

The Real Forms of the Fractional Supergroup $SL(2, C)$

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Abstract: The real forms of complex groups (or algebras) are important in physics and mathematics. The Lie group $SL(2, C)$ is one of these important groups. There are real forms of the classical Lie group $SL(2, C)$ and the quantum group $SL(2, C)$ in the literature. Inspired by this, in our study, we obtain the real forms of the fractional supergroups shown with $A_3^N(SL(2, C))$, for the non-trivial $N = 1$ and $N = 2$ cases, that is, the real forms of the fractional supergroups $A_3^1(SL(2, C))$ and $A_3^2(SL(2, C))$.

Keywords: fractional supergroup; Hopf algebra; star-algebra

1. Introduction

Lie groups and Lie algebras are very important in mathematics and physics. In the field of mathematics, since every Lie group is an analytic manifold and every Lie algebra is a vector space that is tangent to the unit in the manifold, then every innovation in these groups (or algebra) contributes to differential geometry [1–3].

Lie symmetry methods are widely used in the solution of various differential equations that constitute deterministic models of mechanics, engineering, physics, finance and many other fields [4–6]. For example, using Lie symmetry methods in finance, new solutions for stochastic differential equations and stochastic processes have been developed [7,8]. In particle physics or quantum field theories, finite dimensional Lie algebras have been used to explain space-time symmetries and interactions [9].

In addition, super and fractional supersymmetries are important in relativistic quantum field theory [10,11]. The real forms of Lie groups (algebras) such as the Lie group $SL(2, C)$ (or the Lie algebra $sl(2, C)$) also have important and wide application [1]. Every g real Lie algebra corresponds to a complex Lie algebra g_c with the same base and the same structure constant. At the same time, a complex Lie algebra can transition to a real Lie algebra. The process of going from a complex Lie algebra g_c to a real Lie algebra g is called realization. For example, the Lie groups $SU(2)$, $SU(1, 1)$ and $SL(2, R)$ (or Lie algebras $su(2)$, $su(1, 1)$ and $sl(2, R)$), which are the real forms of the classical Lie group $SL(2, C)$ (or classical Lie algebra $sl(2, C)$), have an important place in the literature [1].

Similarly, there are real forms of the quantum Lie group $SL_q(2, C)$ (or quantum Lie algebra $sl_q(2, C)$) [2] such as the Lie group $SU(2)$ and the quantum group $SU_q(2)$, which are very important compact groups in mathematical physics. Hence, many applications exist. For example, in paper [12], a theoretical group approach to generalized oscillator algebra A_k is proposed, which is defined by the compact group $SU(2)$ (for the Morse system) for the case $k < 0$, and $SU(1, 1)$ (for the Harmonic oscillator) for the case $k > 0$. In addition, phase operators and phase states are introduced in the framework of the $SU(2)$ and $SU(1, 1)$ groups in [13].

Inspired by these studies, we aim to obtain the real forms of the fractional supersymmetric group $SL(2, C)$. Fractional supersymmetric groups (or algebras) can be obtained by various methods [11,14–22]. For example, fractional superalgebras based on S_n invariant forms were first introduced in [10]. Then, fractional supergroups (or algebras) based on permutation groups of S_n invariant forms were formulated in Hopf algebra formalism [23].



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One of the most important advantages of this approach is that it is an algebraic approach. Thus, it will always be possible and easy to move from algebra to group or from group to algebra. In our study, we obtain the fractional supergroups $SU(2)$, $SU(1, 1)$ and $SL(2, R)$, which are the real forms of the fractional supergroup $SL(2, C)$, in Hopf algebra formalism based on the permutation group S_3 .

In this context, in the second part, after defining the fractional supergroup by giving some useful definitions, we show with an example that the star operation is consistent with the Hopf algebra structure. In Section 3, we define the real forms of the classical Lie group $SL(2, C)$ and fractional supergroup $SL(2, C)$, respectively.

2. Preliminaries on *- Algebras

In this section, we summarize important definitions and the relations used to calculate the real forms of the fractional supergroup $SL(2, C)$ [1,2,23].

Definition 1. Let $A(G)$ be the Hopf algebra of functions on the Lie group G and Λ_n^N be the Hopf algebra generated by λ and θ_β , where $\beta = 1, 2, \dots, N$, satisfying the following relations:

$$\lambda^n = 1, \lambda\theta_\beta = q\theta_\beta\lambda, \{\theta_{\beta_1}, \theta_{\beta_2}, \dots, \theta_{\beta_n}\} = 0, \beta_k = 1, 2, \dots, N$$

The fractional supergroup $A_n^N(G)$ is the direct product of the algebras $A(G)$ and Λ_n^N for $n = 3, 4, \dots$. It is denoted by $A_n^N(G) = A(G) \times \Lambda_n^N$. The co-algebra operations and antipodes in $A_n^N(G)$ rely on the structure constants of the fractional superalgebra $U_n^N(g)$.

Here, the dual of the Hopf algebra $A_n^N(G)$ is the Hopf algebra $U_n^N(g)$. In our study, we consider the case of $n = 3$ and $N = 1, 2$. The definition and duality relations of the fractional superalgebras $U_3^N(g)$ belonging to this situation are given in the Appendix A.

Definition 2. Let \mathcal{A} be an associative algebra with unit I . The algebra \mathcal{A} is called a *- Algebra if it has the following properties:

1. $(\alpha a + \beta b)^* = \bar{\alpha}a^* + \bar{\beta}b^*$, for $a, b \in \mathcal{A}$
2. $(a^*)^* = a$,
3. $(ab)^* = b^*a^*$, $I^* = I$.

Definition 3. Let \mathcal{A} and \mathcal{B} be two unital *- algebras. If a homomorphism \mathcal{H} satisfies $\mathcal{H}(a^*) = \mathcal{H}(a)^*$ for $a \in \mathcal{A}$, then \mathcal{H} is called a *-homomorphism from \mathcal{A} to \mathcal{B} .

Definition 4. If Hopf algebra \mathcal{A} satisfies Definition 2 and the properties below, then \mathcal{A} is called a *-Hopf algebra.

1. $S((S(a^*))^*) = a$ for $a \in \mathcal{A}$, that is $S \circ * \circ S \circ * = id$
2. $\varepsilon(a^*) = \varepsilon(a)$, $a \in \mathcal{A}$,
3. $\Delta(a^*) = \sum_i b_i^* \otimes c_i^*$, that is $\Delta \circ * = (* \otimes *) \circ \Delta$

3. The Real Forms of the Lie Group $SL(2, C)$ and Fractional Supergroup $SL(2, C)$

3.1. The Real Forms of the Lie Group $SL(2, C)$

Definition 5.

$$SL(2, C) = \left\{ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{11}a_{22} - a_{12}a_{21} = 1, a_{ij} \in C; i, j = 1, 2 \right\}$$

The real forms of the group $SL(2, C)$ are the groups $SU(2)$, $SU(1, 1)$ and $SL(2, R)$ which are defined respectively by the *-operations below:

1. $a_{11}^* = a_{22}, a_{21}^* = -a_{12}$
2. $a_{11}^* = a_{22}, a_{21}^* = a_{12}$
3. $a_{11}^* = a_{11}, a_{12}^* = a_{12}, a_{21}^* = a_{21}, a_{22}^* = a_{22}$

Definition 6. Let the algebra $A(SL(2, C))$ be the algebra of functions of the group $SL(2, C)$. $A(SL(2, C))$ is a Hopf algebra with a co-product, an antipode and a co-unit operation given as follows:

$$\begin{aligned} \Delta a_{11} &= a_{11} \otimes a_{11} + a_{12} \otimes a_{21}, \Delta a_{12} = a_{11} \otimes a_{12} + a_{12} \otimes a_{22}, \\ \Delta a_{21} &= a_{21} \otimes a_{11} + a_{22} \otimes a_{21}, \Delta a_{22} = a_{22} \otimes a_{22} + a_{21} \otimes a_{12}, \\ S(a_{11}) &= a_{11}, S(a_{12}) = -a_{12}, S(a_{21}) = -a_{21}, S(a_{22}) = a_{22}, \\ \varepsilon(a_{11}) &= 1, \varepsilon(a_{12}) = 0, \varepsilon(a_{21}) = 0, \varepsilon(a_{22}) = 1. \end{aligned}$$

Theorem 1. The algebra $A(SU(2))$ is the $*$ -Hopf algebra of $A(SL(2, C))$ with $*$ -operation (1) in Definition 5.

Proof. It is sufficient to show that the Hopf algebra is consistent with (1) in Definition 5. \square

$$\begin{aligned} \Delta a_{11}^* &= \Delta(a_{11})^* = a_{11}^* \otimes a_{11}^* + a_{12}^* \otimes a_{21}^* \\ &= a_{22} \otimes a_{22} + a_{21} \otimes a_{12} \\ &= \Delta a_{22} \\ \Delta a_{21}^* &= \Delta(a_{21})^* = a_{21}^* \otimes a_{11}^* + a_{22}^* \otimes a_{21}^* \\ &= -a_{12} \otimes a_{22} + a_{11} \otimes (-a_{12}) \\ &= -\Delta a_{12} \\ S((S(a_{11}^*))^*) &= S((S(a_{22}))^*) \\ &= S((a_{22})^*) \\ &= S(a_{11}) \\ &= a_{11} \end{aligned}$$

Theorem 2. The algebra $A(SU(1, 1))$ is the $*$ -Hopf algebra of $A(SL(2, C))$ with $*$ -operation (2) in Definition 5.

Proof. The proof can be shown similarly to the proof of Theorem 1. \square

Theorem 3. The algebra $A(SL(2, R))$ is the $*$ -Hopf algebra of $A(SL(2, C))$ with $*$ -operation (3) in Definition 5.

Proof. The proof can be shown similarly to the proof of Theorem 1. \square

3.2. The Real Forms of the Fractional Supergroup $SL(2, C)$

We consider the real forms of the fractional supergroup $SL(2, C)$ for $N = 1, 2$ and $n = 3$ (that is $A_3^N(SL(2, C))$).

3.2.1. The Real Forms of $A_3^1(SL(2, C))$

The fractional supergroup $A_3^1(SL(2, C))$ is the direct product of the Hopf algebras $A(SL(2, C))$ and Λ_3^1 .

Definition 7. The fractional supergroup $A_3^1(SL(2, C))$ is the Hopf algebra generated by a_{mn} , λ and θ_β where $m, n = 1, 2$ and $\beta = 1$, satisfying the co-algebra operations and antipode:

$$\{\theta_\beta, \theta_\alpha, \theta_\gamma\} = 0, \lambda\theta_\beta = q\theta_\beta\lambda, \lambda^3 = 1, q^3 = 1 \tag{1}$$

$$\Delta(a_{mn}) = \sum_{k=1}^2 a_{mk} \otimes a_{kn} \tag{2}$$

$$S(a_{11}) = a_{22}, (a_{22}) = a_{11}, S(a_{12}) = -a_{12}, S(a_{21}) = -a_{21} \tag{3}$$

$$\varepsilon(a_{mn}) = \delta_{mn} \tag{4}$$

$$\Delta(\theta_1) = \theta_1 \otimes 1 + \lambda \otimes \theta_1, \Delta(\lambda) = \lambda \otimes \lambda \tag{5}$$

$$S(\theta_1) = -\lambda^2\theta_1, S(\lambda) = \lambda^2 \tag{6}$$

$$\varepsilon(\theta_1) = 0, \varepsilon(\lambda) = 1 \tag{7}$$

Theorem 4. The fractional supergroup $A_3^1(SU(2))$ is the $*$ -Hopf algebra generated by a_{mn}, λ and θ_β , where $m, n = 1, 2$ and $\beta = 1$ satisfying Definition 7 with the $*$ -operations:

$$a_{11}^* = a_{22}, a_{21}^* = -a_{12}, \theta_1^* = \theta_1, \lambda^* = \lambda$$

Proof. Let us show that the co-algebra structure is consistent with the $*$ -operations. The relations (2–3) in Definition 7 are shown in Theorem 1. Now, let us show that the relations (5–6) are realized. For this, we will use the definitions of the $*$ -Hopf algebra, considering that Δ and ε are homomorphisms and S is an anti-homomorphism. \square

$$\begin{aligned} \Delta\theta^* &= \Delta(\theta)^* = (\theta \otimes 1 + \lambda \otimes \theta)^* \\ &= \theta^* \otimes 1 + \lambda^* \otimes \theta^* \\ &= \theta \otimes 1 + \lambda \otimes \theta \\ &= \Delta\theta \\ \Delta\lambda^* &= (\Delta\lambda)^* = (\lambda \otimes \lambda)^* \\ &= \lambda^* \otimes \lambda^* \\ &= \lambda \otimes \lambda \\ &= \Delta\lambda \\ S((S(\theta^*))^*) &= S((S(\theta))^*) \\ &= S((- \lambda^2\theta)^*) \\ &= -S(\theta^*\lambda^{*2}) \\ &= -S(\lambda^2)S(\theta) \\ &= \lambda \cdot (\lambda^2\theta) \\ &= \theta \\ S((S(\lambda^*))^*) &= S((S(\lambda))^*) \\ &= S((\lambda^2)^*) \\ &= S(\lambda^2) \\ &= \lambda \end{aligned}$$

Theorem 5. The fractional supergroup $A_3^1(SU(1,1))$ is the $*$ -Hopf algebra generated by a_{mn}, λ and θ_β , where $m, n = 1, 2$ and $\beta = 1$, satisfying Definition 7 with the $*$ -operations:

$$a_{11}^* = a_{22}, a_{21}^* = a_{12}, \theta_1^* = \theta_1, \lambda^* = \lambda$$

Proof. The proof is done similarly to the proof of Theorem 4. \square

Theorem 6. The fractional supergroup $A_3^1(SL(2, R))$ is the $*$ -Hopf algebra generated by a_{mn}, λ and θ_β , where $m, n = 1, 2$ and $\beta = 1$, satisfying Definition 7 with the $*$ -operations:

$$a_{11}^* = a_{11}, a_{21}^* = a_{21}, a_{12}^* = a_{12}, a_{22}^* = a_{22}, \theta_1^* = \theta_1, \lambda^* = \lambda$$

Proof. The proof is done similarly to the proof of Theorem 4. \square

3.2.2. The Real Form of $A_3^2(SL(2, C))$

In the case of $N = 2$, we consider the case where the algebra generators transform into spinor representation.

Definition 8. The fractional supergroup $A_3^2(SL(2, C))$ is the $*$ -Hopf algebra generated by a_{mn} , λ and θ_β where $m, n = 1, 2$ and $\beta = 1, 2$, satisfying the relations (1–4) given in Definition 7 and

$$\Delta(\theta_1) = \theta_2 \otimes a_{21} + \theta_1 \otimes a_{11} + \lambda \otimes \theta_1 \tag{8}$$

$$\Delta(\theta_2) = \theta_2 \otimes a_{22} + \theta_1 \otimes a_{12} + \lambda \otimes \theta_2 \tag{9}$$

$$S(\theta_1) = \lambda^2(a_{21}\theta_2 - a_{22}\theta_1), S(\theta_2) = \lambda^2(a_{12}\theta_1 - a_{11}\theta_2) \tag{10}$$

$$\varepsilon(\theta_\beta) = 0, \varepsilon(\lambda) = 1 \tag{11}$$

Theorem 7. The fractional supergroup $A_3^2(SU(2))$ is the $*$ -Hopf algebra generated by a_{mn} , λ and θ_β , where $m, n = 1, 2$ and $\beta = 1, 2$, satisfying Definition 8 with the $*$ -operations:

$$a_{11}^* = a_{22}, a_{21}^* = -a_{12}, \theta_1^* = -\theta_2, \lambda^* = \lambda$$

Proof. Let us show that the co-algebra relations (5) and (8) remain invariant. \square

$$\begin{aligned} \Delta\theta_1^* &= (\Delta\theta_1)^* \text{ From the definition of the homomorphism of } \Delta \\ &= (\theta_2 \otimes a_{21} + \theta_1 \otimes a_{11} + \lambda \otimes \theta_1)^* \\ &= \theta_2^* \otimes a_{21}^* + \theta_1^* \otimes a_{11}^* + \lambda^* \otimes \theta_1^* \\ &= -(\theta_1 \otimes a_{12} + \theta_2 \otimes a_{22} + \lambda \otimes \theta_2) \\ &= -\Delta\theta_2 \end{aligned}$$

$$\begin{aligned} S((S(\theta_1^*))^*) &= S((S(-\theta_2))^*), \text{ Since } S \text{ is an anti-homomorphism} \\ &= S([\lambda^2(a_{12}\theta_1 + a_{11}\theta_2)]^*) \\ &= S((a_{12}\theta_1 + a_{11}\theta_2)^*(\lambda^2)^*) \\ &= S(\lambda^2)[S(-a_{21})S(-\theta_2) - S(a_{22})S(\theta_1)] \\ &= \lambda[-\lambda^2 a_{12} a_{21} \theta_1 + \lambda^2 a_{21} a_{11} \theta_2 - \lambda^2 a_{11} a_{21} + \lambda^2 a_{11} a_{22} \theta_1] \\ &= \lambda^3(a_{11} a_{22} - a_{12} a_{21})\theta_1, \text{ since } \lambda^3 = 1 \text{ and } \det A = 1 \\ &= \theta_1. \end{aligned}$$

Theorem 8. The fractional supergroup $A_3^2(SU(1, 1))$ is the $*$ -Hopf algebra generated by a_{mn} , λ and θ_β , where $m, n = 1, 2$ and $\beta = 1, 2$, satisfying Definition 8 with the $*$ -operations:

$$a_{11}^* = a_{22}, a_{21}^* = a_{12}, \theta_1^* = \theta_2, \lambda^* = \lambda$$

Proof. Let us show here that the common algebra structure is consistent with the $*$ operation of the joint product operation. If $\theta_1^* = \theta_2$, is it possible to get $\Delta\theta_1^* = \Delta\theta_2$? Since Δ is a $*$ -homomorphism, then: \square

$$\begin{aligned} \Delta\theta_1^* &= (\Delta\theta_1)^* \\ &= (\theta_2 \otimes a_{21} + \theta_1 \otimes a_{11} + \lambda \otimes \theta_1)^* \\ &= \theta_2^* \otimes a_{21}^* + \theta_1^* \otimes a_{11}^* + \lambda^* \otimes \theta_1^* \\ &= \theta_1 \otimes a_{12} + \theta_2 \otimes a_{22} + \lambda \otimes \theta_2 \\ &= \Delta\theta_2 \end{aligned}$$

Theorem 9. The fractional supergroup $A_3^2(SL(2, R))$ is the $*$ -Hopf algebra generated by a_{mn} , λ and θ_β , where $m, n = 1, 2$ and $\beta = 1, 2$, satisfying Definition 8 with the $*$ -operations:

$$a_{11}^* = a_{11}, a_{21}^* = a_{21}, a_{12}^* = a_{12}, a_{22}^* = a_{22}, \theta_1^* = \theta_2, \lambda^* = \lambda$$

Proof. This can be done similarly to previous evidence. \square

4. Conclusions

As is known, one can pass from Lie algebras to Lie groups by an exponential map. However, this transition is not always possible, or it may be difficult, but with algebraic approaches, transitions between algebras and groups are easier [1,19,23–26]. For example, the supergroup and fractional supergroup definitions, which are the duals of the superalgebras and fractional superalgebras, have been defined with the algebraic approaches in [23]. In this study, we obtain the real forms of the fractional supergroup $SL(2, C)$ defined in article [23]. That is, we have obtained the fractional supergroups $SU(2)$, $SU(1, 1)$ and $SL(2, R)$ which are denoted with $A_3^N(SU(2))$, $A_3^N(SU(1, 1))$ and $A_3^N(SL(2, R))$, respectively where $N = 1, 2$.

When we compare our results with [23,25,26], we find that we obtained consistent results because the fractional supergroup $SU(2)$ is dual to the fractional superalgebra $su(2)$ given in [26]. Moreover, the definitions in [23,25] are consistent with the fractional supergroups $A_3^N(SU(1, 1))$ and $A_3^N(SL(2, R))$ obtained here. This shows the accuracy of our results.

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Appendix A

The fractional superalgebras $U_3^N(g)$ are the Hopf algebras generated by Y_j, K and Q_α where $\alpha = 1, \dots, N$ and $j = 1, 2, \dots, \dim(g)$, which satisfies the following relations:

$$[Y_i, Y_j] = \sum_{k=1}^{\dim(g)} c_{ij}^k Y_k \tag{A1}$$

$$\{Q_\alpha, Q_\beta, Q_\gamma\} = \sum_{j=1}^{\dim(g)} e_{\alpha\beta\gamma}^j Y_j \tag{A2}$$

$$[Q_\alpha, Y_j] = \sum_{\beta=1}^M d_{\alpha\beta}^j Q_\beta \tag{A3}$$

$$KQ_\alpha = qQ_\alpha K, q^3 = 1, K^3 = 1 \tag{A4}$$

the co-multiplications

$$\Delta(Y_j) = Y_j \otimes 1 + 1 \otimes Y_j, \Delta(K) = K \otimes K, \Delta(Q_\alpha) = Q_\alpha \otimes 1 + K \otimes Q_\alpha \tag{A5}$$

the co-unit and antipode

$$\varepsilon(Y_j) = 0, \varepsilon(K) = 1, \varepsilon(Q_j) = 0 \tag{A6}$$

$$S(Y_j) = -Y_j, S(K) = K^2, S(Q_j) = -K^2 Q_j \tag{A7}$$

The duality relations are defined as follows:

$$\langle \theta_\beta, Q_\alpha \rangle = \delta_{\beta\alpha}, \quad \langle \lambda, K \rangle = q, \quad \langle a_{mn}, K \rangle = \delta_{mn},$$

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