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RESEARCH ARTICLE

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The dynamics of quintessence, the quintessence of dynamics

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Abstract Quintessence theories for cosmic acceleration imbue dark energy with a non-trivial dynamics that offers hope in distinguishing the physical origin of this component. We review quintessence models with an emphasis on the dynamics and discuss classifications of the different physical behaviors. The pros and cons of various parameterizations are examined as well as the extension from scalar fields to other modifications of the Friedmann expansion equation. New results on the ability of cosmological data to distinguish among and between thawing and freezing fields are presented.

1 Introduction

Understanding the acceleration of the cosmic expansion is a landmark problem in physics, impacting gravitation, high energy and quantum physics, and astrophysics, and likely to revolutionize one or more of these fields. The direction in which to look for a solution is almost wholly unknown currently. Though there is no shortage of suggestions, most are far from a first principles explanation of how such physics arises.

Perhaps the simplest proposal—Einstein's cosmological constant Λ [25]—is correct, though even so we have as yet no understanding of why it would arise, with the magnitude needed to explain acceleration occurring near the present epoch. That puzzlement can be broken into two severe problems [12; 71; 91]: the fine tuning problem of how Λ appears with a magnitude (energy density or energy scale) so far from the natural (Planck) scale defined by fundamental constants, and the coincidence problem of why acceleration appears in our recent past, at a cosmic scale factor within 2 of the present value out of perhaps 10^{28} since inflation. The cosmological constant is addressed in far greater detail in the articles by [8; 72] in this special volume.

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To paraphrase Winston Churchill speaking about democracy, it may be that the cosmological constant is the worst form of accelerating physics, except for all those other forms that have been tried from time to time. Nevertheless, this article addresses those other forms, specifically dynamical physics that aims to ameliorate the coincidence, and/or fine tuning, problems. We concentrate on the dynamics, the time evolution of the cosmological expansion physics, (mostly) from a canonical scalar field, given the name quintessence. See [15] for a particle physics perspective.

Section 2 provides a brief historical perspective on the development of quintessence theories. Section 3 reviews key elements of the dynamics of quintessence and the physical origins of structure in the phase space, defining classes of models. Efficient representation of the dynamical behavior through parameterization or principal component analysis is discussed in Sect. 4, and we investigate in detail thawing models, those which approach cosmological constant behavior, in Sect. 7. In Sect. 5, we consider a selection of dynamical models beyond standard quintessence, and briefly mention the effects of expansion dynamics on growth of cosmic structure in Sect. 6. We conclude in Sect. 8.

2 Origins of quintessence

The role of a dynamical scalar field for recent acceleration of the cosmic expansion certainly owes a debt to the use of rolling scalar fields for early universe inflation. A scalar field, and more generally a negative equation of state, were implemented as a substitute for the cosmological constant in a flurry of activity in the 1980s. On the theoretical side [49] proposed a simple extension from the flat potential of the cosmological constant to a tilted, linear potential, that releases the field to roll when the expansion rate of the universe decreases sufficiently, what is now called a thawing field. In 1988, two nearly simultaneous papers by [76; 92] described in more detail cosmology in the presence of a quintessence field.

At the same time, considerable work on the phenomenology of energy density components with an arbitrary (including negative) pressure to density, or equation of state, ratio was being carried out. [90] discussed such generalized cosmology, and [50] then followed up on this with detailed investigation of a variety of cosmological probes of additional components with arbitrary equation of state. These included tests of the expansion dynamics through distance, age, volume, and abundance measurements. Particular attention was paid to light propagation in such a generalized cosmology, including possible inhomogeneities in the components [51] (some results occurred earlier in the unpublished thesis of [41]). General equations of state had been considered in a formal way for the growth of structure within linear perturbation theory by [43]. Implications of general equations of state for growth were presented in [29; 52].

Thus high energy physics theory and cosmology were all ready in the 1980s for data exploring the expansion and growth histories of the universe. It took another 10 years for observations [73; 77] to make the astonishing breakthrough that turned these speculations into a central subject of research into our understanding of gravitation, quantum physics, cosmology, and the fate of the universe.

3 The quintessence of dynamics

3.1 Scalar field basics

If we view the cosmological constant as a quantum zeropoint energy corresponding to the ground state of harmonic modes of a field filling space, we can picture this as an array of identical springs, motionless and each stretched to the same length. By contrast, a scalar field would be a dynamical version of this, with the springs oscillating in time and having different lengths at different points in space. That is, a scalar field is a very simple quantity, a magnitude at each point in space. One can literally picture it as a field: a field of grass where each stalk may have been mown to a different height (a vector field could then be a field of trampled grass, where each stalk has a length and a direction in which it lies).

For quintessence, we take a scalar field ϕ minimally coupled, i.e., feeling only gravity, passively through the spacetime curvature, and a self-interaction described by the scalar field potential $V(\phi)$. Moreover, we consider the kinetic contribution to the Lagrangian (the "bouncing of the springs") to be canonical, i.e., involving only a term linear in the kinetic energy of the field. (We briefly discuss relaxing these conditions in Sect. 5.) So the Lagrangian is about as simple as possible:

$$\mathscr{L}_\phi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{1}$$

Through the Noether prescription we define an energy-momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L})}{\delta g^{\mu\nu}}, \tag{2}$$

where $g_{\mu\nu}$ is the metric and g its determinant. Comparing the result for a homogeneous and isotropic spacetime to the perfect fluid form allows identification of the energy density and pressure:

$$\rho_\phi = \frac{1}{2}\dot\phi^2 + V(\phi) + \frac{1}{2}(\nabla\phi)^2 \tag{3}$$

$$p_\phi = \frac{1}{2}\dot\phi^2 - V(\phi) - \frac{1}{6}(\nabla\phi)^2. \tag{4}$$

Because late time acceleration requires a very light scalar field, with effective mass of order the Hubble parameter, the Compton wavelength of the field will be of order or larger than the Hubble scale and so the field is expected to be spatially smooth within the Hubble scale. Therefore we neglect the spatial gradient terms in the energy density and pressure. These quantities can be put into the usual Friedmann equations to solve for the expansion history of the scale factor vs. time, $a(t)$, from the Hubble parameter $H = \dot a/a$ and acceleration $\ddot a$.

Because both the energy density and pressure enter the equations, it is convenient to define an equation of state ratio,

$$w = p_\phi/\rho_\phi, \tag{5}$$

which is generally time varying. When we refer to dynamical fields, we generally mean time-varying w, i.e., $w \neq$ constant. (Although the energy density of constant

w models varies with time, this happens as well with matter or a frozen network of cosmic strings, say, and so does not capture the flavor of “dynamics”.)

The equation of motion for the scalar field is the Klein–Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} = -dV/d\phi, \tag{6}$$

and is interchangeable with the continuity equation. For example, multiplying through by $\dot{\phi}$ gives the sequence

$$[\dot{\phi}^2/2]^{\cdot} + 6H[\dot{\phi}^2/2] = -\dot{V} \tag{7}$$

$$\dot{\rho}_\phi - \dot{V} + 3H(\rho_\phi + p_\phi) = -\dot{V} \tag{8}$$

$$\frac{d\rho_\phi}{d\ln a} = -3(\rho_\phi + p_\phi) = -3\rho_\phi\,(1+w). \tag{9}$$

where we have turned Eqs. (3)–(4) around to use

$$V = (\rho_\phi - p_\phi)/2 = \rho_\phi(1-w)/2 \tag{10}$$

$$K \equiv \dot{\phi}^2/2 = (\rho_\phi + p_\phi)/2 = \rho_\phi(1+w)/2. \tag{11}$$

From the above equations we can formally go back and forth from the field description to the fluid description or equation of state. From Eqs. (3) to (4) we see that

$$w = \frac{K-V}{K+V}, \tag{12}$$

so for some specified theory we can calculate the equation of state and then the effects on the cosmological expansion. The other direction, starting from observations of the cosmological expansion, is slightly more complicated:

$$\rho_\phi(a) = \Omega_w\,\rho_c\,\exp\left\{3\int_a^1 d\ln a\,[1+w(a)]\right\} \tag{13}$$

$$\phi(a) = \int d\ln a H^{-1}\sqrt{\rho_\phi(a)\,[1+w(a)]} \tag{14}$$

$$V(a) = \rho_\phi(a)\,[1-w(a)]/2 \tag{15}$$

$$K(a) = \dot{\phi}^2/2 = \rho_\phi(a)\,[1+w(a)]/2. \tag{16}$$

Such reconstruction of the scalar field physics is made difficult by a number of issues: noisiness of measurements of the expansion, translation from the measured quantity to density or equation of state through one or two derivatives, and finite range of scale factor, or redshift $z = a^{-1} - 1$, coverage. In particular, from the last of the equations above we see that

$$\dot{\phi} = [\rho_\phi(1+w)]^{1/2} \lesssim HM_P\,(1+w)^{1/2}, \tag{17}$$

so for cases when $1+w \ll 1$ (as seems to be implied by observations), only a small region of the scalar field physics, $\Delta\phi \sim \dot\phi/H \ll M_P$, can be probed. All these issues together makes reconstruction problematic, and we do not consider it further. (For attempts to carry it through, see [82] and references therein.)

While we cannot reconstruct in detail the scalar field potential, we can derive considerable insight into the accelerating physics from study of its dynamics. We can guess from the spring picture at the beginning of this section that there will be at least three basic quantities we want to know: how much energy is there in the field, how springy is it, and how stretchy are the springs? The energy density ρ_ϕ is conveniently written in terms of the dimensionless density $\Omega_w = \rho_\phi/\rho_c$, where $\rho_c = 3H_0^2/(8\pi G)$ is the critical density. For a spatially flat universe, $\Omega_w = 1-\Omega_m$, where Ω_m is the dimensionless matter density. The analog of the springiness is how spacetime curvature reacts to the accelerating component; the passive gravitational mass is given by $\rho+3p$, with acceleration induced by a component possessing $p<-\rho/3$, or $w<-1/3$. So we can regard w as a measure of the springiness. As the universe expands, the springs react, changing their springiness, like stretching the coils of a spring. This time variation can be taken as $w' = dw/d\ln a = \dot w/H$. Thus we are primarily interested in Ω_w, w, w'. The last two quantities give a phase space for the dynamics which we will see is enlightening.

3.2 General dynamical behavior

Scalar fields can at any epoch have one of four behaviors. Their rolling can be fast, slow, more or less steady, or oscillatory.

Fast roll: Fast rollers have kinetic energy exceeding their potential energy, and so by Eq. (12) have $w>0$. These clearly do not act to accelerate the cosmic expansion, but a fast roll epoch ("kination") is a characteristic of tracker models, which follow attractor trajectories in their dynamics such that at certain epochs their equation of state is determined by the dominant energy density component of the universe. Because of the fast roll, the scalar field can rapidly decrease its energy density from an initial, early universe value near the "natural" Planck scale to a much smaller value that will make it suitable for the observed present energy density. Due to the attractor solution for the dynamics, for certain forms of the potential, there is a large variety of initial conditions—"basin of attraction"—that can deliver a reasonable present energy density, thus addressing the fine tuning problem of the cosmological constant. Of course the field must leave both the fast roll regime and the tracking regime if it is to cause acceleration and dominate the energy density, so the coincidence problem is not completely solved. In particular, tracking fields have difficulty reaching equations of state $w \lesssim -0.7$, in tension with observations, and so are no longer considered front runners for explaining the acceleration. For more on trackers (and the earlier "tracers"), see [26; 48; 86; 95].

Slow roll: When the kinetic energy is much smaller than the potential energy, the equation of state is strongly negative, $w \approx -1$. Of course this only leads to acceleration of the expansion if the dark energy also dominates the energy density. The field is nearly frozen, and the dark energy density is nearly constant (while matter and radiation are rapidly diluting due to the expansion), so it would eventually come to dominate the universe if nothing else changed. Note that because matter is not negligible, even today, a field we think of as slowly rolling, $w \approx -1$, may well not have a small value for V'/V (see, e.g., [59]), which is a conventional slow roll parameter for inflation (where the accelerating component is completely dominant). Quintessence models that always have the potential dominating over the kinetic term encounter the same fine tuning and coincidence problems as the cosmological constant, lacking the basin of attraction of tracker models. Thus generically, we want a combination of fast and slow roll behavior for a successful model.

Steady roll: Referring to the original quintessence model of [49] using a linear potential, this category is somewhat of a misnomer since the field does have fast and slow roll epochs over its entire history. However, the linear potential model does have a constant right hand side of the Klein–Gordon equation of motion, and for a long time the dynamics stays reasonably close to the line where the field acceleration $\ddot{\phi}$ (not the cosmic acceleration $\ddot{a}$) is zero (see Sect. 3.3 below). This model is not only the simplest generalization of the cosmological constant but is also interesting in its overall history. It starts generically from a frozen, cosmological constant-like state due to Hubble friction, then thaws and rolls down the potential. However, because the potential has no minimum, the field rolls into territory where the potential goes negative, which actually leads to a collapsing universe, rather than an accelerating expansion. These models therefore have a finite future history, with a "doomsday time" [39; 45].

Oscillation: Common potentials in renormalizable field theories include $V(\phi) \sim \phi^n$, which have a minimum for n even. While the field will have a conventional rolling stage, eventually it will reach the minimum and oscillate around it. If the period for oscillation is much smaller than the Hubble time (as is generally the case) then the effective equation of state becomes [87]

$$w = \frac{n-2}{n+2}. \tag{18}$$

For a quadratic potential, the field acts like nonrelativistic matter, and for a quartic potential it acts like radiation.

One intriguing example of such a field is the axion, or more generally pseudo-Nambu Goldstone bosons (PNGB). If we consider them during the regime when they are still rolling rather than oscillating, they can accelerate the expansion, though this acceleration will eventually fade away as the field evolves to its oscillatory, matter-like phase [28]. PNGB potentials are also radiatively stable against quantum corrections, unlike an ad hoc $V(\phi)$ that might be written down but then acquire a non-zero ground state (cosmological constant) and distortion of its shape. Thus the physics of such pseudoscalar fields offers some promise for a fundamental, high energy physics origin rather than merely a low energy effective potential. The PNGB potential looks like

$$V(\phi) = V_0\,[1+\cos(\phi/f)], \tag{19}$$

where f is a symmetry energy scale. Because the potential is nonmonotonic and the slope of the potential changes from concave to convex, a number of interesting effects can arise, such as mimicking super-negative equations of state $w < -1$ and nontrivial dynamics [18; 31; 40]. For a complex field, one has degrees of freedom in both the modulus and the phase, and researchers have considered making one act as dark energy and the other as dark matter (e.g., [66]), or one giving recent acceleration and one early universe inflation (e.g., [78]. Other elaborations include spintessence [9; 30].

3.3 Fundamental modes of dynamics

While in the previous subsection we considered the behavior of the scalar field dynamics at any one moment, considerably more insight comes from investigating the overall dynamical history given by the trajectory through phase space. In particular, we will be interested not only in the present characteristics, but the asymptotic past and future states.

By examining the physical impact of the three different terms in the Klein–Gordon equation (6) we can identify boundaries in the phase space corresponding to different physical conditions.

- **Phantom line:** This line separates physics obeying the null energy condition [32], $\rho + p \geq 0$ ($w \geq -1$), from physics violating it. Also, consider the friction term $3H\dot{\phi}$. From Eq. (11) one sees that where the sign of this term changes, i.e., $\dot{\phi} = 0$ as the field stops rolling in one direction (and possibly begins rolling in another), corresponds to

$$w = -1. \tag{20}$$

  Canonically the field has $w \geq -1$ but there are various mechanisms (see Sect. 5) for achieving $w < -1$, what is referred to as the phantom regime [10].
- **Null line:** Consider the forcing term of the potential slope. When the field rolls down the potential, $\dot{V} \leq 0$, this corresponds to

$$w' \geq -3(1-w^2), \tag{21}$$

  where we have used Eqs. (9)–(11) to convert the variables $\dot{V}$ and $\dot{\phi}$ to w, w'. If the field has a (noncanonical) negative kinetic energy so it rolls up the potential then the inequality flips but at the same time the sign of w changes so $w < -1$ (one can think of this as the energy density increasing with time, following Eq. 9). Thus the null line passes smoothly through the point $(w, w') = (-1, 0)$.
- **Coasting line:** Consider the acceleration term $\ddot{\phi}$. Generally, at late times, the field accelerates due to the potential forcing dominating over the friction, or decelerates if the friction dominates over the potential slope (note this should not be confused with the acceleration of the cosmic expansion, which holds in either case if w is sufficiently negative). Again from Eq. (11) the dividing line between these dynamics, where the field is freely coasting at constant velocity $\dot{\phi}$, is

$$w' = 3(1+w)^2, \tag{22}$$

  with w' greater (smaller) than this for field acceleration (deceleration).

These three boundaries give general physical divisions for the dynamical behavior of the field. The general equation relating the phase space variables can be derived by taking the derivative of Eq. (10) and using the continuity equation (9) to obtain

$$w' = -3(1-w^2) - (1-w)(1+w)^{1/2}\sqrt{\frac{3\Omega_w(a)}{8\pi}}\frac{M_P V_{,\phi}}{V}. \tag{23}$$

We can readily verify the null line corresponds to $V_{,\phi} = 0$ (and one can specialize to the coasting line with a little more effort). These conditions were defined in [11] and developed further in [59; 83]. The last reference in particular goes into more detail about the derivation and the effect of the ratios of different terms in the Klein–Gordon equation, as well as "slow roll" parameters of the potential.

Without the need for quantitative analysis of the ratios of Klein–Gordon terms, one can broadly understand the dynamics by examining the relative dominance of the driving vs. dragging terms, following [11]. If the Hubble friction dominates at early times, then the field will be pinned and act like a static cosmological constant. As the cosmic expansion reduces the Hubble parameter, eventually the potential slope induces the field to begin rolling: such models are said to be *thawing*, and their dynamics in phase space shows them "leaving Λ", moving to less negative w with positive w'. In particular, fields that thaw during the matter dominated epoch leave Λ along the track $w' = 3(1+w)$.[1] As the matter domination wanes, the trajectory will curve according to the driving force from the potential slope; since the potential (eventually) becomes less steep as it approaches the minimum, the field acceleration decreases and the curve is toward the coasting line, i.e., smaller w'. For broad classes of potentials the condition that dark energy not completely dominate the energy density of the universe by the present means that thawing fields are still accelerating along the potential and the dynamics has a lower bound roughly given by $w' > 1+w$ (for $\Omega_w < 0.8$ and $w < -0.8$). Thus the thawing region of phase space is defined by a dynamical history

$$1+w \lesssim w' \leq 3(1+w). \tag{24}$$

The alternative is that the potential forcing dominates over the Hubble drag at early times, i.e., the potential is sufficiently steep to overcome the friction from cosmic expansion. Such fields will look different from the cosmological constant at early times. Certain forms of potential possess special attractor properties, as discussed in the previous subsection, that during the matter dominated epoch cause the scalar field dynamics to have a constant equation of state determined by the background expansion. As the dark energy density becomes relatively more important, these fields will depart from their tracking behavior and roll according to the dynamics of their potential. As the field rolls toward the minimum, decelerating in its motion (lying below the coasting line), gradually approaching asymptotically a static cosmological constant state, it is said to be *freezing*. In its "approaching Λ", the field contributes an energy density $\rho_w \sim H^{2(1+w)}$, but [59] showed that any H^α model approaching $w = -1$ does so along the asymptotic trajectory $w' = 3w(1+w)$. Conversely, since dark energy dominates (though not fully) today, the field must have departed its matter dominated tracking behavior

[1] Fields whose initial conditions $\dot{\phi}_i$ are fine tuned can avoid this. Also, if the potential driving term is very large, for example in PNGB fields with symmetry energy scale $f \ll M_P$, then one can have $w' > 3(1+w)$.

Fig. 1 The dynamical phase space w-w' is divided by three curves defined by physical conditions: the phantom line $w = -1$, the null line $w' = -3(1-w^2)$ following from a flat potential, and the coasting line $w' = 3(1+w)^2$ following from constant field velocity. These extend across the phase space. In addition, canonical dynamics leads to the distinct regions of the thawing regime bounded by the red dotted lines and the freezing regime bounded between the *green dot-dashed curve* and *the blue dashed curve* (the latter given by the constant pressure condition)

and moved some distance away from the constant w line. For broad classes of potential this leads to a present value $w' \lesssim 0.2w(1+w)$ (for $\Omega_w > 0.6$ and $w < -0.8$). Thus the freezing region of phase space is defined by a dynamical history

$$0 \le w' \le 3w(1+w), \tag{25}$$

with the present value of w' more tightly restricted.

Figure 1 illustrates the three critical dividing lines of the phantom, null, and coasting curves in the dynamical phase space. In addition it shows the upper and lower boundaries of the thawing and freezing regions. Note that the lower boundary of the freezing region coincides with the constant pressure curve (with an adiabatic sound speed $c_a^2 = 0$) discussed in Sect. 5.

Comparing Eqs. (24) and (25), we see that they define narrow, distinct regions in the phase space where scalar field theories obeying a combination of theoretical and observational conditions lie. In particular, there are fairly strongly physically motivated outer boundaries defining the extremes of w'. The exact inner boundaries are more a function of empirical constraints on the present expansion, but there is a distinct intermediary zone unfavorable for habitation. This "desert" lies around the coasting line: only highly fine tuned models would, after the many e-folds of cosmic expansion influencing the scalar field equation of motion, find themselves almost perfectly balanced between field acceleration and deceleration, $\ddot{\phi} \approx 0$.

Two important implications of the physical division into distinct thawing and freezing regions are for the questions of observationally distinguishing dynamical dark energy from Λ and distinguishing the physical origin of the dark energy (e.g., field theories with thawing versus freezing characteristics). Because of the degeneracy
directions of essentially all cosmological probes (see the articles by [46; 69] in this volume), the entire thawing region is difficult to distinguish from the cosmological constant if the data is only at the sensitivity level of a constant, or time averaged, w. For example, the entire thawing region would give an apparent $\langle w\rangle \approx -1 \pm 0.05$. Thus experiments sensitive to w' are necessary for deciding between this half of the dynamical phase space and the cosmological constant. For distinguishing between the classes of effective field theories, one would like to have cosmological sensitivity to the time variation of $\sigma(w') \lesssim 2(1+w)$ to resolve the separation between the thawing and freezing regions. For in depth discussion of mapping the cosmic expansion history, see the review article by [61].

3.4 More complicated dynamics

In the previous subsection we gave physical motivations for bounded regions in phase space but we emphasize these are based on a combination of generic behavior and empirical data, not an absolute exclusion of other possible behaviors. In particular, they relied on a standard matter dominated epoch at high redshift, canonical scalar fields, avoidance of fine tuned initial conditions and potential shapes, and "fundamental modes" of dynamics. We discuss extension of the dynamics to beyond canonical scalar fields in Sect. 5; here we consider initial conditions and fundamental modes.

Initial conditions on the scalar field dynamics are quite important, e.g., one could consider a field so perfectly balanced on a maximum of its potential that it only starts rolling yesterday, or a field that has recently passed a minimum of its potential and is now rolling uphill, or a field with kinetic and potential energies exactly crafted so the dynamics is missing (constant equation of state) or is coasting. Physics does not forbid any of these a priori, but our sense of naturalness disfavors them. If dynamical conditions are set by hand at recent times, rather than the field settling into an evolution following its equation of motion over many e-folds in the early universe and then a matter dominated epoch, then virtually arbitrary behavior can result [37; 47]. One could fine tune the field such that one does not extract general physical precepts on the dynamics, but rather the phase space trajectories would spell out your name.

Under the physics of field evolution through the cosmic expansion history, including a matter dominated epoch, the phase space structure described in the previous subsection generically holds. One further necessary ingredient is that we are talking about fundamental modes, or "atoms", of the dark energy—the quintessence of dynamics. If one combines multiple elements together, such as a scalar field plus a cosmological constant, or plus matter, or plus another scalar field, then one can indeed break the physical boundaries (just as multifield inflation can break consistency relations and other basic predictions). That is, the phase space structure applies to the dynamics of a single, fundamental field, not an effective field of multiple origins.

We can investigate this further by examining the effect on the equation of state when multiple elements are combined. For the simplest approach, we consider adding together two components: a canonical scalar field plus either a cosmological constant, a matter component (e.g., misestimation of Ω_m or dark energy contribution to dark matter), or another scalar field.

The effect of combining two such noninteracting components is given by an effective dynamical equation of state

$$w_{\rm eff} = w_1 \frac{\delta H_1^2}{\delta H_1^2 + \delta H_2^2} + w_2 \frac{\delta H_2^2}{\delta H_1^2 + \delta H_2^2}, \tag{26}$$

where δH_i^2 is the contribution of component i to the Friedmann equation. This approach was used to first point out phantom crossing, evolution across $w = -1$,

Fig. 2 Dynamics involving combination of physics can violate the fundamental phase space regions. To the original thawing scalar field trajectory (*solid black*), we add a cosmological constant ($+\Lambda$), extraneous matter or quartessence component ($+m$), or freezing scalar field ($+V$). We fix $w_0 = -0.8$ for the fields and take the total dimensionless dark energy density to be 0.7. For the second component of Λ or V we take $\Omega_2 = 0.1$ (*darker, black*) or 0.35 (*lighter, red*); for included matter $\Omega_{+m} = 0.01$. Curve endpoints correspond to $z = 0$, with x's at $z = 1$

by two scalar fields [55] (also see [34]). The dynamics is affected as

$$w'_{\rm eff} = 3w_{\rm eff}(1+w_{\rm eff}) + \frac{\delta H_1^2}{\Sigma}[w'_1 - 3w_1(1+w_1)] + \frac{\delta H_2^2}{\Sigma}[w'_2 - 3w_2(1+w_2)], \tag{27}$$

where $\Sigma = \delta H_1^2 + \delta H_2^2$. Note that two constant pressure components (where $w'_i = 3w_i(1+w_i)$) add without affecting the dynamics. In particular, any combination of matter plus Λ keeps the same trajectory, just moving the position along the track.

Furthermore, this formula implies that the sum of components, each of which lies on the same side of the curve $w' = 3w(1+w)$, has effective dynamics doing likewise. For example, two kinetic k-essence components give an effective dynamics that is still kinetic k-essence-like. Similarly, the null condition $w' > -3(1-w^2)$ cannot be overcome by summing components obeying $w'_i > -3(1-w_i^2)$. Other than respecting these two boundaries, the dynamics can change significantly on combining components.

To an initial thawing scalar field we add either a cosmological constant component, a matter component, or a freezing field. Figure 2 shows that such combinations, as opposed to the fundamental modes or "atoms" we discussed in the previous subsection, do not adhere to the restricted thawing and freezing regions of the phase space. Convolutions of different physics can drastically differ from those fundamental behaviors.

Adding a freezing field to a thawing field dramatically alters the trajectory, since at early times the freezing field will dominate. (Adding extra components to a dominant freezer has less effect.) The phase space tracks therefore start off in the freezing regime but curve up toward the thawing regime, possibly lying today in the desert region between the two regimes. A cosmological constant rotates the dynamics toward $w' = 0$ and draws it in toward $w = -1$ (see also [11]); this does not generally move a thawing field out of the thawing region. Including a matter like component with the thawing field has the most severe effect. Adding a mere 0.01 in dimensionless matter-like energy density alters the track wildly – this points up strongly the dangers in attempted direct reconstruction of the dynamics from $H(z)$ or the distance-redshift relation. Misestimation of Ω_m by 0.01 will completely distort the true dark energy dynamics.

4 Describing the dynamics

The phase space dynamics discussed in the previous section presents the dark energy physics in terms of a function $w(a)$ and its derivative w', describing the "springiness" and "stretchiness" of the spacetime in reaction to the dark energy.

Each theoretical model presents its particular description of the function and we can check each against the data to determine whether the model fits. However, there are 10^x theoretical forms (potentials or equation of state functions) already postulated, each with their own parameters. Moreover, we would like to predict the results of experiments, or design experiments, more generally than for a given theory or set of existing theories.

This shows the need for a model independent approach, based on a parametrization of the equation of state function or a similar quantity. Because we want the parametrization to stay close to the underlying physics, of which both the dark energy density and pressure enter, we concentrate on the pressure to density ratio, or equation of state ratio. However parametrization of other quantities such as distances, Hubble parameter, or density alone have been considered (see, e.g., [82] and references therein); two cautions should be stated about this route: certain forms bias the extraction of the underlying physics, see e.g., [38; 60], and if one eventually wants the equation of state then one is forced to take numerical derivatives of a quantity extracted from noisy data.

Numerous parametrizations exist for the equation of state $w(a)$ but the vast majority are purely ad hoc. We here consider a very few that are phenomenological in the best sense, i.e., generalized from the behavior of physically motivated sets of models. From the previous section we have seen that a single parameter model, i.e., $w =$ constant, involves highly fine tuned physics to remove the dynamics. While one way out of this is to invoke a physical symmetry, such as a topological defect origin, which can produce $w = -N/3$ for a frozen network of N-dimensional defects (e.g., $N = 2$ domain walls [94] or $N = 1$ light cosmic strings [89]), such values are not consistent with data.

This leads us to two parameter models as the next simplest alternative. The parametrization

$$w(a) = w_0 + w_a(1-a), \tag{28}$$

where w_0 is the value today ($a = 1$) and w_a is a measure of the time variation w', is widely used in the literature. It is important to realize that it is in no way a mathematical expansion about the present: neither its important introduction by [13] nor the physical foundation work by [53] employed a Taylor expansion, nor would that be mathematically convergent. Therefore w_a is not an expansion parameter about $z = 0$, but rather a fit parameter describing the overall time variation w'. The original convention [53] giving the best description is

$$w_a \equiv (-w'/a)|_{z=1} = -2w'(z=1). \tag{29}$$

Linder [53; 54] give several physical supports for the w_a parametrization: (1) excellent approximation to the exact field equations for a broad range of fundamental or straightforward scalar field potentials, (2) well behaved at both low and high redshift, (3) robust against bias, e.g., if one extends the form to further parameters, the w_0, w_a parameter values estimated are not strongly affected, (4) model independence. For example, a SUGRA inspired model that evolves from $w(a \ll 1) \approx -0.2$ to $w_0 = -0.82$—a substantial variation—has its equation of state reproduced to within 3% back to $z = 1.7$ and the distance-redshift relation in such a cosmology is accurately matched to 0.2% back to CMB last scattering by $w_0 = -0.82$, $w_a = 0.58$.

Of course a two parameter description cannot describe all possible dynamics; in particular it begins to break down for rapid transitions in the equation of state or oscillations. However, for the fundamental modes highlighted in the previous section it serves as an excellent, broad (i.e., model independent, good for both thawing and freezing) parametrization of the physically favored dynamics.

Another two parameter form, which is motivated from the energy density rather than the equation of state, is the bending parametrization of [93]. This was designed to describe early dark energy models where at high redshift (near the CMB last scattering surface, $z \approx 10^3$) the scalar field component has nonnegligible energy density (though it is then acting in a decelerating, rather than accelerating, manner on the expansion, so it is not exactly dark energy). The bending form has

$$\ln\frac{\Omega_w(a)}{\Omega_m(a)} \equiv R_0 - \frac{3w_0 \ln a}{1-b\ln a} \tag{30}$$

$$w(a) = \frac{w_0}{(1-b\ln a)^2}, \tag{31}$$

where $R_0 = \ln(\Omega_m^{-1}-1)$ and b is related to the early dark energy density. The dynamics of this parametrization is that in the past it approaches $w=0$, $w'=0$ (i.e., a finite dark energy density that acts like matter), at some future time $a_* = e^{1/b}$ it runs to $w=-\infty$, $w'=-\infty$, and then returns along the same trajectory to $w=0$, $w'=0$ in the further future. The phase space track is defined by $w' = 2bw_0\,(w/w_0)^{3/2}$. At any given time in the past the variation must be slower than $w' = -(8/27)w_0/\ln a$.

A generalization of the w_a form to three parameters was put forward by [75]. This eases the property of the w_a form where the parameter w_a plays two roles: it describes the characteristic time variation w' but it also determines the asymptotic past value of $w(a \ll 1) \to w_0 + w_a$. The extended form has

$$w(a) = \frac{w_p z + w_0 z_t}{z + z_t}, \tag{32}$$

where w_p is the asymptotic past value and z_t is the transition redshift. When $z_t = 1$, this reduces to the w_a parametrization. The phase space dynamics is a parabola from $(w,w') = (w_p,0)$ to $(w_f,0)$, crossing $w=-1$ if $w_0 < w_p$.

To describe a monotonic $w(a)$ which transitions smoothly from some asymptotic past value w_p to some asymptotic future value w_f requires a minimum of four parameters: w_p, w_f, the epoch of transition a_t, and a rapidity parameter τ. (Note that the previous models are not bounded in the future; this is not overly worrisome because we have no data on the expansion future.) Such forms are particularly successful in describing tracking models which have both asymptotic past and future equations of state. The transition can be described by many functional forms, but the two most common four parameter equations of state both adopt "Fermi-Dirac" transitions. The kink model [16] takes this in scale factor a, obtaining

$$w(a) = w_0 + (w_m - w_0)\frac{1+e^{a_t/\Delta}}{1-e^{1/\Delta}}\,\frac{1-e^{(1-a)/\Delta}}{1+e^{(a_t-a)/\Delta}}, \tag{33}$$

where w_m is the asymptotic value in the matter dominated era and Δ is related to the rapidity, while the e-fold model [63] does the transition in the expansion e-fold factor $\ln a$, obtaining

$$w(a) = w_f + \frac{w_p - w_f}{1 + (a/a_t)^{1/\tau}} \,. \tag{34}$$

One of the advantages of the e-fold model is that it allows an analytic expression for the Hubble parameter $H(a)$.

One could continue developing more complicated forms but sadly even the next generation of experiments will not be able to constrain stringently more than two equation of state parameters [63]. This conclusion holds whether dealing with parameters per se or principal components (see below). Happily, the w_a parametrization is quite satisfactory in giving a model independent, good approximation to the dynamics.

Nevertheless, let us briefly consider principal component analysis (PCA). This approach attempts to gain some independence from the particular form of parametrization, letting the data define the best constrained combination of information. This is a valuable tool; see [36] for its development for the dark energy equation of state, and [35] for an adaptation localizing the principal components in redshift. PCA has the advantage over parametric forms in its nonparametric flavor, and in specifying what a particular survey measures best, however its results are dependent on ingredients other than the underlying physics: the type of cosmological probe, the details of the data, the fiducial cosmology, and priors. That is, a principal component derived from one specific experiment is not exactly comparable to a principal component from another experiment, or the same experiment over a different redshift range. By contrast, w_0 and w_a, say, mean the same thing regardless of probe, survey, cosmology, or priors. (We are talking about the meaning of the variables, not the estimation of the fit values.) Thus, PCA is likely to be of most use as a complementary tool alongside parametric fits.

Note there has been some confusion in the literature regarding the accuracy of PCA fits, with some claims that more than two principal components can be stringently fit by next generation experiments. In the analyses where there appear to be more than two well fit parameters, this arises from consideration only of low noise in the component coefficients α_i, e.g., $\sigma(\alpha_i)$, not high signal to noise criteria $\sigma(\alpha_i)/\alpha_i$.

So we appear restricted to two parameters for our equation of state description. However, a tilt from the cosmological constant value, $1+w$, and a time variation, w', contain rich information on the physics responsible for the acceleration of the universe. Given we have only two parameters, are we sure that w_0 and w_a represent the best, model independent parameters? No, we have no guarantee of this and we should continually be on the lookout for improvements, though to date w_0, w_a have served extremely well.

One idea for an alternate parameter involves the so-called pivot or minimum variance equation of state w_p. This is the equation of state at the scale factor a_p where the variance $\sigma^2(w(a))$ is minimized, i.e., $w_p = w(a_p)$. Note that w_p is also decorrelated with w_a, with zero covariance between their estimations, but this holds only due to the specific linear dependence of the equation of state $w(a)$ on w_a;

generally the minimum variance value is not decorrelated with other equation of state parameters. The pivot parameter possesses many of the same issues as the PCA approach: lack of an invariant physical meaning due to dependence on probe, survey, model, and priors. It is sometimes useful however for the narrow question of whether the data are consistent with a cosmological constant cosmology (in one direction, at least; one can find $w_p = -1$ yet have dynamical dark energy). Note for thawing models the deviation $1 + w$ is greatest at $z = 0$ so a parameter at z_p may not be optimal even for this question. Linder [60] showed that generally w_p is more subject to bias than either w_0 or w_a.

Another suggestion for alternate parametrization involves either so-called statefinder variables (r, s) [81] or combinations of derivatives of the cosmic scale factor such as the deceleration parameter $q = -a\ddot{a}/\dot{a}^2$ and jerk $j = a^2\dddot{a}/\dot{a}^3$ [7]. Note that either parametrization convolves the equation of state parameters with the energy density:

$$q = \frac{1}{2} + \frac{3}{2} w\Omega_w(a) \tag{35}$$

$$j = 1 - \frac{3}{2}\Omega_w(a)\left[w' - 3w(1+w)\right] = q + 2q^2 - q'. \tag{36}$$

(r is the same as j, and $s = [3w(1+w) - w']/(3w) = c_a^2(1+w)/w$, where c_a^2 is the adiabatic sound speed.) These approaches also conflate different physics: $j = 1$, for example, corresponds to an Einstein–de Sitter pure matter universe, or a de Sitter pure cosmological constant universe, or any model that instantaneously lies on the $w' = 3w(1+w)$ line. Of course interpreting q and j as a Taylor expansion about the present expansion behavior would restrict their usage to $z \ll 1$. Also note that while the scale factor can be viewed as a kinematical quantity (e.g., no equation of motion need be specified, just the metric, to know how light is redshifted), this breaks down as soon as time dependence is explicit, e.g., by parametrizing $q = q_0 + q_1 z$. Thus no advantage exists for such a representation over the dynamical phase space.

5 Extending dynamics

We can now investigate whether the dynamics phase space w-w' is useful for physical theories beyond canonical, minimally coupled scalar fields. This includes for modified gravity or other theories where the quantities w and w' are effective quantities, defined in terms of the deviation in the expansion rate from the matter dominated behavior,

$$\delta H^2 \equiv (H/H_0)^2 - \Omega_m a^{-3} \tag{37}$$

$$w_{\rm eff} \equiv -1 - \frac{1}{3}\frac{d\ln\delta H^2}{d\ln a}, \tag{38}$$

possibly distinct from any physical pressure or dark energy density.

Fig. 3 Modifications to the Friedmann equation of the form H^α lie in the freezing regime, despite possibly not arising from a simple scalar field. Moreover, they asymptotically approach Λ along the lower boundary line $w' = 3w(1+w)$. The braneworld curve is shown solid to $z = 0$, with x's indicating $z = 1,2,3$

As already mentioned, phenomenological models such as $\delta H^2 \sim H^\alpha$ [24] fit within the freezing picture and the specific freezing region of the phase space, as illustrated in Fig. 3. Note that the case $\alpha = 1$ corresponds to the dynamics of an extra dimensional braneworld model [19; 23]; such models are discussed in more detail by [44] in this volume.

Since the results of Sect. 3 were discussed in terms of canonical, minimally coupled fields, let us examine the extension to noncanonical or coupled dark energy.

k-essence: If we remove the canonical nature of the scalar field Lagrangian that involves an additive term linear in the kinetic energy, we have a class of theories known as k-essence [2; 14], with Lagrangians of the form

$$\mathscr{L} = V(\phi)\,F(X), \tag{39}$$

where $X = (\partial_\mu \phi\, \partial^\mu \phi)/2$, i.e., in the absence of spatial inhomogeneities X is just the kinetic energy. Such models have some inspirations from field and string theory (for an overview see e.g., [70]), can describe phantom fields with $w < -1$, can have sound speeds less than the speed of light (hence affecting structure formation differently than quintessence) and can have attractor mechanisms to alleviate the fine tuning problem.

Without further specifying the functions V or F, it is difficult to say anything general about k-essence dynamics. Purely kinetic k-essence, where $V =$ constant, does have phase space trajectories limited to one side or the other of the line $w' = 3w(1+w)$ corresponding to constant pressure [59; 83]. However kinetic k-essence can dynamically mimic (or be mimicked by) quintessence as long as the portion of the phase space trajectory of interest does not cross this line [20; 84].

Coupled dark energy: The dark energy could in fact be not dark, that is it could interact non-gravitationally. From the dynamical perspective this creates an effective equation of state shifted from the bare one by the interaction term, e.g.,

$$w_{\rm eff} = w - \frac{\Gamma}{3H}, \tag{40}$$

where Γ is the interaction appearing in the continuity equation

$$\dot\rho_w = -3H\rho_w(1+w) + \Gamma\rho_w, \tag{41}$$

representing a decay/creation process for example. This was set forth in early work by [50; 88]. Such coupling will shift the trajectories in the w-w' phase space, allowing for dynamics outside the thawing and freezing regions. Many different couplings, and their cosmological effects, have been considered; see, e.g., [1; 6; 56]. However, concerns have been raised about the apparent strong effect of quantum corrections on fields coupled to matter [21]. This can be avoided if one postulates that the potential considered is really an effective low energy potential that just happens to take on a simple form as a result of complicated quantum loop corrections to the (in turn necessarily complicated) classical potential; see the article by [22] on low energy effective theories in this volume.

Scalar-tensor gravity: Rather than coupling the dark energy to the matter sector of the Lagrangian, one could make the coupling to gravity nonminimal. These are scalar-tensor theories; see the article by [27] in this volume. Coupling the quintessence field to the Ricci scalar, $R/(8\pi G) \to F(\phi)R$ in the action, these extended quintessence theories [74] can have varied dynamics depending on the form of F, along with an interesting attractor mechanism called the R-boost [3]. For a model with a cosmological constant potential, requiring consistency with solar system tests drives the equation of state very close to $w = -1$ (within 10^{-4}) and with dynamics representative of neither freezing nor thawing fields (C. Baccigalupi et al., 2007, in draft). For another approach, see [68].

Model Zoo: As the fertile imagination of children's author Dr. Seuss envisioned an alphabet and animals "On Beyond Zebra", so has the intense interest in the dark energy mystery led to a zoo of models "On Beyond Λ". The merest glimpse of a small fraction of these includes: *oscillating* (see also *slinky*) models [4; 5] with dynamics corresponding to a circle in phase space [58], *mocker* models that arc from matter like behavior to cosmological constant like behavior along curves of $w' = Cw(1+w)$ [59], closely related to *quartessence* and *Chaplygin gas* models that attempt to unify dark matter and dark energy (see [67] for an overview), *skating* models that arc from free field behavior ($w = +1$), to cosmological constant like behavior along the curve $w' = -3(1-w^2)$, physically corresponding to a field moving across a constant potential [57; 80] (but also related to kinetic k-essence [20]), and *wet fluid* [33] (equivalent to the sum of a constant w component and a cosmological constant; cf. Sect. 3.4) or *leveling* [59] models that approach a cosmological constant as the density nears a limiting value and have parabolic tracks—respectively $w' = 3(1+w)(w-w_*)$ and $w' = -3(1+w)(w+w_*)$.

6 Dynamics and growth

The dynamics of the accelerating component affects the growth of structure in the universe through the expansion rate. This provides a Hubble friction term opposing gravitational instability (e.g., reducing the exponential Jeans growth in a static background to the power law growth in an expanding background). It also affects the matter source term $\Omega_m(a)$, i.e., the evolution of the homogeneous matter density, through the expansion, but to the extent that dark energy remains smooth on the relevant scales it does not directly source growth. Canonical scalar fields are very light, $m \lesssim H$, so they remain smooth on scales less than the Hubble scale [65]. Therefore, within general relativity, the growth effects of dark energy follow directly from the expansion effects discussed in this article. A highly accurate fitting formula for the linear growth can be given in terms of $\Omega_m(a)$ and $w(z=1)$ through the gravitational growth index formalism [56; 62]:

$$g(a) \equiv (\delta\rho_m/\rho_m)/a = e^{\int_0^a (da/a)[\Omega_m(a)^\gamma - 1]} \tag{42}$$

$$\gamma = 0.55 + 0.05\,[1 + w(z=1)], \tag{43}$$

is accurate to 0.2% compared to the numerical solution of the exact second order differential equation. Structure formation in general requires treatment of fully nonlinear growth through N-body numerical computations. Early work with dynamical quintessence included that of [42; 64], with many following investigations.

When the physics of the cosmic acceleration has a gravitational origin, or a dark energy component is not minimally coupled, additional terms enter into the growth, including new source terms such as from anisotropic stress and non-unity sound speed, and varying gravitational coupling. This breaking of the degeneracy between expansion effects and growth effects offers a promising window for identifying the fundamental physics, but is beyond the scope of this article; see, e.g., the review by [61] for more details.

7 Thawing dark energy

Let us now return to the fundamental mode picture of quintessential dynamics, presenting some new results on the specifics of determining the class of dark energy responsible for cosmic acceleration and the ability to zero in on characteristics within that class.

While distinguishing the thawing class of dark energy from the freezing class would be a major accomplishment guiding us toward the fundamental physics behind dark energy, we can also examine thawing models in themselves. These are among the best motivated physics, including radiatively stable PNGB pseudoscalar or axion models and familiar quadratic, quartic, and other renormalizable potentials.

7.1 Thawing physics

Thawing models are defined by their departure from a cosmological constant-like state in the past to a dynamical, $w \neq -1$, behavior today. This property of being frozen into a cosmological constant over much of the history of the universe makes this class difficult to distinguish from a cosmological constant without highly accurate cosmological data. Indeed, current observations are almost wholly degenerate with the entire thawing region as defined in [11], and if an effective, constant w (e.g., a weighted average over the data sensitivity) is determined to equal -1 within 5% then we still have essentially no information on whether this is truly a cosmological constant Λ or any model in the entire half of the physical model space that is categorized as thawing.

This challenge in uncovering the underlying physics makes this class useful as a testbed for the science reach of next generation experiments and for the role of phenomenological parameterization. We will particularly be interested in, of course, distinguishing thawing models from Λ and seeing dynamics such that $w(z) \neq w_{\rm const}$, but we also would learn physics more directly by verifying that the field started in a frozen state at early times and furthermore discerning its trajectory in phase space or at least its dynamical slope parameter $w'/(1+w)$.

Recall from [59] that

$$\frac{w'}{1+w} = 2X + 3(1+w) \quad (44)$$

$$= 3\frac{1-Y}{1+Y} + 3w, \quad (45)$$

where $X = \ddot{\phi}/(H\dot{\phi})$ and $Y = \ddot{\phi}/V_{,\phi}$. Thus, constraining the dynamical slope parameter $w'/(1+w)$ directly leads to information about the field acceleration, friction, and potential tilt terms in the Klein–Gordon equation of motion (6).

7.2 Thawing models

We begin by examining three parameterizations of thawing fields, comparing their behavior and constraints. First is the standard parameterization of $w(a) = w_0 + w_a(1-a)$, reviewed in Sect. 4. If we choose $w_0 + w_a = -1$, then we see that at early times ($a \ll 1$), this possessed $w = -1$. Furthermore, $w' = -aw_a$ so at early times $w' = 0$; thus this parameterization can describe a thawing model. However, we have handcuffed this parameterization by doing this, reducing it to a single parameter, rather than a model with two degrees of freedom, putting it at a disadvantage. It is basically restricted to the trajectory $w' = 1 + w$. Nevertheless, we will see that it is able to describe reasonably most thawing models. The alternative is to retain the two parameters of w_0, w_a but at the price of not matching a cosmological constant at early times; since cosmological data weights the recent universe more heavily, this is not a bad approximation. The energy density of w_a models is

$$\rho_w(a) = \rho_w\, a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}. \quad (46)$$

The second parameterization $w' = F(1+w)$ is motivated by PNGB models, and is an excellent approximation to their dynamics [11], with F inversely proportional to the symmetry breaking energy scale f. For more on PNGB models, see [28; 40; 79]. These have fields starting frozen on their $1+\cos(\phi/f)$ potential and, after the Hubble drag diminishes, are released to roll. Due to the change from convexity to concavity of the potential, they can have interesting dynamics depending on the initial conditions. We assume they are not fine tuned in the sense of starting very near the top of their potential, nor have they already rolled through the minimum and ascended the potential. Eventually the field will oscillate around the minimum (which looks quadratic, i.e., $V \sim \phi^n$ with $n = 2$, so the effective equation of state $w = (n-2)/(n+2) = 0$ [87] as long as the oscillation period is short compared to the Hubble time), acting like dark matter before vanishing as the field comes to rest at zero potential. However, during the accelerating period, $w' = F(1+w)$ accurately describes the dynamics; the equation of state has two parameters, the current equation of state w_0 and the dynamical slope F, with

$$1 + w = (1+w_0)\, a^F. \quad (47)$$

The energy density of these models is

$$\rho_w(a) = \rho_w \, e^{[3(1+w_0)/F](1-a^F)}. \tag{48}$$

For the third parameterization, we craft a new model specifically following the physics of thawing, called the algebraic thawing model:

$$1+w = (1+w_0)\, a^p \left(\frac{1+b}{1+ba^{-3}}\right)^{1-p/3}, \tag{49}$$

with parameters w_0, p (b is fixed). Let us justify this form. As the field is released from the cosmological constant state, still in the matter dominated era $t \sim a^{3/2}$, the dynamics is given by $X = 3/2$, or $w' = 3(1+w)$, as illustrated in [11]. This implies that at early times $1+w \sim a^3$. So far this is identical to the PNGB model with $F = 3$. To add some curvature into the trajectory in the phase plane w-w', let us multiply this by a factor that bends the dynamics away from this line as the scalar field energy density becomes more important, say $\Omega_w(a)^q$. In fact, to preserve the early time behavior, this factor must go to a constant at early times, so we use $[a^3 \Omega_w(a)]^q$, which is indeed constant at early times when $w \to -1$. The only problem with this is that the expression for the equation of state has become non-analytic. Even if we approximate $\Omega_w(a)$ by some fixed function, say $\Omega_\Lambda(a)$, then the equation of state is intertwined with the present energy density parameter Ω_w, or Ω_m, rather than being an independent quantity. For the final form we therefore replace the intruding density ratio—in this one place—with a constant $b = 0.3$. The equation of state is quite insensitive to this specific value, varying by less than 1% as b varies by 50%; of course the value of b is irrelevant as $a \to 1$ and for $a \ll 1$.

The dynamics of the algebraic thawing model is

$$w' = (1+w)\left[3 - \frac{3-p}{1+ba^{-3}}\right], \tag{50}$$

and the energy density is

$$\rho_w(a) = \rho_w \exp\left[\frac{3(1+w_0)}{\alpha p}\left\{1-(\alpha a^3+\beta)^{p/3}\right\}\right], \tag{51}$$

where $\alpha = 1/(1+b)$, $\beta = b/(1+b)$.

In a clever analysis [17] came up with a similar model by analyzing a slow roll-like field expansion, assuming a particular combination $V_{,\phi}/[V(1+X/3)]$ can be Taylor expanded about the present. After some approximations they take $1+w \sim a^p\, \Omega_\Lambda(a)^{1-p/3}$. However, this form still entangles w and the present matter density, and in fact a more exact solution of the field expansion equations works worse! The basic problem is that even for thawing fields there is no reasonable slow roll or field expansion approximation. Even for their less extreme model with $w_0 = -0.8$, the field still traverses $\Delta\phi \sim 0.4 M_P$. The algebraic form Eq. (49) in fact gives more accurate equations of state for the cases they illustrate.

Fig. 4 Together with Figs. 5 and 6, this figure for the w_a thawing model illustrates constraints on the dynamical behavior of three thawing models at four redshift snapshots. While the $z = 0$ behavior is poorly limited by future data, taking into account the dynamical history still allows distinction of the fiducial $w_0 = -0.9$, $w_0' = 0.15$ model from a cosmological constant and from the freezing class of physics

Fig. 5 As Fig. 4, for the PNGB thawing model

Fig. 6 As Fig. 4, for the algebraic thawing model

7.3 Discriminating thawing

We can now use the w_a, PNGB, and algebraic models to examine the constraints, and parameter dependence of the constraints, from future data on the dynamics of quintessence. For each model we have two equation of state fit parameters: (w_0, w_a), (w_0, F), or (w_0, s), where $s = w_0'/(1+w_0)$ is the dynamical slope at present (just as F is the dynamical slope, constant for all times). From the estimation of these parameters (marginalizing over the matter density, in a flat universe, and other parameters such as the supernovae absolute magnitude), and their covariances, we can find the constraints on w and w' at any redshift, giving confidence contours in the w-w' phase plane.

For future data we consider Type Ia supernovae distances from $z = 0 - 1.7$, with systematics, of SNAP quality (see, e.g., [85]), plus the reduced distance to the CMB last scattering surface, of Planck quality (0.7% fractional precision). The fiducial cosmology has $w_0 = -0.9$ and present dynamical slope 1.5, and likelihoods are approximated as Gaussians in a Fisher information analysis. The CMB data in fact has little leverage on the equation of state, because for all the thawing models the high redshift equation of state goes to a cosmological constant. We have checked that adding baryon acoustic oscillation angular distance measurements at 1% precision or a matter density prior of 0.005 (roughly mimicking weak gravitational lensing constraints) does little to improve the constraints.

Figures 4, 5, and 6 show the w-w' constraints for the three models at four redshifts. We exhibit the 68% confidence level contours at $z = 0$, at the redshift where w and w' are decorrelated, giving vertical/horizontal ellipses, and at high redshift, $z \gtrsim 1$. The phase space trajectory is marked by the x's at each of the four redshifts. Note that the confidence contours vary between the models, especially when evaluated at the present, and this may lead to concerns about parameterization dependence. However, as we will see, the qualitative answers to the important physical questions remain independent of the parameterization.

While constraints on the present dynamical state, i.e., w_0 and w_0', are relatively weak, in each of the parameterizations they are still sufficient to distinguish the fiducial model $w_0 = -0.9$, $w_0' = 0.15$ from a cosmological constant. (Note this is despite the uncertainty $\sigma(w_0) \approx 0.14$ from the algebraic thawer, the weakest model at $z = 0$—one must take into account the contour orientation in the phase plane.) At $z = 0$, however, the models cannot distinguish thawing from freezing, or from a constant equation of state $w_{\rm const} = -0.9$. Using the information from throughout the dynamical history greatly improves the situation. At some redshift,

$z \approx 0.2 - -0.3$ in the cases here, the dynamical variables w and w' decorrelate and the contours become vertical. This gives the greatest distance between the constraint contour and the cosmological constant, showing clear distinction, and the intersection of the ellipse with the $w' = 0$ axis also provides the minimal variance estimate on the instantaneous equation of state value. Such a decorrelation redshift is sometimes called a pivot redshift. Generically there can be more than one decorrelation redshift, and for the models where w is not a linear function of the parameters we exhibit the contours at the second of these redshifts, $z \approx 0.7$–1.4. This provides a minimum variance estimate of the instantaneous time variation of the equation of state.

Note that the confidence contours at each redshift are distinct from the cosmological constant, showing that future data can distinguish thawing models from Λ (at least at 1σ for this fiducial cosmology). Furthermore, the early time contours (except in the w_a case) distinguish the thawing model from models with constant equation of state, thus exhibiting the presence of dynamics. The early time contours also draw away from the freezing region of the phase plane, so the data can indeed guide us to the correct class of physical origin. These are all important physical insights that are not parameterization dependent. Gains are more modest in zeroing in on a specific thawing model and these are more sensitive to parameterization. At early times, the form of the algebraic thawer forces the contour to prefer a dynamical slope near 3. However the PNGB and w_a cases do not impose such preferences since the slope is a free fit parameter. They do constrain $w'/(1+w)$ to a subset of the thawing region, rather than the full range of 1–3.

It is heartening that the physical insights can be expected to be as clear as indicated, and not particularly dependent on the specific parameterization. The issue of fitting the dynamical behavior of dark energy (especially when restricted to two parameters, as seems likely from realistic next generation data accuracy), is a fascinating one. Use of a global parameterization like (w_0, w_a) allows a good fit for models over the whole phase plane, but one can imagine that as we close in on the physical origin of dark energy, e.g., narrowing in on thawing models, we may move to more specific parameterizations such as the algebraic thawing model. On the other hand, perhaps specific physical benchmark models, such as PNGB or motivated scalar field potentials, will then be of most use.

8 Conclusion

Dynamics, of quintessence and of the accelerating physics in general, can provide considerable insight into the nature of the new component or new physical law dominating our present universe. Fundamental modes of the physics lead to well defined, distinct regions of w-w' phase space that next generation cosmological probes will be able to test and distinguish. Just as we build our physical intuition in early universe inflation with single field models leading to consistency relations, the fundamental modes of dark energy—the quintessence of dynamics—are a useful foundation.

Model independent parametrization, with a strong physical basis, plays an important role, even if stringent constraints will be limited to two parameters such as the tilt from a cosmological constant, $1+w$, and a variation w'. Nevertheless, this is as much as we expect from inflation as well, while for dark energy we have added complications due to the incomplete dominance of dark energy.

Sensitivity to dynamics is a requirement to make progress in understanding the nature of cosmic acceleration. Once we begin to zero in on a class of physics, model independence may give way to specific discriminating approaches such as the thawing analysis presented here. Models for the equation of state which depend nonlinearly on the time variation parameter also possess minimum variance, or pivot, redshifts for the time variation, $z_{p'}$, and this may prove a useful tool.

Dynamics alone, whether by its characterization or absence, will not fully solve the dark energy enigma. The cosmic expansion history must be properly compared with the cosmic growth history to reveal extensions to gravitational physics or microphysics. We have scarcely addressed this important subject here, nor have we said why in the presence of dynamics Λ should not still exist, at a much larger energy density than the present, causing an abnegation of the universe we observe.

Ten years passed from the time the basic physics and cosmology for the accelerating universe were in place until the first convincing observational evidence for its reality; since then another ten years of work on all fronts have passed. There is clearly still an enormous amount of exciting and challenging work ahead, and the answers, whatever they are and whenever they come, will revolutionize our understanding of gravitation, quantum physics, cosmology, and the fate of our universe.

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