Mac (3.3 GHz 6-Core Intel Core i5 with 72GB of RAM). This computation automatically utilized a polyhedral start system, with the mixed volume 1,367,387, rather than the naive Bézout bound  $8^7 = 2,097,152$ . After approximately 7 hours of compute time the result was a set of 936,047 non-singular solutions, 10,963 of them real.

We are grateful to Jon Hauenstein who was able to run our code in Bertini in parallel on his cluster which found a result (in approximately one day) of 10,971 real solutions.

**Remark 2.11.** *The discrepancy in the number of solutions from different numerical computations applied to the same system is not uncommon and illustrates the fact that solutions can be lost with large computations. We cannot be sure if a numerical computation produces a complete set of solutions. However, this is not a huge concern for us since we only need to find one representative from each  $\tilde{W}$ -orbit. Moreover, if a solution is missing, then all solutions in its  $\tilde{W}$ -orbit must have been missed. For our computations of stationary points of  $|\mathcal{F}_4|$  both sets of numerical solutions produce the same set of representatives.*

Like the previous case we used the numerical results to hypothesize the exact forms of each representative, then we symbolically verified that the forms we report are stationary points on  $\mathbb{S}^7$ . We check numerically that these points are stationary on  $\mathbb{S}^{15}$ . This produced 13 stationary points of  $|\mathcal{F}_3|$  which belong in distinct  $\tilde{G}$ -orbits, listed below as  $\psi_i$  for  $1 \leq i \leq 13$ . As before, we do not normalize the vectors. Note that the first four items  $\psi_1, \dots, \psi_4$  in the list are the same as the first four items  $\varphi_1, \dots, \varphi_4$  in the previous list. Furthermore,  $\psi_5$  and  $\psi_6$  are  $\tilde{G}$ -equivalent to states that appear in Section 2.1.1.

- $\psi_1 = (1, 0, 0, 0) \cong |MP\rangle$
- $\psi_2 = (1, 1, 0, 0) \cong |GHZ\rangle$
- $\psi_3 = (1, 1, 1, 0)$
- $\psi_4 = (2, 1, 1, 0)$
- $\psi_5 = (1, i, 0, 0) \cong |C_1\rangle$
- $\psi_6 = (1, i, e^{-\pi i/4}, e^{\pi i/4}) \cong |BSSB\rangle$
- $\psi_7 = (\sqrt{33}, \sqrt{33}, \sqrt{13} - \sqrt{20}i, \sqrt{13} - \sqrt{20}i)$
- $\psi_8 = (1, ai, ai, 0)$ , where  $a \approx 0.393$  satisfies

$$80a^6 - 91a^4 + 77a^2 - 10 = 0.$$

- $\psi_9 = (1, ai, ai, ai)$ , where  $a \approx 0.342$  satisfies

$$75a^6 - 63a^4 + 49a^2 - 5 = 0.$$

- $\psi_{10} = (1, \frac{1}{2} + ai, \frac{1}{2} - ai, 0)$ , where  $a \approx 0.518$  satisfies

$$640a^6 - 1232a^4 + 2016a^2 - 465 = 0.$$

- $\psi_{11} = (1, ai, b, 0)$ , where  $a \approx 0.920$ ,  $b \approx 0.302$  satisfy

$$35a^4 - 21a^2b^2 - 4b^4 - 21a^2 - 18b^2 - 4 = 0,$$

$$190a^2b^4 - 110b^6 - 237a^2b^2 - 339b^4 + 190a^2 - 339b^2 - 110 = 0.$$

- $\psi_{12} = (1, ai, b, b)$  where  $a \approx 0.879$ ,  $b \approx 0.256$  satisfy

$$35a^4 - 21a^2b^2 + 52b^4 - 21a^2 + 3b^2 - 4 = 0,$$

$$587a^2b^4 + 446b^6 - 918a^2b^2 + 129b^4 + 475a^2 - 732b^2 - 275 = 0.$$

- $\psi_{13} = (1, a + bi, a + bi, 0)$ , where  $a \approx 0.347$ ,  $b \approx 0.716$  satisfy

$$2696a^6 - 24472a^2b^4 + 7792b^6 - 161a^4 - 2030a^2b^2 - 4221b^4 + 679a^2$$

$$+ 1659b^2 + 26 = 0,$$

$$94360a^4b^2 - 213584a^2b^4 + 51096b^6 + 3437a^4 + 2310a^2b^2 - 29687b^4 + 1505a^2$$

$$+ 13405b^2 - 262 = 0.$$

### 2.4.2.6 Overview of stationary points

Given a point  $\varphi \in \mathbb{S}^7$ , choose a chart  $\alpha : U \rightarrow \mathbb{R}^7$  on an open set  $U$  containing  $\varphi$  defined by

$$\alpha^{-1}(x_2, \dots, x_8) = \left( (1 - \sum_{i=2}^8 x_i^2)^{1/2}, x_2, \dots, x_8 \right).$$

Given smooth  $g : \mathbb{S}^7 \rightarrow \mathbb{R}$ , define  $\mathbf{H}(g)$  to be the Hessian matrix of the function  $g \circ \alpha^{-1} : \mathbb{R}^7 \rightarrow \mathbb{R}$ .

In Table 2.2 we list the values attained on  $|\mathcal{F}_1|$ ,  $|\mathcal{F}_3|$ ,  $|\mathcal{F}_4|$ , and  $|\mathcal{F}_6|$  for each of the points introduced in Sections 2.4.2.4 and 2.4.2.5. If  $\varphi \in \mathbb{S}^7$  is a nonvanishing stationary point of  $|\mathcal{F}_i|$ , we also record the 3-tuple  $(\lambda^-, \lambda^0, \lambda^+)$ , where  $\lambda^-$ ,  $\lambda^0$ , and  $\lambda^+$  is the number of negative, zero, and positive eigenvalues respectively of  $\mathbf{H}(|\mathcal{F}_i|)$  evaluated at  $\varphi$ . We write n/a to indicate that a point is not stationary for a function. All Hessian matrices that we calculate have one zero eigenvalue; this is not surprising because  $|\mathcal{F}_i|$  is fixed by the action of the one-dimensional real Lie group  $\mathbb{T}$ . An asterisk indicates that an item of the list also appears in Table 2.1.

## 2.5 Acknowledgements

We thank Jonathan Hauenstein for helping us carry out a computation with Bertini.

## Chapter 3

### Tensor decompositions

Note: This work was written with the candidate's Ph. D. advisor, and has already been published.

Oeding, L., and Tan, I. *Tensor Decompositions with Applications to Local Unitary and Stochastic Local Operations and Classical Communication Equivalence of Multipartite Pure States*. SIAM Journal on Applied Algebra and Geometry 9.1 (2025): 33-57, arXiv:2402.12542.

## Abstract

We introduce a broad lemma, one consequence of which is the higher order singular value decomposition (HOSVD) of tensors (DeLathauwer et al. 2000). By an analogous application of the lemma, we find a complex orthogonal version of the HOSVD. Kraus's algorithm (Kraus 2010) used the HOSVD to compute normal forms of almost all  $n$ -qubit pure states under the action of the local unitary group. Taking advantage of the double cover  $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \rightarrow SO_4(\mathbb{C})$ , we produce similar algorithms (distinguished by the parity of  $n$ ) that compute normal forms for almost all  $n$ -qubit pure states under the action of the SLOCC group.

### 3.1 Introduction

Orbit classification is ubiquitous in mathematics. A group  $G$  acting on a set  $\mathcal{S}$  induces an equivalence relation separating elements in  $\mathcal{S}$  into equivalence classes, or **orbits**. If the number of orbits is finite, we can hope to classify orbits by giving a complete list of representative elements, or **normal forms**, from each orbit. Otherwise, the following possibilities are available:

- Give an effective algorithm that takes as input  $x \in \mathcal{S}$  and computes the unique normal form of all points on its orbit.
- Exhibit a complete, non-redundant list of parameterized families of normal forms.

Usually a complete description of orbits is a lofty goal. In this case a natural first step is to find normal forms for almost all  $x \in \mathcal{S}$ , i.e., for every  $x$  in a full-measure subset of  $\mathcal{S}$ .

We shall be primarily interested in the tensor setting. Specifically, for  $1 \leq i \leq n$  let the subgroup  $G_i \leq \text{GL}_{d_i}(\mathbb{C})$  act on the  $d_i$ -dimensional complex vector space  $V_i$ . The group  $G = G_1 \times \cdots \times G_n$  acts on  $V_1 \otimes \cdots \otimes V_n$  by the natural representation

$$(g_1, \dots, g_n) \mapsto g_1 \otimes \cdots \otimes g_n, \quad g_i \in G_i \text{ for all } 1 \leq i \leq n.$$

Examples of orbit classification problems for tensors arise in quantum information. In this field one studies information-processing tasks that can be carried out by performing operations and measurements on collections of qubits (which are the quantum analog of classical bits). The projective point  $\Phi' \in \mathbb{P}(\mathbb{C}^2)^{\otimes n}$  corresponding to the tensor  $\Phi \in (\mathbb{C}^2)^{\otimes n}$  describes the state of an  $n$ -qubit system. Similarly,  $\Phi' \in \mathbb{P}(\mathbb{C}^3)^{\otimes n}$  represents an  $n$ -qutrit state and in general  $\Phi' \in \mathbb{P}(\mathbb{C}^d)^{\otimes n}$  represents an  $n$ -qudit state. Note that we will work with unnormalized representatives  $\Phi \in (\mathbb{C}^d)^{\otimes n}$  since results can be easily transferred to the projectivization.

One classification problem of interest to us is for the action of the Local Unitary (LU) group  $U_d^{\times n}$  on the  $\mathbb{R}$ -vector space  $(\mathbb{C}^d)^{\otimes n}$ . As the name suggests, two  $n$ -qudit states are LU equivalent if

one is related to another via local unitary operations on the qubits. There is also the coarser classification under the SLOCC group of Stochastic Local Operations and Classical Communication, that is, the action of the Cartesian product of special linear groups  $SL_d(\mathbb{C})^{\times n}$  on the  $\mathbb{C}$ -vector space  $(\mathbb{C}^d)^{\otimes n}$ . In this case, two  $n$ -qudit states are SLOCC equivalent if there is a nonzero probability of transforming one to the other by local operations and classical communication.

### 3.1.1 Historical highlights

Some highlights of orbit classification for the SLOCC and LU groups include the following:

1. *Classification for  $n = 2$ .* The two-qudit SLOCC (resp. LU) classification problem is equivalent to that of  $SL_d(\mathbb{C}) \times SL_d(\mathbb{C})$  (resp.  $U_d \times U_d$ ) acting on the space of  $d \times d$  matrices  $\mathbb{C}^{d \times d}$  by left and right multiplication. In the SLOCC case, this problem is solved by row and column reduction. If  $M \in \mathbb{C}^{d \times d}$  is invertible, the normal form of  $M$  is  $\sqrt[d]{\det(M)} \cdot I_d$ . Otherwise, the normal form of  $M$  is  $\text{diag}(1, \dots, 1, 0, \dots, 0)$  where the number of 1's that appear is equal to the rank of  $M$ . In the LU case, the problem is solved by the singular value decomposition of matrices, which gives real diagonal normal forms. This is also known as **Schmidt decomposition** in the context of quantum information [45, Thm. 2.7].
2. *LU classification for  $n \geq 3$ .*
  - (a) 3 qubits: This case was solved by the generalized Schmidt decomposition of Acín et al. [1]. As with singular value decomposition, this gives a single parameterized family of normal forms.
  - (b)  $n$  qudits: According to [10, pp. 35-36] a complete classification of 4-qubit states is unknown. Normal forms for *almost all*  $n$ -qudit states can be found using the higher order singular value decomposition [37].
3. *SLOCC classification for  $n \geq 3$ .*

- (a) 3 qubits: This case is considered to be classical, but was re-introduced by Gelfand, Kapranov, and Zelevinski [22, Ex. 4.5, p. 478] and famously introduced to the quantum community by Dür, Vidal, and Cirac [15].
- (b) 4 qubits: Chterental and Djokovic classified the orbits for this case [10]. The first complete and irredundant classification was by Dietrich et al. [14], whose methods followed Vinberg’s method and work of Antonyan [2, 48].
- (c) 3 qutrits: Most recently, Di Trani, de Graaf, and Marrani [13] gave a classification of orbits over the real numbers, building on work of Nurmiev, who gave a classification of orbits over the complex numbers [47] using methods of Vinberg and Elasmvili [58]. We also note prior work of [53] and [44] which also considered this case.
- (d) After these cases the situation is considered widely open. In particular we know that the problem of classifying tensors [4] and tensor diagrams [54] is in the “wild” category. However, this should not necessarily stop us from seeking classifications of orbits in small dimensions. We note that Dietrich’s 4-qubit classification and Nurmiev’s 3-qutrit classification rely on Vinberg’s method and utilize connections to the exceptional Lie algebras  $\mathfrak{e}_7$  and  $\mathfrak{e}_6$  respectively. It is not clear how one should generalize, however Holweck and Oeding explored one possible direction in recent works [30, 32] that studied hyperdeterminants in these settings, and [31], which attempted to generalize the Jordan decomposition to larger tensors.

### 3.1.2 Our approach

The strategy of the present article is to provide a novel generalization to both the classical singular value decomposition (SVD) of matrices and higher order singular value decomposition (HOSVD) of tensors [11]. The HOSVD is a tensor generalization of the matrix SVD. Significantly for us, the HOSVD can be used to compute LU normal forms for almost all tensors. On the other hand, Chterental and Djokovic [10] generalize the SVD with an analogous matrix factorization where the unitary group is replaced with the complex orthogonal group. Combining these ideas,

we produce a complex orthogonal version of the HOSVD. The existence of a homomorphism  $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \rightarrow SO_4(\mathbb{C})$  paired with orthogonal HOSVD leads to an algorithm that produces SLOCC normal forms for almost all  $n$ -qubit tensors  $\Phi \in (\mathbb{C}^2)^{\otimes n}$ , just as Kraus’s algorithm [37] achieved for the LU group. The situation is quite different depending on the parity of  $n$ .

### 3.1.3 Wider context and applications

Though the world of tensors and their applications is quite broad, for concreteness we have chosen to cast our results in the language of quantum information. We refer to [39] for general background on the algebra and geometry of tensors, [38, 40, 63] for modern views on tensors in quantum information, and [36] for a classic overview on broader applications of tensor decompositions.

The problem of classifying orbits of tensors is closely related to tensor decompositions. Let  $U_{d_1} \times \cdots \times U_{d_n}$  act on  $V = \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ . If  $\Phi \in V$  has normal form  $\Omega \in V$ , then we can write

$$\Phi = (U_1 \otimes \cdots \otimes U_n)\Omega, \quad U_i \in U_{d_i} \text{ for all } 1 \leq i \leq n.$$

If we add the condition that  $\Omega$  is “all-orthogonal,” then this equation expresses the HOSVD of tensors introduced by De Lathauwer et al. [11]. The SVD of matrices is a special case of HOSVD, which is a special case of Tucker decomposition [36]. However, the HOSVD falls short of classifying unitary orbits since the so-called “core tensor”  $\Omega$  is not unique (see Section 4.3 in [36]). In Section 3.3.2 we discuss an algorithm introduced by Kraus [37] for finding unique  $\Omega$  given general  $\Phi$  when  $V = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ .

The SVD has numerous applications, including the solution to partial least squares problems, principal component analysis, latent variable learning for linear functions, and the classification of 2-qubit quantum states up to local unitary transformation. The HOSVD has also found numerous applications, including in brain science [12], data visualization [43], genetics [49], recommender systems [18], and plant biodiversity [5], to name a few examples.

### 3.1.4 Organization

Here is an overview of the organization of this article. We address our notational conventions in Section 3.1.5. In Section 3.2 we discuss a generalized approach that allows us to obtain not only the well-known HOSVD as a consequence, but also the complex orthogonal HOSVD. After this we discuss normal forms for general qubits in Section 3.3. In particular, we describe the situation for the LU group in Section 3.3.2, for the SLOCC group when  $n$  is even in Section 3.3.3, and for the SLOCC group when  $n$  is odd in Section 3.3.4.

### 3.1.5 Notation and conventions

We will write  $i$  for the imaginary unit and use the symbol “ $\leq$ ” to indicate subgroup containment. The following matrices will appear throughout the text:

$$T := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \end{pmatrix} \quad \text{and} \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These matrices are related by the equation  $T^\top T = J \otimes J$ .

Let  $n > 1$  be a natural number. We write  $\mathcal{H}_n = (\mathbb{C}^2)^{\otimes n}$  to denote the Hilbert space of (unnormalized)  $n$ -qubit state vectors. We follow the physics convention to represent a fixed basis of  $\mathbb{C}^2$  by  $|0\rangle, |1\rangle$ . The induced basis vectors on the tensor product  $\mathcal{H}_n$  are written as

$$|\mathbf{v}\rangle = |v_1 v_2 \dots v_n\rangle = |v_1\rangle \otimes |v_2\rangle \otimes \dots \otimes |v_n\rangle$$

for all  $\mathbf{v} = (v_1, \dots, v_n) \in \{0, 1\}^n$ . Then the coordinates of a tensor  $\Phi \in \mathcal{H}_n$  in this basis are indexed by  $n$ -tuples  $\mathbf{v} \in \{0, 1\}^n$  as in the equation  $\Phi = \sum_{\mathbf{v} \in \{0, 1\}^n} \Phi_{\mathbf{v}} |\mathbf{v}\rangle$ , where  $\Phi_{\mathbf{v}} \in \mathbb{C}$ .

For  $1 \leq i \leq n$  let  $V_i$  be a  $d_i$ -dimensional complex vector space, let  $\text{GL}(V_i)$  denote the group of invertible linear operators on  $V_i$ , and let  $G_i$  denote a group with a representation  $G_i \rightarrow \text{GL}(V_i)$ .

For any vector space  $V$  let  $V^*$  denote the dual vector space of linear functionals on  $V$ . There are natural representations  $G_1 \times \cdots \times G_n \rightarrow \text{GL}(V_1 \otimes \cdots \otimes V_n)$  and  $G_1 \times \cdots \times G_n \rightarrow \text{GL}(V_1 \oplus \cdots \oplus V_n)$  defined respectively by taking the Kronecker product and the direct sum of linear operators. We write  $G_1 \otimes \cdots \otimes G_n$  and  $G_1 \oplus \cdots \oplus G_n$  as the respective images of these representations.

Given  $\Phi \in V_1 \otimes \cdots \otimes V_n$ , let  $\Phi_{(i)}$  denote the  $d_i \times \prod_{j \neq i} d_j$  matrix that is associated with the linear map

$$(V_1 \otimes \cdots \otimes V_{i-1} \otimes V_{i+1} \otimes \cdots \otimes V_n)^* \rightarrow V_i$$

corresponding to  $\Phi$  via a choice of bases. This matrix is called the mode- $i$  flattening in the literature. The actions of  $g = (g_1, \dots, g_n) \in G_1 \times \cdots \times G_n$  on  $\Phi$  and on  $\Phi_{(i)}$  are related by the equation

$$(g \cdot \Phi)_{(i)} = g_i \Phi_{(i)} (g_1 \otimes \cdots \otimes g_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_n)^\top =: g_i \Phi_{(i)} \widehat{g}_i^\top. \quad (3.1)$$

Note the definition of  $\widehat{g}_i$  in Eq. (3.1) above.

## 3.2 HOSVD and its complex orthogonal counterpart

In this section we cast the Higher Order Singular Value Decomposition (HOSVD) in a more general light. In Lemma 3.8 we discuss the existence of core elements associated with a suitable set of reduction maps  $\pi_i$  (see Definition 3.1). This approach allows us to combine the concepts of the complex orthogonal SVD [10, Theorem 2.10] and the HOSVD to produce the (complex) orthogonal HOSVD.

### 3.2.1 Reduction maps

Here we introduce some families of functions that satisfy an important property. We call these functions *reduction maps*. They will play a key role in finding normal forms for various group actions.

**Definition 3.1.** Let  $G_i$  be a group for  $1 \leq i \leq n$  and suppose  $\mathcal{S}$  is a  $G_1 \times \cdots \times G_n$ -set. We say that a function  $\pi : \mathcal{S} \rightarrow \mathcal{S}_i$  to a  $G_i$ -set  $\mathcal{S}_i$  is a **reduction map** if

$$\pi((g_1, \dots, g_n).x) = g_i.\pi(x)$$

for all  $(g_1, \dots, g_n) \in G_1 \times \cdots \times G_n$  and  $x \in \mathcal{S}$ . ■

Many examples of reduction maps appear in the literature around SLOCC and LU equivalence. One reason for this is their ability to generate invariants. By the definition of a reduction map, any  $G_i$ -invariant function  $f$  on  $\mathcal{S}_i$  pulls back to a  $G_1 \times \cdots \times G_n$ -invariant function  $f \circ \pi$  on  $\mathcal{S}$ . The reduction maps that will be useful for us are summarized in Table 3.1. Each map makes use of one of the following bilinear forms

- [ $\mathbb{R}$ -Hermitian]  $(v, w) \mapsto v^*w$ , where  $(v, w) \in \mathbb{C}^d \times \mathbb{C}^d \cong \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ ,
- [ $\mathbb{C}$ -Orthogonal]  $(v, w) \mapsto v^\top w$ , where  $(v, w) \in \mathbb{C}^d \times \mathbb{C}^d$ ,
- [ $\mathbb{C}$ -Symplectic]  $(v, w) \mapsto v^\top Jw$ , where  $(v, w) \in \mathbb{C}^2 \times \mathbb{C}^2$ .

The unitary group  $U_d$  consists of the invertible operators preserving the first bilinear form, the complex orthogonal group  $O_d$  consists of the invertible operators preserving the second bilinear form, and the special linear group  $SL_2 = Sp_2$  consists of the invertible operators preserving the third bilinear form. To see that the last claim is true, consider the equation

$$A^\top J A = \det(A)J, \quad \forall A \in GL_2. \quad (3.2)$$

**Example 3.2.** Let  $\mathcal{S} = \mathbb{C}^{d_1} \oplus \cdots \oplus \mathbb{C}^{d_n}$  be a direct sum of representations of groups  $G_i \rightarrow GL(\mathbb{C}^{d_i})$ . For each  $1 \leq i \leq n$  the projection  $\pi_i : \mathcal{S} \rightarrow \mathbb{C}^{d_i}$  is a reduction map. ◇

**Example 3.3.** (Reduced density matrix [45]) Consider the natural action of  $U_{d_1} \times U_{d_2}$  on  $\mathcal{S} = V \otimes W$ , with  $d_1 = \dim V$  and  $d_2 = \dim W$ . Let  $\mathcal{S}_1$  be the  $\mathbb{R}$ -vector space of  $d_1 \times d_1$  Hermitian

matrices considered as a  $U_{d_1}$ -module by the conjugation action. Define  $\pi: V \otimes W \rightarrow \mathcal{S}_1$  by  $\pi(\Phi) = \Phi_{(1)}\Phi_{(1)}^*$  for  $\Phi \in V \otimes W$ . By Eq. (3.1), if  $(U_1, U_2) \in U_{d_1} \times U_{d_2}$  we have

$$\pi((U_1, U_2) \cdot \Phi) = U_1 \Phi_{(1)} U_2^\top (U_1 \Phi_{(1)} U_2^\top)^* = U_1 \Phi_{(1)} \Phi_{(1)}^* U_1^* = U_1 \cdot \pi(\Phi).$$

Thus  $\pi$  is a reduction map. In this case,  $\pi(\Phi)$  coincides with the reduced density matrix of  $\rho = \Phi\Phi^*$  on the subsystem  $V$ , i.e. the partial trace  $\pi(\Phi) = \text{tr}_W(\rho)$ .  $\diamond$

**Example 3.4.** Consider the natural action of  $O_{d_1} \times O_{d_2}$  on  $\mathcal{S} = V \otimes W$ , with  $d_1 = \dim V$  and  $d_2 = \dim W$ . Let  $\mathcal{S}_1 \cong S^2V$  be the space of  $d_1 \times d_1$  complex symmetric matrices considered as an  $O_{d_1}$ -module by the conjugation action. Define  $\pi: V \otimes W \rightarrow \mathcal{S}_1$  by  $\pi(\Phi) = \Phi_{(1)}\Phi_{(1)}^\top$  for  $\Phi \in V \otimes W$ . By (3.1), if  $(U_1, U_2) \in U_{d_1} \times U_{d_2}$  we have

$$\pi((U_1, U_2) \cdot \Phi) = U_1 \Phi_{(1)} U_2^\top (U_1 \Phi_{(1)} U_2^\top)^\top = U_1 \Phi_{(1)} \Phi_{(1)}^\top U_1^\top = U_1 \cdot \pi(\Phi).$$

Thus  $\pi$  is a reduction map.  $\diamond$

**Example 3.5.** Consider the SLOCC module  $\mathcal{S} = \mathcal{H}_n = V_1 \otimes \cdots \otimes V_n$  where each  $V_i$  is a copy of  $\mathbb{C}^2$ . If  $n$  is odd, let  $\mathcal{S}_i \cong S^2V_i$  be the space of  $2 \times 2$  complex symmetric matrices. Otherwise if  $n$  is even, let  $\mathcal{S}_i \cong \wedge^2V_i$  be the space of  $2 \times 2$  complex skew-symmetric matrices. In either case  $\mathcal{S}_i$  is an  $SL_2$ -module by the action  $A.M = AMA^\top$  for  $A \in SL_2$  and  $M \in \mathcal{S}_i$ . For  $1 \leq i \leq n$  define  $\pi_i: \mathcal{H}_n \rightarrow \mathcal{S}_i$  by  $\pi_i(\Phi) = \Phi_{(i)} J^{\otimes(n-1)} \Phi_{(i)}^\top$  for  $\Phi \in \mathcal{H}_n$ . By Eq. (3.1) and Eq. (3.2), we have

$$\begin{aligned} \pi_i((A_1, \dots, A_n) \cdot \Phi) &= A_i \Phi_{(i)} \widehat{A}_i^\top J^{\otimes(n-1)} (A_i \Phi_{(i)} \widehat{A}_i^\top)^\top \\ &= A_i \Phi_{(i)} \widehat{A}_i^\top J^{\otimes(n-1)} \widehat{A}_i \Phi_{(i)}^\top A_i^\top \\ &= A_i \Phi_{(i)} J^{\otimes(n-1)} \Phi_{(i)}^\top A_i^\top \\ &= A_i \cdot \pi_i(\Phi) \end{aligned}$$

reduction map	source and target	groups
$\pi : \Phi \mapsto \Phi_{(1)} \Phi_{(1)}^*$	$\mathbb{C}^d \otimes \mathbb{C}^e \rightarrow \mathbb{C}^d \otimes \mathbb{C}^{d^*}$	$U_d \times U_e ; U_d$
$\pi : \Phi \mapsto \Phi_{(1)} \Phi_{(1)}^\top$	$\mathbb{C}^d \otimes \mathbb{C}^e \rightarrow S^2 \mathbb{C}^d$	$O_d \times O_e ; O_d$
$\pi_i : \Phi \mapsto \Phi_{(i)} J^{\otimes(n-1)} \Phi_{(i)}^\top$	$\mathcal{H}_n \rightarrow S^2 \mathbb{C}^2$ or $\bigwedge^2 \mathbb{C}^2$	$SL_2^{\times n} ; SL_2$
$\pi_{ij} : \Phi \mapsto T \Phi_{(ij)} J^{\otimes(n-2)} \Phi_{(ij)}^\top T^\top$	$\mathcal{H}_n \rightarrow S^2 \mathbb{C}^4$ or $\bigwedge^2 \mathbb{C}^4$	$SL_2^{\times n} ; SL_2 \times SL_2 \rightarrow SO_4$

Table 3.1: Reduction maps of Examples 3.3 to 3.6.

where  $A_1, \dots, A_n \in SL_2$ . Thus  $\pi_i$  is a reduction map. Note that  $J^{\otimes k}$  is symmetric when  $k$  is even and skew-symmetric otherwise, which explains the two cases for  $\mathcal{S}_i$ .

Since the determinant of  $M \in \mathcal{S}_i$  is an  $SL_2$  invariant, it pulls back to the SLOCC invariant  $\det(\pi_i)$  which was used in [23, 25]. When  $n = 5$ , a calculation shows that these SLOCC invariants span the space of degree 4 invariants in  $\mathbb{C}[\mathcal{H}_5]$ .

Note that the space of  $2 \times 2$  skew-symmetric matrices is one-dimensional. That is, if  $n$  is even, then  $\pi_i(\Phi)$  is a scalar multiple of  $J$  for all  $\Phi \in \mathcal{H}_n$ . In this case the  $SL_2$ -action on  $\mathcal{S}_i$ , by Eq. (3.2), is trivial, and hence the reduction map is not very useful.  $\diamond$

**Example 3.6.** Consider the SLOCC module  $\mathcal{S} = \mathcal{H}_n = V_1 \otimes \dots \otimes V_n$  where  $n \geq 3$  and each  $V_i$  is a copy of  $\mathbb{C}^2$ . If  $n$  is even, let  $\mathcal{S}_{ij} \cong S^2(V_i \otimes V_j)$  be the space of  $4 \times 4$  complex symmetric matrices. Otherwise if  $n$  is odd, let  $\mathcal{S}_{ij} \cong \bigwedge^2(V_i \otimes V_j)$  be the space of  $4 \times 4$  complex skew-symmetric matrices. The  $4 \times 4$  unitary matrix  $T$  used in [55, 10] provides the following isomorphism.

$$\begin{aligned}
SL_2 \otimes SL_2 & \quad > SO_4 \\
(A \otimes B)_\downarrow & \quad > T(A \otimes B)T^*
\end{aligned} \tag{3.3}$$

We consider  $\mathcal{S}_{ij}$  an  $SL_2 \times SL_2$ -module by the representation

$$\begin{aligned}
SL_2 \times SL_2 & \quad > SO_4 & \quad \rho & \quad > GL(\mathcal{S}_{ij}) \\
(A, B)_\downarrow & \quad > T(A \otimes B)T^*_\downarrow & \quad > \rho(T(A \otimes B)T^*)
\end{aligned} \tag{3.4}$$

where  $\rho$  is defined by the conjugation action. For each  $1 \leq i < j \leq n$  let  $\Phi_{(ij)}$  be the  $4 \times 2^{n-2}$  matrix corresponding to the linear map

$$(V_1 \otimes \cdots \otimes V_{i-1} \otimes V_{i+1} \otimes \cdots \otimes V_{j-1} \otimes V_{j+1} \otimes \cdots \otimes V_n)^* \rightarrow V_i \otimes V_j.$$

Finally, define  $\pi_{ij} : V_1 \otimes \cdots \otimes V_n \rightarrow \mathcal{S}_{ij}$  by  $\pi_{ij}(\Phi) = T\Phi_{(ij)}J^{\otimes(n-2)}\Phi_{(ij)}^\top T^\top$ . By Eq. (3.1) and Eq. (3.2), we have

$$\begin{aligned} \pi_{ij}((A_1 \otimes \cdots \otimes A_n)\Phi) &= T(A_i \otimes A_j)\Phi_{(ij)}\widehat{A}_{ij}^\top J^{\otimes(n-2)}((A_i \otimes A_j)\Phi_{(ij)}\widehat{A}_{ij}^\top)^\top T^\top \\ &= T(A_i \otimes A_j)\Phi_{(ij)}\widehat{A}_{ij}^\top J^{\otimes(n-2)}(\widehat{A}_{ij}^\top)^\top \Phi_{(ij)}^\top (A_i \otimes A_j)^\top T^\top \\ &= T(A_i \otimes A_j)T^*(T\Phi_{(ij)}J^{\otimes(n-2)}\Phi_{(ij)}^\top T^\top)T^{*\top} (A_i \otimes A_j)^\top T^\top \\ &= T(A_i \otimes A_j)T^*(T\Phi_{(ij)}J^{\otimes(n-2)}\Phi_{(ij)}^\top T^\top)(T(A_i \otimes A_j)T^*)^\top \\ &= (A_i, A_j).\pi_{ij}(\Phi) \end{aligned}$$

where  $A_1, \dots, A_n \in \text{SL}_2$  and  $\widehat{A}_{ij} = \otimes_{k \notin \{i,j\}} A_k$ . Therefore  $\pi_{ij}$  is a reduction map via an isomorphism  $\text{SL}_2^{\times n} \rightarrow (\text{SL}_2 \times \text{SL}_2) \times \text{SL}_2^{\times(n-2)}$  rearranging factors of the group:

$$(A_1, \dots, A_n) \mapsto ((A_i, A_j), A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_{j-1}, A_{j+1}, \dots, A_n).$$

This reduction map was considered by Li [41]. By the discussion above, if  $\Phi, \Phi' \in \mathcal{H}_n$  are in the same SLOCC orbit, then  $\pi_{ij}(\Phi)$  and  $\pi_{ij}(\Phi')$  are in the same  $\text{SO}_4$ -orbit by conjugation. Li showed the weaker claim that if  $\Phi, \Phi' \in \mathcal{H}_n$  are in the same SLOCC orbit, then  $\pi_{ij}(\Phi)$  and  $\pi_{ij}(\Phi')$  have the same Jordan normal form.  $\diamond$

### 3.2.2 Core elements

In the HOSVD there is a notion of ‘‘core tensor’’ which takes the place of the diagonal matrix in the SVD matrix factorization. Our generalization of this notion (defined in Lemma 3.8 below)

does not rely on tensor properties or linearity. Therefore we call it a “core element,” though we primarily have tensors in mind.

**Definition 3.7.** Suppose a group  $G$  acts on a set  $\mathcal{S}$ . A **normal form function** for the action is a function  $F: \mathcal{S} \rightarrow \mathcal{S}$  such that, for any  $x, y \in \mathcal{S}$

- $x$  and  $F(x)$  are in the same  $G$ -orbit, and
- $F(x) = F(y)$  if  $x$  and  $y$  are in the same  $G$ -orbit.

■

Of course, normal-form functions always exist by the axiom of choice. One may view the orbit classification problem for  $G$  acting on  $\mathcal{S}$  as the problem of finding a “nice” or easily computable normal form function on  $\mathcal{S}$ .

Lemma 3.8 is a “workhorse” lemma. We use it and the associated algorithm (see the comments after the proof) throughout the rest of the paper. Specifically, Lemma 3.8 is used in the proofs of Theorems 3.13, 3.14 and 3.22 and Lemma 3.24.

**Lemma 3.8** (Existence and Uniqueness of Core Elements). *Let  $G_1, \dots, G_n$  be groups. Suppose  $G_1 \times \dots \times G_n$  acts on a set  $\mathcal{S}$  and for each  $1 \leq i \leq n$  there exists a reduction map  $\pi_i: \mathcal{S} \rightarrow \mathcal{S}_i$  to a  $G_i$ -set  $\mathcal{S}_i$ :*

$$\pi_i((g_1, \dots, g_n).x) = g_i.\pi_i(x) \quad \text{for all } g_i \in G_i, x \in \mathcal{S}. \quad (3.5)$$

*Fix a normal form function  $F_i: \mathcal{S}_i \rightarrow \mathcal{S}_i$  for the  $G_i$ -action on  $\mathcal{S}_i$ . Then for each  $x \in \mathcal{S}$  there exists a **core element**  $\omega \in \mathcal{S}$ , which is defined by the properties:*

- $x = (g_1, \dots, g_n).\omega$  for some  $g_i \in G_i$ , and
- $\pi_i(\omega) = F_i(\pi_i(\omega))$  for all  $1 \leq i \leq n$ .

*Moreover, the core element is unique up to the action of  $H_1 \times \dots \times H_n \leq G_1 \times \dots \times G_n$ , where  $H_i = \{g \in G_i : g.\pi_i(\omega) = \pi_i(\omega)\}$  is the stabilizer subgroup of  $\pi_i(\omega)$ .*

*Proof.* Let  $x \in \mathcal{S}$  and for each  $i$  write  $N_i = F_i(\pi_i(x))$ . By definition  $N_i$  and  $\pi_i(x)$  are in the same orbit, hence there exists  $g_i \in G_i$  such that  $\pi_i(x) = g_i.N_i$ . Let  $\omega = (g_1^{-1}, \dots, g_n^{-1}).x$ . Then by Eq. (3.5),

$$\pi_i(\omega) = g_i^{-1}.\pi_i(x) = g_i^{-1}g_i.N_i = N_i.$$

Since  $x$  and  $\omega$  are in the same orbit, by Eq. (3.5) so are  $\pi_i(x)$  and  $\pi_i(\omega)$ . Then  $N_i = F_i(\pi_i(\omega))$ , hence  $\omega$  is a core tensor for  $x$ . This proves existence.

Now for uniqueness, suppose  $\xi$  is another core element for  $x$ , i.e.  $x = (h_1, \dots, h_n).\xi$  for some  $h_i \in G_i$  and  $\pi_i(\xi) = N_i$  for all  $i$ . Then by Eq. (3.5),

$$h_i.N_i = h_i.\pi_i(\xi) = \pi_i(x) = g_i.\pi_i(\omega) = g_i.N_i$$

so  $h_i^{-1}g_i$  fixes  $N_i$  and  $(h_1^{-1}g_1, \dots, h_n^{-1}g_n).\omega = \xi$ . That is,  $\xi$  is in the  $H_1 \times \dots \times H_n$ -orbit of  $\omega$ .  $\square$

The proof of Lemma 3.8 also gives us an algorithm to compute the core element  $\omega$ , supposing that for each  $1 \leq i \leq n$  and  $x_i \in \mathcal{S}_i$  one already has a way to find  $g_i \in G_i$  such that  $g_i.x_i$  is the normal form  $F_i(x_i)$ . Given input  $x \in \mathcal{S}$ , the steps are:

1. For each  $1 \leq i \leq n$  find  $g_i \in G_i$  such that  $F_i(\pi_i(x)) = g_i.\pi_i(x)$ .
2. Set  $\omega = (g_1^{-1}, \dots, g_n^{-1}).x$ .

This fact is used to construct Algorithms 2 and 8 later in the text.

**Proposition 3.9.** *Elements  $x, y \in \mathcal{S}$  have equivalent core elements if and only if  $x$  and  $y$  are in the same  $G_1 \times \dots \times G_n$ -orbit.*

*Proof.* It is clear that if  $x$  and  $y$  have equivalent core elements, then they are in the same orbit. Conversely, suppose  $(g_1, \dots, g_n).x = y$  for some  $g_i \in G_i$ . Set  $N_i = F_i(\pi_i(x)) = F_i(\pi_i(y))$ . A core element  $\omega$  for  $x$  has the properties  $x = (h_1, \dots, h_n).\omega$  for some  $h_i \in G_i$  and  $\pi_i(\omega) = N_i$  for all  $i$ . Then  $\omega$  is also a core element for  $y$ , since  $y = (g_1h_1, \dots, g_nh_n).\omega$  and  $\pi_i(\omega) = N_i$ . By the uniqueness of core elements in Lemma 3.8, we are done.  $\square$

We wish to describe an analogous version of the HOSVD where the complex orthogonal group takes the role of the unitary group. To that end, let us recall two analogous matrix factorizations.

**Proposition 3.10** (Spectral theorem [20]). *Every  $d \times d$  Hermitian matrix  $A$  has a factorization  $A = UDU^*$  where  $U \in U_d$  and  $D$  is diagonal with real entries.*

**Proposition 3.11** (Normal Forms for Complex Symmetric Matrices [21, Section 3]). *Every  $d \times d$  complex symmetric matrix  $A$  has a factorization  $A = UDU^\top$  where  $U \in O_d$  and  $D = S_{k_1}(\lambda_1) \oplus \dots \oplus S_{k_r}(\lambda_r)$ , where  $S_k(\lambda)$ ,  $\lambda \in \mathbb{C}$  is the  $k \times k$  **symmetrized Jordan block***

$$S_k(\lambda) = \begin{pmatrix} \lambda & 1 & \cdots & \cdots & 0 \\ 1 & \lambda & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda & 1 \\ 0 & \cdots & \cdots & 1 & \lambda \end{pmatrix} + i \begin{pmatrix} 0 & \cdots & \cdots & 1 & 0 \\ \vdots & & \ddots & 0 & -1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \ddots & & \vdots \\ 0 & -1 & \cdots & \cdots & 0 \end{pmatrix}.$$

The symmetrized Jordan block  $S_k(\lambda)$  is similar to the standard  $k \times k$  Jordan block with diagonal entries equal to  $\lambda$ .

The spectral theorem gives diagonal normal forms for the action of  $U_d$  by conjugation on the space of Hermitian matrices, whereas Proposition 3.11 gives block diagonal normal forms for the action of  $O_d$  by conjugation on the space of complex symmetric matrices. However, one has to fix an order on the diagonal entries or Jordan blocks. In the unitary case we can use the standard ordering on the reals. For the orthogonal case, we use the lexicographical (lex) order on  $\mathbb{C}$  together with comparing sizes of the blocks - more details in Sections 3.2.3 and 3.2.4.

It is easy to see that Proposition 3.11 remains true if  $O_d$  is replaced with  $SO_d$ . If a  $d \times d$  complex symmetric matrix  $A$  has  $d$  distinct eigenvalues, then Proposition 3.11 implies that  $A = UDU^\top$  where  $U \in SO_d$  and  $D = \text{diag}(\lambda_1, \dots, \lambda_d)$  is diagonal. Given such  $A$ , one can compute  $U$  and  $D$  with Algorithm 1, which very similar to the standard spectral theorem algorithm. Note

that, since the matrix  $U$  computed in step 1 has full rank, the diagonal matrix  $U^\top U$  has nonzero diagonal entries. Then each  $u_i^\top u_i$  in step 2 is not zero.

**Lemma 3.12.** *Let  $A \in \mathbb{C}^{d \times d}$  be symmetric. Suppose  $A$  has an eigenvector  $v$  with eigenvalue  $\lambda$  and an eigenvector  $w$  with eigenvalue  $\mu$ . If  $\lambda \neq \mu$ , then  $v^\top w = 0$ .*

*Proof.* Since  $A$  is symmetric,  $\lambda v^\top w = (Av)^\top w = v^\top A^\top w = v^\top Aw = \mu v^\top w$  which implies  $(\lambda - \mu)v^\top w = 0$ . □

---

**Algorithm 1** Orthogonal spectral theorem for general matrices

---

*Input:* A symmetric matrix  $A \in \mathbb{C}^{d \times d}$  with distinct eigenvalues.

*Output:* Matrices  $U \in \text{SO}_d$  and  $D \in \mathbb{C}^{d \times d}$ , diagonal with weakly decreasing diagonal entries (in lex order), such that  $A = UDU^\top$ .

1. Diagonalize  $A$ , i.e. compute matrices  $U \in \text{GL}_d$  and  $D \in \mathbb{C}^{d \times d}$ , diagonal with weakly decreasing diagonal entries, such that  $A = UDU^{-1}$ .
  2. Applying Lemma 3.12 to the columns of  $U$ , we see that  $U^\top U$  is diagonal. Update  $U$  by replacing each column vector  $u_i$  of  $U$  with  $u_i / (u_i^\top u_i)^{1/2}$ ,  $1 \leq i \leq d$ . This ensures that  $U^\top U = I$ .
  3. If  $\det(U) = -1$ , update  $U \leftarrow U \cdot \text{diag}(-1, 1, 1, \dots, 1)$  so that  $U \in \text{SO}_d$ .
- 

### 3.2.3 HOSVD

For  $1 \leq i \leq n$  let  $U_{d_i} \rightarrow \text{GL}(V_i)$  be the  $d_i$ -dimensional representation of the unitary group. Let  $\mathcal{S}_i$  be the space of  $d_i \times d_i$  Hermitian matrices considered as a  $U_{d_i}$ -module under the conjugation action, and define maps

$$\pi_i: V_1 \otimes \cdots \otimes V_n \rightarrow \mathcal{S}_i \quad \text{where} \quad \pi_i(\Phi) = \Phi_{(i)} \Phi_{(i)}^*$$

for  $\Phi \in V_1 \otimes \cdots \otimes V_n$ . By Example 3.3,  $\pi_i$  is a reduction map.

By the spectral theorem, every  $M \in \mathcal{S}_i$  can be factored as  $M = UDU^*$  for some  $U \in U_{d_i}$  and real diagonal  $D$ . We uniquely specify  $D$  by requiring that its diagonal entries be listed in weakly

decreasing order. Define a normal form function  $F_i: \mathcal{S}_i \rightarrow \mathcal{S}_i$  by setting  $F_i(M) = D$ . In this case, Lemma 3.8 recovers the HOSVD described in [11]. The term ‘‘core element’’ comes from this source. We state the result below. The second item is equivalent to the ‘‘all-orthogonality’’ condition from [11].

**Theorem 3.13** (HOSVD, [11]). *For  $1 \leq i \leq n$  let  $V_i$  be a  $d_i$ -dimensional complex vector space. Then for each tensor  $\Phi \in V_1 \otimes \cdots \otimes V_n$  there exists a core tensor  $\Omega$  such that*

- $\Phi = (U_1 \otimes \cdots \otimes U_n)\Omega$  for some  $U_i \in U_{d_i}$ , and
- $D_i = \Omega_{(i)}\Omega_{(i)}^*$  is real diagonal with weakly decreasing diagonal entries for all  $1 \leq i \leq n$ .

*The core tensor is unique up to the action of  $H_1 \times \cdots \times H_n$ , where  $H_i \leq U_{d_i}$  is the stabilizer subgroup of  $D_i$  by the conjugation action.*

*Proof.* Apply Lemma 3.8. □

### 3.2.4 Orthogonal HOSVD

For  $1 \leq i \leq n$  let  $\text{SO}_{d_i} \rightarrow \text{GL}(V_i)$  be the  $d_i$ -dimensional representation of  $\text{SO}_{d_i}$ . Let  $\mathcal{S}_i \cong S^2V_i$  be the space of  $d_i \times d_i$  complex symmetric matrices considered as a  $\text{SO}_{d_i}$ -module by the conjugation action, and define

$$\pi_i: V_1 \otimes \cdots \otimes V_n \rightarrow \mathcal{S}_i, \quad \text{where} \quad \pi_i(\Phi) = \Phi_{(i)}\Phi_{(i)}^\top$$

for  $\Phi \in V_1 \otimes \cdots \otimes V_n$ . Then  $\pi_i$  is a reduction map.

By Proposition 3.11, every  $M \in \mathcal{S}_i$  can be factored as  $M = UDU^\top$  where  $U \in \text{SO}_{d_i}$  and  $D = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_r}(\lambda_r)$  is a direct sum of symmetrized Jordan blocks. We can uniquely specify  $D$  by ordering the blocks in weakly decreasing order, where order on the set of blocks  $J_k(\lambda)$  is induced by lexicographical order on the triples  $(\text{Re}(\lambda), \text{Im}(\lambda), k)$ . Define a normal form function  $F_i: \mathcal{S}_i \rightarrow \mathcal{S}_i$  by setting  $F_i(M) = D$ .

With this setup, Lemma 3.8 gives us the following tensor factorization. We call it the **orthogonal HOSVD** since it may be viewed as the complex orthogonal version of the HOSVD.

**Theorem 3.14** (Orthogonal HOSVD). *For  $1 \leq i \leq n$  let  $V_i$  be a  $d_i$ -dimensional complex vector space. Then for each tensor  $\Phi \in V_1 \otimes \cdots \otimes V_n$  there exists a core tensor  $\Omega$  such that*

- $\Phi = (U_1 \otimes \cdots \otimes U_n)\Omega$  for some  $U_i \in \text{SO}_{d_i}$ , and
- $D_i = \Omega_{(i)}\Omega_{(i)}^\top$  is a direct sum of symmetrized Jordan blocks in weakly decreasing order for all  $1 \leq i \leq n$ .

The core tensor is unique up to the action of  $H_1 \times \cdots \times H_n$ , where  $H_i \leq \text{SO}_{d_i}$  is the stabilizer subgroup of  $D_i$  by the conjugation action.

*Proof.* Apply Lemma 3.8. □

**Proposition 3.15.** *Let  $G$  be a subgroup of  $\text{GL}_d$  and let  $D = \lambda_1 I_{k_1} \oplus \cdots \oplus \lambda_r I_{k_r}$  where the  $\lambda_i \in \mathbb{C}$  are pairwise distinct and  $k_1 + \cdots + k_r = d$ . Then*

$$H := \{U \in G : UDU^{-1} = D\} = G \cap (\text{GL}_{k_1} \oplus \cdots \oplus \text{GL}_{k_r}).$$

*In particular,  $H = \text{U}_{k_1} \oplus \cdots \oplus \text{U}_{k_r}$  if  $G = \text{U}_d$  and  $H = \text{SO}_{k_1} \oplus \cdots \oplus \text{SO}_{k_r}$  if  $G = \text{SO}_d$ .*

*Proof.* The containment  $\supset$  is clear. For the other direction, suppose  $U \in G$  such that  $UDU^{-1} = D$ . Then  $U$  commutes with  $D$ , which implies that  $U$  preserves the eigenspaces of  $D$ . Thus  $U$  has block diagonal form  $A_1 \oplus \cdots \oplus A_r$  with  $A_i \in \text{GL}_{k_i}$ . □

Proposition 3.15 helps us understand what the stabilizers  $H_i$  of Theorems 3.13 and 3.14 are. In the unitary case (Theorem 3.13) the reduced density matrices  $D_i$  are diagonal, so we have a complete understanding of the possible stabilizers. In the orthogonal case (Theorem 3.14), the matrices  $D_i$  are generically diagonal. We say more about this situation in Section 3.2.5, but will not give a complete description of the possible stabilizers.

Let  $\Phi \in V_1 \otimes \cdots \otimes V_n$  as in the statement of Theorem 3.14. Suppose that for each  $1 \leq i \leq n$  the matrix  $\Phi_{(i)}\Phi_{(i)}^\top$  has distinct eigenvalues. Then the orthogonal HOSVD core tensor  $\Omega$  has the property that  $\Omega_{(i)}\Omega_{(i)}^\top$  is diagonal with distinct diagonal entries. In this case, Algorithm 2 for computing  $\Omega$  mirrors the classical HOSVD algorithm [11]; both follow from the proof of Lemma 3.8. We leave the situation with repeated eigenvalues open.

---

**Algorithm 2** Orthogonal HOSVD for general tensors

---

*Input:* A tensor  $\Phi \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$  such that for each  $1 \leq i \leq n$ ,  $\Phi_{(i)}\Phi_{(i)}^\top$  has distinct eigenvalues.

*Output:* Core tensor  $\Omega$  in the sense of Theorem 3.14.

1. For  $1 \leq i \leq n$  use Algorithm 1 to factorize  $\Phi_{(i)}\Phi_{(i)}^\top = U_i D_i U_i^\top$ , where  $U_i \in \text{SO}_{d_i}$  and  $D_i$  is diagonal with decreasing diagonal entries.
  2. Set  $\Omega \leftarrow (U_1^\top \otimes \cdots \otimes U_n^\top)\Phi$ .
- 

### 3.2.5 Stabilizers of Jordan blocks

In this section, let  $J_k(\lambda)$  denote the Jordan block with diagonal entries equal to  $\lambda$ . Recall also our notation for the symmetrized Jordan block  $S_k(\lambda)$  introduced in Proposition 3.11.

Let  $G \leq \text{GL}_n$ . The stabilizer of a matrix  $M \in \mathbb{C}^{n \times n}$  in  $G$  is

$$\text{Stab}_G(M) = \{A \in G \mid AMA^{-1} = M\}.$$

Proposition 3.15 described  $\text{Stab}_{\text{SO}_n}(M)$  when the Jordan blocks of  $M$  are all  $1 \times 1$ . In general, any  $A \in \text{Stab}_G(M)$  must preserve the flags of generalized eigenspaces of  $M$ . We can be more specific in the case that the Jordan form of  $M$  consists of a single Jordan block. We note that in general, the description of the stabilizer should be more complicated.

**Lemma 3.16.** *Consider the Jordan block  $M = J_k(\lambda)$ . We have*

$$\text{Stab}_{\text{GL}_k}(M) = \left\{ \left( \begin{array}{cccc} a_1 & a_2 & \cdots & a_k \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_2 \\ & & & a_1 \end{array} \right) \mid a_1 \neq 0 \right\}.$$

*Proof.* Suppose  $A$  commutes with  $J_k(\lambda)$ . Then  $A$  preserves the generalized eigenspaces of  $J_k(\lambda)$ , hence  $A$  is upper triangular. Since

$$A(\lambda I_k + J_k(0)) = AJ_k(\lambda) = J_k(\lambda)A = (\lambda I_k + J_k(0))A,$$

it follows that  $AJ_k(0) = J_k(0)A$ . We have

$$\begin{aligned} J_k(0)Ae_i &= A_{2,i}e_1 + A_{3,i}e_2 + \cdots + A_{i,i}e_{i-1}, \quad \text{and} \\ AJ_k(0)e_i &= A_{1,i-1}e_1 + A_{2,i-1}e_2 + \cdots + A_{i-1,i-1}e_{i-1} \end{aligned} \tag{3.6}$$

for all  $2 \leq i \leq k$ . Thus,  $A$  has the desired form.

Conversely, suppose  $A$  has the form above. As in the preceding paragraph, it suffices to show that  $A$  commutes with  $J_k(0)$ . This follows from (3.6).  $\square$

In Lemma 3.16, we showed that  $A$  lies in a  $k$ -dimensional subspace of  $\mathbb{C}^{k \times k}$ . The obvious basis for this space can be written as  $\{I_k, J_k(0), J_k(0)^2, \dots, J_k(0)^{k-1}\}$ . By Proposition 3.11, the symmetrized Jordan block  $S_k(\lambda)$  is similar to  $J_k(\lambda)$ . Since similarity transformations are linear and preserve matrix multiplication, we immediately obtain the following.

**Proposition 3.17.** *Consider the symmetrized Jordan block  $M = S_k(\lambda)$ . We have*

$$\text{Stab}_{\text{GL}_k}(M) = \text{span}\{I_k, S_k(0), S_k(0)^2, \dots, S_k(0)^{k-1}\} \cap \text{GL}_n.$$

*Proof.* Apply Lemma 3.16.  $\square$

**Example 3.18.** Let  $A \in \text{GL}_4$  and  $S = S_4(0)$ . We have

$$S = \begin{pmatrix} 0 & 1 & i & 0 \\ 1 & i & 1 & -i \\ i & 1 & -i & 1 \\ 0 & -i & 1 & 0 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 0 & 2i & 2 & 0 \\ 2i & 0 & 0 & 2 \\ 2 & 0 & 0 & -2i \\ 0 & 2 & -2i & 0 \end{pmatrix}, \quad S^3 = \begin{pmatrix} 4i & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & -4i \end{pmatrix}.$$

In this case, Proposition 3.17 says that  $A$  stabilizes  $S_4(\lambda)$  by the conjugation action if and only if  $A$  has the form below, where  $a, b, c, d \in \mathbb{C}$ .

$$A = \begin{pmatrix} a + di & b + ci & c + bi & d \\ b + ci & a + bi & b & c - bi \\ c + bi & b & a - bi & b - ci \\ d & c - bi & b - ci & a - di \end{pmatrix}$$

◇

**Proposition 3.19.** Let  $M = S_k(\lambda)$ . Then  $\text{Stab}_{\text{O}_k}(M) = \{\pm I\}$ .

*Proof.* One direction is obvious. For the other, suppose  $U \in \text{O}_k$  commutes with  $M$ . By Proposition 3.17, we have

$$U = \alpha_0 I_k + \alpha_1 S_k(0) + \alpha_2 S_k(0)^2 + \cdots + \alpha_{k-1} S_k(0)^{k-1}$$

for some constants  $\alpha_i \in \mathbb{C}$ . Then since  $U$  is symmetric and orthogonal,  $U^2 = UU^\top = I_k$ . It follows that  $\alpha_i = 0$  when  $i > 0$ , and  $\alpha_0 = \pm 1$ . □

### 3.3 Normal forms for almost all qubit states

In this section we present several algorithms for finding normal forms with respect to various group actions on  $\mathcal{H}_n$ . For the LU case we give a detailed exposition of ideas in [37] that lead to Kraus's

Algorithm 4. Our more general perspective allows us to go further and obtain Algorithms 7 and 8 for the SLOCC cases.

### 3.3.1 General points and generic properties

Let  $V$  be a finite-dimensional complex vector space, endowed with the usual Euclidean topology. A **general point** of  $V$  is one contained in an open, full-measure (hence also dense) subset of  $V$ . A property that holds for general points is a **generic property**.

**Lemma 3.20.** *For each  $i = 1, \dots, r$  let  $p_i : W_i \rightarrow \mathbb{C}$  be a nonzero holomorphic function on an open, full-measure subset  $W_i \subset V$ . Then  $D = \{v \in V : p_i(v) \neq 0, \forall i\}$  is an open, full-measure subset of  $V$ .*

*Proof.* For each  $i$  the set of zeros  $Z_i = p_i^{-1}(0)$  of  $p_i$  is a null set (see [28, p. 9]). Then  $W_i \setminus Z_i$  is open and of full measure. Then the intersection  $D = \bigcap_{i=1}^r W_i \setminus Z_i$  is open and of full measure.  $\square$

According to Lemma 3.20, to show that a property  $P$  holds for general  $\Phi \in V$ , it suffices to exhibit a collection  $\{p_i : i = 1, \dots, n\}$  of complex-valued functions, each holomorphic on an open, dense, full-measure subset of  $V$ , such that  $P$  holds for every  $\Phi \in V$  such that  $p_i(\Phi) \neq 0$  for all  $i$ . If the functions  $p_i$  are polynomials in  $\mathbb{C}[V]$ , then  $P$  is generic in the stronger sense of holding on a Zariski open subset of  $V$ . This is the case when  $P$  is the property that a matrix  $M \in \mathbb{C}^{d \times d}$  does not have repeated eigenvalues since  $P$  holds if and only if the discriminant of the characteristic polynomial of  $M$  does not vanish.

### 3.3.2 LU group

The HOSVD of tensors gets us halfway to finding normal forms for almost all tensors under the LU action. Suppose that  $\Phi \in \mathcal{H}_n$  is general: specifically, assume that  $\Phi_{(i)}\Phi_{(i)}^*$  has distinct eigenvalues for all  $1 \leq i \leq n$ . Then the HOSVD core tensor  $\Omega$  associated to  $\Phi$  has the property that  $\Omega_{(i)}\Omega_{(i)}^*$  is real diagonal with distinct diagonal entries. By Proposition 3.15, the core tensor  $\Omega$  is unique up to the action of  $H^{\times n}$ , where

$$H = \{\text{diag}(e^{is}, e^{it}) : s, t \in \mathbb{R}\}.$$

So the problem is reduced to finding an easily computable normal form  $\Omega'$  in the  $H^{\times n}$ -orbit of  $\Omega$  since such  $\Omega'$  also serves as a normal form for the  $U_2^{\times n}$ -orbit of  $\Phi$ . First we present Algorithm 3, which is a simple approach for achieving this, but which assumes that  $\Omega$  is not too sparse. Next we present Kraus's solution [37].

By pulling out scalars, the action of  $H^{\times n}$  is the same as the action of  $\mathbb{T}^{\times n}$  plus multiplication by a global scalar  $e^{is}$ , where  $s \in \mathbb{R}$  and

$$\mathbb{T} = \{\text{diag}(1, e^{it}) : t \in \mathbb{R}\}.$$

The action of  $\mathbb{T}^{\times n}$  can be understood through the following formula for basis vectors, which extends to  $\mathcal{H}_n$  by linearity:

$$\begin{pmatrix} 1 \\ e^{it_1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ e^{it_n} \end{pmatrix} |v_1 \dots v_n\rangle = \exp(i(t_1 v_1 + \cdots + t_n v_n)) |v_1 \dots v_n\rangle \quad (3.7)$$

for each  $(v_1, \dots, v_n) \in \{0, 1\}^n$ .

### 3.3.2.1 Simple normal form under unitary stabilizers

For  $1 \leq i \leq n$  let  $\mathbf{v}^i \in \{0, 1\}^n$  be the ‘‘basis vector’’  $\mathbf{v}^i = (\delta_{1i}, \dots, \delta_{ni})$  with 1 in the  $i$ th entry and 0's elsewhere, then let  $\mathcal{B} = \{\mathbf{v}^1, \dots, \mathbf{v}^n\}$ . Algorithm 3 provides a simple way to compute the  $H^{\times n}$ -normal form  $\Omega'$  for  $\Omega$  assuming that  $\Omega_{\mathbf{v}} \neq 0$  whenever  $\mathbf{v} \in \mathcal{B} \cup \{\mathbf{0}\}$ .

### 3.3.2.2 Kraus's algorithm

[37] Algorithm 4 produces an  $H^{\times n}$ -normal form  $\Omega'$  for any  $0 \neq \Omega \in \mathcal{H}_n$ . The following notation is used. Let  $\text{supp}(\Omega) = \{\mathbf{v} : \Omega_{\mathbf{v}} \neq 0\}$  denote the support. We may assume without loss of generality that  $\mathbf{0} \in \text{supp}(\Omega)$ , i.e. that  $\Omega_{\mathbf{0}} \neq 0$ , since  $\Omega$  must have some nonzero entry and we can relabel the basis vectors  $|0\rangle$  and  $|1\rangle$  in the tensor factors of  $\mathcal{H}_n = (\mathbb{C}^2)^{\otimes n}$ . The function  $\arg$  returns the argument of a complex number.

---

**Algorithm 3** Simple normal form under unitary stabilizers

---

*Input:*  $\Omega \in \mathcal{H}_n$  such that  $\Omega_{\mathbf{v}} \neq 0$  whenever  $\mathbf{v} \in \mathcal{B} \cup \{\mathbf{0}\}$ .

*Output:* The unique  $\Omega'$  in the  $H^{\times n}$ -orbit of  $\Omega$  such that each entry  $\Omega'_{\mathbf{v}}$  is real and positive whenever  $\mathbf{v} \in \mathcal{B} \cup \{0\}$ .

1. Update  $\Omega \leftarrow e^{it}\Omega$ , where  $t \in \mathbb{R}$  such that  $e^{it}\Omega_{\mathbf{0}}$  is real and positive.
  2. For  $1 \leq i \leq n$  choose  $t_i \in \mathbb{R}$  so that  $e^{it_i}\Omega_{\mathbf{v}^i}$  is real and positive.
  3. Compute  $\Omega' = \begin{pmatrix} 1 & \\ & e^{it_1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & \\ & e^{it_n} \end{pmatrix} \Omega$ .
- 

---

**Algorithm 4** Kraus's algorithm

---

*Input:*  $\Omega \in \mathcal{H}_n$  such that  $\Omega_{\mathbf{0}} \neq 0$ .

*Output:* The unique  $\Omega'$  in the  $H^{\times n}$ -orbit of  $\Omega$  such that each entry  $\Omega'_{\mathbf{v}}$  is real and positive whenever  $\mathbf{v} \in \mathcal{B} \cup \{0\}$ , where  $\mathcal{B} \subset \text{supp}(\Omega)$  is constructed from  $\text{supp}(\Omega)$  by the algorithm.

1. Update  $\Omega \leftarrow e^{it}\Omega$ , where  $t \in \mathbb{R}$  such that  $e^{it}\Omega_{\mathbf{0}}$  is real and positive.
2. Construct  $\mathcal{B} = \{\mathbf{v}^1, \dots, \mathbf{v}^m\}$  as follows. First set  $\mathcal{B} \leftarrow \emptyset$ . Then, going over elements  $\mathbf{v} \in \text{supp}(\Omega) \setminus \{\mathbf{0}\}$  in increasing lex order append  $\mathbf{v}$  to  $\mathcal{B}$  if  $\mathbf{v}$  is linearly independent over  $\mathbb{R}$  from the vectors already in  $\mathcal{B}$ . Stop once  $\mathcal{B}$  spans the same space as  $\text{supp}(\Omega)$ .
3. Compute any row vector  $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$  satisfying the system

$$(t_1 \quad \dots \quad t_n) (\mathbf{v}^1 \quad \dots \quad \mathbf{v}^m) = -(\arg(\Omega_{\mathbf{v}^1}) \quad \dots \quad \arg(\Omega_{\mathbf{v}^m})).$$

4. Compute  $\Omega' = \begin{pmatrix} 1 & \\ & e^{it_1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & \\ & e^{it_n} \end{pmatrix} \Omega$ .
-

From step 3 we have  $\mathbf{t}\mathbf{v}^i = t_1\mathbf{v}_1^i + \cdots + t_n\mathbf{v}_n^i = -\arg(\Omega_{\mathbf{v}^i})$  for all  $1 \leq i \leq m$ . Then applying Eq. (3.7),

$$\Omega' = \sum_{\mathbf{v} \in \mathcal{B}} \exp(-i \arg(\Omega_{\mathbf{v}})) \Omega_{\mathbf{v}} |\mathbf{v}\rangle + \sum_{\mathbf{v} \in \text{supp}(\Omega) \setminus \mathcal{B}} \Omega'_{\mathbf{v}} |\mathbf{v}\rangle.$$

This expression for  $\Omega'$  shows that the coefficients corresponding to  $\mathbf{v} \in \mathcal{B}$  are real and positive as claimed. By construction,  $\mathcal{B}$  is a basis for the span of  $\text{supp}(\Omega)$ . Then every  $\mathbf{v} \in \text{supp}(\Omega)$  is a unique real linear combination  $\sum_{i=1}^m \alpha_i \mathbf{v}^i$  of vectors  $\mathbf{v}^i \in \mathcal{B}$ . If  $\Omega$  is the tensor obtained after step 3, then

$$\Omega'_{\mathbf{v}} = \exp(i\mathbf{t}\mathbf{v}) \Omega_{\mathbf{v}} = \exp\left(i \sum \alpha_i \mathbf{t}\mathbf{v}^i\right) \Omega_{\mathbf{v}} = \exp\left(-i \sum \alpha_i \arg(\Omega_{\mathbf{v}^i})\right) \Omega_{\mathbf{v}}$$

for every  $\mathbf{v}$  in the support. Therefore any  $\mathbf{t}$  satisfying the system in step 3 produces the same  $\Omega'$ . It follows that  $\Omega'$  is unique. When  $m < n$ , the solution  $\mathbf{t}$  is not unique; this means that  $\Omega'$  has a nontrivial stabilizer in  $\text{GL}_2^{\times n}$ .

### 3.3.3 SLOCC group, even case

Recall that  $T(A \otimes B)T^* \in \text{SO}_4$  when  $A, B \in \text{SL}_2$  (see Example 3.6). Consider the natural action of  $\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$  on the vector space  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . Applying the  $k$ -fold Kronecker product  $T^{\otimes k}$  we obtain

$$\tilde{O}T^{\otimes k}\Phi = T^{\otimes k}\tilde{A}\Phi, \quad \forall \Phi \in (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k} \quad (3.8)$$

where  $\tilde{A} = (A_1 \otimes B_1) \otimes \cdots \otimes (A_k \otimes B_k) \in (\text{SL}_2 \otimes \text{SL}_2)^{\otimes k}$  is the Kronecker product of  $\text{SL}_2$  matrices and  $\tilde{O} = T^{\otimes k}\tilde{A}T^{*\otimes k} \in \text{SO}_4^{\otimes k}$  is the Kronecker product of  $\text{SO}_4$  matrices. We immediately get the following results.

**Lemma 3.21.** *Two tensors  $\Phi, \Phi' \in (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k}$  are in the same  $(\text{SL}_2 \times \text{SL}_2)^{\times k}$ -orbit if and only if  $T^{\otimes k}\Phi$  and  $T^{\otimes k}\Phi'$  are in the same  $\text{SO}_4^{\times k}$ -orbit.*

*Proof.* Follows from Eq. (3.8). □

**Theorem 3.22.** *Let  $k > 1$ . For each tensor  $\Phi \in (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k}$  there exists a core tensor  $\Omega \in (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k}$  such that*

- $\Phi = (A_1 \otimes B_1 \otimes \cdots \otimes A_k \otimes B_k)\Omega$  for some  $A_i, B_i \in \text{SL}_2$ , and
- $D_i = (T^{\otimes k}\Omega)_{(i)}(T^{\otimes k}\Omega)_{(i)}^\top$  is a direct sum of symmetrized Jordan blocks in weakly decreasing order for all  $1 \leq i \leq k$ .

*The core tensor is unique up to the action of  $H_1 \times \cdots \times H_k$ , where  $H_i \leq \text{SL}_2 \otimes \text{SL}_2$  such that  $TH_iT^* \leq \text{SO}_4$  is the stabilizer of  $D_i$  by the conjugation action.*

*Proof 1.* Given  $\Phi \in (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k}$  let  $\Psi = T^{\otimes k}\Phi$ . Applying Theorem 3.14 to  $\Psi$  in  $(\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k}$  as an  $\text{SO}_4^{\times k}$ -module, we get a core tensor  $\Omega$  with certain properties. These properties are equivalent to the ones above due to Lemma 3.21 and the isomorphism (3.3).  $\square$

*Proof 2.* For  $1 \leq i \leq k$  let  $V_i$  be a copy of the  $\text{SL}_2 \times \text{SL}_2$ -module  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . Let  $\mathcal{S}_i$  be the space of  $4 \times 4$  complex symmetric matrices, considered as an  $\text{SL}_2 \times \text{SL}_2$ -module by the same representation as (3.4) with  $\mathcal{S}_i$  replacing  $\mathcal{S}_{ij}$ . By Example 3.6 the map

$$\pi_i : V_1 \otimes \cdots \otimes V_k \rightarrow \mathcal{S}_i \quad \text{where} \quad \pi_i(\Phi) = T\Phi_{(i)}J^{\otimes 2(k-1)}\Phi_{(i)}^\top T^\top$$

is a reduction map. Observing the alternate expression  $\pi_i(\Phi) = (T^{\otimes k}\Phi)_{(i)}(T^{\otimes k}\Phi)_{(i)}^\top$  and applying Lemma 3.8, we immediately have the existence of a core tensor  $\Omega$  with the desired properties.  $\square$

### 3.3.3.1 Reducing the group action

Let  $k > 1$ . Our goal is to compute SLOCC normal forms for general  $\Phi \in \mathcal{H}_{2k} \cong (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k}$ . By Lemma 3.21, this problem is equivalent to computing normal forms for the action of  $\text{SO}_4^{\times k}$ . That is, we could define the SLOCC normal form  $\Omega$  of  $\Phi$  by the property that  $T^{\otimes k}\Omega$  is the  $\text{SO}_4^{\times k}$ -normal form of  $T^{\otimes k}\Phi$  and vice versa. With this in mind, we now focus on the  $\text{SO}_4^{\times k}$ -action.

Suppose  $\Phi \in (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k}$  such that  $\Phi_{(i)}\Phi_{(i)}^\top$  has distinct eigenvalues for all  $1 \leq i \leq k$ . Let  $\Omega$  be the corresponding core tensor in the sense of orthogonal HOSVD, which will have the property

that  $\Omega_{(i)}\Omega_{(i)}^\top$  is diagonal with distinct diagonal entries. By Proposition 3.15, the core tensor  $\Omega$  is unique up to the action of  $H^{\times k}$ , where

$$H = \{D \in \text{SO}_4 : D \text{ is diagonal}\}.$$

Now the problem is reduced to finding an easily computable  $H^{\times k}$ -normal form  $\Omega'$  for  $\Omega$  since such  $\Omega'$  also serves as an  $\text{SO}_4^{\times k}$ -normal form for  $\Phi$ . As in the LU case, we present two algorithms for achieving this. The first (Algorithm 5) is simpler but assumes  $\Omega$  is not too sparse. The second (Algorithm 6) is modeled after Kraus's Algorithm 4.

Another way to understand  $H$  is as the group of diagonal matrices  $D \in \mathbb{C}^{4 \times 4}$  with diagonal entries from  $\{\pm 1\}$ , since  $DD^\top = I$ , and an even number of appearances of  $-1$ , since  $\det(D) = 1$ . These matrices can be written as

$$H = \{ \pm I_4, \quad \pm Z \otimes I_2, \quad \pm I_2 \otimes Z, \quad \pm Z \otimes Z \}$$

where  $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  is the Pauli matrix. Set  $n = 2k$ . From this expression of  $H$  we see that the action of  $H^{\times k}$  on  $(\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k}$  is equivalent to the action of  $\mathcal{T}^{\times n}$  on  $\mathcal{H}_n = (\mathbb{C}^2)^{\otimes n}$ , where  $\mathcal{T} = \{\pm I_2, \pm Z\}$ . The action of  $\mathcal{T}^{\times n}$  can be understood by the following formula for basis vectors, extending to  $\mathcal{H}_n$  by linearity:

$$\begin{pmatrix} 1 & \\ & (-1)^{t_1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & \\ & (-1)^{t_n} \end{pmatrix} |v_1 \dots v_n\rangle = (-1)^{t_1 v_1 + \dots + t_n v_n} |v_1 \dots v_n\rangle \quad (3.9)$$

for  $t_i \in \{0, 1\}$  and  $(v_1, \dots, v_n) \in \{0, 1\}^n$ . While the case  $n = 2k$  is especially important for us, in Sections 3.3.3.2 and 3.3.3.3  $n$  may be even or odd.

### 3.3.3.2 Simple normal form under orthogonal stabilizers

For  $1 \leq i \leq n$  let  $\mathbf{v}^i \in \{0, 1\}^n$  be the “basis vector”  $\mathbf{v}^i = (\delta_{1i}, \dots, \delta_{ni})$  with 1 in the  $i$ th entry and 0’s elsewhere, then let  $\mathcal{B} = \{\mathbf{v}^1, \dots, \mathbf{v}^n\}$ . Algorithm 5 computes the  $\mathcal{T}^{\times n}$ -normal form  $\Omega'$  for  $\Omega$  with the assumption that  $\text{Re}(\Omega_{\mathbf{v}}) \neq 0$  whenever  $\mathbf{v} \in \mathcal{B} \cup \{\mathbf{0}\}$ .

---

#### **Algorithm 5** Simple normal form under orthogonal stabilizers

---

*Input:*  $\Omega \in \mathcal{H}_n$  such that  $\text{Re}(\Omega_{\mathbf{v}}) \neq 0$  whenever  $\mathbf{v} \in \mathcal{B} \cup \{\mathbf{0}\}$ .

*Output:* The unique  $\Omega'$  in the  $\mathcal{T}^{\times n}$ -orbit of  $\Omega$  such that  $\text{Re}(\Omega'_{\mathbf{v}}) > 0$  whenever  $\mathbf{v} \in \mathcal{B} \cup \{\mathbf{0}\}$ .

1. Update  $\Omega \leftarrow (-1)^t \Omega$ , where  $t \in \{0, 1\}$  such that  $(-1)^t \text{Re}(\Omega_{\mathbf{0}})$  is positive.
  2. For  $1 \leq i \leq n$  choose  $t_i \in \{0, 1\}$  such that  $(-1)^{t_i} \text{Re}(\Omega_{\mathbf{v}^i})$  is positive.
  3. Compute  $\Omega' \leftarrow \begin{pmatrix} 1 & \\ & (-1)^{t_1} \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 1 & \\ & (-1)^{t_n} \end{pmatrix} \Omega$ .
- 

### 3.3.3.3 General normal form under orthogonal stabilizers

Algorithm 6 produces a  $\mathcal{T}^{\times n}$ -normal form  $\Omega'$  for any  $0 \neq \Omega \in \mathcal{H}_n$ . Let  $\text{supp}(\Omega) = \{\mathbf{v} : \Omega_{\mathbf{v}} \neq 0\}$ . We may assume without loss of generality that  $\mathbf{0} \in \text{supp}(\Omega)$ , i.e. that  $\Omega_{\mathbf{0}} \neq 0$  since  $\Omega$  must have some nonzero entry and we can relabel the basis vectors  $|0\rangle$  and  $|1\rangle$  in the tensor factors of  $\mathcal{H}_n = (\mathbb{C}^2)^{\otimes n}$ . Define a function  $s : \mathbb{C} \setminus \{0\} \rightarrow \{0, 1\}$  by

$$s(z) = \begin{cases} 0 & \text{if } \text{Re}(z) > 0 \text{ or if } \text{Re}(z) = 0 \text{ and } \text{Im}(z) > 0 \\ 1 & \text{if } \text{Re}(z) < 0 \text{ or if } \text{Re}(z) = 0 \text{ and } \text{Im}(z) < 0 \end{cases}$$

for  $z \in \mathbb{C} \setminus \{0\}$ . We think of  $s$  as detecting the “sign” of  $z$ .

From step 3 of Algorithm 6 we have  $t_1 \mathbf{v}_1^1 + \dots + t_n \mathbf{v}_n^1 \equiv s(\Omega_{\mathbf{v}^i}) \pmod{2}$  for all  $1 \leq i \leq m$ .

Then applying Eq. (3.9),

$$\Omega' = \sum_{\mathbf{v} \in \mathcal{B}} (-1)^{s(\Omega_{\mathbf{v}})} \Omega_{\mathbf{v}} |\mathbf{v}\rangle + \sum_{\mathbf{v} \in \text{supp}(\Omega) \setminus \mathcal{B}} \Omega'_{\mathbf{v}} |\mathbf{v}\rangle$$

---

**Algorithm 6** General normal form under orthogonal stabilizers

---

*Input:* A tensor  $\Omega \in \mathcal{H}_n$ .

*Output:* The unique  $\Omega'$  in the  $\mathcal{T}^{\times n}$ -orbit of  $\Omega$  such that  $s(\Omega'_{\mathbf{v}}) = 0$  whenever  $\mathbf{v} \in \mathcal{B} \cup \{0\}$ , where  $\mathcal{B} \subset \text{supp}(\Omega)$  is constructed from  $\text{supp}(\Omega)$  by the algorithm.

1. Update  $\Omega \leftarrow (-1)^t \Omega$ , where  $t \in \{0, 1\}$  such that  $s((-1)^t \Omega_{\mathbf{0}}) = 0$ .
2. Construct  $\mathcal{B} = \{\mathbf{v}^1, \dots, \mathbf{v}^m\}$  as follows. First set  $\mathcal{B} \leftarrow \emptyset$ . Then, going over elements  $\mathbf{v} \in \text{supp}(\Omega) \setminus \{0\}$  in increasing lex order append  $\mathbf{v}$  to  $\mathcal{B}$  if  $\mathbf{v}$  is linearly independent over  $\mathbb{F}_2$  from the vectors already in  $\mathcal{B}$ . Stop once  $\mathcal{B}$  spans the same space as  $\text{supp}(\Omega)$ .
3. Compute any row vector  $(t_1, \dots, t_n)$  over  $\mathbb{F}_2$  satisfying the system

$$(t_1 \ \dots \ t_n) (\mathbf{v}^1 \ \dots \ \mathbf{v}^m) = (s(\Omega_{\mathbf{v}^1}) \ \dots \ s(\Omega_{\mathbf{v}^m})).$$

4. Compute  $\Omega' \leftarrow \begin{pmatrix} 1 & \\ & (-1)^{t_1} \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 1 & \\ & (-1)^{t_n} \end{pmatrix} \Omega$ .
- 

which shows that  $s(\Omega'_{\mathbf{v}}) = 0$  whenever  $\mathbf{v} \in \mathcal{B} \cup \{0\}$  as claimed. By construction,  $\mathcal{B}$  is a basis for the span of  $\text{supp}(\Omega)$ . Then every  $\mathbf{v} \in \text{supp}(\Omega)$  is a unique linear combination of vectors in  $\mathcal{B}$ . From this fact, together with Eq. (3.9), it follows that  $\Omega'$  is unique (by an argument similar the one given in Section 3.3.2.2).

### 3.3.3.4 The even SLOCC normal form algorithm

For clarity, in Algorithm 7 we list the full steps of the procedure to compute SLOCC normal forms for general  $\Phi \in \mathcal{H}_n$ , with  $n = 2k \geq 4$  even.

---

**Algorithm 7** SLOCC normal form for general qubits, even case

---

*Input:* A tensor  $\Phi \in \mathcal{H}_{2k} \cong (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k}$  with  $k > 1$  such that for each  $1 \leq i \leq k$  the matrix  $(T^{\otimes k} \Phi)_{(i)} (T^{\otimes k} \Phi)_{(i)}^\top$  has distinct eigenvalues.

*Output:* Normal form  $\Omega$  in the SLOCC orbit of  $\Phi$ .

1. Set  $\Phi' \leftarrow T^{\otimes k} \Phi$ .
  2. Use Algorithm 2 to compute a core tensor  $\Omega'$  for  $\Phi'$  in the sense of Theorem 3.14.
  3. Use Algorithm 6 to compute the normal form  $\Omega''$  in the  $\mathcal{T}^{\times 2k}$ -orbit of  $\Omega'$ .
  4. Set  $\Omega \leftarrow T^{*\otimes k} \Omega''$ .
-

### 3.3.3.5 The 4-qubit case

Let  $V = \mathbb{C}^2 \otimes \mathbb{C}^2$ . Recall that, by Lemma 3.21, the problem of classifying  $(\mathrm{SL}_2 \times \mathrm{SL}_2)^{\times 2}$ -orbits in  $V \otimes V$  is equivalent to that of classifying  $\mathrm{SO}_4 \times \mathrm{SO}_4$ -orbits in  $V \otimes V$ . More generally, let us consider  $\mathrm{SO}_{d_1} \times \mathrm{SO}_{d_2}$ -orbits in  $V_1 \otimes V_2$ , where  $d_1 = \dim V_1$  and  $d_2 = \dim V_2$ . Assume without loss of generality that  $d_1 \leq d_2$ .

**Lemma 3.23.** *Let  $A \in \mathbb{C}^{d_1 \times d_2}$  and  $B \in \mathbb{C}^{d_2 \times d_1}$ . Then  $AB$  and  $BA$  have the same nonzero eigenvalues, counting multiplicity.*

*Proof.* See Theorem 2.8 in [64]. □

Suppose  $M \in \mathbb{C}^{d_1 \times d_2}$  is general in the sense that  $MM^\top$  has distinct eigenvalues, none of them 0. By Lemma 3.23,  $M^\top M$  has  $d_1$  nonzero eigenvalues, so the rank of  $M^\top M$  is  $d_1$  and  $M^\top M$  is semisimple (otherwise the rank would exceed  $d_1$ ). A tensor  $\Phi \in V_1 \otimes V_2$  corresponds to the  $d_1 \times d_2$  matrix  $M = \Phi_{(1)}$ . By this correspondence, Theorem 3.14 implies that there exists a core matrix  $\Omega \in \mathbb{C}^{d_1 \times d_2}$  such that

- $M = U_1 \Omega U_2$ , where  $U_1 \in \mathrm{SO}_{d_1}$  and  $U_2 \in \mathrm{SO}_{d_2}$ ,
- $\Omega \Omega^\top = D_1$  and  $\Omega^\top \Omega = D_2$ , where  $D_1$  and  $D_2$  are diagonal.

From the second property we have  $D_1 \Omega = \Omega \Omega^\top \Omega = \Omega D_2$ . This means that multiplying the rows of  $\Omega$  by the eigenvalues of  $D_1$  is the same as multiplying the columns of  $\Omega$  by the eigenvalues of  $D_2$ . By Lemma 3.23,  $D_1$  and  $D_2$  have the same nonzero eigenvalues. By choice of  $M$ , there are  $d_1$  nonzero eigenvalues and they are distinct. From this we conclude that  $\Omega$  has at most one nonzero entry in each row and in each column. Acting by signed permutation matrices  $P \in \mathrm{SO}_{d_1}$  and  $Q \in \mathrm{SO}_{d_2}$  respectively on the left and right, we can permute rows and columns to obtain a matrix of the form

$$P\Omega Q = \begin{pmatrix} \lambda_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & \lambda_{d_1} & 0 & \dots & 0 \end{pmatrix}$$

where  $\lambda_1, \dots, \lambda_{d_1} \in \mathbb{C}$ . Thus, the orbit of a general  $M \in \mathbb{C}^{d_1 \times d_2}$  intersects the  $d_1$ -dimensional space of such matrices.

This result is a special case of [10, Theorem 2.10]. The theorem of Chterental and Djokovic classifies the various non-general orbits as well, whereas our work generalizes to an arbitrary number of tensor products of  $\mathbb{C}^2$ .

### 3.3.3.6 Failure in a special case

Let us work out an example to illustrate what can happen when  $\Phi$  fails the genericity condition. Consider the matrix multiplication tensor

$$\Phi = \sum_{i,j,k=0}^1 |ik\rangle \otimes |ij\rangle \otimes |jk\rangle \in (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes 3},$$

named so because  $\Phi$  corresponds to the bilinear form  $\mathbb{C}^{2 \times 2} \times \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}^{2 \times 2}$  mapping  $(A, B) \mapsto AB$ . A computation shows that  $\pi_i(\Phi) = 2I_4$  for  $i = 1, 2, 3$ , where  $\pi_i(\Phi) = (T^{\otimes 3}\Phi)_{(i)}(T^{\otimes 3}\Phi)_{(i)}^\top$ . Thus  $\Phi$  is a core tensor for itself in the sense of Theorem 3.22. We do not consider  $\Phi$  to be general since every  $\pi_i(\Phi)$  has only one eigenvalue, not counting multiplicity. The stabilizer subgroup of  $2I_4$  in  $SO_4$  is the entire group  $SO_4$ . Then Theorem 3.22 tells us that the core tensor is unique up to the action of  $H^{\times 3}$ , where  $H = T^*SO_4T = SL_2 \otimes SL_2$ . In other words, every tensor in the SLOCC orbit of  $\Phi$  is a core tensor. This represents the worst-case scenario: Theorem 3.22 does not reduce the problem of detecting tensors in the SLOCC orbit of the matrix multiplication tensor to the action of a smaller group, let alone a finite group.

### 3.3.4 SLOCC group, odd case

Let  $n$  be odd and for  $1 \leq i \leq n$  let  $V_i$  be a copy of the  $SL_2$ -module  $\mathbb{C}^2$ . Let  $\mathcal{S}_i \cong S^2V_i$  be the space of  $2 \times 2$  complex symmetric matrices considered as an  $SL_2$ -module by the action  $A.M = AMA^\top$

for  $A \in \mathrm{SL}_2$  and  $M \in \mathcal{S}_i$ . For each  $i$  define the map  $\pi_i$  as follows:

$$\begin{aligned} \pi_i : V_1 \otimes \cdots \otimes V_n &\rightarrow \mathcal{S}_i \cong S^2 V_i \\ \Phi &\rightarrow \Phi_{(i)} J^{\otimes(n-1)} \Phi_{(i)}^\top. \end{aligned} \quad (3.10)$$

By Example 3.5, the map  $\pi_i$  is a reduction map.

To find normal forms in  $\mathcal{S}_i$ , we use the fact that any complex symmetric matrix  $M \in \mathcal{S}_i$  admits a factorization  $M = ADA^\top$  where  $A \in \mathrm{GL}_2$  and  $D = I_2, \mathrm{diag}(1, 0)$  or  $0$ . Indeed, every nondegenerate complex quadratic form in the variables  $x_1, \dots, x_n$  is equivalent to the form  $x_1^2 + \cdots + x_n^2$ ; see [19, pp. 97-98]. If we require that  $A$  is in the smaller group  $\mathrm{SL}_2$ , we have a factorization  $M = ADA^\top$  where  $D$  equals  $zI_2, \mathrm{diag}(z, 0)$  or  $0$  for some  $z \in \mathbb{C}$  unique up to multiplication by  $-1$ . The parameter  $z$  is not unique because its argument can be flipped by the action of the matrix

$$K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathrm{SL}_2. \quad (3.11)$$

Thus, we define a normal form function  $F: \mathcal{S}_i \rightarrow \mathcal{S}_i$  by setting  $F(M) = zI_2$ , where  $z$  is the greater of the two choices in lexicographical order on  $\mathbb{C}$ .

Alternatively, one can write  $\pi_i(\Phi)$  as  $\Phi_{(i)} T^{*\otimes(n-1)/2} (\Phi_{(i)} T^{*\otimes(n-1)/2})^\top$  and prove that  $\pi_i$  is a reduction map using the fact that (3.3) is an isomorphism. With this setup, we obtain the following lemma.

**Lemma 3.24.** *Let  $n$  be odd. For each tensor  $\Phi \in \mathcal{H}_n$  there exists a core tensor  $\Psi \in \mathcal{H}_n$  such that*

- $\Phi = (A_1 \otimes \cdots \otimes A_n) \Psi$  for some  $A_i \in \mathrm{SL}_2$ , and
- $D_i = \Psi_{(i)} J^{\otimes(n-1)} \Psi_{(i)}^\top$  equals  $z_i I_2, \mathrm{diag}(z_i, 0)$  or  $0$  where  $z_i \in \mathbb{C}$  and  $z_i > -z_i$  in lex order for all  $1 \leq i \leq n$ .

The core tensor  $\Psi$  is unique up to the action of  $H_1 \times \cdots \times H_n$ , where  $H_i \leq \mathrm{SL}_2$  is the stabilizer of  $D_i$  with respect to the action  $A.D_i = AD_i A^\top$  for  $A \in \mathrm{SL}_2$ .

*Proof.* Apply Lemma 3.8. □

**Theorem 3.25.** *Let  $n \geq 5$  be odd. For general  $\Phi \in \mathcal{H}_n$  there exists a core tensor  $\Omega \in \mathcal{H}_n$ , unique up to sign, such that*

- $\Phi = (A_1 \otimes \cdots \otimes A_n)\Omega$  for some  $A_i \in \text{SL}_2$ ,
- For all  $1 \leq i \leq n$ ,  $\Omega_{(i)} J^{\otimes(n-1)} \Omega_{(i)}^\top = z_i I_2$  with  $z_i > -z_i$  in lex order and  $\Omega_{(i)} \Omega_{(i)}^\top = \text{diag}(\lambda_1, \lambda_2)$  with  $\lambda_1 > \lambda_2$  in lex order.

*Proof.* Let  $\pi_i(\Phi) = \Phi_{(i)} J^{\otimes(n-1)} \Phi_{(i)}^\top$ . Suppose  $\Phi \in \mathcal{H}_n$  such that the matrix  $\pi_i(\Phi)$  is invertible for all  $1 \leq i \leq n$ . Then, by Lemma 3.24, there exists a core tensor  $\Psi$  in the SLOCC orbit of  $\Phi$  such that  $\pi_i(\Psi) = z_i I_2$  is a scalar multiple of the identity. The stabilizer of  $z_i I_2$  in  $\text{SL}_2$  is the orthogonal group  $\text{SO}_2$ . Therefore the core tensor  $\Psi$  is unique up to the action of  $\text{SO}_2^{\times n}$ .

Suppose further that  $\Psi_{(i)} \Psi_{(i)}^\top$  has distinct eigenvalues for each  $i$ . Then, by Theorem 3.14, there exists a core tensor  $\Omega$  in the  $\text{SO}_2^{\times n}$ -orbit of  $\Psi$  (hence in the SLOCC orbit of  $\Phi$ ) such that  $\Omega_{(i)} \Omega_{(i)}^\top$  is diagonal with decreasing diagonal entries for all  $i$ . By Proposition 3.15,  $\Omega$  is unique up to the action of  $H^{\times n}$ , where  $H = \{\pm I_2\}$ . Thus  $\Omega$  is unique up to sign.

It remains to prove that the existence of  $\Omega$  is generic. That is,  $\Psi_{(i)} \Psi_{(i)}^\top$  having distinct eigenvalues must be a generic property of  $\Phi \in \mathcal{H}_n$ . We state this fact in the following lemma and postpone the proof to Section 3.3.4.2. □

**Lemma 3.26** (Genericity lemma). *Let  $n \geq 5$  be odd,  $\Phi \in \mathcal{H}_n$  be general, and  $\Psi$  be the corresponding core tensor in the sense of Lemma 3.24. Then each  $\Psi_{(i)} \Psi_{(i)}^\top$  has distinct eigenvalues.*

Theorem 3.25 immediately gives us a way to find normal forms for general  $\Phi \in \mathcal{H}_n$  when  $n \geq 5$ . Given  $\Phi$ , there exists a core tensor  $\Omega$  that is unique up to sign. So pick  $\Omega$  or  $-\Omega$  to be the normal form; one way to make this selection is described in step 5 of Algorithm 8. The case where  $n = 3$  is discussed in Section 3.3.4.3.

### 3.3.4.1 The odd SLOCC normal form algorithm

We now describe an algorithm for computing SLOCC normal forms for general  $\Phi \in \mathcal{H}_n$ , with  $n \geq 5$  odd. First, we need a way to compute the normal form function  $F: \mathcal{S}_i \rightarrow \mathcal{S}_i$  on the space of  $2 \times 2$  complex symmetric matrices. If  $M \in \mathcal{S}_i$  is invertible, we can do this by  $M \mapsto LML^\top = \sqrt{\delta}I_2$ , where  $\delta = \det(M)$  and  $L \in \text{SL}_2$  is given by

$$L = \begin{cases} \begin{pmatrix} M_{11}^{-1/2}\delta^{1/4} & 0 \\ 0 & M_{11}^{1/2}\delta^{-1/4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -M_{12}M_{11}^{-1} & 1 \end{pmatrix} & \text{if } M_{11} \neq 0, \\ \begin{pmatrix} M_{22}^{1/2}\delta^{-1/4} & 0 \\ 0 & M_{22}^{-1/2}\delta^{1/4} \end{pmatrix} \begin{pmatrix} 1 & -M_{12}M_{22}^{-1} \\ 0 & 1 \end{pmatrix} & \text{if } M_{22} \neq 0, \\ \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \frac{i}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } M_{11} = M_{22} = 0. \end{cases}$$

If  $LML^\top$  is not in normal form (i.e. if  $\sqrt{\delta} < -\sqrt{\delta}$  in lex order), then apply  $M \mapsto K L M L^\top K^\top$ , where  $K$  is defined in (3.11).

In Algorithm 8,  $\pi_i$  is the map defined in (3.10). Steps 1-3 compute the core tensor  $\Psi$  in the sense of Lemma 3.24. Step 4 computes the core tensor  $\Omega$  in the sense of Theorem 3.25. Then step 5 picks out the normal form from the choices  $\pm\Omega$ .

### 3.3.4.2 Proof of the genericity lemma

We now turn to the proof of Lemma 3.26. Let  $n > 1$  be odd. Suppose  $\Phi \in \mathcal{H}_n$  such that  $M_i = \Phi_{(i)} J^{\otimes(n-1)} \Phi_{(i)}^\top$  is invertible and  $(M_i)_{11} \neq 0$  for all  $1 \leq i \leq n$ . Feed  $\Phi$  into Algorithm 8. The entries of each matrix  $L_i$  computed in step 1 are functions of  $M_i$ . Moreover, the entries of  $M_i$  are quadratic forms in  $\mathbb{C}[\mathcal{H}_n]$ . Therefore the tensor  $\Psi = (L_1 \otimes \cdots \otimes L_n)\Phi$  computed in step 2 is a function of  $\Phi$ . In step 4, Algorithm 2 is applied; for this to be possible, the discriminant of the

---

**Algorithm 8** SLOCC normal form for general qubits, odd case

---

*Input:* A tensor  $\Phi \in \mathcal{H}_n$  general in the sense of Theorem 3.25,  $n \geq 5$  odd.

*Output:* Normal form  $\Omega$  in the SLOCC orbit of  $\Phi$ .

1. For  $1 \leq i \leq n$  use the formula above to compute  $L_i \in \text{SL}_2$  such that  $L_i \pi_i(\Phi) L_i^\top = \sqrt{\delta_i} I_2$ , where  $\delta_i = \det(\pi_i(\Phi))$ .
  2. Set  $\Psi \leftarrow (L_1 \otimes \cdots \otimes L_n) \Phi$  so that  $\pi_i(\Psi) = \sqrt{\delta_i} I_2$  for all  $i$ .
  3. Update  $\Psi \leftarrow (A_1 \otimes \cdots \otimes A_n) \Psi$ , where  $A_i$  equals  $K$  [see (3.11)] if  $\pi_i(\Psi) = \sqrt{\delta_i} I_2$  is not in normal form, i.e. if  $\sqrt{\delta_i} < -\sqrt{\delta_i}$  in lex order, otherwise  $A_i = I_2$ .
  4. Use Algorithm 2 to compute a core tensor  $\Omega$  for  $\Psi$  in the sense of Theorem 3.14.
  5. If the first nonzero entry  $a \in \mathbb{C}$  of  $\Omega$  is less than  $-a$  in lex order, update  $\Omega \leftarrow -\Omega$ .
- 

characteristic polynomial

$$p_i(\Psi) = \text{disc}(\det(\lambda I_2 - \Psi_{(i)} \Psi_{(i)}^\top)) = \text{tr}(\Psi_{(i)} \Psi_{(i)}^\top)^2 - 4 \det(\Psi_{(i)} \Psi_{(i)}^\top)$$

must not vanish (this is the distinct-eigenvalues condition in Lemma 3.26). This is not affected by step 3. By the discussion above,  $p_i : W_i \rightarrow \mathbb{C}$  is a function on an open, dense, full-measure subset  $W_i \subset \mathcal{H}_n$ . Moreover,  $p_i$  is holomorphic on an open, full-measure subset  $W'_i \subset W_i$ . If  $p_i$  is not identically 0 on  $W'_i$ , then (by Lemma 3.20) Algorithm 8 works for general  $\Phi \in \mathcal{H}_n$  as claimed. To show that  $p_i \neq 0$ , it suffices to exhibit  $\Phi \in \mathcal{H}_n$  such that, for all  $1 \leq i \leq n$ ,

1.  $M_i = \Phi_{(i)} J^{\otimes(n-1)} \Phi_{(i)}^\top$  is a multiple of the identity, and
2.  $\Phi_{(i)} \Phi_{(i)}^\top$  has distinct eigenvalues.

The first condition implies that  $M_i$  is invertible,  $(M_i)_{11} \neq 0$ , and  $\Psi = (L_1 \otimes \cdots \otimes L_n) \Phi = (I \otimes \cdots \otimes I) \Phi = \Phi$ . Then the second condition implies  $p_i(\Psi) \neq 0$ .

In the calculations to follow, let  $n = 2k + 1$ . Given  $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_n) \in \{0, 1\}^n$  let  $|\mathbf{v}|$  be the number of nonzero entries in  $\mathbf{v}$ . Define  $\bar{\mathbf{v}} \in \{0, 1\}^n$  by

$$\bar{\mathbf{v}}_i = \begin{cases} 0 & \text{if } \mathbf{v}_i = 1, \\ 1 & \text{if } \mathbf{v}_i = 0. \end{cases}$$

The **Hamming distance** between  $\mathbf{v}_1, \mathbf{v}_2 \in \{0, 1\}^n$  is the number of occurrences where  $(\mathbf{v}_1)_i \neq (\mathbf{v}_2)_i$  for  $1 \leq i \leq n$ . We claim that the following tensor satisfies properties 1 and 2 above:

$$\Phi = (1 - 2^{2k-1}) |0\rangle + \sum_{\mathbf{v} \in \mathcal{E} \setminus \{0\}} |\mathbf{v}\rangle$$

where  $\mathcal{E}$  is the set of vectors  $\mathbf{v} \in \{0, 1\}^n$  such that  $|\mathbf{v}|$  is even. Splitting terms of  $\Phi$  into two groups depending on whether the first tensor factor is  $|0\rangle$  or  $|1\rangle$ , we write

$$\Phi = \sum_{\mathbf{w} \in \{0,1\}^{2k}} a_{\mathbf{w}} |0\rangle \otimes |\mathbf{w}\rangle + \sum_{\mathbf{w} \in \{0,1\}^{2k}} b_{\mathbf{w}} |1\rangle \otimes |\mathbf{w}\rangle$$

where each  $a_{\mathbf{w}}, b_{\mathbf{w}}$  is equal to 0, 1, or  $1 - 2^{2k-1}$ . Then the mode-1 flattening reads

$$\Phi_{(1)} = \begin{pmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \end{pmatrix} = \begin{pmatrix} a_{\mathbf{w}_1} & a_{\mathbf{w}_2} & \dots & a_{\mathbf{w}_N} \\ b_{\mathbf{w}_1} & b_{\mathbf{w}_2} & \dots & b_{\mathbf{w}_N} \end{pmatrix}$$

where  $N = 2^{2k}$ . The images of the reduction maps are

$$\Phi_{(1)}^\top J^{\otimes 2k} \Phi_{(1)} = \begin{pmatrix} \mathbf{a}^\top J^{\otimes 2k} \mathbf{a} & \mathbf{a}^\top J^{\otimes 2k} \mathbf{b} \\ \mathbf{a}^\top J^{\otimes 2k} \mathbf{b} & \mathbf{b}^\top J^{\otimes 2k} \mathbf{b} \end{pmatrix} \quad \text{and} \quad \Phi_{(1)}^\top \Phi_{(1)} = \begin{pmatrix} \mathbf{a}^\top \mathbf{a} & \mathbf{a}^\top \mathbf{b} \\ \mathbf{a}^\top \mathbf{b} & \mathbf{b}^\top \mathbf{b} \end{pmatrix}.$$

The operator  $J$  maps  $|0\rangle \mapsto -|1\rangle$  and  $|1\rangle \mapsto |0\rangle$ . It follows that  $J^{\otimes 2k} |\mathbf{w}\rangle = (-1)^{|\bar{\mathbf{w}}|} |\bar{\mathbf{w}}\rangle$  and we find

$$\begin{aligned} \mathbf{a}^\top J^{\otimes 2k} \mathbf{b} &= \left( \sum_{i=1}^N a_{\mathbf{w}_i} |\mathbf{w}_i\rangle \right)^\top J^{\otimes 2k} \left( \sum_{j=1}^N b_{\mathbf{w}_j} |\mathbf{w}_j\rangle \right) \\ &= \left( \sum_{i=1}^N a_{\mathbf{w}_i} |\mathbf{w}_i\rangle \right)^\top \left( \sum_{j=1}^N b_{\mathbf{w}_j} (-1)^{|\bar{\mathbf{w}}_j|} |\bar{\mathbf{w}}_j\rangle \right) \\ &= \sum_{\mathbf{w} \in \{0,1\}^{2k}} (-1)^{|\mathbf{w}|} a_{\mathbf{w}} b_{\bar{\mathbf{w}}}. \end{aligned}$$

Notice that  $|0\rangle \otimes |\mathbf{w}\rangle$  and  $|1\rangle \otimes |\mathbf{w}\rangle$  differ in one tensor factor, whereas the Hamming distance between points in  $\mathcal{E}$  is even. Then one of  $a_{\mathbf{w}}$  and  $b_{\mathbf{w}}$  must be 0, hence  $\mathbf{a}^\top \mathbf{b} = \sum a_{\mathbf{w}} b_{\mathbf{w}} = 0$ . Since  $\mathbf{w}$  has  $2k$  entries,  $|\mathbf{w}|$  and  $|\bar{\mathbf{w}}|$  have the same parity. Thus, one of  $a_{\mathbf{w}}$  and  $b_{\bar{\mathbf{w}}}$  is 0 and we similarly have  $\mathbf{a}^\top J^{\otimes 2k} \mathbf{b} = \sum (-1)^{|\mathbf{w}|} a_{\mathbf{w}} b_{\bar{\mathbf{w}}} = 0$ . This shows that  $\Phi_{(1)}^\top J^{\otimes 2k} \Phi_{(1)}^\top$  and  $\Phi_{(1)}^\top \Phi_{(1)}^\top$  are diagonal. Note that, if  $\mathbf{w} \neq \mathbf{0}$ , then  $a_{\mathbf{w}}$  equals 1 if  $|\mathbf{w}|$  is even and 0 otherwise. Thus we compute

$$\begin{aligned} \mathbf{a}^\top J^{\otimes 2k} \mathbf{a} &= \sum (-1)^{|\mathbf{w}|} a_{\mathbf{w}} a_{\bar{\mathbf{w}}} \\ &= \sum a_{\mathbf{w}} a_{\bar{\mathbf{w}}} \\ &= 2(a_{\mathbf{0}} a_{\bar{\mathbf{0}}}) + \sum_{\mathbf{w} \in \{0,1\}^{2k} \setminus \{\mathbf{0}, \bar{\mathbf{0}}\}} a_{\mathbf{w}} a_{\bar{\mathbf{w}}} \\ &= 2(1 - 2^{2k-1}) + (2^{2k-1} - 2). \end{aligned}$$

Similarly,  $b_{\mathbf{w}}$  equals 1 if  $|\mathbf{w}|$  is odd and 0 otherwise, so

$$\mathbf{b}^\top J^{\otimes 2k} \mathbf{b} = \sum (-1)^{|\mathbf{w}|} b_{\mathbf{w}} b_{\bar{\mathbf{w}}} = -2^{2k-1}.$$

It follows that  $\mathbf{a}^\top J^{\otimes 2k} \mathbf{a} = \mathbf{b}^\top J^{\otimes 2k} \mathbf{b}$  so that  $\Omega_{(1)}^\top J^{\otimes 2k} \Omega_{(1)}^\top$  is a multiple of the identity. Additionally, we have

$$\mathbf{a}^\top \mathbf{a} = \sum a_{\mathbf{w}}^2 = (2^{2k-1} - 1)^2 + (2^{2k-1} - 1), \quad \text{and} \quad \mathbf{b}^\top \mathbf{b} = 2^{2k-1}.$$

which gives  $\mathbf{a}^\top \mathbf{a} - \mathbf{b}^\top \mathbf{b} = (2^{2k-1} - 1)^2 - 1$ . Thus  $\mathbf{a}^\top \mathbf{a} \neq \mathbf{b}^\top \mathbf{b}$  so that  $\Phi_{(1)}^\top \Phi_{(1)}^\top$  has distinct eigenvalues when  $k \geq 2$  or  $n = 2k + 1 \geq 5$ . We have equality  $\mathbf{a}^\top \mathbf{a} = \mathbf{b}^\top \mathbf{b}$  when  $k = 1$  or  $n = 3$ . Due to the symmetry of the tensor  $\Phi$ , the same calculations apply for flattenings  $\Phi_{(i)}$  with  $i > 1$ . This concludes the proof of Lemma 3.26.

### 3.3.4.3 The 3-qubit case

Using the methods we have developed, we now examine the orbit classification problem for the SLOCC group action on  $\mathcal{H}_3$ . Suppose  $\Phi \in \mathcal{H}_3$  such that  $\Phi_{(i)}(J \otimes J)\Phi_{(i)}^\top$  is invertible for  $i = 1, 2, 3$ . By Lemma 3.24, there exists  $\Psi$  in the SLOCC orbit of  $\Phi$  such that  $\Psi_{(i)}(J \otimes J)\Psi_{(i)}^\top$  is a multiple of the identity  $\forall i$ . Running the first two steps of Algorithm 8 on a computer with randomly selected  $\Phi$ , we find that the output  $\Psi = av_1 + bv_2$  is always a complex linear combination of the tensors

$$v_1 = |001\rangle + |010\rangle + |100\rangle - |111\rangle, \quad v_2 = |101\rangle + |110\rangle - |000\rangle + |011\rangle.$$

A direct computation shows that

$$U(z_1) \otimes U(z_2) \otimes U(z_3)(av_1 + bv_2) = \begin{pmatrix} v_1 & v_2 \end{pmatrix} U(z_1 + z_2 + z_3) \begin{pmatrix} a \\ b \end{pmatrix},$$

where  $U(z) = \begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}$  for  $z \in \mathbb{C}$ . Therefore the  $\text{SO}_2^{\times 3}$ -representation  $\mathcal{H}_3$  contains a subrepresentation  $\text{SO}_2^{\times 3} \rightarrow \text{SO}_2 \rightarrow \text{GL}(\{v_1, v_2\})$ . Since  $\text{SO}_2$  is abelian, the subrepresentation splits further into two 1-dimensional subrepresentations which correspond to the simultaneous eigenvectors  $w_1 = v_1 + iv_2$  and  $w_2 = v_1 - iv_2$  associated to characters  $e^{-iz}$  and  $e^{iz}$  respectively, i.e.

$$U(z) \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-iz} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad U(z) \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{iz} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \forall z \in \mathbb{C}.$$

Given  $\alpha, \beta \in \mathbb{C}$  we can set  $z = (\arg \alpha + \arg \beta)/2 + i \ln \sqrt{|\alpha\beta|}$  so that

$$e^{-iz} \alpha = e^{iz} \beta = \sqrt{|\alpha\beta|} \exp(i(\arg \alpha + \arg \beta)/2).$$

Then  $I_2 \otimes I_2 \otimes U(z)(\alpha w_1 + \beta w_2)$  is a scalar multiple of  $v_1$ . Hence we obtain normal forms for almost all tensors constituting the line

$$av_1 = a(|001\rangle + |010\rangle + |100\rangle - |111\rangle), \quad a \in \mathbb{C}.$$

The above is true for the specific case when  $\Phi = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$  is the GHZ state. Thus we recover the fact that almost all states in  $\mathbb{P}\mathcal{H}_3$  are SLOCC equivalent to the GHZ state. This normal form for the GHZ state also appears in the Freudenthal triple system classification from [6].

Normal form of $\Phi$	rank $\pi_1(\Phi)$	rank $\pi_2(\Phi)$	rank $\pi_3(\Phi)$
$ 000\rangle +  111\rangle$	2	2	2
$ 001\rangle +  010\rangle +  100\rangle$	1	1	1
$ 001\rangle +  111\rangle$	0	0	1
$ 010\rangle +  111\rangle$	0	1	0
$ 100\rangle +  111\rangle$	1	0	0
$ 000\rangle$	0	0	0

Table 3.2: Ranks of the reduction maps for 3 qubits.

It is known that there are six SLOCC equivalence classes in  $\mathbb{P}\mathcal{H}_3$  with normal forms listed in Table 3.2. We find that the ranks of the matrices  $\pi_i(\Phi) = \Phi_{(i)}(J \otimes J)\Phi_{(i)}^\top$  for  $i = 1, 2, 3$  is enough to distinguish SLOCC orbits in  $\mathbb{P}\mathcal{H}_3$ . It is known that ranks of flattenings and the hyperdeterminant also distinguish these orbits. The multilinear ranks separate all but the top two orbits. The top two orbits both have multilinear rank (2,2,2) and the (non)-vanishing of the hyperdeterminant separates these two (see [22, Ex. 4.5, p. 478]).

## Acknowledgements

The authors recognize partial support from the Thomas Jefferson Foundation, CNRS, and UTBM. The authors would also like to thank Frédéric Holweck and Nick Vannieuwenhoven for useful discussions. We thank the anonymous referees who provided useful feedback.

## References

- [1] Antonio Acín, A Andrianov, L Costa, E Jané, JI Latorre, and Rolf Tarrach, *Generalized Schmidt decomposition and classification of three-quantum-bit states*, Physical Review Letters **85** (2000), no. 7, 1560, arXiv:quant-ph/0003050, doi:10.1103/physrevlett.85.1560.
- [2] L. V. Antonyan, *Classification of four-vectors of an eight-dimensional space*, Trudy Sem. Vektor. Tenzor. Anal. (1981), no. 20, 144–161. MR 622013
- [3] Daniel J. Bates, Jonathan D. Hauenstein, Andrew J. Sommese, and Charles W. Wampler, *Bertini: Software for numerical algebraic geometry*, Available at [bertini.nd.edu](http://bertini.nd.edu), dx.doi.org/10.7274/R0H41PB5.
- [4] Genrich R Belitskii and Vladimir V Sergeichuk, *Complexity of matrix problems*, Linear Algebra and its applications **361** (2003), 203–222, doi:10.1016/s0024-3795(02)00391-9.
- [5] Alessandra Bernardi, Martina Iannacito, and Duccio Rocchini, *High order singular value decomposition for plant diversity estimation*, Bollettino dell’Unione Matematica Italiana **14** (2021), no. 4, 557–591, doi:10.1007/s40574-021-00300-w.
- [6] L Borsten, *Freudenthal ranks: GHZ versus W*, Journal of Physics A: Mathematical and Theoretical **46** (2013), no. 45, 455303, doi:10.1088/1751-8113/46/45/455303.
- [7] Paul Breiding and Sascha Timme, *HomotopyContinuation.jl: A Package for Homotopy Continuation in Julia*, Mathematical Software – ICMS 2018 (Cham) (James H. Davenport,

- Manuel Kauers, George Labahn, and Josef Urban, eds.), Springer International Publishing, 2018, pp. 458–465.
- [8] Iain D. K. Brown, Susan Stepney, Anthony Sudbery, and Samuel L. Braunstein, *Searching for highly entangled multi-qubit states*, Journal of Physics A: Mathematical and General **38** (2005), no. 5, 1119.
- [9] Lin Chen and Dragomir Ž. Đoković, *Proof of the Gour-Wallach conjecture*, Phys. Rev. A **88** (2013), 042307.
- [10] O. Chterental and D. Djokovic, *Normal Forms and Tensor Ranks of Pure States of Four Qubits*, Linear Algebra Research Advances (G. D. Ling, ed.), Nova Science Publishers Inc., 2007, doi:10.48550/arXiv.quant-ph/0612184, p. 133–167.
- [11] Lieven De Lathauwer, Bart De Moor, and Joos Vandewalle, *A multilinear singular value decomposition*, SIAM Journal on Matrix Analysis and Applications **21** (2000), no. 4, 1253–1278, doi:10.1137/s0895479896305696.
- [12] Gopikrishna Deshpande, D Rangaprakash, Luke Oeding, Andrzej Cichocki, and Xiaoping Hu, *A new generation of brain-computer interfaces driven by discovery of latent eeg-fmri linkages using tensor decomposition*, Frontiers in Neuroscience **11** (2017), 246, doi:10.3389/fnins.2017.00246.
- [13] Sabino Di Trani, Willem A de Graaf, and Alessio Marrani, *Classification of real and complex 3-qutrit states*, arXiv preprint arXiv:2305.01270 (2023), doi:10.1063/5.0156805.
- [14] Heiko Dietrich, Willem A de Graaf, Alessio Marrani, and MaFrcos Origlia, *Classification of four qubit states and their stabilisers under SLOCC operations*, Journal of Physics A: Mathematical and Theoretical (2022), arXiv:2111.05488, doi:10.1088/1751-8121/ac4b13.

- [15] Wolfgang Dür, Guifre Vidal, and J Ignacio Cirac, *Three qubits can be entangled in two inequivalent ways*, *Physical Review A* **62** (2000), no. 6, 062314, doi:10.1103/physreva.62.062314.
- [16] M. Enríquez, I. Wintrowicz, and K. Życzkowski, *Maximally entangled multipartite states: A brief survey*, *Journal of Physics: Conference Series* **698** (2016), no. 1, 012003.
- [17] Marco Enríquez, Zbigniew Puchała, and Karol Życzkowski, *Minimal Rényi–Ingarden–Urbanik entropy of multipartite quantum states*, *Entropy* **17** (2015), no. 7, 5063–5084.
- [18] Evgeny Frolov and Ivan Oseledets, *Tensor methods and recommender systems*, *Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery* **7** (2017), no. 3, e1201, doi:10.1002/widm.1201.
- [19] W. Fulton and J. Harris, *Representation theory, a first course*, *Graduate Texts in Mathematics*, vol. 129, New York: Springer-Verlag, 1991, doi:10.2307/3618117.
- [20] F. R. Gantmacher, *Applications of the theory of matrices*, Translated by J. L. Brenner, with the assistance of D. W. Bushaw and S. Evanusa, Interscience Publishers, Inc., New York, 1959, doi:10.1126/science.131.3398.405-b.
- [21] ———, *The theory of matrices*, vol. 2, Chelsea Publishing, 1990.
- [22] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, *Mathematics: Theory & Applications*, Boston: Birkhäuser, Boston, MA, 1994, doi:10.1007/978-0-8176-4771-1\_14.
- [23] Gilad Gour, Barbara Kraus, and Nolan R. Wallach, *Almost all multipartite qubit quantum states have trivial stabilizer*, *J. Math. Phys.* **58** (2017), no. 9, 092204, 14, doi:10.1063/1.5003015.

- [24] Gilad Gour and Nolan R. Wallach, *All maximally entangled four-qubit states*, Journal of Mathematical Physics **51** (2010), 112201.
- [25] Gilad Gour and Nolan R Wallach, *Necessary and sufficient conditions for local manipulation of multipartite pure quantum states*, New Journal of Physics **13** (2011), no. 7, 073013, doi:10.1088/1367-2630/13/7/073013.
- [26] Gilad Gour and Nolan R. Wallach, *On symmetric  $SL$ -invariant polynomials in four qubits*, Symmetry: Representation Theory and Its Applications, Progress in Mathematics, vol. 257, Birkhäuser, New York, NY, 2014.
- [27] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [28] Robert C. Gunning and Hugo Rossi, *Analytic functions of several complex variables*, vol. 368, American Mathematical Society, 2022, doi:10.1090/chel/368.
- [29] A. Higuchi and A. Sudbery, *How entangled can two couples get?*, Physics Letters A **273** (2000), no. 4, 213–217.
- [30] F. Holweck and L. Oeding, *Hyperdeterminants from the  $E_8$  Discriminant*, J. Algebra (2022), arXiv:1810.05857, doi:10.1016/j.jalgebra.2021.10.017.
- [31] Frederic Holweck and Luke Oeding, *Toward jordan decompositions for tensors*, Journal of Computational Science **82** (2024), 102431, arXiv:2206.13662.
- [32] Frédéric Holweck and Luke Oeding, *A hyperdeterminant on Fermionic Fock Space*, Annales de l’Institut Henri Poincaré D: Combinatorics, Physics and their Interactions (2024), arXiv:2301.10660, doi:10.4171/aihpd/187.
- [33] Hamza Jaffali, Frédéric Holweck, and Luke Oeding, *Maximally entangled real states and slocc invariants: the 3-qutrit case*, 2023, arXiv:2307.00970.

- [34] Andrew Jena, *Partitioning pauli operators : in theory and in practice*, Master's thesis, University of Waterloo, 2019.
- [35] C. Kelleher, M. Roomy, and F. Holweck, *Implementing 2-qubit pseudo-telepathy games on noisy intermediate-scale quantum computers*, *Quantum Inf Process* **23** (2024), no. 187.
- [36] T. Kolda and B. Bader, *Tensor decompositions and applications*, *SIAM Rev.* **51** (2009), no. 3, 455–500, doi:10.1137/07070111X.
- [37] B Kraus, *Local unitary equivalence of multipartite pure states*, *Physical Review Letters* **104** (2010), no. 2, 020504, doi:10.1103/physrevlett.104.020504.
- [38] J. Landsberg, *Quantum computation and quantum information*, American Mathematical Society, June 2024, doi:10.1090/gsm/243.
- [39] J.M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012, doi:10.1090/gsm/128.
- [40] Joseph M Landsberg, *A very brief introduction to quantum computing and quantum information theory for mathematicians*, *Quantum Physics and Geometry* (2019), 5–41.
- [41] D. Li, *SLOCC classification of  $n$  qubits invoking the proportional relationships for spectrums and standard Jordan normal forms*, *Quantum Inf Process* **17** (2018), no. 1, doi:10.1007/s11128-017-1770-0.
- [42] Akimasa Miyake, *Classification of multipartite entangled states by multidimensional determinants*, *Physical Review A* **67** (2002).
- [43] Carmeliza Navasca and Deonnia N Pompey, *Random projections for low multilinear rank tensors*, *Visualization and Processing of Higher Order Descriptors for Multi-Valued Data*, Springer, 2015, doi:10.1007/978-3-319-15090-1\_5, pp. 93–106.
- [44] K. O. Ng, *The classification of  $(3, 3, 3)$  trilinear forms*, *J. Reine Angew. Math.* **468** (1995), 49–75, doi:10.1515/crll.1995.468.49.

- [45] Michael A Nielsen and Isaac Chuang, *Quantum computation and quantum information*, 2002, doi:10.1017/cbo9780511976667.
- [46] Michael A. Nielsen and Isaac L. Chuang, *Quantum computation and quantum information*, Massachusetts Institute of Technology, 2010.
- [47] A. G. Nurmiev, *Orbits and invariants of third-order matrices*, Mat. Sb. **191** (2000), no. 5, 101–108, doi:10.1070/sm2000v191n05abeh000478.
- [48] Luke Oeding, *A Translation of “Classification of four-vectors of an 8-dimensional space,” by Antonyan, L. V., with an appendix by the translator*, Tr. Mosk. Mat. Obs. **83** (2022), 269–296, arXiv:2205.09741, <http://mi.mathnet.ru/mmo674>.
- [49] Larsson Omberg, Gene H Golub, and Orly Alter, *A tensor higher-order singular value decomposition for integrative analysis of DNA microarray data from different studies*, Proceedings of the National Academy of Sciences **104** (2007), no. 47, 18371–18376, doi:10.1073/pnas.0709146104.
- [50] Andreas Osterloh and Jens Siewert, *Entanglement monotones and maximally entangled states in multipartite qubit systems*, International Journal of Quantum Information **04** (2006), no. 03, 531–540.
- [51] Andreas Osterloh and Jens Siewert, *The invariant-comb approach and its relation to the balancedness of multipartite entangled states*, New Journal of Physics **12** (2010), no. 7, 075025.
- [52] P.K.Aravind, *A simple demonstration of bell’s theorem involving two observers and no probabilities or inequalities*, arXiv: Quantum Physics (2002).
- [53] R. M. Thrall and J. H. Chanler, *Ternary trilinear forms in the field of complex numbers*, Duke Math. J. **4** (1938), no. 4, 678–690, doi:10.1215/s0012-7094-38-00459-4.
- [54] Jacob Turner, *A finite-tame-wild trichotomy theorem for tensor diagrams*, Advances in Mathematics **370** (2020), 107213, doi:10.1016/j.aim.2020.107213.

- [55] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, *Four qubits can be entangled in nine different ways*, Physical Review A **65** (2002), no. 5, doi:10.1103/physreva.65.052112.
- [56] Frank Verstraete, Jeroen Dehaene, and Bart De Moor, *Normal forms and entanglement measures for multipartite quantum states*, Phys. Rev. A **68** (2003), 012103.
- [57] Guifré Vidal, *Entanglement monotones*, Journal of Modern Optics **47** (2000), no. 2-3, 355–376.
- [58] È. B. Vinberg and A. G. Èlašvili, *A classification of the three-vectors of nine-dimensional space*, Trudy Sem. Vektor. Tenzor. Anal. **18** (1978), 197–233.
- [59] Nolan R. Wallach, *Geometric invariant theory: Over the real and complex numbers*, Springer Cham, 2017.
- [60] P. Walther, K. J. Resch, T. Rudolph, E. Schenck, H. Weinfurter, V. Vedral, M. Aspelmeyer, and A. Zeilinger, *Experimental one-way quantum computing*, Nature **434** (2005), 169–176.
- [61] Alexander Wong and Nelson Christensen, *Potential multiparticle entanglement measure*, Phys. Rev. A **63** (2001), 044301.
- [62] Ye Yeo and Wee Kang Chua, *Teleportation and dense coding with genuine multipartite entanglement*, Phys. Rev. Lett. **96** (2006), 060502.
- [63] Amirul Asyraf Zhahir, Siti Munirah Mohd, Mohd Ilias M Shuhud, Bahari Idrus, Hishamuddin Zainuddin, Nurhidaya Mohamad Jan, and Mohamed Ridza Wahiddin, *Entanglement quantification and classification: A systematic literature review*, International Journal of Advanced Computer Science and Applications (IJACSA) (2022), doi:10.14569/ijacsa.2022.0130527.
- [64] Fuzhen Zhang, *Matrix theory: basic results and techniques*, Springer Science & Business Media, 2011, doi:10.1007/978-1-4614-1099-7.