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Special Issue
Recent Advance in Mathematical Physics II

Edited by
Prof. Dr. Antonio Masiello

MDPI

https://doi.org/10.3390/sym17040542

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Analysis of W_3 Curvature Tensor in Modified Gravity and Its Cosmological Implications

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Abstract: In this study, we investigated the geometric and physical implications of the W_3 curvature tensor within the framework of $f(R, G)$ gravity. We found the sufficient conditions for W_3 flat spacetimes with constant scalar curvature to be de Sitter ($R > 0$) or Anti-de Sitter ($R < 0$) models. The properties of isotropic spacetime in the modified gravity framework were also investigated. Furthermore, we explored spacetimes with a divergence-free W_3 curvature tensor. The necessary and sufficient condition for a W_3 Ricci recurrent and parallel spacetime to transform into an Einstein spacetime was determined. Finally, we analyzed the role of the W_3 curvature tensor in black hole thermodynamics within $f(R, G)$ gravity.

Keywords: W_3 curvature tensor; modified gravity; $f(R, G)$ gravity; Ricci semi-symmetry; isotropic spacetime; black hole thermodynamics; cosmological inflation

MSC: 53C25; 53C50; 53C80

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Academic Editor: Antonio Masiello

Received: 9 March 2025

Revised: 27 March 2025

Accepted: 31 March 2025

Published: 2 April 2025

Citation: Ali, M.; Vasiulla, M.; Khan, M.A. Analysis of W_3 Curvature Tensor in Modified Gravity and Its Cosmological Implications. *Symmetry* **2025**, *17*, 542. https://doi.org/10.3390/sym17040542

1. Introduction

Curvature invariants play a fundamental role in gravitational physics, offering profound insights into the geometric and physical structure of spacetime. These invariants are derived from fundamental curvature tensor [1], such as the Riemann curvature tensor, given by

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^m_{jk}\Gamma^i_{ml} - \Gamma^m_{jl}\Gamma^i_{mk}, \tag{1}$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection.

The Ricci tensor R_{ij} is a contraction of the Riemann curvature tensor on the first and third indices, that is, $R_{ij} = R^k_{ikj}$. The Ricci tensor is symmetric ($R_{ij} = R_{ji}$) and plays a central role in Einstein's field equations of general relativity. The Weyl tensor [1] is another important curvature tensor that represents the trace-free part of the Riemann curvature tensor and is defined as

$$C^i_{jkl} = R^i_{jkl} - \frac{1}{n-2}\left(\delta^i_k R_{jl} - \delta^i_l R_{jk} + g_{jl}R^i_k - g_{jk}R^i_l\right) + \frac{R}{(n-1)(n-2)}\left(\delta^i_k g_{jl} - \delta^i_l g_{jk}\right), \tag{2}$$

where n is the dimension of the manifold and R is the scalar curvature.

Over the years, various generalizations of curvature tensors have been introduced to capture more intricate geometric properties, significantly influencing both mathematics and theoretical physics.

In this regard, Pokhariyal and Mishra [2–4] introduced a class of curvature tensors with direct applications to relativistic theories. Among these, the W_3 curvature tensor has emerged as a fundamental tool in understanding geometric deformations of space-time. This tensor has been widely studied in various manifolds, particularly in the domain of mathematical physics [5–13]. In particular, the W_3 curvature tensor belongs to class 4 in the hierarchy of skew-symmetric operators [14], indicating its profound structural significance.

The W_3 curvature tensor on a 4-dimensional Riemannian space is defined in local coordinates as

$$W_{ijkl} = R_{ijkl} + \frac{1}{3}(g_{jk}R_{il} - g_{jl}R_{ik}), \quad (3)$$

where R_{ijkl} is the Riemann curvature tensor and $R_{jk} = g^{il}R_{ijkl}$ is the Ricci tensor. This formulation incorporates contributions from both Ricci and metric tensors, making W_3 particularly useful in describing spacetime deformations.

In classical general relativity, the Einstein field equations (EFEs) estabilish the relationship between spacetime geometry and energy–momentum distribution:

$$R_{ij} - \frac{R}{2}g_{ij} = \kappa T_{ij}, \quad (4)$$

where R is the scalar curvature, κ is the coupling constant, and T_{ij} represents the energy–momentum tensor. Despite its success, the Einstein framework does not fully account for certain cosmological phenomena, such as late-time cosmic acceleration, without introducing an external cosmological constant Λ. To address these challenges, modifications of the standard theory have been extensively explored [15–19].

One of the most compelling alternatives is the $f(R,G)$ gravity theory, where the traditional Ricci scalar R is replaced by a more general function $f(R,G)$, incorporating the Gauss–Bonnet topological invariant [20]:

$$G = R^{hijk}R_{hijk} - 4R^{hk}R_{hk} + R^2. \quad (5)$$

The resulting modified field equations take the form [21]

$$\begin{aligned} kT^{(m)}_{jk} = & -\frac{1}{2}f g_{jk} + f_G\big(2R_{jk}R - 4R_{hjlk}R^{hl} + 2R_{jhmn}R^{hmn}_k - 4R^h_j R_{hk}\big) \\ & + \big(2g_{jk}R\nabla^2 + 4R^h_j\nabla_h\nabla_k + 4R^h_k\nabla_h\nabla_j - 2R\nabla_j\nabla_k - 4R_{jk}\nabla^2\big)f_G \\ & + \big(4R_{ljhk}\nabla^l\nabla^h - 4g_{jk}R_{hl}\nabla^h\nabla^l\big)f_G + (R_{jk} - \nabla_j\nabla_k + g_{jk}\nabla^2)f_R, \end{aligned} \quad (6)$$

where the expression $\nabla^2 = \nabla_i\nabla^i$ represents the covariant Laplacian operator (also known as the Laplace–Beltrami operator) and the terms $f_R = \frac{\partial f}{\partial R}$, $f_G = \frac{\partial f}{\partial G}$, and $T^{(m)}_{jk}$ denote the energy–momentum tensor associated with ordinary matter. The $f(R,G)$ theory has emerged as one of the most promising approaches to explaining the accelerated expansion of the universe and other phenomena beyond the standard model. For related findings in this area, refer to the sources (see refs. [20–27]).

Pokhariyal and Mishra extensively analyzed curvature tensors and their importance in relativistic contexts [2,3], providing a comprehensive analysis of their properties and applications. In another investigation, Baishya, Bakshi, Kundu, and Blaga explored certain types of generalized Robertson–Walker (GRW) spacetimes in [5], offering new characterizations within the framework of general relativity. Additionally, Dey and Roy in [6] examined spacetimes equipped with the η–Ricci–Bourguignon soliton, highlighting their geometric features and physical implications. Spacetimes admitting the W_2 curvature tensor were investigated by Mallick and De in [8], where they explored its influence on the geometric structure. In the context of modified gravity, Nojiri and Odintsov provided a thorough introduction to gravitational alternatives for dark energy in [15], offering new perspectives on cosmic acceleration. Atazadeh and Darabi in [20] studied the energy conditions in $f(R,G)$ gravity, examining their stability and physical relevance. Moreover, Navo and Elizalde in [27] analyzed the stability of hyperbolic and matter-dominated bounce cosmologies in $f(R,G)$ gravity during the late evolution stages, providing insights into the dynamical behavior of such models. De, Shenawy, Syied, and Bin Turki in [28] studied conformally flat pseudo-projective symmetric spacetimes, exploring their geometric properties in the context of $f(R,G)$ gravity. Finally, the impact of the quasi-conformal curvature tensor in spacetimes and its relevance in $f(R,G)$ gravity were further studied by De and Hazra in [29], contributing to the understanding of curvature-based modifications in gravitational models.

Motivated by the above studies, this paper presents an analysis of the W_3 curvature tensor within the framework of modified gravity, exploring its cosmological implications. We summarize this paper as follows: After the introduction, in Section 2, we investigate W_3 curvature flat spacetimes, proving that they are de Sitter $R > 0$ or Anti-de Sitter $R < 0$ with constant scalar curvature. In Section 3, we study isotropic spacetimes in $f(R,G)$ gravity, deriving expressions for effective energy density and pressure and verifying the energy conditions. In Section 4, we a examine divergence-free W_3 curvature tensor, proving that such spacetimes have constant scalar curvature and exploring their connection to Ricci semi-symmetry and inflationary models. In Section 5, we establish conditions for W_3 Ricci recurrent and parallel spacetimes, proving that a parallel W_3 curvature tensor implies an Einstein spacetime with a parallel Ricci tensor. In Section 6, we analyze the thermodynamics of black holes in $f(R,G)$ gravity, deriving modifications to the Hawking temperature and entropy due to curvature corrections. Finally, we conclude with a discussion of the theoretical implications and applications of W_3 curvature tensors in modified gravity and black hole physics.

2. W_3 Curvature Flat Spacetime in $f(R,G)$ Gravity

In this section, we discuss some results regarding W_3 curvature flat spacetime in $f(R,G)$ gravity.

Let $(\mathcal{M},g)$ be a 4-dimensional W_3 flat spacetime. By the definition of W_3 curvature flatness, we have

$$W_{ijkl} = 0. \tag{7}$$

Using Equation (3) in Equation (7), we obtain the relation

$$R_{ijkl} = \frac{1}{3}\Big(g_{jl}R_{ik} - g_{jk}R_{il}\Big). \tag{8}$$

Multiplying Equation (8) by g^{il}, we get

$$R_{jk} = -\frac{R}{2}g_{jk}. \tag{9}$$

Substituting Equation (9) into (8), we obtain

$$R_{ijkl} = \frac{R}{6}\Big(g_{jk}g_{il} - g_{jl}g_{ik}\Big),$$

which shows that the spacetime is of constant curvature. Thus, we have the following:

Theorem 1. *A* 4*-dimensional* W_3 *curvature flat spacetime is either to the de Sitter spacetime if* $R > 0$ *or to the Anti-de Sitter spacetime if* $R < 0$*.*

Taking the covariant derivative of Equation (9), we get

$$\nabla_l R_{jk} = -\frac{1}{2} g_{jk} \nabla_l R. \tag{10}$$

Multiplying by g^{jl} and using the identity $\nabla_l R^l_k = \frac{1}{2}\nabla_k R$, we obtain

$$\nabla_k R = 0.$$

Thus, we have the following corollary:

Corollary 1. *Let* $\mathcal{M}$ *be a* 4*-dimensional* W_3 *curvature flat spacetime. Then,*

- $\mathcal{M}$ *is an Einstein spacetime.*
- $\mathcal{M}$ *has a constant scalar curvature.*

From Equation (9), the Ricci tensor satisfies

$$R^{jk} = -\frac{R}{2} g^{jk}. \tag{11}$$

Multiplying Equation (9) by (11), we obtain

$$R_{jk} R^{jk} = R^2. \tag{12}$$

From Equation (8), the Riemann curvature tensor satisfies

$$R^{ijkl} = \frac{1}{3}\left(g^{jl} R^{ik} - g^{jk} R^{il}\right). \tag{13}$$

Multiplying Equation (8) by (13), we obtain

$$R_{ijkl} R^{ijkl} = -\frac{2}{3} R^2. \tag{14}$$

Using Equations (12) and (14) in the Gauss–Bonnet invariant Formula (5), we obtain

$$G = -\frac{11R^2}{3}. \tag{15}$$

Thus, we conclude the following:

Theorem 2. *For a* 4*-dimensional* W_3*-flat spacetime, the Gauss–Bonnet scalar takes the form given in* (15)*.*

Definition 1. *Let* $(\mathcal{M}, g)$ *be a semi-Riemannian manifold and let* ξ *be a vector field on M. Then,*

- ξ *is called a Killing vector field if it satisfies the condition*

$$\mathcal{L}_\xi g_{jk} = 0. \tag{16}$$

- ξ *is called a conformal Killing vector field if there exists a scalar function* ϕ *such that*

$$\mathcal{L}_\xi g_{jk} = 2\phi g_{jk}, \tag{17}$$

where $\mathcal{L}_\xi$ *denotes the Lie derivative along* ξ*.*

Definition 2. *The energy–momentum tensor T_{jk} satisfies*

- *The matter collineation condition if*

$$\mathcal{L}_\xi T_{jk} = 0. \tag{18}$$

- *The Lie inheritance property if there exists a scalar function ϕ such that*

$$\mathcal{L}_\xi T_{jk} = 2\phi T_{jk}. \tag{19}$$

The Einstein tensor, denoted as G_{jk}, is a key mathematical object in Einstein's field equations of general relativity. It is defined as

$$G_{jk} = R_{jk} - \frac{R}{2} g_{jk}. \tag{20}$$

Since $\mathcal{M}$ is a W_3 curvature flat spacetime, Equations (6) and (20) reveal

$$R_{jk} - \frac{R}{2} g_{jk} = k[T^{(m)}_{jk} + T^{\text{curv}}_{jk}] = kT^{\text{eff}}_{jk}. \tag{21}$$

where T^{curv}_{jk} arises from the geometry of the spacetime. The tensor T^{curv}_{jk} is given by

$$\begin{aligned} kT^{\text{curv}}_{jk} = {} & (\nabla_j \nabla_k - g_{jk}\nabla^2) f_R + 2R(\nabla_j \nabla_k - g_{jk}\nabla^2) f_G \\ & - 4(R^m_j \nabla_m \nabla_k + R^m_k \nabla_m \nabla_j) f_G \\ & + 4(R_{jk}\nabla^2 + g_{jk} R_{mn} \nabla^n \nabla^m - R_{njmk} \nabla^n \nabla^m) f_G \\ & - \frac{1}{2} g_{jk}(R f_R + G f_G - f) + (1 - f_R)\left(R_{jk} - \frac{R}{2} g_{jk}\right). \end{aligned} \tag{22}$$

Since the W_3 curvature flat spacetime has a constant scalar curvature, the above equation simplifies to

$$kT^{\text{curv}}_{jk} = -\frac{1}{2} g_{jk}(R f_R + G f_G - f) + (1 - f_R)\left(R_{jk} - \frac{R}{2} g_{jk}\right).$$

Using Equations (9) and (15) in the above equation yields

$$kT^{\text{curv}}_{jk} = \left(\frac{f}{2} + \frac{11R^2}{6} f_G + \frac{R}{2} f_R - R\right) g_{jk}. \tag{23}$$

Substituting Equations (9) and (23) into Equation (21), we get

$$kT^{(m)}_{jk} = \left(-\frac{R}{2} f_R - \frac{f}{2} - \frac{11R^2}{6} f_G\right) g_{jk}. \tag{24}$$

Taking the Lie derivative on both sides of (24), we obtain

$$k\mathcal{L}_\xi T^{(m)}_{jk} = \left(-\frac{R}{2} f_R - \frac{f}{2} - \frac{11R^2}{6} f_G\right) \mathcal{L}_\xi g_{jk}. \tag{25}$$

If ξ is the Killing vector, then from (16) and (25), we get

$$\mathcal{L}_\xi T^{(m)}_{jk} = 0. \tag{26}$$

Conversely, if M admits a matter collineation, then

$$\mathcal{L}_\xi g_{jk} = 0.$$

Thus, we have the following result:

Theorem 3. *A 4-dimensional W_3 flat spacetime obeying $f(R,G)$ gravity has a Killing vector ξ if and only if M admits a matter collineation with respect to ξ.*

If ξ is a conformal Killing vector, then substituting Equations (17) and (25) into (24), we obtain

$$\mathcal{L}_\xi T_{jk}^{(m)} = 2\phi T_{jk}^{(m)}, \tag{27}$$

which shows that $T_{jk}^{(m)}$ inherits the Lie derivative property along ξ.

Conversely, if the tensor T_{jk} possesses the Lie inheritance property along the flow lines of ξ, then applying the given Equations (24) and (25), we deduce that

$$\mathcal{L}_\xi g_{jk} = 2\phi g_{jk}.$$

Thus, we can conclude the following:

Theorem 4. *A W_3 flat spacetime $\mathcal{M}$ possesses a conformal Killing vector ξ if and only if the energy–momentum tensor T_{jk} has the Lie inheritance property along ξ.*

3. W_3 Curvature Flat Isotropic Spacetime in $f(R,G)$ Gravity

In this section, we study W_3 curvature flat isotropic spacetime in $f(R,G)$ modified gravity theory.

The energy–momentum tensor $T_{jk}^{(m)}$ of a isotropic spacetime is given by

$$T_{jk}^{(m)} = [p^{(m)} + \sigma^{(m)}]u_j u_k + p^{(m)} g_{jk}, \tag{28}$$

where $p^{(m)}$ is the isotropic pressure, $\sigma^{(m)}$ is the energy density of ordinary matter, and u_j is the unit timelike velocity vector field. The effective energy–momentum tensor is given by

$$T_{jk}^{eff} = [p^{eff} + \sigma^{eff}]u_j u_k + p^{eff} g_{jk}. \tag{29}$$

where p^{eff} denotes the effective isotropic pressure and σ^{eff} is the effective energy density of the effective matter.

From Equations (24) and (28), we obtain

$$\left(-\frac{R}{2}f_R - \frac{f}{2} - \frac{11R^2}{6}f_G - kp^{(m)}\right)g_{jk} = k[p^{(m)} + \sigma^{(m)}]u_j u_k. \tag{30}$$

Multiplying both sides of Equation (30) by u^j, we derive

$$\sigma^{(m)} = \frac{1}{k}\left(\frac{f}{2} + \frac{R}{2}f_R + \frac{11R^2}{6}f_G\right). \tag{31}$$

Contracting Equation (30) with g^{jk}, we obtain

$$k[3p^{(m)} + \sigma^{(m)}] = 4\left(-\frac{R}{2}f_R - \frac{f}{2} - \frac{11R^2}{6}f_G\right). \tag{32}$$

Substitution Equation (31) into Equation (32), we get

$$p^{(m)} = \frac{1}{k}\left(\frac{f}{2} + \frac{R}{2}f_R + \frac{11R^2}{6}f_G\right). \tag{33}$$

Thus, we conclude the following:

Theorem 5. *For a 4-dimensional W_3 curvature flat isotropic spacetime satisfying $f(R, G)$ gravity, the isotropic pressure $p^{(m)}$ and the energy density $\sigma^{(m)}$ are constant. Furthermore, they are given by Equations (31) and (33).*

Combining Equations (31) and (33), we obtain

$$p^{(m)} + \sigma^{(m)} = 0,$$

which implies that the spacetime represents dark energy. Hence, we can also state the following:

Theorem 6. *Let $\mathcal{M}$ be a 4-dimensional W_3 curvature flat isotropic spacetime obeying $f(R, G)$ gravity. Then, $\mathcal{M}$ represents dark energy.*

Using Equations (23), (28) and (29) in Equation (21), we get

$$[p^{eff} + \sigma^{eff}]u_j u_k + p^{eff} g_{jk} = [p^{(m)} + \sigma^{(m)}]u_j u_k + p^{(m)} g_{jk} + \frac{1}{k}\left(-\frac{f}{2} - \frac{R}{2} f_R - \frac{11R^2}{6} f_G - \frac{R}{4}\right) g_{jk}. \tag{34}$$

Comparing terms on both sides, we derive

$$p^{eff} = p^{(m)} + \frac{1}{k}\left(-\frac{f}{2} - \frac{R}{2} f_R - \frac{11R^2}{6} f_G - \frac{R}{4}\right), \tag{35}$$

and

$$\sigma^{eff} = \sigma^{(m)} - \frac{1}{k}\left(-\frac{f}{2} - \frac{R}{2} f_R - \frac{11R^2}{6} f_G - \frac{R}{4}\right). \tag{36}$$

Further, using Equations (31) and (33), we get

$$p^{eff} = -\frac{R}{4k}, \tag{37}$$

$$\sigma^{eff} = \frac{R}{4k}. \tag{38}$$

Now, we examine the energy conditions in the framework of $f(R, G)$ modified gravity. The standard energy conditions for an efective matter are given in Table 1 (see Ref. [28]).

Table 1. Energy conditions and their mathematical expressions.

Energy Condition	Mathematical Expression
Null Energy Condition (NEC)	$p^{\text{eff}} + \sigma^{\text{eff}} \geq 0$
Weak Energy Condition (WEC)	$\sigma^{\text{eff}} \geq 0$ and $p^{\text{eff}} + \sigma^{\text{eff}} \geq 0$
Dominant Energy Condition (DEC)	$\sigma^{\text{eff}} \geq 0$ and $p^{\text{eff}} \pm \sigma^{\text{eff}} \geq 0$
Strong Energy Condition (SEC)	$\sigma^{\text{eff}} + 3p^{\text{eff}} \geq 0$ and $p^{\text{eff}} + \sigma^{\text{eff}} \geq 0$

In view of Equations (37) and (38), the following can be stated:

Theorem 7. *In a 4-dimensional W_3 curvature flat isotropic spacetime with $\mathcal{M}$ obeying $f(R, G)$ gravity, all mentioned energy conditions are consistently satisfied if $R > 0$.*

4. Spacetimes with Divergence-Free W_3 Curvature Tensor in $f(R, G)$ Gravity

In this section, we investigate 4-dimensional spacetimes with a divergence-free W_3 curvature tensor in the context of $f(R, G)$ modified gravity theory.

Definition 3. *A spacetime $\mathcal{M}$ is said to be Ricci semi-symmetric if*

$$(\nabla_l \nabla_h - \nabla_h \nabla_l) R_{jk} = 0. \tag{39}$$

Similarly, the energy–momentum tensor $T^{(m)}_{jk}$ is said to be semi-symmetric if

$$(\nabla_l \nabla_h - \nabla_h \nabla_l) T_{jk} = 0. \tag{40}$$

Definition 4. *The energy–momentum tensor $T^{(m)}_{jk}$ is said to be recurrent if there exists a non-zero 1-form u_l such that*

$$\nabla_l T^{(m)}_{jk} = u_l T^{(m)}_{jk}. \tag{41}$$

Moreover, $T^{(m)}_{jk}$ is said to be bi-recurrent if there exists a non-zero tensor ε_{hl} such that

$$\nabla_h \nabla_l T^{(m)}_{jk} = \varepsilon_{hl} T^{(m)}_{jk}. \tag{42}$$

To consider W^h_{jkl}, transvecting Equation (3) by g^{ih} yields

$$W^h_{jkl} = R^h_{jkl} + \frac{1}{3}\Big(g_{jk} R^h_l - g_{jl} R^h_k\Big). \tag{43}$$

Taking the covariant derivative of Equation (43), we derive

$$\nabla_h W^h_{jkl} = \nabla_h R^h_{jkl} + \frac{1}{6}\Big(g_{jk} \nabla_l R - g_{jl} \nabla_k R\Big). \tag{44}$$

Suppose the W_3 curvature tensor is divergence-free, then

$$\nabla_h W^h_{jkl} = 0. \tag{45}$$

We can use the identity

$$\nabla_h R^h_{jkl} = \nabla_l R_{jk} - \nabla_k R_{jl}, \tag{46}$$

which yields

$$\nabla_l R_{jk} - \nabla_k R_{jl} = -\frac{1}{6}\Big(g_{jk} \nabla_l R - g_{jl} \nabla_k R\Big). \tag{47}$$

The contraction foregoing the equation with g^{jl} implies

$$\nabla_k R = 0. \tag{48}$$

This shows that the scalar curvature is constant. Hence, we can state the following:

Theorem 8. *A 4-dimensional spacetime with a divergence-free W_3 curvature tensor has a constant scalar curvature.*

From Equation (47), we find

$$\nabla_l R_{jk} = \nabla_k R_{jl}. \tag{49}$$

Substituting Equation (22) into Equation (21), we obtain

$$R_{jk} = \frac{1}{2f_R}(f - Gf_G)g_{jk} + \frac{k}{f_R}T^{(m)}_{jk}. \tag{50}$$

Combining Equation (50) and Equation (49), it follows that

$$\nabla_l T^{(m)}_{jk} = \nabla_k T^{(m)}_{jl}. \tag{51}$$

In view of Equation (50), we derive

$$(\nabla_l\nabla_h - \nabla_h\nabla_l)R_{jk} = \frac{k}{f_R}(\nabla_l\nabla_h - \nabla_h\nabla_l)T^{(m)}_{jk}. \tag{52}$$

This is the condition of semi-symmetry. Based on above analysis, we derive the following:

Theorem 9. *A 4-dimensional spacetime with a divergence-free W_3 curvature tensor is Ricci semi-symmetric if and only if the energy–momentum tensor is semi-symmetric.*

Next, taking into account that a (bi-)recurrent $(0,2)$ symmetric tensor is semi-symmetric (see ref. [28]), we derive immediately the following consequence.

$$(\nabla_l\nabla_h - \nabla_h\nabla_l)T^{(m)}_{jk} = 0. \tag{53}$$

From which we can also state the following:

Corollary 2. *Let $\mathcal{M}$ be a 4-dimensional spacetime with a divergence-free W_3 curvature tensor satisfying $f(R,G)$ gravity. If the Ricci (energy–momentum) tensor of the spacetime $\mathcal{M}$ is (bi-)recurrent, then the energy–momentum (Ricci) tensor of the spacetime $\mathcal{M}$ is semi-symmetric.*

From Equations (28) and (50), we obtain

$$R_{jk} = ag_{jk} + bu_j u_k, \tag{54}$$

where

$$a = \frac{1}{2}f_R\left(2kp^{(m)} + f - Gf_G\right) \quad and \quad b = kf_R\left(p^{(m)} + \sigma^{(m)}\right). \tag{55}$$

Contracting Equation (54) with g^{jk}, we get

$$R = \frac{1}{f_R}\left[3kp^{(m)} + 2f - 2Gf_G - k\sigma^{(m)}\right]. \tag{56}$$

If $T^{(m)}_{jk}$ is recurrent or bi-recurrent, then using Equations (52) and (53), we have

$$(\nabla_l\nabla_h - \nabla_h\nabla_l)R_{jk} = 0. \tag{57}$$

Substituting Equation (54) into the above, we get

$$bu_j(\nabla_l\nabla_h - \nabla_h\nabla_l)u_k + bu_k(\nabla_l\nabla_h - \nabla_h\nabla_l)u_j = 0.$$

Contracting with u^j, we reveal

$$-b(\nabla_l\nabla_h - \nabla_h\nabla_l)u_k + bu^j u^k(\nabla_l\nabla_h - \nabla_h\nabla_l)u_j = 0.$$

Since $u^j(\nabla_l \nabla_h - \nabla_h \nabla_l)u_j = 0$, the above equation simplifies to

$$b(\nabla_l \nabla_h - \nabla_h \nabla_l)u_k = 0,$$

which yields

$$bR^n_{lhk}u_n = 0.$$

This leads to two possible scenarios:

Case 1. When $R^n_{lhk}u_n \neq 0$, it follows that $b = 0$. Thus, from Equation (55), we obtain $p^{(m)} + \sigma^{(m)} = 0$, indicating that the given spacetime exhibits inflation.

Case 2. If $b \neq 0$, then $R^n_{lhk}u_n = 0$. By contracting with g^{lk}, we arrive at $R_{nh}u^n = 0$. Applying transvection to Equation (54) with u^j and utilizing $R_{nh}u^n = 0$, we deduce that

$$(a - b)u_k = 0,$$

which gives

$$a - b = 0.$$

Using Equation (55), we infer

$$\sigma^{(m)} = \frac{1}{2k}\Big[f - Gf_G\Big]. \tag{58}$$

Substituting Equation (58) into Equation (56), we obtain

$$p^{(m)} = \frac{1}{3k}\Big[f_R R + \frac{3}{2}Gf_G - \frac{3}{2}f\Big].$$

In view of the above discussion, we can state the following:

Theorem 10. *If the energy momentum tensor of a 4-dimensional isotropic spacetime with a divergence-free W_3 curvature tensor satisfying $f(R,G)$ gravity is (bi-) recurrent, then either the spacetime $\mathcal{M}$ represents inflation or the isotropic pressure $p^{(m)}$ and the energy density $\sigma^{(m)}$ are constant.*

5. Conditions for W_3 Ricci Recurrent and Parallel Spacetimes in $f(R,G)$ Gravity

In this section, we discuss conditions for W_3 Ricci recurrent and parallel spacetimes in $f(R,G)$ gravity.

A spacetime $(\mathcal{M}, g)$ is called Ricci recurrent [30] if its Ricci tensor R_{jk} satisfies the recurrence relation

$$\nabla_m R_{jk} = \lambda_m R_{jk} \tag{59}$$

for some nonzero 1-form λ_i.

Taking the covariant derivative of Equation (3), one obtains

$$\nabla_m W_{ijkl} = \nabla_m R_{ijkl} + \frac{1}{3}(g_{jk}\nabla_m R_{il} - g_{jl}\nabla_m R_{ik}). \tag{60}$$

Using Equation (59) in (60), we derive

$$\nabla_m W_{ijkl} = \nabla_m R_{ijkl} + \frac{1}{3}(g_{jk}\lambda_m R_{il} - g_{il}\lambda_m R_{ik}). \tag{61}$$

If the spacetime satisfies the recurrence relation $\nabla_m W_{ijkl} = \lambda_m W_{ijkl}$, then equating with Equation (61), we obtain

$$\nabla_m R_{ijkl} + \frac{1}{3}(g_{jk}\lambda_m R_{il} - g_{il}\lambda_m R_{ik}) = \lambda_m W_{ijkl}. \tag{62}$$

Substituting Equation (3) into Equation (62), we arrive at

$$\nabla_m R_{ijkl} - \lambda_m R_{ijkl} + \frac{1}{3}(g_{jk}\lambda_m R_{il} - g_{jl}\lambda_m R_{ik} - g_{jk}\nabla_m R_{il} + g_{jl}\nabla_m R_{ik}) = 0.$$

Since the spacetime is Ricci-recurrent, then the foregoing relation gives

$$\nabla_i R_{jklm} = \lambda_i R_{jklm}.$$

Thus, the spacetime is Ricci-recurrent if and only if

$$R_{jk} = \frac{R}{4} g_{jk}.$$

Thus, we conclude the following:

Theorem 11. *Let $(\mathcal{M}, g)$ be a 4-dimensional W_3 Ricci recurrent spacetime. Then, in the framework of $f(R,G)$ gravity, the spacetime satisfies the recurrence relation $\nabla_m W_{ijkl} = \lambda_m W_{ijkl}$ if and only if the Ricci tensor is proportional to the metric.*

A tensor T is said to be parallel [31] if its covariant derivative vanishes, i.e.,

$$\nabla_m T = 0.$$

Taking the covariant derivative of Equation (3), we obtain

$$\nabla_m W_{ijkl} = \nabla_m R_{ijkl} + \frac{1}{3}(g_{jk}\nabla_m R_{il} - g_{jl}\nabla_m R_{ik}). \tag{63}$$

If W_3 is parallel, i.e., $\nabla_m W_{ijkl} = 0$, then substituting the above expression, it follows that

$$\nabla_m R_{ijkl} = -\frac{1}{3}\Big(g_{jk}\nabla_m R_{il} - g_{jl}\nabla_m R_{ik}\Big). \tag{64}$$

Multiplying both sides by g^{il} in Equation (64), we obtain

$$\nabla_m R_{jk} = -\frac{1}{3}\Big(g_{jk}\nabla_m R - g_{jl}\nabla_m R\Big). \tag{65}$$

Further contracting with g^{jk}, we derive

$$\nabla_m R = 0. \tag{66}$$

Thus, the scalar curvature R must be constant. Substituting this into Equation (65), we derive

$$\nabla_m R_{jk} = 0. \tag{67}$$

From the Bianchi identity, we have

$$\nabla^m R_{mjkl} = \nabla_k R_{jl} - \nabla_l R_{jk}. \tag{68}$$

Since $\nabla_m W_{ijkl} = 0$, it follows that $\nabla_k R_{jl} - \nabla_l R_{jk} = 0$, which implies that $\nabla_m R_{jk} = 0$, that is, the Ricci tensor is parallel.

It is well known that in a spacetime where the Ricci tensor is parallel, it must be proportional to the metric tensor

$$R_{jk} = \frac{R}{4} g_{jk}. \tag{69}$$

Thus, the spacetime must be an Einstein spacetime.

Now, suppose the spacetime is an Einstein spacetime, that is, R_{jk} is proportional to g_{jk} and, further, that $\nabla_m R_{jk} = 0$. Taking the covariant derivative of the W_3 curvature tensor, we have

$$\nabla_m W_{ijkl} = \nabla_m R_{ijkl} + \frac{1}{3}(g_{jk} \nabla_m R_{il} - g_{jl} \nabla_m R_{ik}). \tag{70}$$

Since $\nabla_m R_{jk} = 0$, it follows that $\nabla_m R_{ijkl} = 0$ and, thus,

$$\nabla_m W_{ijkl} = 0. \tag{71}$$

This shows that the Einstein condition, along with the parallel Ricci tensor condition, is sufficient for the W_3 curvature tensor to be parallel.

Thus, we can state the following:

Theorem 12. *Let $(\mathcal{M}, g)$ be a 4-dimensional spacetime in $f(R, G)$ gravity. The W_3 curvature tensor is parallel if and only if the spacetime is an Einstein spacetime and the Ricci tensor is parallel.*

6. Example

To investigate black hole solutions using the W_3 curvature tensor in $f(R, G)$ gravity, we systematically derive the geometric properties of a well-known metric. In this example, we consider the Schwarzschild metric [32], which is expressed as

$$ds^2 = g_{ij} dY_1^i dY_1^j = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right),$$

where G is the gravitational constant, M is the mass of the black hole, r is the radial coordinate, and (r, θ, ϕ) are spherical coordinates.

The covariant and contravariant components of the metric are as follows:

$$g_{11} = -\left(1 - \frac{2M}{r}\right), \quad g_{22} = \left(1 - \frac{2M}{r}\right)^{-1}, \quad g_{33} = r^2, \quad g_{44} = r^2 \sin^2\theta,$$

$$g^{11} = -\left(1 - \frac{2M}{r}\right)^{-1}, \quad g^{22} = \left(1 - \frac{2M}{r}\right), \quad g^{33} = \frac{1}{r^2}, \quad g^{44} = \frac{1}{r^2 \sin^2\theta}.$$

The only non-zero components of the Christoffel symbols are as follows:

$$\Gamma^t_{rt} = \frac{M}{r(r-2M)}, \quad \Gamma^r_{tt} = \frac{M(r-2M)}{r^3}, \quad \Gamma^r_{rr} = -\frac{M}{r(r-2M)},$$
$$\Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\phi\theta} = \cot\theta.$$

The Riemann curvature tensor is given by

$$R^i_{jkl} = \partial_j \Gamma^i_{kl} - \partial_k \Gamma^i_{jl} + \Gamma^i_{jm} \Gamma^m_{kl} - \Gamma^i_{km} \Gamma^m_{jl}. \tag{72}$$

The nonzero components of the Riemann curvature tensor for the Schwarzschild metric are given by

$$R^r_{trt} = \frac{2M}{r^3}, \quad R^\theta_{r\theta r} = -\frac{M}{r(r-2M)}, \quad R^\phi_{\theta\phi\theta} = \frac{2M}{r^3}\sin^2\theta. \tag{73}$$

The Ricci tensor is derived by performing a contraction of the Riemann curvature tensor:

$$R_{ij} = R^k_{ikj}. \tag{74}$$

For the Schwarzschild metric,

$$R_{ij} = 0. \tag{75}$$

Since the Ricci tensor vanishes, the Ricci scalar is also zero:

$$R = g^{ij}R_{ij} = 0. \tag{76}$$

This implies that the black hole solution is a Ricci-flat spacetime, satisfying Einstein's vacuum field equations.

The Ricci scalar is derived by performing the contraction of the Ricci tensor:

$$R = g^{ij}R_{ij}. \tag{77}$$

Since $R_{ij} = 0$, we can conclude that

$$R = 0. \tag{78}$$

This indicates that Schwarzschild black hole solutions have zero scalar curvature.

The W_3 curvature tensor is defined as

$$W_{ijkl} = R_{ijkl} + \frac{1}{3}\Big(g_{jk}R_{il} - g_{jl}R_{ik}\Big). \tag{79}$$

Since the Schwarzschild spacetime is Ricci-flat ($R_{ij} = 0$), the W_3 curvature tensor simplifies to

$$W_{ijkl} = R_{ijkl}. \tag{80}$$

Thus, the Schwarzschild metric satisfies the W_3 curvature condition trivially as there are no additional contributions from the Ricci tensor.

For a W_3 flat spacetime, the tensor should satisfy

$$W_{ijkl} = 0. \tag{81}$$

However, since the Schwarzschild black hole solution has a non-zero Riemann curvature tensor, it is not W_3 flat. Instead, we find

$$R_{ij} = 0 \Rightarrow W_{ijkl} = R_{ijkl}. \tag{82}$$

Thus, the Schwarzschild black hole solution is a vacuum solution to Einstein's equations and does not satisfy W_3 flatness. However, since it is Ricci-flat, the W_3 curvature tensor reduces directly to the Riemann curvature tensor, demonstrating that the W_3 curvature tensor effectively generalizes the geometric structure of black hole solutions in modified gravity theories.

7. Application: Black Hole Thermodynamics Using the W_3 Curvature Tensor

In this section, we use the W_3 curvature tensor to study the Hawking temperature and entropy of a black hole, demonstrating how curvature corrections influence black hole thermodynamics.

7.1. Background and Fundamental Laws of Black Hole Thermodynamics

In classical general relativity, a black hole is characterized by three fundamental parameters: its mass M, charge Q, and angular momentum J. The formulation of black hole thermodynamics was pioneered by Bardeen, Carter, and Hawking in their seminal work [33,34], where they established the fundamental laws governing the thermodynamic behavior of black holes. These laws, analogous to the classical laws of thermodynamics, describe the relationship between black hole properties, including surface gravity, horizon area, and entropy:

- **Zeroth Law:** The surface gravity κ is constant across the event horizon of a stationary black hole.
- **First Law:** This relates the change in black hole mass M to the changes in horizon area A, angular momentum J, and charge Q:

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ. \tag{83}$$

- **Second Law:** The area of the event horizon never decreases, analogous to the entropy in classical thermodynamics:

$$dA \geq 0. \tag{84}$$

- **Third Law:** It is impossible to reduce the surface gravity to zero in a finite number of steps.

The foundational work on black hole thermodynamics owes much to the contributions of Hawking, Bardeen, and Carter. Hawking's 1975 discovery of black hole radiation [34] demonstrated that black holes emit thermal radiation due to quantum effects near the event horizon, a phenomenon now known as Hawking radiation. This led to the identification of a black hole's temperature, proportional to its surface gravity. Bardeen and Carter, alongside Hawking, established the analogy between black hole mechanics and classical thermodynamics, formalizing the four laws mentioned above. Their work laid the groundwork for understanding black holes as thermodynamic systems, bridging general relativity and quantum mechanics.

Hawking's theory predicts that black holes emit thermal radiation due to quantum effects, with a temperature given by

$$T_H = \frac{\kappa}{2\pi}, \tag{85}$$

where κ is the surface gravity of the black hole. This result arises from quantum field theory in curved spacetime, where virtual particle–antiparticle pairs near the horizon can result in one particle falling into the black hole while the other escapes, effectively reducing the black hole's mass and emitting radiation.

7.2. Black Hole Thermodynamics in $f(R,G)$ Gravity

In classical general relativity, black holes are characterized by their mass M, charge Q, and angular momentum J. The black hole spacetime is described by the Schwarzschild metric:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{86}$$

where

$$f(r) = 1 - \frac{2M}{r}. \tag{87}$$

The surface gravity at the event horizon $r = r_h$ is given by

$$\kappa = \frac{1}{2}\frac{df(r)}{dr}\Big|_{r=r_h} = \frac{M}{r_h^2}. \tag{88}$$

The corresponding Hawking temperature is

$$T_H = \frac{\kappa}{2\pi} = \frac{M}{2\pi r_h^2}. \tag{89}$$

7.3. Modifications Due to the W_3 Curvature Tensor

In modified gravity, the curvature of spacetime is influenced by higher-order corrections from the W_3 curvature tensor, defined as

$$W_{ijkl} = R_{ijkl} + \frac{1}{3}\Big(g_{jk}R_{il} - g_{jl}R_{ik}\Big). \tag{90}$$

If the black hole spacetime satisfies the W_3 curvature flat condition, the Ricci tensor becomes proportional to the metric tensor:

$$R_{ij} = \frac{R}{4}g_{ij}. \tag{91}$$

For a static, spherically symmetric black hole, the modified metric function is

$$f(r) = 1 - \frac{2M_{eff}}{r}, \quad M_{eff} = M + \frac{R}{12}r^3. \tag{92}$$

The surface gravity in this scenario becomes

$$\kappa = \frac{M_{eff}}{r_h^2} = \frac{M}{r_h^2} + \frac{R}{4}. \tag{93}$$

Thus, the modified Hawking temperature is

$$T_H^{W_3} = \frac{M}{2\pi r_h^2} + \frac{R}{8\pi}. \tag{94}$$

The entropy of a black hole is governed by the Bekenstein–Hawking formula:

$$S = \frac{A}{4}, \tag{95}$$

where $A = 4\pi r_h^2$ is the horizon area. With the W_3 curvature tensor corrections, the entropy is modified as

$$S_{W_3} = \frac{A}{4} + \alpha R A, \tag{96}$$

where α is a proportionality constant determined by the gravitational model. This result suggests that the entropy of a black hole increases in the presence of the W_3 curvature tensor.

7.4. Key Results and Implications

Using the W_3 curvature tensor, we demonstrated how black hole thermodynamics is modified in $f(R,G)$ gravity. The key results include the following:

(i) An increase in the Hawking temperature, implying that black holes evaporate faster.
(ii) A correction to the black hole entropy, indicating higher information storage capacity.
(iii) Potential implications for the information paradox and the nature of quantum gravity.

This approach bridges the gap between classical general relativity and quantum gravity, offering a deeper understanding of black hole thermodynamics in the context of modified curvature theories.

8. Conclusions

In this investigation, we explored the geometric and physical implications of the W_3 curvature tensor in the framework of $f(R,G)$ gravity, yielding significant insights into modified gravity theories and their cosmological and astrophysical applications. First, we established that 4-dimensional W_3 curvature flat spacetimes with constant scalar curvature are either de Sitter ($R > 0$) or Anti-de Sitter ($R < 0$) models, demonstrating their relevance to cosmological frameworks.

Next, we examined isotropic spacetimes within $f(R,G)$ gravity, formulating expressions for both isotropic pressure and energy density and demonstrating that such spacetimes exhibit characteristics consistent with dark energy, as evidenced by the relation $p^{(m)} + \sigma^{(m)} = 0$. The energy conditions, including the null, weak, dominant, and strong conditions were also verified. We also investigated spacetimes with a divergence-free W_3 curvature tensor, proving that they possess a constant scalar curvature and establishing their equivalence to Ricci semi-symmetric spacetimes when the energy–momentum tensor is semi-symmetric.

Finally, we applied these findings to black hole thermodynamics within $f(R,G)$ gravity, demonstrating that W_3 curvature corrections lead to an enhanced Hawking temperature and modified entropy. These modifications suggest faster black hole evaporation and increased information capacity, providing a novel perspective on the interplay between modified gravity and quantum effects.

Author Contributions: Conceptualization, M.A. and M.V.; methodology, M.A., M.V. and M.A.K.; investigation, M.A., M.V. and M.A.K.; writing—original draft preparation, M.A., M.V. and M.A.K.; writing—review and editing, M.V., M.A. and M.A.K. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2502).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors extend their sincere gratitude to the editor and the anonymous referees for their insightful and constructive comments, which greatly contributed to improving the quality of this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

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