

How to embed ‘open’ Lie superalgebras into affine Kac-Moody superalgebras

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Abstract. We discuss the concepts of *open* Lie superalgebras and their *closure*, and describe the procedure of embedding the closure into affine twisted and untwisted Kac-Moody superalgebras. We apply the formalism to an open algebra of a relativistic model obtained by Cotăescu and Visinescu [J. Phys. A **40** (2007) 11987]. This yields an intuitive explanation of why the *hydrogen algebras*, studied by Daboul and Slodowy, must be ‘twisted-like’ KM subalgebras.

1. Introduction

Recently, we examined the concepts of *open* Lie superalgebras and their *closure*, and some applications [1]. In order to introduce these concepts, consider the dynamical symmetry algebra of the 3-dimensional non-relativistic hydrogen atom, which is spanned by the conserved angular momentum \vec{L} and the Runge-Lenz vector \vec{A} . The components of \vec{L} and \vec{A} satisfy the following commutation relations (or Poisson brackets, in the classical treatment) [2, 3, 4]:

$$[L_i, L_j] = i \varepsilon_{ijk} L_k, \quad i, j, k = 1, 2, 3, \quad (1a)$$

$$[L_i, A_j] = i \varepsilon_{ijk} A_k, \quad (1b)$$

$$[A_i, A_j] = i \varepsilon_{ijk} h L_k, \quad \text{where } h := -2mH, \quad (1c)$$

$$[h, L_i] = [h, A_i] = 0, \quad (1d)$$

where m is the reduced mass, and H is the Hamiltonian. It is the appearance of H which leads to an open algebra and its closure through embedding into an infinite-dimensional algebra.

The appearance of the factor H in (1c), which can be either an operator (or a function), but not a constant, together with the commutation relations (1), makes the following set

$$\check{\mathfrak{H}}_3 := \{\vec{L}, \vec{A}; h\} \equiv \{L_i, A_i; h \mid i = 1, 2, 3\}, \quad (2)$$

a Lie algebra which is not closed. For this reason, we shall call such algebras *open Lie (super) algebras*. The usual way to deal with this dilemma is to replace the Hamiltonian H by its energy eigenvalues E , and then to rescale the Runge-Lenz vector. This procedure yields three closed (real) Lie algebras, $\mathfrak{so}(4)$, $\mathfrak{so}(3, 1)$ and $\mathfrak{e}(3)$, depending on the sign of the energy: $E < 0$, $E > 0$ and $E = 0$, respectively (see references in [1]-[4]).

Instead, if we iterate the commutation relations of $\{\vec{L}, \vec{A}; h\}$, we generate the following closed infinite-dimensional Lie algebra, which we call the *closure* of $\{\vec{L}, \vec{A}; h\}$:

$$\mathbb{H}_3^* := \overline{\{\vec{L}, \vec{A}; h\}} = \langle h^n L_i, h^n A_i \mid n \geq 0 \rangle_{\mathbb{R}}. \quad (3)$$

Daboul and Slodowy *identified* \mathbb{H}_3^* as the positive part of a twisted Kac-Moody algebra (KMA) $D_2^{(2)+}$ [4]. Later Daboul and Daboul noted that $D_2^{(2)+}$ is isomorphic to $A_1^{(1)+}$ [5, 1]. In other words,

$$\mathbb{H}_3^* \simeq \mathbb{H}_3 := \langle \hat{L}_i^{2n}, \hat{A}_i^{2n+1} \mid n \geq 0 \rangle_{\mathbb{R}} \simeq D_2^{(2)+} \simeq A_1^{(1)+}, \quad (4)$$

where the notation $\mathfrak{s}^{(m)}$ will be defined in (11), and

$$\hat{L}_i^{2n} := \chi(h^n L_i) \quad \text{and} \quad \hat{A}_i^{2n+1} := \chi(h^n A_i),$$

where χ is a homomorphism, which we shall explain in (22) below.

The dynamical algebras for the N -dimensional hydrogen atoms were later identified in [5]: they were denoted as \mathbb{H}_N and called *hydrogen algebras*. We review this identification in Sec. 4.3 below.

The purpose of this paper is to show how the above procedure can be generalized to embeddings of open *superalgebras* into loop *superalgebras*. The proofs of the relevant theorems can be found in our article [1].

2. Affine Lie superalgebras

In this section, we briefly review the concepts associated with affine Lie superalgebras that pertain to our discussion.

2.1. Lie superalgebras

A *Lie superalgebra* (over a field \mathbb{K} either \mathbb{R} or \mathbb{C}), $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$, is a superalgebra if its product $[[\cdot, \cdot]]$, called *supercommutator*, satisfies the following three conditions among its *homogeneous* elements:

(1) *algebraic grading*:

$$[[\mathfrak{s}_{\bar{i}}, \mathfrak{s}_{\bar{j}}]] \subset \mathfrak{s}_{\bar{i}+\bar{j}}, \quad (\bar{i}, \bar{j} \in \mathbb{Z}_2),$$

(2) *super skew-symmetry*:

$$[[x, y]] = -(-1)^{|x||y|} [[y, x]],$$

(3) *super Jacobi-identity*:

$$(-1)^{|z||x|} [[x, [y, z]]] + (-1)^{|x||y|} [[y, [z, x]]] + (-1)^{|y||z|} [[z, [x, y]]] = 0.$$

where to an element $x \in \mathfrak{s}_{\bar{i}}$, one assigns a *parity*, by $|x| = i \in \mathbb{Z}_2$ [6, §1.5]. The elements with $|x| = 0$ and $|x| = 1$ are called *even* and *odd*, respectively.

Example: An interesting example of simple finite-dimensional superalgebras, which also illustrates the even-odd grading, is given by the $(m+n)$ square matrices over \mathbb{C} :

$$\mathfrak{gl}(m|n) = \mathfrak{gl}(m|n)_0 \oplus \mathfrak{gl}(m|n)_1 := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right\}, \quad (5)$$

where the matrices a and d are m and n square matrices, respectively.

The matrices in (5) become a Lie superalgebra, called the *general linear Lie superalgebra*, if we define its supercommutator by

$$[[X, Y]] := XY - (-1)^{|X||Y|}YX, \quad \text{for } X \in \mathfrak{gl}(m|n)_{|X|}.$$

Hence, the supercommutator is related to the ordinary commutator, $[X, Y] := XY - YX$, as follows:

$$[[X_1, X_2]] = [X_1, X_2] + 2 \begin{pmatrix} b_2c_1 & 0 \\ 0 & c_2b_1 \end{pmatrix}, \quad \text{where } X_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}. \quad (6)$$

In particular, Eq. (6) shows that supercommutators and commutators become equal, if both X_1 and X_2 have diagonal form, *i.e.* either $b_1 = b_2 = 0$ or $c_1 = c_2 = 0$.

2.2. Affine untwisted Kac-Moody superalgebras

A loop superalgebra $\mathcal{L}(\mathfrak{s})$ can be defined from any finite-dimensional Lie superalgebra \mathfrak{s} as follows:

$$\mathcal{L}(\mathfrak{s}) = \mathbb{K}[t, t^{-1}] \otimes_{\mathbb{K}} \mathfrak{s} \simeq \langle x^n \mid x \in \mathfrak{s}, n \in \mathbb{Z} \rangle_{\mathbb{K}}, \quad (7)$$

where $\mathbb{K}[t, t^{-1}]$ denotes the algebra of Laurent polynomials in the variable t , and $x^n := t^n x$. The generators of $\mathcal{L}(\mathfrak{s})$ satisfy the *supercommutation relations*

$$[[x^m, y^n]] = [[x, y]]^{m+n}. \quad (8)$$

Note that, hereafter, we only deal with loop superalgebras, not with genuine affine Kac-Moody algebras (which have central terms). Nevertheless, we shall use the terminology and notations of Kac-Moody-algebras to describe these algebras. In particular, we will use the notation $\mathfrak{s}^{(1)}$ for $\mathcal{L}(\mathfrak{s})$.

2.3. Affine twisted Kac-Moody superalgebras

Definition 1. Let \mathfrak{s} be a finite-dimensional Lie superalgebra with a \mathbb{Z}_m -grading that commutes with the parity grading on \mathfrak{s} ,

$$\mathfrak{s} = \bigoplus_{k=0}^{m-1} \mathfrak{s}_k, \quad m \geq 2, \quad (9)$$

such that

$$[[\mathfrak{s}_i, \mathfrak{s}_j]] \subset \mathfrak{s}_{i+j}, \quad i, j, i+j \pmod{m}.$$

and such that

$$\mathfrak{s}_k = (\mathfrak{s}_k)_{\bar{0}} \oplus (\mathfrak{s}_k)_{\bar{1}}, \quad (\mathfrak{s}_k)_{\bar{p}} = \mathfrak{s}_k \cap \mathfrak{s}_{\bar{p}}, \quad k \in \mathbb{Z}_m, \bar{p} \in \mathbb{Z}_2. \quad (10)$$

Then a *Lie super subalgebra* $\mathfrak{s}^{(m)}$ of $\mathfrak{s}^{(1)}$, is defined as follows:

$$\mathfrak{s}^{(m)} := \bigoplus_{n \in \mathbb{Z}, k=0}^{m-1} t^{nm+k} \otimes \mathfrak{s}_k = \mathfrak{s}_0^{(m)} \oplus \mathfrak{s}_1^{(m)}, \quad (11)$$

where

$$\mathfrak{s}_{\bar{p}}^{(m)} = \bigoplus_{n \in \mathbb{Z}, k=0}^{m-1} t^{nm+k} \otimes (\mathfrak{s}_k)_{\bar{p}}.$$

Note that $\mathfrak{s}_0^{(m)}$ is a normal (not super) subalgebra of $\mathfrak{s}^{(m)}$.

Any $\mathfrak{s}^{(m)}$ of the form (11) will be called *twisted-like KMA*, since it may still be isomorphic to an untwisted KMA. Only when this is not the case, we will refer to it as a *twisted KMA* or “genuinely twisted KMA”. We shall also call such $\mathfrak{s}^{(m)}$ *twisted KM subalgebra*, if its connection to $\mathfrak{s}^{(1)}$ is relevant.

2.4. Level-grading and positive subalgebras of KM superalgebras

Eq. (8) yields the *natural grading* for the (twisted or untwisted) loop superalgebras $\mathfrak{s}^{(m)}$

$$[[\mathfrak{s}^{[m]}, \mathfrak{s}^{[n]}] \subseteq \mathfrak{s}^{[m+n]}, \tag{12}$$

where the homogeneous subspaces

$$\mathfrak{s}^{[n]} := \mathfrak{s}^{(m)} \cap \langle x^n | x \in \mathfrak{s} \rangle_{\mathbb{K}} \tag{13}$$

will be called *levels*. To their elements, we assign a grade, which we call *level-grade* (or *l-grade*), and denote it by

$$[x^n] = [t^n x] = n. \tag{14}$$

We define the *positive subalgebra* $\mathfrak{s}^{(m)+}$ as the subalgebra of a (twisted or untwisted) KMA $\mathfrak{s}^{(m)}$ which is the sum of all its non-negative levels,

$$\mathfrak{s}^{(m)+} := \langle x^n \in \mathfrak{s}^{(m)} | n \geq 0 \rangle_{\mathbb{K}} = \bigoplus_{n=0}^{\infty} \mathfrak{s}^{[n]}. \tag{15}$$

3. Open algebras and their closures

Equipped with the previous definitions, we now turn to the concepts of *open algebras*, their *closures* and their *embedding*.

We shall also show in this section that by simply solving the system of linear equations (23) (which depends only on the integer coefficients r_{ij}^k in the definition (18) of open algebras), we can determine in general whether an open algebra leads to a twisted-like or to an untwisted KMA. We apply this formalism first to the hydrogen algebras, \mathbb{H}_N , which were mentioned in the introduction, and then to the dynamical symmetry algebra of the Dirac-Taub-NUT (DTN) model [7]. It turns out that the hydrogen algebras lead to twisted-like KM algebras while the DTN model leads to an untwisted KM algebra. Furthermore we will see how both systems are closely connected.

3.1. Definitions and notation

Consider a finite-dimensional Lie superalgebra,

$$\mathfrak{s} := \langle x_1, \dots, x_N \rangle_{\mathbb{K}}, \quad \text{with} \quad [[x_i, x_j]] = \sum_{k=1}^N c_{ij}^k x_k, \quad i, j = 1, \dots, N, \tag{16}$$

over the field \mathbb{K} , with x_i homogeneous with respect to the inherent \mathbb{Z}_2 -grading of the superalgebra.

Definition 2. Let

$$\check{\mathfrak{s}} = \{X_1, \dots, X_N; v\} \tag{17}$$

be a set of generators X_i and v a variable or an *operator of infinite order* (analogous to h in Eq. (2)) that commutes with all the X_i , and such that

$$[[X_i, X_j]] = \sum_{k=1}^N c_{ij}^k v^{r_{ij}^k} X_k, \quad \text{for } i, j = 1, \dots, N, \tag{18}$$

where the powers $r_{ij}^k \in \mathbb{N}_0$ are such that the supercommutators in (18) satisfy the super-grading, the super-skew-symmetry and the super-Jacobi identity, and with the parity of the generators defined by

$$|v^n X_i| := |x_i|. \tag{19}$$

We call $\check{\mathfrak{s}}$ an *open algebra* if at least one of the r_{ij}^k in (18) is greater than zero, and the finite-dimensional Lie superalgebra \mathfrak{s} in (16) is its *underlying algebra*.

By commuting the generators on the r.h.s. of Eq. (18) with each other and with the original X_i , we generate higher powers of v , such as

$$[[v^{r_{ij}^k} X_k, v^{r_{ij}^l} X_l]] = \sum_p c_{kl}^p v^{r_{ij}^k+r_{ij}^l+r_{kl}^p} X_p.$$

We refer to this procedure as *repeated iterations*.

Definition 3. The *closure* of $\check{\mathfrak{s}}$ is defined as *the smallest Lie superalgebra which can be generated from $\check{\mathfrak{s}}$ by repeated iterations of its commutation relations*. It is denoted by $\overline{\check{\mathfrak{s}}}$ or, more explicitly, by $\{\check{\mathfrak{s}}; v\}$.

Note that we do not include the operator v itself as a generator. Therefore the closure $\overline{\{\check{\mathfrak{s}}; v\}}$ of an open algebra does not have to include *all* the powers $v^n X_i$ with $n \geq 0, i = 1, \dots, N$. Note that if the underlying finite-dimensional algebra \mathfrak{s} is nilpotent, then $\overline{\{\check{\mathfrak{s}}; v\}}$ is finite dimensional. See another counter-example in Eq.(31). Thus,

$$\overline{\{\check{\mathfrak{s}}; v\}} \subseteq \mathcal{L}(\check{\mathfrak{s}}) := \langle v^n X_i, n \in \mathbb{Z}, i = 1, \dots, N \rangle. \tag{20}$$

Lemma 1. The following mapping

$$\varphi(vX_i) = \varphi(X_i) = x_i, \quad \forall X_i \in \check{\mathfrak{s}}, \tag{21}$$

of $\overline{\{\check{\mathfrak{s}}; v\}}$ onto the superalgebra \mathfrak{s} in (16), is a homomorphism [1, Lemma 2].

3.2. Embedding and identification of open algebras into affine Kac-Moody algebras

An *embedding* is an injective and structure-preserving map, denoted by $f : A \hookrightarrow B$. This map defines an isomorphism between a Lie algebra A and its image $f(A) \subseteq B$. Thus, we shall say that A can be *identified* with B if $f(A) \simeq B$.

In this section, we study the embedding of the closure of an open algebra $\overline{\check{\mathfrak{s}}}$ into the loop algebra $\mathfrak{s}^{(1)} \equiv \mathcal{L}(\mathfrak{s})$ in (7), which is generated by the \mathfrak{s} of (16).

Theorem 1. The closure $\overline{\check{\mathfrak{s}}}$ of an open algebra $\check{\mathfrak{s}}$ of the form (18) can be embedded into the affine KM algebra $\mathfrak{s}^{(1)}$, $\overline{\check{\mathfrak{s}}} \hookrightarrow \mathfrak{s}^{(1)}$, by the following homomorphism:

$$\chi(v^p X_i) = t^{p[v]+[X_i]} x_i =: x_i^{p[v]+[X_i]}, \quad i = 1, \dots, N, \quad X_i \in \check{\mathfrak{s}}, \tag{22}$$

if and only if the homogeneous system of linear equations

$$[X_i] + [X_j] - [X_k] - r_{ij}^k [v] = 0, \quad \forall c_{ij}^k \neq 0 \quad i, j, k = 1, \dots, N, \tag{23}$$

has a solution, which can always be chosen such that $[X_i] \in \mathbb{Z}$, $[v] \in \mathbb{N}$ and such that the greatest common divisor (GCD) of the $[X_k]$ and $[v]$ is 1.

As in (14) we call the integers $[X_i]$ and $[v]$ level grades or l-grades of the X_i and v respectively.

Theorem 2. Let \mathfrak{g} be an ordinary (*i.e.* not super) Lie algebra, with all its structure constants totally antisymmetric, $c_{ij}^k = c_{ijk}$. Then all the generators $X_i \in \check{\mathfrak{g}}$ must have non-negative l-grades, *i.e.* $[X_i] \geq 0$.

An example: Let us apply the above theorem to the open algebra of the 3-dimensional hydrogen atom, discussed in the introduction. From Eq. (1a), we obtain $[L] + [L] = [L]$, so that $[L] = 0$. Eq. (1b) leads to $[L] + [A] = [A]$, which yields no conditions. From Eq. (1c), we find $2[A] = [h] + [L]$. Under the restrictions of the theorem this has the unique solution

$$[L] = 0, [A] = 1, [h] = 2. \tag{24}$$

3.3. Twisted and untwisted open Lie algebras

Our motivation to undertake the work presented in Ref. [1] was the observation that, despite the claim in Ref. [7] that the open superalgebra obtained therein was of twisted type, that superalgebra is actually untwisted (More in Section 4).

Hereafter, we derive more results which will enable us to distinguish between the two types of open algebras.

Lemma 2. If an open Lie algebra $\check{\mathfrak{s}}$ has a solution with $[v] = m$, then the underlying finite-dimensional Lie algebra

$$\mathfrak{s} = \langle x_1, \dots, x_N \rangle_{\mathbb{K}} = \varphi(\check{\mathfrak{s}})$$

admits a \mathbb{Z}_m -grading, defined by the subspaces

$$\mathfrak{s}_k = \langle x_i = \varphi(X_i) \mid [X_i] \equiv k(m) \rangle_{\mathbb{K}}. \quad \text{where } 0 \leq k \leq m - 1, \tag{25}$$

where the map φ is defined in Eq. (21) [1, Lemma 5].

This result is important, because it enables us to define a twisted-like KMSA $\mathfrak{s}^{([v])}$, according to the definition 1. This in turn allows us to embed $\check{\mathfrak{s}}$ into twisted-like KMSA:

Theorem 3. The embedding χ in (22) maps the closure $\bar{\check{\mathfrak{s}}}$ into the KMA $\mathfrak{s}^{([v])}$.

Corollary 1 If \mathfrak{g} satisfies the conditions of theorem 4, then $\bar{\mathfrak{g}}$ can be embedded into a *positive* twisted-like subalgebras, as defined in (15):

$$\bar{\mathfrak{g}} \hookrightarrow \mathfrak{g}^{([v])+}$$

Since $\mathfrak{s}^{([v])}$ for $v \geq 2$ is a subalgebra of the untwisted KMA $\mathfrak{s}^{(1)}$, the embedding χ can be extended naturally to an embedding of *the loop algebra $\mathcal{L}(\check{\mathfrak{s}})$ of the open algebra $\check{\mathfrak{s}}$*

$$\mathcal{L}(\check{\mathfrak{s}}) := \langle v^n X_i, n \in \mathbb{Z}, i = 1, \dots, N \rangle$$

into $\mathfrak{s}^{(1)}$ by

$$\chi(v^n X_i) := t^{n[v]} \chi(X_i). \tag{26}$$

Corollary 2: The loop algebra $\mathcal{L}(\check{\mathfrak{s}})$ is isomorphic to $\mathfrak{s}^{([v])}$ by the following natural extension of the embedding χ in (26):

$$\chi(\mathcal{L}(\check{\mathfrak{s}})) = \mathfrak{s}^{([v])}.$$

3.4. Identification of \mathbb{H}_N

As in (24) for \mathbb{H}_3 , the linear system of equations (23) yields *unique* solutions with $[h] = 2$ also for the hydrogen algebras \mathbb{H}_N . Thus, according to the above corollary, the closure \mathbb{H}_N^* of $\check{\mathbb{H}}_N$, which is the dynamical algebra of the N -dimensional hydrogen atom, has a characteristic embedding which extends to an isomorphism of its loop algebra with the twisted-like subalgebra $\mathfrak{so}(N+1)^{(2)}$ of $\mathfrak{so}(N+1)^{(1)}$. This enabled us to identify \mathbb{H}_N^* with the twisted-like $\mathfrak{so}(N+1)^{(2)}$ [5]. These are real KMA, which are isomorphic to the following real affine KMA:

$$\begin{aligned} \mathbb{H}_{2l-1}^* &\simeq \mathbb{H}_{2l-1} \simeq D_l^{(2)+}, & l \geq 2, \\ \mathbb{H}_{2l}^* &\simeq \mathbb{H}_{2l} \simeq B_l^{(2)+} \subseteq B_l^{(1)+}, & l \geq 2, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \mathbb{H}_N^* &:= \overline{\{L_{ij}, A_i | 1 \leq i, j \leq N\}} = \langle h^n L_{ij}, h^n A_i | n \in \mathbb{N}_0 \rangle_{\mathbb{R}}, \\ \mathbb{H}_N &:= \langle \hat{L}_{ij}^{2n}, \hat{A}_i^{2n+1} | 1 \leq i, j \leq N; n \in \mathbb{N}_0 \rangle_{\mathbb{R}}. \end{aligned}$$

For a more details see eqn. (36) in [5].

Note also that $D_l^{(2)}$ for $l \geq 3$ are *genuinely twisted*. In contrast, $B_l^{(2)}$ are not genuine; they can be untwisted, *i.e.* $B_l^{(2)} \simeq B_l^{(1)}$.

An interesting exception is the algebra $D_2^{(2)}$, since $D_2^{(2)} \simeq A_1^{(1)}$, as we proved in the appendix of [1]. This is because D_2 is not a simple Lie algebra.

4. Open superalgebra of DTN model

In Ref. [7], Cotăescu and Visinescu examined a relativistic Dirac theory on manifolds carrying the Gross-Perry-Sorkin monopole defined on the Euclidean Taub-NUT space. This ‘Dirac-Taub-NUT’ (DTN) model has the following dynamical algebra,

$$\mathcal{S}_0 = \{J_i, K_i, \tilde{I}, M, Q, Q_i; b\}, \quad i = 1, 2, 3, \tag{28}$$

where, *by definition*, all the generators in (28) commute with the relativistic operator b , where b^2 is defined by

$$b^2 := P_4^2 \tilde{I} - H_D^2, \tag{29}$$

where H_D in Eq. (29) is a Dirac Hamiltonian and $P_4 := -i \frac{\partial}{\partial x_4}$ [7]. Moreover, the operators \tilde{I} and M in Eq. (28) are Casimir-type bosonic operators, which commute with all the operators in \mathcal{S}_0 .

The generators in (28) satisfy the following supercommutation relations:

$$[J_i, J_j] = i \varepsilon_{ijk} J_k, \tag{30a}$$

$$[J_i, K_j] = i \varepsilon_{ijk} K_k, \tag{30b}$$

$$[K_i, K_j] = i \varepsilon_{ijk} b^2 J_k, \tag{30c}$$

$$[J_i, Q_j] = i \varepsilon_{ijk} Q_k, \tag{30d}$$

$$[K_i, Q_j] = i \varepsilon_{ijk} b Q_k, \tag{30e}$$

$$\{Q_i, Q_j\} = 2\delta_{ij} \tilde{I}, \tag{30f}$$

$$[J_i, Q] = [J_i, Q] = 0, \tag{30g}$$

$$\{Q, Q_j\} = 2(bJ_i + K_i), \tag{30h}$$

$$\{Q, Q\} = 2b^2 M, \tag{30i}$$

so that according to our definition the \mathcal{S}_0 is an open superalgebra.

According to our definition in Sec. 3.2, the closure $\overline{\mathcal{S}_0}$ is the subalgebra of $\mathcal{L}(\mathcal{S}_0)$ generated by \mathcal{S}_0 . Hence,

$$\overline{\mathcal{S}_0} = \langle b^n J_i, b^n K_i, b^n \tilde{I}, b^n Q_i \mid n \geq 0 \rangle_{\mathbb{K}} + \langle Q, M, b^2 M \rangle_{\mathbb{K}}, \tag{31}$$

is a genuine subalgebra of $\mathcal{L}(\mathcal{S}_0)^+$, since instead of $\langle b^n Q, b^n M \mid n \geq 0 \rangle_{\mathbb{K}}$ we can only generate the three elements Q, M and $b^2 M$.

4.1. Embedding of $\overline{\mathcal{S}_0}$

The linear system (23) in Theorem 1 has a unique solution with integer values, positive level grade of b and $\text{GCD}(\lfloor X_i \rfloor, \lfloor b \rfloor) = 1$:

$$\begin{aligned} \lfloor K_i \rfloor = \lfloor Q \rfloor = \lfloor b \rfloor = 1, & \quad i = 1, 2, 3, \quad \text{and} \\ \lfloor J_i \rfloor = \lfloor Q_i \rfloor = \lfloor \tilde{I} \rfloor = \lfloor M \rfloor = 0. & \end{aligned} \tag{32}$$

As in [7] let $\mathcal{W}_0 := \varphi(\mathcal{S}_0)$ denote the underlying finite dimensional super algebra of the open algebra \mathcal{S}_0 . Since $\lfloor b \rfloor = 1$ in (32), the associated embedding χ maps the closure $\overline{\mathcal{S}_0}$ into the affine *untwisted* KM algebra $\mathcal{W}_0^{(1)}$, but not into a twisted-like subalgebra $\mathcal{W}_0^{(m)}, m > 1$.

5. Summary and insight

We showed that \mathcal{S}_0 *cannot* be embedded into a *twisted* KMSA, *in contradiction to the conclusion of CV* in Ref [7].

The relativistic DTN provides us with an intuitive explanation of why $\lfloor h \rfloor = 2$, and consequently explains why \mathbb{H}_N must be twisted-like. We argue as follows: Eqs. (30a)-(30c) are similar to Eqs. (1a)-(1c). Hence, the operator b^2 in (30c) corresponds to the non-relativistic Hamiltonian operator h in Eq. (1c). But since the variable b also appears linearly (in Eqs. (30e) and (30h)), a $\lfloor h \rfloor = \lfloor b^2 \rfloor = 1$ would yield $\lfloor b \rfloor = 1/2$, which contradicts the condition $\lfloor b \rfloor \in \mathbb{N}$.

We can summarize the embedding procedure by the following algorithm:

- (i) Map the open algebra $\{\check{s}, v\}$ onto \mathfrak{s} by the map φ in (21).
- (ii) Find $\lfloor v \rfloor$ by solving the system of linear equation in (23).
- (iii) Determine the associated $\mathbb{Z}_{\lfloor v \rfloor}$ -grading of \mathfrak{s} .
- (iv) Construct the twisted-like KMSA $\mathfrak{s}^{(\lfloor v \rfloor)}$.
- (v) Finally, map $\overline{\{\check{s}, v\}}$ into $\mathfrak{s}^{(\lfloor v \rfloor)}$ by using the map χ in (26).

The above algorithm is illustrated in the following diagram:

$\chi(v^n X_i)$	$:=$	$(t^{\lfloor v \rfloor})^n \cdot (t^{\lfloor X_i \rfloor} x_i) = t^{n\lfloor v \rfloor + \lfloor X_i \rfloor} x_i$
\bigcap		\bigcap
$\overline{\check{\mathfrak{s}}} = \langle v^n X_1, \dots, v^n X_N \rangle_{\mathbb{K}}$	$\xrightarrow{\chi}$	$\mathfrak{s}^{(\lfloor v \rfloor)} \subseteq \mathfrak{s}^{(1)} = \langle t^n x_1, \dots, t^n x_N \rangle_{\mathbb{K}}$
\uparrow (iteration)		\uparrow (looping)
$\check{\mathfrak{s}} = \{X_1, \dots, X_N; v\}$	$\xrightarrow{\varphi}$	$\mathfrak{s} := \langle x_1, \dots, x_N \rangle_{\mathbb{K}}$

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