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The Fokker–Planck formalism for closed bosonic strings

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Every Riemann surface with genus g and n punctures admits a hyperbolic metric, if $2g - 2 + n > 0$. Such a surface can be decomposed into pairs of pants whose boundaries are geodesics. We construct a string field theory for closed bosonic strings based on this pants decomposition. In order to do so, we derive a recursion relation satisfied by the off-shell amplitudes, using Mirzakhani's scheme for computing integrals over the moduli space of bordered Riemann surfaces. The recursion relation can be turned into a string field theory via the Fokker–Planck formalism. The Fokker–Planck Hamiltonian consists of kinetic terms and three-string vertices. Unfortunately, the worldsheet BRST symmetry is not manifest in the theory thus constructed. We will show that the invariance can be made manifest by introducing auxiliary fields.

Subject Index B28

1. Introduction

To construct a string field theory (SFT), we should specify a rule to cut worldsheets into fundamental building blocks, i.e., propagators and vertices. A few simple rules have been proposed and SFTs for bosonic strings have been constructed following these rules [1–4]. Construction of an SFT for superstrings is more complicated because of the spurious singularities [5].

The worldsheets of closed strings describing scattering amplitudes are punctured Riemann surfaces. In mathematics, there exists a convenient way to decompose them into fundamental building blocks. On a Riemann surface with genus g and n boundaries or punctures, one can introduce a metric with constant negative curvature, if $2g - 2 + n > 0$. Such a metric is called a hyperbolic metric and surfaces with hyperbolic metrics are called hyperbolic surfaces. With a hyperbolic metric, one can decompose the surface into pairs of pants with geodesic boundaries. It may be possible to consider the pair of pants as the fundamental building block of the surface.

The hyperbolic metric was used to construct an SFT in Refs. [6–8], in which the kinetic term of the action was taken to be the conventional one

$$\int \Psi c_0^- Q\Psi, \tag{1}$$

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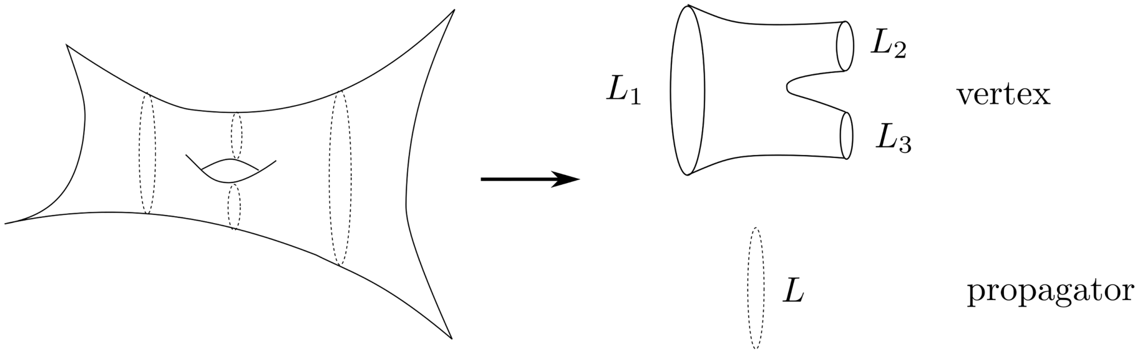

Fig. 1. A pants decomposition.

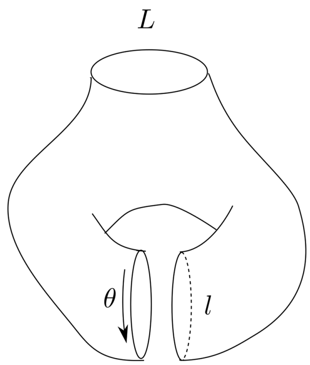

Fig. 2. One-loop one-point function.

so that the propagators correspond to cylinders. The theories include infinitely many vertices besides the three-string vertex and the Feynman graphs have nothing to do with the pants decomposition. In this paper, we would like to construct an SFT based on the pants decomposition. Namely, we will construct an SFT for closed bosonic strings regarding the pair of pants as the three-string vertex and the cylinders with vanishing heights as the propagator, as depicted in Fig. 1.

In such a theory, a string state will correspond to the boundary of a pair of pants. Accordingly, the string field should be labeled by an element of the Hilbert space of the first quantized strings and the length L of the boundary. The external states of the scattering amplitudes are regarded as the limit $L \to 0$ of such states. The off-shell amplitudes may correspond to Riemann surfaces that have geodesic boundaries with fixed lengths and will be expressed by integrals over the moduli spaces of such surfaces.

Unfortunately, such an approach suffers from a problem addressed in Ref. [9] (Sect. IV.E). The three-string vertex will be given by the correlation function of the worldsheet theory on hyperbolic pants with the boundary lengths specified. Suppose that one calculates the one-loop one-point function following the conventional Feynman rules. The amplitude corresponds to the worldsheet in Fig. 2 and we should integrate over the length l and the twist angle θ. By doing so, the fundamental domain of the modular group is covered infinitely many times, as will be seen in Sect. 4.3. The same happens for all the other amplitudes. Therefore, the conventional Feynman rule with the vertex and the propagator in Fig. 1 does not yield the correct amplitudes.

In order to overcome this problem, we formulate the theory using Mirzakhani's scheme [10,11] for computing integrals over moduli space of bordered Riemann surfaces. Mirzakhani derived a recursion relation for the volume of the moduli space. Applying her method to the off-shell amplitudes of closed bosonic strings, we derive a recursion relation satisfied by these amplitudes.

As was pointed out in Refs. [12,13], Mirzakhani's recursion relation is related to the loop equation of minimal string theory. On the other hand, the loop equations for minimal strings can be described by an SFT via the Fokker–Planck formalism [14,15]. We will show that the recursion relation of the off-shell amplitudes can be described by an SFT using the Fokker–Planck formalism. The Fokker–Planck Hamiltonian consists of kinetic terms and three-string vertices. One can develop perturbation theory that does not suffer from the above-mentioned problem. Unfortunately, the worldsheet BRST symmetry is not manifest in the SFT thus constructed. We will show that we can make the invariance manifest by introducing auxiliary fields.

The organization of this paper is as follows. In Sect. 2, we define the off-shell amplitudes of closed bosonic string theory based on the moduli space of bordered Riemann surfaces. In Sect. 3, we derive recursion relations satisfied by the off-shell amplitudes. In Sect. 4, we prove that the off-shell amplitudes defined in Sect. 2 can be derived from the Fokker–Planck formalism for string fields. We show that the solution of the recursion relations in Sect. 3 satisfies the Schwinger–Dyson equation derived from the Fokker–Planck Hamiltonian. In Sect. 5, we modify the theory by introducing auxiliary fields and make it manifestly BRST invariant. Section 6 is devoted to discussions and comments. In Appendix A, we present formulas for the local coordinates on hyperbolic pants. In Appendix B, we prove the BRST identity.

2. Off-shell amplitudes

The off-shell amplitudes of the theory that we will study should correspond to hyperbolic surfaces that have geodesic boundaries with fixed lengths. In this section, we would like to define such amplitudes. The formulation is a modification of the conventional ones [4,16–18].

2.1. *The moduli space* $\mathcal{M}_{g,n,\mathbf{L}}$

Let $\Sigma_{g,n,\mathbf{L}}$ with $\mathbf{L} = (L_1, \ldots, L_n)$ be a genus-g hyperbolic surface with n geodesic boundaries (labeled by an index $a = 1,\ldots, n$) whose lengths are $L_1,\ldots, L_n$. Cutting the surface $\Sigma_{g,n,\mathbf{L}}$ along nonperipheral simple closed geodesics, we can decompose it into pairs of pants S_i ($i = 1,\ldots,2g - 2 + n$). There are many choices for such decomposition and here we pick one. The hyperbolic structure of the surface is specified by the lengths of the nonperipheral simple closed geodesics and the way how boundaries of S_i are identified. Therefore the hyperbolic structure of $\Sigma_{g,n,\mathbf{L}}$ can be parametrized by the Fenchel–Nielsen coordinates $(l_s; \tau_s)$ ($s = 1,\ldots, 3g - 3 + n$), where l_s are the lengths of the nonperipheral boundaries of S_i and τ_s denote the twist parameters that specify how boundaries of different pairs of pants are identified. The Teichmüller space $\mathcal{T}_{g,n,\mathbf{L}}$ corresponds to the region $0 < l_s < \infty$, $-\infty < \tau_s < \infty$. A volume form $\Omega_{g,n,\mathbf{L}}$ on $\mathcal{T}_{g,n,\mathbf{L}}$ called the Weil–Petersson volume form is given by

$$\Omega_{g,n,\mathbf{L}} = \bigwedge_{s=1}^{3g-3+n} [dl_s \wedge d\tau_s] .$$

$\Omega_{g,n,\mathbf{L}}$ does not depend on the choice of the pants decomposition. The moduli space $\mathcal{M}_{g,n,\mathbf{L}}$ is defined as

$$\mathcal{M}_{g,n,\mathbf{L}} \equiv \mathcal{T}_{g,n,\mathbf{L}}/\Gamma ,$$

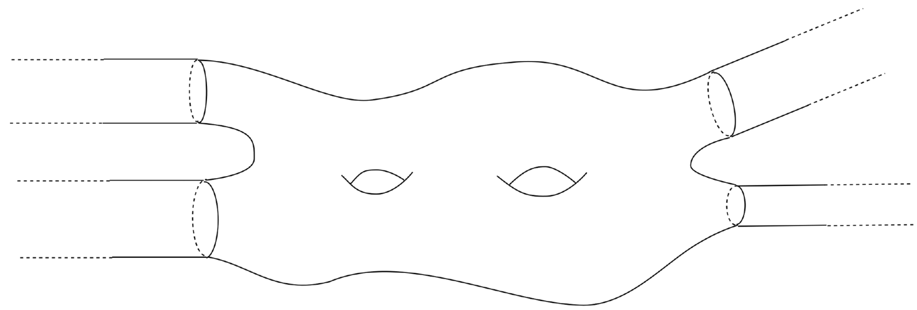

Fig. 3. Attaching flat semi-infinite cylinders to $\Sigma_{g,n,\mathbf{L}}$.

where Γ denotes the mapping class group. The Fenchel–Nielsen coordinates $(l_s; \tau_s)$ can be used as local coordinates on $\mathcal{M}_{g,n,\mathbf{L}}$. We will define the off-shell amplitudes as integrals over $\mathcal{M}_{g,n,\mathbf{L}}$. The space of all inequivalent hyperbolic structures on a surface is the same as that of the complex structures. Hence the definition of the off-shell amplitudes here can be regarded as the traditional one for the case where the lengths of the external strings are specified.

2.2. *b-ghost insertions*

Let us consider an element $\Sigma_{g,n,\mathbf{L}}$ of $\mathcal{M}_{g,n,\mathbf{L}}$. One can attach a flat semi-infinite cylinder to each boundary [8] as depicted in Fig. 3 and obtain a punctured Riemann surface. The cylinder is conformally equivalent to a disk with a puncture. Letting $w_a\,(a = 1,\ldots,n)$ be a local coordinate on the ath disk D_a such that D_a corresponds to the region $|w_a| \le 1$, the flat metric is given as

$$ds^2 = \frac{L_a^2}{(2\pi)^2}\frac{|dw_a|^2}{|w_a|^2},$$

and the ath puncture corresponds to $w_a = 0$. By these conditions, w_a is fixed up to a phase rotation. w_a can be expressed as a function $w_a(z)$ of a local coordinate z on $\Sigma_{g,n,\mathbf{L}}$. $w_a(z)$ is holomorphic in a neighborhood of ∂D_a.

In this way, from $\Sigma_{g,n,\mathbf{L}}$, we obtain a punctured Riemann surface $\Sigma_{g,\,n}$ with local coordinates around punctures, which are specified up to phase rotations. With $\Sigma_{g,\,n}$ thus obtained, one can associate a surface state, picking a local coordinate w_a as above for each D_a. Let us denote this surface state by $\langle\Sigma_{g,n,\mathbf{L}}|$. By definition, we have

$$\langle\Sigma_{g,n,\mathbf{L}}|\Psi_1\rangle\cdots|\Psi_n\rangle = \left\langle\prod_{a=1}^{n} w_a^{-1}\circ\mathcal{O}_{\Psi_a}(0)\right\rangle_{\Sigma_{g,n}}, \tag{2}$$

where $\mathcal{O}_{\Psi_a}$ denotes the operator corresponding to the state $|\Psi_a\rangle$ and $\langle\cdot\rangle_{\Sigma_{g,n}}$ denotes the correlation function on $\Sigma_{g,\,n}$. Under a phase rotation $w_a \to e^{i\alpha_a}w_a$, $\langle\Sigma_{g,n,\mathbf{L}}|$ transforms as

$$\langle\Sigma_{g,n,\mathbf{L}}| \to \langle\Sigma_{g,n,\mathbf{L}}|\prod_a e^{i\alpha_a\left(L_0^{(a)}-\bar{L}_0^{(a)}\right)}.$$

The correlation function $\langle\Sigma_{g,n,\mathbf{L}}|\Psi_1\rangle\cdots|\Psi_n\rangle$ is invariant under the phase rotation, if

$$(L_0-\bar{L}_0)|\Psi_a\rangle = 0.$$

In order to define the amplitudes, we need to construct a top form on the moduli space $\mathcal{M}_{g,n,\mathbf{L}}$ from the b-ghost. A deformation of the hyperbolic structure of a surface induces that of the complex structure. Therefore we can construct the b-ghost insertion corresponding to a tangent vector of $\mathcal{M}_{g,n,\mathbf{L}}$, following the procedure given in Refs. [4,16,18,19]. Let z_i be a local coordinate on the pair of pants S_i, such that the hyperbolic metric on S_i is in the form

$$ds^2 = e^{\varphi}|dz_i|^2.$$

Each boundary of S_i is either shared by another pair of pants S_j ($j \neq i$) or is equal to one of the boundaries of $\Sigma_{g,n,\mathbf{L}}$. In the former case, the local coordinates z_j on S_j and z_i are related by

$$z_i = F_{ij}(z_j), \tag{3}$$

in a neighborhood of $S_i \cap S_j = C_{ij}$. If the boundary of S_i coincides with ∂D_a, z_i and w_a are related by

$$z_i = f_{ia}(w_a), \tag{4}$$

in a neighborhood of ∂D_a. The transition functions F_{ij}, f_{ia} describe the moduli of $\Sigma_{g,n,\mathbf{L}}$.

Suppose that under an infinitesimal change of moduli, z_i, w_a, F_{ij}, f_{ia} change as

$$\begin{aligned} z_i &\to z_i + \varepsilon v_i, \\ w_a &\to w_a, \\ F_{ij} &\to F_{ij} + \delta F_{ij}, \\ f_{ia} &\to f_{ia} + \delta f_{ia}. \end{aligned}$$

Equations (3), (4) imply

$$\begin{aligned} z_i + \varepsilon v_i &= \left(F_{ij} + \delta F_{ij}\right)(z_j + \varepsilon v_j). \\ z_i + \varepsilon v_i &= (f_{ia} + \delta f_{ia})(w_a), \end{aligned}$$

and we obtain

$$\begin{aligned} \varepsilon\left(v_i - \frac{dz_i}{dz_j} v_j\right) &= \delta F_{ij}(z_j), \\ \varepsilon v_i &= \delta f_{ia}(w_a), \end{aligned}$$

in a neighborhood of C_{ij}, ∂D_a respectively. One can take v_i to be holomorphic in neighborhoods of boundaries of S_i and smooth inside. For such v_i, we define

$$b(v) \equiv \sum_i \left[\oint_{\partial S_i} \frac{dz_i}{2\pi i} v_i(z_i) b(z_i) - \oint_{\partial S_i} \frac{d\bar{z}_i}{2\pi i} \bar{v}_i(\bar{z}_i) \bar{b}(\bar{z}_i)\right]. \tag{5}$$

Here the integration contours are taken so that they run along ∂S_i keeping S_i on the left for z_i.

For our purpose, we need to make the formulas (3), (4), and (5) more explicit. S_i itself is a hyperbolic surface with three boundaries and by attaching flat semi-infinite cylinders to the boundaries as above, we get a three-punctured sphere with local coordinates W_k ($k = 1, 2, 3$). Therefore S_i is conformally equivalent to $\mathbb{C} - \bigcup_{k=1}^{3} D_k$ where D_k are the disks corresponding to the cylinders. We choose the local coordinate z_i on S_i to be the complex coordinate z on $\mathbb{C}$ such that the three punctures are at $z = 0, 1, \infty$. The explicit forms of $W_k(z_i)$ are given in Refs. [20,21] and are presented in Appendix A. There is a freedom in choosing which of the ∂D_k corresponds to each boundary of S_i, but Eq. (A3) implies that the $W_k(z_i)$ are related by SL(2, $\mathbb{C}$) transformation of z_i and a phase rotation and the choice does not change the result. If the boundary ∂D_a of $\Sigma_{g,n,\mathbf{L}}$ coincides with $|W_k(z_i)| = 1$, we can take w_a to be equal to $W_k(z_i)$. Then the explicit form of Eq. (4) becomes

$$z_i = W_k^{-1}(w_a). \tag{6}$$

C_{ij} should coincide with $|W_k(z_i)| = 1$ and $\left|W_{k'}(z_j)\right| = 1$ for some k, k' and we obtain the explicit form of Eq. (3) as

$$z_i = W_k^{-1}\left(\frac{e^{i\theta_{ij}}}{W_{k'}(z_j)}\right), \tag{7}$$

where θ_{ij} is the twist angle.

We take the Fenchel–Nielsen coordinates l_s, τ_s ($s = 1,\ldots, 3g - 3 + n$) on $\mathcal{M}_{g,n,\mathbf{L}}$. Changes of the transition functions (6) and (7) under the variation $l_s \to l_s + \delta l_s$, $\tau_s \to \tau_s + \delta\tau_s$ describe those of the hyperbolic structure of $\Sigma_{g,n,\mathbf{L}}$. If l_s is the length of C_{ij}, $\tau_s = \frac{l_s}{2\pi}\theta_{ij}$. For $l_s \to l_s + \varepsilon$, we can take

$$v_i = -\frac{\partial W_k(z_i)}{\partial l_s}\left(\frac{\partial W_k}{\partial z_i}\right)^{-1},$$

$$v_j = -\frac{\partial W_k(z_j)}{\partial l_s}\left(\frac{\partial W_k}{\partial z_j}\right)^{-1},$$

for $k = 1, 2, 3$, in neighborhoods of boundaries $|W_k(z_i)| = 1$, $|W_k(z_j)| = 1$ of S_i, S_j respectively. Therefore we define

$$\begin{aligned} b(\partial_{l_s}) &\equiv b(v) = b_{S_i}(\partial_{l_s}) + b_{S_j}(\partial_{l_s}),\\ b_{S_i}(\partial_{l_s}) &= -\oint_{\partial S_i}\frac{dz_i}{2\pi i}\frac{\partial W_k}{\partial l_s}\left(\frac{\partial W_k}{\partial z_i}\right)^{-1} b(z_i) + \oint_{\partial S_i}\frac{d\bar{z}_i}{2\pi i}\frac{\partial \bar{W}_k}{\partial l_s}\left(\frac{\partial \bar{W}_k}{\partial \bar{z}_i}\right)^{-1}\bar{b}(\bar{z}_i),\\ b_{S_j}(\partial_{l_s}) &= -\oint_{\partial S_j}\frac{dz_j}{2\pi i}\frac{\partial W_k}{\partial l_s}\left(\frac{\partial W_k}{\partial z_j}\right)^{-1} b(z_j) + \oint_{\partial S_j}\frac{d\bar{z}_j}{2\pi i}\frac{\partial \bar{W}_k}{\partial l_s}\left(\frac{\partial \bar{W}_k}{\partial \bar{z}_i}\right)^{-1}\bar{b}(\bar{z}_j). \end{aligned} \tag{8}$$

Here k ($k = 1, 2, 3$) for W_k in each term is chosen so that the relevant component of the boundary corresponds to $|W_k| = 1$. For $\tau_s \to \tau_s + \varepsilon$, we define

$$b(\partial_{\tau_s}) = -\frac{2\pi}{l_s}\left[\oint_{C_{ij}}\frac{dz_i}{2\pi i} iW_k(z_i)\left(\frac{\partial W_k}{\partial z_i}\right)^{-1} b(z_i) + \oint_{C_{ij}}\frac{d\bar{z}_i}{2\pi i} i\bar{W}_k\left(\frac{\partial \bar{W}_k}{\partial \bar{z}_i}\right)^{-1}\bar{b}(\bar{z}_i)\right], \tag{9}$$

where k for W_k is chosen so that C_{ij} coincides with $|W_k(z_i)| = 1$. The contours run along C_{ij} so that S_j lies to its left for z_j.

In the same way, for a pair of pants S_i one of whose boundaries coincides with ∂D_a, we define

$$b_{S_i}(\partial_{L_a}) = -\oint_{\partial S_i}\frac{dz_i}{2\pi i}\frac{\partial W_k}{\partial L_a}\left(\frac{\partial W_k}{\partial z_i}\right)^{-1} b(z_i) + \oint_{\partial S_i}\frac{d\bar{z}_i}{2\pi i}\frac{\partial \bar{W}_k}{\partial L_a}\left(\frac{\partial \bar{W}_k}{\partial \bar{z}_i}\right)^{-1}\bar{b}(\bar{z}_i). \tag{10}$$

2.3. *Off-shell amplitudes*

Now we define the connected g-loop n-point amplitude $A_{g,n}((|\Psi_1\rangle, L_1),\ldots, (|\Psi_n\rangle, L_n))$ by

$$A_{g,n}\left((|\Psi_1\rangle, L_1), \ldots, (|\Psi_n\rangle, L_n)\right) = 2^{-\delta_{g,1}\delta_{n,1}}\int_{\mathcal{M}_{g,n,\mathbf{L}}}(2\pi i)^{-3g+3-n}\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}|\Psi_1\rangle\cdots|\Psi_n\rangle. \tag{11}$$

Here $\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}$ is defined so that

$$\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}|\Psi_1\rangle\cdots|\Psi_n\rangle = \left\langle B_{6g-6+2n}\prod_{a=1}^{n} w_a^{-1}\circ\mathcal{O}_{\Psi_a}(0)\right\rangle_{\Sigma_{g,n}}$$

holds for any $|\Psi_a\rangle$, with

$$B_{6g-6+2n} = \prod_{s=1}^{3g-3+n} \left[b(\partial_{l_s}) b(\partial_{\tau_s}) \right] \bigwedge_{s=1}^{3g-3+n} \left[dl_s \wedge d\tau_s \right]. \tag{12}$$

The factor $2^{-\delta_{g,1}\delta_{n,1}}$ is due to the fact that $\Sigma_{1,\,1,\,L}$ has a $\mathbb{Z}_2$ symmetry. The state $|\Psi_a\rangle$ is taken to be an element of $\mathcal{H}_0$ that consists of the states $|\Psi\rangle$ satisfying

$$b_0^-|\Psi\rangle = (L_0 - \bar{L}_0)|\Psi\rangle = 0, \tag{13}$$

where $b_0^\pm \equiv b_0 \pm \bar{b}_0$.

$B_{6g\,-\,6\,+\,2n}$ is defined by using the Fenchel–Nielsen coordinate l_s, τ_s associated with a pants decomposition of $\Sigma_{g,n,\mathbf{L}}$. We should check if the amplitude (11) does not depend on the choice of the pants decomposition. Suppose that we have two pants decompositions, in which $\Sigma_{g,n,\mathbf{L}}$ is decomposed into pairs of pants $S_i\,(i = 1,\ldots, 2g - 2 + n)$ and S'_j $(j = 1, \ldots, 2g-2+n)$. Let z_i and z'_j be the local coordinates on S_i, S'_j respectively. There should be a function G_{ij} holomorphic on $S_i \cap S'_j$ such that

$$z_i = G_{ij}\left(z'_j\right).$$

If a boundary of $S_i \cap S'_j$ coincides with ∂D_a, we have functions g_{ia}, g'_{ja} such that

$$\begin{aligned} z_i &= g_{ia}(w_a), \\ z'_j &= g'_{ja}(w_a), \end{aligned} \tag{14}$$

in a neighborhood of ∂D_a. Suppose that under an infinitesimal change of moduli, $z_i, z'_j, w_a, G_{ij}, g_{ia}, g'_{ia}$ change as

$$\begin{aligned} z_i &\to z_i + \varepsilon v_i, \\ z'_j &\to z'_j + \varepsilon v'_j, \\ w_a &\to w_a, \\ G_{ij} &\to G_{ij} + \delta G_{ij}, \\ g_{ia} &\to g_{ia} + \delta g_{ia}, \\ g'_{ja} &\to g'_{ja} + \delta g'_{ja}. \end{aligned}$$

We can derive

$$\varepsilon\left(v_i - \frac{\partial z_i}{\partial z'_j} v'_j\right) = \delta G_{ij}\left(z'_j\right), \tag{15}$$

$$\begin{aligned} \varepsilon v_i &= \delta g_{ia}(w_a), \\ \varepsilon v'_j &= \delta g'_{ja}(w_a). \end{aligned} \tag{16}$$

Equation (15) implies

$$\oint_{\partial(S_i \cap S'_j)} \frac{dz_i}{2\pi i}\left(v_i - \frac{\partial z_i}{\partial z'_j} v'_j\right) b(z_i) = 0.$$

If we take v_i, v'_j to be holomorphic in neighborhoods of the boundaries of S_i, S'_j respectively, we get

$$0 = \sum_{i,j} \oint_{\partial(S_i \cap S'_j)} \frac{dz_i}{2\pi i} \left(v_i - \frac{\partial z_i}{\partial z'_j} v'_j \right) b(z_i)$$

$$= \sum_i \oint_{\partial S_i} \frac{dz_i}{2\pi i} v_i(z_i) b(z_i) - \sum_j \oint_{\partial S'_j} \frac{dz'_j}{2\pi i} v'_j\left(z'_j\right) b\left(z'_j\right).$$

Therefore the b-ghost insertion (5) satisfies

$$b(v) = b(v'), \tag{17}$$

if v and v' correspond to the same change of moduli.

Let $(l_s; \tau_s)$, $(l'_t; \tau'_t)$ be the Fenchel–Nielsen coordinates associated with the two different pants decompositions. Using Eq. (17), we may be able to express $b(\partial_{l_s})$, $b(\partial_{\tau_s})$ in terms of $b(\partial_{l'_t})$, $b(\partial_{\tau'_t})$. In doing so, there is one thing that one should be careful about. In defining $b(\partial_{l_s})$, we have taken the coordinate on D_a to be $W_k(z_i)$, if ∂D_a coincides with a boundary of S_i. If one of the boundaries of S'_j coincides with ∂D_a,

$$W_k(z_i) = e^{i\alpha_a} W_{k'}(z'_j)$$

should hold with some k'. Here α_a is a real function of moduli. If we fix w_a in Eq. (14) to be $W_k(z_i)$, Eq. (16) implies

$$\varepsilon \oint_{\partial D_a} \frac{dz'_j}{2\pi i} v'_j\left(z'_j\right) b\left(z'_j\right) = -\oint_{\partial D_a} \frac{dz'_j}{2\pi i} \delta W_{k'}\left(z'_j\right) \left(\frac{\partial W_{k'}}{\partial z'_j}\right)^{-1} b(z'_j) - i\delta\alpha_a \oint_{\partial D_a} \frac{dw_a}{2\pi i} w_a b(w_a).$$

Therefore the relations between $b(\partial_{l_s})$, $b(\partial_{\tau_s})$ and $b(\partial_{l'_t})$, $b(\partial_{\tau'_t})$ should be

$$b(\partial_{l_s}) = \sum_t \left[\frac{\partial l'_t}{\partial l_s} b(\partial_{l'_t}) + \frac{\partial \tau'_t}{\partial l_s} b(\partial_{\tau'_t}) \right] - i \sum_{a=1}^{n} \frac{\partial \alpha_a}{\partial l_s} b_0^{-(a)},$$

$$b(\partial_{\tau_s}) = \sum_t \left[\frac{\partial l'_t}{\partial \tau_s} b(\partial_{l'_t}) + \frac{\partial \tau'_t}{\partial \tau_s} b(\partial_{\tau'_t}) \right] - i \sum_{a=1}^{n} \frac{\partial \alpha_a}{\partial \tau_s} b_0^{-(a)}.$$

Here $b_0^{-(a)}$ denotes b_0^- acting on the ath Hilbert space. Substituting these into the amplitude (11), we can see that it is independent of the pants decomposition, if $|\Psi_a\rangle$ $(a = 1,\ldots, n)$ satisfy the condition (13).

By the BRST identity proved in Appendix B, we have

$$\langle \Sigma_{g,n,\mathbf{L}} | B_{6g-6+2n} \sum_a Q^{(a)} |\Psi_1\rangle \cdots |\Psi_n\rangle = d\left[\langle \Sigma_{g,n,\mathbf{L}} | B_{6g-7+2n} |\Psi_1\rangle \cdots |\Psi_n\rangle \right], \tag{18}$$

and the amplitude $A_{g,n}((|\Psi_1\rangle, L_1),\ldots, (|\Psi_n\rangle, L_n))$ is BRST invariant if one treats the boundary contributions appropriately. By construction, $A_{g,n}((|\Psi_1\rangle, L_1),\ldots, (|\Psi_n\rangle, L_n))$ exists for $2g - 2 + n > 0$.

The amplitude (11) is not something we usually deal with in string theory. In the limit $L_a \to 0$, $\mathcal{M}_{g,n,\mathbf{L}}$ coincides with the moduli space $\mathcal{M}_{g,n}$ of punctured Riemann surfaces and $(l_s; \tau_s)$ become the Fenchel–Nielsen coordinates on $\mathcal{M}_{g,n}$. Therefore

$$\lim_{L_a \to 0} A_{g,n}\left((|\Psi_1\rangle, L_1), \ldots, (|\Psi_n\rangle, L_n)\right) \tag{19}$$

is equal to the on-shell amplitude when $|\Psi_a\rangle$ are taken to be on-shell physical states. In Sect. 5, we will show that the off-shell amplitudes of the kind studied in Refs. [22–24] can also be derived in our formalism.

3. A recursion relation of the off-shell amplitudes

Given a propagator, one can construct the string field action that reproduces the off-shell amplitudes defined in the previous section order by order in the string coupling constant g_s. If we take the propagator to be the one depicted in Fig. 1, we run into the difficulty mentioned in the introduction. In this paper, as a workaround, we construct an SFT by studying equations satisfied by the off-shell amplitudes.

In order to calculate the right-hand side of Eq. (11), we need to specify the integration region in terms of the Fenchel–Nielsen coordinates. However, no concrete description of the fundamental domain of the mapping class group in $\mathcal{T}_{g,n,\mathbf{L}}$ is known in general. Mathematicians tried to calculate the the Weil–Petersson volume $V_{g,n}(\mathbf{L})$ of $\mathcal{M}_{g,n,\mathbf{L}}$ defined by[1]

$$V_{g,n}(\mathbf{L}) \equiv 2^{-\delta_{g,1}\delta_{n,1}} \int_{\mathcal{M}_{g,n,\mathbf{L}}} \Omega_{g,n,\mathbf{L}}, \tag{20}$$

and encountered the same problem. Mirzakhani discovered [10,11] a way to overcome this difficulty. In this section, we would like to explain her method (for reviews, see, e.g., Refs. [7,25,26]) and apply it to the off-shell amplitudes.

3.1. *Mirzakhani's scheme*

Mirzakhani's idea is to transform an integral over $\mathcal{M}_{g,n,\mathbf{L}}$ into the one over its covering space. Suppose that X_1 are X_2 are manifolds and

$$\pi : X_1 \to X_2$$

is a covering map. Let dv_2 be a volume form on X_2, and we define dv_1 to be the pull-back, i.e.,

$$dv_1 = \pi^* dv_2.$$

For a function f on X_1, one can define the push-forward $\pi_* f$ by

$$(\pi_* f)(x) = \sum_{y \in \pi^{-1}(x)} f(y).$$

Then

$$\int_{X_2} (\pi_* f) dv_2 = \int_{X_1} f dv_1 \tag{21}$$

holds.

Equation (21) can be used to calculate the volume of the moduli space $\mathcal{M}_{1,1,0}$, for instance. We take X_2 to be $\mathcal{M}_{1,1,0}$ and X_1 to be the following space of pairs

$$\{(\Sigma_{1,1,0}, \gamma)\,|\,\Sigma_{1,1,0} \in \mathcal{M}_{1,1,0} \text{ and } \gamma \text{ is a simple closed geodesic on } \Sigma_{1,1,0}\}.$$

The set of simple closed geodesics γ on $\Sigma_{1,1,0}$ is a discrete set with infinitely many elements and a mapping class group orbit. X_1 can be described by the pair (l_γ, τ_γ) where l_γ is the length of γ and τ_γ is the twist parameter corresponding to it. X_1 corresponds to the region

$$0 < l_\gamma < \infty,\ 0 \le \tau_\gamma \le l_\gamma,$$

with $(l_\gamma, 0) \sim (l_\gamma, l_\gamma)$. The projection π can be defined by

$$\pi(\Sigma_{1,1,0}, \gamma) = \Sigma_{1,1,0},$$

and for $dv_2 = \Omega_{1,1,0}$, we have

$$dv_1 = \pi^* dv_2 = dl_\gamma \wedge d\tau_\gamma.$$

[1] There are two conventions for $V_{1,1}(L)$ due to the presence of $\mathbb{Z}_2$ symmetry. Here we adopt Eq. (20) so as to make Eq. (28) look simple.

If one takes the function f to be a function of l_γ, the value of $\pi_* f$ at $\Sigma_{1,1,0} \in \mathcal{M}_{1,1,0}$ becomes

$$\sum_\gamma f(l_\gamma),$$

where the sum is over the set of simple geodesics on $\Sigma_{1,1,0}$. In Ref. [27], McShane proved that for $f(l) = \frac{2}{1+e^l}$,

$$\sum_\gamma f(l_\gamma) = 1 \tag{22}$$

holds. Equation (22) is called the McShane identity. For this choice of f, Eq. (21) becomes

$$\int_{\mathcal{M}_{1,1,0}} \Omega_{1,1,0} = \int_{X_2} (\pi_* f) dv_2 = \int_{X_1} f dv_1 = \int_0^\infty dl_\gamma \frac{2l_\gamma}{1+e^{l_\gamma}} = \frac{\pi^2}{6}, \tag{23}$$

and we get the volume of $\mathcal{M}_{1,1,0}$.

Mirzakhani generalized this procedure to general (g, n), by discovering a generalization of the McShane identity. For $\Sigma_{g,n,\mathbf{L}} \in \mathcal{M}_{g,n,\mathbf{L}}$, let $\beta_1, \ldots, \beta_n$ be the boundaries so that the lengths of $\beta_1, \ldots, \beta_n$ are $L_1, \ldots, L_n$ respectively. The generalized McShane identity derived in Ref. [10] is

$$L_1 = \sum_{\{\gamma,\delta\}\in\mathscr{C}_1} \mathsf{D}_{L_1 l_\gamma l_\delta} + \sum_{a=2}^{n} \sum_{\gamma\in\mathscr{C}_a} (\mathsf{T}_{L_1 L_a l_\gamma} + \mathsf{D}_{L_1 L_a l_\gamma}), \tag{24}$$

where

$$\mathsf{D}_{LL'L''} = 2\left(\log(e^{\frac{L}{2}} + e^{\frac{L'+L''}{2}}) - \log(e^{-\frac{L}{2}} + e^{\frac{L'+L''}{2}})\right), \tag{25}$$

$$\mathsf{T}_{LL'L''} = \log \frac{\cosh\frac{L''}{2} + \cosh\frac{L+L'}{2}}{\cosh\frac{L''}{2} + \cosh\frac{L-L'}{2}}, \tag{26}$$

$$\mathscr{C}_1 \equiv \left\{ \begin{array}{l} \text{the collection of unordered pairs of nonperipheral simple closed geodesics } \{\gamma, \delta\} \\ \text{on } \Sigma_{g,n,\mathbf{L}} \text{ which bounds a pair of pants along with the boundary } \beta_1 \end{array} \right\},$$

$$\mathscr{C}_a \equiv \left\{ \begin{array}{l} \text{the collection of simple closed geodesics } \gamma \text{ on } \Sigma_{g,n,\mathbf{L}} \\ \text{which bounds a pair of pants along with the boundaries } \beta_1 \text{ and } \beta_a \end{array} \right\},$$

and l_γ, l_δ are the lengths of γ, δ respectively. For $L, L', L'' > 0$, $\mathsf{D}_{LL'L''}$, $\mathsf{T}_{LL'L''}>0$ and

$$\begin{aligned} \mathsf{D}_{LL'L''} &= \mathsf{D}_{LL''L'}, \\ \mathsf{T}_{LL'L''} &= \mathsf{T}_{L'LL''}, \\ \mathsf{D}_{LL'L''} + \mathsf{T}_{LL'L''} + \mathsf{T}_{LL''L'} &= L. \end{aligned} \tag{27}$$

Multiplying Eq. (24) by $\Omega_{g,n,\mathbf{L}}$ and integrating it over $\mathcal{M}_{g,n,\mathbf{L}}$, one obtains Mirzakhani's recursion relation:

$$\begin{aligned} LV_{g,n+1}(L,\mathbf{L}) = &\frac{1}{2}\int_0^\infty dL'L' \int_0^\infty dL''L''\mathsf{D}_{LL'L''} \\ &\times \left(V_{g-1,n+2}(L', L'', \mathbf{L}) + \sum_{\text{stable}} V_{g_1,n_1}(L', \mathbf{L}_1) V_{g_2,n_2}(L'', \mathbf{L}_2) \right) \\ &+ \sum_{a=1}^{n} \int_0^\infty dL'L' \left(\mathsf{T}_{L_1 L_a L'} + \mathsf{D}_{L_1 L_a L'}\right) V_{g,n}(L, \mathbf{L}\backslash L_a), \end{aligned} \tag{28}$$

which holds for $2g-2+n>0$. The sum $\sum_{\text{stable}}$ here means the sum over $g_1, g_2, n_1, n_2, \mathbf{L}_1, \mathbf{L}_2$ such that[2]

$$\begin{aligned} g_1+g_2&=g,\\ n_1+n_2&=n+2,\\ \mathbf{L}_1\cup\mathbf{L}_2&=\{L_1,\ldots,L_n\},\\ \mathbf{L}_1\cap\mathbf{L}_2&=\phi,\\ 2g_1-2+n_1&>0,\\ 2g_2-2+n_2&>0. \end{aligned} \tag{29}$$

With the information $V_{0,3}(L_1,L_2,L_3)=1$, $V_{1,1}(L)=\frac{\pi^2}{12}+\frac{L^2}{48}$, it is possible to calculate $V_{g,n}(\mathbf{L})$ for all the other g, n by the recursion relation (28).

3.2. *Recursion relation of the off-shell amplitudes*

A recursion relation of the off-shell amplitudes (11) is derived in the same way as Mirzakhani's recursion relation. Multiplying Eq. (24) by $(2\pi i)^{-3g+3-n}\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}|\Psi_1\rangle\cdots|\Psi_n\rangle$ and integrating it over $\mathcal{M}_{g,n,\mathbf{L}}$, we obtain

$$\begin{aligned} &L_1\int_{\mathcal{M}_{g,n,\mathbf{L}}}(2\pi i)^{-3g+3-n}\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}|\Psi_1\rangle\cdots|\Psi_n\rangle\\ &\quad=\int_{\mathcal{M}_{g,n,\mathbf{L}}}\sum_{\{\gamma,\delta\}\in\mathscr{C}_1}\mathsf{D}_{L_1l_\gamma l_\delta}\cdot(2\pi i)^{-3g+3-n}\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}|\Psi_1\rangle\cdots|\Psi_n\rangle\\ &\qquad+\sum_{a=2}^{n}\int_{\mathcal{M}_{g,n,\mathbf{L}}}\sum_{\gamma\in\mathscr{C}_a}(\mathsf{T}_{L_1L_al_\gamma}+\mathsf{D}_{L_1L_al_\gamma})\cdot(2\pi i)^{-3g+3-n}\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}|\Psi_1\rangle\cdots|\Psi_n\rangle. \end{aligned}\tag{30}$$

The left-hand side yields

$$L_1A_{g,n}\left((|\Psi_1\rangle,L_1),\ldots,(|\Psi_n\rangle,L_n)\right).$$

Here we restrict ourselves to the case $2g-2+n>1$. We will rewrite the terms on the right-hand side by using the formula (21). Let us first consider the integral

$$\int_{\mathcal{M}_{g,n,\mathbf{L}}}\sum_{\gamma\in\mathscr{C}_a}(\mathsf{T}_{L_1L_al_\gamma}+\mathsf{D}_{L_1L_al_\gamma})\cdot(2\pi i)^{-3g+3-n}\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}|\Psi_1\rangle\cdots|\Psi_n\rangle. \tag{31}$$

In order to unfold this integral, we take X_2 to be $\mathcal{M}_{g,n,\mathbf{L}}$ and X_1 to be the space of pairs

$$\left\{(\Sigma_{g,n,\mathbf{L}},\gamma)\;\middle|\;\begin{array}{c}\Sigma_{g,n,\mathbf{L}}\in\mathcal{M}_{g,n,\mathbf{L}}\text{ and }\gamma\text{ is a simple closed geodesic on }\Sigma_{g,n,\mathbf{L}}\\ \text{that bounds a pair of pants along with the boundaries }\beta_1\text{ and }\beta_a\end{array}\right\}.$$

The set of possible γ on $\Sigma_{g,n,\mathbf{L}}$ is exactly $\mathscr{C}_a$ and is a mapping class group orbit. X_1 can be described by the triple $(l_\gamma,\tau_\gamma,\Sigma'_{g,n-1,\mathbf{L}'})$ where τ_γ is the twist parameter corresponding to γ and $\Sigma'_{g,n-1,\mathbf{L}'}$ is the complement of the pair of pants bounded by β_1,β_a,γ with $\mathbf{L}'=(l_\gamma,L_2,\ldots,\hat{L}_a,\ldots,L_n)$. X_1 corresponds to the region

$$0<l_\gamma<\infty,\;\;0\le\tau_\gamma\le l_\gamma,$$

with $(l_\gamma,0,\Sigma'_{g,n-1,\mathbf{L}'})\sim(l_\gamma,l_\gamma,\Sigma'_{g,n-1,\mathbf{L}'})$. The projection π can be defined by

$$\pi(\Sigma_{g,n,\mathbf{L}},\gamma)=\Sigma_{g,n,\mathbf{L}},$$

[2]Here we consider $\mathbf{L}_1, \mathbf{L}_2$ as unordered subsets of $\mathbf{L}=\{L_1,\cdots,L_n\}$.

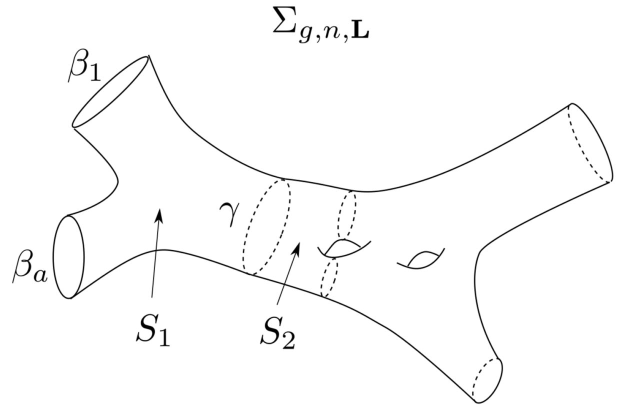

Fig. 4. $\Sigma_{g,n,\mathbf{L}}$ and γ.

and for $dv_2 = \Omega_{g,n,\mathbf{L}}$, we have

$$dv_1 = \pi^* dv_2 = dl_\gamma \wedge d\tau_\gamma \wedge \Omega'_{g,n-1,\mathbf{L}'},$$

where $\Omega'_{g,n-1,\mathbf{L}'}$ is the Weil–Petersson volume form on $\Sigma'_{g,n-1,\mathbf{L}'}$.

Now, if one takes

$$f dv_1 = (\mathsf{T}_{L_1 L_a l_\gamma} + \mathsf{D}_{L_1 L_a l_\gamma}) \cdot (2\pi i)^{-3g+3-n} \langle \Sigma_{g,n,\mathbf{L}} | B_{6g-6+2n} | \Psi_1 \rangle \cdots | \Psi_n \rangle,$$

Eq. (21) becomes

$$\int_{X_1} (\mathsf{T}_{L_1 L_a l_\gamma} + \mathsf{D}_{L_1 L_a l_\gamma}) \cdot (2\pi i)^{-3g+3-n} \langle \Sigma_{g,n,\mathbf{L}} | B_{6g-6+2n} | \Psi_1 \rangle \cdots | \Psi_n \rangle$$
$$= \int_{\mathcal{M}_{g,n,\mathbf{L}}} \sum_{\gamma \in \mathscr{C}_a} (\mathsf{T}_{L_1 L_a l_\gamma} + \mathsf{D}_{L_1 L_a l_\gamma}) \cdot (2\pi i)^{-3g+3-n} \langle \Sigma_{g,n,\mathbf{L}} | B_{6g-6+2n} | \Psi_1 \rangle \cdots | \Psi_n \rangle. \quad (32)$$

Therefore Eq. (31) is obtained by evaluating the left-hand side of Eq. (32).

Let us consider a pants decomposition of $\Sigma_{g,n,\mathbf{L}}$ such that one pair of the pants has boundaries β_1, β_a, γ (Fig. 4). We denote this pair of pants by S_1 and the adjacent one by S_2. Based on the pants decomposition we define the Fenchel–Nielsen coordinates l_s, τ_s ($s = 1, \ldots, 3g - 3 + n$) such that $(l_1, \tau_1) = (l_\gamma, \tau_\gamma)$. Cutting $\Sigma_{g,n,\mathbf{L}}$ along γ, we get a three-holed sphere $\Sigma_{0,3,(L_1,L_a,l_1)}$ and $\Sigma'_{g,n-1,\mathbf{L}'}$. $\Sigma'_{g,n-1,\mathbf{L}'}$ inherits the pants decomposition of $\Sigma_{g,n,\mathbf{L}}$. Then $(2\pi i)^{-3g+3-n} \langle \Sigma_{g,n,\mathbf{L}} | B_{6g-6+2n} | \Psi_1 \rangle \cdots | \Psi_n \rangle$ can be expressed as

$$(2\pi i)^{-3g+3-n} \langle \Sigma_{g,n,\mathbf{L}} | B_{6g-6+2n} | \Psi_1 \rangle \cdots | \Psi_n \rangle$$
$$= (2\pi i)^{-3g+3-n} dl_1 \wedge d\tau_1 \langle \Sigma_{g,n,\mathbf{L}} | \left[b_{S_1}(\partial_{l_1}) + b_{S_2}(\partial_{l_1}) \right] b(\partial_{\tau_1}) B'_{6g-8+2n} | \Psi_1 \rangle \cdots | \Psi_n \rangle, \quad (33)$$

where $B'_{6g-8+2n}$ denotes the $6g - 6 + 2(n-1)$ form on $\Sigma'_{g,n-1,\mathbf{L}'}$ defined through Eq. (12).

$\Sigma_{g,n,\mathbf{L}}$ can be generated by gluing a pair of pants $\Sigma_{0,3,(L_1,L_a,l_1)}$ and $\Sigma'_{g,n-1,\mathbf{L}'}$ (Fig. 5) using the plumbing fixture relation (7). Hence the correlation function on the right-hand side of Eq. (33) can be factorized into those on $\Sigma_{0,3,(L_1,L_a,l_1)}$ and $\Sigma'_{g,n-1,\mathbf{L}'}$. Let $|\varphi_i\rangle$ be a basis of the Hilbert space $\mathcal{H}$ of the worldsheet theory of the strings and $\langle \varphi_i^c|$ be the conjugate state of $|\varphi_i\rangle$ such that

$$\langle \varphi_i^c | \varphi_j \rangle = \delta_{ij},$$
$$\langle \varphi_j | \varphi_i^c \rangle = (-1)^{n_{\varphi_i}} \delta_{ij},$$
$$\sum_i |\varphi_i\rangle \langle \varphi_i^c| = \sum_i |\varphi_i^c\rangle \langle \varphi_i| (-1)^{n_{\varphi_i}} = \mathbf{1}.$$

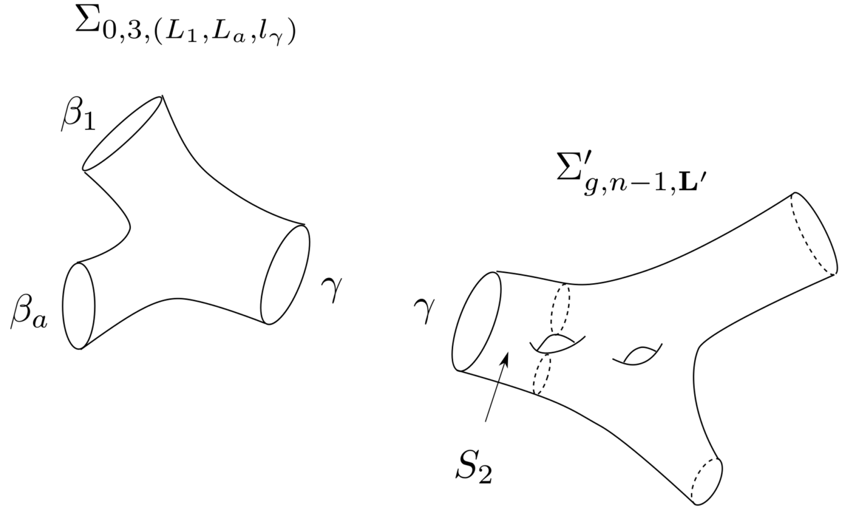

Fig. 5. The decomposition of $\Sigma_{g,n,\mathbf{L}}$ corresponding to Eq. (34).

Here $\langle\varphi_i|$ is the BPZ conjugate of $|\varphi_i\rangle$ and n_{φ_i} is the ghost number of $|\varphi_i\rangle$. Then we have

$$\begin{aligned}
&\langle\Sigma_{g,n,\mathbf{L}}|\left[b_{S_1}(\partial_{l_1})+b_{S_2}(\partial_{l_1})\right]b(\partial_{\tau_1})B_{6g-6+2n}|\Psi_1\rangle\cdots|\Psi_n\rangle\\
&=-\frac{2\pi i}{l_1}\varepsilon_a\sum_{i,j}\Big[\langle\Sigma_{0,3,(L_1,L_a,l_1)}|b_{\Sigma_{0,3,(L_1,L_a,l_1)}}(\partial_{l_1})b_0^{-(0)}|\Psi_1\rangle_1|\Psi_a\rangle_a e^{i\theta(L_0-\bar{L}_0)}|\varphi_i\rangle_0\\
&\quad\times 2^{-\delta_{g,1}\delta_{n,2}}\langle\Sigma'_{g,n-1,\mathbf{L}'}|B'_{6g-8+2n}|\varphi_j\rangle|\Psi_2\rangle\cdots\widehat{|\Psi_a\rangle}\cdots|\Psi_n\rangle\\
&\quad+\langle\Sigma_{0,3,(L_1,L_a,l_1)}|\Psi_1\rangle|\Psi_a\rangle|\varphi_i\rangle\\
&\quad\times 2^{-\delta_{g,1}\delta_{n,2}}\langle\Sigma'_{g,n-1,\mathbf{L}'}|B'_{6g-8+2n}b_{S_2}(\partial_{l_1})b_0^{-}e^{i\theta(L_0-\bar{L}_0)}|\varphi_j\rangle|\Psi_2\rangle\cdots\widehat{|\Psi_a\rangle}\cdots|\Psi_n\rangle\Big]\\
&\quad\times\langle\varphi_i^c|\varphi_j^c\rangle(-1)^{n_{\varphi_j}},
\end{aligned}\tag{34}$$

where

$$\varepsilon_a=(-1)^{n_a(n_2+\cdots+n_{a-1})},$$

n_b denotes the ghost number of $|\Psi_b\rangle$, and θ denotes the twist angle. The factor $2^{-\delta_{g,1}\delta_{n,2}}$ is due to the fact that $\Sigma'_{g,n-1,\mathbf{L}'}$ has a $\mathbb{Z}_2$ symmetry for $g=1$, $n=2$.

Substituting Eqs. (33) and (34) into Eq. (32), we can see that the second term on the right-hand side of Eq. (30) becomes

$$\begin{aligned}
&\sum_{a=2}^{n}\int_{\mathcal{M}_{g,n,\mathbf{L}}}\sum_{\gamma\in\mathscr{C}_a}(\mathsf{T}_{L_1L_al_\gamma}+\mathsf{D}_{L_1L_al_\gamma})\cdot(2\pi i)^{-3g+3-n}\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}|\Psi_1\rangle\cdots|\Psi_n\rangle\\
&=-\sum_{a=2}^{n}\sum_{i,j}\varepsilon_a\Big[\int_0^\infty dl_1(\mathsf{T}_{L_1L_al_1}+\mathsf{D}_{L_1L_al_1})\langle\Sigma_{0,3,(L_1,L_a,l_1)}|b_{\Sigma_{0,3,(L_1,L_a,l_1)}}(\partial_{l_1})b_0^{-(0)}P^{(0)}|\Psi_1\rangle_1|\Psi_a\rangle_a|\varphi_i\rangle_0\\
&\quad\times 2^{-\delta_{g,1}\delta_{n,2}}\int_{\mathcal{M}_{g,n-1,\mathbf{L}'}}(2\pi i)^{-3g+4-n}\langle\Sigma'_{g,n-1,\mathbf{L}'}|B'_{6g-8+2n}|\varphi_j\rangle|\Psi_2\rangle\cdots\widehat{|\Psi_a\rangle}\cdots|\Psi_n\rangle\\
&\quad+\int_0^\infty dl_1(\mathsf{T}_{L_1L_al_1}+\mathsf{D}_{L_1L_al_1})\langle\Sigma_{0,3,(L_1,L_a,l_1)}|\Psi_1\rangle|\Psi_a\rangle|\varphi_i\rangle\\
&\quad\times 2^{-\delta_{g,1}\delta_{n,2}}\int_{\mathcal{M}_{g,n-1,\mathbf{L}'}}(2\pi i)^{-3g+4-n}\langle\Sigma'_{g,n-1,\mathbf{L}'}|B'_{6g-8+2n}b_{S_2}(\partial_{l_1})b_0^{-}P|\varphi_j\rangle|\Psi_2\rangle\cdots\widehat{|\Psi_a\rangle}\cdots|\Psi_n\rangle\Big]\\
&\quad\times\langle\varphi_i^c|\varphi_j^c\rangle(-1)^{n_{\varphi_j}},
\end{aligned}\tag{35}$$

with $P = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(L_0 - \bar{L}_0)}$.

Equation (35) implies that it will be convenient to consider the recursion relation of the amplitudes of the form

$$\begin{aligned} &A_{g,n}\left((|\varphi_{i_1}\rangle, \alpha_1, L_1), \ldots, (|\varphi_{i_n}\rangle, \alpha_n, L_n)\right) \\ &\quad = 2^{-\delta_{g,1}\delta_{n,1}} \int_{\mathcal{M}_{g,n,\mathbf{L}}} (2\pi i)^{-3g+3-n} \langle \Sigma_{g,n,\mathbf{L}} | B_{6g-6+2n} B^1_{\alpha_1} \cdots B^n_{\alpha_n} |\varphi_{i_1}\rangle_1 \cdots |\varphi_{i_n}\rangle_n. \end{aligned} \tag{36}$$

Here the indices α_a ($a = 1,\ldots, n$) take values $\pm$ and

$$B^a_{\alpha_a} \equiv \begin{cases} 1 & \alpha_a = + \\ b_0^{-(a)} b_{S_a}(\partial_{L_a}) P^{(a)} & \alpha_a = - \end{cases}. \tag{37}$$

S_a for $b_{S_a}(\partial_{L_a})$ in Eq. (37) denotes the pair of pants that has a boundary corresponding to the ath external line in a pants decomposition of $\Sigma_{g,n,\mathbf{L}}$. $b_{S_a}(\partial_{L_a})$ depends on the choice of the pants decomposition, because it corresponds to the variation $L_a \to L_a + \varepsilon$ with l_s, τ_s fixed. However, $b_{S_a}(\partial_{L_a}) B_{6g-6+2n}$ and the amplitude in Eq. (36) is independent of the choice of S_a.

Equation (35) can be recast into

$$\begin{aligned} &\sum_{a=2}^{n} \int_{\mathcal{M}_{g,n,\mathbf{L}}} \sum_{\gamma \in \mathscr{C}_a} (\mathsf{T}_{L_1 L_a l_\gamma} + \mathsf{D}_{L_1 L_a l_\gamma}) \cdot (2\pi i)^{-3g+3-n} \langle \Sigma_{g,n,\mathbf{L}} | B_{6g-6+2n} B^1_{\alpha_1} \cdots B^n_{\alpha_n} |\varphi_{i_1}\rangle_1 \cdots |\varphi_{i_n}\rangle_n \\ &\quad = \sum_{a=2}^{n} \int_0^\infty dL (\mathsf{T}_{L_1 L_a L} + \mathsf{D}_{L_1 L_a L}) \\ &\qquad \times \sum_{i,j,\alpha} \varepsilon_a \Big[\langle \Sigma_{0,3,(L_1,L_a,L)} | B^1_{\alpha_1} B^a_{\alpha_a} B^0_{-\alpha} |\varphi_{i_1}\rangle_1 |\varphi_{i_a}\rangle_a |\varphi_i\rangle_0 \langle \varphi_i^c | \varphi_j^c \rangle (-1)^{n_{\varphi_j}} \\ &\qquad \times A_{g,n-1}\Big((|\varphi_j\rangle, \alpha, L), (|\varphi_{i_2}\rangle, \alpha_2, L_2), \ldots, (|\varphi_{\widehat{i_a}\rangle, \alpha_a,} L_a), \ldots, (|\varphi_{i_n}\rangle, \alpha_n, L_n) \Big). \end{aligned} \tag{38}$$

We simplify the formula by introducing the following notation. The external states are labeled by i (for $|\varphi_i\rangle$), α, and L. We denote these collectively by I and rewrite Eq. (38) in the following way:

$$\begin{aligned} &\sum_{a=2}^{n} \int_{\mathcal{M}_{g,n,\mathbf{L}}} \sum_{\gamma \in \mathscr{C}_a} (\mathsf{T}_{L_1 L_a l_\gamma} + \mathsf{D}_{L_1 L_a l_\gamma}) \cdot (2\pi i)^{-3g+3-n} \langle \Sigma_{g,n,\mathbf{L}} | B_{6g-6+2n} B^1_{\alpha_1} \cdots B^n_{\alpha_n} |\varphi_{i_1}\rangle \cdots |\varphi_{i_n}\rangle \\ &\quad = \sum_{a=2}^{n} \varepsilon_a (T^{I_1 I_a J} + D^{I_1 I_a J}) G_{JI} A^{I I_2 \cdots \hat{I}_a \cdots I_n}_{g,n-1}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} T^{I_1 I_2 I_3} &\equiv \mathsf{T}_{L_1 L_2 L_3} \langle \Sigma_{0,3,(L_1,L_2,L_3)} | B^1_{\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\varphi_{i_1}\rangle_1 |\varphi_{i_2}\rangle_2 |\varphi_{i_3}\rangle_3, \\ D^{I_1 I_2 I_3} &\equiv \mathsf{D}_{L_1 L_2 L_3} \langle \Sigma_{0,3,(L_1,L_2,L_3)} | B^1_{\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\varphi_{i_1}\rangle_1 |\varphi_{i_2}\rangle_2 |\varphi_{i_3}\rangle_3, \\ G_{I_1 I_2} &\equiv \langle \varphi^c_{i_1} | \varphi^c_{i_2} \rangle (-1)^{n_{\varphi_{i_2}}} \delta(L_1 - L_2) \delta_{\alpha_1, -\alpha_2}, \\ A^{I_1 \cdots I_n}_{g,n} &\equiv A_{g,n}\left((|\varphi_{i_1}\rangle, \alpha_1, L_1), \cdots, (|\varphi_{i_n}\rangle, \alpha_n, L_n)\right), \end{aligned}$$

and for $X_I = X(i, \alpha, L)$ and $Y^I = Y(i, \alpha, L)$

$$X_I Y^I = \sum_i \sum_{\alpha=\pm} \int_0^\infty dL X(i, \alpha, L) Y(i, \alpha, L).$$

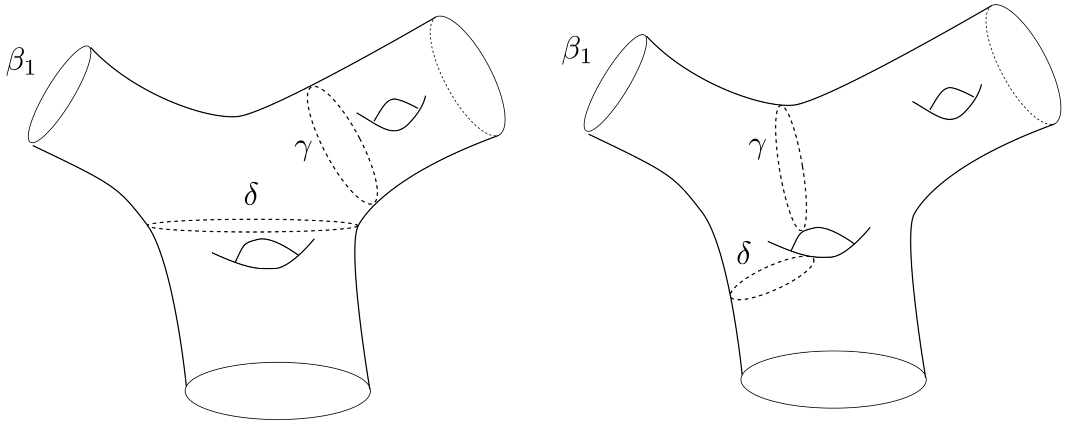

Fig. 6. Examples of $\{\gamma, \delta\}$ in $\mathscr{C}_1$.

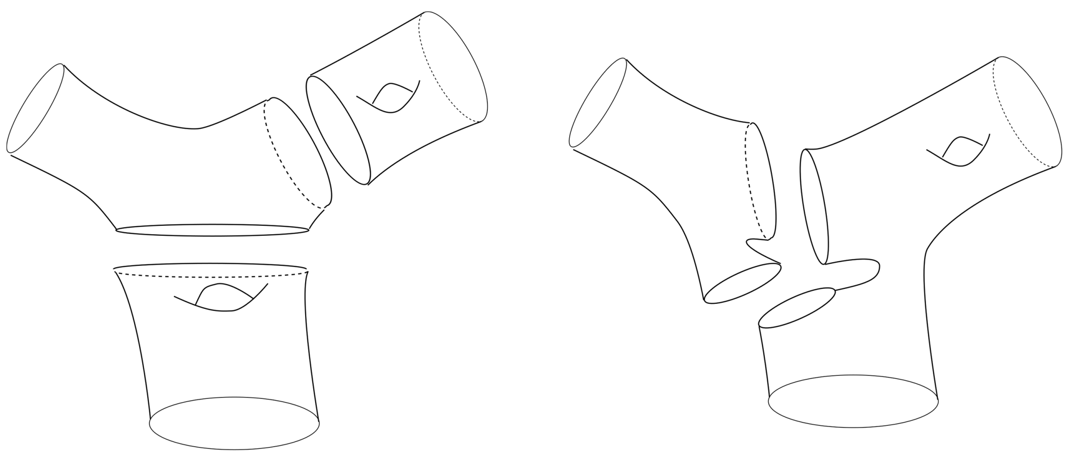

Fig. 7. Factorizations of the surfaces in Fig. 6.

The integral

$$\int_{\mathcal{M}_{g,n,\mathbf{L}}} \sum_{\{\gamma,\delta\}\in\mathscr{C}_1} \mathsf{D}_{L_1 l_\gamma l_\delta} \cdot (2\pi i)^{-3g+3-n} \langle \Sigma_{g,n,\mathbf{L}} | B_{6g-6+2n} | \Psi_1 \rangle \cdots | \Psi_n \rangle \tag{40}$$

on the right-hand side of Eq. (30) can be dealt with in the same way. In this case, there can be topologically distinct configurations of the pair $\{\gamma, \delta\}$ in $\mathscr{C}_1$ as depicted in Fig. 6. They belong to different mapping class group orbits. For each orbit, we take X_1 to be the space of the combination $(\Sigma_{g,n,\mathbf{L}}, \gamma, \delta)$, where (γ, δ) is in the orbit, and express the contribution to Eq. (40) by an integral over X_1. The amplitudes are factorized as in Fig. 7. A formula similar to Eq. (39) can be derived for each contribution.

Putting everything together, we can see that Eq. (30) is transformed into

$$\begin{aligned} L_1 A^{I_1\cdots I_n}_{g,n} = \frac{1}{2} D^{I_1 J' J} G_{JI} G_{J'I'} \left[A^{II' I_2 \cdots I_n}_{g-1,n+1} + \sum_{\text{stable}} \frac{\varepsilon_{\mathcal{I}_1\mathcal{I}_2}}{(n_1-1)!(n_2-1)!} A^{I\mathcal{I}_1}_{g_1,n_1} A^{I'\mathcal{I}_2}_{g_2,n_2} \right] \\ + \sum_{a=2}^{n} \varepsilon_a (T^{I_1 I_a J} + D^{I_1 I_a J}) G_{JI} A^{II_2\cdots \hat{I}_a \cdots I_n}_{g,n-1}, \end{aligned} \tag{41}$$

which holds for $2g-2+n>1$. Here $\mathcal{I}_1, \mathcal{I}_2$ are ordered sets of indices with n_1-1, n_2-1 elements respectively. The sum $\sum_{\text{stable}}$ means the sum over $g_1, g_2, n_1, n_2, \mathcal{I}_1, \mathcal{I}_2$ such that

$$\begin{aligned} g_1+g_2&=g,\\ n_1+n_2&=n+1,\\ \mathcal{I}_1\cup\mathcal{I}_2&=\{I_2,\ldots,I_n\},\\ \mathcal{I}_1\cap\mathcal{I}_2&=\phi,\\ 2g_1-2+n_1&>0,\\ 2g_2-2+n_2&>0. \end{aligned} \tag{42}$$

$\varepsilon_{\mathcal{I}_1\mathcal{I}_2}=\pm 1$ is the sign that will appear when we change the order of the product $II'I_2\cdots I_n$ to $I\mathcal{I}_1 I'\mathcal{I}_2$, if we regard the indices as Grassmann numbers with Grassmannality of the corresponding string state.

Equation (41) can be made more tractable by introducing $A^{I_1I_2}_{0,2}$. Since we define the amplitudes for surfaces with $2g-2+n>0$, amplitudes for $g=0, n=2$ do not exist. We here introduce a fictitious amplitude

$$A^{I_1I_2}_{0,2}=G^{I_1I_2},$$

where

$$G^{I_1I_2}\equiv\langle\varphi_{i_1}|\varphi_{i_2}\rangle\delta(L_1-L_2)\delta_{\alpha_1,-\alpha_2},$$

which satisfies

$$G_{I_1I_2}G^{I_2I_3}=\delta_{I_1}^{\ I_3}=\delta_{i_1,i_3}\delta_{\alpha_1,\alpha_3}\delta(L_1-L_3).$$

Taking this into account, we can turn Eq. (41) into

$$\begin{aligned} L_1A^{I_1\cdots I_n}_{g,n}&=L_1G^{I_1I_2}\delta_{g,0}\delta_{n,2}\\ &\quad+\frac{1}{2}D^{I_1J'J}G_{JI}G_{J'I'}\left[A^{II'I_2\cdots I_n}_{g-1,n+1}+\sum{}'\frac{\varepsilon_{\mathcal{I}_1\mathcal{I}_2}}{(n_1-1)!(n_2-1)!}A^{I\mathcal{I}_1}_{g_1,n_1}A^{I'\mathcal{I}_2}_{g_2,n_2}\right]\\ &\quad+\sum_{a=2}^{n}\varepsilon_aT^{I_1I_aJ}G_{JI}A^{II_2\cdots\hat{I}_a\cdots I_n}_{g,n-1}, \end{aligned} \tag{43}$$

which holds for $2g-2+n>0$ or $g=0$, $n=2$. Here the summation $\sum'$ is over $g_1, g_2, n_1, n_2, \mathcal{I}_1, \mathcal{I}_2$ such that

$$\begin{aligned} g_1+g_2&=g,\\ n_1+n_2&=n+1,\\ \mathcal{I}_1\cup\mathcal{I}_2&=\{I_2,\ldots,I_n\},\\ \mathcal{I}_1\cap\mathcal{I}_2&=\phi,\\ 2g_1-2+n_1&\geq 0,\\ 2g_2-2+n_2&\geq 0. \end{aligned}$$

Let us check if Eq. (43) is valid for $(g, n)=(0, 2), (0, 3), (1, 1)$. For $g=0$, $n=2$, Eq. (43) becomes[3]

$$L_1A^{I_1I_2}_{0,2}=L_1G^{I_1I_2}. \tag{44}$$

[3]Notice that $A^{I}_{0,1}$ does not exist.

The first term on the right-hand side of Eq. (43) is introduced so that $A_{0,2}^{I_1 I_2} = G^{I_1 I_2}$ holds. For $g = 0$, $n = 3$, we have

$$\begin{aligned} L_1 A_{0,3}^{I_1 I_2 I_3} &= \frac{1}{2} D^{I_1 J' J} G_{JI} G_{J'I'} \left[(-1)^{|I_2||I_3|} G^{II_2} G^{I'I_3} + (-1)^{|I_3|(|I_2|+|I_2|)} G^{II_3} G^{I'I_2} \right] \\ &\quad + T^{I_1 I_2 J} G_{JI} G^{II_3} + (-1)^{|I_2||I_3|} T^{I_1 I_3 J} G_{JI} G^{II_2} \\ &= \left(\frac{1}{2} \mathsf{D}_{L_1 L_2 L_3} + \frac{1}{2} \mathsf{D}_{L_1 L_3 L_2} + \mathsf{T}_{L_1 L_2 L_3} + \mathsf{T}_{L_1 L_3 L_2} \right) \\ &\qquad \times \langle \Sigma_{0,3,(L_1,L_2,L_3)} | B^1_{\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\varphi_{i_1}\rangle_1 |\varphi_{i_2}\rangle_2 |\varphi_{i_3}\rangle_3, \end{aligned}$$

where $|I|$ denotes the Grassmannality of $|\varphi_i\rangle$. Substituting Eq. (27) into this, we obtain

$$L_1 A_{0,3}^{I_1 I_2 I_3} = L_1 \langle \Sigma_{0,3,(L_1,L_2,L_3)} | B^1_{\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\varphi_{i_1}\rangle_1 |\varphi_{i_2}\rangle_2 |\varphi_{i_3}\rangle_3. \tag{45}$$

Notice that $\mathcal{M}_{0,3,\mathbf{L}}$ is a point and Eq. (36) implies

$$A_{0,3}^{I_1 I_2 I_3} = \langle \Sigma_{0,3,(L_1,L_2,L_3)} | B^1_{\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\varphi_{i_1}\rangle_1 |\varphi_{i_2}\rangle_2 |\varphi_{i_3}\rangle_3, \tag{46}$$

which is consistent with the above equation. For $g = 1$, $n = 1$, Eq. (43) becomes

$$\begin{aligned} L_1 A_{1,1}^{I} &= \frac{1}{2} D^{IJ'J} G_{J'J} \\ &= -\frac{1}{2} \int dl_\gamma \, \mathsf{D}_{L l_\gamma l_\gamma} \sum_j \langle \Sigma_{0,3,(L,L_2,L_3)} | B^1_\alpha (b(\partial_{L_2}) + b(\partial_{L_3})) b_0^{-(2)} P^{(2)} |\varphi_i\rangle_1 |\varphi_j\rangle_2 |\varphi_j^c\rangle_3 \Big|_{L_2=L_3=l_\gamma} . \end{aligned} \tag{47}$$

On the other hand, $A_{1,1}^I$ can be given as

$$\begin{aligned} A_{1,1}^{I} &= \frac{1}{2} \int_{\mathcal{M}_{1,1,L}} \left\langle \Sigma_{1,1,L} | B_\alpha B_2 | \varphi_i^\alpha \right\rangle (2\pi i)^{-1} \\ &= -\frac{1}{2} \int_{\mathcal{M}_{1,1,L}} \sum_j \Big\langle \Sigma_{0,3,(L,L_2,L_3)} | B^1_\alpha (b(\partial_{L_2}) + b(\partial_{L_3})) b_0^{-(2)} \frac{1}{2\pi} \\ &\qquad \times e^{i\theta_\gamma (L_0^{(2)} - \bar{L}_0^{(29)})} |\varphi_i\rangle_1 |\varphi_j\rangle_2 |\varphi_j^c \Big\rangle_3 \Big|_{L_2=L_3=l_\gamma} dl_\gamma \wedge d\theta_\gamma . \end{aligned}$$

The integral on the last line can be unfolded by using the McShane identity and we get Eq. (47) exactly.

3.3. *The solution of the recursion relation*

The recursion relation (41) is derived from the properties of the off-shell amplitudes $A_{g,n}^{I_1 \cdots I_n}$. Conversely, $A_{g,n}^{I_1 \cdots I_n}$ can be derived by solving Eq. (41).

$A_{g,n}^{I_1 \cdots I_n}$ is the order-g_s^{2g-2+n} contribution to the n-point amplitude. Equation (43) can be solved order by order in g_s, because the right-hand side of Eq. (43) consists of lower-order terms compared with the $A_{g,n}^{I_1 \cdots I_n}$ on the left-hand side. For example, the equation for $g = 0$, $n = 3$ becomes Eq. (45) and the solution is Eq. (46) because $A_{0,3}^{I_1 I_2 I_3}$ is defined for $L_1 > 0$. Equation (43) can be solved in the same way for general g, n. The solution is unique, because $A_{g,n}^{I_1 \cdots I_n}$ is defined for $L_1 > 0$. This unique solution should coincide with the $A_{g,n}^{I_1 \cdots I_n}$ in Eq. (36). Therefore Eq. (43) can be used to derive the off-shell amplitudes of closed bosonic string theory.

For later convenience, let us define the generating functional of the off-shell amplitudes:

$$W_A[J] \equiv \sum_{g=0}^{\infty} \sum_{n=2}^{\infty} g_s^{2g-2+n} \frac{1}{n!} J_{I_n} \cdots J_{I_1} A_{g,n}^{I_1 \cdots I_n}. \tag{48}$$

J_I is taken to have the same Grassmannality as that of ϕ^I. It is straightforward to show that the recursion relation (43) is equivalent to the following identity:

$$L\frac{\delta W_A[J]}{\delta J_I} = LJ_{I'}G^{I'I} + \frac{1}{2}g_s D^{II'I''}G_{I''K''}G_{I'K'}\left[\frac{\delta^2 W_A[J]}{\delta J_{K''}\delta J_{K'}} + \frac{\delta W_A[J]}{\delta J_{K''}}\frac{\delta W_A[J]}{\delta J_{K'}}\right]$$
$$+ g_s T^{II'I''}G_{I''K''}J_{I'}\frac{\delta W_A[J]}{\delta J_{K''}}(-1)^{|I||I'|}. \tag{49}$$

Here all the functional derivatives are left derivatives.

4. The Fokker–Planck formalism

In this section, we would like to develop the Fokker–Planck formalism for the string theory from which we can derive the recursion relation (43) through the Schwinger–Dyson equation.

4.1. *The Fokker–Planck formalism for conventional field theory*

Let $\phi(x)$ be a scalar field with action S[ϕ]. The Euclidean correlation functions are defined by

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \int [d\phi]P[\phi]\phi(x_1)\cdots\phi(x_n), \tag{50}$$

where

$$P[\phi] = \frac{e^{-S[\phi]}}{\int[d\phi]e^{-S[\phi]}}. \tag{51}$$

In order to describe this quantum field theory, we consider a system governed by the following Fokker–Planck equation:

$$-\frac{\partial}{\partial\tau}P[\phi,\tau] = H_{\mathrm{FP}}P[\phi,\tau]. \tag{52}$$

Here H_{FP} is the Fokker–Planck Hamiltonian defined by

$$H_{\mathrm{FP}} = -\int dx\frac{\delta}{\delta\phi(x)}\left(\frac{\delta}{\delta\phi(x)} + \frac{\delta S[\phi]}{\delta\phi(x)}\right). \tag{53}$$

It is possible to show that, for a solution of Eq. (52) with an appropriate initial condition,

$$\lim_{\tau\to\infty} P[\phi,\tau] = P[\phi]$$

holds. The Fokker–Planck equation with the Fokker–Planck Hamiltonian (53) appears in the context of stochastic quantization [28] where τ coincides with the fictitious time.

The Fokker–Planck Hamiltonian can be realized as an operator acting on a Hilbert space. Let $\hat{\pi}(x)$, $\hat{\phi}(x)$ be operators satisfying the commutation relations

$$[\hat{\pi}(x),\hat{\phi}(y)] = \delta(x-y),$$
$$[\hat{\pi}(x),\hat{\pi}(y)] = [\hat{\phi}(x),\hat{\phi}(y)] = 0,$$

and $|0\rangle$, $\langle 0|$ be states satisfying

$$\hat{\pi}(x)|0\rangle = \langle 0|\hat{\phi}(x) = 0,$$
$$\langle 0|0\rangle = 1.$$

Then

$$P[\phi,\tau] = \langle 0|e^{-\tau\hat{H}_{\mathrm{FP}}}\prod_x \delta(\hat{\phi}(x)-\phi(x))|0\rangle,$$

with

$$\hat{H}_{\mathrm{FP}} = -\int dx\left(\hat{\pi}(x) - \frac{\delta S}{\delta\phi(x)}[\hat{\phi}]\right)\hat{\pi}(x),$$

gives a solution to Eq. (52) with initial condition $P[\phi, 0] = \prod_x \delta(\phi(x))$. Assuming that this is a good initial condition, we get

$$P[\phi] = \lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\rm FP}} \prod_x \delta(\hat{\phi}(x) - \phi(x))|0\rangle. \tag{54}$$

The correlation function in Eq. (50) is given by

$$\lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\rm FP}} \hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle.$$

In Ref. [14], a string field theory for the (2,3) minimal string theory using this kind of operator formalism was proposed. The string fields are labeled by the length l of the string and we define the operators $\hat{\pi}(l)$, $\hat{\phi}(l)$ accordingly. The Fokker–Planck Hamiltonian is given by

$$\begin{aligned}\hat{H}_{\rm FP} = 2\int_0^\infty dl_1 \int_0^\infty dl_2 \hat{\phi}(l_1) w(l_2)\hat{\pi}(l_1+l_2)(l_1+l_2) \\ + \int_0^\infty dl_1 \int_0^\infty dl_2 w(l_1+l_2)\hat{\pi}(l_1) l_1 \hat{\pi}((l_2) l_2 \\ + g_{\rm s}\int_0^\infty dl_1 \int_0^\infty dl_2 \hat{\phi}(l_1)\hat{\phi}(l_2)\hat{\pi}(l_1+l_2)(l_1+l_2) \\ + g_{\rm s}\int_0^\infty dl_1 \int_0^\infty dl_2 \hat{\phi}(l_1+l_2)\hat{\pi}(l_1) l_1 \hat{\pi}((l_2) l_2,\end{aligned}$$

where $w(l)$ is the disk amplitude for the (2,3) minimal string theory[4]. The correlation functions of the string fields are given by

$$\lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\rm FP}} \hat{\phi}(l_1)\cdots\hat{\phi}(l_n)|0\rangle. \tag{55}$$

One can prove that the correlation functions thus defined coincide with the loop amplitudes of the (2,3) minimal string theory in the following way. In order for the limit (55) to exist,

$$\lim_{\tau\to\infty} \partial_\tau \langle 0| e^{-\tau \hat{H}_{\rm FP}} \hat{\phi}(l_1)\cdots\hat{\phi}(l_n)|0\rangle = -\lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\rm FP}} \hat{H}_{\rm FP}\hat{\phi}(l_1)\cdots\hat{\phi}(l_n)|0\rangle = 0 \tag{56}$$

should hold. Equation (56) yields the Schwinger–Dyson equation satisfied by the correlation functions of the minimal string theory. It can be shown that the loop equation of the minimal string theory is equivalent to this Schwinger–Dyson equation. Moreover, this string field theory can be derived from the stochastic quantization of the one-matrix model [15]. The Fokker–Planck formalism was applied to construct string field theories for general (p, q) minimal string theories in Refs. [29–31].

4.2. *The Fokker–Planck Hamiltonian for closed bosonic strings*

In Ref. [12], it was shown that Mirzakhani's recursion relation (28) is a special case of random matrix recursion relations. In Ref. [13], Eq. (28) is identified with a limit $p \to \infty$ of the loop equation of the (2,p) minimal string theory. Since (2,p) minimal string theory is a close cousin of the (2,3) one, it is possible to develop the Fokker–Planck formalism of string field theory corresponding to Eq. (28). The recursion relation in Eq. (41) is a (not so close) cousin of Eq.

[4]The correspondence between our notation and that in Ref. [14] is given by

$$\frac{1}{g_{\rm s}} w(l) + \hat{\phi}(l) \leftrightarrow \Psi^\dagger(l),$$

$$\hat{\pi}(l) \leftrightarrow \Psi(l).$$

(28), so it is conceivable that the same approach is applicable to this equation. In this subsection, we would like to show that this is the case.

We introduce operators $\hat{\phi}^I$, $\hat{\pi}_I$ that satisfy the commutation relations

$$[\hat{\pi}_I, \hat{\phi}^K] = \delta_I{}^K,$$
$$[\hat{\pi}_I, \hat{\pi}_K] = [\hat{\phi}^I, \hat{\phi}^K] = 0.$$

Here we define

$$[X^I, Y^K] \equiv X^I Y^K - (-1)^{|I||K|} Y^K X^I.$$

Let $|0\rangle\rangle$, $\langle\langle 0|$ be states that satisfy

$$\langle\langle 0|\hat{\phi}^I = \hat{\pi}_I|0\rangle\rangle = 0. \tag{57}$$

We define the correlation functions of ϕ^I as

$$\langle\langle \phi^{I_1} \cdots \phi^{I_n}\rangle\rangle \equiv \lim_{\tau\to\infty} \langle\langle 0|e^{-\tau\hat{H}}\hat{\phi}^{I_1} \cdots \hat{\phi}^{I_n}|0\rangle\rangle, \tag{58}$$

with the Hamiltonian

$$\begin{aligned}\hat{H} = &-L\hat{\pi}_I\hat{\pi}_{I'}G^{I'I} + L\hat{\phi}^I\hat{\pi}_I \\ &- \frac{1}{2}g_{\rm s}D^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'}\hat{\pi}_I \\ &- g_{\rm s}T^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}\hat{\pi}_I.\end{aligned} \tag{59}$$

As we will see, the right-hand side of Eq. (58) can be calculated perturbatively with respect to $g_{\rm s}$. We define the connected correlation functions $\langle\langle \phi^{I_1} \cdots \phi^{I_n}\rangle\rangle^{\rm c}$ in the usual way and they can be expanded as

$$\langle\langle \phi^{I_1} \cdots \phi^{I_n}\rangle\rangle^{\rm c} = \sum_{g=0}^{\infty} g_{\rm s}^{2g-2+n}\langle\langle \phi^{I_1} \cdots \phi^{I_n}\rangle\rangle^{\rm c}_g.$$

It is possible to show that

$$\langle\langle \phi^{I_1} \cdots \phi^{I_n}\rangle\rangle^{\rm c}_g = A^{I_1\cdots I_n}_{g,n} \tag{60}$$

holds.

In order to prove Eq. (60), we define the generating functional $W[J]$ of the connected correlation functions

$$W[J] = \sum_{n=2}^{\infty}\frac{1}{n!}J_{I_n}\cdots J_{I_1}\langle\langle \phi^{I_1} \cdots \phi^{I_n}\rangle\rangle^{\rm c}. \tag{61}$$

such that

$$e^{W[J]} = \lim_{\tau\to\infty}\langle\langle 0|e^{-\tau\hat{H}}e^{J_I\hat{\phi}^I}|0\rangle\rangle.$$

Since the limit $\tau\to\infty$ of $\langle\langle 0|e^{-\tau\hat{H}}\hat{\phi}^{I_1}\cdots\hat{\phi}^{I_n}|0\rangle\rangle$ exists, we have[5]

$$0 = \lim_{\tau\to\infty}\partial_\tau\langle\langle 0|e^{-\tau\hat{H}}e^{J_I\hat{\phi}^I}|0\rangle\rangle = -\lim_{\tau\to\infty}\langle\langle 0|e^{-\tau\hat{H}}\hat{H}e^{J_I\hat{\phi}^I}|0\rangle\rangle. \tag{62}$$

Using Eq. (57), we get the following equation from Eq. (62):

$$\begin{aligned}0 = J_I\Bigg\{&L\frac{\delta W[J]}{\delta J_I} - LJ_{I'}G^{I'I} - \frac{1}{2}g_{\rm s}D^{II'I''}G_{I''K''}G_{I'K'}\left[\frac{\delta^2 W[J]}{\delta J_{K''}\delta J_{K'}} + \frac{\delta W[J]}{\delta J_{K''}}\frac{\delta W[J]}{\delta J_{K'}}\right] \\ &- g_{\rm s}T^{II'I''}G_{I''K''}J_{I'}\frac{\delta W[J]}{\delta J_{K''}}(-1)^{|I||I'|}\Bigg\}.\end{aligned} \tag{63}$$

[5]Equation (62) can be proved perturbatively in $g_{\rm s}$.

It is possible to solve Eq. (63) order by order in g_s and obtain $\langle\langle\phi^{I_1}\cdots\phi^{I_n}\rangle\rangle_g^{\mathrm{c}}$. For example, at $\mathcal{O}(g_s^0)$, Eq. (63) implies

$$J_I J_{I'}(L+L')\left(\langle\langle\phi^{I'}\phi^I\rangle\rangle_0^{\mathrm{c}} - G^{I'I}\right) = 0.$$

Since $\langle\langle\phi^{I'}\phi^I\rangle\rangle_0^{\mathrm{c}}$ is defined for $L, L' > 0$, we obtain the unique solution

$$\langle\langle\phi^{I'}\phi^I\rangle\rangle_0^{\mathrm{c}} = G^{I'I}. \tag{64}$$

In general, Eq. (63) implies an equation in which $(L_1+\cdots+L_n)\langle\langle\phi^{I_1}\cdots\phi^{I_n}\rangle\rangle_g^{\mathrm{c}}$ is expressed in terms of lower-order correlation functions. Since $\langle\langle\phi^{I_1}\cdots\phi^{I_n}\rangle\rangle_g^{c}$ is defined for $L_1,\ldots, L_n > 0$, one can solve the equation and the solution is unique. Hence all the coefficients of the expansion (61) are uniquely fixed by Eq. (63). On the other hand,

$$W[J] = W_A[J] \tag{65}$$

yields a solution to Eq. (63) because $W_A[J]$ satisfies Eq. (49). Since the solution of Eq. (63) should be unique, we obtain Eq. (60).

Equation (65) implies that $W[J]$ satisfies Eq. (49), which can be expressed as

$$\lim_{\tau\to\infty}\langle\langle 0|e^{-\tau\hat{H}}\hat{\mathcal{T}}^I e^{J_I\hat{\phi}^I}|0\rangle\rangle = 0 \tag{66}$$

in the Fokker–Planck formalism. Here

$$\begin{aligned}\hat{\mathcal{T}}^I \equiv & -L\hat{\pi}_{I'}G^{II'} + L\hat{\phi}^I \\ & -\frac{1}{2}g_s D^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'} \\ & -g_s T^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'},\end{aligned} \tag{67}$$

and we have

$$\hat{H} = \hat{\mathcal{T}}^I\hat{\pi}_I.$$

Since every ket vector is expressed as a linear combination of states of the form

$$\langle\langle 0|\hat{\pi}^{I_1}\cdots\hat{\pi}^{I_n},$$

Eq. (58) means that $\lim_{\tau\to\infty}\langle\langle 0|e^{-\tau\hat{H}}$ is expressed as

$$\lim_{\tau\to\infty}\langle\langle 0|e^{-\tau\hat{H}} = \sum_{n=0}^{\infty}\frac{1}{n!}\langle\langle\phi^{I_1}\cdots\phi^{I_n}\rangle\rangle\langle\langle 0|\hat{\pi}_{I_n}\cdots\hat{\pi}_{I_1}.$$

In the same way, we can deduce from Eqs. (62), (66) that

$$\begin{aligned}\left[\lim_{\tau\to\infty}\langle\langle 0|e^{-\tau\hat{H}}\right]\hat{H} &= 0, \\ \left[\lim_{\tau\to\infty}\langle\langle 0|e^{-\tau\hat{H}}\right]\hat{\mathcal{T}}^I &= 0.\end{aligned} \tag{68}$$

4.3. *String field action $S[\phi^I]$*

In the case of conventional field theory, the Fokker–Planck formalism is an alternative to the path integral formalism. Let us discuss whether the theory that we have can be formulated using a path integral with action $S[\phi^I]$. It is possible to define the weight $P[\phi^I]$ following Eq. (54):

$$P[\phi^I] = \frac{e^{-S[\phi^I]}}{\int[d\phi^I]e^{-S[\phi^I]}} = \lim_{\tau\to\infty}\langle\langle 0|e^{-\tau\hat{H}}\prod_I\delta(\hat{\phi}^I - \phi^I)|0\rangle\rangle.$$

From Eq. (68), we obtain an equation for $S[\phi^I]$:

$$[LG^{IJ}+g_sT^{IJI'}G_{I'J'}\phi^{J'}]\frac{\delta S}{\delta\phi^J}$$
$$=L\phi^I-\frac{1}{2}g_sD^{II'I''}G_{I'J'}G_{I''J''}\phi^{J''}\phi^{J'}+g_sT^{II'I''}G_{I'I''}. \quad (69)$$

Using Eq. (27), the last term on the right-hand side of Eq. (69) is expressed as

$$T^{II'I''}G_{I'I''}=\frac{1}{2}L\int_0^\infty dl_\gamma\sum_j\langle\Sigma_{0,3,(L,L_2,L_3)}|B^1_\alpha(b(\partial_{L_2})+b(\partial_{L_3}))b_0^{-(2)}P^{(2)}|\varphi_i\rangle_1|\varphi_j\rangle_2|\varphi_j^c\rangle_3\Big|_{L_2=L_3=l_\gamma}$$
$$-\frac{1}{2}\int_0^\infty dl_\gamma\mathsf{D}_{Ll_\gamma l_\gamma}\sum_j\langle\Sigma_{0,3,(L,L_2,L_3)}|B^1_\alpha(b(\partial_{L_2})$$
$$+b(\partial_{L_3}))b_0^{-(2)}P^{(2)}|\varphi_i\rangle_1|\varphi_j\rangle_2|\varphi_j^c\rangle_3\Big|_{L_2=L_3=l_\gamma}.$$

The integrand of the first term on the right-hand side coincides with that of $A^I_{1,1}$, but the integration region includes infinitely many fundamental domains of the mapping class group. The second term is equal to $-LA^I_{1,1}$. Therefore the last term on the right-hand side of Eq. (69) may be given as

$$L\left[\infty\times A^I_{1,1}-A^I_{1,1}\right],$$

and is divergent. Hence Eq. (69) is not well defined.

Still, Eq. (69) can be solved formally order by order in g_s and we get

$$S[\phi^I]=\frac{1}{2}G_{IJ}\phi^I\phi^J-\frac{g_s}{6}A^{II'I''}_{0,3}G_{IJ}G_{I'J'}G_{I''J''}\phi^{J''}\phi^{J'}\phi^J+\frac{g_s}{L}T^{II'I''}G_{I'I''}G_{IJ}\phi^J+\mathcal{O}\left(g_s^2\right). \quad (70)$$

This should be the action that yields the off-shell amplitudes with the propagator G^{IJ}. Let us compute the one-loop one-point function using the path integral formalism. The contribution from the three-string vertex becomes

$$\frac{g_s}{2}\int dl_\gamma\sum_j\langle\Sigma_{0,3,(L,L_2,L_3)}|B^1_\alpha(b(\partial_{L_2})+b(\partial_{L_3}))b_0^{-(2)}P^{(2)}|\varphi_i\rangle_1|\varphi_j\rangle_2|\varphi_j^c\rangle_3\Big|_{L_2=L_3=l_\gamma},$$

and diverges. The contribution from the divergent term $\frac{g_s}{L}T^{II'I''}G_{I'I''}G_{IJ}\phi^J$ in the action cancels this divergence and we get the correct answer $g_sA^I_{1,1}$. This pattern seems to continue forever. If one computes the four-point amplitude using the three-string vertex in Eq. (70), one gets a divergent result. The four-string vertex cancels the divergence and makes the amplitude finite.

Therefore, the Fokker–Planck formalism is necessary for a well-defined formulation of the theory in our setup. On the other hand, the formally defined action (70) will be useful in studying various aspects of our formulation.

4.4. *SFT notation*

In order to discuss various properties of the theory, it is more convenient to express the Fokker–Planck Hamiltonian (59) in terms of the variables in the Hilbert space of strings. Let us define

$$|\phi^\alpha(L)\rangle\equiv\sum_i\hat{\phi}^I|\varphi_i^c\rangle, \quad (71)$$

$$|\pi_\alpha(L)\rangle\equiv\sum_i|\varphi_i\rangle\hat{\pi}_I. \quad (72)$$

The string fields $|\phi^{\alpha}(L)\rangle$, $|\pi_{\alpha}(L)\rangle$ are taken to satisfy

$$|\pi_{+}(L)\rangle, |\varphi^{-}(L)\rangle \in \mathcal{H}_{0},$$
$$|\pi_{-}(L)\rangle, |\varphi^{+}(L)\rangle \in \mathcal{H}_{0}^{c}, \tag{73}$$

where $\mathcal{H}_0^c$ consists of the states $|\Psi\rangle$ satisfying

$$c_0^-|\Psi\rangle = (L_0 - \bar{L}_0)|\Psi\rangle = 0,$$

where $c_0^{\pm} = c_0 \pm \bar{c}_0$. We also impose the reality condition [4,32]

$$|\phi^{+}(L)\rangle^{\dagger} = \langle\phi^{+}(L)|,$$
$$|\phi^{-}(L)\rangle^{\dagger} = -\langle\phi^{-}(L)|. \tag{74}$$

The reality condition for $|\pi_{\alpha}(L)\rangle$ will not be so simple, as is always the case in the Fokker–Planck formalism. Conditions (73) and (74) have been implicitly assumed in the previous subsection.

Notice that $|\phi^{\alpha}(L)\rangle$, $|\pi_{\alpha}(L)\rangle$ are Grassmann even. They satisfy the canonical commutation relation

$$[|\pi_{\alpha}(L)\rangle_1, |\phi^{\alpha'}(L')\rangle_2] = \delta_{\alpha}^{\alpha'}\delta(L-L')P_{\alpha}^{(1)}|R_{12}\rangle, \tag{75}$$

where

$$|R_{12}\rangle = \sum_i |\varphi_i\rangle_1|\varphi_i^c\rangle_2 = \sum_i |\varphi_i\rangle_2|\varphi_i^c\rangle_1 = |R_{21}\rangle = |R\rangle$$

is the reflector and

$$P_{\alpha} = \begin{cases} \frac{1}{2}b_0^-c_0^-P & \alpha = + \\ \frac{1}{2}c_0^-b_0^-P & \alpha = - \end{cases}.$$

The states $|0\rangle\rangle$, $\langle\langle 0|$ satisfy

$$|\pi_{\alpha}(L)\rangle \left|0\right\rangle\rangle = \left\langle\langle 0\right| |\phi^{\alpha}(L)\rangle = 0.$$

In terms of these string fields, the Fokker–Planck Hamiltonian (59) is expressed as

$$\begin{aligned}\hat{H} = &\int_0^{\infty} dLL\,[\langle R|\phi^{\alpha}(L)\rangle|\pi_{\alpha}(L)\rangle - \langle R|\pi_{\alpha}(L)\rangle|\pi_{-\alpha}(L)\rangle]\\ &- g_s\int dL_1dL_2dL_3\langle T_{L_2L_3L_1}|B^1_{-\alpha_1}B^2_{\alpha_2}B^3_{\alpha_3}|\phi^{\alpha_1}(L_1)\rangle_1|\pi_{\alpha_2}(L_2)\rangle_2|\pi_{\alpha_3}(L_3)\rangle_3\\ &- \frac{1}{2}g_s\int dL_1dL_2dL_3\langle D_{L_3L_1L_2}|B^1_{-\alpha_1}B^2_{-\alpha_2}B^3_{\alpha_3}|\phi^{\alpha_1}(L_1)\rangle_1|\phi^{\alpha_2}(L_2)\rangle_2|\pi_{\alpha_3}(L_3)\rangle_3,\end{aligned} \tag{76}$$

where

$$\langle T_{L_2L_3L_1}| \equiv \mathsf{T}_{L_2L_3L_1}\langle\Sigma_{0,3,(L_1,L_2,L_3)}|,$$
$$\langle D_{L_3L_1L_2}| \equiv \mathsf{D}_{L_3L_1L_2}\langle\Sigma_{0,3,(L_1,L_2,L_3)}|,$$

and the sum over repeated indices $\alpha_1, \alpha_2, \alpha_3$ is understood. ϕ^I and π_I are given by

$$\hat{\phi}^I = \langle\varphi_i|\phi^{\alpha}(L)\rangle,$$
$$\hat{\pi}_I = \langle\varphi_i^c|\pi_{\alpha}(L)\rangle,$$

and the correlation functions of $|\phi^{\alpha}(L)\rangle$ are expressed as

$$\langle\langle|\phi^{\alpha_1}(L_1)\rangle_1\cdots|\phi^{\alpha_n}(L_n)\rangle_n\rangle\rangle_g^{\mathrm{c}} = \int_{\mathcal{M}_{g,n,\mathbf{L}}} {}_{1'\cdots n'}\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}B^{1'}_{\alpha_1}\cdots B^{n'}_{\alpha_n}P^{(1)}_{-\alpha_1}|R_{1'1}\rangle\cdots P^{(n)}_{-\alpha_n}|R_{n'n}\rangle. \tag{77}$$

The correlation functions can be calculated perturbatively. The Euclidean action corresponding to the Fokker–Planck Hamiltonian (76) becomes

$$I = \int_0^\infty d\tau \left[-\int_0^\infty dL \langle R|\pi_\alpha(\tau, L)\rangle \frac{\partial}{\partial \tau} |\phi^\alpha(\tau, L)\rangle + H(\tau) \right], \tag{78}$$

where

$$\begin{aligned} H(\tau) = &\int_0^\infty dLL \left[\langle R|\phi^\alpha(\tau, L)\rangle |\pi_\alpha(\tau, L)\rangle - \langle R|\pi_\alpha(\tau, L)\rangle |\pi_{-\alpha}(\tau, L)\rangle \right] \\ &- g_s \int dL_1 dL_2 dL_3 \langle T_{L_2L_3L_1}|B^1_{-\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\phi^{\alpha_1}(\tau, L_1)\rangle_1 |\pi_{\alpha_2}(\tau, L_2)\rangle_2 |\pi_{\alpha_3}(\tau, L_3)\rangle_3 \\ &- \frac{1}{2} g_s \int dL_1 dL_2 dL_3 \langle D_{L_3L_1L_2}|B^1_{-\alpha_1} B^2_{-\alpha_2} B^3_{\alpha_3} |\phi^{\alpha_1}(\tau, L_1)\rangle_1 |\phi^{\alpha_2}(\tau, L_2)\rangle_2 |\pi_{\alpha_3}(\tau, L_3)\rangle_3. \end{aligned}$$

$|\phi^\alpha(\tau, L)\rangle$, $|\pi_\alpha(\tau, L)\rangle$ satisfy the boundary conditions

$$\lim_{\tau\to\infty} |\phi^\alpha(\tau, L)\rangle = |\pi_\alpha(0, L)\rangle = 0.$$

The correlation functions are expressed as

$$\langle\langle |\phi^{\alpha_1}(L_1)\rangle \cdots |\phi^{\alpha_n}(L_n)\rangle \rangle\rangle = \frac{\int [d\pi d\phi] e^{-I} |\phi^{\alpha_1}(0, L_1)\rangle \cdots |\phi^{\alpha_n}(0, L_n)\rangle}{\int [d\pi d\phi] e^{-I}}, \tag{79}$$

using the path integral. To develop the perturbation theory, we decompose the action as $I = I_0 + g_s V$, and we get propagators by Wick's theorem:

$$\begin{aligned} &\overbrace{|\phi^\alpha(\tau, L)\rangle_1 |\phi^{\alpha'}}(\tau', L')\rangle_2 = e^{-|\tau-\tau'|L} \delta(L - L') \delta_{\alpha.-\alpha'} P^{(1)}_{-\alpha} |R_{12}\rangle, \\ &\overbrace{|\pi_\alpha(\tau, L)\rangle_1 |\pi_{\alpha'}}(\tau', L')\rangle_2 = 0, \\ &\overbrace{|\pi_\alpha(\tau, L)\rangle_1 |\phi^{\alpha'}}(\tau', L')\rangle_2 = e^{-(\tau-\tau')L} \theta(\tau - \tau') \delta(L - L') \delta^{\alpha'}_\alpha P^{(1)}_\alpha |R_{12}\rangle. \end{aligned}$$

With the propagators and the vertex, it is straightforward to compute Eq. (79). By construction, the results are given by the integral

$$\int_{\mathcal{M}_{g,n,\mathbf{L}}} {}_{1'\cdots n'}\langle \Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n} B^1_{\alpha_1} \cdots B^n_{\alpha_n} P^{(1)}_{-\alpha_1} |R_{1'1}\rangle \cdots P^{(n)}_{-\alpha_n} |R_{n'n}\rangle,$$

unfolded by Mirzakhani's method. The integrations can be done taking account of the contributions from the boundaries of the moduli space appropriately.

5. BRST invariant formulation

With the Fokker–Planck formalism developed in the previous section, one can express the off-shell amplitudes of the bosonic string theory. In order to describe the string theory, we need the BRST symmetry on the worldsheet to specify which states of strings are physical. Unfortunately, the Fokker–Planck Hamiltonian (76) and the action (78) are not invariant under the BRST symmetry, although the amplitudes are. We will modify the action (78) so that the BRST symmetry becomes manifest in our formalism.

5.1. *BRST transformation*

As is proved in Appendix B, $\langle\Sigma_{g,n,\mathbf{L}}|$ satisfies the BRST identity

$$\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}B^1_{\alpha_1}\cdots B^n_{\alpha_n}\sum_{a=1}^{n}Q^{(a)} = d\left(\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-7+2n}B^1_{\alpha_1}\cdots B^n_{\alpha_n}\right)$$
$$-\sum_{a=1}^{n}\delta_{\alpha_a,-}\partial_{L_a}\left(\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}B^1_{\alpha_1}\cdots b_0^{-(a)}P^{(a)}\cdots B^n_{\alpha_n}\right). \tag{80}$$

Integrating this over $\mathcal{M}_{g,n,\mathbf{L}}$, we obtain

$$\sum_{a=1}^{n}P^{(a)}_{-\alpha_a}Q^{(a)}\langle\langle|\phi^{\alpha_1}(L_1)\rangle_1\cdots|\phi^{\alpha_n}(L_n)\rangle_n\rangle\rangle^{\mathrm{c}}_g$$
$$=\sum_{a=1}^{n}\delta_{\alpha_a,-}\langle\langle|\phi^{\alpha_1}(L_1)\rangle_1\cdots b_0^{-(a)}P^{(a)}\partial_{L_a}|\phi^{+}(L_a)\rangle_a\cdots|\phi^{\alpha_n}(L_n)\rangle_n\rangle\rangle^{\mathrm{c}}_g. \tag{81}$$

Equation (81) implies that the correlation functions of $|\phi^\alpha(L)\rangle$ is invariant under

$$\delta_\epsilon|\phi^+(L)\rangle = \epsilon P_-Q|\phi^+(L)\rangle,$$
$$\delta_\epsilon|\phi^-(L)\rangle = \epsilon Q|\phi^-(L)\rangle - \epsilon b_0^- P\partial_L|\phi^+(L)\rangle, \tag{82}$$

with a Grassmann-odd parameter ϵ. This can be identified with the BRST transformation of $|\phi^\alpha(L)\rangle$. It is easily checked that the two-point function

$$\langle\langle|\phi^{\alpha_1}(L_1)\rangle_1|\phi^{\alpha_2}(L_2)\rangle_2\rangle\rangle^{\mathrm{c}}_0 = \delta(L_1-L_2)\delta_{\alpha_1,-\alpha_2}P^{(1)}_{-\alpha_1}|R_{12}\rangle$$

is also BRST invariant. The transformation of $|\pi_\alpha(L)\rangle$ is fixed by requiring that the commutation relation (75) is invariant and we obtain

$$\delta_\epsilon|\pi_+(L)\rangle = \epsilon Q|\pi_+(L)\rangle - \epsilon b_0^- P\partial_L|\pi_-(L)\rangle,$$
$$\delta_\epsilon|\pi_-(L)\rangle = \epsilon P_-Q|\pi_-(L)\rangle. \tag{83}$$

The generator of the transformation is given by

$$\hat{Q} = \int dL\left[\langle R|Q|\phi^+(L)\rangle|\pi_+(L)\rangle + \langle R|Q|\phi^-(L)\rangle|\pi_-(L)\rangle\right.$$
$$\left. - \langle R|b_0^- P\partial_L|\phi^+(L)\rangle|\pi_-(L)\rangle\right],$$

which satisfies

$$\hat{Q}|0\rangle\rangle = \langle\langle 0|\hat{Q} = 0,$$
$$\left[\lim_{\tau\to\infty}\langle\langle 0|e^{-\tau\hat{H}}\right]\hat{Q} = 0. \tag{84}$$

5.2. *The BRST variation of $\hat{H}$*

Although the correlation functions are invariant under Eq. (82), the Hamiltonian (76) is not. The BRST variations of the correlation functions that appear on the right-hand side of Eq. (76) yield total derivatives with respect to the length variables, but they come with the coefficients $\mathsf{T}_{LL'L''}$, $\mathsf{D}_{LL'L''}$ and do not vanish upon integration.

The Hamiltonian (76) can be expressed as

$$\hat{H} = \int_0^\infty dL\langle R|\mathcal{T}^\alpha(L)\rangle|\pi_\alpha(L)\rangle, \tag{85}$$

where

$$
\begin{aligned}
|\mathcal{T}^{\alpha}(L)\rangle_{3'} = & L|\phi^{\alpha}(L)\rangle_{3'} - L|\pi_{-\alpha}(L)\rangle_{3'} \\
& - g_{\mathrm{s}} \int dL_1 dL_2 \langle T_{L_2 L L_1}|B^1_{-\alpha_1} B^2_{\alpha_2} B^3_{\alpha} |\phi^{\alpha_1}(L_1)\rangle_1 |\pi_{\alpha_2}(L_2)\rangle_2 |R_{33'}\rangle \\
& - \frac{1}{2} g_{\mathrm{s}} \int dL_1 dL_2 \langle D_{L L_1 L_2}|B^1_{-\alpha_1} B^2_{-\alpha_2} B^3_{\alpha} |\phi^{\alpha_1}(L_1)\rangle_1 |\phi^{\alpha_2}(L_2)\rangle_2 |R_{33'}\rangle .
\end{aligned}
$$

$|\mathcal{T}^{\alpha}(L)\rangle$ is the SFT version of $\mathcal{T}^I$ and we have

$$
\left[\lim_{\tau\to\infty} \langle\langle 0| e^{-\tau \hat{H}} \right] |\mathcal{T}^{\alpha}(L)\rangle = 0. \tag{86}
$$

The BRST variation of $\hat{H}$ is given by

$$
[\hat{Q}, \hat{H}] = \int_0^\infty dL \left(\langle R|\mathcal{Q}^{\alpha}(L)\rangle |\pi_{\alpha}(L)\rangle + \langle R|\mathcal{T}^{\alpha}(L)\rangle [\hat{Q}, |\pi_{\alpha}(L)\rangle] \right),
$$

where

$$
|\mathcal{Q}^{\alpha}(L)\rangle \equiv [\hat{Q}, |\mathcal{T}^{\alpha}(L)\rangle].
$$

From Eqs. (86) and (84), we obtain

$$
\left[\lim_{\tau\to\infty} \langle\langle 0| e^{-\tau \hat{H}} \right] |\mathcal{Q}^{\alpha}(L)\rangle = 0. \tag{87}
$$

5.3. *BRST invariant formulation*

Although $[\hat{Q}, \hat{H}]$ does not vanish, Eqs. (86) and (87) imply that it consists of "null" quantities. Using this fact, we will make the theory manifestly invariant under the BRST transformation by introducing auxiliary fields.

We modify the Euclidean action (78) by adding terms involving auxiliary fields $|\lambda^{\mathcal{T}}_{\alpha}(\tau, L)\rangle$, $|\lambda^{\mathcal{Q}}_{\alpha}(\tau, L)\rangle$ as follows:

$$
\begin{aligned}
I_{\mathrm{BRST}} = & \int_0^\infty d\tau \left[- \int_0^\infty dL \langle R|\pi_{\alpha}(\tau, L)\rangle \frac{\partial}{\partial\tau} |\phi^{\alpha}(\tau, L)\rangle + H(\tau) \right. \\
& \left. + \int_0^\infty dL \left(\langle R|\mathcal{Q}^{\alpha}(\tau, L)\rangle |\lambda^{\mathcal{Q}}_{\alpha}(\tau, L)\rangle + \langle R|\mathcal{T}^{\alpha}(\tau, L)\rangle |\lambda^{\mathcal{T}}_{\alpha}(\tau, L)\rangle \right) \right].
\end{aligned} \tag{88}
$$

Here $|\mathcal{Q}^{\alpha}(\tau, L)\rangle (|\mathcal{T}^{\alpha}(\tau, L)\rangle)$ is equal to $|\mathcal{Q}^{\alpha}(L)\rangle (|\mathcal{T}^{\alpha}(L)\rangle)$ with $|\phi^{\alpha}(L)\rangle$, $|\pi_{\alpha}(L)\rangle$ replaced by classical fields $|\phi^{\alpha}(\tau, L)\rangle$, $|\pi_{\alpha}(\tau, L)\rangle$ respectively. $|\lambda^{\mathcal{T}}_{\alpha}(\tau, L)\rangle$ and $|\lambda^{\mathcal{Q}}_{\alpha}(\tau, L)\rangle$ are taken to satisfy the boundary conditions

$$
|\lambda^{\mathcal{T}}_{\alpha}(0, L)\rangle = |\lambda^{\mathcal{Q}}_{\alpha}(0, L)\rangle = 0.
$$

I_{BRST} is invariant under the BRST transformation

$$
\begin{aligned}
\delta_\epsilon |\phi^{+}(\tau, L)\rangle &= \epsilon P_{-} Q |\phi^{+}(\tau, L)\rangle, \\
\delta_\epsilon |\phi^{-}(\tau, L)\rangle &= \epsilon Q |\phi^{-}(\tau, L)\rangle - \epsilon b_0^{-} P \partial_L |\phi^{+}(\tau, L)\rangle, \\
\delta_\epsilon |\pi_{+}(\tau, L)\rangle &= \epsilon Q |\pi_{+}(\tau, L)\rangle - \epsilon b_0^{-} P \partial_L |\pi_{-}(\tau, L)\rangle, \\
\delta_\epsilon |\pi_{-}(\tau, L)\rangle &= \epsilon P_{-} Q |\pi_{-}(\tau, L)\rangle, \\
\delta_\epsilon |\lambda^{\mathcal{Q}}_{\alpha}(\tau, L)\rangle &= \epsilon \left[|\pi_{\alpha}(\tau, L)\rangle + |\lambda^{\mathcal{T}}_{\alpha}(\tau, L)\rangle \right], \\
\delta_\epsilon |\lambda^{\mathcal{T}}_{\alpha}(\tau, L)\rangle &= -\delta_\epsilon |\pi_{\alpha}(\tau, L)\rangle .
\end{aligned}
$$

The correlation functions are defined by

$$
\frac{\int [d\pi\, d\phi\, d\lambda^{\mathcal{Q}} d\lambda^{\mathcal{T}}] e^{-I_{\mathrm{BRST}}} |\phi^{\alpha_1}(0, L_1)\rangle \cdots |\phi^{\alpha_n}(0, L_n)\rangle}{\int [d\pi\, d\phi\, d\lambda^{\mathcal{Q}} d\lambda^{\mathcal{T}}] e^{-I_{\mathrm{BRST}}}}. \tag{89}
$$

We would like to show that the correlation functions in this BRST invariant theory coincide with those given in Eq. (79). The numerator of Eq. (89) is computed as

$$\int [d\pi d\phi d\lambda^{\mathcal{Q}} d\lambda^{\mathcal{T}}] e^{-I_{\mathrm{BRST}}} |\phi^{\alpha_1}(0, L_1)\rangle \cdots |\phi^{\alpha_n}(0, L_n)\rangle$$

$$= \int [d\lambda^{\mathcal{Q}} d\lambda^{\mathcal{T}}] \int [d\pi d\phi] e^{-I} \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\times \left[-\int_0^\infty d\tau \int_0^\infty dL \left(\langle R|\mathcal{Q}^\alpha(\tau, L)\rangle |\lambda^{\mathcal{Q}}_\alpha(\tau, L)\rangle + \langle R|\mathcal{T}^\alpha(\tau, L)\rangle |\lambda^{\mathcal{T}}_\alpha(\tau, L)\rangle \right) \right]^n$$

$$\times |\phi^{\alpha_1}(0, L_1)\rangle \cdots |\phi^{\alpha_n}(0, L_n)\rangle.$$

The $n \neq 0$ terms on the right-hand side vanish because of Eqs. (86) and (87), and this becomes

$$\left[\lim_{\tau\to\infty} \langle\langle 0| e^{-\tau \hat{H}} \right] |\phi^{\alpha_1}(0, L_1)\rangle \cdots |\phi^{\alpha_n}(0, L_n)\rangle |0\rangle\rangle \int [d\pi d\phi d\lambda^{\mathcal{Q}} d\lambda^{\mathcal{T}}] e^{-I}.$$

The denominator is evaluated in the same way and we obtain

$$\int [d\pi d\phi d\lambda^{\mathcal{Q}} d\lambda^{\mathcal{T}}] e^{-I_{\mathrm{BRST}}} = \int [d\pi d\phi d\lambda^{\mathcal{Q}} d\lambda^{\mathcal{T}}] e^{-I}.$$

Therefore the correlation function (89) coincides with Eq. (79).

Now the theory is invariant under the BRST symmetry, and we regard the BRST invariant quantities as physical. One type of BRST invariant observable is of the form

$$\langle \varphi | \phi^+(L) \rangle$$

for $|\varphi\rangle$ satisfying

$$Q|\varphi\rangle = 0.$$

As we mentioned in Sect. 2,

$$\lim_{L_a\to 0} \langle \varphi_1| \cdots \langle \varphi_n| \langle\langle |\phi^+(L_1)\rangle \cdots |\phi^+(L_n)\rangle\rangle\rangle$$

gives the on-shell amplitude if we take $|\varphi_a\rangle$ to be on-shell physical states.

Another type of BRST invariant observable would be of the form

$$\int_0^\infty dL \langle \varphi | \phi^-(L) \rangle, \tag{90}$$

with $Q|\varphi\rangle = 0$. The amplitude

$$\int_0^\infty dL_1 \cdots \int_0^\infty dL_n \langle \varphi_1| \cdots \langle \varphi_n| \langle\langle |\phi^-(L_1)\rangle \cdots |\phi^-(L_n)\rangle\rangle\rangle \tag{91}$$

for these observables is in the form of an integration over the moduli space of complex structures of Riemann surfaces with boundaries. Therefore it is natural to take $\langle\varphi|$ to be

$$\langle \varphi | = \langle B | (c_0 - \bar{c}_0),$$

where $\langle B|$ is the boundary state corresponding to some D-brane configuration. For example, taking the states $|\varphi_a\rangle$ to be a point-like string state with the appropriate ghost part, we obtain off-shell amplitudes of the kind studied in Refs. [22–24]. Such amplitudes involve external leg contributions coming from the integration region $L_a \sim 0$. Indeed, using Eq. (A5), the contribution of $\langle \varphi_a | \phi^-(L_a) \rangle$ for $L_a \sim 0$ can be approximated as

$$\int_0 dL_a b_0^- b(\partial_{L_a}) P |\varphi_a\rangle \sim \int_0 dL_a \frac{\pi^2}{L_a^2} b_0^+ b_0^- P e^{-(c+\frac{\pi^2}{L_a})(L_0+\bar{L}_0)} |\varphi_a\rangle$$

$$\sim \frac{b_0^+ b_0^-}{L_0 + \bar{L}_0} P e^{-c(L_0+\bar{L}_0)} |\varphi_a\rangle.$$

This type of observable is suitable for studying mass renormalization [33,34].

6. Discussions

In this paper, we have constructed a string field theory for closed bosonic strings based on the pants decomposition of hyperbolic surfaces. In such a setup, the Fokker–Planck formalism is indispensable, as discussed in Sect. 4.3. We have introduced auxiliary fields to make the theory manifestly BRST invariant. The action (88) that we obtain consists of kinetic terms and three-string vertices.

There are many interesting points that deserve further study. The most obvious one would be to construct an SFT for superstrings based on the same idea. It is straightforward to generalize our formalism to the Type 0 superstring case, using the supersymmetric version of Eq. (24) derived in Ref. [35]. We will present these results elsewhere.

The formulation that we get in Sect. 5 is invariant under the worldsheet BRST symmetry. We should clarify the meaning of this symmetry from the point of view of string fields. In the ordinary formulation of SFT, the worldsheet BRST symmetry is utilized to define the gauge or BRST transformation for string fields that is nonlinear with respect to these fields. In our case, the worldsheet BRST symmetry will not be related to the gauge or BRST symmetry of the string fields in the usual way, because the theory is not based on a triangulation of the moduli space. The similarity between the structure of our formalism and that of the covariantized light-cone SFT [36] may provide a clue to this problem.

Our formalism is based on the one constructed for minimal strings. In the minimal string case, the operator corresponding to $\hat{\mathcal{T}}^I$ in Eq. (67) becomes

$$\hat{T}(l) = -2\int_0^l dl' w(l')\hat{\phi}(l-l') - \int_0^\infty dl' w(l+l')\hat{\pi}(l')l' \\ - g_s \int_0^l dl' \hat{\phi}(l')\hat{\phi}(l-l') - g_s \int_0^\infty dl' \hat{\phi}(l+l')\hat{\pi}(l')l',$$

and satisfies

$$\left[\lim_{\tau\to\infty}\langle 0|e^{-\tau \hat{H}_{\mathrm{FP}}}\right] l\hat{T}(l) = 0. \tag{92}$$

Equation (92) is equivalent to the Virasoro constraints [37,38]. $\hat{T}(l)$ satisfies an algebra

$$[l_1\hat{T}(l_1), l_2\hat{T}(l_2)] = g_s l_1 l_2 (l_1 - l_2)\hat{T}(l_1 + l_2), \tag{93}$$

which serves as the integrability condition of Eq. (92). On Laplace transforming Eq. (93), we obtain the Virasoro algebra. We are not sure if $\hat{\mathcal{T}}^I$ satisfies a similar algebra. Exploring the algebra of $\hat{\mathcal{T}}^I$ will be crucial to understanding the structure of the theory. It may also be important to point out that one can take the Fokker–Planck Hamiltonian to be

$$\int_0^\infty dL f(L)\langle R|\mathcal{T}^\alpha(L)\rangle|\pi_\alpha(L)\rangle,$$

instead of Eq. (85). Here $f(L)$ is a function of L satisfying $f(L) \neq 0$ for $L > 0$. The recursion relation can be derived from this modified Hamiltonian. Such a Hamiltonian was constructed in Ref. [39] in the minimal string case.

The classical equation of motion for string fields can be derived in our formalism. It is possible to assign the target space ghost number g^{t} such that [4]

$$g^{\mathrm{t}}(\phi^I) = \begin{cases} 4 - n_{\varphi_i^c} & \alpha = + \\ 2 - n_{\varphi_i^c} & \alpha = - \end{cases},$$

$$g^{\mathrm{t}}(\pi_I) = \begin{cases} 2 - n_{\varphi_i} & \alpha = + \\ 4 - n_{\varphi_i} & \alpha = - \end{cases}.$$

The fields with $g^{\mathrm{t}} = 0$ can be considered as classical fields. Although the action $S[\phi^I]$ is not well defined, Eq. (69) implies that the equation

$$L\phi^I - \frac{1}{2} g_{\mathrm{s}} D^{II'I''} G_{I'J'} G_{I''J''} \phi^{J''} \phi^{J'} = 0$$

may be identified with the classical equation of motion for string fields. In the BRST invariant formulation in Sect. 5, this equation coincides with

$$|\mathcal{T}^\alpha(\tau, L)\rangle = 0, \tag{94}$$

under the conditions

$$\begin{aligned} |\pi_\alpha(\tau, L)\rangle &= 0, \\ \partial_\tau |\phi^\alpha(\tau, L)\rangle &= 0. \end{aligned} \tag{95}$$

For BRST invariance, we may also have to impose

$$|\mathcal{Q}^\alpha(\tau, L)\rangle = 0. \tag{96}$$

Indeed, Eqs. (94), (95), and (94) solve the equation of motion derived from the action (88), if the auxiliary fields vanish.

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Appendix A. Hyperbolic metric on the three-holed sphere

Let us consider a hyperbolic pair of pants whose boundaries are geodesics. The pair of pants is conformally equivalent to $\mathbb{C} - \bigcup_{k=1}^{3} D_k$ where D_1, D_2, D_3 are disks around $z = 0, 1, \infty$ respectively. We take the length of ∂D_k to be $L_k = 2\pi\lambda_k$ $(k = 1, 2, 3)$. Around ∂D_k, a local coordinate ρ_k is taken so that the metric becomes

$$ds^2 = \frac{\lambda_k^2}{|\rho_k| \sin^2(\lambda_k \log|\rho_k|)} |d\rho_k|^2 .$$

The boundary ∂D_k corresponds to

$$|\rho_k| = \exp\left[\frac{\pi}{\lambda_k}\left(\tilde{l}_k + \frac{1}{2}\right)\right],$$

where $\tilde{l}_k$ is an integer. ρ_k can be expressed as a function $\rho_k(z)$ of the complex coordinate z on $\mathbb{C}$. Although $\rho_k(z)$ have singularities in $\mathbb{C} - \bigcup_{k=1}^{3} D_k$, it is well defined around ∂D_k and the three-holed sphere corresponds to the region

$$|\rho_k(z)| > \exp\left[\frac{\pi}{\lambda_k}\left(\tilde{l}_k + \frac{1}{2}\right)\right].$$

The explicit forms of $\rho_k(z)$ are given by [20,21]

$$\rho_1(z) = e^{\frac{v(\lambda_1,\lambda_2,\lambda_3)}{\lambda_1}} z(1-z)^{-\frac{\lambda_2}{\lambda_1}} \left[\frac{{}_2F_1\left(\frac{1+i\lambda_1-i\lambda_2+i\lambda_3}{2}, \frac{1+i\lambda_1-i\lambda_2-i\lambda_3}{2}; 1+i\lambda_1; z\right)}{{}_2F_1\left(\frac{1-i\lambda_1+i\lambda_2-i\lambda_3}{2}, \frac{1-i\lambda_1+i\lambda_2+i\lambda_3}{2}; 1-i\lambda_1; z\right)}\right]^{\frac{1}{i\lambda_1}},$$

$$\rho_2(z) = e^{\frac{v(\lambda_2,\lambda_1,\lambda_3)}{\lambda_2}} (1-z)z^{-\frac{\lambda_1}{\lambda_2}} \left[\frac{{}_2F_1\left(\frac{1+i\lambda_2-i\lambda_1+i\lambda_3}{2}, \frac{1+i\lambda_2-i\lambda_1-i\lambda_3}{2}; 1+i\lambda_2; 1-z\right)}{{}_2F_1\left(\frac{1-i\lambda_2+i\lambda_1-i\lambda_3}{2}, \frac{1-i\lambda_2+i\lambda_1+i\lambda_3}{2}; 1-i\lambda_2; 1-z\right)}\right]^{\frac{1}{i\lambda_2}},$$

$$\rho_3(z) = e^{\frac{v(\lambda_3,\lambda_2,\lambda_1)}{\lambda_3}} \frac{1}{z}\left(1-\frac{1}{z}\right)^{-\frac{\lambda_2}{\lambda_3}} \left[\frac{{}_2F_1\left(\frac{1+i\lambda_3-i\lambda_2+i\lambda_1}{2}, \frac{1+i\lambda_3-i\lambda_2-i\lambda_1}{2}; 1+i\lambda_3; \frac{1}{z}\right)}{{}_2F_1\left(\frac{1-i\lambda_3+i\lambda_2-i\lambda_1}{2}, \frac{1-i\lambda_3+i\lambda_2+i\lambda_1}{2}; 1-i\lambda_3; \frac{1}{z}\right)}\right]^{\frac{1}{i\lambda_3}}, \quad \text{(A1)}$$

where

$$e^{2iv(\lambda_1,\lambda_2,\lambda_3)} = \frac{\Gamma(-i\lambda_1)^2}{\Gamma(i\lambda_1)^2}\frac{\gamma\left(\frac{1+i\lambda_1+i\lambda_2+i\lambda_3}{2}\right)\gamma\left(\frac{1+i\lambda_1-i\lambda_2+i\lambda_3}{2}\right)}{\gamma\left(\frac{1-i\lambda_1-i\lambda_2+i\lambda_3}{2}\right)\gamma\left(\frac{1-i\lambda_1+i\lambda_2+i\lambda_3}{2}\right)}, \quad \text{(A2)}$$

and

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}.$$

Notice that

$$\begin{aligned}
\rho_2(z) &= \rho_1(1-z)|_{\lambda_1\leftrightarrow\lambda_2}, \\
\rho_3(z) &= \rho_1\left(\frac{1}{z}\right)\bigg|_{\lambda_1\leftrightarrow\lambda_3}, \\
\rho_1(z)|_{\lambda_2\leftrightarrow\lambda_3} &= -\rho_1\left(\frac{z}{z-1}\right), \\
\rho_2(z)|_{\lambda_1\leftrightarrow\lambda_3} &= -\rho_2\left(\frac{1}{z}\right), \\
\rho_3(z)|_{\lambda_1\leftrightarrow\lambda_3} &= -\rho_3(1-z) \quad \text{(A3)}
\end{aligned}$$

hold. Attaching flat semi-infinite cylinders to the boundaries of the three-holed sphere, we get a surface conformally equivalent to a three-punctured sphere. The local coordinates on the cylinders are given by

$$W_k(z) = \exp\left[-\frac{\pi}{\lambda_k}\left(\tilde{l}_k + \frac{1}{2}\right)\right]\rho_k(z) \quad \text{(A4)}$$

up to a phase rotation and the metrics on the cylinders become

$$ds^2 = \lambda_k^2 \frac{|dW_k|^2}{|W_k|^2}.$$

Notice that $\mathrm{Im} z = 0$ gives geodesics connecting the boundary components and perpendicular to them. Therefore $W_k = \pm 1$ become the basepoints that are used to define the twist parameters of the Fenchel–Nielsen coordinates [40].

For studying various properties of the amplitudes, it is useful to examine the limits $L_k \to 0, \infty$ of the formula (A4). Since the behavior does not depend on k because of Eq. (A3), we consider the case $k = 1$. In the limit $L_1 = 2\pi\lambda_1 \to 0$, Eq. (A2) implies

$$v(\lambda_1, \lambda_2, \lambda_3) = n\pi + c\lambda_1 + \mathcal{O}(\lambda_1^2),$$

where

$$c = 2\gamma + \frac{1}{2}\left[\psi\left(\frac{1 + i\lambda_2 + i\lambda_3}{2}\right) + \psi\left(\frac{1 - i\lambda_2 - i\lambda_3}{2}\right) + \psi\left(\frac{1 - i\lambda_2 + i\lambda_3}{2}\right) + \psi\left(\frac{1 + i\lambda_2 - i\lambda_3}{2}\right)\right] \in \mathbb{R},$$

and $n \in \mathbb{Z}$. From Eqs. (A1) and (A4), we get

$$z \sim e^{-c+\frac{\pi}{\lambda_1}(\tilde{l}_1+\frac{1}{2}-n)} W_1.$$

$\tilde{l}_1$ should be taken [20] so that $\tilde{l}_1 - n = -1$ and we eventually obtain

$$z \sim e^{-c-\frac{\pi^2}{L_1}} W_1. \tag{A5}$$

The limit $L_1 = 2\pi\lambda_1 \to \infty$ can be obtained from the Fuchsian equation. We get

$$z \sim 4W_1 + \mathcal{O}\left(\frac{1}{\lambda_1}\right). \tag{A6}$$

Appendix B. BRST identity

In this appendix, we would like to prove

$$\begin{aligned}\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}\tilde{B}^1_{\alpha_1}\cdots\tilde{B}^n_{\alpha_n}\sum_{a=1}^{n}Q^{(a)} &= d\Bigg[\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-7+2n}\tilde{B}^1_{\alpha_1}\cdots\tilde{B}^n_{\alpha_n}\\ &\quad + \sum_{a=1}^{n}\langle\Sigma_{g,n,\mathbf{L}}|B_{6g-6+2n}\tilde{B}^1_{\alpha_1}\cdots\tilde{b}^a_{\alpha_a}\cdots\tilde{B}^n_{\alpha_n}\Bigg],\end{aligned} \tag{B1}$$

where

$$B_{6g-7+2n} = \sum_{t=1}^{3g-3+n}\left[\prod_{s\neq t}\left(b(\partial_{l_s})b(\partial_{\tau_s})\right)\bigwedge_{s\neq t}(dl_s \wedge d\tau_s)\right]\left[b(\partial_{l_t})dl_t + b(\partial_{\tau_t})d\tau_t\right]$$

$$\tilde{B}^a_{\alpha_a} = \begin{cases}1 & \alpha_a = + \\ b_S(\partial_{L_a})b(\partial_{\theta_a})e^{i\theta_a(L_0^{(a)}-\bar{L}_0^{(a)})}dL_a \wedge d\theta_a & \alpha_a = -\end{cases},$$

$$\tilde{b}^a_{\alpha_a} = \begin{cases}0 & \alpha_a = + \\ b_S(\partial_{L_a})e^{i\theta_a(L_0^{(a)}-\bar{L}_0^{(a)})}dL_a + b(\partial_{\theta_a})e^{i\theta_a(L_0^{(a)}-\bar{L}_0^{(a)})}d\theta_a & \alpha_a = -\end{cases}.$$

It is straightforward to derive Eqs. (18) and (80) from Eq. (B1).

Decomposing $\Sigma_{g,n,\mathbf{L}}$ into pairs of pants, $\langle\Sigma_{g,n,\mathbf{L}}|$ can be expressed in terms of $\langle\Sigma_{0,3,\mathbf{L}}|$. In order to prove Eq. (B1), we will study some of the properties of $\langle\Sigma_{0,3,\mathbf{L}}|$. $\langle\Sigma_{0,3,\mathbf{L}}|$ satisfying

$$\langle\Sigma_{0,3,\mathbf{L}}|\Psi_1\rangle|\Psi_2\rangle|\Psi_3\rangle = \left\langle W_1^{-1}\circ\mathcal{O}_{\Psi_1}(0)W_2^{-1}\circ\mathcal{O}_{\Psi_2}(0)W_3^{-1}\circ\mathcal{O}_{\Psi_3}(0)\right\rangle_{\mathbb{C}\cup\{\infty\}}.$$

Here we take the local coordinate on $\mathbb{C}\cup\{\infty\}$ to be the z in Appendix A and $W_k(z)$ is given in Eq. (A4). We introduce the twist angles θ_a by deforming $\langle\Sigma_{0,3,\mathbf{L}}|$ as

$$\langle\Sigma_{0,3,\mathbf{L}}| \to \langle\Sigma_{0,3,\mathbf{L}}|\prod_{a=1}^{3}e^{i\theta_a(L_0^{(a)}-\bar{L}_0^{(a)})},$$

so that we have

$$\langle\Sigma_{0,3,\mathbf{L}}|\prod_{a=1}^{3}e^{i\theta_a(L_0^{(a)}-\bar{L}_0^{(a)})}|\Psi_1\rangle|\Psi_2\rangle|\Psi_3\rangle=\left\langle f_1\circ\mathcal{O}_{\Psi_1}(0)f_2\circ\mathcal{O}_{\Psi_2}(0)f_3\circ\mathcal{O}_{\Psi_3}(0)\right\rangle_{\mathbb{C}\cup\{\infty\}}, \tag{B2}$$

where

$$f_a(w_a)=W_a^{-1}(e^{i\theta_a}w_a).$$

Equation (B2) implies

$$\langle\Sigma_{0,3,\mathbf{L}}|\prod_{a=1}^{3}e^{i\theta_a(L_0^{(a)}-\bar{L}_0^{(a)})}=\sum_{i_1,i_2,i_3}\left\langle f_1\circ\mathcal{O}_{\varphi_{i_1}}(0)f_2\circ\mathcal{O}_{\varphi_{i_2}}(0)f_3\circ\mathcal{O}_{\varphi_{i_3}}(0)\right\rangle_{\mathbb{C}\cup\{\infty\}}\langle\varphi_{i_3}^c|\langle\varphi_{i_2}^c|\langle\varphi_{i_1}^c|,$$

which can be regarded as the definition of the state $\langle\Sigma_{0,3,\mathbf{L}}|$.

Following the formula (5), we define

$$b(\partial_{L_a})=\sum_{a'=1}^{3}b^{(a')}(\partial_{L_a}),$$

$$T(\partial_{L_a})\equiv\sum_{a'=1}^{3}T^{(a')}(\partial_{L_a}),$$

with

$$b^{(a')}(\partial_{L_a})\equiv-\oint_0\frac{dw_{a'}}{2\pi i}\frac{\partial f_{a'}}{\partial L_a}\frac{\partial w_{a'}}{\partial z}b(w_{a'})-\oint_0\frac{d\bar{w}_{a'}}{2\pi i}\frac{\partial f_{a'}}{\partial L_a}\frac{\partial\bar{w}_{a'}}{\partial\bar{z}}\bar{b}(\bar{w}_{a'}),$$
$$T^{(a')}(\partial_{L_a})\equiv-\oint_0\frac{dw_{a'}}{2\pi i}\frac{\partial f_{a'}}{\partial L_a}\frac{\partial w_{a'}}{\partial z}T(w_{a'})-\oint_0\frac{d\bar{w}_{a'}}{2\pi i}\frac{\partial f_{a'}}{\partial L_a}\frac{\partial\bar{w}_{a'}}{\partial\bar{z}}\bar{T}(\bar{w}_{a'}).$$

Here $T(z)$, $\bar{T}(\bar{z})$ are the stress tensors of the worldsheet theory and we have

$$\left\{Q,b(\partial_{L_a})\right\}=T(\partial_{L_a}), \tag{B3}$$

with

$$Q\equiv\sum_{a'}Q^{(a')}.$$

It is possible to show that, for any state $|\Psi\rangle$,

$$\left[-\oint_0\frac{dw_{a'}}{2\pi i}\frac{\partial f_{a'}}{\partial L_a}\frac{\partial w_{a'}}{\partial z}T(w_{a'})-\oint_0\frac{d\bar{w}_{a'}}{2\pi i}\frac{\partial f_{a'}}{\partial L_a}\frac{\partial\bar{w}_{a'}}{\partial\bar{z}}\bar{T}(\bar{w}_{a'})\right]f_{a'}\circ\mathcal{O}_\Psi(0)$$
$$=-\partial_{L_a}[f_{a'}\circ\mathcal{O}_\Psi(0)]$$

holds. Hence we obtain

$$\begin{aligned}\langle\Sigma_{0,3,\mathbf{L}}|T(\partial_{L_a})\prod_{a=1}^{3}e^{i\theta_a(L_0^{(a)}-\bar{L}_0^{(a)})}&=-\sum_{i_1,i_2,i_3}\langle\Sigma_{0,3,\mathbf{L}}|T(\partial_{L_a})\prod_{a=1}^{3}e^{i\theta_a(L_0^{(a)}-\bar{L}_0^{(a)})}|\varphi_{i_1}\rangle|\varphi_{i_2}\rangle|\varphi_{i_3}\rangle\langle\varphi_{i_3}^c|\langle\varphi_{i_2}^c|\langle\varphi_{i_1}^c|\\&\quad-\sum_{i_1,i_2,i_3}\partial_{L_a}\left\langle f_1\circ\mathcal{O}_{\Psi_1}(0)f_2\circ\mathcal{O}_{\Psi_2}(0)f_3\circ\mathcal{O}_{\Psi_3}(0)\right\rangle_{\mathbb{C}\cup\{\infty\}}\\&\quad\times\langle\varphi_{i_3}^c|\langle\varphi_{i_2}^c|\langle\varphi_{i_1}^c|\\&=-\partial_{L_a}\left[\langle\Sigma_{0,3,\mathbf{L}}|\prod_{a=1}^{3}e^{i\theta_a\left(L_0^{(a)}-\bar{L}_0^{(a)}\right)}\right].\end{aligned} \tag{B4}$$

We can also define

$$b(\partial_{\theta_a}) = -i(b_0^{(a)} - \bar{b}_0^{(a)}),$$
$$T(\partial_{\theta_a}) = -i(L_0^{(a)} - \bar{L}_0^{(a)}),$$

and it is easy to prove

$$\left\{Q, b(\partial_{\theta_a})\right\} = T(\partial_{\theta_a}),$$
$$\langle\Sigma_{0,3,\mathbf{L}}|T(\partial_{\theta_a})\prod_{a=1}^{3} e^{i\theta_a(L_0^{(a)}-\bar{L}_0^{(a)})} = -\partial_{\theta_a}\left[\langle\Sigma_{0,3,\mathbf{L}}|\prod_{a=1}^{3} e^{i\theta_a(L_0^{(a)}-\bar{L}_0^{(a)})}T(\partial_{\theta_a})\right]. \tag{B5}$$

$b(\partial_{L_a})$, $T(\partial_{L_a})$, $b(\partial_{\theta_a})$, and $T(\partial_{\theta_a})$ satisfy the following commutation relations:

$$[T(\partial_{L_a}), b(\partial_{L_{a'}})] = \partial_{L_a} b(\partial_{L_{a'}}) - \partial_{L_{a'}} b(\partial_{L_a}),$$
$$[T(\partial_{L_a}), b(\partial_{\theta_{a'}})] = -\partial_{\theta_{a'}} b(\partial_{L_a}),$$
$$[b(\partial_{L_a}), T(\partial_{\theta_{a'}})] = -\partial_{\theta_{a'}} b(\partial_{L_a}),$$
$$[T(\partial_{\theta_a}), b(\partial_{\theta_{a'}})] = 0. \tag{B6}$$

Using Eqs. (B3), (B4), (B5), and (B6), it is straightforward to show

$$\langle\Sigma_{0,3,\mathbf{L}}|\tilde{B}^1_{\alpha_1}\tilde{B}^2_{\alpha_2}\tilde{B}^3_{\alpha_3}\sum_{a=1}^{3} Q^{(a)}$$
$$= d\left[\langle\Sigma_{0,3,\mathbf{L}}|\left(\tilde{b}^1_{\alpha_1}\tilde{B}^2_{\alpha_2}\tilde{B}^3_{\alpha_3} + \tilde{B}^1_{\alpha_1}\tilde{b}^2_{\alpha_2}\tilde{B}^3_{\alpha_3} + \tilde{B}^1_{\alpha_1}\tilde{B}^2_{\alpha_2}\tilde{b}^3_{\alpha_3}\right)\right], \tag{B7}$$

which is Eq. (B1) for $g = 0$, $n = 3$.

Other cases can be proved by using Eq. (B7). Let us consider the next simplest case $g = 1$, $n = 1$. The surface state $\langle\Sigma_{1,\,1,\,L}|$ can be expressed as

$$\langle\Sigma_{1,1,L}| = \sum_{i,j} {}_{123}\langle\Sigma_{0,3,(L,l_\gamma,l_\gamma)}|e^{i\theta_\gamma(L_0^{(2)}-\bar{L}_0^{(2)})}|\varphi_i\rangle_2|\varphi_j\rangle_3\langle\varphi_i^c|\varphi_j^c\rangle(-1)^{n_{\varphi_j}}$$
$$= {}_{123}\langle\Sigma_{0,3,(L,l_\gamma,l_\gamma)}|e^{i\theta_\gamma(L_0^{(2)}-\bar{L}_0^{(2)})}|R_{23}\rangle. \tag{B8}$$

Using this, we obtain

$$\langle\Sigma_{1,1,L}|B_2\tilde{B}_\alpha Q = {}_{123}\langle\Sigma_{0,3,(L,l_\gamma,l_\gamma)}|B_2\tilde{B}^1_\alpha Q^{(1)}e^{i\theta_\gamma(L_0^{(2)}-\bar{L}_0^{(2)})}|R_{23}\rangle$$
$$= {}_{123}\langle\Sigma_{0,3,(L,l_2,l_3)}|(b(\partial_{l_2}) + b(\partial_{l_3}))b(\partial_{\theta_\gamma})\tilde{B}^1_\alpha$$
$$\times \sum_{a=1}^{3} Q^{(a)}e^{i\theta_\gamma(L_0^{(2)}-\bar{L}_0^{(2)})}|R_{23}\rangle\Bigg|_{l_2=l_3=l_\gamma} dl_\gamma\wedge d\theta_\gamma. \tag{B9}$$

In going from the first line to the second line, we have used

$$(Q^{(2)} + Q^{(3)})|R_{23}\rangle = 0. \tag{B10}$$

Using Eq. (B7), we eventually get

$$\begin{aligned}\langle\Sigma_{1,1,L}|B_2\tilde{B}_\alpha Q = &\ \partial_{l_2}\left[{}_{123}\langle\Sigma_{0,3,(L,l_2,l_3)}|b(\partial_{\theta_\gamma})\tilde{B}^1_\alpha e^{i\theta_\gamma(L_0^{(2)}-\bar{L}_0^{(2)})}|R_{23}\rangle\right]\Big|_{l_2=l_3=l_\gamma} dl_\gamma\wedge d\theta_\gamma \\ &+\ \partial_{l_3}\left[{}_{123}\langle\Sigma_{0,3,(L,l_2,l_3)}|b(\partial_{\theta_\gamma})\tilde{B}^1_\alpha e^{i\theta_\gamma(L_0^{(2)}-\bar{L}_0^{(2)})}|R_{23}\rangle\right]\Big|_{l_2=l_3=l_\gamma} dl_\gamma\wedge d\theta_\gamma \\ &+\ \partial_{\theta_\gamma}\left[{}_{123}\langle\Sigma_{0,3,(L,l_2,l_3)}|(b(\partial_{l_2})+b(\partial_{l_3}))\tilde{B}^1_\alpha e^{i\theta_\gamma(L_0^{(2)}-\bar{L}_0^{(2)})}|R_{23}\rangle\right]\Big|_{l_2=l_3=l_\gamma} dl_\gamma\wedge d\theta_\gamma \\ &+\ {}_{123}\langle\Sigma_{0,3,(L,l_2,l_3)}|(b(\partial_{l_2})+b(\partial_{l_3}))b(\partial_{\theta_\gamma})\tilde{b}^1_\alpha e^{i\theta_\gamma(L_0^{(2)}-\bar{L}_0^{(2)})}|R_{23}\rangle\Big|_{l_2=l_3=l_\gamma} dl_\gamma\wedge d\theta_\gamma \\ =&\ d\left[\langle\Sigma_{1,1,L}|(B_1\tilde{B}_\alpha+B_2\tilde{b}_\alpha)\right]. \end{aligned} \tag{B11}$$

The proof for all the other cases goes in the same way. We use induction with respect to $2g-2+n$. So far we have shown Eq. (B1) for $2g-2+n=1$. Assuming that Eq. (B1) is true for $2g-2+n=K>0$, let us prove Eq. (B1) for $2g-2+n=K+1$. $\langle\Sigma_{g,n,\mathbf{L}}|$ can be expressed by $\langle\Sigma_{0,3,\mathbf{L}}|$ and surface states with $2g-2+n\le K$ by factorizing the surface as in Figs. 5 or 7. Using the induction hypothesis, we obtain Eq. (B1) for $\langle\Sigma_{g,n,\mathbf{L}}|$ in the same way as we did for $\langle\Sigma_{1,1,L}|$.

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