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Parity and Time Reversal Violation in Two Nucleons Systems

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Chapter 1

Introduction

The understanding of the violations of discrete symmetries as parity (P), charge conjugation (C) and time reversal (T) presents interesting aspect and it is far to be complete still nowadays. Many experiments, starting from Madame Wu's experiment, have confirmed the presence of "parity violation" (PV) in weak interaction accompanied by a corresponding C-violation, combined so that to have conservation of the CP symmetry. These effects are well described in the Standard Model (SM) via fermion currents having both vector and axial components. More complicated and more recent experiments have confirmed the presence of tiny "parity violating" and "C-conserving" terms. From the CPT theorem these terms must be "T-violating" (TV through this work). This violation is one of the ingredients needed to explain why the visible universe seems to be made predominantly of matter, without a significant fraction of antimatter [1]. The violation observed in K and B decays are well described by the complex phase in the Cabibbo-Kobaiashi-Maskawa (CKM) matrix, the mixing matrix of the quarks [2]. However the CKM phase gives very small contributions to observables that do not involve flavour change between the initial and final states. In particular it is insuffcent to account for the observed matter-antimatter asymmetry [3]. Other TV terms can be introduced in the foundamental Lagrangian of the SM: the one we are most interested in, it is the so called θ-term in the quantum chromodynamics (QCD) sector. The interaction term is estimated in terms of gluon fields and it has a strength given by a parameter $\overline{\theta}$ (further terms involving other gauge bosons should play a neglegible role in hadron physics) [4]. It contributes directly to the permanent electric dipole moments (EDM) of the neutron $d_n = \overline{\theta} \cdot (2.7 \pm 1.2) \cdot 10^{-16}\, e$ cm [5]. From the current experimental estimate of the neutron EDM, however the value of the $\overline{\theta}$ angle is estimated to be $\lesssim 10^{-10}$ [6]. The presence in the SM of a so small constant is the so called "strong CP problem" related also to the existence of the axions [7]. TV could originate also from further terms beyond the standard model (BSM), as complex phase(s) in the the Pontecorvo-Naki-Makagawa-Sakata (PNMS) matrix, the leptons mixing matrix counterpart of the CKM or also from more exotic mechanisms [6]. Therefore any measurement of a non-vanishing TV effect above the CKM "background" would be a signal of new physics.

In this respect the light nuclei could be good laboratories to study the violation of these discrete symmetries. The PV and TV interactions contribute also to the parity conserving (PC) nucleon-nucleon (NN) potential via second order effects. This contribution is obviously totally "hidden" by the strong and electromagnetic interactions and it is therefore not accesible experimentally. However, the PV and TV effects can be highlighted considering physical observables that would be zero if the nuclear interaction had only the standard strong interaction and the electromagnetic interaction. These studies represent also a new "window" to have information on the properties of the systems composed by light quarks and gluons at low energies, and so in a highly non-perturbative regime.

PV effects in low-energy hadronic processes have been measured in few experiments. Finite signals were obtained in proton-proton (pp) and $p\alpha$ scattering, radiative decays of ^{19}F, and other experiments. Strong upper bounds are found in radiative np capture, radiative decays of ^{18}F as well as for the spin rotation of a polarized proton beam moving through a ^{4}He gas [8]. Other experiments at ultra-cold neutron facilities are being completed or are in advanced stage. The interpretation of the experiments that involve medium-heavy nuclei is difficult due to the complexity of the structure of these systems. For this reason most of the new experiments to study PV effects involve light nuclei, where the calculation of the nuclear structure is under control. The main objective of these experiments is to determine the constants entering the PV NN potential.

The search of TV observables is another hot topic in modern Physics. The most studied observable is the EDM, which require both P and T violation. The current upper bounds on the neutron and proton EDMs are $|d_n| < 2.9 \cdot 10^{-13}e$ fm [9] and $|d_p| < 7.9 \cdot 10^{-12}e$ fm [10], respectively, where the upper bound on the proton EDM has been inferred from a measurement of the diamagnetic ^{199}Hg atom. In general a single EDM measurement will not be sufficent to identify the source of TV and new experiments to measure the EDMs of light nuclei directly at dedicated storage rings with an accurancy of $\sim 10^{-16}e$ fm have been proposed [11]. Other observables could test the TV effects in light nuclei framework, as for example, from measurement of particular polarized neutron-polarized nucleus forward scattering amplitudes [12]. In particular, in this work we will focus on the rotation of the spin of polarized neutrons along the y axis [13, 14] which can be used as an unambiguous TV observable.

A systematic description of PV and TV effects at nuclear level can be obtained using the so-called chiral effective field theory (χEFT), the low energy effective field theory of the QCD [15]. The χEFT approach is based on the observation that the restrictions imposed by chiral symmetry in QCD has a noticeable impact in the low energy regime [16, 17]. In particular the form of the interactions among the nucleons will have tight constrains due to this symmetry. This method has put the nuclear physics on a more foundamental basis by provinding a direct connection between the QCD symmetries and the strong and electroweak interaction in nuclei and also a systematic scheme to construct the interactions, the so-called chiral perturbation theory (ChPT) (see, for example, the review papers [18, 19]). This method allows to order the different contributions as an expansion on Q/Λ_χ where Q is the energy scale of the nuclear processes ($\sim 10 \div 100$ MeV) and $\Lambda_\chi \simeq 1$ GeV specifies the symmetry-breaking scale.

There are different versions of the χEFT:

- the "pionless" theory: it assumes that the pionic degrees of freedom are freezed and all the interactions reduces to nucleon-nucleon contact terms (see for example [20]). Such a theory is valid at energy much lower than the pion mass. For example it has been used to study PV effects in low energy NN scattering, and in ultracold neutron experiments as PV asymmetries in $^1\mathrm{H}(\vec{n},\gamma)^2\mathrm{H}$ capture, spin rotation $\vec{n}p$ and $\vec{n}d$ scattering, as well as other observables [21];

- the "pionfull theory: both nucleons and pions are considered as dynamical degrees of freedom. The energy range of the validity of this approach extends up to energy of the order of the pion mass. This theory has been found to work well in the PC sector [22, 23]. It is reasonable that the same approach will also work for the PV and TV interactions. We will use this approach in this work.

To each term of the nuclear Lagrangian is associated a low energy constant (LEC) that takes into account the high energy physics. Usually these LECs are fixed from observable experimental data but they can be also extimated using Lattice QCD, or other non perturbative methods in terms of the parameters of the SM, or of a for more fundamental theory. One of the aim of these studies of Nuclear Physics is actually devoted to the determination of the parameters of the fundamental theory, using experimental data taken in low-energy nuclear processes. To achieve this result one needs to take into account the dynamics of the nucleons and to obtain the LECs from present or future experimental data. This is the aim of the present Thesis, where we focus on PV and TV observables which can be measured in the study of the scattering of two-nucleons. In a successive step, further studies will be required to relate the LECs to the parameters of the fundamental theory using Lattice QCD or other methods (as the renormalization group method), see, for example, Ref. [5] for a discussion of the methods used to study the TV interaction.

In most of the works in literature, the calculation of the observables goes through a sort of non-relativistic expansion of the nucleon field entering the Lagrangian, the so called"heavy barion chiral perturbation theory" (HBChPT) [24].

This expansion is obtained dividing the nucleon field in a "light" and a "heavy" part in order to isolate the mass term in the nucleon four-momentum, because it is of the same magnitude of our symmetry-breaking scale Λ_χ. In this work, we will not use this method but we will work with the "normal" nucleon field; terms proportional to the nucleon mass will be subtracted explicitly.

Many derivations of the *PV* potential agree that this potential include, up to next-to-next-to leading order (N2LO), a long range one pion exchange (OPE) component at the leading order (LO), a medium range component originating from two pion exchanges (TPE), and five indipendent *NN* contact terms [25]. The *PV* potential at N2LO included in total 6 LECs, the pion-nucleon *PV* coupling constant and 5 LECs coming from the contact terms. This version of the potential was used in particular to study the longitudinal analyzing power (A_z) in $\vec{p}p$ scattering. However for this observable the LO contribution vanishes and therefore possible corrections that come from the next-to-next-to-next leading order (N3LO) could be very important and not negligible. The price that we pay introducing the N3LO is to add five new LECs to the six that comes at N2LO. Therefore two of the aims of the present work are:

- to derive the N3LO component of the *PV* potential between two nucleons (a first derivation of this potential was alredy given by de Vries et *al.* [26]);
- to study the $\vec{p}p$ A_z and the $\vec{n}p$ spin rotation observables with the new potential in order to investigate the effect of the N3LO components.

The *TV* potential have been derived using χEFT in a few works up to now [6]. Up to the next-to-leading order it includes a OPE terms and a three pions exchange due to a three-pion interaction vertex in the *TV* Lagrangian. It is possible to build also two contact terms which formally belongs to the N2LO. In total we have five LECs directly connected with the $\overline{\theta}$ angle. So the other two aims of this Thesis are:

- to derive the *TV* potential between two nucleons up to NLO using our framework. This potential will depend on 5 LECs;
- to study a different observable from the classical EDM. In particular we will focus on the $\vec{n}\vec{p}$ spin rotation in order to fix some LECs entering the *TV* potential.

This work will be organized as follows. In Chapter 2 we will introduce the chiral symmetry. Then we will discuss how we can build the χEFT Lagrangian with only nucleons and pions as degrees of freedom. The *PV* terms will be constructed so that, under chiral transformations, they transform under chiral symmetry as the *PV* weak interaction terms involving the quarks u and d in the SM.

In Chapter 3 we will explain how to build a potential starting from the χEFT Lagrangian. In particular we will discuss how to construct the T-matrix using the "time ordered perturbation theory" order by order in the Q/Λ_χ expansion and then how to derive the potential. The derivation of the PV NN potential up to the N3LO with all the time-ordered diagrams will be discussed in Chapter 4.

The TV interaction will be discussed in Chapter 5. We will briefly explain how the θ-term can be included in the mass term of the QCD Lagrangian via a $U(1)_A$ rotation and then incorporated in the χEFT. In the last section of the chapter we will present the explicit derivation of the TV potential up to NLO.

In Chapter 6 we will explain how to calculate the observables like A_z for $\vec{p}p$ elastic scattering and $\vec{n}p$ spin rotation angle. After an introduction on the two body scattering problem, we will present an algorithm that allows to solve the Schrödinger equation with a PC plus a PV or TV potential. Then we will introduce the M-matrix formalism in order to calculate the observables of interest. In particular we will discuss the A_z observable in $\vec{p}p$ scattering and the spin rotation in $\vec{n}p$ scattering to reveal PV effects. Moreover, we will study the $\vec{n}\vec{p}$ spin rotation along the y-axis to reveal TV effects. In Chapter 7 we will present the results concerning the studied observables and in Chapter 8 we will discuss the conclusions and perspectives of the present work. Finally, a number of technical details will be given in several Appendices.

Chapter 2

The chiral symmetry

In this Chapter we will discuss how to construct a theory of nuclear forces using the constraints of the symmetries (in some cases approximates) of the fundamental theory of the strong interactions of quarks, the QCD. In particular, we will show that the QCD Lagrangian is almost symmetric under the so-called chiral symmetry. Then we will see how it is possible to build an effective field theory with pions and nucleons as degrees of freedom, being as well almost symmetric under chiral symmetry. In this way it has been possible in the last years to construct an accurate theory of the PC strong interaction between nucleons. Inspecting the transformation properties of the interaction terms of the weak interaction in the SM, in particular the terms that are PV, we will introduce in our χEFT new terms that transform in the same way and are PV. These Lagrangian terms will be our starting point to build the PV NN potential. The construction of the TV interaction will be addressed in Chapter 5.

The Chapter is organized as follows. In Section 2.1 we will introduce the chiral symmetry as an approximate symmetry of QCD and we will explain its properties. In Section 2.2 we will present the ideas on which the χEFT is based. We will present the PC Lagrangian in Section 2.3 and 2.4 for the pions and the nucleons, respectively. In the last Section we will analyse the structure of the weak interaction among the quarks and then, focusing on the PV interaction terms, we will build the PV EFT Lagrangian.

2.1 The chiral symmetry in QCD

Let's consider the standard QCD Lagrangian density. Our aim is to obtain a theory that describes processes of two nucleons at low energy (under the threshold of the pion production), thus we only consider the lighter quarks u and d [27]:

$$\mathcal{L}_{\rm QCD} = \sum_{f=u,d} \overline{q}_f(x)\big(i\gamma^\mu D_\mu - m_f\big)q_f(x) - \frac{1}{4}\mathcal{G}_{\mu\nu,a}(x)\mathcal{G}^{\mu\nu}_a(x)$$

$$\equiv \ \overline{q}(x)\big(i\gamma^\mu D_\mu - \mathcal{M}\big)q(x) - \frac{1}{4}\mathcal{G}_{\mu\nu,a}(x)\mathcal{G}_a^{\mu\nu}(x), \tag{2.1}$$

where

$$q(x) = \begin{pmatrix} q_u(x) \\ q_d(x) \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}, \tag{2.2}$$

q_u, q_d being the fields of the quarks, m_u, m_d their masses, $D_\mu = \partial_\mu - ig\sum_{a=1}^{8} T_a G_{\mu,a}(x)$ is the covariant derivative of quark fields, $G_{\mu,a}(x)$ the gluon fields, and $2T_a$ are the Gell-Mann matrices, that are the generators of the color gauge group $SU(3)$ in the fundamental representation. Above, $\mathcal{G}_a^{\mu\nu}(x)$ are the force tensor of the gluonic fields,

$$\mathcal{G}_a^{\mu\nu}(x) = \partial^\mu G_a^\nu(x) - \partial^\nu G_a^\mu(x) + g f_{abc} G_b^\mu(x) G_c^\nu(x), \tag{2.3}$$

with f_{abc} structure constants of the gauge group $SU(3)$, defined by the commutation properties of the Gell-Mann matrices

$$[T_a, T_b] = i f_{abc} T_c. \tag{2.4}$$

The quark fields $q(x)$ are actually vectors in the color space and the $SU(3)$ Gell-Mann matrices act on them. The Lagrangian in Eq. (2.1) is invariant under local transformations of $SU(3)$ on the color degrees of freedom of the quark and gluon fields as [28]

$$q_f(x) \to q'_f(x) = U(\theta(x)) q_f(x), \tag{2.5}$$

$$G_{\mu a}(x)T_a \to G'_\mu = U(\theta(x)) G_{\mu a} T_a(x) U(\theta(x))^\dagger + \frac{i}{g}\big(\partial_\mu U(\theta(x))\big)U^\dagger(\theta(x)), \tag{2.6}$$

where

$$U(\theta(x)) = e^{-i\sum_{a=1}^{8}\theta_a(x)T_a}. \tag{2.7}$$

The QCD Lagrangian has a $U(1)_u \otimes U(1)_d$ (global) symmetry where the field of each flavour transforms as

$$q_f(x) \to q'_f(x) = e^{-i\theta_f} q_f(x). \tag{2.8}$$

From the above symmetry it follows the conservation of the flavour number and also, assigning the values $1/3$ to quarks and $-1/3$ to the antiquarks, the conservation of the baryon number.

If we neglect the mass term (the so-called chiral limit), the $U(1)_u \otimes U(1)_d$ symmetry group can be extended. Let's rewrite the Lagrangian in Eq. (2.1) in terms of the left and right spinors of the quarks:

$$q_R(x) = \frac{1+\gamma^5}{2} q(x) = \begin{pmatrix} q_{u,R}(x) \\ q_{d,R}(x) \end{pmatrix}, \tag{2.9a}$$

$$q_L(x) = \frac{1-\gamma^5}{2} q(x) = \begin{pmatrix} q_{u,L}(x) \\ q_{d,L}(x) \end{pmatrix}, \tag{2.9b}$$

we find

$$\mathcal{L}_{\text{QCD}}^{\mathcal{M}=0} = \bar{q}_L(x)i\gamma^\mu D_\mu q_L(x) + \bar{q}_R(x)i\gamma^\mu D_\mu q_R(x) - \frac{1}{4}\mathcal{G}_{\mu\nu,a}(x)\mathcal{G}_a^{\mu\nu}(x). \quad (2.10)$$

We can consider the chiral limit because the quarks' masses are much smaller than the hadronic mass (~ 1 GeV) [29]

$$m_u \simeq 2.3 \text{ MeV}, \ \ m_d \simeq 4.8 \text{ MeV}. \quad (2.11)$$

The Lagrangian in Eq. (2.10) is invariant under global transformations that transform the left and the right components independently. The complete group of transformations is $G = U(1)_R \otimes U(1)_L \otimes SU(2)_R \otimes SU(2)_L$, corresponding to

$$U(1)_R : q_R(x) \to q_R'(x) = e^{-i\Theta_R} q_R(x), \quad (2.12a)$$
$$U(1)_L : q_L(x) \to q_L'(x) = e^{-i\Theta_L} q_L(x), \quad (2.12b)$$
$$SU(2)_R : q_R(x) \to q_R'(x) = e^{-i\vec{\epsilon}_R\cdot\vec{\tau}/2} q_R(x) \equiv R q_R(x), \quad (2.12c)$$
$$SU(2)_L : q_L(x) \to q_L'(x) = e^{-i\vec{\epsilon}_L\cdot\vec{\tau}/2} q_L(x) \equiv L q_L(x), \quad (2.12d)$$

where with $\vec{\tau}$ we indicate a vector that has as components the Pauli's matrices τ_a with $a = x, y, z$. This group is isomorphic to the group $U(1)_V \otimes U(1)_A \otimes SU(2)_V \otimes SU(2)_A$:

$$U(1)_V : q(x) \to q'(x) = e^{-i\Theta_V} q(x), \quad (2.13a)$$
$$U(1)_A : q(x) \to q'(x) = e^{-i\gamma^5\Theta_A} q(x), \quad (2.13b)$$
$$SU(2)_V : q(x) \to q'(x) = e^{-i\vec{\epsilon}_V\cdot\vec{\tau}/2} q(x) \equiv V q(x), \quad (2.13c)$$
$$SU(2)_A : q(x) \to q'(x) = e^{-i\gamma^5\vec{\epsilon}_A\cdot\vec{\tau}/2} q(x) \equiv A q(x). \quad (2.13d)$$

The transformations $SU(2)_V$ are obtained from $SU(2)_L \otimes SU(2)_R$ choosing $L = R = V$, while imposing $L = R^\dagger = A$ we get $SU(2)_A$. In the same way, if we take the same rotation angle in the transformation $U(1)_R \otimes U(1)_L$ we have $U(1)_V$, while if we take opposite angles we have $U(1)_A$.

Using the Noether's theorem for each transformation we get the following currents [28]

$$J^\mu(x) = \bar{q}(x)\gamma^\mu q(x), \quad (2.14a)$$
$$J_5^\mu(x) = \bar{q}(x)\gamma^\mu\gamma^5 q(x), \quad (2.14b)$$
$$V_a^\mu(x) = \bar{q}(x)\gamma^\mu \frac{\tau_a}{2} q(x), \quad (2.14c)$$
$$A_a^\mu(x) = \bar{q}(x)\gamma^\mu\gamma^5 \frac{\tau_a}{2} q(x). \quad (2.14d)$$

Calculating the derivative of the currents and using the classical equations of motion, not neglecting the mass matrix yet, one obtains

$$\partial_\mu J^\mu(x) = 0, \tag{2.15a}$$

$$\partial_\mu J_5^\mu(x) = 2i\overline{q}(x)\gamma^5\mathcal{M}q(x), \tag{2.15b}$$

$$\partial_\mu V_a^\mu(x) = i\overline{q}(x)\Big[\mathcal{M}, \frac{\tau_a}{2}\Big]q(x), \tag{2.15c}$$

$$\partial_\mu A_a^\mu(x) = i\overline{q}(x)\Big\{\mathcal{M}, \frac{\tau_a}{2}\Big\}q(x). \tag{2.15d}$$

As we expect, the first current is always conserved, since it represents the conservation of the baryon number as confirmed by the experiments. The other three currents are classically conserved in the chiral limit ($\mathcal{M} = 0$). The requirement for the conservation of the current $V_\mu^a(x)$ is however less restrictive: we need only that the masses of the two quarks were equals, namely $\mathcal{M} \propto \mathbb{1}$. This corresponds to the isospin symmetry known to be reather well verified in Nature. If $\mathcal{M} = 0$ we would have $\partial_\mu J_5^\mu = \partial_\mu A^\mu = 0$. It is well-known that the $U(1)_A$ symmetry is broken at quantum level by a quantum effect (anomaly) [28]. We will treat the symmetry $U(1)_A$ later, for the moment we limit our discussion to the chiral group:

$$G_\chi \equiv U_V(1) \otimes SU_V(2) \otimes SU_A(2). \tag{2.16}$$

The Lagrangian $\mathcal{L}_{\rm QCD}^{\mathcal{M}=0}$ given in Eq. (2.10) is invariant under G_χ.

As we have seen, the chiral symmetry is a global symmetry but it is useful to upgrade it to a local one; in this way we can couple the quarks with external source fields. In particular we can introduce the coupling with external Hermitean isoscalar and isovector currents

$$v_\mu^{(s)}(x), \qquad v_\mu(x) = \sum_{a=x,y,z} \frac{\tau_a}{2} v_\mu^a(x), \qquad a_\mu(x) = \sum_{a=x,y,z} \frac{\tau_a}{2} a_\mu^a(x). \tag{2.17}$$

We introduce also the couplings with scalar and pseudoscalar density,

$$s(x) = \sum_{a=0}^{3} \tau_a s^a(x), \qquad p(x) = \sum_{a=0}^{3} \tau_a p^a(x), \tag{2.18}$$

which are Hermitean matrices in isospin space and $\tau_0 \equiv \mathbb{1}$. The QCD Lagrangian with the external fields reads [28]

$$\begin{aligned} \mathcal{L}_{\rm QCD}^{\rm EXT} \;=\; & \mathcal{L}_{\rm QCD}^{\mathcal{M}=0} + \overline{q}(x)\gamma^\mu\big(v_\mu(x) + \frac{1}{3}v_\mu^{(s)}(x) + \gamma^5 a_\mu(x)\big)q(x) \\ & -\overline{q}(x)\big(s(x) - i\gamma^5 p(x)\big)q(x). \end{aligned} \tag{2.19}$$

Imposing that the Lagrangian be invariant under the local transformations induced by the group G_χ, we deduce the transformation properties of the external fields. Defining,

$$r_\mu(x) = v_\mu(x) + a_\mu(x), \qquad l_\mu(x) = v_\mu(x) - a_\mu(x), \tag{2.20}$$

and rewriting the Lagrangian in terms of the left and right components of the quark fields

$$\begin{aligned}\mathcal{L}_{\rm QCD}^{\rm EXT} &= \mathcal{L}_{\rm QCD}^{\mathcal{M}=0} + \overline{q}_L(x)\gamma^\mu\big(l_\mu(x) + \frac{1}{3}v_\mu^{(s)}(x)\big)q_L(x) + \overline{q}_R(x)\gamma^\mu\big(r_\mu(x) + \frac{1}{3}v_\mu^{(s)}(x)\big)q_R(x) \\ &\quad -\overline{q}_R(x)\big(s(x)+ip(x)\big)q_L(x) - \overline{q}_L(x)\big(s(x)-ip(x)\big)q_R(x), \end{aligned} \tag{2.21}$$

we find the transformation properties of the external source fields

$$r_\mu(x) \to R(x)r_\mu(x)R^\dagger(x) + iR(x)\partial_\mu R^\dagger(x), \tag{2.22a}$$

$$l_\mu(x) \to L(x)l_\mu(x)L^\dagger(x) + iL(x)\partial_\mu L^\dagger(x), \tag{2.22b}$$

$$v_\mu^{(s)}(x) \to v_\mu^{(s)}(x) - \partial_\mu\Theta(x), \tag{2.22c}$$

$$s(x)+ip(x) \to R(x)(s(x)+ip(x))L^\dagger(x), \tag{2.22d}$$

$$s(x)-ip(x) \to L(x)(s(x)-ip(x))R^\dagger(x)\ , \tag{2.22e}$$

with $\Theta = 3\Theta_V$.

For example, the vector and axial source fields are useful to take into account the coupling of the quarks with the electromagnetic field or the $W^\pm$ and the Z^0 fields. We will not discuss anymore the coupling with these vector fields. More interesting for this Thesis it is the coupling with the scalar fields $s(x)$ and $p(x)$ which we will use to introduce in the nucleon Lagrangian the mass and the TV terms. The procedure is the following [5]. i) First we assume that the source fields transform as in Eqs. (2.22) and we construct $\mathcal{L}_{\rm QCD}^{\rm EXT}$ to be completely symmetric under the G_χ transformations. ii)As a second step, we can include in the source fields, terms which are not invariant, as for example the mass terms (see below). When we will perform the same procedure for the nuclear Lagrangian $\mathcal{L}_{\rm EFT}$, we obtain automatically all terms which violates chiral symmetry in the same ways as in the quark Lagrangian $\mathcal{L}_{\rm QCD}^{\rm EXT}$.

For example rewriting the mass matrix as

$$\mathcal{M} = \overline{m}\mathbb{1} + \epsilon\overline{m}\tau_3\ , \tag{2.23}$$

where

$$\overline{m} = \frac{m_u+m_d}{2}, \qquad \epsilon = \frac{m_u-m_d}{m_u+m_d}, \tag{2.24}$$

it is easy to identify $s_0 = \overline{m}$ and $s_3 = \epsilon\overline{m}$. In this way, using the scalar field $s(x)$ it is possible to reintroduce the mass term, explicitly violating the chiral symmetry, in the Lagrangian $\mathcal{L}_{\rm QCD}^{\rm EXT}$. In the first part of this work, in the treatment of the PV interactions we will neglect the isospin violating term s_3 (imposing $\epsilon = 0$) and the pseudoscalar $p(x) = 0$. In the second part we will take into account these terms in order to introduce the TV interaction.

Even if the symmetry $SU(2)_A$ is a symmetry of the chiral Lagrangian $\mathcal{L}_{QCD}^{\mathcal{M}=0}$, it is not realized in the low energy hadronic spectrum. Hadrons show themselves as a multiplet of the isospin subgroup $SU(2)_V$, but not of $SU(2)_A$. If so, we will expect for each hadronic state, another degenerate state (same mass, charge, ...) but with opposite parity. Since no multiplet with different parities is observed, the chiral symmetry suffers a spontaneus breaking with the pattern

$$SU(2)_V \otimes SU(2)_A \otimes U(1)_V \to SU(2)_V \otimes U(1)_V. \tag{2.25}$$

Using the Goldstone theorem we can deduce the properties of the Goldstone bosons [27]:

- the Goldstone bosons are three as the broken generators of the quotient group $SU(2)_A$;
- their mass must be zero;
- they have the same quantum numbers associated with the charge operator of the broken generators, in this case the charge operator of the three non Abelian axial currents in Eq. (2.14d)

$$Q^a_A(t) = \int d^3x q^\dagger(x)\gamma^5 \frac{\tau_a}{2} q(x), \tag{2.26}$$

  so they must be pseudoscalar, their baryonic number must be zero, and they must transform under $SU(2)_V$ as a isospin triplet.

The three Goldstone bosons are identified with the pions. The fact that they have a non zero mass can be explained saying that the chiral symmetry is only an approximate symmetry of the QCD, due to the mass term which explicitly breaks the symmetry.

2.2 The Effective Field Theory

The description of hadrons and their interactions using QCD is very complicated. The interactions of gluons and quarks is very intense because the coupling constant of the theory α_S increases when the energy decreases; therefore each possibility to describe perturbatively the low energy regime fails. In order to solve this problem, it is possible to build an EFT which describes the low energy dynamics of the hadrons and which allows for an expansion in terms of small momentum Q instead of a coupling constant [15]. In this kind of theory the degrees of freedom are the hadrons which are not elementary particles, but considering processes of energies not allowing the excitations of them, they can be treated as elementary constituents.

The symmetries used to build the effective Lagrangian of the χEFT are i) the chiral symmetry, seen in the previous Section, ii) the Lorentz invariance and iii) the discrete symmetries of charge conjugation C and parity P (clearly in this Thesis we are also interested in PV and TV Lagrangians). With this Lagrangians it is possible to treat processes of momenta $Q \ll \Lambda_\chi$, with $\Lambda_\chi \sim 4\pi f_\pi \sim 1$ GeV [30], where $f_\pi \simeq 92.4$ MeV is identified as the charge pions decay constant [31]. If the chiral symmetry was an exact symmetry of the theory, the momentum Q would be the only expansion parameter. As we have seen before, this is not true; the chiral symmetry is explicitly broken by the mass term of the quarks that generates the mass of the pion m_π. This quantity reappears in the EFT as a new expansion parameter. However also m_π is a small parameter compared to Λ_χ, so we have two expansion scales: Q/Λ_χ and m_π/Λ_χ. From now, we will indicate with Q both the typical momentum scale and the mass of the pion. If we limit the range of Q between zero and the mass difference between the baryon $\Delta(1232)$ and the nucleon, we can take as effective degrees of freedom only the pions and the nucleons, without including heavier mesons or barions. Thus we will include explicitly in the scale Λ_χ, all the quantities which are out of our energy range and which represent all those degrees of freedom we consider to be integrated out: Δ, ρ, etc.

In order to build the chiral effective Lagrangian we need to consider all the possible terms compatible with the restrictions imposed by the symmetries [19, 32]. What we get is an infinite number of operators, each of them multiplied by a free parameter. These parameters contain all the underlying high energy physics and they represent our inability to understand it. Fortunately we can associate each Lagrangian term with a power $(Q/\Lambda_\chi)^\nu$, where ν is an integer index named "chiral order". There is a finite number of Lagrangian terms for each chiral order. The terms that appear in the Lagrangian can be organized depending on their index ν. It is one of the nice property related to the chiral symmetry that all pion-nucleons interaction terms includes at least one derivative, so $\nu_{\min}$ is always finite. The terms with index $\nu = \nu_{\min}$ are named "leading order" (LO) terms and usually they bring the largest contribution to the observables. The terms of order $\nu = \nu_{\min} + 1$, named "next-to-leading order" (NLO) terms, give a first correction to the values calculated using the LO terms only, and so on. Therefore the chiral effective Lagrangian can be written as

$$\mathcal{L}_{\rm EFT} = \mathcal{L}_{\rm LO} + \mathcal{L}_{\rm NLO} + \mathcal{L}_{\rm N2LO} + \dots \,. \tag{2.27}$$

The theory build in this way can be renormalized order by order; considering terms up to a given order, the constants that appear in the Lagrangian are used to reabsorb the divergences of the loops diagrams. Removed the divergences, a certain number of renormalized LECs remain. In principle, these LECs can be calculated from the underlying physics, but practically they are fixed by (eventually) available experimental data. At this point we can use the theory to make predictions.

2.3 The ChPT for the pions

In order to describe the pions we must introduce a 2×2 "pionic field" matrix $U(x)$ [28] which belongs to the quotient group $SU(2)_A$ as prescribed by the spontaneous symmetry breaking discussed in Section 2.1. This matrix field is identified by three coordinates that are necessary to parametrize the group $SU(2)$. We have infinite ways to parametrize the matrix field $U(x)$ [30]. Introducing the Hermitean fields of the pion of type $a = x, y, z$, $\pi_a(x)$, see Chapter 3 for their definition, we can use them as coordinates and write the matrix field as

$$U(x) = 1 + \frac{i}{f_\pi}\vec{\tau}\cdot\vec{\pi}(x) - \frac{1}{2f_\pi^2}\,\vec{\pi}^{\,2}(x) - \frac{i\alpha}{f_\pi^3}\,\vec{\pi}^{\,2}(x)\,\vec{\tau}\cdot\vec{\pi}(x) + \frac{8\alpha-1}{8f_\pi^4}\vec{\pi}^{\,4}(x) + \dots\ , \quad (2.28)$$

where α is an arbitrary coefficient reflecting our freedom in the choice of the pion field [30]. Choosing $\alpha = 1/6$ we get the canonical form for the matrix field $U(x)$:

$$U(x) = e^{\frac{i}{f_\pi}\vec{\pi}(x)\cdot\vec{\tau}}. \quad (2.29)$$

The transformation law of the pionic field under the chiral group $SU(2)_R \otimes SU(2)_L$ is given by [28]

$$U(x) \rightarrow U'(x) = RU(x)L^\dagger\ . \quad (2.30)$$

On the other hand the pion fields transform linearly only under the group $SU(2)_V$, indeed taking

$$\begin{cases} L &= A^\dagger V, \\ R &= AV, \end{cases}$$

we can write

$$\begin{aligned} U'(x) &= RU(x)L^\dagger, \\ &= AVU(x)V^\dagger A, \\ &= Ae^{iV\vec{\pi}(x)V^\dagger\cdot\vec{\tau}/f_\pi}A, \end{aligned} \quad (2.31)$$

so considering a pure infinitesimal vector transformation ($A = \mathbb{1}$) and expanding in series, it is easy to find that the fields $\vec{\pi}$ transform linearly (as a isospin triplet). For infinitesimal transformation:

$$\vec{\pi}(x) \rightarrow \vec{\pi}'(x) = \vec{\pi}(x) - \epsilon_V \times \vec{\pi}(x). \quad (2.32)$$

Performing the same thing using a pure axial transformation ($V = \mathbb{1}$) in Eq. (2.31) the transformation is not linear. For infinitesimal transformations:

$$\vec{\pi}(x) \rightarrow \vec{\pi}'(x) = \vec{\pi}(x) + \vec{\epsilon}_A f_\pi. \quad (2.33)$$

as we expect from the spontaneous breaking of the symmetry $SU(2)_A$ and the non vanishing expectation value on the vacuum of the charge in Eq. (2.26). Let's note that the field $U(x)$ is invariant under transformations $U(1)_V$ because, being the baryonic number of the pions zero, we have $\pi_a(x) \to \pi_a(x)$.

As in the quarks case, we can upgrade the symmetry from global to local, considering the coupling with external fields. Given the transformation of field [28]

$$U(x) \to U'(x) = R(x)U(x)L(x)^\dagger \ . \tag{2.34}$$

we can define the covariant derivative

$$D_\mu U(x) = \partial_\mu U(x) - ir_\mu(x)U(x) + il_\mu U(x). \tag{2.35}$$

where the external fields r_μ and l_μ are those introduced in Section 2.1. Taking into account of their transformation laws, given in Eq. (2.22a) and Eq. (2.22b), it is easy to show that $D_\mu U(x)$ transforms as

$$D_\mu U(x) \to R(x)(D_\mu U(x))L^\dagger(x). \tag{2.36}$$

As in the Lagrangian in Eq. (2.19) we can introduce a coupling with the scalar fields $s(x)$ and $p(x)$ via the operators χ and $\chi^\dagger$ defined as

$$\chi(x) = 2B\big(s(x) + ip(x)\big), \tag{2.37a}$$

$$\chi^\dagger(x) = 2B\big(s(x) - ip(x)\big). \tag{2.37b}$$

Assuming that $s(x)$ and $p(x)$ transform as in Eq. (2.22d) and (2.22e) then

$$\chi(x) \to R(x)\chi(x)L^\dagger(x), \tag{2.38}$$

$$\chi^\dagger(x) \to L(x)\chi^\dagger(x)R(x). \tag{2.39}$$

The LEC B is related to the order parameter for the breaking of the chiral symmetry $\langle 0 |\bar{q}q| 0\rangle$ [28]

$$B = -\frac{1}{3f_\pi^2}\langle 0 |\bar{q}q| 0\rangle \, . \tag{2.40}$$

The building blocks for the construction of the Lagrangian has the follow chiral order

$$U \sim O(Q^0), \quad D_{\mu_1}...D_{\mu_n}U \sim O(Q^n), \quad \chi \sim O(Q^2). \tag{2.41}$$

So the most general effective Lagrangian at the order Q^2, invariant under the Lorentz transformations, local chiral transformations, parity and charge conjugation including the source fields is

$$\mathcal{L}_\pi^{(2)} = \frac{f_\pi^2}{4}\left\langle D_\mu U(x)(D^\mu U(x))^\dagger\right\rangle + \frac{f_\pi^2}{4}\left\langle \chi(x)U^\dagger(x) + U(x)\chi^\dagger(x)\right\rangle , \tag{2.42}$$

where $\langle \dots \rangle$ indicates the trace of the matrices. The 7 terms entering the Q^4 order Lagriangian are discussed in Ref. [32], we only report the terms we will use to construct the TV Lagrangian:

$$\mathcal{L}_\pi^{(4)} = \frac{l_3}{16}\langle\chi(x)U^\dagger(x) + U(x)\chi^\dagger(x)\rangle^2 - \frac{l_7}{16}\langle\chi(x)U^\dagger(x) - U(x)\chi^\dagger(x)\rangle^2 + \dots . \tag{2.43}$$

2.4 The ChPT for the nucleons

As in the case of the pions, we can choose the representation of the nucleon which is the simplest to build the Lagrangian, because it is irrelevant when we compute the observables. We will consider the most common representation used in the chiral perturbation theory. In the isospin formalism we denote the nucleonic field as

$$N(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix} \tag{2.44}$$

where $p(x)$ and $n(x)$ are the proton and neutron fields respectively. It is useful to define a matrix $u(x)$ such that $u(x) = \sqrt{U(x)}$ [19, 28]. Remembering the transformation of $U(x)$ under global chiral transformation in Eq. (2.30), the $u(x)$ matrix transform as

$$u(x) \to u'(x) = Ru(x)h^\dagger(x) = h(x)u(x)L^{-1} \tag{2.45}$$

where the function $h(x)$ is defined

$$h(x) \equiv h[L, R, U(x)] = \sqrt{RU(x)L^\dagger}^{-1} R\sqrt{U(x)}, \tag{2.46}$$

which is non linear in the pion field. Neglecting for the moment the subgroup $U(1)_V$, the transformation law of the nucleonic field under the global transformation $SU(2)_V \otimes SU(2)_A$ can be shown to be [28]

$$N(x) \to N'(x) = h[L, R, U(x)]N(x). \tag{2.47}$$

If the transformation is a pure vector transformation $L = R = V$, from the definition of $u(x)$ it follows that $u' = VuV^\dagger$, and so from Eq. (2.46) $h = V$ and the nucleonic field transforms as a isospin doublet. Conversely the axial transformation results to be non linear [28].

In order to build a Lagrangian that contains the interactions among pions, nucleons and external fields, we upgrade the chiral symmetry from global to local, introducing the external field $r_\mu(x)$, $l_\mu(x)$, $v_\mu^{(s)}(x)$, $s(x)$ and $p(x)$ with the transformation properties given in Eq. (2.22a)-(2.22e). Under the local group G_χ the doublet of the nucleons N transforms as

$$N(x) = e^{-i\theta(x)}h[L(x), R(x), U(x)]N(x). \tag{2.48}$$

Let's introduce the covariant derivative for the nucleonic field [19, 28]

$$D_\mu N = (\partial_\mu + \Gamma_\mu - iv_\mu^{(s)})N, \tag{2.49}$$

where Γ_μ is the "connection" given by

$$\Gamma_\mu = \frac{1}{2}[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger]. \tag{2.50}$$

Using the properties transformation of $u(x)$, $r_\mu(x)$ and $l_\mu(x)$ we find the properties transformation of Γ_μ:

$$\Gamma_\mu \to \Gamma'_\mu = h\Gamma_\mu h^\dagger + h\partial_\mu h^\dagger \ , \tag{2.51}$$

in this way when the derivative acts on the field $N(x)$, it transforms covariantly,

$$D_\mu N(x) \to h(x) D_\mu N(x). \tag{2.52}$$

We can introduce other building blocks which has simple transformation properties under G_χ and as well [19, 28]:

$$\begin{aligned}
u_\mu &= i(u^\dagger \partial_\mu u - u\,\partial_\mu u^\dagger) + u^\dagger r_\mu u - u\,\ell_\mu u^\dagger \ , && (2.53)\\
\chi_\pm &= u^\dagger \chi\, u^\dagger \pm u\,\chi^\dagger u \ , && (2.54)\\
F^\pm_{\mu\nu} &= u^\dagger F^R_{\mu\nu} u \pm u\, F^L_{\mu\nu} u^\dagger \ , && (2.55)\\
& && (2.56)
\end{aligned}$$

with

$$\begin{aligned}
F^R_{\mu\nu} &= \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu] \ , && (2.57)\\
F^L_{\mu\nu} &= \partial_\mu \ell_\nu - \partial_\nu \ell_\mu - i[\ell_\mu, \ell_\nu] \ , && (2.58)
\end{aligned}$$

which has the following properties under the group $SU(2)_L \otimes SU(2)_R$

$$\begin{aligned}
u_\mu &\to u'_\mu = h u_\mu h^\dagger \ , && (2.59)\\
\chi_\pm &\to \chi'_\pm = h\chi_\pm h^\dagger \ , && (2.60)\\
F^\pm_{\mu\nu} &\to F'^\pm_{\mu\nu} = hF^\pm_{\mu\nu} h^\dagger \ . && (2.61)
\end{aligned}$$

We can now write the most general term of the πN Lagrangian invariant under parity, charge conjugation, Lorentz and local chiral transformations. At the leading order (order Q) it reads

$$\mathcal{L}^{(1)}_{\pi N} = \overline{N}\left(i\gamma^\mu D_\mu - M + \frac{g_A}{2}\gamma^\mu\gamma^5 u_\mu\right) N, \tag{2.62}$$

A complete list up to order Q^3 is reported in Ref. [33]; here we report only the terms we will use later in the thesis:

$$\begin{aligned}
\mathcal{L}^{(2)}_{\pi N} = \; & c_1 \overline{N}\langle\chi_+\rangle N \\
& - \frac{c_2}{8M^2}\overline{N}\langle u_\mu u_\nu\rangle D^{\mu\nu} N + \text{h.c.} \\
& + \frac{c_3}{2}\overline{N}\langle u_\mu u^\mu\rangle N \\
& + \frac{ic_4}{4}\overline{N}[u_\mu, u_\nu]\sigma^{\mu\nu} N \\
& + c_5 \overline{N}\hat{\chi}_+ N + \dots \ , && (2.63)
\end{aligned}$$

and

$$\mathcal{L}_{\pi N}^{(3)} = d_{16}\overline{N}\frac{1}{2}\gamma^{\mu}\gamma^{5}u_{\mu}\langle\chi_{+}\rangle N + d_{18}\overline{N}\frac{i}{2}\gamma^{\mu}\gamma^{5}[D_{\mu},\chi_{-}]N + \dots \,, \tag{2.64}$$

where we define

$$\hat{A} = A - \frac{1}{2}\langle A\rangle. \tag{2.65}$$

Other terms exist that involves four nucleonic fields. These are called "contact terms" (CT) and they give point interaction vertices between nucleons which include implicitly the effects of heavier meson exchanges. From all the possible contact terms we can write down, imposing the Lorentz invariance and the conservation of the discrete symmetries, the non relativistic expansion identifies at the leading order only two independent terms [22], so the contact Lagrangian reads

$$\mathcal{L}_{CT} = -\frac{1}{2}C_S(\overline{N}N)(\overline{N}N) + \frac{1}{4}C_T(\overline{N}\gamma_{\mu}\gamma^5 N)(\overline{N}\gamma^{\mu}\gamma^5 N). \tag{2.66}$$

It is also possible to write four-nucleons terms involving the quantities χ_+ and χ_- that are invariant under the transformations cited above. We list only those terms we will use later to build the TV interaction [5]

$$\begin{aligned}\mathcal{L}_{4N}^{(2)} = {} & \overline{C}_1\langle\chi_+\rangle\overline{N}N\overline{N}N + \overline{C}_2\langle\chi_+\rangle\overline{N}\gamma_{\mu}\gamma^5 N\overline{N}\gamma^{\mu}\gamma^5 N \\ & +\overline{C}_3\overline{N}\hat{\chi}_+ N\overline{N}N + \overline{C}_4\overline{N}\hat{\chi}_+\gamma_{\mu}\gamma^5 N\overline{N}\gamma^{\mu}\gamma^5 N \\ & +i\overline{C}_5\langle\chi_-\rangle\overline{N}N\partial_{\mu}\big(\overline{N}\gamma^{\mu}\gamma^5 N\big) \\ & +i\overline{C}_6\langle\chi_-\rangle\overline{N}\vec{\tau}N\partial_{\mu}\big(\overline{N}\vec{\tau}\gamma^{\mu}\gamma^5 N\big) \\ & +i\overline{C}_7\overline{N}\hat{\chi}_- N\partial_{\mu}\big(\overline{N}\gamma^{\mu}\gamma^5 N\big) \\ & +i\frac{1}{2}\overline{C}_8\overline{N}\big\{\hat{\chi}_+,\vec{\tau}\big\}N\partial_{\mu}\big(\overline{N}\vec{\tau}\gamma^{\mu}\gamma^5 N\big) + \dots \,.\end{aligned} \tag{2.67}$$

2.5 The PV interaction

In the previous Section we have used the transformations properties of the terms appearing in the QCD Lagrangian in order to determine the effective Lagrangian describing the strong interaction between nucleons and pions. In the same way, we can study the transformation properties of the terms that violate the P symmetry in the SM to construct a chiral effective Lagrangian that violates parity and transform under G_χ in the same way as the terms of the SM.

In the SM, in the limit of low energies, the weak interaction is described by the Lagrangian density [34]

$$\mathcal{L}_{weak} = \frac{G_F}{\sqrt{2}}\left(J_W^{\mu\dagger}J_{W\mu} + J_{W\mu}J_W^{\mu\dagger} + J_Z^{\mu\dagger}J_{Z\mu}\right), \tag{2.68}$$

where J_W is the current related to the exchange of the $W^{\pm}$ bosons, and J_Z is the neutral current, related to the exchange of the Z^0 boson. The two currents explicitly read

$$J_W^\mu = \cos\theta_c \overline{d}\gamma^\mu (1-\gamma_5)\, u = \cos\theta_c \overline{q}\gamma^\mu (1-\gamma_5)\, \tau_- q \ , \tag{2.69}$$

$$\begin{aligned} J_Z^\mu &= \frac{1}{\sqrt{2}c_W}\left[\overline{u}\gamma^\mu\left(1-\frac{8}{3}s_W^2-\gamma^5\right)u - \overline{d}\gamma^\mu\left(1-\frac{4}{3}s_W^2-\gamma^5\right)d\right] \ , \\ &= \frac{1}{\sqrt{2}c_W}\overline{q}\left[\gamma^\mu\left(1-\frac{8}{3}s_W^2-\gamma^5\right)\frac{1+\tau_z}{2} - \gamma^\mu\left(1-\frac{4}{3}s_W^2-\gamma^5\right)\frac{1-\tau_z}{2}\right]q, \end{aligned} \tag{2.70}$$

where θ_c is the Cabibbo angle, $s_W = \sin\theta_W$ and $c_W = \cos\theta_W$, where θ_W is the Weinberg angle, and $\tau_- = (\tau_x - i\tau_y)/2$. Writing the expressions in terms of the right and left components of the quark fields and using the definitions in Eq. (2.9a) and (2.9b), we get

$$J_W^\mu = 2\cos\theta_c \overline{q}_L \gamma^\mu \tau_- q_L \ , \tag{2.71}$$

$$\begin{aligned} J_Z^\mu = \frac{1}{\sqrt{2}c_W}\Big\{ &-\frac{2}{3}s_W^2\left(\overline{q}_R\gamma^\mu q_R + \overline{q}_L\gamma^\mu q_L\right) \\ &+ \left(2-2s_W^2\right)\overline{q}_L\gamma^\mu\tau_z q_L - 2s_W^2\overline{q}_R\gamma^\mu\tau_z q_R\Big\}. \end{aligned} \tag{2.72}$$

In order to simplify the notations let's define

$$A_R^\mu = \overline{q}_R\gamma^\mu q_R, \qquad A_L^\mu = \overline{q}_L\gamma^\mu q_L, \tag{2.73a}$$

$$\vec{B}_R^\mu = \overline{q}_R\gamma^\mu\vec{\tau} q_R, \qquad \vec{B}_L^\mu = \overline{q}_L\gamma^\mu\vec{\tau} q_L, \tag{2.73b}$$

which transform under the group $SU(2)_L \otimes SU(2)_R$ as follow

$$A_R^\mu \ \rightarrow \ A_R'^\mu = \overline{q}_R R^\dagger \gamma^\mu R q_R = A_R, \tag{2.74}$$

$$A_L^\mu \ \rightarrow \ A_L'^\mu = \overline{q}_L L^\dagger \gamma^\mu L q_L = A_L, \tag{2.75}$$

$$\tag{2.76}$$

so they are invariant over G_χ, namely they transform as isoscalar, and

$$\vec{B}_R^\mu \ \rightarrow \ \vec{B}_R'^\mu = \overline{q}_R\gamma^\mu R^\dagger\vec{\tau} R q_R, \tag{2.77}$$

$$\vec{B}_L^\mu \ \rightarrow \ \vec{B}_L'^\mu = \overline{q}_L\gamma^\mu L^\dagger\vec{\tau} L q_L, \tag{2.78}$$

$$\tag{2.79}$$

which transform as isovectors under the transformation R and L, respectively. When we apply the parity operator, we exchange $q_L \leftrightarrow q_R$ and so $A_R \leftrightarrow A_L$

and $\vec{B}_R \leftrightarrow \vec{B}_L$. Rewriting the coupling terms in the Lagrangian in Eq. (2.68) as follows

$$\begin{aligned}
J_W^\dagger J_W &= \cos^2\theta_c (B_{x,L} - iB_{y,L})(B_{x,L} + iB_{y,L}) \\
&= \frac{\cos^2\theta_c}{2}\Big[\frac{1}{2}\Big((\vec{B}_L\cdot\ \vec{B}_L + \vec{B}_R\cdot\ \vec{B}_R) + (\vec{B}_L\cdot\ \vec{B}_L - \vec{B}_R\cdot\ \vec{B}_R)\Big) \\
&\quad -\mathcal{I}^{ij}(B_{i,L}B_{j,L} + B_{i,R}B_{j,R}) - \mathcal{I}^{ij}(B_{i,L}B_{j,L} - B_{i,R}B_{j,R})\Big) \\
&\quad +i\Big((B_{y,L}B_{x,L} + B_{y,R}B_{x,R}) - (B_{x,L}B_{y,L} + B_{x,R}B_{y,R}) \\
&\quad +(B_{y,L}B_{x,L} - B_{y,R}B_{x,R}) - (B_{x,L}B_{y,L} - B_{x,R}B_{y,R})\Big)\Big] , \qquad (2.80)
\end{aligned}$$

where

$$\mathcal{I}_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +2 \end{pmatrix} , \qquad (2.81)$$

and also

$$\begin{aligned}
J_Z^\dagger J_Z &= \frac{1}{2c_W^2}\Big[\frac{4}{9}s_W^2(A_R + A_L)^2 - \frac{4}{3}s_W^2(2 - 2s_W^2)(A_R + A_L)B_{z,L} + \frac{8}{3}s_W^2(A_R + A_L)B_{z,R} \\
&\quad +(2 - 2s_W^2)^2 B_{z,L}B_{z,L} - 2s_W^2(2 - 2s_W^2)(B_{z,L}B_{z,R} + B_{z,R}B_{z,L}) + (2s_W)^2 B_{z,R}B_{z,R}\Big] \\
&= \frac{1}{2c_W^2}\Big[\frac{4}{9}s_W^2(A_R + A_L)^2 - \frac{4}{3}s_W^2(2 - 4s_W^2)(A_R + A_L)\frac{B_{z,L} + B_{z,R}}{2} \\
&\quad -\frac{8}{3}s_W^4(A_R + A_L)\frac{B_{z,L} - B_{z,R}}{2} + (2 - 2s_W^2)^2\frac{B_{z,L}B_{z,R} + B_{z,R}B_{z,L}}{2} \\
&\quad +(4 - 4s_W^2)\Big(\frac{\vec{B}_L\cdot\vec{B}_L - \vec{B}_R\cdot\vec{B}_R}{2} + \mathcal{I}^{ij}\frac{B_{i,L}B_{j,L} - B_{i,R}B_{j,R}}{2}\Big) \\
&\quad +(2 - 4s_W^2 + 8s_W^4)\frac{B_{z,L}B_{z,L} + B_{z,R}B_{z,R}}{2}\Big] .
\end{aligned} \qquad (2.82)$$

where the PV terms are the terms like $(A_R + A_L)(B_L - B_R)$ or $B_LB_L - B_RB_R$. From the expressions above, there are various PV terms which transform in different way under the chiral group $SU(2)_R \otimes SU(2)_L$, in particular

$$(\vec{B}_L\cdot\ \vec{B}_L - \vec{B}_R\cdot\ \vec{B}_R) \qquad \text{isoscalar}\ \Delta I = 0, \qquad (2.83)$$

$$(A_R + A_L)(B_{z,L} - B_{z,R}) \qquad \text{isovector}\ \Delta I = 1, \qquad (2.84)$$

$$(B_{y,L}B_{x,L} - B_{y,R}B_{x,R}) - (B_{x,L}B_{y,L} - B_{x,R}B_{y,R}) \qquad \text{isovector}\ \Delta I = 1, \qquad (2.85)$$

$$\mathcal{I}^{ij}(B_{i,L}B_{j,L} - B_{i,R}B_{j,R}) \qquad \text{isotensor}\ \Delta I = 2. \qquad (2.86)$$

Let's note that no terms like $B_R B_L - B_L B_R$ appear in the weak Lagrangian and so we cannot have a term like this in the effective Lagrangian.

At the hadronic level we reproduce the effect of the chiral symmetry breaking using the following quantities [35]

$$X_L^i = u\tau_i u^\dagger \ , \qquad X_R^i = u^\dagger \tau_i u \ , \tag{2.87}$$

where their transformations under the chiral group are found using Eq. (2.45). Therefore we obtain

$$\begin{aligned} (X_L^i)' &= (hu^\dagger L^\dagger)\tau_i(Luh^\dagger) = hu^\dagger(L^\dagger\tau_i L)uh^\dagger \ , &(2.88)\\ (X_R^i)' &= (huR^\dagger)\tau_i(Ru^\dagger h^\dagger) = hu^\dagger(R^\dagger\tau_i R)uh^\dagger \ , &(2.89)\end{aligned}$$

therefore terms like $\overline{N}X_L^i N$ and $\overline{N}X_R^i N$ have exactly the same transformations of $\vec{B}_L$ and $\vec{B}_R$. Regarding the isoscalar terms proportional to $\vec{X}_L \cdot \vec{X}_L - \vec{X}_R \cdot \vec{X}_R$, reproducing the term in Eq. (2.83), it vanishes, since

$$X_L^i X_L^j = \delta_{ij} + i\epsilon_{ijk}X_L^k \quad , \quad X_R^i X_R^j = \delta_{ij} + i\epsilon_{ijk}X_R^k. \tag{2.90}$$

So it is not possible to construct isoscalar terms in terms of X_L or X_R but they have to be constructed using PV terms like $\overline{N}u_\mu\gamma^\mu N$ [35] (see Appendix A for a detailed discussion). As regarding the isovector terms in Eqs. (2.84) and (2.85), using again the properties in Eq. (2.90), we see that they must be proportional to terms like $X_L^3 - X_R^3$ or $X_L^3 + X_R^3$. The isotensor terms will read $\mathcal{I}_{ij}(X_L^i\Theta X_L^j - X_R^i\Theta X_R^j)$, with Θ one of the possible operators which transform as $\Theta \to h\Theta h^\dagger$ under the chiral symmetry (if $\Theta = 1$ the term is zero).

In order to obtain the most general effective PV Lagrangian we will use X_L^i and X_R^i with the other building blocks introduced in Section 2.3 and 2.4 reproducing the isoscalar, isovector and isotensor terms discussed above and satisfying the properties of the violation of both parity and charge conjugation. We will consider also PV contact terms. In Appendix A a detailed discussion of the terms used in this work is reported. For a complete list of the terms up to order Q^2 see Ref. [25]. The terms of the PV Lagrangian entering the PV potential up to Q^3 are the following [25, 35]

$$\begin{aligned} \mathcal{L}^{(0)}_{\pi N,PV} &= \frac{h_\pi^1}{2\sqrt{2}} f_\pi \overline{N} X_-^3 N &(2.91)\\ \mathcal{L}^{(1)}_{\pi N,PV} &= \frac{h_V^0}{2}\overline{N}\gamma^\mu u_\mu N + \frac{h_V^1}{4}\overline{N}\gamma^\mu N\langle u_\mu X_+^3\rangle \\ &+ \frac{h_A^1}{4}\overline{N}\gamma^\mu\gamma^5 N\langle u_\mu X_-^3\rangle \\ &- \frac{1}{3}\mathcal{I}_{ab}\Bigg[\frac{h_V^2}{2}\overline{N}\left(X_R^a u_\mu X_R^b + X_L^a u_\mu X_L^b\right)\gamma^\mu N \end{aligned}$$

$$+\frac{h_A^2}{4}\overline{N}\left(X_R^a u_\mu X_R^b - X_L^a u_\mu X_L^b\right)\gamma^\mu\gamma^5 N\Big] \tag{2.92}$$

$$\begin{aligned}\mathcal{L}^{(2)}_{\pi N,PV} &= -\frac{h_2^1}{f_\pi}\bar{N}X_-^3 N\langle\chi_+\rangle - \frac{h_3^1}{f_\pi}\bar{N}[\hat{\chi}_-, X_+^3]N \\ &- \frac{h_{12}^1}{f_\pi}\Big(\frac{1}{M}\bar{N}[h_{\mu\nu}, X_+^3]\gamma^\mu D^\nu N + \text{h.c.}\Big) + \dots \end{aligned} \tag{2.93}$$

$$\begin{aligned}\mathcal{L}^{(3)}_{\pi N,PV} &= -\frac{\tilde{h}_1^0}{f_\pi^2}\overline{N}u_\mu\gamma^\mu N\langle\chi_+\rangle - \frac{\tilde{h}_1^1}{f_\pi^2}\overline{N}\{u_\mu\,,\,X_+^3\}\gamma^\mu N\langle\chi_+\rangle \\ &+ \frac{\tilde{h}_1^2}{3f_\pi^2}\mathcal{I}_{ab}\overline{N}\left(X_R^a u_\mu X_R^b + X_L^a u_\mu X_L^b\right)\gamma^\mu N\langle\chi_+\rangle + \dots\,. \end{aligned} \tag{2.94}$$

We will need also the following terms with only pionic degrees of freedom [25]

$$\begin{aligned}\mathcal{L}^{(2)}_{\pi\pi,PV} &= f_\pi^2 h_{3\pi}^1\langle u_\mu X_-^3 u^\mu\rangle \\ &+ f_\pi^2 h_{3\pi}^2\mathcal{I}^{ab}\langle X_R^a u_\mu X_R^b u^\mu - (R\to L)\rangle\,, \end{aligned} \tag{2.95}$$

Above the various parameters $h_n^{\Delta I}$ are unknown LECs. The superscript ΔI distinguishes the constant which multiply the isoscalar ($\Delta I = 0$), isovector ($\Delta I = 1$) or isotensor ($\Delta I = 2$) terms. The magnitude of the costants $h_n^{\Delta I}$ can be estimated to be

$$h_n^{\Delta I} \sim G_F f_\pi^2 \simeq 10^{-7}\,, \tag{2.96}$$

which is the typical order of magnitude of the PV interactions. In the next Chapters, the derivation of the PV potential from these interaction terms will be discussed.

Chapter 3

From χEFT to potentials

In this Chapter we will present the construction of a nuclear potential, operating in a non-relativistic framework, starting from the χEFT Lagrangian. In the first Section we will present our conventions. In Section 3.2 we will use time order perturbation theory to compute a T-matrix amplitude from the χEFT. Finally, in Section 3.3, we will define the non relativistic nuclear potential by imposing that solving the Lippmann-Schwinger (LS) equation in the non-relativistic regime, one can obtain the same T-matrix calculated before from the field theory.

3.1 Notations

In this Section we briefly summarize the conventions used in the Thesis.

- We will use the natural system $\hbar = c = 1$. We will work in a finite volume $\Omega = L^3$, so the momenta are discretized

$$k_i = \frac{2\pi n_i}{L}, \tag{3.1}$$

where $i = x,\ y,\ z$ and $n_i = 0, \pm 1, \pm 2, \ldots$. In the infinite volume limit the sum over the discretized momentum values is substituted by an integral as follows

$$\sum_{\mathbf{k}} \to \Omega \int \frac{d\mathbf{k}}{(2\pi)^3}. \tag{3.2}$$

- The creation and annihilation operators for pions and nucleons satisfy the following commutation rules

$$[a_{\boldsymbol{k},i}, a^\dagger_{\boldsymbol{k}',j}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{i,j} \ , \qquad \{b_{\boldsymbol{p},s,t}, b^\dagger_{\boldsymbol{p}',s',t'}\} = \delta_{\boldsymbol{p},\boldsymbol{p}'}\delta_{s,s'}\delta_{t,t'} \ , \tag{3.3}$$

where $i = x, y, z$ are the (cartesian) isospin components of the pion and s, t are the z components of the spin/isospin of the nucleon. The cartesian operators are defined as

$$a_{\boldsymbol{k},x} = \frac{a^{(+)}_{\boldsymbol{k}} + a^{(-)}_{\boldsymbol{k}}}{\sqrt{2}}, \qquad a_{\boldsymbol{k},y} = \frac{a^{(+)}_{\boldsymbol{k}} - a^{(-)}_{\boldsymbol{k}}}{\sqrt{2}}, \qquad a_{\boldsymbol{k},z} = a^{(0)}_{\boldsymbol{k}}, \tag{3.4}$$

where $a^{(+)}_{\boldsymbol{k}}$, $a^{(-)}_{\boldsymbol{k}}$ and $a^{(0)}_{\boldsymbol{k}}$ are the annihilation operators of π^+, π^-, and π^0 respectively. In the rest of the Thesis we will also use $\alpha \equiv \mathbf{p}, s, t$ to indicate the nucleon quantum numbers.

- The free Hamiltonian is [31]

$$H_0 = \sum_{\boldsymbol{p},s,t} E_p \left(b^\dagger_{\boldsymbol{p},s,t} b_{\boldsymbol{p},s,t} + d^\dagger_{\boldsymbol{p},s,t} d_{\boldsymbol{p},s,t} \right) + \sum_{\boldsymbol{q},i} \omega_q a^\dagger_{\boldsymbol{q},i} a_{\boldsymbol{q},i} \ , \tag{3.5}$$

with $E_p = \sqrt{M^2 + p^2}$ and $\omega_q = \sqrt{m^2 + q^2}$ are the energies of the nucleons and the pions, respectively. $d^\dagger_{\boldsymbol{p},s,t}$ is the creation operator of the antinucleons.

- The pion and nucleon fields in interaction picture read

$$N(x) = \sum_{\boldsymbol{p},s} \frac{1}{\sqrt{2E_p\Omega}} \left(b_{\boldsymbol{p},s,t} u(\boldsymbol{p}, s) e^{-ip\cdot x} + d^\dagger_{\boldsymbol{p},s,t} v(\boldsymbol{p}, s) e^{ip\cdot x} \right) \ , \tag{3.6}$$

$$\pi_i(x) = \sum_{\boldsymbol{q}} \frac{1}{\sqrt{2\omega_q\Omega}} \left(a_{\boldsymbol{q},i} e^{-iq\cdot x} + a^\dagger_{\boldsymbol{q},i} e^{iq\cdot x} \right) \ , \tag{3.7}$$

where the Dirac spinors $u(\mathbf{p}, s)$, $v(\mathbf{p}, s)$, and the Dirac matrices γ^μ and γ^5 are defined as in Ref. [31].

3.2 Time ordered diagrams

In field theory, the transition probability from an initial state $|i\rangle$ to a final state $|f\rangle$ is given by the matrix element $\langle f |S| i\rangle$, where S in the interaction picture reads [36]

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 ... d^4x_n \ \mathcal{T} \left(\mathcal{H}_I(x_1) ... \mathcal{H}_I(x_n) \right), \tag{3.8}$$

where $x \equiv x^\mu = (t, \mathbf{x})$, $\mathcal{T}$ indicates the time-ordered product and $\mathcal{H}_I(x)$ is the Hamiltonian density in interaction picture. Usually $\mathcal{H}_I$ is written as a sum of terms given by products of fields and their derivatives

$$\mathcal{H}_I(x) = \sum \overline{N}(x) ... \pi_i(x) ... \partial_\mu \pi_j(x) ... N(x) \ , \tag{3.9}$$

which is defined from the Lagrangian density discussed in the previous Chapter. Working in Heisenberg picture, the Hamiltonian density $\mathcal{H}$ is related to the corresponding Lagrangian $\mathcal{L}$ via the Legendre transformation

$$\mathcal{H} = \Pi_a \partial_0 \pi_a + \Pi_N \partial_0 N - \mathcal{L}, \tag{3.10}$$

where

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial(\partial \pi_a)}, \qquad \Pi_N = \frac{\partial \mathcal{L}}{\partial(\partial \pi_a)}, \tag{3.11}$$

are the conjugate momenta of pion fields and nucleon fields, respectively. Performing the calculation subtracting the unperturbed Hamiltonian density and returning in interaction picture it is possible to prove that (see Ref. [37])

$$\mathcal{H}_I(x) = -\mathcal{L}_I(x) + \cdots , \tag{3.12}$$

where the correction terms can be neglected since of high order in ChPT [37]. The Hamiltonian in interaction picture is defined as

$$H_I(t) = \int d\mathbf{x}\ \mathcal{H}_I(x), \tag{3.13}$$

and is related to H_I^{SR} in Schrödinger picture by

$$H_I(t) = e^{iH_0 t} H_I^{\text{SR}} e^{-iH_0 t}, \tag{3.14}$$

where H_0 is the free Hamiltonian. Integrating analytically over all the time variables, Eq. (3.8) can be written as [38]

$$\langle f\,|S|\,i\rangle = \delta_{f,i} - 2\pi\delta(E_f - E_i)\,\langle f\,|T|\,i\rangle\,, \tag{3.15}$$

where the operator T (the so-called T-matrix) is explicitly given by

$$\begin{aligned} T \;=\; & H_I^{SR} + H_I^{SR}\frac{1}{E_0 - H_0 + i\epsilon}H_I^{SR} + \\ & + H_I^{SR}\frac{1}{E_0 - H_0 + i\epsilon}H_I^{SR}\frac{1}{E_0 - H_0 + i\epsilon}H_I^{SR} + \cdots\ , \end{aligned} \tag{3.16}$$

being ϵ an infinitesimal positive quantity.

In order to obtain the Hamiltonian terms expressed in Schrödinger picture, we must write the fields in this representation using

$$\psi^{\text{SR}}(\mathbf{x}) = e^{-iH_0 t}\psi(x)e^{iH_0 t}. \tag{3.17}$$

In this way the fields of the pion and nucleon and their derivatives become

$$N^{SR}(\boldsymbol{x}) = \sum_{\boldsymbol{p},s} \frac{1}{\sqrt{2E_p\Omega}} \left(b_{\boldsymbol{p},s,t} u(\boldsymbol{p},s) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + d^\dagger_{\boldsymbol{p},s,t} v(\boldsymbol{p},s) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right) , \quad (3.18)$$

$$\partial_\mu N^{SR}(\boldsymbol{x}) = \sum_{\boldsymbol{p},s} \frac{1}{\sqrt{2E_p\Omega}} \Big(b_{\boldsymbol{p},s,t} u(\boldsymbol{p},s)(-ip_\mu) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + d^\dagger_{\boldsymbol{p},s,t} v(\boldsymbol{p},s)(ip_\mu) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \Big) , \quad (3.19)$$

$$\pi_i^{SR}(\boldsymbol{x}) = \sum_{\boldsymbol{q}} \frac{1}{\sqrt{2\omega_q\Omega}} \left(a_{\boldsymbol{q},i} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{q},i} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \right) , \quad (3.20)$$

$$\partial_\mu \pi_i^{SR}(\boldsymbol{x}) = \sum_{\boldsymbol{q}} \frac{1}{\sqrt{2\omega_q\Omega}} \left(a_{\boldsymbol{q},i}(-iq_\mu) e^{i\boldsymbol{q}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{q},i}(iq_\mu) e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \right) . \quad (3.21)$$

and thus we can write the Hamiltonian in the Schrödinger picture (see Eq. (3.9))

$$H_I^{\rm SR} = \sum \int d\mathbf{x}\ \overline{N}^{\rm SR}(\mathbf{x}) ... \pi_j^{\rm SR}(\mathbf{x}) ... \partial_\mu \pi_i^{\rm SR}(\mathbf{x}) ... \partial_\mu N^{\rm SR}(\mathbf{x}) . \quad (3.22)$$

Inserting the expressions (3.18)-(3.21) in Eq. (3.22) and integrating over $\mathbf{x}$ the interaction Hamiltonian can be written as

$$H_I = H^{CT,00} + H^{\pi NN,01} + H^{\pi NN,10} + H^{\pi\pi NN,02} + H^{\pi\pi NN,11} + H^{\pi\pi NN,20} + \cdots , \quad (3.23)$$

where we have eliminated the SR superscript because from now on we will work only in Schröedinger picture. The term $H_I^{v,nm}$ derives from an interaction term of type v and it has n creation and m annihilation pion operators. Explicitly,

$$H^{CT,00} = \frac{1}{\Omega} \sum_{\alpha'_1,\alpha_1\alpha'_2,\alpha_2} b^\dagger_{\alpha'_1} b_{\alpha_1} b^\dagger_{\alpha'_2} b_{\alpha_2}\ M^{CT,00}_{\alpha'_1\alpha_1\alpha'_2\alpha_2}\ \delta_{\boldsymbol{p}'_1+\boldsymbol{p}'_2,\boldsymbol{p}_1+\boldsymbol{p}_2} , \quad (3.24)$$

$$H^{\pi NN,01} = \frac{1}{\sqrt{\Omega}} \sum_{\alpha',\alpha} \sum_{\boldsymbol{q},i} b^\dagger_{\alpha'} b_\alpha a_{\boldsymbol{q},i}\ M^{\pi NN,01}_{\alpha'\alpha,\boldsymbol{q}i}\ \delta_{\boldsymbol{q}+\boldsymbol{p},\boldsymbol{p}'} , \quad (3.25)$$

$$H^{\pi NN,10} = \frac{1}{\sqrt{\Omega}} \sum_{\alpha',\alpha} \sum_{\boldsymbol{q},i} b^\dagger_{\alpha'} b_\alpha a^\dagger_{\boldsymbol{q},i}\ M^{\pi NN,10}_{\alpha'\alpha,\boldsymbol{q}i}\ \delta_{\boldsymbol{q}+\boldsymbol{p}',\boldsymbol{p}} , \quad (3.26)$$

$$H^{\pi\pi NN,02} = \frac{1}{\Omega} \sum_{\alpha',\alpha} \sum_{\boldsymbol{q}'i',\boldsymbol{q}i} b^\dagger_{\alpha'} b_\alpha a_{\boldsymbol{q}',i'} a_{\boldsymbol{q},i}\ M^{\pi\pi NN,02}_{\alpha'\alpha,\boldsymbol{q}'i'\boldsymbol{q}i}\ \delta_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p},\boldsymbol{p}'} , \quad (3.27)$$

$$H^{\pi\pi NN,11} = \frac{1}{\Omega} \sum_{\alpha',\alpha} \sum_{\boldsymbol{q}'i',\boldsymbol{q}i} b^\dagger_{\alpha'} b_\alpha a^\dagger_{\boldsymbol{q}',i'} a_{\boldsymbol{q},i}\ M^{\pi\pi NN,11}_{\alpha'\alpha,\boldsymbol{q}'i'\boldsymbol{q}i}\ \delta_{\boldsymbol{q}+\boldsymbol{p},\boldsymbol{q}'+\boldsymbol{p}'} , \quad (3.28)$$

$$H^{\pi\pi NN,20} = \frac{1}{\Omega} \sum_{\alpha',\alpha} \sum_{\boldsymbol{q}'i',\boldsymbol{q}i} b^\dagger_{\alpha'} b_\alpha a^\dagger_{\boldsymbol{q}',i'} a^\dagger_{\boldsymbol{q},i}\ M^{\pi\pi NN,20}_{\alpha'\alpha,\boldsymbol{q}'i'\boldsymbol{q}i}\ \delta_{\boldsymbol{p},\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}'} . \quad (3.29)$$

In the following, we need also the three pions interaction terms,

$$H^{3\pi,30} = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{q}'i',\boldsymbol{q}i,\boldsymbol{p}j} a^\dagger_{\boldsymbol{q}',i'} a^\dagger_{\boldsymbol{q},i} a^\dagger_{\boldsymbol{p},j} \; M^{3\pi,30}_{\boldsymbol{q}'i'\boldsymbol{q}i\boldsymbol{p},j} \; \delta_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p},0} \;, \tag{3.30}$$

$$H^{3\pi,21} = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{q}'i',\boldsymbol{q}i,\boldsymbol{p}j} a^\dagger_{\boldsymbol{q}',i'} a^\dagger_{\boldsymbol{q},i} a_{\boldsymbol{p},j} \; M^{3\pi,21}_{\boldsymbol{q}'i'\boldsymbol{q}i\boldsymbol{p},j} \; \delta_{\boldsymbol{q}+\boldsymbol{q}',\boldsymbol{p}} \;, \tag{3.31}$$

$$H^{3\pi,12} = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{q}'i',\boldsymbol{q}i,\boldsymbol{p}j} a^\dagger_{\boldsymbol{q}',i'} a_{\boldsymbol{q},i} a_{\boldsymbol{p},j} \; M^{3\pi,12}_{\boldsymbol{q}'i'\boldsymbol{q}i\boldsymbol{p},j} \; \delta_{\boldsymbol{q}',\boldsymbol{q}+\boldsymbol{p}} \;, \tag{3.32}$$

$$H^{3\pi,03} = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{q}'i',\boldsymbol{q}i,\boldsymbol{p}j} a_{\boldsymbol{q}',i'} a_{\boldsymbol{q},i} a_{\boldsymbol{p},j} \; M^{3\pi,03}_{\boldsymbol{q}'i'\boldsymbol{q}i\boldsymbol{p},j} \; \delta_{0,\boldsymbol{q}'+\boldsymbol{q}+\boldsymbol{p}} \;, \tag{3.33}$$

$$\tag{3.34}$$

where the quantities $M^{CT,00}$, $M^{\pi NN,10}$, $M^{\pi NN,01}$, etc. are the so called "vertex functions". Appendix C reports the explicit expression of the vertex functions for the PC, PV and TV Hamiltonian terms we need to determine the PV potential up to order Q^2 and the TV potential up to order Q^0. For each of them it is possible to perform a non relativistic expansion of these functions in power of $Q/M \sim Q/\Lambda_\chi$, where Q is a typical value of the momentum, in order to control the chiral order. For example from the term,

$$\frac{g_A}{2} \overline{N} \gamma^\mu \gamma^5 u_\mu N \;, \tag{3.35}$$

expanding the fields as in Eqs. (3.18)-(3.21) we find

$${}^{PC}M^{\pi NN,01}_{\alpha'\alpha,\boldsymbol{q}\,a} = -i \frac{g_A}{2f_\pi} \frac{\xi^\dagger_{t'} \tau_a \xi_t}{\sqrt{2\omega_k}} \frac{\overline{u}_{\alpha'}}{\sqrt{2E'}} \not{q} \gamma^5 \frac{u_\alpha}{\sqrt{2E}} = \frac{g_A}{2f_\pi} \frac{\tau_a}{\sqrt{2\omega_q}} i\boldsymbol{q}\cdot\boldsymbol{\sigma} + \mathcal{O}(Q^{3/2}) \tag{3.36}$$

$${}^{PC}M^{\pi NN,10}_{\alpha'\alpha,\boldsymbol{q}\,a} = +i \frac{g_A}{2f_\pi} \frac{\xi^\dagger_{t'} \tau_a \xi_t}{\sqrt{2\omega_k}} \frac{\overline{u}_{\alpha'}}{\sqrt{2E'}} \not{q} \gamma^5 \frac{u_\alpha}{\sqrt{2E}} = -\frac{g_A}{2f_\pi} \frac{\tau_a}{\sqrt{2\omega_q}} i\boldsymbol{q}\cdot\boldsymbol{\sigma} + \mathcal{O}(Q^{3/2}) \;, \tag{3.37}$$

where the superscript "PC" remembers that the vertex functions come from a PC vertex.

Let's now consider for example the calculation of the T-matrix element for a scattering process $NN \to NN$ from an initial state $|i\rangle = |\alpha_1\alpha_2\rangle$ and a final state $|f\rangle = |\alpha'_1\alpha'_2\rangle$. To compute the matrix element $\langle f\,|T|\,i\rangle$ we consider the expression of T reported in Eq. (3.16), we replace the expression of H_I with Eq. (3.23), and insert where necessary a sum over intermediate states, obtaining

$$\begin{aligned}
\langle\alpha_1'\alpha_2'|T|\alpha_1\alpha_2\rangle &= \langle\alpha_1'\alpha_2'|H_I|\alpha_1\alpha_2\rangle + \sum_{\rm INT}\frac{\langle\alpha_1'\alpha_2'|H_I|{\rm INT}\rangle\langle|{\rm INT}|H_I|\alpha_1\alpha_2\rangle}{E_{\alpha_1}+E_{\alpha_2}-E_{\rm INT}+i\epsilon}+\cdots\ , \\
&= \langle\alpha_1'\alpha_2'|H^{CT,00}|\alpha_1\alpha_2\rangle \\
&+\sum_{\beta_1\beta_2}\frac{\langle\alpha_1'\alpha_2'|H^{CT,00}|\beta_1\beta_2\rangle\langle\beta_1\beta_2|H^{CT,00}|\alpha_1\alpha_2\rangle}{E_{\alpha_1}+E_{\alpha_2}-E_{\beta_1}-E_{\beta_2}+i\epsilon} \\
&+\sum_{\beta_1\beta_2\boldsymbol{q}i}\frac{\langle\alpha_1'\alpha_2'|H^{\pi NN,01}|\beta_1\beta_2\boldsymbol{q}i\rangle\langle\beta_1\beta_2\boldsymbol{q}i|H^{\pi NN,10}|\alpha_1\alpha_2\rangle}{E_{\alpha_1}+E_{\alpha_2}-E_{\beta_1}-E_{\beta_2}-\omega_q+i\epsilon} \\
&+\cdots\ , \qquad (3.38)
\end{aligned}$$

where $|\boldsymbol{q}i\rangle$ is the state of a pion with momentum $\boldsymbol{q}$ and type i. Making explicit H_I as in Eq. (3.23), we can select the intermediate states which will give a non vanishing contribution after the contraction of all the creation/annihilation operators of the Hamiltonian.

Each matrix element, after using the creation/annihilation operator algebra, contains only the vertex functions and Kronecker δ expressing the momentum conservation at each vertex. For example

$$\begin{aligned}
\langle\beta_1\beta_2\boldsymbol{q}i|H^{\pi NN,10}|\alpha_1\alpha_2\rangle &= M^{\pi NN,10}_{\beta_1\alpha_1\boldsymbol{q}i}\delta_{\boldsymbol{q}_{\beta_1}+\boldsymbol{q},\boldsymbol{p}_{\alpha_1}}\delta_{\beta_2,\alpha_2} - M^{\pi NN,10}_{\beta_1\alpha_2\boldsymbol{q}i}\delta_{\boldsymbol{q}_{\beta_1}+\boldsymbol{q},\boldsymbol{p}_{\alpha_2}}\delta_{\beta_2,\alpha_1} \\
&- M^{\pi NN,10}_{\beta_2\alpha_1\boldsymbol{q}i}\delta_{\boldsymbol{q}_{\beta_2}+\boldsymbol{q},\boldsymbol{p}_{\alpha_1}}\delta_{\beta_1,\alpha_2} + M^{\pi NN,10}_{\beta_2\alpha_2\boldsymbol{q}i}\delta_{\boldsymbol{q}_{\beta_2}+\boldsymbol{q},\boldsymbol{p}_{\alpha_2}}\delta_{\beta_1,\alpha_1} , \qquad (3.39)
\end{aligned}$$

where $\delta_{\beta_2,\alpha_2}\equiv\delta_{\boldsymbol{p}_{\beta_2},\boldsymbol{p}_{\alpha_2}}\ \delta_{s_{\beta_2},s_{\alpha_2}}\ \delta_{t_{\beta_2},t_{\alpha_2}}$, etc. Using the δ's to eliminate either all or part of the sums over the intermediate states, we have the final expression for the T matrix as a series of terms which can be represented by "time ordered" diagrams. For example the third line in Eq. (3.38) gives

$$\frac{1}{\Omega}\left(\frac{M^{\pi NN,01}_{\alpha_2',\alpha_2,\boldsymbol{k}_2,a}M^{\pi NN,10}_{\alpha_1',\alpha_1,-\boldsymbol{k}_1,a}}{E_{\alpha_1}-E_{\alpha_1'}-\omega_k}+\frac{M^{\pi NN,01}_{\alpha_1',\alpha_1,\boldsymbol{k}_1,a}M^{\pi NN,10}_{\alpha_2',\alpha_2,-\boldsymbol{k}_2,a}}{E_{\alpha_2}-E_{\alpha_2'}-\omega_k}\right)\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'}-(\alpha_1'\leftrightarrow\alpha_2')\ , \qquad (3.40)$$

which can be represented by the "time ordered" diagrams reported in Fig. (3.1)

There is one-to-one correspondence between the diagrams and the corresponding expressions of the contributions to the T-matrix. If we consider the time running upward, the various factors are ordered starting from the top of the diagrams and going "backward". Every time we meet a vertex, we associate the corresponding vertex function; between two vertices we have a "propagator" or better an energy denominator which takes into account of the "flying" particles in the intermediate state. For each vertex there is a δ's conservation of momenta. After the elimination of the sum over the intermediate state momenta with the δ's, it remains an integrations over a momentum for each loop. It's important to notice that these diagrams are not Feynman diagrams. Unlike the latters, we

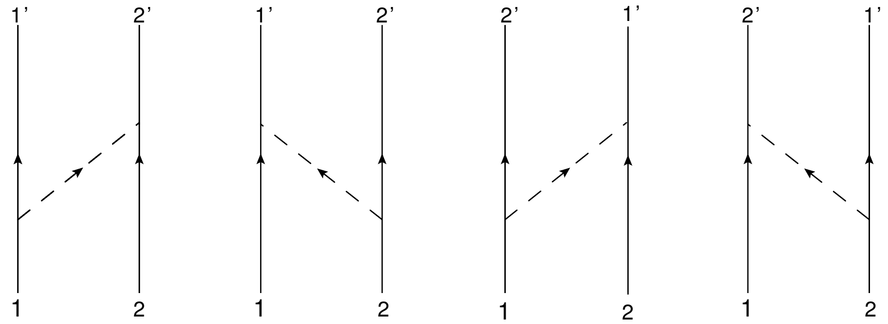


Figure 3.1: Graphical representation, using the time ordered diagrams, for the one pion exchange in the NN scattering.

must consider here all time orderings: thus, for example, when we have a one pion exchange we must consider the diagrams where the pion is emitted by the first nucleon and absorbed by the second and the diagrams where the pion is emitted by the second nucleon and absorbed by the first. This is due to the analytical integration over time we have performed when we have written the S-matrix in terms of the T-matrix.

The reason of using time-ordered diagrams is connected to possibility to identify the chiral order $(Q/\Lambda_\chi)^\nu$ for each of them. The order of a diagram depends simply on:

1. the chiral order of the non relativistic expansion of the vertex functions;
2. the energy denominators. We note that typical momenta $\boldsymbol{p}$ of the nucleons are much smaller than the mass of the nucleons, so we can treat them non relativistically. Namely $E_\alpha \simeq M + \frac{p^2}{2M} \sim O(Q^0) + O(Q^2)$. Regarding the pion energies, $\omega_{\mathbf{k}} = \sqrt{m_\pi^2 + k^2} \sim O(Q)$. Usually in the energy denominator all the nucleon masses M cancel out and therefore we have two cases:
   - if there are no pions in flight, the energy denominator has only nucleon energy terms so it result of order $1/Q^2$.
   - if there are pions in flight the energy denominator reads
   $$\frac{1}{\Delta E - \omega_k} \sim -\frac{1}{\omega_k}\left(1 + \frac{\Delta E}{\omega_k} + \cdots\right) , \tag{3.41}$$
   where the term $\Delta E = E_1 + \cdots - E_0$ where $E_1, \ldots$ are the energies of the nucleons and E_0 is the initial scattering energy. In the Taylor expansion the first term is of order Q^{-1}, the second of order Q^0 and so on.
3. The number of loops, or better the number of the sums over the intermediate state momenta that remains after using the conservation δ's. Each loop at the end will give a contribution of order Q^3.

For example let's consider the OPE diagrams in the *PC* case in Fig. 3.1. From the expression in Eqs. (3.37) and (3.36) the vertex functions at the LO are of order $Q^{1/2}$ while the energy denominator is of order Q^{-1}. Therefore the lowest order of the diagrams is

$$\sim Q^{\frac{1}{2}} Q^{\frac{1}{2}} \frac{1}{Q} \sim \mathcal{O}(Q^0) \ . \tag{3.42}$$

From an analysis of the diagrams contributing to the *PC* potential it is easy to see that the LO has order $\mathcal{O}(Q^0)$ as calculated above for OPE. Also the Lagrangian in Eq. (2.66) is of order $\mathcal{O}(Q^0)$ so the contact term in the *PC* case contribute to LO. The *PV* and *TV* LO is $\mathcal{O}(Q^{-1})$ and it is obtained from the corresponding *PV* or *TV* OPE.

3.3 The NN potential

As we have seen before, the T-matrix can be written as a sum over contributions (diagrams) each of them of a given chiral index ν. For the T-matrix of NN scattering we have

$$T = \sum_n T^{(n)}, \tag{3.43}$$

where $T^{(n)} \sim Q^n$. In all cases the sum starts from a minimum value of n, $n = 0$ for the *PC* and $n = -1$ for the *PV* and *TV* amplitude. Assuming that the non relativistic nuclear potential V has the same Q expansion as the T matrix,

$$V = \sum_n V^{(n)} \ , \qquad V^{(n)} \sim Q^n \ , \tag{3.44}$$

we can build V from the T-matrix using the LS equation [39]. In fact, if we consider the state $|\phi\rangle$ to be a non interacting two nucleon state, i. e. the solution of the free Schrödinger equation

$$(H_0 - E_i)\,|\phi\rangle = 0, \tag{3.45}$$

and $|\psi\rangle$ the NN state solution of the full Schrödinger equation

$$(H_0 + V - E_i)\,|\psi\rangle = 0, \tag{3.46}$$

then we can write the LS equation as

$$|\psi\rangle = |\phi\rangle + \frac{1}{E_i - H_0 + i\epsilon} V\,|\psi\rangle \,, \tag{3.47}$$

where $\epsilon \to 0^+$ have been inserted in order to eliminate the singularity. Defining the operator T_V as $T_V\phi = V\psi$ we derive the integral equation for T_V,

$$T_V = V + V\frac{1}{E_0 - H_0 + i\epsilon} T_V \ , \tag{3.48}$$

which we can be solved iterating

$$T_V = V + V\frac{1}{E_0 - H_0 + i\epsilon}V + V\frac{1}{E_0 - H_0 + i\epsilon}V\frac{1}{E_0 - H_0 + i\epsilon}V + \cdots . \quad (3.49)$$

We will construct V so

$$\langle\alpha_1'\alpha_2'\,|T_V|\,\alpha_1\alpha_2\rangle = \langle\alpha_1'\alpha_2'\,|T|\,\alpha_1\alpha_2\rangle\,, \quad (3.50)$$

order by order in the Q power expansion.

We will now explain how to perform in practice the power expansion of $\langle\alpha_1'\alpha_2'\,|T_V|\,\alpha_1\alpha_2\rangle$. In the calculation of the amplitude $\langle\alpha_1'\alpha_2'\,|T_V|\,\alpha_1\alpha_2\rangle$ in Eq. (3.49), we must consider intermediate states of only two nucleons, because the potential couples only nucleonic degrees of freedom. Each term of the free Green functions

$$G_0 \equiv \frac{1}{E_i - H_0 + i\epsilon} \quad (3.51)$$

then brings a contribution of order Q. In fact, let us consider for example a generic term like

$$\langle\alpha_1'\alpha_2'|\,V^{(n')}G_0V^{(n)}\,|\alpha_1\alpha_2\rangle = \sum_{\beta_1\beta_2}\frac{\langle\alpha_1'\alpha_2'|\,V^{(n')}\,|\beta_1\beta_2\rangle\,\langle\beta_1\beta_2|\,V^{(n)}\,|\alpha_1\alpha_2\rangle}{E_{\alpha_1}+E_{\alpha_2}-E_{\beta_1}-E_{\beta_2}+i\epsilon}. \quad (3.52)$$

As discussed in Section 3.2 this energy denominator is of order Q^{-2}. Concerning the matrix element of the potential, there is always a delta related to the momentum conservation

$$\langle\alpha_1'\alpha_2'|\,V^{(n)}\,|\alpha_1\alpha_2\rangle = \frac{1}{\Omega}\,\langle\alpha_1'\alpha_2'|\,v^{(n)}\,|\alpha_1\alpha_2\rangle\,\delta_{\mathbf{p}_1'+\mathbf{p}_2',\mathbf{p}_1+\mathbf{p}_2}, \quad (3.53)$$

with $v^{(n)}$ of order Q^n. In Eq. (3.52) we have two deltas: one fixes the value of one of the two momenta of the sum, the second brings an overall momentum conservation. Thus, one integration over one of the intermediate momentum remains contributing to a factor of order Q^3. Therefore the total order of the term in Eq. (3.52) will be $Q^{n+n'+1}$. Similarly we obtain

$$V^{(n)}G_0V^{(n')}G_0V^{(n'')} \sim Q^{n+n'+n''+2}\,, \quad (3.54)$$

$$V^{(n)}G_0V^{(n')}G_0V^{(n'')}G_0V^{(n''')} \sim Q^{n+n'+n''+n'''+3}\,, \quad (3.55)$$

etc.

Let us discuss the case where the two nucleons interact through a PC potential plus a very small non conserving part X, which can be either PV or TV. The T-matrix calculated from the χEFT results in the following expansion in powers of Q for $T = T_{PC} + T_X$:

$$T_{PC} = T_{PC}^{(0)} + T_{PC}^{(1)} + T_{PC}^{(2)} + T_{PC}^{(3)} + \ldots, \quad (3.56)$$

$$T_X = T_X^{(-1)} + T_X^{(0)} + T_X^{(1)} + T_X^{(2)} + \ldots . \quad (3.57)$$

We assume that $V = V_{PC} + V_X$, with the two having a similar expansion

$$V_{PC} = V_{PC}^{(0)} + V_{PC}^{(1)} + V_{PC}^{(2)} + T_{PC}^{(3)} + \cdots \tag{3.58}$$
$$V_X = V_X^{(-1)} + V_X^{(0)} + V_X^{(1)} + T_X^{(2)} + \cdots \,. \tag{3.59}$$

Using this form of the potential in Eq. (3.49) and neglecting the terms like $(V_X)^2$ or of higher power we find

$$\begin{aligned} T &= V + VG_0V + VG_0VG_0V + VG_0VG_0VG_0V + \cdots \\ &= V_{PC} + V_X + V_{PC}G_0V_{PC} + V_XG_0V_{PC} + V_{PC}G_0V_X \\ &+ V_{PC}G_0V_{PC}G_0V_{PC} + V_XG_0V_{PC}G_0V_{PC} \\ &+ V_{PC}G_0V_XG_0V_{PC} + V_{PC}G_0V_{PC}G_0V_X + \cdots \,. \end{aligned} \tag{3.60}$$

Matching order by order the two sides of Eq. (3.60), taking into account the "rule" $G_0 \sim Q$, we obtain for the PC potential up to order Q^3

$$V_{PC}^{(0)} = T_{PC}^{(0)} \,, \tag{3.61}$$
$$V_{PC}^{(1)} = T_{PC}^{(1)} - \left[V_{PC}^{(0)}G_0V_{PC}^{(0)}\right] \,, \tag{3.62}$$
$$\begin{aligned} V_{PC}^{(2)} &= T_{PC}^{(2)} - \left[V_{PC}^{(0)}G_0V_{PC}^{(1)}\right] - \left[V_{PC}^{(1)}G_0V_{PC}^{(0)}\right] \\ &- \left[V_{PC}^{(0)}G_0V_{PC}^{(0)}G_0V_{PC}^{(0)}\right] \,, \end{aligned} \tag{3.63}$$
$$\cdots$$

and for the non-conserving part

$$V_X^{(-1)} = T_X^{(-1)} \,, \tag{3.64}$$
$$V_X^{(0)} = T_X^{(0)} - \left[V_{PV}^{(-1)}G_0V_{PC}^{(0)}\right] - \left[V_{PC}^{(0)}G_0V_{PV}^{(-1)}\right] \,, \tag{3.65}$$
$$\begin{aligned} V_X^{(1)} &= T_X^{(1)} - \left[V_X^{(0)}G_0V_{PC}^{(0)}\right] - \left[V_{PC}^{(0)}G_0V_X^{(0)}\right] \\ &- \left[V_{PV}^{(-1)}G_0V_{PC}^{(1)}\right] - \left[V_{PC}^{(1)}G_0V_X^{(-1)}\right] \\ &- \left[V_X^{(-1)}G_0V_{PC}^{(0)}G_0V_{PC}^{(0)}\right] - \left[V_{PC}^{(0)}G_0V_X^{(-1)}G_0V_{PC}^{(0)}\right] \\ &- \left[V_{PC}^{(0)}G_0V_{PC}^{(0)}G_0V_X^{(-1)}\right] \,, \end{aligned} \tag{3.66}$$
$$\cdots$$

For the study of the TV potential we will stop to $V_{TV}^{(0)}$. This formal expression will allow us to calculate the matrix element $\langle \alpha_1'\alpha_2' |V| \alpha_1\alpha_2\rangle$ from the various contributions of the matrix elements $\langle \alpha_1'\alpha_2' |T| \alpha_1\alpha_2\rangle$.

The generic amplitude $\langle \alpha_1'\alpha_2' |T| \alpha_1\alpha_2\rangle$ has "direct" terms and "exchange" terms. The first corresponds to the diagrams where there is the transition $\alpha_i \to \alpha_i'$

with $i = 1, 2$, for example the first two terms in Eq. (3.40). The second represents the transition $\alpha_i \to \alpha'_j$ with $i \neq j = 1, 2$, for example the terms we have denoted with "$(\alpha'_1 \leftrightarrow \alpha'_2)$" in Eq. (3.40). Now we can write

$$\langle \alpha'_1 \alpha'_2 | V | \alpha_1 \alpha_2 \rangle = \int d^3r_1 d^3r_2 d^3r'_1 d^3r'_2 \; \langle \alpha'_1 \alpha'_2 | \boldsymbol{r}'_1 \boldsymbol{r}'_2 \rangle \langle \boldsymbol{r}'_1 \boldsymbol{r}'_2 | V | \boldsymbol{r}_1 \boldsymbol{r}_2 \rangle \langle \boldsymbol{r}_1 \boldsymbol{r}_2 | \alpha_1 \alpha_2 \rangle \; , \tag{3.67}$$

where $|\alpha_1 \alpha_2\rangle$ is an antisymmetric state of two nucleons which can be projected on r-space obtaining

$$\langle \boldsymbol{r}_1 \boldsymbol{r}_2 | \alpha_1 \alpha_2 \rangle = \frac{1}{\sqrt{\Omega}} \frac{e^{i\boldsymbol{p}_1 \cdot \boldsymbol{r}_1 + i\boldsymbol{p}_2 \cdot \boldsymbol{r}_2} |s_1 t_1\rangle |s_2 t_2\rangle - e^{i\boldsymbol{p}_2 \cdot \boldsymbol{r}_1 + i\boldsymbol{p}_1 \cdot \boldsymbol{r}_2} |s_2 t_2\rangle |s_1 t_1\rangle}{\sqrt{2}} \; , \tag{3.68}$$

where $|s_i t_i\rangle$ is the spin-isospin state of the i-th particle. Substituing this equation in Eq. (3.67) we find

$$\begin{aligned} \langle \alpha'_1 \alpha'_2 | V | \alpha_1 \alpha_2 \rangle \;\; = \;\; & \frac{1}{\Omega} \int d^3r_1 d^3r_2 d^3r'_1 d^3r'_2 \left[\left(e^{-i\boldsymbol{p}'_1 \cdot \boldsymbol{r}'_1 - i\boldsymbol{p}'_2 \cdot \boldsymbol{r}'_2} \langle s'_1 t'_1 | \langle s'_2 t'_2 | \right) \right. \\ & \times \langle \boldsymbol{r}'_1 \boldsymbol{r}'_2 | V | \boldsymbol{r}_1 \boldsymbol{r}_2 \rangle \left(e^{i\boldsymbol{p}_1 \cdot \boldsymbol{r}_1 + i\boldsymbol{p}_2 \cdot \boldsymbol{r}_2} |s_1 t_1\rangle |s_2 t_2\rangle \right) \\ & \left. - (\alpha'_1 \leftrightarrow \alpha'_2) \right) \; . \end{aligned} \tag{3.69}$$

As in the T-matrix also in the potential matrix elements we obtain a "direct" term and an "exchange" term. It will be sufficent to match the direct part of the T-matrix elements to the direct potential matrix elements via Eq. (3.62)–(3.66) to obtain the potential.

Chapter 4

The PV potential

In this chapter we will discuss in detail the derivation of the PV potential up to N3LO. In the first section the notation used in this and next Chapter will be introduced. In the second and third Sections we derive the PV potential up to N2LO and N3LO, respectively. In Section 4.4 we will regularize the divergences coming from the loops and in Section 4.5 we will give the complete expression of the potential in momentum space. In the last Section we will give the potential in configuration space. This is a very technical Chapter, the reader could skip Sections 4.2, 4.3 and 4.4 and refers to the last two Sections for the final expression of the potential.

4.1 Notations

In this and in the next chapter we will use the following notation. The process under consideration is the scattering of two nucleons from an initial state $|\boldsymbol{p}_1\boldsymbol{p}_2\rangle$ to the final state $|\boldsymbol{p}'_1\boldsymbol{p}'_2\rangle$. It is convenient to define the momenta

$$\boldsymbol{K}_j = \frac{\boldsymbol{p}'_j + \boldsymbol{p}_j}{2} \ , \quad \boldsymbol{k}_j = \boldsymbol{p}'_j - \boldsymbol{p}_j \ , \tag{4.1}$$

where $\mathbf{p_j}$ and $\mathbf{p'_j}$ are the initial and the final momenta of the nucleon j. Furthermore is useful to define

$$\boldsymbol{\sigma}_j \equiv (\boldsymbol{\sigma})_{s'_j,s_j} \equiv \langle \frac{1}{2}s'_j|\boldsymbol{\sigma}|\frac{1}{2}s_j\rangle \ , \quad \vec{\tau}_j \equiv (\vec{\tau})_{t'_j,t_j} \equiv \langle \frac{1}{2}t'_j|\vec{\tau}|\frac{1}{2}t_j\rangle \ , \tag{4.2}$$

which are the spin (isospin) matrix element between the final state s'_j (t'_j) and the intial state s_j (t_j) of the nucleon j.

Because $\boldsymbol{k}_1 = -\boldsymbol{k}_2 \equiv \boldsymbol{k}$ from the overall momentum conservation $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}'_1 + \boldsymbol{p}'_2$, the momentum-space potential V is a function of the momentum variables $\boldsymbol{k}$, $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, namely

$$\langle \alpha'_1\alpha'_2|V_{PV}|\alpha_1\alpha_2\rangle = \frac{1}{\Omega}V_{PV}(\boldsymbol{k},\boldsymbol{K}_1,\boldsymbol{K}_2)\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2} \ , \tag{4.3}$$

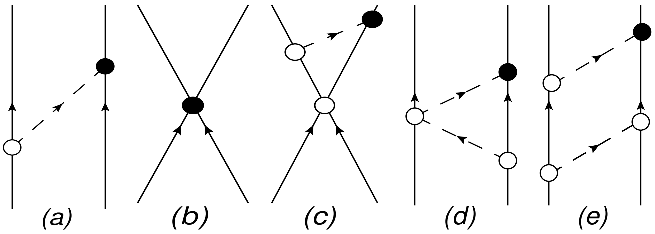


Figure 4.1: Diagrams contributing up to N2LO to the PV potential. Nucleons and pions are denoted by solid and dashed lines, respectively. The open (solid) circles represent PC (PV) vertices.

where $\alpha_j = \{\boldsymbol{p}_j, s_j, t_j\}$ and the dependence on the spin-isospin quantum number is understood. Moreover, we can write

$$V_{PV}(\boldsymbol{k}, \boldsymbol{K}_1, \boldsymbol{K}_2) = V_{PV}^{(\mathrm{CM})}(\boldsymbol{k}, \boldsymbol{K}) + V_{PV}^{(\boldsymbol{P})}(\boldsymbol{k}, \boldsymbol{K}) \ , \tag{4.4}$$

where $\boldsymbol{K} = (\boldsymbol{K}_1 - \boldsymbol{K}_2)/2$, $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{K}_1 + \boldsymbol{K}_2$, and the term $V_{PV}^{(\boldsymbol{P})}(\boldsymbol{k}, \boldsymbol{K})$ represents a boost correction to $V_{PV}^{(\mathrm{CM})}(\boldsymbol{k}, \boldsymbol{K})$, the potential in the center-of-mass frame (CM). Below we will ignore the boost correction and provide expressions for $V_{PV}^{(\mathrm{CM})}(\boldsymbol{k}, \boldsymbol{K})$ only.

4.2 The PV potential up to N2LO

In this section we will give a detailed derivation of the PV potential from the time ordering diagrams as discussed in Chapter 3. The diagrams contributing to the T-matrix up to N2LO are shown in Fig. 4.1 in panels $(a)-(e)$. The one pion exchange diagrams (a) give a contribution to the T-matrix of order Q^{-1} (that will be our LO) and then other contributions of higher order coming from the successive orders of the NR expansion of the vertex functions ${}^{PV}M^{\pi NN,01}$ and ${}^{PC}M^{\pi NN,01}$. The diagrams (b) represent a PV contact interaction of order Q; also the diagrams (c) with the PC contact vertex and one pion exchange give a contribution of order Q. The triangle diagrams (d) with a PC $\pi\pi NN$ vertex is of order Q, while if we consider the PV $\pi\pi NN$ vertex, the diagrams will be of order Q^2, so it will be considered in the next section. The box diagrams (e) includes contribution of order Q^0 and Q; as we will see the contribution of order Q^0 is cancelled exactly by the terms $V_{PV}^{(-1)}G_0V_{PC}^{(0)} + V_{PC}^{(0)}G_0V_{PV}^{(-1)}$ in Eq. (3.65). We don't consider vertex corrections or where there are dressed propagators as in panels (1), (2), (3) of Fig. 4.2, they give simply a renormalization of the coupling constant h_π^1 and of the masses, see Ref. [25] for more details. For our aim it is just sufficent to say that the formulas below are given in terms of the renormalized (physical) LECs and masses. The contribution of diagram (4) is cancelled by

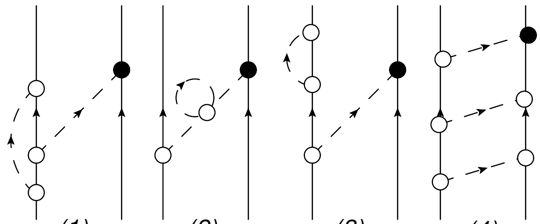


Figure 4.2: Other diagrams that give contribution to the N2LO. These diagrams contribute to the renoramlization of the LECs (panels (1), (2) and (3)) or give a vanishing contribution to the potential (panel (4)) due to the subtraction terms given in Eqs. (3.64)-(3.66). Notation as in Fig. 4.1.

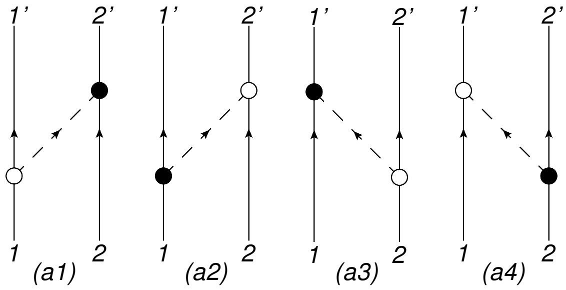


Figure 4.3: Time ordered diagrams that contribute to the PV OPE. Notation as in Fig. 4.1.

the last three terms of Eq. (3.66). Therefore $V_{PV} = V_{PV}^{(-1)} + V_{PV}^{(1)}$. The detailed calculation of each diagram is given in the next subsections.

4.2.1 One pion exchange

The complete set of time ordered diagrams that contribute are shown in Fig. 4.3. Using Eq. (3.38) from these diagrams we can derive the following expressions to the T-matrix

$$
\begin{aligned}
T(a1+a2) &= \frac{1}{\Omega}\sum_{\boldsymbol{q}j} \frac{1}{E_0-(E_{1'}+E_2+\omega_q)} \Big({}^{PV}M^{\pi NN,01}_{\alpha_2'\alpha_2,\boldsymbol{q}j}\; {}^{PC}M^{\pi NN,10}_{\alpha_1'\alpha_1,\boldsymbol{q}j} \\
&+ {}^{PC}M^{\pi NN,01}_{\alpha_2'\alpha_2,\boldsymbol{q}j}\; {}^{PV}M^{\pi NN,10}_{\alpha_1'\alpha_1,\boldsymbol{q}j}\Big) \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'}\delta_{-\boldsymbol{k},\boldsymbol{q}}\ , \qquad (4.5)\\
T(a3+a4) &= \frac{1}{\Omega}\sum_{\boldsymbol{q}j} \frac{1}{E_0-(E_1+E_{2'}+\omega_q)} \Big({}^{PV}M^{\pi NN,01}_{\alpha_1'\alpha_1,\boldsymbol{q}j}\; {}^{PC}M^{\pi NN,10}_{\alpha_2'\alpha_2,\boldsymbol{q}j} \\
&+ {}^{PC}M^{\pi NN,01}_{\alpha_1'\alpha_1,\boldsymbol{q}j}\; {}^{PV}M^{\pi NN,10}_{\alpha_2'\alpha_2,\boldsymbol{q}j}\Big) \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'}\delta_{\boldsymbol{k},\boldsymbol{q}}\ , \qquad (4.6)
\end{aligned}
$$

where $E_1 \equiv E_{\alpha_1}$, $E_{1'} \equiv E_{\alpha'_1}$ etc. In the energy denominator in Eqs. (4.5) and (4.6) we have neglected the term $+i\epsilon$ because at low energy $\Delta E \ll \omega_q$, where $\Delta E = E_0 - E_{1'} - E_2$ or $\Delta E = E_0 - E_1 - E_{2'}$, and since $\omega_q \geq m_\pi$ the denominator cannot vanish. Above, E_0 is defined as the initial scattering energy, $E_0 = E_1 + E_2$. Morover, the conservation of energy enforces $E_1 + E_2 = E_{1'} + E_{2'}$ which is the final energy. In the CM we have also $E_1 = E_{1'}$ and E_2=$E_{2'}$, so in the present case $\Delta E = 0$. Integrating over $\boldsymbol{q}$ we have finally

$$\begin{aligned} T(a1+a2) &= -\frac{1}{\Omega}\frac{1}{\omega_k}\Big({}^{PV}M^{\pi NN,01}_{\alpha'_2\alpha_2,-\boldsymbol{k}j}\ {}^{PC}M^{\pi NN,10}_{\alpha'_1\alpha_1,-\boldsymbol{k}j} + \\ &\qquad {}^{PC}M^{\pi NN,01}_{\alpha'_2\alpha_2,-\boldsymbol{k}j}\ {}^{PV}M^{\pi NN,10}_{\alpha'_1\alpha_1,-\boldsymbol{k}j}\Big)\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2}\ , \qquad (4.7) \\ T(a3+a4) &= -\frac{1}{\Omega}\frac{1}{\omega_k}\Big({}^{PV}M^{\pi NN,01}_{\alpha'_1\alpha_1,\boldsymbol{k}j}\ {}^{PC}M^{\pi NN,10}_{\alpha'_2\alpha_2,\boldsymbol{k}j} + \\ &\qquad {}^{PC}M^{\pi NN,01}_{\alpha'_1\alpha_1,\boldsymbol{k}j}\ {}^{PV}M^{\pi NN,10}_{\alpha'_2\alpha_2,\boldsymbol{k}j}\Big)\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2}\ , \qquad (4.8) \end{aligned}$$

where the sum over j is implied (remember that $\boldsymbol{k}_1 = \boldsymbol{p}'_1 - \boldsymbol{p}_1$). We can now use the expressions for the vertex functions given in Appendix C and obtain the various terms of order Q^n. The lowest order (Q^{-1}) is a non relativistic (NR) term which reads

$$T^{(-1)}_{PV}(\mathrm{NR}-a) = \frac{g_A h^1_\pi}{2\sqrt{2}f_\pi}\left(\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2\right)_z \frac{i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)}{\omega_k^2}\ . \qquad (4.9)$$

From Eq. (3.64) we can derive the expression $V^{(-1)}_{PV} = T^{(-1)}_{PV}$

The contributions to the T-matrix at order Q^0 and Q^2 vanish. In fact, at these orders, the contribution of the time ordered diagram (a1) cancels out the contribution of diagram (a3) and analogously for diagrams (a2) and (a4). At order Q^1 we obtain the term

$$T^{(1)}_{PV}(\mathrm{REN}-a) = T^{(-1)}(\mathrm{NR})\Big[\frac{2m_\pi^2}{g_A}(2d_{16}-d_{18}) - \frac{8\sqrt{2}m_\pi^2}{h^1_\pi f_\pi^2}(h^1_2 - h^1_3)\Big]\ , \qquad (4.10)$$

which simply gives results to a renormalization of the LECs h^1_π, and the term

$$\begin{aligned} T^{(1)}_{PV}(\mathrm{RC}-a) &= \frac{g_A h^1_\pi}{2\sqrt{2}f_\pi}\frac{1}{4M^2}(\vec{\tau}_1\times\vec{\tau}_2)_z\ \frac{1}{\omega_k^2}[-4iK^2\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) \\ &\quad +\boldsymbol{k}\cdot\boldsymbol{\sigma}_1\ (\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}_2 + \boldsymbol{k}\cdot\boldsymbol{\sigma}_2\ (\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}_1]\ , \qquad (4.11) \end{aligned}$$

which can be interpreted as a relativistic correction (RC). To obtain the potential we can now use Eqs. (3.64)-(3.66). The subtraction terms in Eq. (3.65) will be effective for cancelling the Q^0 contributions of diagrams (e) of Fig. 4.1 (see later), and since $T^{(0)} = 0$ we have $V^{(0)} = 0$. From Eq. (3.66), we obtain $V^{(1)}_{PV}(a) = T^{(1)}_{PV}(a)$, since the subtraction terms are effective only to cancel (partially) the contributions of the diagrams with only nucleons in at least one of the intermediate states.

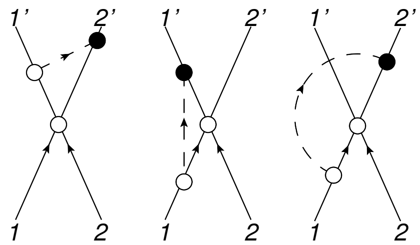


Figure 4.4: Time-ordered diagrams with a PC contact term and a exchange of a pion. Notation as in Fig. 4.1.

4.2.2 Contact terms

From the diagram (b) in Fig. 4.1 we obtain

$$T(b) = \frac{1}{\Omega} 2\ {}^{PV}M^{00}_{\alpha_1'\alpha_1\alpha_2'\alpha_2}\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'}\ , \tag{4.12}$$

where ${}^{PV}M^{00}$ is given in Eq. (A.40) of Appendix A. The T-matrix derived from this contribution is

$$\begin{aligned} T^{(1)}_{PV}(b) &= \frac{1}{\Lambda_\chi m_\pi^2}[C_1 i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k} + C_2(\vec{\tau}_1\cdot\vec{\tau}_2)i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k} \\ &\quad + C_3(\vec{\tau}_1\times\vec{\tau}_2)_z i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k} + C_4(\tau_{1z}+\tau_{2z})i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k} \\ &\quad + C_5\mathcal{I}^{ab}(\tau_1)_a(\tau_2)_b i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}]\ . \end{aligned} \tag{4.13}$$

Also in this case from Eq. (3.66) we find $V^{(1)}_{PV}(b) = T^{(1)}_{PV}(b)$.

The diagram (c) in Fig. 4.1 is representative of the three kinds of time ordered diagrams with a contact term and a OPE shown in Fig. 4.4. However all these diagrams give a vanishing contribution after the integration over the loop variable.

4.2.3 Two pions exchange: triangle diagrams

The terms $H^{01}G_0H^{01}G_0H^{20} + H^{01}G_0H^{11}G_0H^{10} + H^{02}G_0H^{01}G_0H^{01}$ in the expression of the T-matrix give 12 two pions exchange (TPE) diagrams, six of them given explicitly in Fig. 4.5, plus six other diagrams with the $\pi\pi NN$ vertex attached to nucleon 2. From these diagrams we obtain the following contribution to the T-matrix:

$$\begin{aligned} T(d1) &= \sum_{\boldsymbol{q}_1 j_1,\boldsymbol{q}_2 j_2\beta} \frac{{}^{PV}M^{\pi NN,01}_{\alpha_2'\beta,\boldsymbol{q}_2 j_2}\ {}^{PC}M^{\pi\pi NN,11}_{\alpha_1'\alpha_1,\boldsymbol{q}_2 j_2\boldsymbol{q}_1 j_1}\ {}^{PC}M^{\pi NN,10}_{\beta\alpha_2,\boldsymbol{q}_1 j_1}}{(E_0-(E_\beta+E_1+\omega_{q_1}))(E_0-(E_\beta+E_{1'}+\omega_{q_2}))} \\ &\quad \times\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'}\delta_{\boldsymbol{q}_2+\boldsymbol{p}_1',\boldsymbol{q}_1+\boldsymbol{p}_1}\delta_{\boldsymbol{p}_\beta+\boldsymbol{q}_1,\boldsymbol{p}_2} \\ T(d2) &= \sum_{\boldsymbol{q}_1 j_1,\boldsymbol{q}_2 j_2\beta} \frac{{}^{PC}M^{\pi NN,01}_{\alpha_2'\beta,\boldsymbol{q}_2 j_2}\ {}^{PC}M^{\pi\pi NN,11}_{\alpha_1'\alpha_1,\boldsymbol{q}_2 j_2\boldsymbol{q}_1 j_1}\ {}^{PV}M^{\pi NN,10}_{\beta\alpha_2,\boldsymbol{q}_1 j_1}}{(E_0-(E_\beta+E_1+\omega_{q_1}))(E_0-(E_\beta+E_{1'}+\omega_{q_2}))} \end{aligned}$$

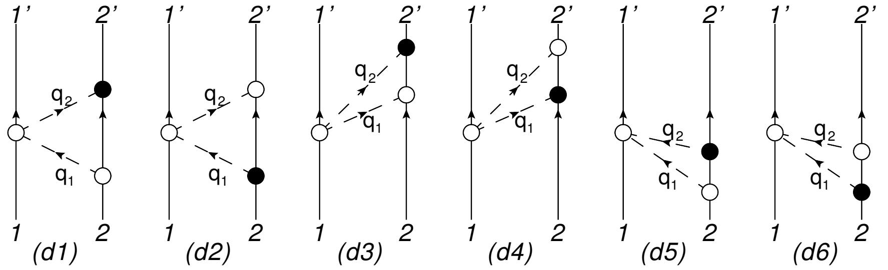


Figure 4.5: Triangle diagrams that contribute to the PV TPE. Notation as in Fig. 4.1.

$$
\begin{aligned}
&\times \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2}\delta_{\boldsymbol{q}_2+\boldsymbol{p}'_1,\boldsymbol{q}_1+\boldsymbol{p}_1}\delta_{\boldsymbol{p}_\beta+\boldsymbol{q}_1,\boldsymbol{p}_2} \\
T(d3) \;=\; &\sum_{\boldsymbol{q}_1 j_1,\boldsymbol{q}_2 j_2 \beta} \frac{{}^{PV}M^{\pi NN,01}_{\alpha'_2\beta,\boldsymbol{q}_2 j_2}\;{}^{PC}M^{\pi NN,01}_{\beta\alpha_2,\boldsymbol{q}_1 j_1}\;{}^{PC}\overline{M}^{\pi\pi NN,20}_{\alpha'_1\alpha_1,\boldsymbol{q}_2 j_2 \boldsymbol{q}_1 j_1}}{(E_0-(E_{1'}+E_\beta+\omega_{q_2}))(E_0-(E_2+E_{1'}+\omega_{q_1}+\omega_{q_2}))} \\
&\times \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2}\delta_{\boldsymbol{p}_1,\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{p}'_1}\delta_{\boldsymbol{p}_\beta,\boldsymbol{q}_1+\boldsymbol{p}_2} \\
T(d4) \;=\; &\sum_{\boldsymbol{q}_1 j_1,\boldsymbol{q}_2 j_2 \beta} \frac{{}^{PC}M^{\pi NN,01}_{\alpha'_2\beta,\boldsymbol{q}_2 j_2}\;{}^{PV}M^{\pi NN,01}_{\beta\alpha_2,\boldsymbol{q}_1 j_1}\;{}^{PC}\overline{M}^{\pi\pi NN,20}_{\alpha'_1\alpha_1,\boldsymbol{q}_2 j_2 \boldsymbol{q}_1 j_1}}{(E_0-(E_{1'}+E_\beta+\omega_{q_2}))(E_0-(E_2+E_{1'}+\omega_{q_1}+\omega_{q_2}))} \\
&\times \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2}\delta_{\boldsymbol{p}_1,\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{p}'_1}\delta_{\boldsymbol{p}_\beta,\boldsymbol{q}_1+\boldsymbol{p}_2} \\
T(d5) \;=\; &\sum_{\boldsymbol{q}_1 j_1,\boldsymbol{q}_2 j_2 \beta} \frac{{}^{PC}\overline{M}^{\pi\pi NN,02}_{\alpha'_1\alpha_1,\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2}\;{}^{PV}M^{\pi NN,10}_{\alpha'_2\beta,\boldsymbol{q}_2 j_2}\;{}^{PC}M^{\pi NN,10}_{\beta\alpha_2,\boldsymbol{q}_1 j_1}}{(E_0-(E_\beta+E_1+\omega_{q_1}))(E_0-(E_{2'}+E_1+\omega_{q_1}+\omega_{q_2}))} \\
&\times \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2}\delta_{\boldsymbol{p}_1+\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{p}'_1}\delta_{\boldsymbol{p}_\beta+\boldsymbol{q}_1,\boldsymbol{p}_2} \\
T(d6) \;=\; &\sum_{\boldsymbol{q}_1 j_1,\boldsymbol{q}_2 j_2 \beta} \frac{{}^{PC}\overline{M}^{\pi\pi NN,02}_{\alpha'_1\alpha_1,\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2}\;{}^{PC}M^{\pi NN,10}_{\alpha'_2\beta,\boldsymbol{q}_2 j_2}\;{}^{PV}M^{\pi NN,10}_{\beta\alpha_2,\boldsymbol{q}_1 j_1}}{(E_0-(E_\beta+E_1+\omega_{q_1}))(E_0-(E_{2'}+E_1+\omega_{q_1}+\omega_{q_2}))} \\
&\times \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2}\delta_{\boldsymbol{p}_1+\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{p}'_1}\delta_{\boldsymbol{p}_\beta+\boldsymbol{q}_1,\boldsymbol{p}_2}\;,
\end{aligned}
\tag{4.14}
$$

where

$$
{}^{PC}\overline{M}^{\pi\pi NN,02}_{\alpha'_1\alpha_1,\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} = {}^{PC}M^{\pi\pi NN,02}_{\alpha'_1\alpha_1,\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} + {}^{PC}M^{\pi\pi NN,02}_{\alpha'_1\alpha_1,\boldsymbol{q}_2 j_2 \boldsymbol{q}_1 j_1}\;, \tag{4.15}
$$

$$
{}^{PC}\overline{M}^{\pi\pi NN,20}_{\alpha'_1\alpha_2,\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} = {}^{PC}M^{\pi\pi NN,20}_{\alpha'_1\alpha_1,\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} + {}^{PC}M^{\pi\pi NN,20}_{\alpha'_1\alpha_1,\boldsymbol{q}_2 j_2 \boldsymbol{q}_1 j_1}\;, \tag{4.16}
$$

since we have two ways to contract the creation-annihilation operators in the T-matrix elements. Above β indicates the quantum numbers of the nucleons in the loop. We can neglect again the term $+i\epsilon$ in the energy denominators. Expanding non relativistically the energies of the nucleons, summing over β, j_1, j_2 and using the vertex functions $M^{\pi\pi NN,02}$, $M^{\pi\pi NN,11}$, $M^{\pi\pi NN,20}$ given in Eqs. (C.24)-(C.26) of the Appendix C, at the lowest order we obtain:

$$
\begin{aligned}
T(d1+d2) &= \frac{1}{\Omega^2} i \frac{g_A h_\pi^1}{32\sqrt{2} f_\pi^3} (\vec{\tau}_1 \times \vec{\tau}_2)_z \sum_{\boldsymbol{q}_1, \boldsymbol{q}_2} \frac{\omega_{q_1} + \omega_{q_2}}{\omega_{q_1}^2 \omega_{q_2}^2} (\boldsymbol{q}_1 - \boldsymbol{q}_2) \boldsymbol{\sigma}_2 \\
& \quad \times \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}'_1 + \boldsymbol{p}'_2} \delta_{\boldsymbol{q}_2 + \boldsymbol{p}'_1, \boldsymbol{q}_1 + \boldsymbol{p}_1} , \qquad (4.17) \\
T(d3+d4) &= \frac{1}{\Omega^2} i \frac{g_A h_\pi^1}{32\sqrt{2} f_\pi^3} (\vec{\tau}_1 \times \vec{\tau}_2)_z \sum_{\boldsymbol{q}_1, \boldsymbol{q}_2} \frac{\omega_{q_2} - \omega_{q_1}}{\omega_{q_2}^2 \omega_{q_1} (\omega_{q_1} + \omega_{q_2})} (\boldsymbol{q}_2 + \boldsymbol{q}_1) \boldsymbol{\sigma}_2 \\
& \quad \times \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}'_1 + \boldsymbol{p}'_2} \delta_{\boldsymbol{p}_1 + \boldsymbol{q}_1 + \boldsymbol{q}_2, \boldsymbol{p}'_1} , \qquad (4.18) \\
T(d5+d6) &= \frac{1}{\Omega^2} i \frac{g_A h_\pi^1}{32\sqrt{2} f_\pi^3} (\vec{\tau}_1 \times \vec{\tau}_2)_z \sum_{\boldsymbol{q}_1, \boldsymbol{q}_2} \frac{\omega_{q_2} - \omega_{q_1}}{\omega_{q_1}^2 \omega_{q_2} (\omega_{q_1} + \omega_{q_2})} (\boldsymbol{q}_2 + \boldsymbol{q}_1) \boldsymbol{\sigma}_2 \\
& \quad \times \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}'_1 + \boldsymbol{p}'_2} \delta_{\boldsymbol{p}_1, \boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{p}'_1} . \qquad (4.19)
\end{aligned}
$$

We can redefine $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ to obtain the same δ in the different expressions and then we have

$$
\begin{aligned}
T(d1+\cdots+d6) &= \frac{1}{\Omega^2} i \frac{g_A h_\pi^1}{8\sqrt{2} f_\pi^3} (\vec{\tau}_1 \times \vec{\tau}_2)_z \sum_{\boldsymbol{q}_1, \boldsymbol{q}_2} \frac{1}{\omega_{q_1} \omega_{q_2} (\omega_{q_1} + \omega_{q_2})} (\boldsymbol{q}_1 + \boldsymbol{q}_2) \boldsymbol{\sigma}_2 \\
& \quad \times \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}'_1 + \boldsymbol{p}'_2} \delta_{\boldsymbol{k}, \boldsymbol{q}_1 + \boldsymbol{q}_2} . \qquad (4.20)
\end{aligned}
$$

We note that this contribution is of order Q. Let us define

$$
\boldsymbol{Q} = \frac{\boldsymbol{q}_1 + \boldsymbol{q}_2}{2} , \qquad \boldsymbol{q} = \boldsymbol{q}_2 - \boldsymbol{q}_2 , \qquad (4.21)
$$

and perform a change in the integration variables

$$
\sum_{\boldsymbol{q}_1, \boldsymbol{q}_2} \longrightarrow \int \frac{d^3 Q}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} . \qquad (4.22)
$$

This type of variable change will be used frequently in the integration of diagrams with one loop.

Thus our integral becomes, after exploiting the delta function $\delta_{\boldsymbol{k}, \boldsymbol{q}_1 + \boldsymbol{q}_2}$:

$$
\begin{aligned}
T(d1+\cdots+d6) &= \frac{1}{\Omega^2} \frac{g_A h_\pi^1}{8\sqrt{2} f_\pi^3} (\vec{\tau}_1 \times \vec{\tau}_2)_z i \boldsymbol{k} \cdot \boldsymbol{\sigma}_2 \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\omega_+ \omega_- (\omega_+ + \omega_-)} \\
& \quad \times \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}'_1 + \boldsymbol{p}'_2} , \qquad (4.23)
\end{aligned}
$$

where $\omega_\pm = \sqrt{(\boldsymbol{q} \pm \boldsymbol{k})^2 + 4m_\pi^2}$.

Summing also the contribution of the other six diagrams, the total contribution from the triangle diagrams is

$$
T_{PV}^{(1)}(d) = \frac{g_A h_\pi^1}{8\sqrt{2} f_\pi^3} (\vec{\tau}_1 \times \vec{\tau}_2)_z i \boldsymbol{k} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\omega_+ \omega_- (\omega_+ + \omega_-)} . \qquad (4.24)
$$

From Eq. (3.66) we find $V_{PV}^{(1)}(d) = T_{PV}^{(1)}(d)$.

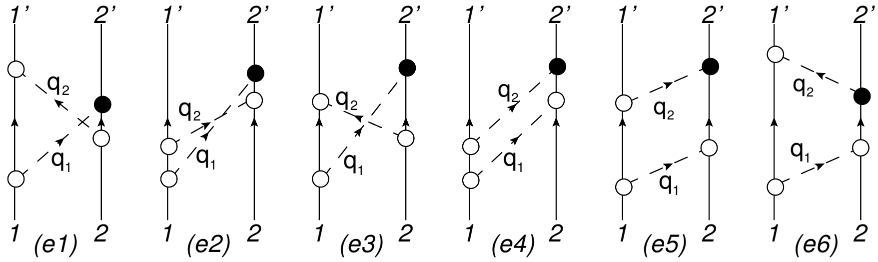


Figure 4.6: Box diagrams that contribute to the PV TPE. Notation as in Fig. 4.1.

4.2.4 Two pions exchange: box diagrams

Let's consider the terms $H^{01}G_0H^{01}G_0H^{10}G_0H^{10}$ and $H^{01}G_0H^{10}G_0H^{01}G_0H^{10}$ in the expression of T-matrix. These terms give 48 diagrams represented by the diagram of type (e) in Fig. 4.1 plus all possible time ordered. A representative group of them is shown in Fig. 4.6 where we have depicted a single time-ordered for each type. Moreover for each diagram, one needs to consider the possible diagrams taking into account all possible positions of the PV vertex.

The diagrams in Fig. 4.6, approximating $E_{\boldsymbol{p}_i} \approx M$, correspond to the following expressions:

$$
\begin{aligned}
T(e1) &= \frac{1}{\Omega^2} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2 \beta\gamma} \frac{-1}{\omega_{q_1}\omega_{q_2}\left(\omega_{q_1}+\omega_{q_2}\right)} \\
&\quad \left[{}^{PC}M^{\pi NN,01}_{\alpha_1'\beta,\boldsymbol{q}_2 j_2}\; {}^{PV}M^{\pi NN,01}_{\alpha_2'\gamma,\boldsymbol{q}_1 j_1}\; {}^{PC}M^{\pi NN,10}_{\gamma\alpha_2,\boldsymbol{q}_2 j_2}\; {}^{PC}M^{\pi NN,10}_{\beta\alpha_1,\boldsymbol{q}_1 j_1} \right] \\
&\quad \times \delta_{\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{p}_2'-\boldsymbol{p}_2}\, \delta_{\boldsymbol{p}_\beta+\boldsymbol{q}_2,\boldsymbol{p}_1'}\, \delta_{\boldsymbol{p}_\gamma+\boldsymbol{q}_1,\boldsymbol{p}_2'}\, \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'} \\
T(e2) &= \frac{1}{\Omega^2} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2 \beta\gamma} \frac{-1}{\omega_{q_1}^2\left(\omega_{q_1}+\omega_{q_2}\right)} \\
&\quad \left[{}^{PV}M^{\pi NN,01}_{\alpha_2'\gamma,\boldsymbol{q}_1 j_1}\; {}^{PC}M^{\pi NN,01}_{\gamma\alpha_2,\boldsymbol{q}_2 j_2}\; {}^{PC}M^{\pi NN,10}_{\alpha_1'\beta,\boldsymbol{q}_2 j_2}\; {}^{PC}M^{\pi NN,10}_{\beta\alpha_1,\boldsymbol{q}_1 j_1} \right] \\
&\quad \times \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{p}_2'-\boldsymbol{p}_2}\, \delta_{\boldsymbol{p}_\beta,\boldsymbol{p}_1'+\boldsymbol{q}_2}\, \delta_{\boldsymbol{p}_\gamma+\boldsymbol{q}_1,\boldsymbol{p}_2'}\, \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'} \\
T(e3) &= \frac{1}{\Omega^2} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2 \beta\gamma} \frac{-1}{\omega_{q_1}^2\left(\omega_{q_1}+\omega_{q_2}\right)} \\
&\quad \left[{}^{PV}M^{\pi NN,01}_{\alpha_2'\gamma,\boldsymbol{q}_1 j_1}\; {}^{PC}M^{\pi NN,10}_{\alpha_1'\beta,\boldsymbol{q}_2 j_2}\; {}^{PC}M^{\pi NN,01}_{\gamma\alpha_2,\boldsymbol{q}_2 j_2}\; {}^{PC}M^{\pi NN,10}_{\beta\alpha_1,\boldsymbol{q}_1 j_1} \right] \\
&\quad \times \delta_{\boldsymbol{q}_1-\boldsymbol{q}_2,\boldsymbol{p}_2'-\boldsymbol{p}_2}\, \delta_{\boldsymbol{p}_\beta+\boldsymbol{q}_2,\boldsymbol{p}_1'}\, \delta_{\boldsymbol{p}_\gamma+\boldsymbol{q}_1,\boldsymbol{p}_2'}\, \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'} \\
T(e4) &= \frac{1}{\Omega^2} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2 \beta\gamma} \frac{-1}{\omega_{q_1}\omega_{q_2}\left(\omega_{q_1}+\omega_{q_2}\right)} \\
&\quad \left[{}^{PV}M^{\pi NN,01}_{\alpha_2'\gamma,\boldsymbol{q}_2 j_2}\; {}^{PC}M^{\pi NN,01}_{\gamma\alpha_2,\boldsymbol{q}_2 j_2}\; {}^{PC}M^{\pi NN,10}_{\alpha_1'\beta,\boldsymbol{q}_2 j_2}\; {}^{PC}M^{\pi NN,10}_{\beta\alpha_1,\boldsymbol{q}_1 j_1} \right] \\
&\quad \times \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{p}_2'-\boldsymbol{p}_2}\, \delta_{\boldsymbol{p}_\beta,\boldsymbol{p}_1'+\boldsymbol{q}_2}\, \delta_{\boldsymbol{p}_\gamma,\boldsymbol{p}_2+\boldsymbol{q}_1}\, \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'}
\end{aligned}
$$

$$. \tag{4.25}$$

For the diagrams (e5) and (e6) the calculation is a little bit more complicated because we have a pure nucleonic intermediate state. The expression are the following

$$\begin{aligned} T(e5) &= \frac{1}{\Omega^2} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2, \beta\gamma} \frac{1}{(E_0 - (E_\beta + E_2 + \omega_{q_1}))\,(E_0 - (E_\gamma + E_{1'} + \omega_{q_2}))} \\ &\times \frac{1}{(E_0 - (E_\beta + E_\gamma) + i\epsilon)} \delta_{\boldsymbol{q}_1 + \boldsymbol{q}_2, \boldsymbol{p}'_2 - \boldsymbol{p}_2} \delta_{\boldsymbol{p}_\beta, \boldsymbol{p}'_1 + \boldsymbol{q}_2} \delta_{\boldsymbol{p}_\gamma, \boldsymbol{q}_1 + \boldsymbol{p}_2} \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}'_1 + \boldsymbol{p}'_2} \\ &\left[{}^{PV}M^{\pi NN,01}_{\alpha'_2\gamma, \boldsymbol{q}_2 j_2} \; {}^{PC}M^{\pi NN,10}_{\alpha'_1\beta, \boldsymbol{q}_2 j_2} \; {}^{PC}M^{\pi NN,01}_{\gamma\alpha_2, \boldsymbol{q}_1 j_1} \; {}^{PC}M^{\pi NN,10}_{\beta\alpha_1, \boldsymbol{q}_1 j_1} \right] \,, \end{aligned} \tag{4.26}$$

$$\begin{aligned} T(e6) &= \frac{1}{\Omega^2} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2, \beta\gamma} \frac{1}{(E_0 - (E_\beta + E_2 + \omega_{q_1}))\,(E_0 - (E_\beta + E_{2'} + \omega_{q_2}))} \\ &\times \frac{1}{(E_0 - (E_\beta + E_\gamma) + i\epsilon)} \delta_{\boldsymbol{q}_1 - \boldsymbol{q}_2, \boldsymbol{p}'_2 - \boldsymbol{p}_2} \delta_{\boldsymbol{p}_\beta + \boldsymbol{q}_2, \boldsymbol{p}'_1} \delta_{\boldsymbol{p}_\gamma, \boldsymbol{q}_1 + \boldsymbol{p}_2} \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}'_1 + \boldsymbol{p}'_2} \\ &\left[{}^{PC}M^{\pi NN,01}_{\alpha'_1\beta, \boldsymbol{q}_2 j_2} \; {}^{PV}M^{\pi NN,10}_{\alpha'_2\gamma, \boldsymbol{q}_2 j_2} \; {}^{PC}M^{\pi NN,01}_{\gamma\alpha_2, \boldsymbol{q}_1 j_1} \; {}^{PC}M^{\pi NN,10}_{\beta\alpha_1, \boldsymbol{q}_1 j_1} \right] \,. \end{aligned} \tag{4.27}$$

Note that expanding the vertex function at the leading order, $T(e5)$ and $T(e6)$ would be of order Q^0. In one intermediate state there are no pions, therefore we cannot eliminate the term $+i\epsilon$ because $E_0 - (E_\beta + E_\gamma)$ could be zero. To obtain V_{PV}, we must subtract the contribution of $V^{(-1)}_{PV} G_0 V^{(0)}_{PC} + V^{(0)}_{PC} G_0 V^{(-1)}_{PV}$ as given in Eq. (3.65). The matrix element $\langle \alpha_{1'}\alpha_{1'} | V^{(-1)}_{PV} G_0 V^{(0)}_{PC} | \alpha_1\alpha_2 \rangle$ gives a series of contributions, two of them are

$$\begin{aligned} T(e5') &= \frac{1}{\Omega^2} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2, \beta\gamma} \frac{\left[{}^{PV}M^{\pi NN,01}_{\alpha'_2\gamma, \boldsymbol{q}_2 j_2} \; {}^{PC}M^{\pi NN,10}_{\alpha'_1\beta, \boldsymbol{q}_2 j_2} \; {}^{PC}M^{\pi NN,01}_{\gamma\alpha_2, \boldsymbol{q}_1 j_1} \; {}^{PC}M^{\pi NN,10}_{\beta\alpha_1, \boldsymbol{q}_1 j_1} \right]}{\omega_{q_1}\omega_{q_2}\,(E_0 - (E_\beta + E_\gamma) + i\epsilon)} \\ &\times \delta_{\boldsymbol{q}_1 + \boldsymbol{q}_2, \boldsymbol{p}'_2 - \boldsymbol{p}_2} \delta_{\boldsymbol{p}_\beta, \boldsymbol{p}'_1 + \boldsymbol{q}_2} \delta_{\boldsymbol{p}_\gamma, \boldsymbol{q}_1 + \boldsymbol{p}_2} \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}'_1 + \boldsymbol{p}'_2} \,, \end{aligned} \tag{4.28}$$

$$\begin{aligned} T(e6') &= \frac{1}{\Omega^2} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2, \beta\gamma} \frac{\left[{}^{PC}M^{\pi NN,01}_{\alpha'_1\beta, \boldsymbol{q}_2 j_2} \; {}^{PV}M^{\pi NN,10}_{\alpha'_2\gamma, \boldsymbol{q}_2 j_2} \; {}^{PC}M^{\pi NN,01}_{\gamma\alpha_2, \boldsymbol{q}_1 j_1} \; {}^{PC}M^{\pi NN,10}_{\beta\alpha_1, \boldsymbol{q}_1 j_1} \right]}{\omega_{q_1}\omega_{q_2}\,(E_0 - (E_\beta + E_\gamma) + i\epsilon)} \\ &\times \delta_{\boldsymbol{q}_1 - \boldsymbol{q}_2, \boldsymbol{p}'_2 - \boldsymbol{p}_2} \delta_{\boldsymbol{p}_\beta + \boldsymbol{q}_2, \boldsymbol{p}'_1} \delta_{\boldsymbol{p}_\gamma, \boldsymbol{q}_1 + \boldsymbol{p}_2} \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}'_1 + \boldsymbol{p}'_2} \,. \end{aligned} \tag{4.29}$$

As we can see, these corrections have the same vertex function product but differ in two of the energy denominators. Therefore the cancellation of the contribution (e5) and (e6) is not exact. Assuming however that $\Delta E_1 = E_0 - E_\beta - E_1$ and $\Delta E_2 = E_0 - E_\gamma - E_{1'}$ are of order Q^2, we have $\Delta E_i \ll \omega_{q_i}$. Therefore we can use the following Taylor expansion of these energy denominators,

$$\frac{1}{\Delta E_1 - \omega_{q_1}} \approx -\frac{1}{\omega_{q_1}} \left[1 + \frac{\Delta E_1}{\omega_{q_1}} + \cdots \right] \,. \tag{4.30}$$

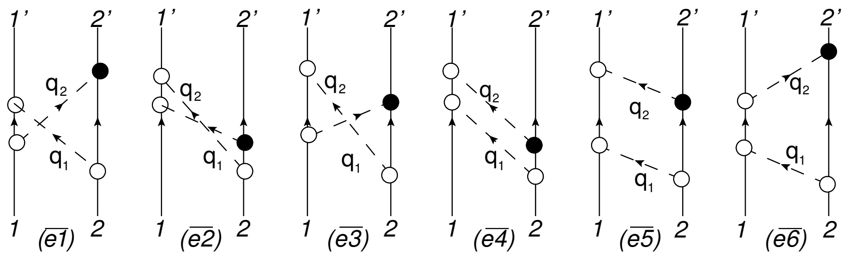


Figure 4.7: Box diagrams that summed with the diagrams in Fig. 4.6 permits to obtain a simple form for the T-matrix. Notation as in Fig. 4.1.

In summary, the subtraction of the terms $(e5)-(e5')$ reads

$$\begin{aligned} T(e5-e5') &= [\ldots]\Bigg[\frac{1}{(\Delta E_1-\omega_{q_1})}\frac{1}{(\Delta E_2-\omega_{q_2})}\frac{1}{(E_0-(E_\beta+E_\gamma)+i\epsilon)} \\ &\qquad -\frac{1}{\omega_{q_1}\omega_{q_2}\,(E_0-(E_\beta+E_\gamma)+i\epsilon)}\Bigg] \\ &= [\ldots]\,\frac{1}{\omega_{q_1}\omega_{q_2}}\frac{1}{(E_0-(E_\beta+E_\gamma)+i\epsilon)}\left(\frac{\Delta E_1}{\omega_{q_1}}+\frac{\Delta E_2}{\omega_{q_2}}\right)+\mathcal{O}(Q^2)\ , \end{aligned} \tag{4.31}$$

where $[\ldots]$ indicates schematically the product of the vertex functions. Practically the final contribution is due to a "recoil" of the nucleons in the intermediate states. Note that the contribution to V_{PV} is of order Q. As stated before no contribution of order Q^0 is found.

Considering also the corresponding contributions of the 6 diagrams where $\alpha_1 \rightleftharpoons \alpha_2$ and $\alpha_{1'} \rightleftharpoons \alpha_{2'}$ but where the PV vertex remains on the line of the nucleon on the right (diagrams $(\overline{e1})$-$(\overline{e6})$ in Fig. 4.7), the product of the δ and the vertex functions can be reduced to the same form as for the diagrams $(e1)$-$(e6)$. In particular

$$\begin{aligned} T(\overline{e5}) &= \frac{1}{\Omega^2}\sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2,\beta\gamma}\frac{1}{(E_0-(E_\gamma+E_1+\omega_{q_1}))\,(E_0-(E_\beta+E_{2'}+\omega_{q_2}))} \\ &\quad\times\frac{1}{(E_0-(E_\beta+E_\gamma)+i\epsilon)}\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{p}_2-\boldsymbol{p}'_2}\delta_{\boldsymbol{p}_\beta,\boldsymbol{p}_1+\boldsymbol{q}_1}\delta_{\boldsymbol{p}_\gamma,\boldsymbol{p}_2-\boldsymbol{q}_1}\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2} \\ &\quad\left[{}^{PC}M^{\pi NN,01}_{\alpha'_1\beta,\boldsymbol{q}_2 j_2}\;{}^{PV}M^{\pi NN,10}_{\alpha 2'\gamma,\boldsymbol{q}_2 j_2}\;{}^{PC}M^{\pi NN,01}_{\beta\alpha_1,\boldsymbol{q}_1 j_1}\;{}^{PC}M^{\pi NN,10}_{\gamma\alpha_2,\boldsymbol{q}_1 j_1}\right]\ , \end{aligned} \tag{4.32}$$

considering also the contribution coming from the subtraction term $V^{(-1)}_{PV}G_0V^{(0)}_{PC}$

$$T(\overline{e5'}) = \frac{1}{\Omega^2}\sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2,\beta\gamma}\frac{\left[{}^{PC}M^{\pi NN,01}_{\alpha'_1\beta,\boldsymbol{q}_2 j_2}\;{}^{PV}M^{\pi NN,10}_{\alpha 2'\gamma,\boldsymbol{q}_2 j_2}\;{}^{PC}M^{\pi NN,01}_{\beta\alpha_1,\boldsymbol{q}_1 j_1}\;{}^{PC}M^{\pi NN,10}_{\gamma\alpha_2,\boldsymbol{q}_1 j_1}\right]}{\omega_{q_1}\omega_{q_2}\,(E_0-(E_\beta+E_\gamma)+i\epsilon)}$$

$$\times \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{p}_2-\boldsymbol{p}'_2}\delta_{\boldsymbol{p}_\beta,\boldsymbol{p}_1+\boldsymbol{q}_1}\delta_{\boldsymbol{p}_\gamma,\boldsymbol{p}_2-\boldsymbol{q}_1}\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2} \ , \tag{4.33}$$

we see that the product of the vertex functions is always the same. Therefore

$$\begin{aligned} T(E5) &= T(e5) - T(e5') + T(\overline{e5}) - T(\overline{e5'}) \\ &\simeq [\ldots]\frac{1}{\omega_{q_1}\omega_{q_2}}\frac{\Delta E_1 + \Delta E_2}{(E_0 - (E_\beta + E_\gamma) + i\epsilon)}\left[\frac{1}{\omega_{q_1}} + \frac{1}{\omega_{q_2}}\right] \ , \\ &= [\ldots]\frac{1}{\omega_{q_1}\omega_{q_2}}\left[\frac{1}{\omega_{q_1}} + \frac{1}{\omega_{q_2}}\right] \ , \end{aligned} \tag{4.34}$$

since $\Delta E_1 + \Delta E_2 = E_0 - E_\beta - E_\gamma$ and the "dangerous" denominator cancels out. The same happens for $(e6)$, $(e6')$ and $(\overline{e6})$, $(\overline{e6'})$. Including also the contribution of the diagrams $(e4)$ and $(\overline{e4})$ we find

$$\begin{aligned} T(E5 + E6 + e4 + \overline{e4}) &= \frac{1}{\Omega^2}\sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2,\beta\gamma} \frac{2}{\omega_{q_1}\omega_{q_2}}\left(\frac{\omega_{q_1}^2 + \omega_{q_1}\omega_{q_2} + \omega_{q_2}^2}{\omega_{q_1}\omega_{q_2}\left(\omega_{q_1} + \omega_{q_2}\right)}\right) \\ &\times \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,-\boldsymbol{k}}\delta_{\boldsymbol{p}_\beta,\boldsymbol{p}'_1+\boldsymbol{q}_2}\delta_{\boldsymbol{p}_\gamma,\boldsymbol{q}_1+\boldsymbol{p}_2}\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2} \\ &\times \left[{}^{PV}M^{\pi NN,01}_{\alpha'_2\gamma,\boldsymbol{q}_2 j_2}\ {}^{PC}M^{\pi NN,10}_{\alpha'_1\beta,\boldsymbol{q}_2 j_2}\ {}^{PC}M^{\pi NN,01}_{\gamma\alpha_2,\boldsymbol{q}_1 j_1}\ {}^{PC}M^{\pi NN,10}_{\beta\alpha_1,\boldsymbol{q}_1 j_1}\right] \ . \end{aligned} \tag{4.35}$$

Similarly the vertex functions for the diagrams $(e1)$, $(e2)$, $(e3)$, $(\overline{e1})$, $(\overline{e2})$ and $(\overline{e3})$ can be summed up to obtain

$$\begin{aligned} T(e1 + \cdots + \overline{e3}) &= \frac{1}{\Omega^2}\sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2,\beta\gamma} -\frac{2(\omega_{q_1}^2 + \omega_{q_1}\omega_{q_2} + \omega_{q_2}^2)}{\omega_{q_1}^2\omega_{q_2}^2(\omega_{q_1} + \omega_{q_2})} \\ &\times \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{p}'_2-\boldsymbol{p}_2}\delta_{\boldsymbol{p}_\beta,\boldsymbol{p}'_1+\boldsymbol{q}_2}\delta_{\boldsymbol{p}_\gamma+\boldsymbol{q}_1,\boldsymbol{p}'_2}\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2} \\ &\times \left[{}^{PV}M^{\pi NN,01}_{\alpha'_2\gamma,\boldsymbol{q}_1 j_1}\ {}^{PC}M^{\pi NN,01}_{\gamma\alpha_2,\boldsymbol{q}_2 j_2}\ {}^{PC}M^{\pi NN,10}_{\alpha'_1\beta,\boldsymbol{q}_2 j_2}\ {}^{PC}M^{\pi NN,10}_{\beta\alpha_1,\boldsymbol{q}_1 j_1}\right] \ . \end{aligned} \tag{4.36}$$

Doing the same procedure for all the other possible time ordering diagrams, and integrating as in triangle diagrams we obtain the complete contribution of diagrams of type (e) of Fig. 4.1 at order Q.

$$\begin{aligned} V^{(1)}_{PV}(e) &= \frac{h^1_\pi g_A}{2\sqrt{2}f_\pi}\frac{g_A^2}{4f_\pi^2}\int\frac{d^3q}{(2\pi)^3}\frac{\omega_+^2 + \omega_+\omega_- + \omega_-^2}{\omega_+^3\omega_-^3\left(\omega_+ + \omega_-\right)} \\ &\{-2i\left(\tau_{1z} + \tau_{2z}\right)\left[\boldsymbol{q}\cdot\boldsymbol{\sigma}_1(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_2 - \boldsymbol{q}\cdot\boldsymbol{\sigma}_2(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_1\right] \\ &-2i\left(\tau_{1z} - \tau_{2z}\right)\left[\boldsymbol{q}\cdot\boldsymbol{\sigma}_2(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_1 + \boldsymbol{q}\cdot\boldsymbol{\sigma}_1(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_2\right] \\ &+i\left(\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2\right)_z\left(k^2 - q^2\right)\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\} \ . \end{aligned} \tag{4.37}$$

We remember that in this case we have alredy subtracted the term $V^{(-1)}_{PV}G_0V^{(0)}_{PC} + V^{(0)}_{PC}G_0V^{(-1)}_{PV}$ in Eq. (3.66) and therefore we have obtained directly the expression of $V^{(1)}_{PV}(e)$.

4.2.5 Vertex corrections and dressed propagators

The diagrams in panels (1),(2) and (3) in Fig. 4.2 are discussed in Ref. [25]. To be completed, here we report the final results

$$\begin{aligned} T^{(1)}_{PV}(1-2-3) &= -T^{(-1)}_{PV}(\mathrm{NR})\left[\frac{2}{3}\frac{g_A^2}{f_\pi^2}J_{13} + \frac{20\,\alpha-1}{4f_\pi^2}J_{01} - 2\,\ell_4\frac{m_\pi^2}{f_\pi^2}\right. \\ &\quad \left. + \frac{1-10\,\alpha}{2f_\pi^2}J_{01} - \frac{1}{4f_\pi^2}J_{01}\right], \end{aligned} \tag{4.38}$$

where

$$J_{mn} = \int \frac{d\boldsymbol{q}}{(2\pi)^3}\,\frac{q^{2m}}{\omega_q^n}. \tag{4.39}$$

From Eq. (3.66), we obtain $V^{(1)}_{PV}(1-2-3) = T^{(1)}_{PV}(1-2-3)$. All these contributions can be reabsorbed in the term $V^{(-1)}_{PV}(\mathrm{NR})$ by a renormalization of the LEC h^1_π.

4.3 The N3LO potential

In this Section we calculate the components of the PV potential of order Q^2. The diagrams that give a contribution to this order are those reported in Fig. 4.8. At this order two new types of diagrams appear: the "bubble" diagrams and the three pions exchange diagrams (see panels (f) and (g)). We have contributions of order Q^2 also by the triangle diagrams in three ways: from NLO terms of the vertex functions (diags. (h), (i), (j)), from NLO terms in the expansion of the energy denominator (diag. (l)) and from the diagrams with the $PV\ \pi\pi NN$ vertex (diag. (k)). From the box diagrams we have contribution from the NLO term of the vertex functions (diag. (m), (n)) and energy denominators (diag (o), (p), (q), (r)). As already discussed in Section 4.2.1, the one pion exchange diagrams do not give any contribution at this order.

4.3.1 Bubble diagrams

Let's consider the term $H_{02}G_0H_{20}$ in the expansion of the T-matrix. It gives the diagrams of type (f), explicitly depicted in Fig. 4.9. The expressions we obtain

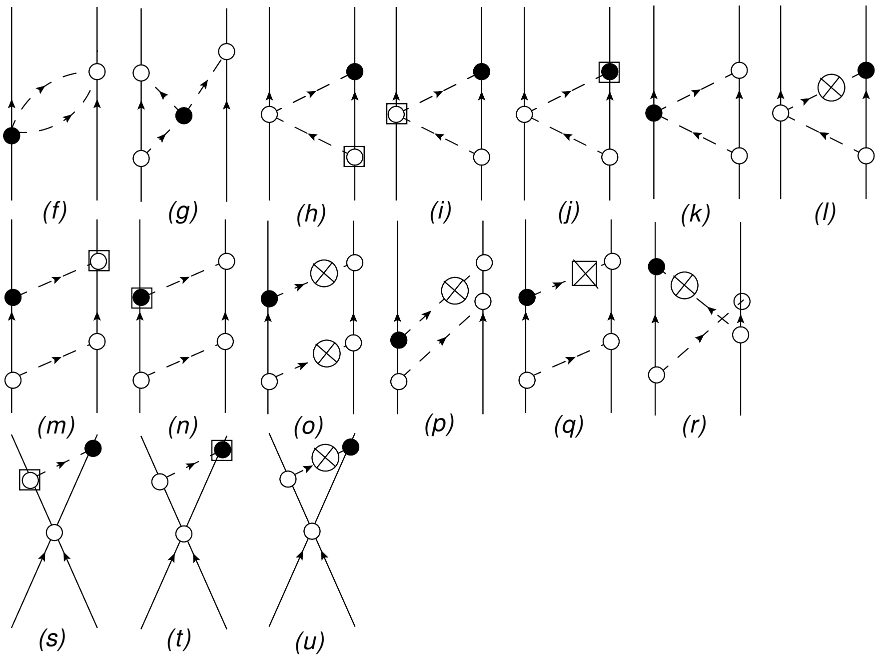


Figure 4.8: Time ordered diagrams that contribute to the N3LO T-matrix. The vertices depicted by a square surrounding a circle indicate the NLO terms in the expansion of the vertex functions and the crossed circle (square) on a pion propagator indicates the NLO (N2LO) term in the energy denominator expansion given in Eq. (4.30). For the other notation see Fig. 4.1.

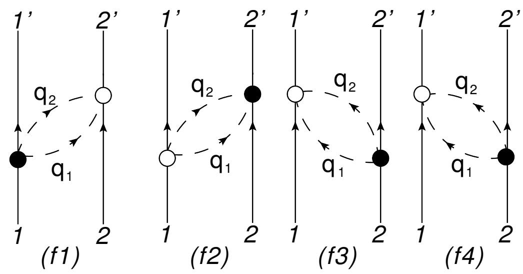


Figure 4.9: Time ordered bubble diagrams that contribute at N3LO. Notation as in Fig. 4.1.

for the first two diagrams are:

$$T(f1) = -\frac{2}{\Omega^2} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} \frac{1}{\omega_{q_1} + \omega_{q_2}} \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2} \delta_{\boldsymbol{p}_1+\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{p}'_1} \times \left[{}^{PC}M^{\pi\pi NN,02}_{\alpha_1\alpha_{1'},\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} \, {}^{PV}M^{\pi\pi NN,02}_{\alpha_2\alpha_{2'},\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} + {}^{PC}M^{\pi\pi NN,02}_{\alpha_1\alpha_{1'},\boldsymbol{q}_2 j_2 \boldsymbol{q}_1 j_1} \, {}^{PV}M^{\pi\pi NN,02}_{\alpha_2\alpha_{2'},\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} \right], \tag{4.40}$$

$$T(f2) = -\frac{2}{\Omega^2} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} \frac{1}{\omega_{q_1} + \omega_{q_2}} \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2} \delta_{\boldsymbol{p}_1+\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{p}'_1} \times \left[{}^{PV}M^{\pi\pi NN,02}_{\alpha_1\alpha_{1'},\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} \, {}^{PC}M^{\pi\pi NN,02}_{\alpha_2\alpha_{2'},\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} + {}^{PV}M^{\pi\pi NN,02}_{\alpha_1\alpha_{1'},\boldsymbol{q}_2 j_2 \boldsymbol{q}_1 j_1} \, {}^{PC}M^{\pi\pi NN,02}_{\alpha_2\alpha_{2'},\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} \right]. \tag{4.41}$$

Making explicit the vertex functions we obtain

$$T(f1) = -\frac{1}{16 f_\pi^4}(2h_A^1 \tau_{1z} + h_A^2 I^b \tau_{1b}\tau_{2b}) \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} \frac{\omega_{q_1} - \omega_{q_2}}{\omega_{q_1}\omega_{q_2}(\omega_{q_2} + \omega_{q_1})} \times (\boldsymbol{q}_2 - \boldsymbol{q}_1) \cdot \boldsymbol{\sigma}_2 \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2} \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,-\boldsymbol{k}}, \tag{4.42}$$

$$T(f2) = -\frac{1}{16 f_\pi^4}(2h_A^1 \tau_{2z} + h_A^2 I^b \tau_{1b}\tau_{2b}) \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2} \frac{\omega_{q_1} - \omega_{q_2}}{\omega_{q_1}\omega_{q_2}(\omega_{q_2} + \omega_{q_1})} \times (\boldsymbol{q}_2 - \boldsymbol{q}_1) \cdot \boldsymbol{\sigma}_1 \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2} \delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,-\boldsymbol{k}}. \tag{4.43}$$

From these expressions it is easy to obtain those for the other two time ordering diagrams $(f3)$ and $(f4)$ of Fig. 4.9 exchanging $\boldsymbol{k} \to -\boldsymbol{k}$, $\tau_1 \leftrightarrow \tau_2$ and $\boldsymbol{\sigma}_1 \leftrightarrow \boldsymbol{\sigma}_2$. Summing up all together, the various contributions cancel out and therefore these diagrams do not give any contribution to $V^{(2)}_{PV}$.

4.3.2 Three pions exchange

The expansion in terms of pion fields of the term given in Eq. (2.95) of the χEFT Lagrangian, give two terms proportional to π^3 (see Eq. (B.22)). Expanding the pion fields in terms of the creation/annihilation operators, the vertex function proportional to $h^1_{3\pi}$ cancels out. The other term gives, after reordering the creation/annihilation operators, a Hamiltonian term like

$$H^{3\pi\,12} = \frac{16}{\Omega^{3/2}} \sum_{\boldsymbol{q}_1 j_1 \boldsymbol{q}_2 j_2 \boldsymbol{q}_3 j_3} a^\dagger_{\boldsymbol{q}_1 j_1} a_{\boldsymbol{q}_2 j_2} a_{\boldsymbol{q}_3 j_3} \frac{h^2_{3\pi} I^{j_3}}{f_\pi} \epsilon_{j_1 j_2 j_3} \frac{(\boldsymbol{q}_2 \cdot \boldsymbol{q}_3)}{\sqrt{8\omega_{\boldsymbol{q}_1}\omega_{\boldsymbol{q}_2}\omega_{\boldsymbol{q}_3}}} \delta_{\boldsymbol{q}_1,\boldsymbol{q}_2+\boldsymbol{q}_3} \delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}'_1+\boldsymbol{p}'_2},$$

with

$$I^{j_3} = (-1,-1,2). \tag{4.44}$$

This Hamiltonian term contributes to the scattering of two nucleons via diagrams of type (g) of Fig. 4.8. However, after summing over all possible time orderings, the final contribution vanishes.

4.3.3 Two pions exchange: triangle diagrams

As discussed at the beginning of this Section, the N3LO contributions of these diagrams come from different origins.

Diagrams like (h) take into account the NLO term of the πNN PC vertex functions, given by

$$ {}^{PC}M^{\pi NN,01}_{\alpha'\alpha,\boldsymbol{q}j} = -\frac{g_A}{2f_\pi M}\frac{\tau_j}{\sqrt{2\omega_q}} i\omega_q \boldsymbol{K}\cdot\boldsymbol{\sigma} \tag{4.45} $$

and

$$ {}^{PC}M^{\pi NN,10}_{\alpha'\alpha,\boldsymbol{q}j} = \frac{g_A}{2f_\pi M}\frac{\tau_j}{\sqrt{2\omega_q}} i\omega_q \boldsymbol{K}\cdot\boldsymbol{\sigma} \ , \tag{4.46} $$

where $\boldsymbol{K} = (\boldsymbol{p}+\boldsymbol{p}')/2$.

Substituing the LO term of the πNN PC vertex function in Eq. (4.14), with the second order given in Eqs. (4.45)-(4.46), and using the Fierz transformations (see Eqs. (A.39)), we obtain

$$ \begin{aligned} V^{(2)}_{PV}(h) &= -\frac{h^1_\pi g_A}{32\sqrt{2}f^3_\pi M}\left[i(\tau_{1z}+\tau_{2z})\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2) - \frac{1}{2}i(\tau_1\times\tau_2)_z\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\right] \\ &\quad \times \int \frac{d^3q}{(2\pi)^3}\frac{1}{(q^2+4m^2_\pi)} \ , \end{aligned} \tag{4.47} $$

however this term can be reabsorbed in the CT potential given in Eq. (4.13). It consists in fact in a renormalization of the contact LECs C_3 and C_4.

In diagrams like (i) of Fig. 4.8, substituting the LO part of the $\pi\pi NN$ PC vertex function with the NLO term (see Eqs. (4.45)-(4.46)) we find two contributions to the potential: a term where the LEC c_4 appears,

$$ \begin{aligned} V^{(2)}_{PV}(i1) &= -i\frac{c_4 h^1_\pi g_A}{2\sqrt{2}f^3_\pi}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\omega^2_+\omega^2_-} \times \\ &\quad [(\boldsymbol{q}\cdot\boldsymbol{\sigma}_1(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_2)\tau_{2z} - (\boldsymbol{q}\cdot\boldsymbol{\sigma}_2(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_1)\tau_{1z}] \ , \end{aligned} \tag{4.48} $$

and a second term coming from a relativistic correction of ${}^{PC}M^{\pi\pi NN}$

$$ \begin{aligned} V^{(2)}_{PV}(i2) &= -\frac{h^1_\pi g_A}{8\sqrt{2}f^3_\pi M}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\omega^2_+\omega^2_-}\Big[2[(\boldsymbol{K}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{\sigma}_2)\tau_{1z} - (\boldsymbol{K}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{\sigma}_1)\tau_{2z}] \\ &\quad -i[(\boldsymbol{q}\cdot\boldsymbol{\sigma}_2(\boldsymbol{k}\times\boldsymbol{\sigma}_1)\cdot\boldsymbol{q})\tau_{1z} - (\boldsymbol{q}\cdot\boldsymbol{\sigma}_1(\boldsymbol{k}\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{q})\tau_{2z}] \ . \end{aligned} \tag{4.49} $$

The contribution of diagrams (j) of Fig. 4.8 is calculated substituting the LO part of the πNN PV vertex with the NLO one. We obtain the following expression

$$ V^{(2)}_{PV}(j) = -\frac{h^1_v g_A}{32f^4_\pi}(\tau_1\times\tau_2)_z i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\int \frac{d^3q}{(2\pi)^3}\frac{1}{(q^2+4m^2_\pi)} \ , \tag{4.50} $$

which again can be reabsorbed by a redefinition of the C_3 LEC.

To obtain the expressions for the diagrams (k), one needs the vertex function ${}^{PV}M^{\pi\pi NN}$ given in Eqs. (C.30)-(C.32). The final result is

$$
\begin{aligned}
V_{PV}^{(2)}(k) &= -\frac{g_A^2}{8f_\pi^4}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega_+^2\omega_-^2}\times \\
&\{2h_A^1[(\boldsymbol{q}\cdot\boldsymbol{\sigma}_1(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_2)\tau_{2z}-(\boldsymbol{q}\cdot\boldsymbol{\sigma}_2(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_1)\tau_{1z}]+ \\
&h_A^2 I^b\tau_{1b}\tau_{2b}[(\boldsymbol{q}\cdot\boldsymbol{\sigma}_1(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_2)-(\boldsymbol{q}\cdot\boldsymbol{\sigma}_2(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_1)]\}\ , \quad (4.51)
\end{aligned}
$$

where h_A^1 and h_A^2 are the two new LECs that appear explicitly in ${}^{PV}M^{\pi\pi NN}$.

In the triangle diagrams (l) we use the first and second order terms in the expansion of the energy denominator given in Eq. (4.30). Remembering that $E_i \sim M + p_i^2/2M$ we finally obtain

$$
\begin{aligned}
V_{PV}^{(2)}(l) &= \frac{h_\pi^1 g_A}{16\sqrt{2}f_\pi^3 M}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega_+^2\omega_-^2}\Big[4(\boldsymbol{K}\cdot\boldsymbol{q})\boldsymbol{q}\cdot(\boldsymbol{\sigma}_1\tau_{2z}-\boldsymbol{\sigma}_2\tau_{1z}) \\
&\quad +(\tau_1\times\tau_2)_z i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\Big(\frac{k^2-q^2}{2}\Big)\Big]\ . \qquad (4.52)
\end{aligned}
$$

4.3.4 Two pions exchange: box diagrams

As for the triangle diagrams, the first N3LO contribution is obtained by taking the NLO vertex functions like in diagrams (m) and (n). Some of the diagrams to be taken into account are those shown in Figs. 4.6 and 4.7, diagrams $(e1)-(e6)$ and $(\overline{e1})-(\overline{e6})$. We have to take into account again of the subtraction of terms $(e5')$, $(e6')$, $(\overline{e5}')$ and $(\overline{e6}')$. Inserting the NLO vertex functions in diagrams $(e5)$ and $(e6)$, both for the PV vertex and for the PC vertex, the combination of δ functions and vertex functions take the same form as in $(\overline{e5})$ and $(\overline{e6})$ but with a different sign. Therefore Eq. (4.34) becomes

$$
T(e5-e5'+\overline{e5}-\overline{e5'}) \simeq [\ldots]\frac{1}{\omega_{q_1}\omega_{q_2}}\frac{\Delta E_1-\Delta E_2}{(E_0-(E_\beta+E_\gamma)+i\epsilon)}\left[\frac{1}{\omega_{q_1}}+\frac{1}{\omega_{q_2}}\right] . \qquad (4.53)
$$

Using the condition that in the CM the difference $\Delta E_1 - \Delta E_2 = 0$ we find that the contribution of these diagrams vanishes. This condition is valid for all the time-ordering diagrams of this type, so they don't give any contribution at N3LO. Also the N3LO contribution of diagrams $(e4)$ and $(\overline{e4})$ cancel out.

The only contribution we have from the NLO vertex functions comes from diagrams like $(e1)$, $(e2)$ and $(e3)$ of Fig. 4.6. Taking into account the NLO PC

vertex we obtain

$$\begin{aligned}V^{(2)}_{PV}(m) &= \frac{h^1_\pi g^3_A}{32\sqrt{2}f^3_\pi M}\Big[(\tau_1\times\tau_2)_z\Big(-i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega^2_+\omega^2_-}\frac{q^2+3k^2}{2}+\\ &4[\boldsymbol{k}\cdot\boldsymbol{\sigma}_1\,(\boldsymbol{K}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_2+\boldsymbol{k}\cdot\boldsymbol{\sigma}_2\,(\boldsymbol{K}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_1]\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega^2_+\omega^2_-}\Big)+\\ &(\tau_{1z}+\tau_{2z})\Big(-8\int\frac{d^3q}{(2\pi)^3}\frac{(\boldsymbol{K}\cdot\boldsymbol{q})\boldsymbol{q}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)}{\omega^2_+\omega^2_-}+4\boldsymbol{K}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega^2_+\omega^2_-}-\\ &2i\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega^2_+\omega^2_-}[(\boldsymbol{q}\cdot\boldsymbol{\sigma}_1(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_2)-(\boldsymbol{q}\cdot\boldsymbol{\sigma}_2(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_1)]\Big)\Big]\ . \qquad (4.54)\end{aligned}$$

In the same way taking into account of the NLO term of the PV vertex we obtain

$$\begin{aligned}V^{(2)}_{PV}(n) &= \frac{g^3_A}{32f^4_\pi}\Big[\Big(h^0_V(3+2\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_1)-\frac{4}{3}h^2_V I^b\tau_{1b}\tau_{2b}\Big)i\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega^2_+\omega^2_-}[(\boldsymbol{q}\cdot\boldsymbol{\sigma}_1(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_2)\\ &-(\boldsymbol{q}\cdot\boldsymbol{\sigma}_2(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_1)]-2ih^1_V\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega^2_+\omega^2_-}[(\boldsymbol{q}\cdot\boldsymbol{\sigma}_1(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_2)\boldsymbol{\tau}_1\\ &-(\boldsymbol{q}\cdot\boldsymbol{\sigma}_2(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_1)\boldsymbol{\tau}_2]+ih^1_V(\tau_1\times\tau_2)_z\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\int\frac{d^3q}{(2\pi)^3}\frac{q^2-k^2}{\omega^2_+\omega^2_-}\Big]\ . \qquad (4.55)\end{aligned}$$

The second N3LO contribution of the box diagrams comes from the diagrams (o), (p), (q) and (r). We need to compute

$$\begin{aligned}V^{(2)}_{PV} &= T^{(2)}_{PV}-V^{(-1)}_{PV}G_0V^{(2)}_{PC}-V^{(2)}_{PC}G_0V^{(-1)}_{PV}\\ &\quad -V^{(1)}_{PV}G_0V^{(0)}_{PC}-V^{(0)}_{PC}G_0V^{(1)}_{PV}\ , \qquad (4.56)\end{aligned}$$

where $V^{(2)}_{PC}$ and $V^{(1)}_{PV}$ in this expression are N2LO contributions coming from the PC and PV OPE terms due to the intermediate states.

Let us explain in detail this issue. For example in Eq. (4.56) we need to include the matrix element $\langle\beta\gamma|V^{(2)}_{PC}|\alpha_1\alpha_2\rangle$ where β and γ are the quantum numbers of two-nucleon intermediate states. Now we have to sum over all values of $\boldsymbol{p}_\beta$ (for example $\boldsymbol{p}_\gamma$ is fixed by momentum conservation) and therefore we cannot assume $E_1+E_2=E_\beta+E_\gamma$. Let us consider again from the beginning the diagrams contributing to the OPE between states $|\alpha_1\alpha_2\rangle$ and $\langle\beta\gamma|$. They are reported in Fig. 4.10, where V_1, V_2, $\overline{V_1}$ and $\overline{V_2}$ are the LO vertex functions. So we obtain:

$$V_{PC}(\text{OPE})=\Big(\frac{V_1V_2}{E_2-E_\beta-\omega_k}+\frac{\overline{V_2V_1}}{E_1-E_\gamma-\omega_k}\Big)\delta_{\boldsymbol{k},\boldsymbol{p}_\beta-\boldsymbol{p}_\gamma}\ . \qquad (4.57)$$

Now at LO the product $\overline{V_2V_1}$ is equal to V_1V_2. Expanding the denominator up to the second order

$$V_{PC}(\text{OPE})=V^{(0)}_{PC}(\text{OPE})\Big(1+\frac{E_1+E_\beta-E_2-E_\gamma}{\omega_k}+\frac{(E_1-E_\beta)^2+(E_2-E_\gamma)^2}{2\omega^2_k}+\ldots\Big)\ , \qquad (4.58)$$

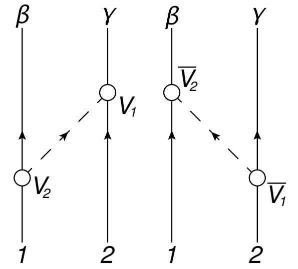


Figure 4.10: Diagrams that contribute to the PC OPE. Note that the final states here are "intermediate" states β and γ. Notation as in Fig. 4.1.

where $V^{(0)}_{PC}(\text{OPE}) = -2\frac{V_1 V_2}{\omega_k}$. Assuming as usual that $\frac{\Delta E}{\omega} \sim Q$, the Q^2 term is given by

$$V^{(2)}_{PC}(\text{OPE}) = V^{(0)}_{PC}(\text{OPE})\frac{(E_1 - E_\beta)^2 + (E_2 - E_\gamma)^2}{2\omega_k^2} \ . \tag{4.59}$$

where $\boldsymbol{k} = \boldsymbol{p}_\beta - \boldsymbol{p}_1$. Analogously for the $V_{PV}(\text{OPE})$, we find

$$V^{(1)}_{PV}(\text{OPE}) = V^{(-1)}_{PV}(\text{OPE})\frac{(E_1 - E_\beta)^2 + (E_2 - E_\gamma)^2}{2\omega_k^2} \ . \tag{4.60}$$

Let us return to our problem. In the calculation of $T^{(2)}_{PV}$ the diagrams of type (q) are exactly cancelled by the subtraction terms

$$V^{(-1)}_{PV} G_0 V^{(2)}_{PC} + V^{(2)}_{PC} G_0 V^{(-1)}_{PV} + V^{(1)}_{PV} G_0 V^{(0)}_{PC} + V^{(0)}_{PC} G_0 V^{(1)}_{PV} \ , \tag{4.61}$$

where the expressions (4.59) and (4.60) are used for $V^{(2)}_{PC}$ and $V^{(1)}_{PV}$. So no contributions for diagram (q) is found. Moreover, the contribution to the T-matrix given by diagrams of type (o) is cancelled by the contribution that comes from the diagrams (p). The only recoil correction comes from the diagrams (r); making explicit the vertex functions and using the delta functions to eliminate an integration over a loop momentum we obtain at order Q^2

$$\begin{aligned} V^{(2)}_{PV}(r) \;=\; & \frac{h^1_\pi g_A^3}{32\sqrt{2} f_\pi^3 M}\Big[i(\tau_1 \times \tau_2)_z \boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \int \frac{d^3q}{(2\pi)^3}(q^2 - k^2)^2 \frac{\omega_+^2 + \omega_-^2}{\omega_+^4 \omega_-^4} \\ & + 8(\tau_{1z} + \tau_{2z}) \int \frac{d^3q}{(2\pi)^3}(q^2 - k^2)\frac{\omega_+^2 + \omega_-^2}{\omega_+^4\omega_-^4} \\ & \times [(\boldsymbol{q}\cdot\boldsymbol{\sigma}_1(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_2) - (\boldsymbol{q}\cdot\boldsymbol{\sigma}_2(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}_1)]\Big] \end{aligned} \tag{4.62}$$

4.3.5 Contact terms with a OPE

Diagrams like (s), (t) and (u) of Fig. 4.8 do not give any contribution to the potential, but only corrections to the LECs of the contact terms. In fact the (t) type diagrams, vanish directly due to the integration over the loop momentum. Diagram (s) sums up three kind of diagrams showed in Fig. 4.4 where we take the NLO part of the $PC\ \pi NN$ vertex. If we perform this on the first type, all the time-ordering contributions cancel out, while if we take all the time-orderings of the other two, we obtain:

$$V_{PV}^{(2)}(s) = -\frac{h_\pi g_A}{8\sqrt{2} f_\pi M}(C_S - 3C_T)(\tau_{1z} + \tau_{2z}) i\boldsymbol{k} \cdot (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \int \frac{d^3q}{(2\pi)^3} \frac{1}{\omega_{\boldsymbol{q}}^2} \ , \quad (4.63)$$

which can be reabsorbed in the CT potential. If we consider the recoil corrections of diagrams in Fig. 4.4 (diagrams (u)), from the first diagram we obtain

$$T_{PV}^{CT}(u) = V_{PV}^{(1)} G_0 V_{PC}^{(0)}(\mathrm{CT}) + V_{PC}^{(0)}(\mathrm{CT}) G_0 V_{PV}^{(1)} \ , \quad (4.64)$$

which is exactly eliminated by the subtracting term in Eq. (3.66). On the other hand, from the other two we have

$$V_{PV}^{(2)}(u) = +\frac{h_\pi^1 g_A}{12\sqrt{2} f_\pi M}(C_S + C_T)(\tau_{1z} + \tau_{2z}) i\boldsymbol{k} \cdot (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \int \frac{d^3q}{(2\pi)^3} \frac{q^2}{\omega_{\boldsymbol{q}}^4} \ , \quad (4.65)$$

which redefine again the C_4 constant.

4.4 Regularization of the PV potential

Now we have to deal with the divergences due to the loops. We obtain two types of divergences: a logarithmic divergence presents in the N2LO terms and a linear divergence in the N3LO terms. We will treat the first kind of divergence with the dimensional regularization (DR) method [27]. The second kind of divergence is more complicate to treat because DR does not work with this kind of divergence, even if it gives the exact result for the non divergent part. Therefore in this last case we will discuss the regularization via a cut-off momentum. A detailed discussion of the DR and cut-off regularization can be found in Appendix D.

4.4.1 Regularization of the N2LO divergences

In the DR method, the integrals are defined in d dimensions and computed for a generic value of d. Successively one takes the limit $d \to 3$. Alternatively, defined $\epsilon = 3 - d$, we will take the limit $\epsilon \to 0$, isolating in this way the divergent part.

The details of the method are reported in Appendix D. Defining $s = \sqrt{4\,m_\pi^2 + k^2}$, and the functions

$$L(k) = \frac{1}{2}\frac{s}{k}\ln\frac{s+k}{s-k}\,, \qquad H(k) = \frac{m_\pi^2}{s^2}L(k)\,, \qquad d_\epsilon = \frac{2}{\epsilon} - \gamma + \ln\pi - \ln\frac{m_\pi^2}{\mu^2}\,, \tag{4.66}$$

and taking into account the results obtained in Section D.2, the contributions of box and triangle diagrams given in Eq. (4.67) and (4.68), respectively, in DR is the following

$$V_{PV}^{(1)}(d) = -\frac{g_A h_\pi^1}{2\sqrt{2}f_\pi}\frac{1}{\Lambda_\chi^2}(\vec{\tau}_1 \times \vec{\tau}_2)_z i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\Big(L(k) - \frac{1}{2}d_\epsilon - 1\Big)\,, \tag{4.67}$$

$$\begin{aligned} V_{PV}^{(1)}(e) = & -\frac{g_A h_\pi^1}{2\sqrt{2}f_\pi}\frac{g_A^2}{\Lambda_\chi^2}\Big[4(\vec{\tau}_1 + \vec{\tau}_2)_z\, i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)\,\Big(L(k) - \frac{1}{2}d_\epsilon - \frac{2}{3}\Big) \\ & +(\vec{\tau}_1 \times \vec{\tau}_2)_z i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\,\Big(H(k) - 3L(k) + \frac{1}{2}d_\epsilon + 1\Big)\Big]\,, \end{aligned} \tag{4.68}$$

where $\Lambda_\chi = 4\pi f_\pi$. Redefining the constants C_3 and C_4 entering the CT potential given in Eq. (4.13), in order to reabsorb the divergent part proportional to d_ϵ, and considering only the finite part, we obtain

$$V_{PV}^{(1)}(d - \mathrm{FIN}) = -\frac{g_A h_\pi^1}{2\sqrt{2}f_\pi}\frac{1}{\Lambda_\chi^2}(\vec{\tau}_1 \times \vec{\tau}_2)_z i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)L(k)\,, \tag{4.69}$$

$$\begin{aligned} V_{PV}^{(1)}(e - \mathrm{FIN}) = & -\frac{g_A h_\pi^1}{2\sqrt{2}f_\pi}\frac{g_A^2}{\Lambda_\chi 2}\Big[4(\tau_1 + \tau_2)_z\, i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)\,L(k) \\ & +(\vec{\tau}_1 \times \vec{\tau}_2)_z i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\,\Big(H(k) - 3L(k)\Big)\Big]\,. \end{aligned} \tag{4.70}$$

Let's note that the chiral order is always Q, also after the regularization. These terms compared to the LO are suppressed by a factor $(m_\pi/\Lambda_\chi)^2$ which someway justifies the idea of chiral expansion.

4.4.2 Regularization of the N3LO divergences

As we have already anticipated at the beginning of this Section, the regularization of linear divergences is more complicated. Indeed, even if the DR gives the exact result of the non divergent part, it does not identify the divergent part. For this reason we impose a cut-off Λ on the integrals. For example, let us consider the potential term

$$V_{PV}^{(2)} = \frac{g_A^3 h_V^1}{32 f_\pi^4} i(\tau_1 \times \tau_2)_z i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)K^{(2)}(k) \tag{4.71}$$

where

$$K^{(2)}(k) \quad = \quad \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \, \frac{q^2}{\omega_+^2 \, \omega_-^2} \tag{4.72}$$

which appears in Eq. (4.55). Performing the integrals as shown in Appendix D.3 we get

$$K^{(2)}(k) = -\frac{s^2 \, A(k)}{4\pi} - \frac{m_\pi}{2\pi} + \frac{\Lambda}{4\pi^2} + \mathcal{O}\big(\frac{k^2}{\Lambda}\big), \tag{4.73}$$

where

$$A(k) = \frac{1}{2k} \arctan\Big(\frac{k}{2m_\pi}\Big), \tag{4.74}$$

therefore,

$$V_{PV}^{(2)} = \frac{g_A^3 h_V^1}{32 f_\pi^4} i(\tau_1 \times \tau_2)_z i\boldsymbol{k} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\Big(- \frac{s^2 \, A(k)}{4\pi} - \frac{m_\pi}{2\pi} + \frac{\Lambda}{4\pi^2} + \mathcal{O}\big(\frac{k^2}{\Lambda}\big)\Big) \tag{4.75}$$

where the term proportional to Λ carries all the divergence. This part is independent on $\boldsymbol{k}$. The term proportional to Λ therefore can be reabsorbed in the CT potential given in Eq. (4.13) as we will see explicitly below. For what concern the non divergent part we find a term independent on Λ which is exactly the same obtained using the DR method, plus a number of other terms give as a power series of Q/Λ, starting with the order Q^2/Λ. Sending Λ to infinity this latter part would disappear. In general we must fix Λ at a value greater than the typical energies of the χEFT, then the terms must be included in the potential. However it carries at least an additional power of Q (considering that Λ doesn't influence the chiral counting), which means it gives contribution to the N4LO or beyond to the potential. Somehow the price we pay using a simple cut-off is to "dirty" the next order in the chiral expansion.

From a practical point of wiew, we neglect the terms Q^2/Λ and we find for the various N3LO terms

$$V_{PV}^{(2)}(i1) \quad = \quad -\frac{c_4 h_\pi^1 g_A}{\sqrt{2} f_\pi} \frac{\pi}{\Lambda_\chi^2} i\boldsymbol{k} \cdot (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)(\tau_1 + \tau_2)_z \Big(s^2 A(k) - \frac{2}{3\pi}\Lambda + m_\pi\Big) \,, \tag{4.76}$$

$$\begin{aligned} V_{PV}^{(2)}(i2) \quad = \quad & \frac{h_\pi^1 g_A}{4\sqrt{2} f_\pi^3} \frac{\pi}{\Lambda_\chi^2 M} \Big[2\boldsymbol{K} \cdot (\boldsymbol{\sigma}_2 \tau_{1z} - \boldsymbol{\sigma}_1 \tau_{2z}) \\ & + i\boldsymbol{k} \cdot (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)(\tau_{1z} + \tau_{2z}) \Big] \Big(s^2 A(k) - \frac{2}{3\pi}\Lambda + m_\pi\Big) \,, \end{aligned} \tag{4.77}$$

$$V_{PV}^{(2)}(k) \quad = \quad -\frac{g_A^2}{2 f_\pi^2} \frac{\pi}{\Lambda_\chi^2} \Big[h_A^1 (\tau_{1z} + \tau_{2z}) + h_A^2 I^b \tau_{1b} \tau_{2b} \Big] i\boldsymbol{k} \cdot (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \Big(s^2 A(k) - \frac{2}{3\pi}\Lambda + m_\pi\Big) \,, \tag{4.78}$$

$$V_{PV}^{(2)}(l) \quad = \quad \frac{g_A h_\pi^1}{4\sqrt{2} f_\pi} \frac{\pi}{\Lambda_\chi^2 M} \Big\{ i\boldsymbol{k} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)(\vec{\tau}_1 \times \vec{\tau}_2)_z \Big[\Big(\frac{k^2 + s^2}{2}\Big) A(k) - \frac{\Lambda}{2\pi} + m_\pi \Big]$$

$$
\begin{aligned}
&+2\boldsymbol{K}\cdot(\boldsymbol{\sigma}_2\tau_{1z}-\boldsymbol{\sigma}_1\tau_{2z})\Big(s^2A(k)-\frac{2}{3\pi}\Lambda+m_\pi\Big)\Big\}\ , && (4.79)\\
V_{PV}^{(2)}(m) \;=\;& \frac{g_A^3h_\pi^1}{16\sqrt{2}f_\pi}\frac{\pi}{\Lambda_\chi^2M}\Big\{-8\boldsymbol{K}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)(\tau_1+\tau_2)_z\Big(k^2A(k)-\frac{\Lambda}{3\pi}+m_\pi\Big)\\
&-4i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)(\vec{\tau}_1\times\vec{\tau}_2)_z\Big(s^2A(k)-\frac{2}{3\pi}\Lambda+m_\pi\Big)\\
&+i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)(\vec{\tau}_1\times\vec{\tau}_2)_z\Big[(s^2-3k^2)A(k)-\frac{\Lambda}{\pi}+2m_\pi\Big]\\
&-8(\vec{\tau}_1\times\vec{\tau}_2)_z[\boldsymbol{k}\cdot\boldsymbol{\sigma}_1\;(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}_2+\boldsymbol{k}\cdot\boldsymbol{\sigma}_2\;(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}_1]A(k)\Big\}\ , && (4.80)\\
V_{PV}^{(2)}(n) \;=\;& \frac{g_A^3}{2f_\pi^2}\frac{\pi}{\Lambda_\chi^2}\Big\{\Big[\frac{h_V^0}{4}(3+2\vec{\tau}_1\cdot\vec{\tau}_2)-\frac{2}{3}h_V^2I^b\tau_{1b}\tau_{2b}+h_V^1\frac{(\tau_{1z}+\tau_{2z})}{2}\Big]\\
&\times\Big(s^2A(k)-\frac{2}{3\pi}\Lambda+m_\pi\Big)-h_V^1i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)(\vec{\tau}_1\times\vec{\tau}_2)_z\\
&\times\Big[\Big(1-\frac{2m_\pi^2}{s^2}\Big)s^2A(k)-\frac{\Lambda}{\pi}+m_\pi\Big)\Big]\Big\}\ , && (4.81)\\
V_{PV}^{(2)}(r) \;=\;& \frac{g_A^3h_\pi^1}{2\sqrt{2}f_\pi}\frac{\pi}{\Lambda_\chi^2M}\Big\{-i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)(\vec{\tau}_1\times\vec{\tau}_2)_z\Big[\Big(1-\frac{2m_\pi^2}{s^2}\Big)s^2A(k)\\
&-\frac{m_\pi^3}{2s^2}-\frac{3}{4\pi}\Lambda+\frac{11}{8}m_\pi\Big]+i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)(\tau_{1z}+\tau_{2z})\\
&\times\Big[\Big(3-\frac{4m_\pi^2}{s^2}\Big)s^2A(k)-\frac{2\Lambda}{3\pi}+3m_\pi\Big]\Big\}\ . && (4.82)
\end{aligned}
$$

Reabsorbing the divergent terms proportional to Λ in the contact term LECs we finally obtain the contribution to the potential coming from the N3LO diagrams.

4.5 The PV potential in k-space

In summary, the PV potential up to N3LO derived from χEFT is the following:

$$
\begin{aligned}
V_{PV}^{(-1)}(\mathrm{OPE}) \;=\;& \frac{g_Ah_\pi^1}{2\sqrt{2}f_\pi}\left(\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2\right)_z\frac{i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)}{\omega_k^2}\ , && (4.83)\\
V_{PV}^{(1)}(\mathrm{CT}) \;=\;& \frac{1}{\Lambda_\chi m_\pi^2}[C_1i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}+C_2(\vec{\tau}_1\cdot\vec{\tau}_2)i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}\\
&+C_3(\vec{\tau}_1\times\vec{\tau}_2)_zi(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}+C_4(\tau_{1z}+\tau_{2z})i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}\\
&+C_5\mathcal{I}^{ab}(\tau_1)_a(\tau_2)_bi(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}]\ , && (4.84)\\
V_{PV}^{(1)}(\mathrm{TPE}) \;=\;& -\frac{g_Ah_\pi^1}{2\sqrt{2}f_\pi}\frac{1}{\Lambda_\chi^2}(\vec{\tau}_1\times\vec{\tau}_2)_zi\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)L(k)\\
&-\frac{g_Ah_\pi^1}{2\sqrt{2}f_\pi}\frac{g_A^2}{\Lambda_\chi2}\Big[4(\tau_1+\tau_2)_z\;i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\;L(k)
\end{aligned}
$$

$$
\begin{aligned}
& +(\vec{\tau}_1 \times \vec{\tau}_2)_z i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\,\Big(H(k)-3L(k)\Big)\Big]\;, \qquad (4.85)\\
V^{(1)}_{PV}(\text{RC-OPE}) &= \frac{g_A h^1_\pi}{2\sqrt{2}f_\pi}\frac{1}{4M^2}(\vec{\tau}_1\times\vec{\tau}_2)_z\,\frac{1}{\omega_k^2}[-4iK^2\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\\
&\quad +\boldsymbol{k}\cdot\boldsymbol{\sigma}_1\,(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}_2+\boldsymbol{k}\cdot\boldsymbol{\sigma}_2\,(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}_1]\;, \qquad (4.86)\\
V^{(2)}_{PV}(c_4) &= -\frac{c_4 h^1_\pi g_A}{\sqrt{2}f_\pi}\frac{\pi}{\Lambda_\chi^2}i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)(\tau_1+\tau_2)_z s^2A(k)\;, \qquad (4.87)\\
V^{(2)}_{PV}(\text{LEC}) &= \frac{g_A^2}{2f_\pi^2}\frac{\pi}{\Lambda_\chi^2}\Big\{\Big[\frac{3g_Ah^0_V}{4}+\frac{g_Ah^0_V}{2}\vec{\tau}_1\cdot\vec{\tau}_2+\Big(\frac{g_Ah^1_V}{4}-h^1_A\Big)(\tau_{1z}+\tau_{2z})\\
&\quad -\Big(h^2_A+\frac{g_Ah^2_V}{3}\Big)I^b\tau_{1b}\tau_{2b}\Big]i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\\
&\quad -\frac{g_Ah^1_V}{2}(\vec{\tau}_1\times\vec{\tau}_2)_z i\boldsymbol{k}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\Big(1-\frac{2m_\pi^2}{s^2}\Big)\Big\}s^2A(k)\;, \qquad (4.88)\\
V^{(2)}_{PV}(\text{RC-TPE}) &= \frac{g_Ah^1_\pi}{4\sqrt{2}f_\pi}\frac{\pi}{\Lambda_\chi^2 M}\Big[i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)(\vec{\tau}_1\times\vec{\tau}_2)_z\Big(1-\frac{2m_\pi^2}{s^2}\Big)\\
&\quad +i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)(\tau_{1z}+\tau_{2z})\\
&\quad +2\boldsymbol{K}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)(\tau_{1z}-\tau_{2z})-2\boldsymbol{K}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)(\tau_{1z}+\tau_{2z})\Big]s^2A(k)\\
&\quad +\frac{g_A^3h^1_\pi}{2\sqrt{2}f_\pi}\frac{\pi}{\Lambda_\chi^2M}\Big[-\boldsymbol{K}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)(\tau_{1z}+\tau_{2z})\Big(1-\frac{4m_\pi^2}{s^2}\Big)s^2A(k)\\
&\quad +\frac{1}{2}(\tau_{1z}+\tau_{2z})i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\Big(5-\frac{8m_\pi^2}{s^2}\Big)s^2A(k)\\
&\quad -\frac{1}{4}(\vec{\tau}_1\times\vec{\tau}_2)_z i\boldsymbol{k}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\Big(5-\frac{14m_\pi^2}{s^2}\Big)s^2A(k)\\
&\quad +\frac{1}{2}(\vec{\tau}_1\times\vec{\tau}_2)_z i\boldsymbol{k}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\frac{m_\pi^3}{s^2}\\
&\quad -(\vec{\tau}_1\times\vec{\tau}_2)_z[\boldsymbol{k}\cdot\boldsymbol{\sigma}_1\,(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}_2+\boldsymbol{k}\cdot\boldsymbol{\sigma}_2\,(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}_1]A(k)\Big]\;. \qquad (4.89)
\end{aligned}
$$

Let's note that we have in total 11 LECs that must be determined from the experimental data: one in the LO term, six in the subleading order and five in the N3LO. In the RC-TPE terms it appears a strange factor that goes like $1/s^2$ which has the same form of the one pion exchange but with twice the mass of the pion. The terms $V^{(2)}_{PV}(c_4)$ and $V^{(2)}_{PV}(\text{LEC})$ are exactly the same found in Ref. [26]. In addiction to them, we have also obtained for the first time the contributions of the RC terms.

4.6 The PV potential in r-space

In order to perform calculation with this potential for the NN system, we need the potential in configuration space. Remembering that

$$\langle \alpha_1'\alpha_2'|V|\alpha_1\alpha_2\rangle = \frac{1}{\Omega} V(\boldsymbol{k},\boldsymbol{K})\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'} \ ,$$

and performing the following change of variables

$$\boldsymbol{r} = \boldsymbol{r}_1-\boldsymbol{r}_2 \ , \qquad \boldsymbol{R} = \frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2} \ , \tag{4.90}$$

$$\boldsymbol{r}' = \boldsymbol{r}_1'-\boldsymbol{r}_2' \ , \qquad \boldsymbol{R}' = \frac{\boldsymbol{r}_1'+\boldsymbol{r}_2'}{2} \ , \tag{4.91}$$

we obtain

$$\langle \boldsymbol{r}_1'\boldsymbol{r}_2'|V|\boldsymbol{r}_1\boldsymbol{r}_2\rangle = \delta^3(\boldsymbol{R}-\boldsymbol{R}')\int \frac{d^3k}{(2\pi)^3}\frac{d^3K}{(2\pi)^3}\, e^{i(\boldsymbol{K}+\frac{1}{2}\boldsymbol{k})\cdot\boldsymbol{r}'}V(\boldsymbol{k},\boldsymbol{K})e^{-i(\boldsymbol{K}-\frac{1}{2}\boldsymbol{k})\cdot\boldsymbol{r}} \ , \tag{4.92}$$

where we use the fact that the potential does not depend on the total momentum $\boldsymbol{P}$. In order to compute the integral we must multiply V by a "cut -off" function parametrized by Λ_F. This modification is necessary since our potential is valid only for small momentum values. The cut-off function we use is:

$$C_{\Lambda_F}(k) = \exp\Big(-(k/\Lambda_F)^4\Big) \ , \tag{4.93}$$

where $\Lambda_F = 400 \div 700$ MeV. The parameter Λ_F represents a "cut-off" energy. The physics above Λ_F is taken into accounts through the LECs. In fact, the LECs well depend on Λ_F, while the physical observables should not depend on it (this should be verified when more and more order in ChPT are included in our ChPT expansion).

This choice of the cut-off function and the fact that the potential contains at most terms linear in $\boldsymbol{k}$ makes that the final potential will be *local*, namely:

$$\langle \boldsymbol{r}_1'\boldsymbol{r}_2'|V|\boldsymbol{r}_1\boldsymbol{r}_2\rangle = \delta^3(\boldsymbol{R}-\boldsymbol{R}')\delta^3(\boldsymbol{r}-\boldsymbol{r}')V(\boldsymbol{r}) \ . \tag{4.94}$$

For example, if we have a potential like $V(\boldsymbol{k},\boldsymbol{K}) = g(k) + K_i f_i(k)$, using the identities

$$e^{i(\boldsymbol{K}+\frac{1}{2}\boldsymbol{k})\cdot\boldsymbol{r}'}\boldsymbol{k}e^{-i(\boldsymbol{K}-\frac{1}{2}\boldsymbol{k})\cdot\boldsymbol{r}} = (-i\boldsymbol{\nabla}_{\boldsymbol{r}'} - i\boldsymbol{\nabla}_{\boldsymbol{r}})\Big(e^{i(\boldsymbol{K}+\frac{1}{2}\boldsymbol{k})\cdot\boldsymbol{r}'}e^{-i(\boldsymbol{K}-\frac{1}{2}\boldsymbol{k})\cdot\boldsymbol{r}}\Big) \ , \tag{4.95}$$

and

$$e^{i(\boldsymbol{K}+\frac{1}{2}\boldsymbol{k})\cdot\boldsymbol{r}'}\boldsymbol{K}e^{-i(\boldsymbol{K}-\frac{1}{2}\boldsymbol{k})\cdot\boldsymbol{r}} = \frac{-i\boldsymbol{\nabla}_{\boldsymbol{r}'} + i\boldsymbol{\nabla}_{\boldsymbol{r}}}{2}\Big(e^{i(\boldsymbol{K}+\frac{1}{2}\boldsymbol{k})\cdot\boldsymbol{r}'}e^{-i(\boldsymbol{K}-\frac{1}{2}\boldsymbol{k})\cdot\boldsymbol{r}}\Big) \ , \tag{4.96}$$

we obtain that

$$\langle \boldsymbol{r}_1'\boldsymbol{r}_2'|V|\boldsymbol{r}_1\boldsymbol{r}_2\rangle = \delta^3(\boldsymbol{R}-\boldsymbol{R}')\delta^3(\boldsymbol{r}-\boldsymbol{r}')g(r)+\delta^3(\boldsymbol{R}-\boldsymbol{R}')\frac{-i\boldsymbol{\nabla}_{\boldsymbol{r}'}+i\boldsymbol{\nabla}_{\boldsymbol{r}}}{2}\Big[\delta^3(\boldsymbol{r}-\boldsymbol{r}')f_i(r)\Big] , \tag{4.97}$$

where

$$g(r) = \int \frac{d^3k}{(2\pi)^3}\, g(k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \; , \qquad f_i(r) = \int \frac{d^3k}{(2\pi)^3}\, f_i(k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \; . \tag{4.98}$$

The action of the derivative on the δ function is a problem, but our final goal is the calculation of $\langle A|V|B\rangle$, so we can write

$$\begin{aligned} \langle A|V|B\rangle &= \int d^3r \int d^3r'\; \psi_B^\dagger(\boldsymbol{r}')g(r)\delta^3(\boldsymbol{r}-\boldsymbol{r}')\psi_A(\boldsymbol{r}) \\ &\qquad +\int d^3r \int d^3r'\; \psi_B^\dagger(\boldsymbol{r}')\Big[\frac{-i\boldsymbol{\nabla}_{\boldsymbol{r}'}+i\boldsymbol{\nabla}_{\boldsymbol{r}}}{2}(\delta^3(\boldsymbol{r}-\boldsymbol{r}')f_i(r))\psi_A(\boldsymbol{r})\Big] \\ &= \int d^3r\; \psi_B^\dagger(\boldsymbol{r})g(r)\psi_A(\boldsymbol{r}) + \int d^3r \int d^3r'\; (i\boldsymbol{\nabla}_{\boldsymbol{r}'}\psi_B^\dagger(\boldsymbol{r}'))\delta^3(\boldsymbol{r}-\boldsymbol{r}')f_i(r))\psi_A(\boldsymbol{r}) \\ &\qquad -\psi_B(\boldsymbol{r})^\dagger\delta^3(\boldsymbol{r}-\boldsymbol{r}')f_i(r)(i\boldsymbol{\nabla}_{\boldsymbol{r}}\psi_A) \\ &\qquad = \int d^3r\; \psi_B(\boldsymbol{r})^\dagger\Big(g(r) + \Big\{\frac{-i\nabla_i}{2}, f_i(r)\Big\}\Big)\psi_A(\boldsymbol{r}) \; , \end{aligned} \tag{4.99}$$

where we use the integration by parts in order to move the derivatives from the delta to the wave functions and $\{\cdots\}$ is the anticommutator. Therefore

$$\langle \boldsymbol{r}_1'\boldsymbol{r}_2'|V|\boldsymbol{r}_1\boldsymbol{r}_2\rangle = \delta^3(\boldsymbol{R}-\boldsymbol{R}')\delta^3(\boldsymbol{r}-\boldsymbol{r}')\Big(g(r) + \Big\{\frac{-i\nabla_i}{2}, f_i(r)\Big\}\Big). \tag{4.100}$$

Applying this relation to the various terms of the potential given in Eqs. (4.83)-(4.89), we obtain the expression of various term of the potential in r-space:

$$\langle \boldsymbol{r}_1'\boldsymbol{r}_2'|V|\boldsymbol{r}_1\boldsymbol{r}_2\rangle = \delta^3(\boldsymbol{R}-\boldsymbol{R}')\delta^3(\boldsymbol{r}-\boldsymbol{r}')V(\boldsymbol{r}), \tag{4.101}$$

with

$$\begin{aligned} V(\boldsymbol{r}) &= V^{(\mathrm{OPE})}(\boldsymbol{r}) + V^{(\mathrm{RC-OPE})}(\boldsymbol{r}) + V^{(\mathrm{CT})}(\boldsymbol{r}) + V^{(\mathrm{TPE})}(\boldsymbol{r}) \\ &\quad +V^{(\mathrm{c_4})}(\boldsymbol{r}) + V^{(\mathrm{LEC})}(\boldsymbol{r}) + V^{(\mathrm{RC-TPE})}(\boldsymbol{r}) \end{aligned} \tag{4.102}$$

$$V^{(\mathrm{OPE})}(\boldsymbol{r}) = \frac{g_A h_\pi^1}{2\sqrt{2} f_\pi} (\vec{\tau}_1 \times \vec{\tau}_2)_z \, (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \, g_1'(r) \; , \tag{4.103}$$

$$\tag{4.104}$$

$$\begin{aligned} V^{(\mathrm{RC-OPE})}(\boldsymbol{r}) &= -\frac{g_A h_\pi^1}{2\sqrt{2} f_\pi 4M^2} (\vec{\tau}_1 \times \vec{\tau}_2)_z \\ & \Big[\big\{ p_j \, , \, \big\{ p_j \, , \, (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \, g_1'(r) \big\} \big\} \\ &+ \frac{\epsilon_{j\ell m}}{2} \Big(\sigma_{1i} \, \sigma_{2j} + \sigma_{1j} \, \sigma_{2i} \Big) \Big\{ p_m \, , \, \partial_i \, \partial_\ell g_1(r) \Big\} \Big] \; , \end{aligned} \tag{4.105}$$

$$\begin{aligned} V^{(\mathrm{CT})}(\boldsymbol{r}) &= \frac{m_\pi^2}{\Lambda_\chi^2 f_\pi} \Big[C_1 (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \\ & +C_2 \, \vec{\tau}_1 \cdot \vec{\tau}_2 \, (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \\ & +C_3 \, (\vec{\tau}_1 \times \vec{\tau}_2)_z \, (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \\ & +C_4 \, (\tau_{1z} + \tau_{2z}) \, (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \\ & +C_5 \, \mathcal{I}^{ab} \, \tau_{1a} \, \tau_{2b} \, (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \, \Big] Z'(r) \; , \end{aligned} \tag{4.106}$$

$$\begin{aligned} V^{(\mathrm{TPE})}(\boldsymbol{r}) &= -\frac{g_A h_\pi^1}{2\sqrt{2} f_\pi} \frac{m_\pi^2}{\Lambda_\chi^2} (\vec{\tau}_1 \times \vec{\tau}_2)_z (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \, L'(r) \\ & -\frac{g_A h_\pi^1}{2\sqrt{2} f_\pi} \frac{g_A^2 m_\pi^2}{\Lambda_\chi^2} \Big[4 \, (\tau_{1z} + \tau_{2z}) \, (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \, L'(r) \\ & +(\vec{\tau}_1 \times \vec{\tau}_2)_z (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \Big[H'(r) - 3L'(r) \Big] \Big] \; , \end{aligned} \tag{4.107}$$

$$V^{(c_4)}(\boldsymbol{r}) = -\frac{c_4 h_\pi^1 g_A}{\sqrt{2} f_\pi} \frac{\pi \, m_\pi^3}{\Lambda_\chi^2} (\tau_{1z} + \tau_{2z}) \, (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \, A_1'(r) \; , \tag{4.108}$$

$$\begin{aligned} V^{(\mathrm{LEC})}(\boldsymbol{r}) &= \frac{g_A^2}{2 f_\pi^2} \frac{\pi \, m_\pi^3}{\Lambda_\chi^2} \Big[\Big[\frac{3 g_A h_V^0}{4} + \frac{g_A h_V^0}{2} \vec{\tau}_1 \cdot \vec{\tau}_2 + \Big(\frac{g_A h_V^1}{4} - h_A^1 \Big) (\tau_{1z} + \tau_{2z}) \\ & - \Big(h_A^2 + \frac{g_A h_V^2}{3} \Big) I^b \tau_{1b} \tau_{2b} \Big] (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \, A_1'(r) \\ & - \frac{g_A h_V^1}{2} (\vec{\tau}_1 \times \vec{\tau}_2)_z (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \, (A_1'(\boldsymbol{r}) - 2A_2'(\boldsymbol{r})) \Big] \; , \end{aligned} \tag{4.109}$$

$$\begin{aligned} V^{(\mathrm{RC-TPE})}(\boldsymbol{r}) &= \frac{g_A h_\pi^1}{4\sqrt{2} f_\pi} \frac{\pi \, m_\pi^3}{\Lambda_\chi^2 M} \Big[(\vec{\tau}_1 \times \vec{\tau}_2)_z (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \, (A_1'(\boldsymbol{r}) - 2A_2'(\boldsymbol{r})) \\ & +i (\tau_{1z} + \tau_{2z}) \, (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \hat{\boldsymbol{r}} \, A'(r) \\ & +(\tau_{1z} - \tau_{2z}) \, (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)_j \{ p_j \, , \, A_1(r) \} \\ & -(\tau_{1z} + \tau_{2z}) \, (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)_j \{ p_j \, , \, A_1(r) \} \Big] \end{aligned}$$

$$+\frac{g_A^3 h_\pi^1}{2\sqrt{2}f_\pi}\frac{\pi\, m_\pi^3}{\Lambda_\chi^2 M}\Bigg[-\frac{1}{2}(\tau_{1z}+\tau_{2z})\,(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)_j\{p_j\,,\,(A_1(r)-4A_2(r))\}$$
$$+\frac{1}{2}(\tau_{1z}+\tau_{2z})\,(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}\,(5A_1'(r)-8A_2'(r))$$
$$-\frac{1}{4}(\vec{\tau}_1\times\vec{\tau}_2)_z\,(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}\,(5A_1'(r)-14A_2'(r))$$
$$+\frac{1}{2}(\vec{\tau}_1\times\vec{\tau}_2)_z\,(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}\,g_2'(r)$$
$$-(\vec{\tau}_1\times\vec{\tau}_2)_z\frac{\epsilon_{j\ell m}}{2m_\pi^2}\Big(\sigma_{1i}\,\sigma_{2j}+\sigma_{1j}\,\sigma_{2i}\Big)\Big\{p_m\,,\,\partial_i\,\partial_\ell A_2(r)\Big\}\Bigg]\,, \qquad (4.110)$$

with $\boldsymbol{p}=-i\nabla$ and

$$g_1(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{C_{\Lambda_F}(k)}{k^2+m_\pi^2}\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,, \qquad (4.111)$$
$$g_2(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{C_{\Lambda_F}(k)}{k^2+4m_\pi^2}\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,, \qquad (4.112)$$
$$L(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{C_{\Lambda_F}(k)}{m_\pi^2}L(k)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,, \qquad (4.113)$$
$$H(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{C_{\Lambda_F}(k)}{m_\pi^2}H(k)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,, \qquad (4.114)$$
$$Z(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{C_{\Lambda_F}(k)}{m_\pi^2}\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,, \qquad (4.115)$$
$$A_1(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{C_{\Lambda_F}(k)}{m_\pi^3}\,(k^2+4m_\pi^2)A(k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,, \qquad (4.116)$$
$$A_2(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{C_{\Lambda_F}(k)}{m_\pi}\,A(k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,. \qquad (4.117)$$
$$(4.118)$$

Note that

$$\big\{p_j\,,\,\{p_j\,,\,O\}\big\} = -\big(\boldsymbol{\nabla}^2O\big)-4\,\big[(\boldsymbol{\nabla}O)\cdot\boldsymbol{\nabla}+4\,O\boldsymbol{\nabla}^2\big]\,,$$
$$\big\{p_j\,,\,O\big\} = -i(\boldsymbol{\nabla}O)-2iO\boldsymbol{\nabla}\,, \qquad (4.119)$$

and

$$\boldsymbol{\nabla}^2(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}\,g'(r)$$
$$=\ (\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}\left[g'''(r)+2\,\frac{g''(r)}{r}-2\,\frac{g'(r)}{r^2}\right]\,, \qquad (4.120)$$

$$\partial_j(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}\,g'(r)=(\sigma_{1j}+\sigma_{2j})\,\frac{g'(r)}{r}$$
$$+(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}\left[\frac{g''(r)}{r}-\frac{g'(r)}{r^2}\right]\frac{r_j}{r}\,. \qquad (4.121)$$

It's convenient to define the operators,

$$
\begin{aligned}
S_r^{\pm} &= (\boldsymbol{\sigma}_1 \pm \boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}} \ , && (4.122)\\
S_p^{\pm} &= (\boldsymbol{\sigma}_1 \pm \boldsymbol{\sigma}_2)\cdot\boldsymbol{p} \ , && (4.123)\\
S_r^{\times} &= (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}} \ , && (4.124)\\
S_L &= \boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{r}}\ \boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{L}} + \boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{r}}\ \boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{L}} \ , && (4.125)
\end{aligned}
$$

where $\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}}\times\boldsymbol{p}$ is the "reduced" orbital angular momentum operator. Using these definitions we can rewrite the relativistic corrections as

$$
\begin{aligned}
V^{(\mathrm{RC-OPE})}(\boldsymbol{r},\boldsymbol{p}) &= \frac{g_A h_\pi^1}{4\sqrt{2}F_\pi M^2}(\vec{\tau}_1\times\vec{\tau}_2)_z\Bigg[\left[g_1'''(r) + 2\frac{g_1''(r)}{r} - 2\frac{g_1'(r)}{r^2}\right]S_r^+\\
&+4\,i\left[\frac{g_1'(r)}{r}\,S_p^+ + \left[g_1''(r) - \frac{g_1'(r)}{r}\right]S_r^+\,\hat{\boldsymbol{r}}\cdot\boldsymbol{p}\right]\\
&-4\,g_1'(r)\ S_r^+\ \boldsymbol{p}^2\ - \left[g_1''(r) - \frac{g_1'(r)}{r}\right]S_L\Bigg] \ , && (4.126)\\
V^{(\mathrm{RC-TPE})}(\boldsymbol{r},\boldsymbol{p}) &= \frac{g_A h_\pi^1}{4\sqrt{2}f_\pi}\frac{\pi\, m_\pi^3}{\Lambda_\chi^2 M}\Bigg[(\vec{\tau}_1\times\vec{\tau}_2)_z S_r^+\left(A_1'(\boldsymbol{r}) - 2A_2'(\boldsymbol{r})\right)\\
&+i(\tau_{1z}+\tau_{2z})S_r^\times\, A_1'(r)\\
&+i((\tau_{1z}+\tau_{2z})S_r^- - (\tau_{1z}-\tau_{2z})S_r^+)\,A_1'(r)\\
&-2A(r)((\tau_{1z}+\tau_{2z})S_p^- - (\tau_{1z}-\tau_{2z})S_p^+)\Bigg]\\
&+\frac{g_A^3 h_\pi^1}{2\sqrt{2}f_\pi}\frac{\pi\, m_\pi^3}{\Lambda_\chi^2 M}\Bigg[i\frac{1}{2}(\tau_{1z}+\tau_{2z})\,S_r^-\,(A_1'(r) - 4A_2'(r))\\
&-(A_1(r) - 4A_2(r))(\tau_{1z}+\tau_{2z})S_p^-\\
&+\frac{1}{2}(\tau_{1z}+\tau_{2z})\,S_r^\times\,(5A_1'(r) - 8A_2'(r))\\
&-\frac{1}{4}(\vec{\tau}_1\times\vec{\tau}_2)_z\,S_r^+\,(5A_1'(r) - 14A_2'(r))\\
&+\frac{1}{2}(\vec{\tau}_1\times\vec{\tau}_2)_z\,S_r^+\,g_2'(r)\\
&+\frac{1}{m_\pi^2}(\vec{\tau}_1\times\vec{\tau}_2)_z S_L\left(\frac{A_2''(r)}{r} - \frac{A_2'(r)}{r^2}\right)\Bigg] \ . && (4.127)
\end{aligned}
$$

Chapter 5

The TV interaction

In this Chapter we will analyse the TV interaction terms appearing in the QCD Lagrangian and how we can model them at the hadronic level using the χEFT. We will concentrate in particular on the so called θ-term. As we will see, it is possible to rewrite this T-violating term via a $U(1)_A$ transformation as a complex mass term [5, 6]. In this way, it can be interpreted as an external pseudoscalar field $p(x)$, already introduced in Chapter 2, and include it in the χEFT. We will study here only the θ-term, possible further BSM TV Lagrangian terms can be treated in a similar way [5]. From the derived Lagrangian, using the technique described in Chapter 3, we will build the NN TV potential.

This Chapter is organized as follow. In Section 5.1 we will introduce the fundamental concepts regarding the θ-term in the QCD and how it is related to the mass matrix via the $U(1)_A$ transformation. In Section 5.2 we will derive the nuclear Lagrangian terms induced by the θ-term. The derived Hamiltonian at the classical level admits a ground state which does not coincide with the void. Therefore we have to redefine the fields expanding them around the new ground state: we will perform this in Section 5.3. In the last Section we will discuss the time-ordered diagrams that contribute to the NN TV potential and we will find the potential in momentum and configuration space.

5.1 The θ-term

Within the SM, it is possible to build in the QCD Lagrangian a term that violate P but not C, therefore for the CPT invariance, it must violate T. As it is known, this term, named θ-term, is the only source of the P and T violation in the strong interaction sector of the SM [4]. The θ-term is given by the full contraction of the gluon field-strength tensor $\mathcal{G}^a$ (see Chapter 2 Eq. (2.3)) with its dual, and it is parametrized by an angle θ. Therefore the Lagrangian of the QCD with the new term reads

$$\overline{\mathcal{L}}_{QCD} = \mathcal{L}_{QCD} + \mathcal{L}^{\theta}_{QCD} = \mathcal{L}_{QCD} - \theta \frac{g^2}{64\,\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathcal{G}^a_{\mu\nu} \mathcal{G}^a_{\rho\sigma} \,, \tag{5.1}$$

where $\mathcal{L}_{QCD}$ is given in Eq. (2.1) and g is the strong coupling constant.

This term is not the only possible source of P and T violation in the SM. Another well-known term comes from the complex phase of the CKM matrix. However this phase plays a role only for flavour changing processes [5] and therefore we will not consider it hereafter. Moreover, beyond the SM it is possible to write other TV effective operators composed by SM degrees of freedom, still verifying the gauge symmetries of the SM [40]. In particular, it is possible to write new Lagrangian terms of energy dimension $d = 6$ that can generate CP violation in nuclear systems down to the energy scale $\Lambda_\chi \sim$1 GeV. We will not treat these terms and we focus only on the θ-term. A complete discussion can be found in [5].

Before starting to discuss how we can include the θ-term in our χEFT let's return to the chiral group $G = SU(2)_V \otimes SU(2)_A \otimes U(1)_V \otimes U(1)_A$. In Section 2.1 we disregarded the symmetry $U(1)_A$ saying that at quantum level it is broken by an anomaly. Now we will present how to take in account the anomalous behaviour of the axial current.

First of all, let us study the divergence of the axial current given in Eq. (2.14b). To this aim, we must take care that the axial current is an operator built from fermionic fields and the product of these local operators often have singularities. So we rewrite the current keeping separated the two fields by a distance ϵ and then take the limit $\epsilon \to 0$. The axial current can be defined as [27]

$$J^\mu_5 = \operatorname{sym}\lim_{\epsilon\to 0} \Big\{ \overline{q}\big(x + \frac{\epsilon}{2}\big)\gamma^\mu\gamma^5 \exp\big[-ig\int_{x-\epsilon/2}^{x+\epsilon/2} \mathrm{d}z\, G^a(z)T^a\big] q\big(x - \frac{\epsilon}{2}\big)\Big\}, \tag{5.2}$$

where we have introduced the exponential term in order to preserve the $SU(3)$ color gauge invariance of the current, G^a being the gluons field and T^a the matrix defined in Eq. (2.4). The symbol $\operatorname{sym}\lim_{\epsilon\to 0}$ means that we take the symmetric limit to have the correct properties under Lorentz transformation, for example [27]

$$\operatorname{sym}\lim_{\epsilon\to 0}\left(\frac{\epsilon^\mu}{\epsilon^2}\right) = 0, \qquad \operatorname{sym}\lim_{\epsilon\to 0}\left(\frac{\epsilon^\mu\epsilon^\nu}{\epsilon^2}\right) = \frac{1}{4} g^{\mu\nu}. \tag{5.3}$$

Now we can use the equation of motion neglecting the mass term $\mathcal{M}$ (classicaly, as discussed in Chapter 2, one would aspect $\partial_\mu J^\mu_5 = 0$). From $\mathcal{L}_{QCD}$ we have

$$\gamma^\mu \partial_\mu q(x) = -ig\gamma^\mu G^a_\mu T^a\, q(x). \tag{5.4}$$

Performing the calculation as discussed in [27], we find

$$\partial_\mu J^\mu_5 = -\frac{g^2 N_f}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathcal{G}^a_{\mu\nu} \mathcal{G}^a_{\rho\sigma}, \tag{5.5}$$

where N_f is the number of flavours involved in the process, in our case $N_f = 2$.

Therefore the axial current is not conserved also for the case $\mathcal{M} = 0$ and this has some consequences on the generating functional of the QCD,

$$\mathcal{Z}[\theta] = \int [\mathrm{d}G][\mathrm{d}\overline{q}\mathrm{d}q] \exp\Big[i \int \mathrm{d}^4x \, \overline{\mathcal{L}}_{QCD}(q, \overline{q}, G) - \theta \frac{g^2}{64\,\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathcal{G}^a_{\mu\nu} \mathcal{G}^a_{\rho\sigma}\Big]. \tag{5.6}$$

If we apply an $U(1)_A$ transformation,

$$q(x) \to q'(x) = q(x) e^{-i\alpha\gamma^5}, \tag{5.7}$$

the measure of the generating functional integral change due to the anomaly related to the axial current, obtaining

$$[\mathrm{d}G][\mathrm{d}\overline{q}\mathrm{d}q] \to [\mathrm{d}G][\mathrm{d}\overline{q}'\mathrm{d}q'] = [\mathrm{d}G][\mathrm{d}\overline{q}\mathrm{d}q] \exp\Big[- 2N_f\alpha \int \mathrm{d}^4x \, \frac{g^2}{64\,\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathcal{G}^a_{\mu\nu} \mathcal{G}^a_{\rho\sigma}\Big], \tag{5.8}$$

which exactly cancel out the θ-term if we take $\alpha = -\frac{\theta}{2N_f}$. The $U(1)_A$ transformation however changes the mass term of the QCD Lagrangian

$$\mathcal{L}^{\mathcal{M}}_{QCD} = -\overline{q}(x)\overline{\mathcal{M}}q(x) = -q_R(x)\overline{\mathcal{M}}q_L(x) + \text{h.c.} \tag{5.9}$$

where the most general mass matrix $\overline{\mathcal{M}}$ can be written as

$$\overline{\mathcal{M}} = e^{i\rho}\mathcal{M} = e^{i\rho}\,\overline{m}(\mathbb{1} + \epsilon\,\tau_3) \tag{5.10}$$

where

$$\overline{m} = \frac{m_u + m_d}{2}, \qquad \epsilon = \frac{m_u - m_d}{m_u + m_d}, \tag{5.11}$$

and ρ is an arbitrary phase. Performing the $U(1)_A$ axial transformation given in Eq. (5.7) imposing $\alpha = -\frac{\theta}{4}$, one obtains

$$\begin{aligned} \mathcal{L}^{\mathcal{M}}_{QCD} &\to -\overline{q}(x) e^{i\frac{\theta}{4}\gamma^5} \overline{\mathcal{M}} e^{-i\frac{\theta}{4}\gamma^5} q(x) \\ &= -\overline{q}_R(x) e^{-i\left(\frac{\theta}{2}-\rho\right)} \overline{m}(\mathbb{1} + \epsilon\,\tau_3) q_L(x) + \text{h.c.} \\ &= -\overline{q}(x) e^{i\left(\rho - \frac{\theta}{2}\right)} \mathcal{M} q(x) \end{aligned} \tag{5.12}$$

so the physical observable is not θ but $\overline{\theta} = 2\rho - \theta$. If $\overline{\theta}$ is small we can expand the exponential and the new mass Lagrangian reads

$$\mathcal{L}^{\mathcal{M}}_{QCD} = -\overline{q}(x)(\overline{m}\mathbb{1} + \epsilon\,\overline{m}\,\tau_3 - i\frac{\overline{\theta}\,\overline{m}}{2}\gamma^5\mathbb{1} - i\frac{\overline{\theta}\,\epsilon\,\overline{m}}{2}\tau_3\gamma^5)q(x). \tag{5.13}$$

In this way we have redefined all the θ dependence in the mass term. According to the Lagrangian given in Eq. (2.19) and Eq. (5.13), we can identify the sources s_0, p_0, s_i and p_i with $i = 1, 2, 3$ of Eq. (2.18) with

$$s_0 \to \overline{m}, \quad s_{1,2} \to 0, s_3 \to \epsilon\,\overline{m}. \quad p_0 \to \frac{\overline{\theta}\,\overline{m}}{2}, \quad p_{1,2} \to 0, \quad p_3 \to \frac{\overline{\theta}\,\epsilon\,\overline{m}}{2}. \tag{5.14}$$

It is possible to perform another transformation of quark fields via the group $SU(2)_A$ such that the P and T violating term [5]

$$\begin{aligned}\mathcal{L}_{QCD}^{TV} &= \frac{\overline{\theta}\,\overline{m}}{2}\left(\overline{q}(x)i\gamma^5 q(x) + \epsilon\overline{q}(x)i\gamma^5\tau_3 q(x)\right) \\ &\xrightarrow{SU(2)_A} \overline{\theta}m^{\star}\left(\overline{q}(x)i\gamma^5 q(x)\right) .\end{aligned} \tag{5.15}$$

The second axial rotation moves the θ-term completly into a P and T violating and isospin conserving quark-mass term with a reduced mass

$$m^{\star} = \frac{m_u m_d}{(m_u + m_d)} = \frac{\overline{m}}{2}(1 - \epsilon^2). \tag{5.16}$$

This fact must be reflected in the absence of P and T violating terms that violate isospin in the χEFT Lagrangian [5]. Therefore the final substitution of the external source fields will be

$$s_0 \to \overline{m}, \quad s_{1,2} \to 0, s_3 \to \epsilon\,\overline{m}. \quad p_0 \to \frac{\overline{\theta}\,\overline{m}}{2}(1 - \epsilon^2), \quad p_{1,2} \to 0, \quad p_3 \to 0. \tag{5.17}$$

5.2 The χEFT Lagrangian induced by the θ term

The inclusion of the TV terms deriving from the θ-term is obtained simply by including in χ (Eq. 2.37a) the new sources p_0 and p_3 [5],

$$\chi = 2B\Big(s_0 + \tau_3 s_3 + ip_0 + i\tau_3 p_3\Big) \tag{5.18}$$

where B is defined as in Eq. (2.40). Substituting the expression of χ in the Lagrangian term (2.42)

$$\mathcal{L}_{\pi}^{(2)} = \frac{f_\pi^2}{4}\left\langle \chi(x)U^\dagger(x) + U(x)\chi^\dagger(x)\right\rangle \tag{5.19}$$

and performing the expansion in term of the pion fields as in Eq. (B.9), we obtain

$$\mathcal{L}_{\pi}^{(TV,2)} = (2Bp_3)f_\pi\,\pi_3\Big(1 - \alpha\frac{\pi^2}{f_\pi^2}\Big) + \cdots , \tag{5.20}$$

where the elipses denote - hereafter in this section - terms which are either of higher order in the pion-field expansion or P and T conserving. Above, α represents the arbitrariness in the choice of the $U(x)$ field, following the definition given in Eq. (2.28).

From the fourth order pion-sector Lagrangian given in Eq. (2.43)

$$\mathcal{L}_\pi^{(4)} = \frac{l_3}{16}\langle\chi(x)U^\dagger(x) + U(x)\chi^\dagger(x)\rangle^2 - \frac{l_7}{16}\langle\chi(x)U^\dagger(x) - U(x)\chi^\dagger(x)\rangle^2 \qquad (5.21)$$

developing in powers of pion field we obtain the following TV terms

$$\begin{aligned}\mathcal{L}_\pi^{(TV,4)} &= 2l_3(2Bs_0)(2Bp_3)\frac{\pi_3}{f_\pi}\Big(1-\big(\frac{1}{2}+\alpha\big)\frac{\pi^2}{f_\pi^2}\Big) \\ &\quad -2l_7(2Bs_3)(2Bp_0)\frac{\pi_3}{f_\pi}\Big(1-\big(\frac{1}{2}+\alpha\big)\frac{\pi^2}{f_\pi^2}\Big)+\cdots\ . \end{aligned} \qquad (5.22)$$

The second order pion-nucleon sector Lagrangian with χ operator is

$$\mathcal{L}_{\pi N}^{(2)} = c_1\overline{N}\langle\chi_+\rangle N + c_5\overline{N}\hat{\chi}_+N \qquad (5.23)$$

where χ_+ is given in Eq. (B.9) which, after developing in pion fields, gives

$$\begin{aligned}\mathcal{L}_{\pi N}^{(TV,2)} &= 4c_1(2Bp_3)\frac{\pi_3}{f_\pi}\Big(1-\alpha\frac{\pi^2}{f_\pi^2}\Big)\overline{N}N \\ &\quad +2c_5(2Bp_0)\overline{N}\frac{\vec{\tau}\cdot\vec{\pi}}{f_\pi}\Big(1-\alpha\frac{\pi^2}{f_\pi^2}\Big)N+\cdots\ . \end{aligned} \qquad (5.24)$$

There are also several four-nucleon terms induced by the theta terms that come from the leading order four-nucleon Lagrangian in Eq. (2.67)

$$\begin{aligned}\mathcal{L}_{4N}^{(TV,2)} &= \overline{C}_1 4(2Bp_3)\overline{N}N\overline{N}N/f_\pi \\ &\quad +\overline{C}_2 4(2Bp_3)\overline{N}\gamma_\mu\gamma^5N\overline{N}\gamma^\mu\gamma^5N/f_\pi \\ &\quad +\overline{C}_3 2(2Bp_0)\overline{N}(\vec{\tau}\cdot\vec{\pi})N\overline{N}N/f_\pi \\ &\quad +\overline{C}_4 2(2Bp_0)\overline{N}(\vec{\tau}\cdot\vec{\pi})\gamma_\mu\gamma^5N\overline{N}\gamma^\mu\gamma^5N \\ &\quad -\overline{C}_5 4(2Bp_0)\overline{N}N\partial_\mu\big(\overline{N}\gamma^\mu\gamma^5N\big)/f_\pi \\ &\quad -\overline{C}_6 4(2Bp_0)\overline{N}\vec{\tau}N\partial_\mu\big(\overline{N}\vec{\tau}\gamma^\mu\gamma^5N\big)/f_\pi \\ &\quad -\overline{C}_7 2(2Bp_3)\overline{N}\tau_3N\partial_\mu\big(\overline{N}\gamma^\mu\gamma^5N\big)/f_\pi \\ &\quad -\overline{C}_8 2(2Bp_3)\overline{N}N\partial_\mu\big(\overline{N}\tau_3\gamma^\mu\gamma^5N\big)/f_\pi+\cdots\ . \end{aligned} \qquad (5.25)$$

A more complete list of P and T violating terms induced by the θ term can be find in [5].

5.3 Selection of the ground state

The Lagrangian terms in Eqs. (5.20)-(5.22) are linear in the pion fields. This mean that we are expanding the EFT around the wrong ground state. In particular the minimum of the energy will be for $\pi_3 \neq 0$.

The ground state is identified minimizing the potential of the pion-sector Lagrangian of the χEFT:

$$V = -\int d^3x \Big\{ \frac{f_\pi^2}{4}\langle \chi U^\dagger + U\chi^\dagger\rangle + \frac{l_3}{16}\langle \chi U^\dagger + U\chi^\dagger\rangle^2 - \frac{l_7}{16}\langle \chi U^\dagger - U\chi^\dagger\rangle^2 \Big\} \tag{5.26}$$

with χ given in Eq. (5.14). Here we assume that the fields are classical. In term of the pion field the potential reads,

$$V = -\int d^3x \Big\{ f_\pi^2 2Bs_0\Big(1-\frac{\pi^2}{2f_\pi^2}\Big) + 4l_3B^2s_0^2\Big(1-\frac{\pi^2}{f_\pi^2}\Big) - 4l_7B^2\Big[\frac{s_3^2}{f_\pi}\pi_3^2 + \frac{2s_3p_0}{f_\pi}\pi_3\Big(1-(\frac{1}{2}+\alpha)\frac{\pi^2}{f_\pi^2}\Big)\Big]\Big\}, \tag{5.27}$$

where we have used the fact that the field source $p_3 = 0$ and we have neglected the terms proportional to $\overline{\theta}^2$ like p_0^2, etc. We have also neglected higher terms in the pion field expansion supposing that the value of the minimum for the pions is proportional to $\overline{\theta}$ and therefore very small (hypothesis we will verify a posteriori). The minimum of the potential is identified by a variation of the field $\vec{\pi}$ which reads in term of its components

$$\pi_i \to \pi_i + \delta\pi_i. \tag{5.28}$$

Performing this transformation, the variation of the potential reads

$$\delta V = i\int d^3x \Big\{ (2Bf_\pi)^2 s_0\pi_i + 2(4B^2l_3)s_0^2\pi_i + 2(4B^2l_7)(p_0s_3\delta_{i3} - s_3p_0\epsilon_{3ij}\pi_j - s_3^2\pi_3\delta_{i3})\Big\}\delta\pi_i = 0\ . \tag{5.29}$$

Imposing this condition for $i = 1, 2$ we get two equations

$$\begin{cases} \pi_1 A + \pi_2 C = 0 \\ \pi_2 A - \pi_1 C = 0 \end{cases}$$

where

$$A = f_\pi^2(2Bs_0) + 2l_3(2Bs_0)^2 \tag{5.30}$$

$$C = 2(2B)^2 l_7 p_0 s_3\ . \tag{5.31}$$

Solving the system we get that $\pi_2(A^2+C^2)=0 \Rightarrow \pi_2=0$ and $\pi_1=0$. For $i=3$ we find the following condition

$$\pi_3 = -\frac{4Bl_7 p_0 s_3}{f_\pi^2 s_0 + 4Bl_3 s_0^2 - 4Bl_7 s_3^2}, \tag{5.32}$$

thus $\pi_3 \propto \overline{\theta}$ and so the condition that $\vec{\pi}$ is very small is verified. Picking up a term $f_\pi^2 s_0$ in the denominator we can expand it in a Taylor series assuming $Bl_3 s_0/f_\pi^2 \ll 1$. We obtain, keeping the first order,

$$\pi_3 = (\pi_3)_{min} \simeq -\frac{4Bl_7 s_3}{f_\pi^2 s_0} p_0 \tag{5.33}$$

The approximation above is based on the assumption that the LECs l_3 and l_7 are higher order in the chiral expansion (see later).

We can now redefine the pion field $\pi_3 \to \pi_3 + (\pi_3)_{\rm min}$ and evaluate the Lagrangian terms. Taking into account explicitly that $p_3 = 0$, the Lagrangian term in Eq. (5.19) becomes (we neglect the terms depending only on $(\pi_3)_{\rm min}$, since constant terms do not play a role in the dynamics)

$$\mathcal{L}_\pi^{(TV,2)} = \frac{(4B)^2 l_7 s_3 p_0}{2f_\pi}\pi_3\Big(1-\alpha\frac{\pi^2}{f_\pi^2}\Big) + \cdots \ . \tag{5.34}$$

Using the same procedure for the Lagrangian term in Eq. (2.43), we obtain

$$\mathcal{L}_\pi^{(TV,4)} \ = \ -\frac{(4B)^2 l_7 s_3 p_0}{2f_\pi}\pi_3\Big(1-\big(\frac{1}{2}+\alpha\big)\frac{\pi^2}{f_\pi^2}\Big) + \cdots \ , \tag{5.35}$$

where we have neglected higher order terms in the LECs l_7 and l_3 as before. Summing these Lagrangian terms, it remains only the term

$$\mathcal{L}_\pi^{(TV)} = \frac{4B^2 l_7 s_3 p_0}{f_\pi^3}\pi_3\pi^2 + \cdots \ , \tag{5.36}$$

which is exactly the term obtained in Ref. [5]. Let's note that the terms with α cancel out removing the arbitrariness on the choice of U.

In the same way, after the redefinition of the ground state and the imposition $p_3 = 0$, the second order pion-nucleon sector Lagrangian reads

$$\mathcal{L}_{\pi N}^{(TV)} \ = \ 8\mathrm{c}_1\frac{(2B)^2 l_7 s_3 p_0}{f_\pi^3}\overline{N}\pi_3 N + 2\mathrm{c}_5(2Bp_0)\overline{N}\frac{\vec{\tau}\cdot\vec{\pi}}{f_\pi}N \ . \tag{5.37}$$

From the leading order four-nucleon Lagrangian we will get only two contact interaction terms which reads

$$\begin{aligned}\mathcal{L}_{4N}^{(TV)} \ = \ & -8B\overline{C}_5 p_0 \overline{N}N\partial_\mu\big(\overline{N}\gamma^\mu\gamma^5 N\big)/f_\pi \\ & -8B\overline{C}_6 p_0 \overline{N}\vec{\tau}N\partial_\mu\big(\overline{N}\vec{\tau}\gamma^\mu\gamma^5 N\big)/f_\pi \ .\end{aligned} \tag{5.38}$$

A more formal derivation of these expression can be found in [5].

The final P and T violating Lagrangian at the lowest order can be written in general as

$$\begin{aligned}\mathcal{L}^{(TV)} &= -\frac{1}{f_\pi}\overline{N}(g_0^\theta\,\vec{\tau}\cdot\vec{\pi}+g_1^\theta\pi_3)N-\frac{\Delta^\theta}{f_\pi}\pi_3\pi^2 \\ &\quad +C_1^\theta\overline{N}N\partial_\mu\big(\overline{N}\gamma^\mu\gamma^5N\big)+C_2^\theta\overline{N}\vec{\tau}N\partial_\mu\big(\overline{N}\vec{\tau}\gamma^\mu\gamma^5N\big)+\ldots\ . \end{aligned} \tag{5.39}$$

where with the dots we indicates terms with higher power of pion field that are of no interest here.

The parameters g_0^θ, g_1^θ, Δ^θ, C_1^θ, C_2^θ are LECs, which from our derivation can be written in terms of the parameter $\overline{\theta}$. First of all, from Eq. (5.36), we identify

$$\frac{\Delta^\theta}{Mf_\pi}=-\frac{4Bl_7s_3p_0}{Mf_\pi^3}, \tag{5.40}$$

where the LEC l_7 is related to the square of the strong mass difference between the charged and the neutral pions [41],

$$(\delta m_\pi^2)^{str}=(m_{\pi^+}^2-m_{\pi^0}^2)^{str}\simeq\frac{2B^2}{f_\pi^2}l_7(m_u-m_d)^2. \tag{5.41}$$

The relation [41]

$$(\delta m_\pi^2)^{str}=\frac{\epsilon^2}{4}\frac{m_\pi^4}{m_K^2-m_\pi^2} \tag{5.42}$$

with the averaged kaon mass $m_K = 494.98$ MeV [42]. Substituting the terms in Eq. (5.40) and making explicit p_0 as defined in Eq. (5.17) we obtain

$$\frac{\Delta^\theta}{Mf_\pi}=-\frac{\epsilon(1-\epsilon^2)}{16f_\pi M}\frac{m_\pi^4}{m_K^2-m_\pi^2}\overline{\theta}=(0.37\pm0.09)\cdot10^{-3}\overline{\theta}, \tag{5.43}$$

where the prediction for the quark-mass ratio, $m_u/m_d = 0.46 \pm 0.03$, has been used here to compute ϵ [43]. Similarly, from Eq. (5.37) it is possible to estimate

$$\frac{g_1^\theta}{f_\pi}=-\frac{32B^2c_1l_7s_3p_0}{f_\pi^3}=\frac{2c_1(\delta m_\pi^2)^{str}(1-\epsilon^2)}{f_\pi\epsilon}\overline{\theta}=(0.0034\pm0.0011)\overline{\theta}, \tag{5.44}$$

where we use $c_1 = (-1.0 \pm 0.3)\,\mathrm{GeV}^{-1}$ as derived from the NN scattering data [44]. The LEC c_5 is related to the proton-neutron mass difference [45],[46]

$$\delta M_{np}^{str}=(M_n-M_p)^{str}=4B(m_u-m_d)c_5=(2.44\pm0.18)\mathrm{MeV}. \tag{5.45}$$

Using this in Eq. (5.37), it is possible to estimate g_0^θ/f_π as

$$\frac{g_0^\theta}{f_\pi}=-\frac{4Bc_5p_0}{f_\pi}=\frac{\delta M_{np}^{str}(1-\epsilon^2)}{4f_\pi\epsilon}\overline{\theta}=(0.0155\pm0.0019)\overline{\theta}. \tag{5.46}$$

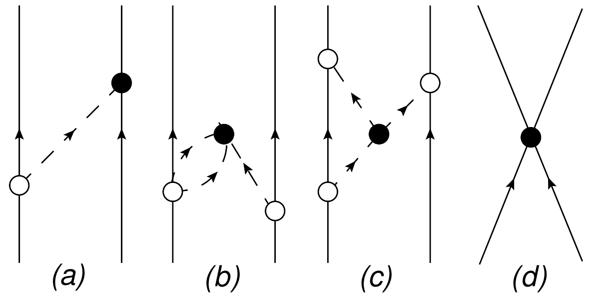


Figure 5.1: Diagrams that give contribution to the TV T matrix up to NLO. Nucleons and pions are denoted by solid and dashed lines, respectively. The open (solid) circles represent PC (TV) vertices.

The estimates of the contact LECs $C^{\theta}_{1,2}$, related to the $\overline{\theta}$ angle via Eq. (5.38), is more complicate [47], so here we report only the results,

$$C^{\theta}_{1,2} \simeq (2 \cdot 10^{-3})\overline{\theta}\,\mathrm{fm}^3 \,. \tag{5.47}$$

A more complete derivation of these results can be found in Ref. [5].

5.4 The TV potential

From $\mathcal{L}^{(TV)}$ given in Eq. (5.39), we can define the Hamiltonian and following the procedure described in Chapter 3, we can now obtain the potential. Using the time-ordered diagrams we will derive it up to the NLO. We will also add the contact terms that nominally contribute to N2LO. The diagrams that give contributions are shown in Fig. 5.1. As in the PV case the LO is given by the OPE diagrams which give a contribution to order Q^{-1}. At NLO, in the TV T-matrix a new class of diagrams appears with a three pion exchange vertex, coming from the Δ^{θ}-terms in the Lagrangian, see Eq. (5.39). These diagrams will contribute to order Q^0. In the following we will explain the details of the calculation. We will use the same notation as in Chapter 4.

5.4.1 One pion exchange

The time ordered diagrams that contribute are shown in Fig. 5.2. From these diagrams we derive exactly the same formulas given in Eqs. (4.5) and (4.6) but with the PV vertex function replaced by the TV vertex function. Using the

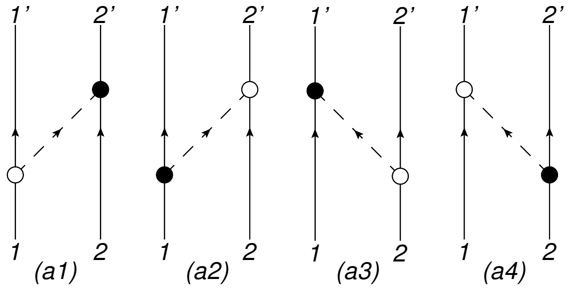


Figure 5.2: Diagrams that contribute to OPE term of the T-matrix. Notation as in Fig. 5.1.

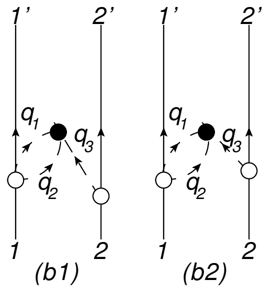


Figure 5.3: Example of three pions exchange with two pions emission vertex. Notation as in Fig. 5.1.

expression of the vertices given in Eqs. (C.33)-(C.34) we obtain

$$\begin{aligned} V^{(-1)}(\mathrm{NR}-a) &= -\frac{g_A g_0^\theta}{2f_\pi^2}(\vec{\tau}_1\cdot\vec{\tau}_2)\frac{i(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}}{\omega_k^2} \\ &\quad -\frac{g_A g_1^\theta}{4f_\pi^2}\left[(\tau_{1z}+\tau_{2z})\frac{i(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}}{\omega_k^2}+(\tau_{1z}-\tau_{2z})\frac{i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}}{\omega_k^2}\right] \end{aligned} \tag{5.48}$$

5.4.2 Three pions exchange

We have two possibilities to build a diagram with a three pion vertex at order Q^0. The first one is to consider a one pion emission/absorption vertex on a nucleon and a two pion emission/absorption vertex on the other nucleon as in Fig. 5.3.

The expression we derive from these diagrams is

$$\begin{aligned} T(b1+b2) &= -\frac{6}{\Omega}\sum_{\boldsymbol{q}_1 j_1,\boldsymbol{q}_2 j_2,\boldsymbol{q}_3 j_3} -\frac{1}{(\omega_{q_1}+\omega_{q_2})\omega_{q_3}}\Big[{}^{TV}M^{3\pi,03}_{\boldsymbol{q}_1 j_1,\boldsymbol{q}_2 j_2,\boldsymbol{q}_3 j_3}\,{}^{PC}M^{\pi NN,10}_{\alpha_2'\alpha_2,\boldsymbol{q}_3 j_3} \\ &\quad \times {}^{PC}M^{\pi\pi NN,20}_{\alpha_1'\alpha_1,\boldsymbol{q}_2 j_2,\boldsymbol{q}_1 j_1}\Big]\delta_{\boldsymbol{q}_3,\boldsymbol{k}}\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{k}}\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'}\ , \end{aligned} \tag{5.49}$$

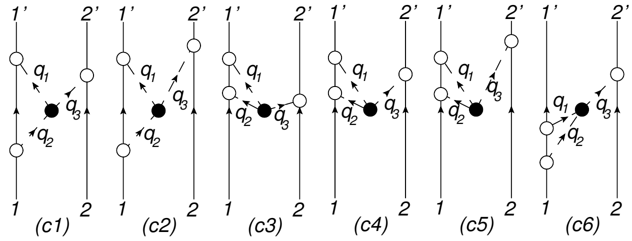


Figure 5.4: Three pions exchange diagrams with only one pion emission/absorption vertex. Notation as in Fig. 5.1.

where the factor 6 comes from the different possible contractions of the creation/annihilation operators. Making explicit the vertex functions, the isospin operators automatically cancel out and so these diagrams do not give any contribution.

The second type of diagrams is given in Fig. 5.4. To these we have to add the diagrams where the pion of momentum $\boldsymbol{q}_3$ is emitted by the second nucleon and the diagrams where we exchange $\alpha_1 \rightleftharpoons \alpha_2$ and $\alpha_{1'} \rightleftharpoons \alpha_{2'}$. In total we have 24 time-ordered diagrams. Summing up all the contributions the final expression is

$$V^{(0)}(c) = -\frac{25}{16}\frac{g_A^3\Delta^\theta}{f_\pi^4}\frac{i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_2\tau_{2z}-\boldsymbol{\sigma}_1\tau_{1z})}{\omega_k^2}\int\frac{d^3q}{(2\pi)^3}\frac{k^2-q^2}{\omega_+^2\omega_-^2} \ . \tag{5.50}$$

As we see the integral diverges linearly and so applying the same prescription used for the regularization of the linear divergences in Section 4.4.2 we obtain

$$\begin{aligned} V^{(0)}(c) &= \frac{25}{32}\frac{g_A^3\Delta^\theta}{f_\pi^4}\Big[(\tau_{1z}+\tau_{2z})\frac{i(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}}{\omega_k^2}+(\tau_{1z}-\tau_{2z})\frac{i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}}{\omega_k^2}\Big] \\ &\times\Big[\big(2-\frac{4m_\pi^2}{s^2}\big)\frac{s^2A(k)}{4\pi}-\frac{\Lambda}{4\pi^2}+\frac{m_\pi}{2\pi}+\mathcal{O}\big(\frac{k^2}{\Lambda}\big)\Big] \ , \end{aligned} \tag{5.51}$$

where $A(k)$ is defined in Eq. (4.74) and Λ is the cut-off. The divergences and the term proportional to m_π can be reabsorbed in g_1^θ, and we neglect the terms $\mathcal{O}\big(\frac{k^2}{\Lambda}\big)$ for the reasons discussed in Section 4.4.2. Therefore the final result is

$$\begin{aligned} V^{(0)}(c) &= \frac{25g_A^3\Delta^\theta}{2f_\pi^2}\frac{\pi}{\Lambda_\chi^2}\Big[(\tau_{1z}+\tau_{2z})\frac{i(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}}{\omega_k^2}+(\tau_{1z}-\tau_{2z})\frac{i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}}{\omega_k^2}\Big] \\ &\times\Big(1-\frac{2m_\pi^2}{s^2}\Big)s^2A(k) \ . \end{aligned} \tag{5.52}$$

5.4.3 Contact terms

From the diagram (d) in Fig. 5.1 we obtain

$$(d) = \frac{1}{\Omega}2\,{}^{TV}M^{00}_{\alpha_1'\alpha_1\alpha_2'\alpha_2}\delta_{\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_1'+\boldsymbol{p}_2'} \ , \tag{5.53}$$

whence the contact (CT) potential reads,

$$V^{(1)}(d) = \frac{\overline{C_1}^{\theta}}{\Lambda_\chi^2 f_\pi} i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) + \frac{\overline{C_2}^{\theta}}{\Lambda_\chi^2 f_\pi} i\boldsymbol{k}\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)(\vec{\tau}_1\cdot\vec{\tau}_2) \tag{5.54}$$

where we have defined the constants $\overline{C_1}^{\theta}$ and $\overline{C_2}^{\theta}$ in such a way they are adimensional

$$\overline{C_1}^{\theta} = \Lambda_\chi^2 f_\pi C_1^\theta\ , \quad \overline{C_2}^{\theta} = \Lambda_\chi^2 f_\pi C_2^\theta\ . \tag{5.55}$$

5.4.4 The TV potential in r-space

Using the same procedure explained in Section 4.6 the potential in coordinate space reads

$$\begin{aligned} V^{(\mathrm{OPE})}(\boldsymbol{r}) &= -\frac{g_A\overline{g}_0^{\theta}}{2f_\pi}(\vec{\tau}_1\cdot\vec{\tau}_2)(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}\,g_1'(r) \\ &\quad -\frac{g_A\overline{g}_1^{\theta}}{2f_\pi}\Big[\frac{(\tau_{1z}+\tau_{2z})}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}} + \frac{(\tau_{1z}-\tau_{2z})}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}\Big]g_1'(r) \end{aligned} \tag{5.56}$$

$$\begin{aligned} V^{(3\pi)}(\boldsymbol{r}) &= \frac{25 g_A^3 M\overline{\Delta}^{\theta}}{2f_\pi}\frac{m_\pi\,\pi}{\Lambda_\chi^2}\Big[(\tau_{1z}+\tau_{2z})(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}} + (\tau_{1z}-\tau_{2z})(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}\Big] \\ &\quad \times\big(\tilde{A}_1'(r) - 2\tilde{A}_2'(r)\big) \end{aligned} \tag{5.57}$$

$$V^{(CT)}(\boldsymbol{r}) = \frac{m_\pi^2}{\Lambda_\chi^2 f_\pi}\big[\overline{C_1}^{\theta}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}} + \overline{C_2}^{\theta}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}(\vec{\tau}_1\cdot\vec{\tau}_2)\big]Z'(r)\ , \tag{5.58}$$

where

$$g_1(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{C_{\Lambda_F}(k)}{k^2+m_\pi^2}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ , \tag{5.59}$$

$$Z(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{C_{\Lambda_F}(k)}{m_\pi^2}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ , \tag{5.60}$$

$$\tilde{A}_1(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{C_{\Lambda_F}(k)}{m_\pi(k^2+m_\pi^2)}(k^2+4m_\pi^2)A(k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ , \tag{5.61}$$

$$\tilde{A}_2(r) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{C_{\Lambda_F}(k)}{(k^2+m_\pi^2)}m_\pi\,A(k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ , \tag{5.62}$$

(5.63)

and we have defined

$$\overline{g}_0^{\theta} = \frac{g_0^\theta}{f_\pi},\quad \overline{g}_1^{\theta} = \frac{g_1^\theta}{f_\pi},\quad \overline{\Delta}^{\theta} = \frac{\Delta^\theta}{Mf_\pi}, \tag{5.64}$$

in order to have adimensional constants. In total we have 5 LECs to be extracted from the comparison with experimental data. As we have seen the LECs are directly related to the angle $\overline{\theta}$, thus from an experimental measurement of them it would be possible to have an estimate of the $\overline{\theta}$ angle via the estimate of the LECs given in Section 5.3.

Chapter 6

PV and TV observables in two nucleon systems

In this Chapter we will study some PV and TV observables in the two nucleon systems (in particular, in this Thesis we will focus our attention to PV and TV observables in two nucleon scattering only). First of all, we will present the general two-body scattering problem (Section 6.1) and discuss a method of solution based on the Kohn variational Principle (Section 6.2). In Section 6.3 we will introduce the M matrix formalism in order to compute the cross-section and the scattering observables. In the last Section we will discuss the observables we are interested in: the A_z in the $\vec{p}p$ scattering and the neutron spin rotation along the z-axis in the $\vec{n}p$ scattering in order to reveal PV effects and the neutron spin rotation along the y-axis in the $\vec{n}\vec{p}$ scattering to reveal TV effects. In order to perform the calculation, in the same Section we relate these observables to the M-matrix.

6.1 Scattering wave functions

In order to find the wave function for the two body system, we have to solve the Schrödinger equation[1]:

$$\Big(-\frac{\nabla_1^2}{2M_1} - \frac{\nabla_2^2}{2M_2} + V(\mathbf{r_1} - \mathbf{r_2})\Big)\psi_{NN}(\mathbf{r_1},\mathbf{r_2}) = E\psi_{NN}(\mathbf{r_1},\mathbf{r_2}). \tag{6.1}$$

Using the relative coordinate and the coordinate of the center of mass (CM) ,

$$\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}, \quad \mathbf{R} = \frac{M_1\mathbf{r_1} + M_2\mathbf{r_2}}{M_1 + M_2}, \tag{6.2}$$

[1]In this section we present the formalism in case of NN potentials of local form. The extension to non local potentials is easily obtained, however for the sake of simplicity we have not reported it here.

where we have considered $M_1 \approx M_2 \approx M = 938$ MeV, we can separate the CM motion from the relative motion. So we can write the wave function as:

$$\Psi_{NN}(\mathbf{R}, \mathbf{r}) = \frac{e^{i\mathbf{P}_{cm}\cdot\mathbf{R}}}{\sqrt{\Omega}}\psi(\mathbf{r}), \tag{6.3}$$

and the equation for the relative motion becomes,

$$\Big(-\frac{\nabla^2}{2\mu} + V(\mathbf{r})\Big)\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{6.4}$$

where μ is the reduced mass of the system and $\mu \approx M/2$. In our case, the potential $V(\mathbf{r})$ is composed by two terms,

$$V = V_{PC} + V_Z, \tag{6.5}$$

where V_{PC} is the parity conserving nuclear potential and V_Z is either the PV or TV potential. We can write the wave function as

$$\psi(\mathbf{r}) = \sum_{JJ_z}\sum_{LS} f_{LS}(r) i^L \Big[Y_L(\hat{r})\chi_S\Big]_{J,J_z} \xi_{T,T_z} \ , \tag{6.6}$$

where χ_{S,S_z} (ξ_{T,T_z}) is the spin (isospin) state of the two nucleons. In the previous formula $\Big[Y_L(\hat{r})\chi_S\Big]_{J,J_z}$ is the state with a total angular momentum J (and projection on the z-axis J_z) built with the spherical harmonics $Y_{LM}(\hat{r})$ and the spin states χ_{S,S_z}

$$\Big[Y_L(\hat{r})\chi_S\Big]_{JJ_z} = \sum_{M,S_z} \langle L, M, S, S_z | J, J_z\rangle Y_{LM}(\hat{r})\chi_{SS_z} \ , \tag{6.7}$$

and $f_{LS}(r)$ is a radial function.

For given total angular momentum J, the sum over L and S runs over all their possible combinations permitted by Pauli's exclusion principle. Pauli's principle requires that the wave function of a fermionic system must be completely antisymmetric, in our case we have $(-)^{L+S+T} = -1$. Since T is 0 or 1, for given L and S, T is fixed. For the np system $T_z = 0$ and so T can assume both values. In this case all the values of L, S compatible with a given J are possible. On the other hand, for the pp system, $T_z = +1$ and so only the $T = 1$ isospin state is possible. The asymmetry condition $(-1)^{L+S+T} = -1$ simplifies in this case to $(-)^{L+S} = +1$. All the possible combinations allowed by the exclusion principle are summarized in Table 6.1.

We now focus on the scattering problem. We consider a collision between two nucleons of energy $E = k^2/2\mu$. The wave function of our system can be written as a sum on a "inner" part ϕ (when the particles are close) and an "asymptotic"

case np ($T_z=0$)				
J	$S=0$	$S=1$	$^{2S+1}L_J$ parity +	$^{2S+1}L_J$ parity −
0	$L=0$	$L=1$	$^1S_0$	$^3P_0$
1	$L=1$	$L=0,1,2$	$^3S_1, ^3D_1$	$^1P_1, ^3P_1$
2	$L=2$	$L=1,2,3$	$^1D_2, ^3D_2$	$^3P_2, ^3F_2$
3	$L=3$	$L=2,3,4$	$^3D_3, ^3G_3$	$^1F_3, ^3F_3$
4	$L=4$	$L=3,4,5$	$^1G_4, ^3G_4$	$^3F_4, ^3H_4$
case pp ($T_z=1$)				
J	$S=0$	$S=1$	$^{2S+1}L_J$ parity +	$^{2S+1}L_J$ parity −
0	$L=0$	$L=1$	$^1S_0$	$^3P_0$
1	−	$L=1$	−	$^3P_1$
2	$L=2$	$L=1,3$	$^1D_2$	$^3P_2, ^3F_2$
3	−	$L=3$	−	$^3F_3$
4	$L=4$	$L=3,5$	$^1G_4$	$^3F_4, ^3H_4$

Table 6.1: Values of the orbital angular momentum L and spin S for a two nucleon system for a given state of angular momentum J (the spin S refers to the nucleon pair). In the last two columns the possible combinations of L,S and J are given in spectroscopic notation, separated in odd and even parity states. In the pp system, the Pauli's principle imposes $(-1)^{L+S}=1$ and this reduces the possible states.

part. The latter part is the solution of the free Schrödinger equation in the region where $r \gg r_0$ (where r_0 is the typical nuclear interaction radius). In the pp case we must consider also the Coulomb interaction in the asymptotic region. The component ϕ goes to zero for $r \gg r_0$ by definition.

For the np scattering, the asymptotic function can be written in terms of the quantities Ω^F_{LS} and Ω^G_{LS}, which are defined as

$$\Omega^F_{LS} = Ci^L \left[Y_L(\hat{r})\chi_S\right]_{JJ_Z} \xi_{TT_z} j_L(kr) \ , \tag{6.8}$$

$$\Omega^G_{LS} = Ci^L \left[Y_L(\hat{r})\chi_S\right]_{JJ_Z} \xi_{TT_z} n_L(kr) \left(1-e^{-\beta r}\right)^{2L+1} \ , \tag{6.9}$$

where j_L is the regular Riccati-Bessel (spherical Bessel) function, while n_L is the irregular one. C is a constant that will be determined later. In the pp case, the two particles have a long distance interaction due to the Coulomb potential. In this case the asymptotic functions are defined as

$$\Omega^F_{LS} = Ci^L \left[Y_L(\hat{r})\chi_S\right]_{JJ_Z} \xi_{TT_z} \frac{F_L(\eta, kr)}{kr} \ , \tag{6.10}$$

$$\Omega^G_{LS} = Ci^L \left[Y_L(\hat{r})\chi_S\right]_{JJ_Z} \xi_{TT_z} \frac{G_L(\eta, kr)}{kr} \left(1-e^{-\beta r}\right)^{2L+1} \ , \tag{6.11}$$

where F_L (G_L) is the Coulomb regular (irregular) function. The η parameter is defined as

$$\eta = \frac{\mu e^2}{k} \ , \tag{6.12}$$

where e is the unit of the electric charge. The factor $\left(1 - e^{-\beta r}\right)^{2L+1}$ was introduced to avoid the singularity of $n_L(kr)$ o $G_L(\eta, kr)$ for $r \to 0$. The Bessel functions have the following asymptotic behaviour

$$x j_L(x) \to \sin\left(x - \frac{L\pi}{2}\right) \ , \qquad -x n_L(x) \to \cos\left(x - \frac{L\pi}{2}\right) \ , \tag{6.13}$$

while the Coulomb functions have

$$F_L(\eta, x) \ \to \ \sin\left(x - \frac{L\pi}{2} - \eta \log(2x) + \sigma_L\right) \ , \tag{6.14}$$

$$G_L(\eta, x) \ \to \ \cos\left(x - \frac{L\pi}{2} - \eta \log(2x) + \sigma_L\right) \ , \tag{6.15}$$

$$\tag{6.16}$$

and σ_L is the Coulomb phase shift defined as $\sigma_L = \arg\Big[\Gamma(L+1+i\eta)\Big]$, or better

$$\sigma_0 = -\eta\gamma \sum_{n=0}^{\infty} \left[\frac{\eta}{n+1} - \arctan\Big(\frac{\eta}{n+1}\Big)\right] \ , \quad \sigma_{L+1} = \sigma_L + \arctan\Big(\frac{\eta}{L+1}\Big) \ , \tag{6.17}$$

where $\gamma = 0.57721566\ldots$. The F and G functions are the solutions of the Schrödinger radial equation with the Coulomb potential only. The C constant was determined in order to satisfy the relation

$$\langle \Omega^F_{LS} | H - E | \Omega^G_{LS} \rangle - \langle \Omega^G_{LS} | H - E | \Omega^F_{LS} \rangle = 1 \ , \tag{6.18}$$

that gives $|C|^2 = kM$. This identity mainly comes from the Wronskian property of the Bessel and Coulomb functions.

We consider now the specific case of two nucleons (with T_z fixed) that collide in a state of total angular momentum J, orbital angular momentum L and total spin S (in the following the quantum numbers J and T_z, which are fixed in the collision, will be understood). The *exact* wave function will have the following form

$$\psi_{LS} = \phi_{LS} + \Omega^F_{LS} + \sum_{L'S'} R_{LS,L'S'} \Omega^G_{L'S'} \ . \tag{6.19}$$

The coefficients $R_{LS,L'S'}$, which form the R-matrix, are the relative weights between the regular and irregular components and they are the goals of our calculations. From these coefficients we can easily obtain the S matrix (which represents the relative weight between the incoming wave $\Omega^G_{LS} - i\Omega^F_{LS}$ and the outcoming wave $\Omega^G_{LS} + i\Omega^F_{LS}$) or the T-matrix. Alternatively, in fact, we can impose the following boundary conditions

$$\begin{aligned}
\psi^U_{LS} &\rightarrow \Omega^G_{LS} + \sum_{L'S'} U_{LS,L'S'}\Omega^F_{L'S'} \ , && (6.20)\\
\psi^S_{LS} &\rightarrow -\Big(\Omega^G_{LS} - i\Omega^F_{LS}\Big) + \sum_{L'S'} S_{LS,L'S'}\Big(\Omega^G_{L'S'} + i\Omega^F_{L'S'}\Big) \ , && (6.21)\\
\psi^T_{LS} &\rightarrow \Omega^F_{LS} + \sum_{L'S'} T_{LS,L'S'}\Big(\Omega^G_{L'S'} + i\Omega^F_{L'S'}\Big) \ . && (6.22)
\end{aligned}$$

These functions are linear combination of each other and it is possible to verify that

$$U = R^{-1} \ , \qquad S = (I + iR)(I - iR)^{-1} \ , \qquad T = (I - iR)^{-1} R \ . \qquad (6.23)$$

6.2 Construction of the wave function

In our calculation we will try to obtain the best approximation to the *exact* wave function in Eq. (6.19) using a "test" wave function

$$\overline{\psi}_{LS} = \overline{\phi}_{LS} + \Omega^F_{LS} + \sum_{L'S'} \overline{R}_{LS,L'S'}\Omega^G_{L'S'} \ . \qquad (6.24)$$

where the "inner" part $\overline{\phi}_{LS}$ and the coefficients $\overline{R}_{LS,L'S'}$ are to be calculated. From now on, we indicate with L_0S_0 the orbital angular momentum and spin of the nucleon pair in the incoming state. The "inner" part of the "test" wave function can be written as

$$\overline{\phi}_{L_0S_0} = \sum_{LS} f^{(L_0S_0)}_{LS}(r)\, i^L \left[Y_L(\hat{r})\,\chi_S\right]_{JJ_z} \xi_{TT_z} \ , \qquad (6.25)$$

where the $f^{(L_0S_0)}_{LS}(r)$ is here evaluated using an expansion over a complete set of functions, more precisely the set of order 2 Laguerre polynomials $L^{(2)}_n$ multiplied by an exponential,

$$f^{(L_0S_0)}_{LS}(r) = \sum_{n=0}^{N_L-1} a^{(L_0S_0)}_{LS,n} N_n L^{(2)}_n(\gamma r)\, e^{-\gamma r/2} \ , \qquad (6.26)$$

where the coefficients $a^{(L_0S_0)}_{LS,n}$ have to be determined and N_L is the number of the expansion terms. Increasing N_L the accuracy of the wave function will increase, but for obvious motivations we have to truncate the expansion. The parameter

γ is used to optimize the expansion, if N_L is large enough the results become almost indipendent of it. N_n is a normalization factor, chosen to be

$$N_n = \sqrt{\frac{n!}{(n+2)!}\gamma^{\frac{3}{2}}} \ , \tag{6.27}$$

so,

$$\int_0^\infty dr\, r^2 \left[N_n L_n^{(2)}(\gamma r) \exp\Big(-\frac{\gamma r}{2}\Big)\right]^2 = 1 \ , \tag{6.28}$$

following from the Laguerre polynomials' properties. The factor $e^{-\gamma r/2}$ guarantees that the "inner" wave function goes to zero for $r \to \infty$. We can rewrite Eq. (6.25) as an expansion over the states

$$|\phi_{LS,n}\rangle = N_n L_n^{(2)}(\gamma r) i^L \left[Y_L(\hat{r})\chi_S\right]_{JJ_z} \xi_{TT_z} \ , \tag{6.29}$$

so,

$$\overline{\phi}_{L_0S_0} = \sum_{LS} \sum_{n=0}^{N_L-1} a_{LS,n}^{(L_0S_0)} |\psi_{LS,n}\rangle \ . \tag{6.30}$$

At the end the "test" function is

$$\overline{\psi}_{L_0S_0} = \sum_{LS,n} a_{LS,n}^{(L_0S_0)} |\psi_{LSn}\rangle + \Omega^F_{L_0S_0} + \sum_{LS} \overline{R}_{L_0S_0,LS}\Omega^G_{LS} \ . \tag{6.31}$$

Now we briefly discuss the Kohn variational Principle. Let's consider the following quantities:

$$I_{LS,LS} = \langle \psi_{LS}|H-E|\overline{\psi}_{LS}\rangle - \langle \overline{\psi}_{LS}|H-E|\psi_{LS}\rangle \ , \tag{6.32}$$

where ψ_{LS} is the exact wave function given in Eq. (6.19) and $\overline{\psi}_{LS}$ the trial wave function. Replacing in ψ_{LS} and $\overline{\psi}_{LS}$ the expressions (6.19) and (6.24) and using the identity (6.18) we obtain:

$$I_{LS,LS} = \overline{R}_{LS,LS} - R_{LS,LS} \ . \tag{6.33}$$

The exact wave function is the solution of the equation $(H-E)|\psi_{LS}\rangle = 0$, thus we obtain the exact relation

$$R_{LS,LS} = \overline{R}_{LS,LS} - \langle \psi_{LS}|H-E|\overline{\psi}_{LS}\rangle. \tag{6.34}$$

Defining $\delta\psi_{LS} = \psi_{LS} - \overline{\psi}_{LS}$ the "error" wave function, $R_{LS,LS}$ takes the form

$$R_{LS,LS} = \overline{R}_{LS,LS} - \langle \overline{\psi}_{LS}|H-E|\overline{\psi}_{LS}\rangle + \langle \delta\psi_{LS}|H-E|\delta\psi_{LS}\rangle \ . \tag{6.35}$$

We can now build the functional $[R_{LS,LS}]$:

$$[R_{LS,LS}] = \overline{R}_{LS,LS} - \langle \overline{\psi}_{LS} | H - E | \overline{\psi}_{LS} \rangle \ . \tag{6.36}$$

This functional differs from the exact value of $R_{LS,LS}$ by a term which is quadratic in the "error" wave function and so it is stationary for small variation of $\overline{\psi}_{LS}$ around the exact value $\overline{\psi}_{LS} = \psi_{LS}$ (this is called Kohn variational Principle).

Asking that the functional $R_{L_0S_0,L_0S_0}$ is stationary for a variation of the coefficients $a^{(L_0S_0)}_{LS,n}$ and $\overline{R}_{L_0S_0,LS}$, one obtains

$$0 = \frac{\delta\,[R_{L_0S_0,L_0S_0}]}{\delta a^{(L_0S_0)}_{LS,n}} \ , \qquad 0 = \frac{\delta\,[R_{L_0S_0,L_0S_0}]}{\delta \overline{R}_{L_0S_0,LS}} \ . \tag{6.37}$$

From the first condition we obtain

$$0 = \sum_{L'S',n'} \widetilde{H}_{LSn,L'S'n'}\; a_{L'S',n'} + T^F_{LSn,L_0S_0} + \sum_{L'S'} \overline{R}_{L_0S_0,L'S'} T^G_{LSn,L'S'} \ , \tag{6.38}$$

where

$$\begin{cases} \widetilde{H}_{LSn,L'S'n'} = \langle \psi_{LS,n} | H - E | \psi_{L'S'n'} \rangle \\ T^X_{LSn,L'S'} = \langle \psi_{LS,n} | H - E | \Omega^X_{L'S'} \rangle \end{cases} \tag{6.39}$$

The second condition gives

$$\begin{aligned} 0 \ &= \ \delta_{L_0S_0,LS} - \Bigg[\sum_{L'S'n'} T^G_{L'S'n',LS} a^{(L_0S_0)}_{L'S',n'} + T^{GF}_{LS,L_0S_0} + \sum_{L'S'} \overline{R}_{L_0S_0,L'S'} T^{GG}_{LS,L'S'} \\ &\quad + \sum_{L'S'n'} T^G_{L'S'n',LS} a^{(L_0S_0)}_{L'S'n'} + T^{FG}_{L_0S_0,LS} + \sum_{L'S'} \overline{R}_{L_0S_0,L'S'} T^{GG}_{L'S',LS} \Bigg], \end{aligned} \tag{6.40}$$

where $T^{XY}_{LS,L'S'} = \langle \Omega^X_{LS} | H - E | \Omega^Y_{L'S'} \rangle$. We can now sum up all these equations in a system where $a^{(L_0S_0)}_{LS,n}$ and $\overline{R}_{L_0S_0,L'S'}$ are unknown:

$$\begin{bmatrix} \widetilde{H}_{LSn,L'S'n'} & T^G_{LSn,L'S'} \\ T^G_{L'S'n',LS} & \frac{1}{2}(T^{GG}_{LS,L'S'} + T^{GG}_{L'S',LS}) \end{bmatrix} \begin{bmatrix} a^{(L_0S_0)}_{L'S',n'} \\ \overline{R}_{L_0S_0,L'S'} \end{bmatrix} = \begin{bmatrix} -T^F_{LSn,L_0S_0} \\ \frac{1}{2}(\delta_{L_0S_0,LS} - T^{GF}_{LS,L_0S_0} - T^{FG}_{L_0S_0,LS}) \end{bmatrix} , \tag{6.41}$$

We have to compute all the quantities $\widetilde{H}_{LSn,L'S'n'}$, $T^X_{LSn,L'S'}$ and $T^{XY}_{LS,L'S'}$ (where X, Y indicates F, G), for all possible combinations of L, S and n for a given J and T_z. We give now a summary of the explicit formulas for the elements of the system in Eq. (6.41).

The quantity $\widetilde{H}_{LSn,L'S'n'} = H_{LSn,L'S'n'} - E\delta_{LL'}\delta_{SS'}\delta_{nn'}$, where $H_{LSn,L'S'n'} = \langle\psi_{LS,n}|H|\psi_{L'S'n'}\rangle$. The Hamiltonian can be written as $H = K + V_{PC} + V_Z$, where Z stands for PV or TV term. The kinetic matrix element can be calculated analitically getting:

$$\begin{aligned}\langle\psi_{LS,n}|K|\psi_{L'S',n'}\rangle &= \frac{\gamma^2}{M}[(L\,(L+1)+n')\,I^2_{n,n'} + (n'+1)\,I^1_{n,n'} \\ &\quad -\frac{1}{4}\delta_{n,n'} - \sqrt{n'\,(n'+1)}I^2_{n,n'-1}],\end{aligned} \tag{6.42}$$

with $n \leq n'$ and where

$$\begin{aligned}I^1_{n,n'} &= \int_0^\infty drr^2\frac{L^{(2)}_n\,(\gamma r)\,L^{(2)}_{n'}\,(\gamma r)}{\gamma r}e^{-\gamma r}N_nN_{n'}\ , \\ I^2_{n,n'} &= \int_0^\infty drr^2\frac{L^{(2)}_n\,(\gamma r)\,L^{(2)}_{n'}\,(\gamma r)}{(\gamma r)^2}e^{-\gamma r}N_nN_{n'}\ .\end{aligned} \tag{6.43}$$

The potentials which we use have also terms with ∇ and ∇^2 at maximum, so the matrix element on the angular, spin and isospin variables is in general:

$$\begin{aligned}&\langle i^L[Y_L\chi_S]_{JJ_z}\xi_{TT_z}|V_Z|i^{L'}[Y_{L'}\chi_{S'}]_{JJ_z}\xi_{T'T_z}\rangle_{\theta,\phi} = \\ &\quad = \left[v^{Z0,JT_z}_{LS,L'S'}(r) + v^{Z1,JT_z}_{LS,L'S'}(r)\frac{d}{dr} + v^{Z2,JT_z}_{LS,L'S'}(r)\frac{d^2}{dr^2}\right]\ ,\end{aligned} \tag{6.44}$$

where Z stands here for the PC, PV, or TV part of the potential and $\langle\rangle_{\theta,\phi}$ points out that the matrix element is calculated performing the integration on $d\hat{r}$ and evaluating the spin-isospin traces. The final result for the potential matrix element is:

$$\begin{aligned}\langle\psi_{LS,n}|V_Z|\psi_{L'S',n'}\rangle &= \int_0^\infty dr\ r^2N_nL^{(2)}_n(\gamma r)e^{-\gamma r/2}\left[v^{Z0,JT_z}_{LS,L'S'}(r) + v^{Z1,JT_z}_{LS,L'S'}(r)\frac{d}{dr}\right. \\ &\quad \left.+v^{Z2,JT_z}_{LS,L'S'}(r)\frac{d^2}{dr^2}\right]N_{n'}L^{(2)}_{n'}(\gamma r)e^{-\gamma r/2}\ .\end{aligned} \tag{6.45}$$

The derivatives of the Laguerre polynomials are calculated using recurrence formulas. The integrals are numerically computed using the Gauss-Laguerre integration formula with N_p points.

The quantities T^X and T^{XY} are evaluated considering that the Bessel or Coulomb functions are solutions of the Schrödinger equation without the nuclear potential. And so:

$$
\begin{aligned}
T^F_{LSn,L'S'} &= \int_0^\infty dr\; r^2 N_n L_n^{(2)}(\gamma r) e^{-\gamma r/2} \Big[v^{PC,JT_z}_{LS,L'S'}(r) + v^{Z0,JT_z}_{LS,L'S'}(r) \\
&\quad + v^{Z1,JT_z}_{LS,L'S'}(r)\frac{d}{dr} + v^{Z2,JT_z}_{LS,L'S'}(r)\frac{d^2}{dr^2}\Big] C\frac{F_{L'}(\eta,kr)}{kr} \;, \qquad (6.46)\\
T^G_{LSn,L'S'} &= -\frac{1}{M}\int_0^\infty dr\; r^2 N_n L_n^{(2)}(\gamma r) e^{-\gamma r/2} C\frac{2G'_{L'}y'_\beta + G_{L'}y''_\beta}{kr} \\
&+ \int_0^\infty dr\; r^2 N_n L_n^{(2)}(\gamma r) e^{-\gamma r/2} \Big[v^{PC,JT_z}_{LS,L'S'}(r) + v^{Z0,JT_z}_{LS,L'S'}(r) \\
&\quad + v^{Z1,JT_z}_{LS,L'S'}(r)\frac{d}{dr} + v^{Z2,JT_z}_{LS,L'S'}(r)\frac{d^2}{dr^2}\Big] C\frac{G_{L'}(\eta,kr)}{kr} y_\beta(r) \;, \qquad (6.47)
\end{aligned}
$$

$$
\begin{aligned}
T^{XF}_{LS,L'S'} &= \int_0^\infty dr\; r^2 C\frac{X_L(\eta,kr)}{kr} \Big[v^{PC,JT_z}_{LS,L'S'}(r) + v^{Z0,JT_z}_{LS,L'S'}(r) \\
&\quad + v^{Z1,JT_z}_{LS,L'S'}(r)\frac{d}{dr} + v^{Z2,JT_z}_{LS,L'S'}(r)\frac{d^2}{dr^2}\Big] C\frac{F_{L'}(\eta,kr)}{kr} \;, \qquad (6.48)\\
T^{XG}_{LS,L'S'} &= -\frac{1}{M}\int_0^\infty dr\; r^2 C\frac{X_L(\eta,kr)}{kr} C\frac{2G'_{L'}y'_\beta + G_{L'}y''_\beta}{kr} \\
&+ \int_0^\infty dr\; r^2 C\frac{X_L(\eta,kr)}{kr} \Big[v^{PC,JT_z}_{LS,L'S'}(r) + v^{Z0,JT_z}_{LS,L'S'}(r) \\
&\quad + v^{Z1,JT_z}_{LS,L'S'}(r)\frac{d}{dr} + v^{Z2,JT_z}_{LS,L'S'}(r)\frac{d^2}{dr^2}\Big] C\frac{G_{L'}(\eta,kr)}{kr} y_\beta(r) \;, \qquad (6.49)
\end{aligned}
$$

where $y_\beta(r) = \left(1 - e^{-\beta r}\right)^{2L+1}$, $X = F$ or G, $Z = PC$, PV or TV and $f' \equiv df(r)/dr$, etc. Let's note that the integrands go to zero quickly for $r \to \infty$, so there are no difficulties in evaluating these integrals. As for the Hamiltonian matrix elements, T^X and T^{XY} are computed using the Gauss-Laguerre numerical integration using N_p grid points. After having obtained all the quantities entering Eq. (6.41), we can determine the coefficients $\overline{R}_{L_0S_0,LS}$ and the "test" functions. These functions are the "first order" approximation (indeed they differ from the exact $\psi_{L_0S_0}$ by a quantity $\delta\psi_{L_0S_0}$). If we use the first order functions to evaluate the quantities $[R_{L_0S_0L_0S_0}]$ in Eq. (6.36) we will obtain a new estimate of $R_{L_0S_0L_0S_0}$ which differs to terms that go like $\sim (\delta\psi_{L_0S_0})^2$ from the exact ones.

For the "non diagonal" R-matrix elements there is a similar functional

$$
\begin{aligned}
[R_{L_1S_1,L_2S_2}] &= \frac{1}{2}\Big(\overline{R}_{L_1S_1,L_2S_2} + \overline{R}_{L_2S_2,L_1S_1}\Big) \\
&- \frac{1}{2}\Big(\langle\overline{\psi}_{L_1S_1}|H-E|\overline{\psi}_{L_2S_2}\rangle + \langle\overline{\psi}_{L_2S_2}|H-E|\overline{\psi}_{L_1S_1}\rangle\Big) \;. \qquad (6.50)
\end{aligned}
$$

Also these quantities differ by a quadratic term from the exact coefficients. Therefore $[R_{L_1S_1L_2S_2}]$ are a "second order" approximations and usually they are more precise than $\overline{R}_{L_1S_1L_2S_2}$ and converge faster.

Let's note an important difference between the TV and PV potential matrix elements of the type

$$\langle\overline{\psi}_{L'S'}|V|\overline{\psi}_{LS}\rangle \ , \tag{6.51}$$

Both potentials are P-violating so they connect states with different parities, thus $L+L'$ should be odd. When we calculate the matrix elements we have to take care of the factor $(-i)^{L'}$ in the bra and $(i)^L$ in the ket. In the case of the PV potential all the terms of the potential have a vector product and this always produces an extra i factor (see Appendix E). This extra factor, taking into account that $L+L'$ is odd, says that the matrix element is real. On the other hand in the TV potential there is no vector products so the matrix element will be imaginary. The final result will be that the S-matrix will not be anymore symmetric.

6.3 The M matrix

Using Eq. (6.23) we can compute the $T_{LS,L'S'}$ matrix directly from the R matrix. In this section we reintroduce the explicit dependence on J and T_z, so $T_{LS,L'S'} \equiv T^{JT_z}_{LS,L'S'}$. The T matrix represents the transition probability between an initial "plane wave" with an orbital angular momentum L and a total spin S to a final outgoing spherical wave with an orbital angular momentum L' and a total spin S'. For the calculation of the observables we introduce the M matrix which represents the transition probability from two nucleons initial state of relative momentum $\boldsymbol{k}$ and given projections of the spin and isospin states, to a final state of relative momentum $\boldsymbol{k}'$ where the two particles have other projections of the spin and isospin. Usually the z direction is chosen parallel to the incoming momentum $\boldsymbol{k}$.

We have to distinguish the np and the pp cases. In the np case the particles are non interacting in the long range sector, so we can write the "unperturbed" wave function as

$$\Phi_{\boldsymbol{k},s_1,s_2,t_1,t_2} = \frac{1}{\sqrt{2}}\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\chi_{s_1}(1)\chi_{s_2}(2)\xi_{t_1}(1)\xi_{t_2}(2) - e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\chi_{s_2}(1)\chi_{s_1}(2)\xi_{t_2}(1)\xi_{t_1}(2)\right) \ , \tag{6.52}$$

where $\chi_s(i)$ ($\xi_t(i)$) is the spin (isospin) particle state i with projection s (t) along z. Expanding in partial waves and coupling spin and isospin we obtain

$$\begin{aligned}\Phi_{\boldsymbol{k},s_1,s_2,t_1,t_2} \quad &= \sum_{LMSS_zJJ_zTT_z} 4\pi(\frac{1}{2}s_1\frac{1}{2}s_2|SS_z)(LMSS_z|JJ_z)(\frac{1}{2}t_1\frac{1}{2}t_2|TT_z) \\ &\qquad \times\sqrt{2}\epsilon_{LSJ}Y^*_{LM}(\hat{k})\Omega^F_{LS} \ ,\end{aligned} \tag{6.53}$$

where Ω^F_{LS} is defined in Eq. (6.8) and $\epsilon_{LSJ} = (1-(-)^{L+S+T})/2$ in order to respect Pauli's principle.

If we "turn on" the interaction the solution of the Schrödinger equation is

$$\begin{aligned}\Psi_{\boldsymbol{k},s_1,s_2,t_1,t_2} &= \sum_{LMSS_zJJ_zTT_z} 4\pi(\frac{1}{2}s_1\frac{1}{2}s_2|SS_z)(LMSS_z|JJ_z)(\frac{1}{2}t_1\frac{1}{2}t_2|TT_z) \\ &\quad\times\epsilon_{LST}\sqrt{2}Y^*_{LM}(\hat{k})\psi^T_{LS}(\boldsymbol{r}) \ , \qquad (6.54)\end{aligned}$$

where $\psi^T_{LST}(\boldsymbol{r})$ are the solution defined in Eq. (6.22). We consider only the asymptotic part of the wave function because the experimental apparat are very far from the interacting point. So asymptotically the wave function becomes

$$\Psi_{\boldsymbol{k},s_1,s_2,t_1,t_2} \to \Phi_{\boldsymbol{k},s_1,s_2,t_1,t_2}+\sqrt{2}\sum_{s'_1s'_2t'_1t'_2} M_{s'_1s'_2t'_1t'_2;s_1,s_2,t_1,t_2}(\hat{r})\frac{e^{ikr}}{r}\chi_{s'_1}(1)\chi_{s'_2}(2)\xi_{t'_1}(1)\xi_{t'_2}(2) \ . \qquad (6.55)$$

Explicitly the M matrix for the np process reads

$$\begin{aligned}M_{s'_1s'_2t'_1t'_2;s_1,s_2,t_1,t_2}(\hat{r}) &= \sum_{LST,L'S'T'JJ_zT_z} (\frac{1}{2}s_1\frac{1}{2}s_2|SS_z)(L0SJ_z|JJ_z)(\frac{1}{2}t_1\frac{1}{2}t_2|TT_z) \\ &\quad(\frac{1}{2}s'_1\frac{1}{2}s'_2|S'S'_z)(L'M'S'S'_z|JJ_z)(\frac{1}{2}t'_1\frac{1}{2}t'_2|T'T_z) \\ &\quad\epsilon_{LST}\epsilon_{L'S'T'}\sqrt{4\pi}\hat{L}\frac{T^{JT_z}_{LS,L'S'}}{k}Y_{L'M'}(\hat{r}) \ , \qquad (6.56)\end{aligned}$$

where $Y_{LM}(\hat{k} \parallel \hat{z}) = \delta_{M0}\hat{L}/\sqrt{4\pi}$ and $\hat{L} = \sqrt{2L+1}$.

For the pp case the wave function in the $r \to \infty$ is not simply an incoming plane wave plus a outcoming wave, due to the Coulomb potential in the long range region. In fact the solution of the Schröedinger equation with the Coulomb potential for $r \to \infty$ is [48]

$$\psi_c(\boldsymbol{k},\boldsymbol{r}) \to e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\eta\ln(kr-\boldsymbol{k}\cdot\boldsymbol{r}))} + f_c(\theta)\frac{e^{i(kr+\eta\ln(2kr))}}{r} \ , \qquad (6.57)$$

where $f_c(\theta)$ is the scattering function for a collision of two charged point particles (without spin) and

$$f_c(\theta) = -\eta\frac{e^{2i\sigma_0-i\eta\ln(sin^2(\theta/2))}}{2k\sin^2(\theta/2)} \ , \qquad (6.58)$$

where θ is the scattering angle defined as $\cos\theta = \hat{k}\cdot\hat{r}$. We can write a formula similar to Eq. (6.52) considering the unperturbed wave function of the pp case, where the plane waves are substituted by $\psi_c(\boldsymbol{k},\boldsymbol{r})$

$$\begin{aligned}\Phi^C_{\boldsymbol{k},s_1,s_2} &= \frac{1}{\sqrt{2}}\Big(\psi_c(\boldsymbol{k},\boldsymbol{r})\chi_{s_1}(1)\chi_{s_2}(2)\xi_{+1/2}(1)\xi_{+1/2}(2) \\ &\quad-\psi_c(\boldsymbol{k},-\boldsymbol{r})\chi_{s_2}(1)\chi_{s_2}(2)\xi_{+1/2}(1)\xi_{+1/2}(2)\Big) \ . \qquad (6.59)\end{aligned}$$

Expanding in partial wave using

$$\psi_c(\boldsymbol{k},\boldsymbol{r}) = \sum_{LM} 4\pi i^L Y^*_{LM}(\hat{k}) Y_{LM}(\hat{r}) e^{i\sigma_L} F_L(\eta, kr) \ , \tag{6.60}$$

where $F_L(\eta, kr)$ is the regular Coulomb function, we obtain

$$\Phi^C_{\boldsymbol{k},s_1,s_2} = \sum_{LMSS_zJJ_z} 4\pi(\frac{1}{2}s_1\frac{1}{2}s_2|SS_z)(LMSS_z|JJ_z)\sqrt{2}\epsilon_{LS1}Y^*_{LM}(\hat{k})e^{i\sigma_L}\Omega^F_{LS} \ , \tag{6.61}$$

and here Ω^F_{LS} is given by (6.10). When we add the nuclear potential the wave function becomes

$$\Psi^C_{\boldsymbol{k},s_1,s_2} = \sum_{LMSS_zJJ_z} 4\pi(\frac{1}{2}s_1\frac{1}{2}s_2|SS_z)(LMSS_z|JJ_z)\sqrt{2}\epsilon_{LS1}Y^*_{LM}(\hat{k})e^{i\sigma_L}\psi^T_{LS} \ , \tag{6.62}$$

where $\psi^T_{LS}(\boldsymbol{r})$ has the behaviour defined by Eq. (6.22), but with the functions $\Omega^{F,G}_{LS}$ given in Eq. (6.10) and (6.11). Asymptotically the term Ω^F_{LS} in $\psi^T_{LS}(\boldsymbol{r})$ reconstructs $\Phi^C_{\boldsymbol{k},s_1,s_2}$. We want isolate the scattering wave function and therefore we have to subtract the unperturbed one, which is not simply the wave function $\Phi^C_{\boldsymbol{k},s_1,s_2}$ because it also includes the scattering amplitude f_C due to the Coulomb repulsion. We define the unperturbed part as the function that asymptotically goes like

$$\begin{aligned}\overline{\Phi}^C_{\boldsymbol{k},s_1,s_2} &= \frac{1}{\sqrt{2}}\Big[e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\eta\ln(kr-\boldsymbol{k}\cdot\boldsymbol{r}))}\chi_{s_1}(1)\chi_{s_2}(2)\xi_{+1/2}(1)\xi_{+1/2}(2)\\ &\quad -e^{i(-\boldsymbol{k}\cdot\boldsymbol{r}-\eta\ln(kr+\boldsymbol{k}\cdot\boldsymbol{r}))}\chi_{s_2}(1)\chi_{s_2}(2)\xi_{+1/2}(1)\xi_{+1/2}(2)\Big] \ ,\end{aligned} \tag{6.63}$$

as we can deduce from Eq. (6.57). So we can define

$$\Psi_{\boldsymbol{k},s_1,s_2} \to \overline{\Phi}^C_{\boldsymbol{k},s_1,s_2} + \sqrt{2}\sum_{s_1's_2'} M^C_{s_1's_2';s_1,s_2}(\hat{r})\frac{e^{ikr-\eta\ln(2kr)}}{r}\chi_{s_1'}(1)\chi_{s_2'}(2)\xi_{+1/2}(1)\xi_{+1/2}(2) \ , \tag{6.64}$$

with

$$\begin{aligned}M^C_{s_1's_2't_1't_2';s_1,s_2,t_1,t_2}(\hat{r}) &= f_c(\theta)\delta_{s_1s_1'}\delta_{s_2s_2'} - f_c(\pi-\theta)\delta_{s_1s_2'}\delta_{s_2s_1'}\\ &\quad + \sum_{LS,L'S'JJ_z}(\frac{1}{2}s_1\frac{1}{2}s_2|SS_z)(L0SJ_z|JJ_z)\\ &\quad (\frac{1}{2}s_1'\frac{1}{2}s_2'|S'S_z')(L'M'S'S_z'|JJ_z)\\ &\quad \epsilon_{LS1}\epsilon_{L'S'1}\sqrt{4\pi}\hat{L}e^{i\sigma_L}\frac{T^{JT_z}_{LS,L'S'}}{k}e^{i\sigma_{L'}}Y_{L'M'}(\hat{r}) \ .\end{aligned} \tag{6.65}$$

Now, let's suppose to have a detector at distance $r/2$ from the interaction point covering a solid angle $d\Omega$. The particle flow that hits the detector with the particle 1 (2) in the spin state s'_1 (s'_2) and isospin state t'_1 (t'_2) is given by

$$\varphi_{\rm usc} = r^2 d\Omega\, \frac{k}{\mu} \frac{|M_{s'_1,s'_2,t'_1,t'_2;s_1,s_2,t_1,t_2}(\hat{r})|^2}{r^2} \ . \tag{6.66}$$

The incoming flow per unit area is $\varphi_{\rm entr} = k/\mu$, so from the definition of cross-section we obtain

$$\frac{d\sigma}{d\Omega} = |M_{s'_1,s_2,'t'_1,t'_2;s_1,s_2,t_1,t_2}(\hat{r})|^2 \ . \tag{6.67}$$

This is the cross section for the process $|s_1 s_2, t_1 t_2\rangle \rightarrow |s'_1 s'_2, t'_1 t'_2\rangle$, where the incoming particles have the momentum in the z direction while the outcoming particles have $\boldsymbol{k}' \parallel \hat{r}$. In the pp case the cross-section contains three terms: the Rutherford cross section, a purely nuclear term and an interference term. If we have an unpolarized beam which hits an unpolarized target and we do not measure the polarization of the reactants, the unpolarized differential cross-section reads

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \sum_{s_2,s'_1,s'_2} |M_{s_1,'s'_2,t_1,'t'_2;s_1,s_2,t_1,t_2}(\hat{r})|^2 \ . \tag{6.68}$$

6.4 Physical observables

Before starting to discuss about the physical observables, we need to define a coordinate system for the projectile polarization and another for the outgoing particle polarization. For the projectile we adopt the usual frame where the z-axis is along the $\boldsymbol{k}$; the y-axis along the normal to the scattering plane $\boldsymbol{k} \times \boldsymbol{k}'$ (let us remember the $\boldsymbol{k}$ ($\boldsymbol{k}'$) is the relative momentum of the incident (outgoing) particles); and x is chosen to form a right-handed system. For the outgoing particles we define the x',y',z' coordinate system similarly with the z'-axis along $\boldsymbol{k}'$; the y'-axis still along $\boldsymbol{k} \times \boldsymbol{k}'$; and x' again chosen to form a right-handed system (see Fig. 6.1).

As we have shown in the previous Section, the M matrix represents the transition probability from a given initial state to a given final state. Let us describe the initial system with a spinor χ_i and the final state with χ_f which are related by

$$\chi_f = M \chi_i \ . \tag{6.69}$$

where

$$\chi_i = \Big|\frac{1}{2} s_1\Big\rangle \Big|\frac{1}{2} s_2\Big\rangle, \qquad \chi_f = \Big|\frac{1}{2} s'_1\Big\rangle \Big|\frac{1}{2} s'_2\Big\rangle. \tag{6.70}$$

We can write the density matrix for the initial and the final beams of particles as [48]

$$\rho_i = \sum_k \omega_k \chi_i^{(k)} (\chi_i^{(k)})^\dagger \ , \qquad \rho_f = \sum_k \omega_k \chi_f^{(k)} (\chi_f^{(k)})^\dagger \tag{6.71}$$

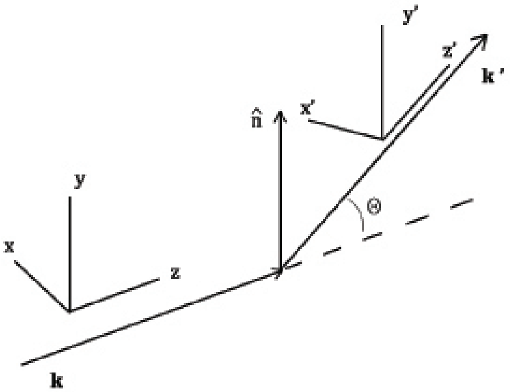


Figure 6.1: Frame system for the incoming and the outcoming particles.

for an initial beam with a fraction ω_k of the pairs in the spin state $\chi_i^{(k)}$. It is possible also to write the density matrix distinguishing the contribution of the two particles for example $\rho_i = \rho_i^{(1)} \rho_i^{(2)}$ where $\rho_i^{(j)}$ represents the density matrix of particle j. Clearly $\rho^{(j)}$ in our case is a matrix 2×2. So from Eq. (6.69)

$$\rho_f = M \rho_i M^\dagger \ . \tag{6.72}$$

If $\mathrm{Tr}(\rho_i)$ is normalized to 1, from the definition of cross-section we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\mathrm{Tr}(\rho_f)}{\mathrm{Tr}(\rho_i)} = \mathrm{Tr}(M \rho_i M^\dagger) \ . \tag{6.73}$$

If the beam is not polarized $\rho_i = \mathbb{1}/4$ and the cross-section will read

$$\frac{d\sigma_0}{d\Omega} = \frac{1}{4} \mathrm{Tr}(M M^\dagger) \ , \tag{6.74}$$

which is the same form of Eq. (6.68).

Let's consider the process where particle 2 (the target) is unpolarized. Now in general $\rho_i^{(i)}$ is a 2×2 matrix so we can write it as a linear expansion over the Pauli matrix

$$\rho_i^{(1)} = \sum_{j=0}^{3} a_j^{(1)} \sigma_j^{(1)}, \qquad \rho_i^{(2)} = \frac{1}{2} \mathbb{1}, \tag{6.75}$$

where $\sigma_0 = \mathbb{1}$. Defining $p_j = \langle \sigma_j \rangle$, which is the expectation value of the i-th Pauli matrix on the initial state, we can write the identity

$$\mathrm{Tr}(\rho_i^{(1)} \sigma_j) = 2 a_j^{(1)} = p_j \ , \tag{6.76}$$

whence

$$\rho_i^{(1)} = \frac{1}{2}\Big(\mathbb{1} + \sum_j p_j \sigma_j^{(1)}\Big) \ . \tag{6.77}$$

Inserting Eq. (6.77) in Eq. (6.73) we obtain

$$\frac{d\sigma}{d\Omega}(\theta,E) = \frac{d\sigma_0}{d\Omega}(\theta,E)\Big(1 + p_x A_x(\theta,E) + p_y A_y(\theta,E) + p_z A_z(\theta,E)\Big) \ , \tag{6.78}$$

where $d\sigma_0/d\Omega$ is the unpolarized cross-section and the quantities $A_j(\theta)$ are called analyzing powers and are defined as

$$A_j = \frac{\mathrm{Tr}(M\sigma_j^{(1)}M^\dagger)}{\mathrm{Tr}(MM^\dagger)} \ , \tag{6.79}$$

and are related to the different polarization of the beam, refered to the given axis frame. The superscript (1) remembers that $\sigma_j^{(1)}$ operates on the spin states of particle 1.

6.4.1 PV observables in $\vec{p}p$ system

We can now study how the analyzing powers transform under the parity operator. The parity operator reverses all the polar vectors but leaves invariant the axial vectors like the spin. Thus $\boldsymbol{k} \to -\boldsymbol{k}$, $\boldsymbol{k}' \to -\boldsymbol{k}'$ and $\boldsymbol{k} \times \boldsymbol{k}' \to \boldsymbol{k} \times \boldsymbol{k}'$.

Because the coordinates x, z, x', z' are linear combination of $\boldsymbol{k}$ and $\boldsymbol{k}'$, they reverse under parity, but the y, y' axes do not. If parity is conserved, the transformed system must be identical to the initial system; this means that all coefficents must remain the same. In Fig. (6.2) we study for example the scattering of a neutron beam polarized along z. In part (a) the original system is shown; in (b) under parity $\boldsymbol{k}$ and $\boldsymbol{k}'$ are reversed in direction, taking care not to reverse the role namely their meaning of initial/final momenta; in (c) the entire system is rotated by 180° around the y-axis. So if parity is a good symmetry, the scattering of particles of helicity +1 and −1 would give the same cross-section, namely $A_z = 0$. Therefore a value of A_z different from zero is a signal of the presence of PVin the nuclear interaction.

If we consider a beam with the spin parallel to the z-axis, Eq. (6.78) becomes

$$\frac{d\sigma_+}{d\Omega}(\theta,E) = \frac{d\sigma_0}{d\Omega}(\theta,E)\Big(1 + A_z(\theta,E)\Big) \ , \tag{6.80}$$

while for spin antiparallel to the z-axis we obtain

$$\frac{d\sigma_-}{d\Omega}(\theta,E) = \frac{d\sigma_0}{d\Omega}(\theta,E)\Big(1 - A_z(\theta,E)\Big) \ . \tag{6.81}$$

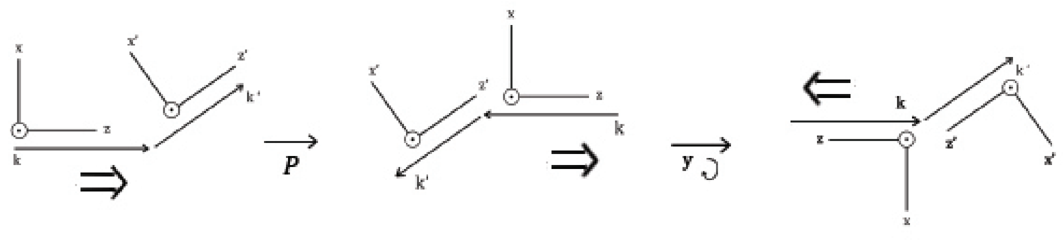


Figure 6.2: Pictorial demonstration that A_z violates parity. In part (a) the original system is shown; in (b) under parity $\boldsymbol{k}$ and $\boldsymbol{k}'$ are reversed in direction, taking care not to reverse the role namely their meaning of initial/final momenta; in (c) the entire system is rotated by 180° around the y-axis.

Redefining $\sigma_+ = d\sigma_+/d\Omega$ and $\sigma_- = d\sigma_-/d\Omega$ we can write

$$A_z(E,\theta) = \frac{\sigma_+(\theta,E) - \sigma_-(\theta,E)}{\sigma_+(\theta,E) + \sigma_-(\theta,E)} \, , \tag{6.82}$$

where θ is the scattering angle and E the energy in the laboratory frame.

From a practical point of view, the experiments detect the particles scattered in angular range $[\theta_1, \theta_2]$ and the measured quantity is an "average" of the asymmetry over the total cross-section in this range, explicitly

$$\overline{A}_z(E) = \frac{\int_{\theta_1 \le \theta \le \theta_2} d\hat{r} \; A_z(\theta,E)\sigma(\theta,E)}{\int_{\theta_1 \le \theta \le \theta_2} d\hat{r} \; \sigma(\theta,E)} \, , \tag{6.83}$$

where

$$\sigma(\theta,E) = \frac{1}{2}\left(\sigma_+(\theta,E) + \sigma_-(\theta,E)\right) , \tag{6.84}$$

is the unpolarized differential cross-section for the process. This quantity is also called longitudinal asymmetry.

6.4.2 Observables in np system: the spin rotation

Another possible parity violating observable is related to the spin rotation in $\vec{n}p$ elastic scattering. The transmission of a neutron beam through a slab of matter of width d and density N is described in term of a refraction index.

To introduce the index of refraction we will follow Ref. [49]. In order to make valid the following approximation, the neutron momenta must be such that $d \ll \lambda$ where $\lambda = 1/|\boldsymbol{p}_n|$ and $\boldsymbol{p}_n = 2\boldsymbol{p}$ is the initial momentum of the neutrons in the

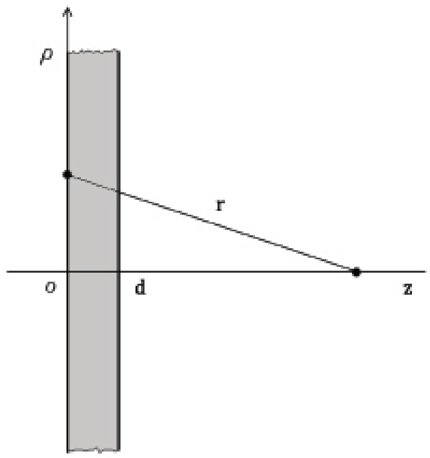


Figure 6.3: Transmission of a neutron beam through a slab of matter of width d. Note that $r^2 = \rho^2 + z^2$ and we are supposing that $d \ll r$.

laboratory system. Considering the momentum $\boldsymbol{p}_n$ along the z-axis, the incoming wave function can be written as

$$\psi_{\text{in}} = e^{ip_n z} \ . \tag{6.85}$$

Passing through the material the plane wave will move with a momenta $n\boldsymbol{p}_n$ where n is the refraction index we are looking for. Thus, the wave after the target will be

$$\begin{aligned} \psi_{\text{out}} &= e^{i(np_n d + p_n(z-d))} \ , & (6.86) \\ &\sim e^{ip_n z}(1 + ip_n d(n-1)) \ , \ z > d \ . & (6.87) \end{aligned}$$

From the scattering theory, we can also write the outcoming wave function as a sum of the incoming wave plus the scattering wave which we can expand in partial waves as

$$\psi_{\text{out}} = e^{ip_n z} + \sum_{l=0}^{\infty} \int_0^{+\infty} \frac{C_l}{r} e^{ip_n r} P_l(\cos\theta) 2\pi N d\rho \mathrm{d}\rho \ , \tag{6.88}$$

where $2\pi N d\rho \mathrm{d}\rho$ represents the average number of collision of the neutron and r is defined as in Fig. 6.3. C_l represents the scattering parameters for different l. Namely, the final wave function along the z axis is given by the unperturbed plane wave plus all the contributions of the scattering waves coming from the neutron passing in the circular corona of radius ρ and width $d\rho$. Because we are at low energies the only wave that contributes is $l = 0$ and so the outcoming wave results

$$\psi_{\rm out} = e^{ip_n z} + 2\pi N d\rho C_0 \int_z^{+\infty} e^{ip_n r} {\rm d}r \ , \tag{6.89}$$

where we have used the fact that $\rho{\rm d}\rho = r{\rm d}r$ as it is possible to deduce from Fig. 6.3. The integral we get is divergent and in order to calculate it we use a regularization function $\exp(-b^2 r)$ which goes fast to zero for $r \to +\infty$ (this function has a physical meaning because the beam is not infinitely large), and then we take the limit $b \to 0$. The final result is

$$\psi_{\rm out} = e^{ip_n z} + 2\pi N d\rho C_0 \frac{i}{p_n} e^{ip_n z} \tag{6.90}$$

from which is simple to identify the refraction index with (see Eq. (6.87))

$$n - 1 = \frac{2\pi N}{p_n^2} C_0 \ . \tag{6.91}$$

However we can identify $C_0 = f(0)$ [50] where $f(0)$ is the forward scattering amplitude therefore

$$n - 1 = \frac{2\pi N}{p_n^2} f(0) \ . \tag{6.92}$$

This formula defines the refraction index in term of the forward scattering amplitude.

The most general form for the forward scattering amplitude for two spin 1/2 particles, taking into account the dependence on the polarization of both initial particles, can be written as

$$f(0) = f_0 + f_M(\boldsymbol{\sigma} \cdot \boldsymbol{S}) + f_P(\boldsymbol{\sigma} \cdot \boldsymbol{p}_n) + f_T \boldsymbol{\sigma} \cdot (\boldsymbol{p}_n \times \boldsymbol{S}) \ , \tag{6.93}$$

where f_0, f_M, f_P and f_T are general function of $\boldsymbol{p}_n^2$ only. Above $\boldsymbol{\sigma}$ is the spin operator of the beam, $\boldsymbol{S}$ is the spin operator of the target and $\boldsymbol{p}_n$ the impulse of the beam. The first term is the spin-indipendent forward scattering amplitude. The second term explicits the dependence between the spin of the target and the spin of the neutron. The third term is a PV term describing the dependence of the forward amplitude if the neutron spins are aligned either along or opposite the direction of propagation. The fourth term appears if there is a TV effect (see later).

If we define the z-axis along $\boldsymbol{p}_n$ and the other axes as defined in Fig. (6.3) and the target is not polarized, $\langle \overline{S} \rangle = 0$ we get from Eq. (6.93)

$$f = f_0 + f_P \sigma_z \ , \tag{6.94}$$

so the index of refraction changes if the polarization of the beam changes, as follows from Eq. (6.92). In term of the M-matrix the forward scattering amplitude, for a non polarized target, can be written as

$$f_{m_n} = \frac{1}{2} \sum_{m_p} \langle m_n\, m_p | M | m_p\, m_n \rangle |_{\theta=0}, \tag{6.95}$$

where m_n represent the polarization of the initial neutron state along the z-axis. Since $f_+ = f_0 + f_P$ and $f_- = f_0 - f_P$, from Eq. (6.95) we identify f_0 and f_p in term of the M-matrix. If we consider a neutron beam initially polarized along the x axis namely

$$\psi_{in} = e^{ip_n z} \frac{|+\rangle + |-\rangle}{\sqrt{2}} \tag{6.96}$$

the state of the neutron after the interaction with the matter will be

$$\psi_{out} = \frac{e^{ip_n(z-d)}}{\sqrt{2}} \left(e^{ip_n d n_+} |+\rangle + e^{ip_n d n_-} |-\rangle \right) , \tag{6.97}$$

which we can be rewritten as

$$\psi_{out} = \frac{e^{ip_n(z-d)}}{\sqrt{2}} e^{ip_n d \frac{n_+ + n_-}{2}} \left(e^{ip_n d \frac{n_+ - n_-}{2}} |+\rangle + e^{-ip_n d \frac{(n_+ - n_-)}{2}} |-\rangle \right) , \tag{6.98}$$

where $n_+ - n_-$ is proportional to f_P as follows from Eq. (6.94), explicitly

$$n_+ - n_- = \frac{2\pi N}{p_n^2} 2 f_P = \frac{2\pi N}{p_n^2} \frac{1}{2} \sum_{m_p} \left[\langle +\, m_p | M | m_p\, + \rangle - \langle -\, m_p | M | m_p\, - \rangle \right]_{\theta=0} , \tag{6.99}$$

while the phase proportional to $n_+ + n_-$ is unobservable. This is correlated to a rotation of the spin along the z-axis. In fact, if we consider a spin state $|A\rangle$ we can define the rotated state of an angle α along a direction $\hat{n}$ as

$$|B\rangle = e^{-i(\boldsymbol{\sigma}\cdot\hat{n})\alpha/2} |A\rangle . \tag{6.100}$$

If we start with a spin directed on the x-axis and we consider a rotation around the z-axis of an angle ϕ_z we obtain

$$|B\rangle = \frac{1}{\sqrt{2}} \left(e^{-i\phi_z/2} |+\rangle + e^{+i\phi_z/2} |-\rangle \right) , \tag{6.101}$$

where ϕ_z, the rotation angle, can be measured experimentally. From Eq. (6.101) it is easy to identify the phase ϕ_z with the real part of the phase in Eq. (6.98), thus

$$\phi_z = -\frac{2\pi N}{p_n} \frac{d}{2} \sum_{m_p} \mathrm{Re} \left[\langle +\, m_p | M | m_p\, + \rangle - \langle -\, m_p | M | m_p\, - \rangle \right]_{\theta=0} \tag{6.102}$$

where d is the width of the target, and so the rotation for unit length is given by:

$$\frac{\mathrm{d}\phi_z}{\mathrm{d}z} = -\frac{2\pi N}{p_n}\frac{1}{2}\sum_{m_p}\mathrm{Re}\big[\langle + \, m_p|M|m_p \, +\rangle - \langle - \, m_p|M|m_p \, -\rangle\big]_{\theta=0}. \tag{6.103}$$

In Eq. (6.93) it appears a TV terms. If we consider also the target polarized along the x-axis either before and after the scattering, the forward amplitude can be written as

$$f(0) = f_0 + f_M\sigma_x + f_P\sigma_z + f_T\sigma_y \ . \tag{6.104}$$

In analogy with Eq. (6.94), we can see that the real part of f_T is proportional to the rotation angle around the y-axis. So we can derive a formula for ϕ_y similar to Eq. (6.102)

$$\phi_y = -\frac{4\pi N d}{p_n}\mathrm{Re}f_T \tag{6.105}$$

Let's derive the expression of f_T in terms of the M-matrix. $f(0)$ is a matrix, and when the initial and final polarizations of the proton target are along the x-axis, its elements can be expressed in term of the M-matrix as

$$f(0)_{m_n',m_n} = \frac{1}{2}\sum_{m_p',m_p}\langle m_p' \, m_n'|M|m_p \, m_n\rangle|_{\theta=0} \tag{6.106}$$

where m_p,m_p' are the initial and final state of the spin proton and m_n,m_n' are the initial and final state of the spin neutron along the z axis. Writing the expression (6.104) in a matricial form we obtain

$$f(0) = \begin{pmatrix} f_0 + f_P & f_M - if_T \\ f_M + if_T & f_0 - f_P \end{pmatrix} \ . \tag{6.107}$$

Because both the expressions are written in the basis of the enginestates of σ_z it's easy to get the expression for f_T

$$f_T = \frac{f_{-+} - f_{+-}}{2i} \tag{6.108}$$

and from Eq. (6.106) it is easy to obtain the spin rotation along y for unit length

$$\frac{\mathrm{d}\phi_y}{\mathrm{d}z} = -\frac{2\pi N}{p_n}\frac{1}{2}\sum_{m_p',m_p}\mathrm{Im}\big[\langle + \, m_p'|M|m_p \, -\rangle - \langle - \, m_p'|M|m_p \, +\rangle\big]_{\theta=0}. \tag{6.109}$$

Chapter 7

Results

In this Chapter we will present the results for the observables we have studied. We have written a program in FORTRAN language in order to solve Eq. (6.41) and to evaluate the M-matrix.

As PC potential we will use the chiral potential derived in Ref. [51] by Entem and Machleidt at N3LO. This potential is regularized with a cut-off function depending on a parameter Λ_F, its functional form, however, is different from that adopted here for the PV and TV potentials. We will consider four versions of this PC potential corresponding to $\Lambda_F = 414$ MeV, $\Lambda_F = 450$ MeV, $\Lambda_F = 500$ MeV, and $\Lambda_F = 600$ MeV. In spite of the use different cut-off each case, we use the same value for the Λ_F for our PV and TV potentials.

This Chapter is divided in two Sections. In the first Section we will present the results for the A_z observable in the $\vec{p}p$ scattering, studying in particular the possibility of extracting the LECs from the available experimental data. In the second Section we will present the results for the $n\vec{p}$ spin rotation PV observable and the $\vec{n}\vec{p}$ spin rotation TV observable discussed in the previous Chapter. In the latter case we will discuss the possibility of estimating the $\overline{\theta}$ angle.

7.1 The $\vec{p}p$ longitudinal asymmetry

There exist three accurate measurements of the angle-averaged $\vec{p}p$ longitudinal asymmetry $\overline{A}_z(E)$, see Eq. (6.83), obtained at different laboratory energies E [52, 53, 54]. The measurements are:

$$\begin{aligned}
\overline{A}_z(13.6\,\mathrm{MeV}) &= (-0.97 \pm 0.20) \times 10^{-7}\ , \\
\overline{A}_z(45\,\mathrm{MeV}) &= (-1.53 \pm 0.21) \times 10^{-7}\ , \\
\overline{A}_z(221\,\mathrm{MeV}) &= (+0.84 \pm 0.34) \times 10^{-7}\ .
\end{aligned} \tag{7.1}$$

The errors reported above include both statistical and systematic errors added in quadrature. In these experiments, the asymmetry was measured averaging over

$E\,(\text{MeV})$	(θ_1, θ_2)
13.6	$(20^\circ, 78^\circ)$
45	$(23^\circ, 52^\circ)$
221	$(5^\circ, 90^\circ)$

Table 7.1: Angle ranges used to compute the $\overline{A}_z$ asymmetry in the different experiments.

a range (θ_1, θ_2) of (laboratory) scattering angles as defined in Eq. (6.83). The $A_z(E, \theta)$, defined in Eq. (6.82) is approximately constant except at small angles $\lesssim 15^\circ$ where the Coulomb scattering dominates. In Table 7.1, the angle ranges used to compute the asymmetry are reported.

In the case of pp scattering the isospin state is $|pp\rangle \equiv |T = 1, T_z = 1\rangle$ and so the matrix elements of the isospin operators which appear in the expression of the PV potential are the following:

$$\langle T = 1, T_z = +1|(\vec{\tau}_1 \cdot \vec{\tau}_2)|T = 1, T_z = +1\rangle = 1 \ , \tag{7.2}$$
$$\langle T = 1, T_z = +1|(\vec{\tau}_1 \times \vec{\tau}_2)_z|T = 1, T_z = +1\rangle = 0 \ , \tag{7.3}$$
$$\langle T = 1, T_z = +1|(\tau_{1z} + \tau_{2z})|T = 1, T_z = +1\rangle = 2 \ , \tag{7.4}$$
$$\langle T = 1, T_z = +1|(\mathcal{I}_{ij}\tau_{1i}\tau_{2j})|T = 1, T_z = +1\rangle = 2 \ , \tag{7.5}$$

therefore the LO contribution that comes from the OPE in Eq. (4.83) vanishes. The LEC h^1_π will contribute to the observable only via the TPE box diagrams that appear at N2LO and N3LO. Regarding the contact terms, the term with C_3 does not contribute. From Eq. (4.84) we note that the other terms differ only for the isospin part so,

$$\langle V^{(1)}_{PV}(\text{CT})\rangle = (C_1 + C_2 + 2C_4 + 2C_5)\langle(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \boldsymbol{r}\rangle Z'(r). \tag{7.6}$$

Performing a similar analysis for the terms of the potential in Eq. (4.88) and taking into account the matrix elements given in Eqs. (7.2)-(7.5), the longitudinal asymmetry can be written as

$$\overline{A}_z(E) = h^1_\pi\, a_0(E) + C\, a_1(E) + \tilde{h}\, a_2(E), \tag{7.7}$$

where

$$C = C_1 + C_2 + 2\,(C_4 + C_5) \ , \tag{7.8}$$
$$\tilde{h} = \frac{3g_A}{4} h^0_V + \frac{g_A}{2} h^0_V + 2\Big(\frac{g_A}{4} h^1_V - h^1_A\Big) - 2\Big(\frac{g_A}{3} h^2_V + h^2_A\Big) \ , \tag{7.9}$$

and $a_0(E)$, $a_1(E)$, $a_2(E)$ are numerical coefficients independent of the LEC values. The linear relation between the observables and the three LECs follows simply

N_L	γ [fm^{-1}]	N_p	β [fm^{-1}]	J_{max}	a_0	a_1	a_2
40	3	100	0.25	6	0.70038	−0.07994	−0.43854
50	3	100	0.25	6	0.70035	−0.07994	−0.43853
40	**4**	100	0.25	6	0.70034	−0.07994	−0.43853
40	3	**120**	0.25	6	0.70038	−0.07994	−0.43854
40	3	100	**0.30**	6	0.69949	−0.07974	−0.43874
40	3	100	0.25	**8**	0.70039	−0.07994	−0.43855

Table 7.2: Numerical stability and convergence of the coefficients a_0, a_1, a_2 calculated at $E = 45$ MeV and for $\Lambda_f = 500$ MeV. The first line is calculated using the parameters selected for this work. In the other lines we change one of the parameters (bold) in order to compare the results with our choice. See the text for more details.

from their smallness, they are of the order of 10^{-7}. Therefore quadratic or higher power dependence on the LEC can be safely disregarded. The coefficients a_0, a_1, a_2, are clearly dependent on the cut-off Λ_F in the PV and PC potential. To calculate one of the three numerical coefficients we just put to zero the other two LECs exploiting the linear dependence of the $\overline{A}_z$ from them.

Before discussing the results, let us study the dependence of the coefficients a_0, a_1, a_2 on the numerical code. The program requires to set different parameters: N_L the number of Laguerre polynomials, γ used to optimize the expansion in the Laguerre polynomials, N_p the number of points used for the Gauss-Laguerre numerical integrations, β the parameter of the regularization factor of the Coulomb irregular function and J_{max} the maximum total angular momentum in the expansion of the M-matrix, Eq. (6.65). These parameters have been chosen to optimize the convergence in the calculation of the observables. In Table 7.2 the dependence of the coefficients a_0, a_1, and a_2 from the values of the parameters entering the numerical code is discussed. The first line in the Table 7.2 represents the choice of the parameters used through this work. As we can see the modification of the parameters N_L, γ, N_p, β, J_{max} give corrections only to the fourth decimal digit. Therefore we can conclude that the coefficients are calculated very accurately. An analogous accuracy is found for all the other cases we have studied.

The values of the coefficients a_0, a_1, and a_2 obtained for the four choices of the cut-off are reported in Table 7.3. As it is possible to see the values of the a_i for $E = 13.6$ MeV are approximately a factor 2 smaller than those obtained at $E = 45$ MeV. At these energies the asymmetry is dominated by the contribution of the $S - P$ matrix elements, so it is sensitive only to the matrix element

$$\begin{aligned}&\langle\overline{\psi}_{11}(J=0)|V_{PV}|\overline{\psi}_{00}(J=0)\rangle\\&\qquad\sim\int_0^\infty dr\; r^2\frac{F_1(\eta,kr)}{kr}v^{PV}_{11,00}\frac{F_0(\eta,kr)}{kr}\sim\epsilon_0 k\ ,\end{aligned} \tag{7.10}$$

E [MeV]	a_0(N2LO)	a_0(N3LO)	a_1	a_2
		$\Lambda_F = 414$ MeV		
13.6	0.27435	0.35523	−0.04127	−0.20997
45	0.56804	0.70048	−0.07545	−0.43652
221	−0.14068	−0.16771	0.01815	0.12816
		$\Lambda_F = 450$ MeV		
13.6	0.28172	0.36334	−0.04233	−0.21555
45	0.55321	0.72237	−0.07795	−0.45308
221	−0.20976	−0.25520	0.02709	0.19039
		$\Lambda_F = 500$ MeV		
13.6	0.26992	0.33560	−0.04159	−0.19719
45	0.55528	0.70038	−0.07994	−0.43854
221	−0.24340	−0.30487	0.03134	0.21941
		$\Lambda_F = 600$ MeV		
13.6	0.25441	0.31705	−0.03990	−0.17869
45	0.53438	0.67215	−0.07841	−0.40520
221	−0.19342	−0.25520	0.02743	0.14378

Table 7.3: Values of the coefficients a_i at the three energies corresponding to the experimental data points for the four choices of cut-off parameters Λ_F. The calculations include contributions up to $J_{\max} = 6$ in the expansion of the pp scattering state. For the coefficient a_0 we give the calculation with N2LO only and then adding the N3LO.

where $\overline{\psi}_{LS}$ are the wave function introduced in the previous chapter ϵ_0 is a constant independent of the energy. Therefore the a_i at low energy scale as $\sqrt{E}$. This explains why the values of the a_i obtained at $E = 45$ MeV are approximately twice larger than the values obtained at $E = 13.6$ MeV. Because of this scaling, the experimental points at $E = 45$ MeV and $E = 13.6$ MeV do not provide independent constraints on the LECs h^1_π, C and $\tilde{h}$.

The presence of only two independent data points makes problematic to fix the three LECs. To have an idea of the possible values of the LECs, we can fix one of the three LECs and perform a χ^2 analysis in order to define a region of the "most probable values" of the other two for the given value of the first one. In order to compare the results with Ref. [25] and [26] we will restrict our study to $\Lambda_F = 500$ MeV and $\Lambda_F = 600$ MeV. The value of χ^2 is calculated as

$$\chi^2 = \sum_i \frac{(\overline{A}_z(i) - \overline{A}_z^{exp}(i))^2}{(\Delta\overline{A}_z^{exp}(i))^2}, \tag{7.11}$$

where the sum i is over the three energies of which we have the data, $A_z^{exp}(i)$

and $\Delta\overline{A}_z^{exp}(i)$ are respectively the data and the correspondent experimental uncertainty.

For the first analysis we choose $\tilde{h} = 0$. The results obtained are reported in Fig. 7.1. As we can see in both plots, the regions for different Λ_F almost

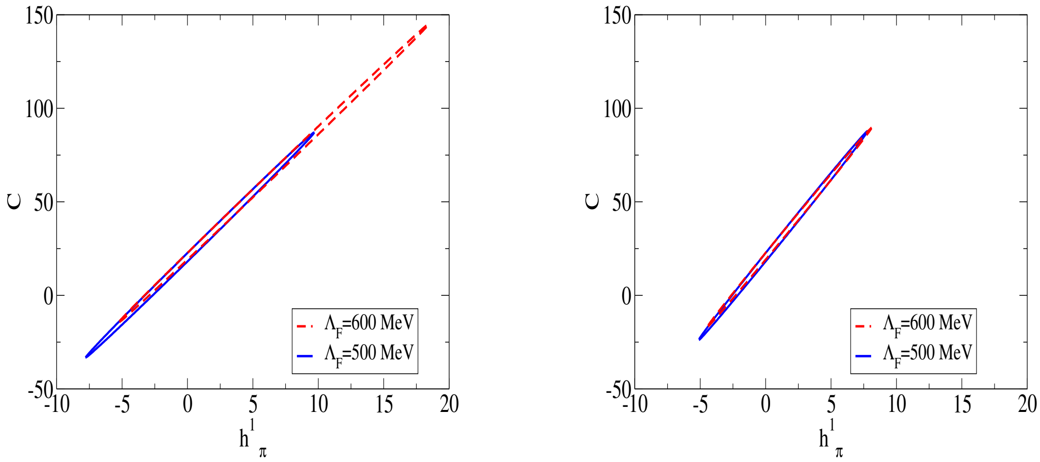


Figure 7.1: Countours for h_π^1 and C values (in units of 10^{-7}) corresponding to $\chi^2 = 2$ for the $\vec{p}p$ longitudinal asymmetry with $\tilde{h} = 0$. The region $\chi^2 < 2$ is the region inside the narrow ellipses. The blue solid (red dashed) countour is relative to Λ_F=500 (600) MeV. In the left (right) panel, the χ^2 values obtained using the N2LO (N2LO+N3LO) potential are shown.

coincide. Comparing the two figures we can see that, adding the N3LO component of the potential, the region $\chi^2 < 2$ was reduced for both Λ_F (in particular for $\Lambda_F = 600$ MeV). The regions for the two different Λ_F using the potential at N3LO completely overlap. The strictly correlation between h_π^1 and C is still present, but their mutual dependence changes as we can see by the increase of inclination of the ellipses. The range of allowed h_π^1 and C are

$$h_\pi^1 = (2.5 \pm 7.5) \cdot 10^{-7} \tag{7.12}$$

$$C = (3.5 \pm 6.5) \cdot 10^{-6} \tag{7.13}$$

which are perfectly in agreement with the results obtained in Ref. [26] and the DDH (another potential model)"reasonable range" [55]

$$0 < h_\pi^1 < 11.2 \cdot 10^{-7} \ . \tag{7.14}$$

To study the dependence on $\tilde{h}$ we have to fix one of the other two constants. There exist some independent estimates of the LEC h_π^1 from the DDH potential model, from which it has been possible to extrapolate the range reported in Eq. (7.14), or from Lattice QCD which has given the prediction $h_\pi^1 = 1 \cdot 10^{-7}$ [56]. In the following we will perform the calculations using:

1. $h_\pi^1 = 1 \cdot 10^{-7}$ (lattice estimate);

2. $h_\pi^1 = 4.56 \cdot 10^{-7}$ (DDH "best value");

The $\chi^2 = 2$ contours in the plane of parameters C and $\tilde{h}$ are reported in Fig. 7.2. From the plots it is possible to note that there is a strictly correlation also between

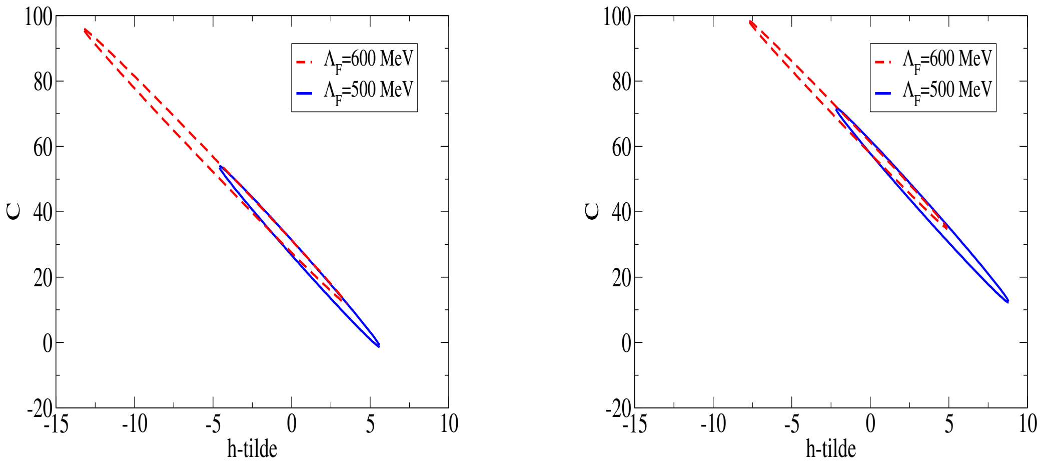


Figure 7.2: χ^2 countours as function of $\tilde{h}$ and C (in units of 10^{-7}) corresponding to $\chi^2 = 2$ for the $\vec{p}p$ longitudinal asymmetry. The blue solid (red dashed) countours is relative to Λ_F=500 (600) MeV. The left figure is obtained considering $h_\pi^1 = 1 \cdot 10^{-7}$. The right figure is obtained with $h_\pi^1 = 4.56 \cdot 10^{-7}$.

C and $\tilde{h}$. The values of C are included within the range found before with $\tilde{h} = 0$. From the comparison of the two plots it is possible to note that increasing h_π^1 the center of the ellipses move towards higher values of C and $\tilde{h}$ but they maintain a similar size and the same inclination. This shows that the correlation between C and $\tilde{h}$ is independent of the choice of h_π^1. The estimate of the range for $\tilde{h}$ are

1. $\tilde{h} = (-0.5 \pm 1) \cdot 10^{-6}$ for $h_\pi^1 = 1 \cdot 10^{-7}$;

2. $\tilde{h} = (0 \pm 0.9) \cdot 10^{-6}$ for $h_\pi^1 = 4.56 \cdot 10^{-7}$.

These range of values for $\tilde{h}$ are in agreement with the result obtained in Ref. [26]. Larger values of this LEC could be a problem for the EFT as discussed in the next section.

Let us study the energy dependence of the coefficients a_i. From now on, we will use an angle range between 15° and 90° to compute the average asymmetry for all energies. The coefficient a_0 receives various contributions, so we can write it as

$$a_0 = a_0(\text{NR}) + a_0(\text{TPE}) + a_0(\text{RC-OPE}) + a_0(\text{RC-TPE}) + a_0(c_4) \tag{7.15}$$

where the coefficient a_0(X) derives from the part proportional to h^1_π of potential $V_{PV}^{(x)}$ as given in Eq. (4.103)-(4.110). The contribution $a_0(\text{NR}) = a_0(\text{RC-OPE}) = 0$ because the corresponding components are proportional to $(\vec{\tau}_1 \times \vec{\tau}_2)_z$. For convenience let's define also

$$a_0(\text{N2LO}) = a_0(\text{NR}) + a_0(\text{TPE}) + a_0(\text{RC-OPE}) \tag{7.16}$$

$$a_0(\text{N3LO}) = a_0(\text{RC-TPE}) + a_0(c_4). \tag{7.17}$$

The behaviour of these coefficients in function of the laboratory energy is shown in Fig. 7.3. The contribution of a_0(N3LO) is found to be $\sim 12\%$ of a_0(N2LO),

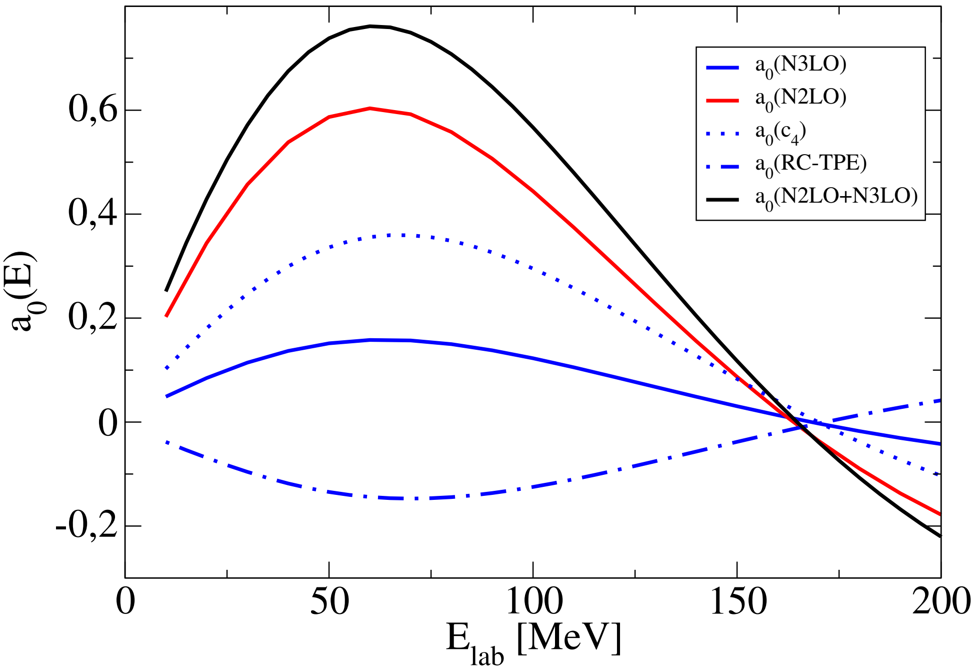
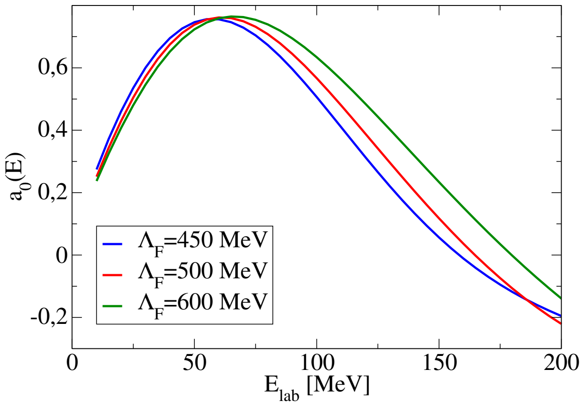


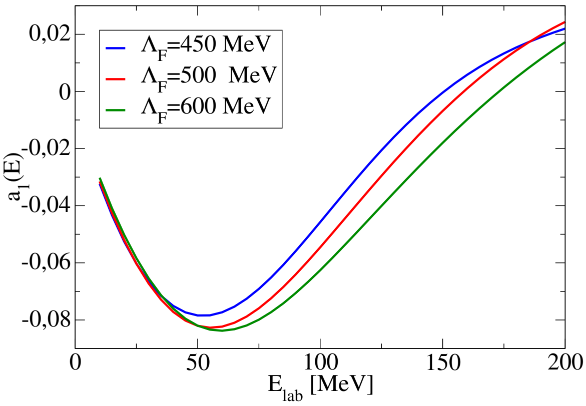
Figure 7.3: The energy dependence of the a_0 coefficients introduced in the main text for $\Lambda_F = 500$ MeV. The black solid line is the total contribution. a_0(N2LO)-red line, a_0(N3LO)-blue line, $a_0(c_4)$-blue dotted line, a_0(RC-TPE)-blue dash-dotted line.

in line with what we expect from the chiral expansion where the importance of the contributions should be of the order of $m_\pi/\Lambda_\chi \sim \frac{1}{7}$. The contribution given by $a_0(c_4)$ to a_0 is however a bit anomalous. The presence of the LEC $c_4 = 3.4\,\text{GeV}^{-1}$ as deduced from the NN scattering data [57] make this factor somewhat larger than the contributions of the other N3LO order terms. However, the term a_0(RC-TPE) reduce the impact of $a_0(c_4)$ bringing $a_0(N3LO)$ in line with what expected from the chiral expansion.

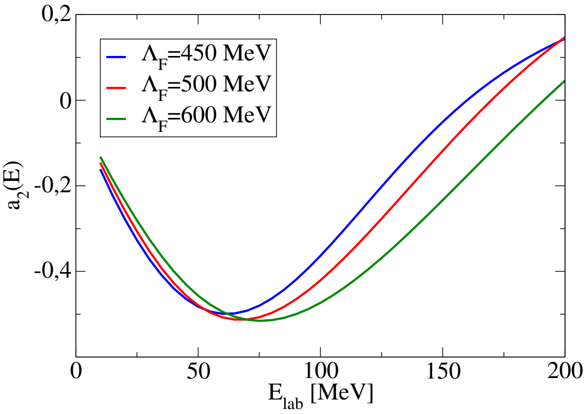
Let us study the dependence of the coefficients on the cut-off Λ_F, reported in Fig. 7.4. For energies smaller than 50 MeV the coefficients a_i are rather



(a)



(b)



(c)

Figure 7.4: Energy dependence of the coefficients $a_0(E)$ (panel a), $a_1(E)$ (panel b), $a_2(E)$ (panel c) obtained for $\Lambda_F = 450$ MeV (blue line), $\Lambda_F = 500$ MeV (red line) and $\Lambda_F = 600$ MeV (green line) .

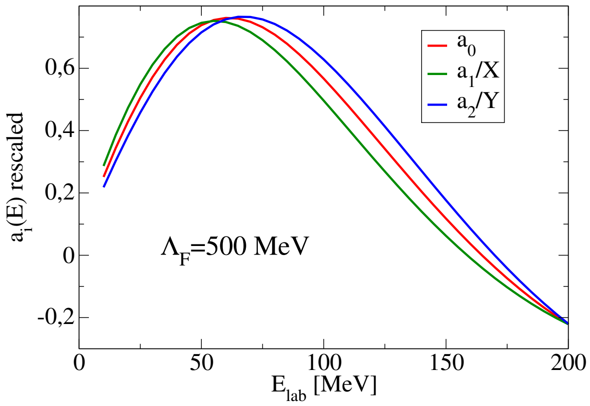


Figure 7.5: Coefficients a_0, a_1, a_2 calculated for $\Lambda_F = 500$ MeV rescaled by suitable factors. As we can see the three functions are very similar. The values of the scaling factors are reported in the main text.

insensitive to Λ_F. We have already discussed that at low energies the asymmetry is dominated by the $S-P$ matrix elements, see Eq. (7.10). At this energies, the P wave function is suppressed at short distances by the centrifugal barrier. The matrix element therefore receive contribution from the long range region, where the various potential terms are independent of Λ_F (the cut-off only modifies the short range part of the potential). On the other hand, at higher energies the particles come closer and closer, and the asymmetry starts to depend on Λ_F.

Looking with more attention to Fig. 7.4 we observe that the energy behaviour of the coefficients $a_i(E)$ is similar. Rescaling $a_i(E)$ by proper factors, the three functions becomes almost identical, as can be seen from Fig. 7.5. In order to understand this behaviour let us study the potential components that contribute to the three coefficients. It is possible to write the PV potential that contributes to the $\vec{p}p$ scattering as

$$V_{PV}(pp) = h^1_\pi\, V(h^1_\pi) + C\, V(\mathrm{CT}) + \tilde{h}\, V(\mathrm{LEC}), \tag{7.18}$$

where we have used the isospin matrix element given in Eqs. (7.2)-(7.5), and we have defined

$$V(h^1_\pi) = -\frac{4g_A^3 m_\pi^2}{\sqrt{2} f_\pi \Lambda_\chi^2} L'(r)\, (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)\cdot \hat{\boldsymbol{r}} + \mathcal{O}(\mathrm{N3LO}), \tag{7.19}$$

$$V(\mathrm{CT}) = \frac{m_\pi^2}{\Lambda_\chi^2 f_\pi} Z'(r)\, (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)\cdot \hat{\boldsymbol{r}}, \tag{7.20}$$

$$V(\mathrm{LEC}) = \frac{g_A^2}{2f_\pi^2} \frac{\pi\, m_\pi^3}{\Lambda_\chi^2} A'(r)\, (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)\cdot \hat{\boldsymbol{r}}. \tag{7.21}$$

Only the terms proportionals to $(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)\cdot \hat{\boldsymbol{r}}$ contributes to the $\vec{p}p$ longitudinal asymmetry for the parts $V(\mathrm{CT})$ and $V(\mathrm{LEC})$. For the term proportional to

$V(h_\pi^1)$ the main contribution comes at N2LO again proportional to $(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}}$ while the N3LO component has also other angular and spin operators. However these terms give negligible corrections. Therefore, the only difference between the three potential terms comes from the radial dependence. However, as can be seen in Fig. 7.6, the three potentials for the same Λ_F have a similar radial dependence. Therefore we can say that $V(\text{CT}) \approx X\ V(h_\pi^1)$ and $V(\text{LEC}) \approx Y\ V(h_\pi^1)$. Consequently

$$a_1(E) \approx X a_0(E), \qquad a_2(E) \approx Y a_0(E). \tag{7.22}$$

Let us define

$$a(E) \equiv a_0(E), \tag{7.23}$$

so,

$$\overline{A}_z = (h_\pi^1 + X\,C + Y\tilde{h})a(E) + f(E) \tag{7.24}$$

where $f(E)$ is a correction which results to be almost negligible. For example, the numerical values for X and Y for $\Lambda_F = 500$ MeV are,

$$X = -0.128, \qquad\qquad Y = -0.571, \tag{7.25}$$

and the correction reads,

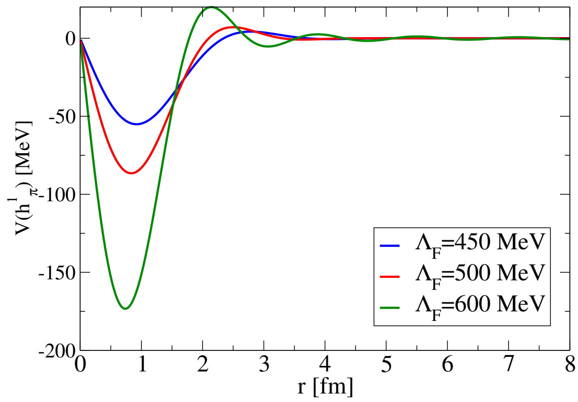
$$f(E) = (0.34C - 1.3\tilde{h})(E\cdot 10^{-3}) - (0.20C - 4.1\tilde{h})(E\cdot 10^{-3})^2 + \mathcal{O}((E\cdot 10^{-3})^3) \tag{7.26}$$

which gives a correction smaller than 1% in the energy range of interest.

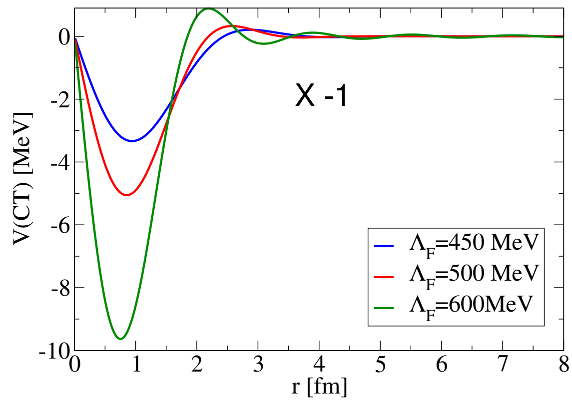
Therefore we can safely neglect $f(E)$, and it is possible to explain the almost linear correlation found in Figs. 7.1 and 7.2. In fact, it is evident from the above relation, that the only possibility is to extract the value of $h = h_\pi^1 + XC + Y\tilde{h}$, also with the availability of more numerous and accurate experimental data. Fixing for example h_π^1, the region of allowed values for C and $\tilde{h}$ would be the line

$$\tilde{h} = \frac{(h - h_\pi^1)}{Y} - \frac{X\,C}{Y}. \tag{7.27}$$

From the available experiments, and the calculated $a(E)$, we can in any case extract h, for the various values of Λ_F. The results are reported in Table 7.4. In Fig. 7.7, $\overline{A}_z$ calculated as $h\,a(E)$, for the various choices of the cut-off Λ_F, is compared with the data. Let us note however that the curves are calculated computing the average asymmetry between 15° and 90° while the data have been obtained averaging over the angle ranges given in Table 7.1, so the comparison can be only qualitative. In Table 7.5 we report the values of $\overline{A}_z$ calculated as $h\,a(E)$ for $\Lambda_F = 500$ MeV at the three energies where the data are available using the correct angle ranges as specified in Table 7.1. The theoretical uncertainty comes from the error obtained in the calculation of h using the mean square method. Taking into account the experimental and theoretical errors, the results are in agreement with the experimental data to fit the data.



(a)



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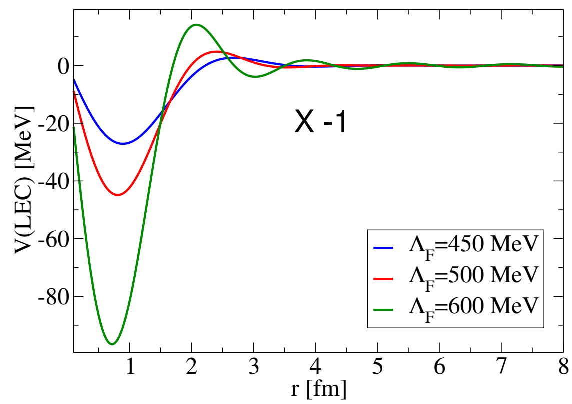

(c)

Figure 7.6: Radial behaviour of the potential terms $V(h^1_\pi)$ (panel a), $V(\mathrm{CT})$ (panel b), $V(\mathrm{LEC})$ (panel c) obtained for $\Lambda_F = 450$ MeV (blue lines), $\Lambda_F = 500$ MeV (red lines) and $\Lambda_F = 600$ MeV (green lines). In panels (b) and (c) the potentials are multiplied $\times - 1$ to have an easier comparison with the curves of panel (a).

Λ_F(MeV)	h
414	$(-2.35 \pm 0.26) \cdot 10^{-7}$
450	$(-2.28 \pm 0.25) \cdot 10^{-7}$
500	$(-2.35 \pm 0.26) \cdot 10^{-7}$
600	$(-2.47 \pm 0.27) \cdot 10^{-7}$

Table 7.4: Values obtained for $h = h^1_\pi + X\,C + Y\tilde{h}$ using the least mean square method for different choice of the cut off Λ_F.

E(MeV)	$\overline{A}_z$(N3LO)	$\overline{A}_z^{exp}$
13.6	$(-0.79 \pm 0.09) \times 10^{-7}$	$(-0.97 \pm 0.20) \times 10^{-7}$
45	$(-1.65 \pm 0.18) \times 10^{-7}$	$(-1.53 \pm 0.21) \times 10^{-7}$
221	$(+0.72 \pm 0.08) \times 10^{-7}$	$(+0.84 \pm 0.34) \times 10^{-7}$

Table 7.5: Values for $\overline{A}_z$ calculated as $h\,a(E)$ for $\Lambda_F = 500$ MeV using the correct angle range as specified inn Table 7.1. $\overline{A}_z^{exp}$ are the experimental value. The theoretical error comes from the error obtained in the calculation of h using the mean square method.

7.2 The $\vec{n}p$ spin rotation

Let us study the $\vec{n}p$ spin rotation for a very small incident neutron energy. Measurements of this observable are in fact performed using ultracold neutron beams. In the following we assume that the beam energy is $E = 0.0001$ MeV, in any case for these energies the observable does not depend on the energy. The density for the hydrogen target is assumed to be $N = 0.4 \cdot 10^{23}\,\mathrm{cm}^{-1}$.

Let's consider first the PV effects in the $\vec{n}p$ spin rotation along the z-axis which we can calculate using Eq. (6.102). In general the rotation angle depends linearly on the PV LECs, in fact second order effects (quadratic in the PV LECs) are surely negligible, therefore

$$\begin{aligned}\frac{\mathrm{d}\phi_z}{\mathrm{d}z} &= h^1_\pi\, b_0 + C_1\, b_1 + C_2\, b_2 + C_3\, b_3 + C_4\, b_4 + C_5\, b_5 \\ &\qquad + h^0_V\, b_6 + h^1_V\, b_7 + h^2_V\, b_8 + h^1_A\, b_9 + h^2_A\, b_{10}\ , \end{aligned} \tag{7.28}$$

where the b_i for $i = 0, \ldots, 10$ are numerical coefficients. The coefficient b_0 receives contributions from different chiral orders, in particular:

$$b_0 = b_0(\mathrm{LO}) + b_0(\mathrm{N2LO}) + b_0(\mathrm{N3LO}). \tag{7.29}$$

Their calculated values for the four choices of cut-off Λ_F are listed in Table 7.6.

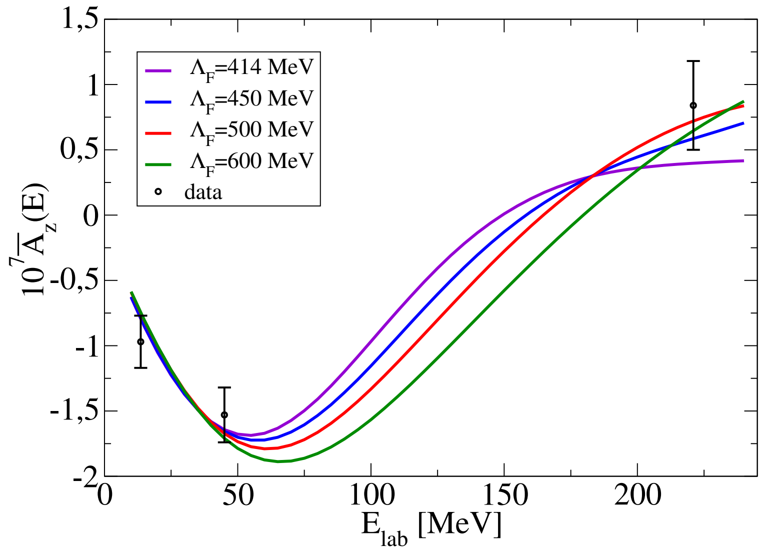

Figure 7.7: Longitudinal asymmetry calculated as $h\,a(E)$ (see text) as function of the energy in the laboratory frame compared with the experimental data. The calculations are performed for $\Lambda_F = 500$ MeV (red line), $\Lambda_F = 600$ MeV (green line), $\Lambda_F = 450$ MeV (blue line) and $\Lambda_F = 414$ MeV (purple line) using the values of h reported in Table 7.4.

As can be seen the N2LO and N3LO components of the potential gives smaller and smaller contributions. More in detail, in Table 7.7 we report the contribution of the OPE, TPE, OPE-RC and TPE-RC components of the PV potential. As shown the contributions of the N2LO component via the TPE and OPE-RC represent a correction of the 10% and 1%, respectively. The N3LO give a contribution via the TPE-RC of 0.1% as we expect from the chiral expansion. No contributions come from the c_4 term in this case since it is proportional to $\tau_{1z}+\tau_{2z}$ which is zero for an np system. Looking closely to Table 7.6, it is possible to see that the $\vec{n}p$ spin rotation is not sensitive to LECs C_4 and h_A^1 since they are both proportional to $\tau_{1z}+\tau_{2z}$; the small sensitivity to the LECs h_V^1, compared to the other LECs that appear at N3LO, is related to the fact that one of the contribution to b_7 comes from a term proportional to $\tau_{1z}+\tau_{2z}$. On the other hand there is a large sensitivity to C_5 and h_A^2, which multiply the isotensor term of the PV potential.

Let's discuss in more details the contributions that come from the N3LO terms. As we can see from the Table 7.6, the coefficients b_i with $i = 6,\ldots,10$ are of the same order of magnitude as b_0 (except for b_7 and b_9). These contributions comes from the N3LO component and so, from the chiral expansion, we expect that they would give a small correction, around 1% of the contribution given by the LO. However, this clearly should be true for the overall N3LO contribution, given by the second line in Eq. (7.28). The lack of experimental data does not

Λ_F [MeV]	b_0(LO)	b_0(N2LO)	b_0(N3LO)	b_0	b_1	b_2
414	1.18505	0.12026	0.00181	1.30712	0.24503	0.17358
450	1.21510	0.11560	0.00060	1.33131	0.25202	0.17617
500	1.23758	0.12239	0.00152	1.36149	0.24399	0.17391
600	1.18923	0.07379	0.01553	1.27854	0.23549	0.15845

b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
0.08978	0.00000	−0.90865	1.62707	−0.16967	1.82856	0.00000	4.32967
0.09993	0.00000	−0.93226	1.66419	−0.15664	1.87308	0.00000	4.43508
0.10537	0.00000	−0.90588	1.49517	−0.17195	1.69261	0.00000	4.00776
0.08464	0.00000	−0.86493	1.31925	−0.04357	1.52293	0.00000	3.60599

Table 7.6: Values of the coefficients b_i in units of Rad m^{-1} for the $\vec{n}p$ spin rotation along the z-axis calculated for the four choices of cutoff Λ_F at vanishing neutron beam energy. For b_0 we give explicitly the contribution of the different orders, the sum of the three contributions is given in fourth column.

Λ_F [MeV]	OPE	OPE-RC	TPE	TPE-RC
414	1.18505	−0.00636	0.12662	0.00181
450	1.21510	−0.00878	0.12438	0.00060
500	1.23758	−0.01202	0.13341	0.00152
600	1.18923	−0.00793	0.08172	0.01553

Table 7.7: Contributions given by the different term of the PV potential to the coefficient b_0 for the different choice of the cut-off Λ_F.

permit us to speculate about the magnitude of the N3LO terms. On the other hand, we have already seen how the corrections coming from the N2LO and N3LO to b_0(LO) are in line of what expected.

Let us consider the spin rotation along the y-axis, due to TV effects calculated using Eq. (6.109). As for the neutron spin rotation along the z-axis, the rotation angle along the y-axis is linearly dependent on the TV LECs,

$$\frac{\mathrm{d}\phi_y}{\mathrm{d}z} = \overline{g}_0^\theta d_0 + \overline{g}_1^\theta d_1 + \overline{\Delta}^\theta d_2 + \overline{C}_1^\theta d_3 + \overline{C}_2^\theta d_4 \,, \tag{7.30}$$

where d_i with $i = 1, \ldots, 5$ are numerical coefficients. Their values calculated with our TV potential at $E = 0.0001$ MeV for the four choices of the cut-off are reported in Table 7.8. As we can see the $\vec{n}\vec{p}$ spin rotation along the y-axis does not receive any contribution from the LECs $\overline{g}_1^\theta$ and $\overline{\Delta}^\theta$. The main contribution comes from the term of the potential proportional to $\overline{g}_0^\theta$ while the corrections of the contact terms are $\sim 5\%$ of the main contribution.

Λ_F [MeV]	d_0	d_1	d_2	d_3	d_4
414	−3.85569	0.00000	0.00000	0.25291	0.14987
450	−3.88891	0.00000	0.00000	0.25920	0.15438
500	−3.84718	0.00000	0.00000	0.25444	0.14262
600	−3.83276	0.00000	0.00000	0.23944	0.14688

Table 7.8: Values of the coefficients d_i in units of Rad m^{-1} for the $\vec{n}p$ spin rotations along the y-axis calculated for the four choices of cutoff Λ_F at vanishing neutron beam energy.

Using the estimates of the LECs given in Eqs. (5.46)-(5.47), we obtain the following results:

$$\left.\frac{\mathrm{d}\phi_y}{\mathrm{d}z}\right|_{\Lambda_F=414} = (-5.98 \pm 0.73)\cdot 10^{-2}\,\overline{\theta}\,\mathrm{Rad\,m^{-1}}, \tag{7.31}$$

$$\left.\frac{\mathrm{d}\phi_y}{\mathrm{d}z}\right|_{\Lambda_F=450} = (-6.02 \pm 0.74)\cdot 10^{-2}\,\overline{\theta}\,\mathrm{Rad\,m^{-1}}, \tag{7.32}$$

$$\left.\frac{\mathrm{d}\phi_y}{\mathrm{d}z}\right|_{\Lambda_F=500} = (-5.96 \pm 0.73)\cdot 10^{-2}\,\overline{\theta}\,\mathrm{Rad\,m^{-1}}, \tag{7.33}$$

$$\left.\frac{\mathrm{d}\phi_y}{\mathrm{d}z}\right|_{\Lambda_F=600} = (-5.94 \pm 0.73)\cdot 10^{-2}\,\overline{\theta}\,\mathrm{Rad\,m^{-1}}. \tag{7.34}$$

The contributions that come from the contact terms is smaller than the error that comes from the estimate of $\overline{g}_0^{\theta}$, for this reason we have neglected that correction. This observable is calculated for a very low energy and so, as discussed for the PV case, it depends only on the long range part of the potential which have a small dependence on Λ_F. For this reason the theoretical error due to the choice of Λ_F is much smaller than the error related to the estimate of $\overline{g}_0^{\theta}$.

A measurement of this observable could permit an estimate of the θ angle. But the value of $\overline{\theta}$ from the neutron EDMs is known to be $\lesssim 10^{-10}$ [6] and so we expect $\mathrm{d}\phi_y/\mathrm{d}z \lesssim 10^{-11}$, a value very difficult to be measured experimentally. Any measurement finding a larger value of $\mathrm{d}\phi_y/\mathrm{d}z$ would be a signal of BSM physics. However, this rotation is expected to be magnified for neutrons moving through other materials, due to the presence of close resonances of different parities [12]. Experiments of this type are currently under study [59].

Chapter 8

Conclusions

In this work we have derived the NN PV and TV potentials within the framework of an EFT based on the chiral symmetry (χEFT), with only pions and nucleons as degrees of freedom. The potentials are obtained as an expansion in terms of a small momentum or the pion mass. In particular the PV potential has been obtained at N3LO and the TV potential at NLO for the first time using the S-matrix method. Then these potentials have been used to study the longitudinal asymmetry A_z in $\vec{p}p$ scattering, the $\vec{n}p$ spin rotation, and the $\vec{n}\vec{p}$ spin rotation as TV observable.

First of all, we have introduced the concept of χEFT as the low energy theory of QCD, the underlying theory that describes the strong interaction. Then we have considered a PV Lagrangian density which describes the PV weak interactions among the nucleons, imposing the condition to have the same transformation properties under the chiral group as the PV part of the weak interaction in the Standard Model. Regarding the TV interactions between nucleons and pions we have taken into account the so-called θ-term as the only possible source of TV in the strong interactions sector within the SM (other possible TV terms have dimension 6 which however should be suppressed by a factor M_H^{-2}, where $M_H \geq 200$ TeV is an energy scale where the BSM physics should start to manifest [5]). The TV terms deriving from the θ-term can be included in the EFT using the external source method introduced in Ref. [5]. Starting form these Lagrangian interaction terms, we have studied the $NN \to NN$ transition amplitude using the field theoretic method and then, exploiting the Lippman-Schwinger equation, we have defined the "effective" PV and TV nuclear potentials.

We have now considered the two nucleon system, in particular their scattering states. In order to study scattering observables we have discussed the general formalism to solve the scattering problem between two particles. We have presented a numerical algorithm based on the Kohn variational principle for solving the problem exactly. In the numerical program we have implemented the derived PV potential with the aim of investigating the longitudinal asymmetry A_z in $\vec{p}p$

scattering and the $\vec{n}p$ spin rotation. The existence of only three experimental data for the A_z does not permit to fix the LECs of the potential but only to determine a linear correlation between them. The absence of data for the $\vec{n}p$ spin rotation does not allow to have new information. As discussed, new measurements of A_z would hardly provide more information about the LECs due to the similar energy dependence of the cofficients a_i. Only the study of other observables in systems with $A \geq 3$ could bring new information about the LECs: for example the $\vec{n}d$ spin rotation, the $\vec{n}^3$He longitudinal asymmetry or the $\vec{n}^4$He spin rotation. Calculation for these reactions are under way. Another future topic is to develop the PV electromagnetic nucleon-photon interaction, in order to study PV observables in $\vec{n}p$ radiative capture or in other electromagnetic processes.

We have then implemented the TV potential in our two nucleon code and we have studied the $\vec{n}\vec{p}$ spin rotation. Here the interest is to obtain information on the 5 LECs entering the NN TV potential. From the theory it is possible to relate these LECs to the $\overline{\theta}$ angle. The very small value of the $\overline{\theta}$ makes difficult the experimental measurement of this observable. However it represents a demonstration that other nuclear observables, over the EDMs, can be used to test the TV. Future goals will be the study of TV observables in medium-heavy nuclei where TV effects can be magnified by the presence of particular resonant states [12]. In this case clearly we need to implement our TV potential in nuclear shell model codes which allows reliable calculations of the structure of heavy nuclei [60]. We plan also to extend our study to TV observables in radiative capture and to compute with our potential the EDMs of light and heavy nuclei.

Appendix A

PV interactions terms

In this Appendix we will discuss the construction of the independent PV πN interactions terms used in this Thesis. In the first part we will recall the transformation properties of the building blocks discussed in Chapter 2 under Hermitian conjugation (H), parity (P) and charge conjugation (C). Then we will present some useful relation used to reduce the number of independent terms. In the last Section we will build the terms we need.

A.1 Transformation properties of the various field under P and C

We list the transformation properties under Hermitian conjugation (H), parity (P) and charge conjugation (C). For a generic combination O of fields, we have

$$\begin{aligned} O^\dagger &= s_H\, O \ , \\ O_{\mu_1\,\mu_2\,\dots} &\xrightarrow{P} s_P\, \sigma_{\mu_1}\, \sigma_{\mu_2} \cdots O_{\mu_1\,\mu_2\,\dots} \ , \\ O &\xrightarrow{C} s_C\, O^T \ , \end{aligned} \tag{A.1}$$

where s_H, s_P, and s_C are ± 1 phase factors, σ_μ is $+1$ when $\mu = 0$ (time-like) and -1 when $\mu = 1, 2, 3$ (space-like), and no summation is implied here over the repeated indices μ_i. The phase factors s_H, s_P, and s_C in the case of bilinears $O = \overline{N}\,\Gamma N$, where Γ is one of the elements of the Clifford algebra, are listed in Table A.1. When an operator also includes the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ as in $\epsilon^{\mu\nu\rho\sigma} O_{\mu\nu\rho\sigma}$, then $\epsilon^{\mu\nu\rho\sigma} O_{\mu\nu\rho\sigma} \xrightarrow{P} -s_P\, \epsilon^{\mu\nu\rho\sigma} O_{\mu\nu\rho\sigma}$ since the Lorentz indices μ, ν, ρ, and σ must be all different, and hence $\epsilon_{\mu\nu\rho\sigma}$ may be considered odd under parity. In reference to combinations of pion fields, we have under parity

$$u \xrightarrow{P} u^\dagger \ , \qquad u_\mu \xrightarrow{P} -\sigma_\mu u_\mu \ ,$$

	1	$i\,\gamma_5$	γ_μ	$\gamma_\mu\gamma_5$	$\sigma_{\mu\nu}$
s_H	$+$	$+$	$+$	$+$	$+$
s_P	$+$	$-$	$+$	$-$	$+$
s_C	$+$	$+$	$-$	$+$	$-$

Table A.1: Transformation properties of the elements of the Clifford algebra under H, P and C.

and

$$X^a_L \stackrel{P}{\longleftrightarrow} X^a_R\ , \qquad \chi \stackrel{P}{\longrightarrow} \chi^\dagger\ ,$$

and under charge conjugation

$$u \stackrel{C}{\longrightarrow} u^T\ , \qquad u_\mu \stackrel{C}{\longrightarrow} u^T_\mu\ , \tag{A.2}$$

$$X^2_L \stackrel{C}{\longrightarrow} -(X^2_R)^T, \quad X^2_R \stackrel{C}{\longrightarrow} -(X^2_L)^T\ , \tag{A.3}$$

$$X^{1,3}_L \stackrel{C}{\longrightarrow} (X^{1,3}_R)^T, \quad X^{1,3}_R \stackrel{C}{\longrightarrow} (X^{1,3}_L)^T\ , \tag{A.4}$$

$$\chi \stackrel{C}{\longrightarrow} \chi^T\ , \tag{A.5}$$

because $(\tau_2)^T = -\tau_2$. When considering terms involving O and the covariant derivative D_μ, it is convenient to introduce the combinations

$$\{D_\mu\,,\,O\} = D_\mu O + O D_\mu\ , \qquad [D_\mu\,,\,O] = D_\mu O - O D_\mu\ , \tag{A.6}$$

and determine how $\{D_\mu\,,\,\ldots\}$ and $[D_\mu\,,\,\ldots]$ transform under hermitian conjugation, P, and C independently of O. It is also usefull to introduce the combinations

$$X^a_\pm = X^a_L \pm X^a_R\ , \tag{A.7}$$

which transform in as simple way under C and P. The transfromation properties of the blocks discussed in this section is listed in Table A.2.

	u_μ	Γ_μ	$\{D_\mu\,,\,\ldots\}$	$[D_\mu\,,\,\ldots]$	X^a_+	X^a_-	χ_+	χ_-
s_H	$+$	$-$	$-$	$+$	$+$	$+$	$+$	$-$
s_P	$-$	$+$	$+$	$+$	$+$	$-$	$+$	$-$
s_C	$+$	$-$	$-$	$+$	$(-)^{a+1}$	$-(-)^{a+1}$	$+$	$+$

Table A.2: Transformation properties of the building blocks used to build the Lagrangian under H, P and C.

A.2 Useful relations

We list some relations we will use in the next section to reduce the number of PV tems. From the Lagrangian in Eqs. (2.62) and (2.63) we get the following equation of motion (EOM)

$$i\gamma^\mu D_\mu N = \Big(M + \frac{g_A}{2}\gamma_5\gamma^\mu u_\mu + \mathrm{c}_1\langle\chi_+\rangle + \cdots \Big) N, \tag{A.8}$$

where the dots indicate terms either proportional to the pion field of order Q^3 or more or proportional to higher power of the pion field at least of order Q. This notation will be used through all this Appendix. Other usefull relations are the pion EOM,

$$[D_\mu, u^\mu] = \frac{i}{2}\hat{\chi}_- + \mathcal{O}(Q^4) \ , \tag{A.9}$$

and the identities,

$$[D_\mu, D_\nu] = \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2}F^+_{\mu\nu} \ , \tag{A.10}$$

$$[D_\mu, u_\nu] - [D_\nu, u_\mu] = F^-_{\mu\nu} \ . \tag{A.11}$$

Note that covariant derivatives of u_μ can only appear in the symmetrized form

$$h_{\mu\nu} = [D_\mu, u_\nu] + [D_\nu, u_\mu] \ , \tag{A.12}$$

and that further simplifications follow from the Cayley-Hamilton relation, valid for any 2×2 matrices A and B,

$$AB + BA = A\langle B\rangle + B\langle A\rangle + \langle AB\rangle - \langle A\rangle\langle B\rangle \ , \tag{A.13}$$

and from the traceless property of u_μ and $X^a_{L/R}$. Care must be taken when constructing combinations of terms like $D_\mu X^a_{L/R}$, since they do not transform as given in Eqs. (2.88) and (2.89), see discussion in [25]. There it is also shown that it is convenient to work instead with the following quantities

$$(X^a_R)_\mu = \left[D_\mu + iu^\dagger r_\mu u \, , \, X^a_R\right] \ , \tag{A.14}$$

$$(X^a_L)_\mu = \left[D_\mu + iu\,\ell_\mu u^\dagger \, , \, X^a_L\right] \ . \tag{A.15}$$

These, in turn, reduce to

$$(X^a_R)_\mu = \frac{i}{2}\left[u_\mu \, , \, X^a_R\right] \ , \qquad (X^a_L)_\mu = -\frac{i}{2}\left[u_\mu \, , \, X^a_L\right] \ . \tag{A.16}$$

A.2.1 Properties of the γ matrices

The γ^μ matrices are in the standard form as given, for example, in Ref. [31]. They satisfy the following identities:

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] , \tag{A.17}$$

$$\gamma^5\sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta} , \tag{A.18}$$

$$i\,\sigma^{\mu\nu} = g^{\mu\nu} - \gamma^\mu\gamma^\nu , \tag{A.19}$$

$$\frac{1}{2}\{\sigma^{\mu\nu}, \gamma^\alpha\} = \epsilon^{\mu\nu\alpha\beta}\gamma^5\gamma_\beta , \tag{A.20}$$

$$\frac{1}{2}\left[\sigma^{\mu\nu}, \gamma^\alpha\right] = -i\,g^{\mu\alpha}\gamma^\nu + i\,g^{\nu\alpha}\gamma^\mu , \tag{A.21}$$

$$\sigma^{\mu\nu}\gamma^\alpha = \epsilon^{\mu\nu\alpha\beta}\gamma^5\gamma_\beta - i\,g^{\mu\alpha}\gamma^\nu + i\,g^{\nu\alpha}\gamma^\mu , \tag{A.22}$$

$$\gamma^\alpha\sigma^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\gamma^5\gamma_\beta + i\,g^{\mu\alpha}\gamma^\nu - i\,g^{\nu\alpha}\gamma^\mu . \tag{A.23}$$

A.3 Independent interaction terms

The terms of order Q^0 and of order Q are simply derived using the transformation properties of the building blocks and are given in Eqs. (2.91) and (2.92). In this Section we limit our discussion to the πN interaction term type of order Q^2 and Q^3. Terms contributing to the vertices $\pi\pi NN$ terms of order Q^2 or higher are not consider because they are not necessary to the present work. Also, we will not consider terms that contain $F_\pm^{\mu\nu}$. Moreover, the following power counting is assumed

$$u_\mu \sim Q , \qquad \chi_\pm \sim Q^2 . \tag{A.24}$$

The covariant derivative D_μ is taken as of order Q, except when it acts on a nucleon field, in which case it is of order Q^0 due to the presence of the heavy mass scale. We can also develop the building blocks in terms of pion fields (see Appendix B)

$$\chi_+ \sim \pi^0 + \pi^2 + \mathcal{O}(\pi^4) \qquad \chi_- \sim \pi + \mathcal{O}(\pi^3) , \tag{A.25}$$

$$X_+^a \sim \pi^0 + \pi^2 + \mathcal{O}(\pi^4) \qquad X_-^a \sim \pi + \mathcal{O}(\pi^3) , \tag{A.26}$$

$$u_\mu \sim \pi + \mathcal{O}(\pi^3) . \tag{A.27}$$

We will exclude from our discussions term like $\sim u_\mu u_\nu$ because they contain at least terms quadratic in the pion field of order Q^2. The (independent) isoscalar (ΔI=0), isovector (ΔI=1), and isotensor (ΔI=2) interaction terms are constructed in the next three subsections.

A.3.1 The $\Delta I = 0$ sector

1. *Terms with* $\chi_\pm$. These are already of order Q^2, so at this order we can not build new P-odd and C-odd terms. We have two ways to obtain terms of order Q^3: adding derivatives or multiply by u_μ. The terms we can build are

$$\begin{aligned} &\bar{N}i\{D_\mu\,,\,\chi_+\}\gamma^\mu\gamma^5 N\ ,\\ &\bar{N}i[D_\mu\,,\,\chi_-]\gamma^\mu N\ ,\\ &\bar{N}i\{u_\mu\,,\,\chi_+\}\gamma^\mu N\ ,\\ &\bar{N}i[u_\mu\,,\,\chi_-]\gamma^\mu\gamma^5 N\ , \end{aligned} \tag{A.28}$$

   The first, the second and the last combinations contain only quadratic with terms in the pion field of at least order Q^3 and so we can neglect them. The third combination, using Eq. (2.65) and the fact that the traceless part of χ_+ is zero (if we consider only non isospin-violating terms) reduces to the term proportional to $\tilde{h}_1^0$ given in Eq. (2.94). Additional terms must involve more derivatives and can be reduced via the EOM.

2. *Terms with a single* u_μ *plus one or more* D^μ*'s.* With a single D^μ we can form the combinations $\bar{N}\{D^\mu\,,\,u_\mu\}N$ and $\bar{N}[D_\mu\,,\,u_\nu]\sigma^{\mu\nu}N$. Using the EOM up to order Q^2—see Eq. (A.8)—the first expression can be reduced to a combination of terms proportional to h_V^0 and $\tilde{h}_1^0$ (Eq. (2.92) and (2.94)) (ignoring terms of order Q^4 and that contains $F_-^{\mu\nu}$). The second expression is seen to be identical to twice the sum of the terms proportional to h_V^0 and $\tilde{h}_1^0$. Terms with two or more D^μ's can be reduced using the EOM. In general, each $i\not{D}\psi$ gives a term $M\psi$ plus terms of order Q proportional to $\not{u}$ and terms of order Q^2 proportional to $c_1\langle\chi_+\rangle$. Terms with the nucleon mass are found to be proportional to those without covariant derivatives, which have already been accounted for, while terms with the additional u_μ are at least quadratic in the pion fields and thus neglected. Therefore, at order Q^3, no new (independent) terms with a single u_μ and one or more D^μ's appear.

A.3.2 The $\Delta I = 1$ sector

1. *Terms with* $\chi_\pm$. We can combine these quantities with $X_\pm^3$ to form the following P-odd and C-odd combinations

$$\bar{N}\{\chi_+\,,\,X_-^3\}N\ ,\quad \bar{N}\,[\chi_-\,,\,X_+^3]N\ . \tag{A.29}$$

As per the isospin structure, for each of these terms one needs to consider the following possibilities:

$$\bar{N}_t A_{tt'} B_{t't''} N_{t''} \ , \quad \bar{N}_t A_{tt'} B_{t''t''} N_{t'} \ , \quad \bar{N}_t A_{t't'} B_{tt''} N_{t''} \ ,$$
$$\bar{N}_t A_{t't''} B_{t''t'} N_t \ , \quad \bar{N}_t A_{t't'} B_{t''t''} N_t \ ,$$

where A and B denote schematically the various pairs of isospin matrices corresponding to $\chi_+ X^3_-$ (or $X^3_- \chi_+$) and so on. Obviously, if both A and B are traceless, only the first and the fourth are non vanishing. Recall that $\langle u_\mu \rangle = \langle X^a_\pm \rangle = \langle \chi_- \rangle = 0$. The other quantities ($\chi_+$ and D_μ) are conveniently written as $A = \hat{A} + \langle A \rangle I/2$ with $\hat{A}$ traceless. A number of manipulations allow one to express the terms in Eq. (A.29) as the combinations of terms proportional to h^1_1 and h^1_3 in Eq. (2.93) plus other terms that we can safely neglect. We can also add a term u_μ which give the combination

$$\bar{N}\{X^3_+ \, , \, u_\mu\}\gamma^\mu N \langle \chi_+ \rangle, \tag{A.30}$$

all the other possible combinations of these three elements can be reduced to this one. Combinations of $\chi_\pm$ with one or more D_α's (in the form $\{D_\alpha \, , \, \ldots\}$ or $[D_\alpha \, , \, \ldots]$) can be eliminated using the EOM.

2. *Terms with a single u_μ and one or more D_μ's.* First consider terms with the anticommutator of the type $\bar{N}\{D_\mu, \{X^3_\pm, u_\nu\}\} \ldots N$, which involve a D_μ acting on the nucleon fields $\bar{N}$ or N. These terms can always be reduced via the EOM to one of the terms of order Q proportional to h^1_V or h^1_A given in Eq. (2.92) plus a term $\sim u_\mu u_\nu$, which we neglect. Next, we consider terms with the commutator of type $[D_\mu, X^3_{L/R}]$ or $[D_\mu, u_\nu]$. As discussed above, combinations of D_μ with $X^a_{L/R}$ must be included via $(X^a_{L/R})_\mu$ defined in Eqs. (A.14)–(A.15). However, by using the identities (A.16), combinations with a single u_μ and a $(X^a_{R/L})_\nu$ reduce to terms $\propto u_\mu u_\nu X^a_{R/L}$ discarded here. Turning our attention to terms including a commutator $[D_\mu, u_\nu]$, we note that, since $[D_\mu, u_\nu] - [D_\nu, u_\mu] = F^-_{\mu\nu}$, we only need to consider operators involving $h_{\mu\nu}$, as defined in Eq. (A.12), which is odd under P and even under C. In combination with $X^3_\pm$ we can form the 4 operators listed in Table A.3, along with their transformation properties under P and C. Note that $h_{\mu\nu}$ is of order $\mathcal{O}(Q^2)$. Since $h_{\mu\nu} = h_{\nu\mu}$, without any additional covariant derivatives we can construct the terms:

$$\bar{N} i [h_{\mu\nu}, X^3_+] g^{\mu\nu} N \ , \qquad \bar{N}\{h_{\mu\nu}, X^3_-\} g^{\mu\nu} \gamma^5 N \ . \tag{A.31}$$

However, using the pion EOM in Eq. (A.9), these terms are the same, up to additional terms of order $\mathcal{O}(Q^4)$, as those constructed with χ_- (see

	$\{h_{\mu\nu}, X^3_+\}$	$i[h_{\mu\nu}, X^3_+]$	$\{h_{\mu\nu}, X^3_-\}$	$i[h_{\mu\nu}, X^3_-]$
s_H	+	+	+	+
s_P	−	−	+	+
s_C	+	−	−	+

Table A.3: Transformation properties under hermitian conjugation (H), parity (P), and charge conjugation (C) of quantities constructed in terms of $h_{\mu\nu}$.

point 1 above). Operators with $h_{\mu\nu}$ and an additional covariant derivative enter only in combinations with $\{D_\mu, \ldots\}$ or $[D_\mu, \ldots]$. These combinations give the terms proportional to h^1_{12} plus other we neglect. Terms with more derivatives can be reduced via the EOM to terms already taken into account.

A.3.3 The $\Delta I = 2$ sector

1. *Terms with* $\chi_\pm$. We can form the following P-odd and C-odd combinations

$$\begin{aligned} &\mathcal{I}_{ab}\bar{N}\Big(X^a_R\chi_+X^b_R - X^a_L\chi_+X^b_L\Big)N \ , \\ &\mathcal{I}_{ab}\bar{N}\Big(X^a_R\chi_-X^b_R - X^a_L\chi_-X^b_L\Big)i\gamma^5 N \ . \end{aligned} \tag{A.32}$$

The second combination is of order Q^3 but developing it in term of the pion field it doesn't give any term linear in the pion field. The remaning term contains only terms quadratic in the pion field. We can add u_μ in order to reach the order Q^3; the only indipendent term that gives contribution is

$$\mathcal{I}_{ab}\bar{N}\Big(X^a_R u_\mu X^b_R - X^a_L u_\mu X^b_L\Big)N\langle\chi_+\rangle \tag{A.33}$$

As for the $\Delta I = 1$ case, combinations of $\chi_\pm$ with one or more operators D_α (in the form $\{D_\alpha, \ldots\}$ and $[D_\alpha, \ldots]$) can be eliminated using the EOM.

2. *Terms with a single* u_μ *and one or more* D_μ*'s.* First, we consider combinations with the anticommutator like $\bar{N}\,\mathcal{I}_{ab}\{D_\mu, X^a_R u_\nu X^b_R \pm (L \longrightarrow R)\}\cdots N$, namely with D_μ acting on the nucleon fields $\bar{N}$ or N. Using the EOM, these can always be reduced to one of the order Q terms given in Eq. (2.92), terms involving $u_\mu u_\nu$ which we don't consider, and terms with the commutator of D_μ, as shown below. For example, for

$$O_\nu = \mathcal{I}_{ab}(X^a_R u_\nu X^b_R + X^a_L u_\nu X^b_L) \ , \tag{A.34}$$
$$\overline{O}_\nu = \mathcal{I}_{ab}(X^a_R u_\nu X^b_R - X^a_L u_\nu X^b_L) \ , \tag{A.35}$$

we have, respectively

$$
\begin{aligned}
&\bar{N}\{D_\mu\,,\,O_\nu\}g^{\mu\nu}N\\
&\quad= \; -2iMO^{(2)}_{2V} + \bar{N}\Big[D_\mu\,,\,O_\nu\Big]i\sigma^{\mu\nu}N\\
&\qquad +(\text{terms with } u_\mu, u_\nu)\ ,\\
&\bar{N}\{D_\mu\,,\,\overline{O}_\nu\}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}N\\
&\quad= \; 4iMO^{(2)}_{2A} - 2\bar{N}\Big[D_\mu\,,\,\overline{O}^\mu\Big]N\\
&\qquad +(\text{terms with } u_\mu, u_\nu)\ ,
\end{aligned}
$$

where the operators $O^{(2)}_{2V}$ and $O^{(2)}_{2A}$ are the terms proportional to h_V^2 and h_A^2 given in Eqs. (2.92).

Next, we consider the terms with the commutator of type $[D_\mu, X^3_{L/R}]$. As discussed previously, combinations of D_μ with $X^a_{L/R}$ must be included via $(X^a_{L/R})_\mu$. However, by using the identities (A.14) and (A.15), terms with a single u_μ and a $(X^a_{R/L})_\nu$ are $\propto u_\mu u_\nu X^a_{R/L}$, and therefore neglected. Turning our attention to terms including a commutator $[D_\mu, u_\nu]$, we note that, since $[D_\mu, u^\mu] = (i/2)\hat{\chi}_- + \mathcal{O}(Q^4)$ and $[D_\mu, u_\nu] - [D_\nu, u_\mu] = F^-_{\mu\nu}$,. we need only consider operators involving $h_{\mu\nu}$. We can form two combinations:

$$\bar{N}\,\mathcal{I}_{ab}\Big(X^a_R h_{\mu\nu} X^b_R + X^a_L h_{\mu\nu} X^b_L\Big)\sigma^{\mu\nu}N\ , \tag{A.36}$$

$$\bar{N}\,\mathcal{I}_{ab}\Big(X^a_R h_{\mu\nu} X^b_R - X^a_L h_{\mu\nu} X^b_L\Big)g^{\mu\nu}i\gamma^5 N\ , \tag{A.37}$$

which both can be disregarded. Combinations with additional D_α's don't give contributions.

A.4 Contact terms

The contact terms are products of a pair of bilinears of nucleon fields, which are odd under P and even under CP. We must build isoscalar, isovector and isotensor terms as discussed in Section 2.5. The operators moreover have to conserve the electric charge, so namely they must commute with the third component of the isospin operator. The most general bilinear product reads

$$\widetilde{O}^k_{AB} = \sum_{i,j=0}^{4} F^k_{ij}(\overline{N}\,\tau_i\Gamma_A\,N)\,(\overline{N}\,\tau_j\Gamma_B\,N)\ , \tag{A.38}$$

where Γ_A and Γ_B are elements of the Clifford's algebra with the possible addition of 4-gradients. There are 6 possible choice for the coefficents F^k_{ij}, as discussed in Table A.4. In Ref. [58], 58 operators that can contribute to order Q are listed.

k	F^k_{ij}	Operatore NN	$\mathcal{C}$
1	$\delta_{i,0}\delta_{j,0}$	1	+
2	$\delta_{i,j} - \delta_{i,0}\delta_{j,0}$	$\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$	+
3	$\delta_{i,3}\delta_{j,0} + \delta_{i,0}\delta_{j,3}$	$\tau_{1z} + \tau_{2z}$	+
4	$\delta_{i,3}\delta_{j,0} - \delta_{i,0}\delta_{j,3}$	$\tau_{1z} - \tau_{2z}$	+
5	$\delta_{i,1}\delta_{j,1} + \delta_{i,2}\delta_{j,2} - 2\delta_{i,3}\delta_{j,3}$	$\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 - 3\tau_{1z}\tau_{2z}$	+
6	$i\left[\delta_{i,1}\delta_{j,2} - \delta_{i,2}\delta_{j,1}\right]$	$i\left[\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2\right]_z$	−

Table A.4: Possible combinations for the coefficents F^k_{ij}. In the third coloumn the corresponding operator which would contribute to the NN potential. In the last coloumn their transformation properties under charge conjugation is reported.

Analyzing the NR limit and the respective vertex functions it is possible using the Fierz transformations for the Pauli's matrices:

$$\begin{aligned}
(1)_{s_1',s_1}(1)_{s_2',s_2} &= \frac{1}{2}(1)_{s_2',s_1}(1)_{s_1',s_2} + \frac{1}{2}(\boldsymbol{\sigma})_{s_2',s_1}\cdot(\boldsymbol{\sigma})_{s_1',s_2}\ , \\
(\boldsymbol{\sigma})_{s_1',s_1}\cdot(\boldsymbol{\sigma})_{s_2',s_2} &= \frac{3}{2}(1)_{s_2',s_1}(1)_{s_1',s_2} - \frac{1}{2}(\boldsymbol{\sigma})_{s_2',s_1}\cdot(\boldsymbol{\sigma})_{s_1',s_2}\ , \\
(\sigma_a)_{s_1',s_1}(1)_{s_2',s_2} &= \frac{1}{2}(\sigma_a)_{s_2',s_1}(1)_{s_1',s_2} + \frac{1}{2}(1)_{s_2',s_1}(\sigma_a)_{s_1',s_2} \\
&\quad + \frac{i}{2}\epsilon_{abc}(\sigma_b)_{s_2',s_1}(\sigma_c)_{s_1',s_2}\ , \\
(1)_{s_1',s_1}(\sigma_a)_{s_2',s_2} &= \frac{1}{2}(\sigma_a)_{s_2',s_1}(1)_{s_1',s_2} + \frac{1}{2}(1)_{s_2',s_1}(\sigma_a)_{s_1',s_2} \\
&\quad - \frac{i}{2}\epsilon_{abc}(\sigma_b)_{s_2',s_1}(\sigma_c)_{s_1',s_2}\ , \\
(\sigma_a)_{s_1',s_1}(\sigma_b)_{s_2',s_2} &= \frac{1}{2}\delta_{ab}(1)_{s_2',s_1}(1)_{s_1',s_2} - \frac{i}{2}\epsilon_{abc}\left[(\sigma_c)_{s_2',s_1}(1)_{s_1',s_2} - (1)_{s_2',s_1}(\sigma_c)_{s_1',s_2}\right] \\
&\quad + \frac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{dc} + \delta_{ad}\delta_{bc})(\sigma_c)_{s_2',s_1}(\sigma_d)_{s_1',s_2}\ ,
\end{aligned} \tag{A.39}$$

to reduce to 5 the number of indipendent contact terms at order Q. So the most general vertex function is the following (alredy reported in NR limit):

$$\begin{aligned}
{}^{PV}M^{00}_{\alpha_1'\alpha_1\alpha_2'\alpha_2} &= \frac{1}{2\Lambda_\chi^2 f_\pi}\Big[C_1(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k} \\
&+ C_2\,\vec{\tau}_1\cdot\vec{\tau}_2\,(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k} \\
&+ C_3\,(\vec{\tau}_1\times\vec{\tau}_2)_z\,(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k} \\
&+ C_4\,(\tau_{1z}+\tau_{2z})\,(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k} \\
&+ C_5\,\mathcal{I}^{ab}\,\tau_{1a}\,\tau_{2b}\,(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}\,\Big]\ ,
\end{aligned} \tag{A.40}$$

where $\Lambda_\chi = 4\pi f_\pi$. We have chosen a definition of the LECs C_i, $i = 1, \ldots, 5$ in such a way they are adimensional.

Appendix B

Development in terms of pion fields

In this section we report the expansion of the building blocks u, u_μ, Γ_μ and $X^a_{L/R}$, $X^a_\pm$ and of the various Lagrangian vertices in term of pion fields. Remembering Eq. (2.28) we have:

$$
\begin{aligned}
u &\longrightarrow 1+\frac{i}{2f_\pi}\vec{\tau}\cdot\vec{\pi}-\frac{1}{8f_\pi^2}\pi^2+\frac{i(8\alpha-1)}{16f_\pi^3}\pi^2\,\vec{\tau}\cdot\vec{\pi}+\mathcal{O}(\pi^4), &\text{(B.1)}\\
u_\mu &\longrightarrow -\frac{1}{f_\pi}\vec{\tau}\cdot\partial_\mu\vec{\pi}+\mathcal{O}(\pi^3), &\text{(B.2)}\\
\Gamma_\mu &\longrightarrow \frac{1}{4f_\pi^2}(\vec{\tau}\times\vec{\pi})\cdot\partial_\mu\vec{\pi}, &\text{(B.3)}\\
X^a_R &\longrightarrow \tau^a-\frac{1}{f_\pi}(\vec{\pi}\times\vec{\tau})^a+\frac{1}{2f_\pi^2}\pi^a(\vec{\pi}\cdot\vec{\tau}-\tau^a\pi^2)+\mathcal{O}(\pi^3), &\text{(B.4)}\\
X^a_L &\longrightarrow \tau^a+\frac{1}{f_\pi}(\vec{\pi}\times\vec{\tau})^a+\frac{1}{2f_\pi^2}\pi^a(\vec{\pi}\cdot\vec{\tau}-\tau^a\pi^2)+\mathcal{O}(\pi^3), &\text{(B.5)}\\
X^a_- &\longrightarrow \frac{2}{f_\pi}(\vec{\pi}\times\vec{\tau})^a+\mathcal{O}(\pi^3), &\text{(B.6)}\\
X^a_+ &\longrightarrow 2\tau^a+\frac{1}{f_\pi^2}\pi^a(\vec{\pi}\cdot\vec{\tau}-\tau^a\pi^2)+\mathcal{O}(\pi^4). &\text{(B.7)}
\end{aligned}
$$

For $\chi_\pm$ we report the complete expansion in the pion field considering

$$\chi = 2B(s_0+s_3\tau_3+ip_0+ip_3\tau_3), \qquad \text{(B.8)}$$

explicitly,

$$
\begin{aligned}
\chi_+ = {} & 2B\Big[s_0\Big(2-\frac{\pi^2}{f_\pi^2}\Big)+\frac{2p_0}{f_\pi}(\vec{\tau}\cdot\vec{\pi})\Big(1-\frac{\alpha\pi^2}{f_\pi^2}\Big)\\
&+s_3\Big(2\tau_3-\frac{\pi_3(\vec{\tau}\cdot\vec{\pi})}{f_\pi^2}\Big)+\frac{2p_3}{f_\pi}\tau_3\Big(1-\frac{\alpha\pi^2}{f_\pi^2}\Big)\Big]+\mathcal{O}(\pi^4) \qquad \text{(B.9)}
\end{aligned}
$$

$$\chi_- = 4iB\Big[-\frac{s_0}{f_\pi}(\vec{\tau}\cdot\vec{\pi})\Big(1-\frac{\alpha\pi^2}{f_\pi^2}\Big) + p_0\Big(1-\frac{\pi^2}{2f_\pi^2}\Big)$$
$$-\frac{s_3}{f_\pi}\pi_3\Big(1-\frac{\alpha\pi^2}{f_\pi^2}\Big) + p_3\Big(\tau_3 - \frac{\pi_3(\vec{\tau}\cdot\vec{\pi})}{2f_\pi^2}\Big)\Big] + \mathcal{O}(\pi^4) \;, \qquad (B.10)$$
$$(B.11)$$

where α represents the arbitrariness in the choice of the $U(x)$ field, following the definition given in Eq. (2.28). Let's remember that in the study of the PV interaction we have $s_3 = p_0 = p_3 = 0$ and that $2Bs_0 \simeq m_\pi^2$. In the next two subsections we will report the Lagrangian vertices expanded in terms of the pion field we need to build the PV potential. The TV terms will be reported in Section 5.3.

B.0.1 PC Lagrangian in pion fields

Taking into account only the interactions terms and substituting the building blocks in Eqs. (2.62)–(2.64) we obtain

$$\mathcal{L}^{(1)}_{\pi N} = -\frac{g_A}{2f_\pi}\overline{N}(\vec{\tau}\cdot\partial_\mu\vec{\pi})\gamma^\mu\gamma^5 N + \ldots, \qquad (B.12)$$
$$\mathcal{L}^{(3)}_{\pi N} = \frac{d_{16}}{2f_\pi}\overline{N}(\vec{\tau}\cdot\partial_\mu\vec{\pi})\gamma^\mu\gamma^5 N$$
$$+\frac{d_{18}m_\pi^2}{f_\pi}\overline{N}(\vec{\tau}\cdot\partial_\mu\vec{\pi})\gamma^\mu\gamma^5 N + \ldots, \qquad (B.13)$$
$$\mathcal{L}^{(1)}_{\pi\pi N} = \frac{i}{4f_\pi^2}\overline{N}(\partial_\mu\vec{\pi})\cdot(\vec{\tau}\times\vec{\pi})\gamma^\mu N + \ldots, \qquad (B.14)$$
$$\mathcal{L}^{(2)}_{\pi\pi N} = -c_1\frac{2m_\pi^2}{f_\pi^2}\overline{N}\pi^2 N + \frac{c_2}{f_\pi^2}\overline{N}(\partial_0\vec{\pi}\partial_0\vec{\pi})N$$
$$+\frac{c_3}{f_\pi^2}\overline{N}(\partial_\mu\vec{\pi}\partial^\mu\vec{\pi})N$$
$$-\frac{c_4}{2f_\pi^2}\overline{N}\vec{\tau}\cdot(\partial_\mu\vec{\tau}\times\partial_\nu\vec{\tau})\sigma^{\mu\nu}N + \ldots. \qquad (B.15)$$
$$(B.16)$$

B.0.2 PV Lagrangian in pion fields

Performing the substitution in Eqs. (2.91)–(2.95) we get

$$\mathcal{L}^{PV\,(0)}_{\pi N} = \frac{h^1_\pi}{\sqrt{2}}\overline{N}(\vec{\pi}\times\vec{\tau})_3 N + \ldots, \qquad (B.17)$$
$$\mathcal{L}^{PV\,(1)}_{\pi N} = -\frac{h^0_V}{2f_\pi}\overline{N}(\vec{\tau}\cdot\partial_\mu\vec{\pi})\gamma^\mu N - \frac{h^2_V}{f_\pi}\overline{N}(\partial_\mu\pi_3)\gamma^\mu N$$

$$+\frac{2h_V^2}{3f_\pi}\mathcal{I}^{ab}\overline{N}\tau_a(\partial_\mu\pi_b)\gamma^\mu N+\ldots \quad \text{(B.18)}$$

$$\mathcal{L}_{\pi N}^{PV\,(2)} = -\frac{8h_2^1}{f_\pi^2}\overline{N}(\vec{\pi}\times\vec{\tau})_3 N-\frac{8h_2^1}{f_\pi^2}\overline{N}(\vec{\pi}\times\vec{\tau})_3 N$$
$$+\frac{16h_{12}^1}{f_\pi^2 M}\overline{N}(\vec{\tau}\times(\partial_\mu\partial_\nu\vec{\pi}))_3\gamma^\mu\partial^\nu N+\ldots, \quad \text{(B.19)}$$

$$\mathcal{L}_{\pi N}^{PV\,(3)} = +\frac{4m_\pi^2\tilde{h}_1^0}{f_\pi^3}\overline{N}(\vec{\tau}\cdot\partial_\mu\vec{\pi})\gamma^\mu N+\frac{8m_\pi^2\tilde{h}_1^1}{f_\pi^3}\overline{N}(\partial_\mu\pi_3)\gamma^\mu N$$
$$-\frac{16m_\pi^2\tilde{h}_1^2}{3f_\pi^3}\mathcal{I}^{ab}\overline{N}\tau_a(\partial_\mu\pi_b)\gamma^\mu N+\ldots \quad \text{(B.20)}$$

$$\mathcal{L}_{\pi\pi N}^{PV\,(1)} = \frac{h_A^1}{f_\pi^2}\overline{N}(\vec{\pi}\times\partial_\mu\vec{\pi})_3\gamma^\mu\gamma^5 N$$
$$-\frac{h_A^2}{3f_\pi^2}\mathcal{I}^{ab}\overline{N}(\vec{\pi}\times\partial_\mu\vec{\pi})_a\tau_b\gamma^\mu N+\ldots \quad \text{(B.21)}$$

$$\mathcal{L}_{3\pi}^{PV\,(2)} = -\frac{16h_{3\pi}^2}{f_p i}\mathcal{I}^{ab}(\partial_\mu\pi)_b(\vec{\pi}\times\partial_\mu\vec{\pi})_a+\ldots\ , \quad \text{(B.22)}$$

where in the last expression the three pions exchange term proportional to $h_{3\pi}^1$ vanishes.

Appendix C

Interaction vertices

In this Appendix we will report the explicit forms of the vertex functions. When we substitute the explicit expression of the nucleons and pions fields as given in Eqs. (3.24)–(3.33), in the various Hamiltonian terms reported in Eqs. (3.18)–(3.19), the creation and annihilation operators are usually not normal ordered. Therefore, after normal ordering them, tadpole type contributions result, which can be relevant when discussing renormalization. In this chapter we only display the vertex functions used in Chapter 4 and 5 for the derivation of the potentials.

C.1 Bilinears

The vertex functions involve the bilinears

$$\frac{\overline{u}(\boldsymbol{p}', s')}{\sqrt{2E_{\boldsymbol{p}'}}}\Gamma\frac{u(\boldsymbol{p}, s)}{\sqrt{2E_{\boldsymbol{p}}}} = \chi^\dagger_{s'} B(\Gamma)_{\alpha'\alpha}\chi_s \ , \tag{C.1}$$

where Γ denotes generically an element of the Clifford algebra and χ_s, $\chi_{s'}$ are spin states. These bilinears are expanded non-relativistically in powers of momenta, and terms up to order Q^3 are included. We obtain (subscripts are suppressed for brevity):

$$B(1) = 1 - \frac{F^{(2)}(1)}{4M^2} \ , \tag{C.2}$$

$$B(i\gamma^5) = -\frac{i\boldsymbol{\sigma}\cdot\boldsymbol{k}}{2M} + \frac{F^{(3)}(\gamma^5)}{16M^3} \ , \tag{C.3}$$

$$B(\gamma^0) = 1 + \frac{F^{(2)}(\gamma^0)}{4M^2} \ , \tag{C.4}$$

$$\boldsymbol{B}(\boldsymbol{\gamma}) = \frac{2\,\boldsymbol{K} - i\,\boldsymbol{k}\times\boldsymbol{\sigma}}{2M} - \frac{\boldsymbol{G}^{(3)}(\boldsymbol{\gamma})}{16M^3} \ , \tag{C.5}$$

$$B(\gamma^0\gamma^5) = \frac{\boldsymbol{K}\cdot\boldsymbol{\sigma}}{M} - \frac{F^{(3)}(\gamma^0\gamma^5)}{8M^3} \ , \tag{C.6}$$

$$\begin{aligned}
\boldsymbol{B}(\boldsymbol{\gamma}\gamma^5) &= \boldsymbol{\sigma} + \frac{\boldsymbol{G}^{(2)}(\boldsymbol{\gamma}\gamma^5)}{4M^2} , && \text{(C.7)}\\
\boldsymbol{B}(\sigma^{0i}) &= -\frac{i\,\boldsymbol{k} + 2\boldsymbol{K}\times\boldsymbol{\sigma}}{2M} + \frac{\boldsymbol{G}^{(3)}(\sigma^{0i})}{16M^3} , && \text{(C.8)}\\
B(\sigma^{ij}) &= \epsilon_{ijl}\Big[\sigma_l - \frac{G_l^{(2)}(\sigma^{ij})}{4M^2}\Big], && \text{(C.9)}
\end{aligned}$$

where $F^{(n)}(\Gamma)$ and $\boldsymbol{G}^{(n)}(\Gamma)$ are, respectively, scalar and vector quantities of order Q^n, explicitly given by

$$\begin{aligned}
F^{(2)}(1) &= 2K^2 + i(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma} , && \text{(C.10)}\\
F^{(3)}(\gamma^5) &= i\big[\boldsymbol{\sigma}\cdot\boldsymbol{k}\,(4K^2+k^2) \\
&\qquad \times + 4\,\boldsymbol{\sigma}\cdot\boldsymbol{K}\,\boldsymbol{k}\cdot\boldsymbol{K}\big] , && \text{(C.11)}\\
F^{(2)}(\gamma^0) &= -k^2/2 + i(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma} , && \text{(C.12)}\\
\boldsymbol{G}^{(3)}(\boldsymbol{\gamma}) &= (2\boldsymbol{K} - i\boldsymbol{k}\times\boldsymbol{\sigma})\left(4K^2+k^2\right) \\
&\quad +2\,(\boldsymbol{k} - 2i\boldsymbol{K}\times\boldsymbol{\sigma})\;\boldsymbol{K}\cdot\boldsymbol{k} , && \text{(C.13)}\\
F^{(3)}(\gamma^0\gamma^5) &= \boldsymbol{k}\cdot\boldsymbol{\sigma}\,\boldsymbol{k}\cdot\boldsymbol{K} + \boldsymbol{K}\cdot\boldsymbol{\sigma}(4K^2+k^2) , && \text{(C.14)}\\
\boldsymbol{G}^{(2)}(\boldsymbol{\gamma}\gamma^5) &= 2\;(\boldsymbol{K}\cdot\boldsymbol{\sigma})\;\boldsymbol{K} - (\boldsymbol{k}\cdot\boldsymbol{\sigma})\;\boldsymbol{k}/2 \\
&\quad -2K^2\boldsymbol{\sigma} - i\,\boldsymbol{k}\times\boldsymbol{K} , && \text{(C.15)}\\
\boldsymbol{G}^{(3)}(\sigma^{0i}) &= (i\boldsymbol{k} + 2\boldsymbol{K}\times\boldsymbol{\sigma})\left(4K^2+k^2\right) \\
&\quad +2\,(2i\boldsymbol{K} + \boldsymbol{k}\times\boldsymbol{\sigma})\;\boldsymbol{K}\cdot\boldsymbol{k} , && \text{(C.16)}\\
\boldsymbol{G}^{(2)}(\sigma^{ij}) &= 2\,(\boldsymbol{K}\cdot\boldsymbol{\sigma})\,\boldsymbol{K} - (\boldsymbol{k}\cdot\boldsymbol{\sigma})\,\boldsymbol{k}/2 + \boldsymbol{\sigma}\,k^2/2 \\
&\quad -i\,\boldsymbol{k}\times\boldsymbol{K} , && \text{(C.17)}
\end{aligned}$$

with the momenta $\boldsymbol{K} = (\boldsymbol{p}' + \boldsymbol{p})/2$ and $\boldsymbol{k} = \boldsymbol{p}' - \boldsymbol{p}$. We also expand K^0 and $K_\mu K^\mu$ as

$$\begin{aligned}
K^0 &= \frac{E+E'}{2} \to M\left(1 + \frac{2K^2 + k^2/2}{4M^2}\right) , && \text{(C.18)}\\
K_\mu K^\mu &= (K^0)^2 - K^2 \to M^2\left(1 + \frac{k^2}{4M^2}\right) . && \text{(C.19)}
\end{aligned}$$

Note that in the power counting of these vertices below, we do not include the $1/\sqrt{\omega_k}$ normalization factors present in the pion fields.

C.2 PC vertex functions

The LO PC interaction term (of order Q) in Eq. (B.12) contains the following vertex functions:

$$ {}^{PC}M^{01}_{\alpha'\alpha,\boldsymbol{q}\,a} = -i\frac{g_A}{2f_\pi}\frac{\xi^\dagger_{t'}\tau_a\xi_t}{\sqrt{2\omega_k}}\frac{\overline{u}_{\alpha'}}{\sqrt{2E'}}\not{q}\,\gamma^5\frac{u_\alpha}{\sqrt{2E}} , \qquad \text{(C.20)}$$

$$ {}^{PC}M^{10}_{\alpha'\alpha,\boldsymbol{q}\,a} = +i\frac{g_A}{2f_\pi}\frac{\xi^\dagger_{t'}\tau_a\xi_t}{\sqrt{2\omega_k}}\frac{\overline{u}_{\alpha'}}{\sqrt{2E'}}\not{q}\gamma^5\frac{u_\alpha}{\sqrt{2E}}\ , \tag{C.21}$$

where $u_\alpha \equiv u(\boldsymbol{p}, s)$, etc., and ξ_t, $\xi_{t'}$ are isospin states. The non-relativistic (NR) expansion of these amplitudes is needed up to order Q^3. Other $PC\ \pi NN$ vertices follow from the interactions terms in $\mathcal{L}^{(3)}_{\pi N}$ given in Eq. (B.13) proportional to the LEC's d_{16} and d_{18}. The NR expansion is needed up to order Q^2. No one pion exchange vertex functions come from $\mathcal{L}^{(4)}_{\pi N}$. Thus, up to order Q^4 we find (spin-isospin states are suppressed for brevity):

$$\begin{aligned} {}^{PC}M^{\pi NN,01}_{\alpha'\alpha,\boldsymbol{q}\,a} &= \frac{g_A}{2f_\pi}\frac{\tau_a}{\sqrt{2\omega_q}}\Big[i\,\boldsymbol{q}\cdot\boldsymbol{\sigma} - \frac{i}{M}\omega_q\,\boldsymbol{K}\cdot\boldsymbol{\sigma} + \frac{i}{4M^2}\Big(2\boldsymbol{K}\cdot\boldsymbol{q}\,\boldsymbol{K}\cdot\boldsymbol{\sigma} \\ &\quad -2K^2\,\boldsymbol{q}\cdot\boldsymbol{\sigma} - \frac{1}{2}\boldsymbol{k}\cdot\boldsymbol{\sigma}\,\boldsymbol{q}\cdot\boldsymbol{k}\Big) - \frac{\omega_q}{8M^3}\Big(\boldsymbol{k}\cdot\boldsymbol{\sigma}\,\boldsymbol{k}\cdot\boldsymbol{K} \\ &\quad +\boldsymbol{K}\cdot\boldsymbol{\sigma}\,(4K^2+k^2)\Big)\Big] \\ &+ \frac{m_\pi^2}{f_\pi}(2d_{16}-d_{18})\frac{\tau_a}{\sqrt{2\omega_q}}\Big[i\boldsymbol{q}\cdot\boldsymbol{\sigma} - \frac{i}{M}\omega_q\,\boldsymbol{K}\cdot\boldsymbol{\sigma}\Big]\ , \end{aligned} \tag{C.22}$$

$$ {}^{PC}M^{\pi NN,10}_{\alpha'\alpha,\boldsymbol{q}\,a} = -{}^{PC}M^{\pi NN,01}_{\alpha'\alpha,\boldsymbol{q}\,a}\ . \tag{C.23}$$

The $PC\ \pi\pi NN$ interaction is due to the Weinberg-Tomozawa term in Eq. (B.14) where terms up to NLO are needed. At NLO also terms proportional to c_1, c_2, c_3 and c_4 from Eq. (B.15) where the NR expansion is needed up to order Q. The corresponding vertex functions read

$$\begin{aligned} {}^{PC}M^{\pi\pi NN,02}_{\alpha'\alpha,\boldsymbol{q}'a'\,\boldsymbol{q}\,a} &= \frac{i}{8f_\pi^2}\frac{\epsilon_{aa'b}\tau_b}{\sqrt{2\omega_q}\sqrt{2\omega_{q'}}}\Big[(\omega_q-\omega_{q'}) \\ &\quad -\frac{2\boldsymbol{K}\cdot(\boldsymbol{q}-\boldsymbol{q}') - i(\boldsymbol{k}\times\boldsymbol{\sigma})\cdot(\boldsymbol{q}-\boldsymbol{q}')}{2M}\Big] \\ &\quad +\frac{\delta_{aa'}}{f_\pi^2\sqrt{2\omega_q}}\sqrt{2\omega_{q'}}\Big[2c_1m_\pi^2 + (c_2+c_3)\omega_q\omega_{q'} - c_3\boldsymbol{q}\cdot\boldsymbol{q}'\Big] \\ &\quad -\frac{c_4}{2f_\pi^2}\frac{\epsilon_{aa'b}\tau_b}{\sqrt{2\omega_q}\sqrt{2\omega_{q'}}}(\boldsymbol{q}\times\boldsymbol{q}')\cdot\boldsymbol{\sigma}\ , \end{aligned} \tag{C.24}$$

$$\begin{aligned} {}^{PC}M^{\pi\pi NN,11}_{\alpha'\alpha,\boldsymbol{q}'a'\,\boldsymbol{q}\,a} &= \frac{i}{4f_\pi^2}\frac{\epsilon_{aa'b}\tau_b}{\sqrt{2\omega_q}\sqrt{2\omega_{q'}}}\Big[(\omega_q+\omega_{q'}) \\ &\quad -\frac{2\boldsymbol{K}\cdot(\boldsymbol{q}+\boldsymbol{q}') - i(\boldsymbol{k}\times\boldsymbol{\sigma})\cdot(\boldsymbol{q}+\boldsymbol{q}')}{2M}\Big] \\ &\quad +\frac{2\delta_{aa'}}{f_\pi^2\sqrt{2\omega_q}}\sqrt{2\omega_{q'}}\Big[2c_1m_\pi^2 - (c_2+c_3)\omega_q\omega_{q'} + c_3\boldsymbol{q}\cdot\boldsymbol{q}'\Big] \\ &\quad +\frac{c_4}{f_\pi^2}\frac{\epsilon_{aa'b}\tau_b}{\sqrt{2\omega_q}\sqrt{2\omega_{q'}}}(\boldsymbol{q}\times\boldsymbol{q}')\cdot\boldsymbol{\sigma}\ , \end{aligned} \tag{C.25}$$

$$
\begin{aligned}
{}^{PC}M^{\pi\pi NN,20}_{\alpha'\alpha,\boldsymbol{q}'a'\,\boldsymbol{q}\,a} &= \frac{i}{8f_\pi^2}\frac{\epsilon_{aa'b}\tau_b}{\sqrt{2\omega_q}\sqrt{2\omega_{q'}}}\Big[(\omega_{q'}-\omega_q) \\
&\quad -\frac{2\boldsymbol{K}\cdot(\boldsymbol{q}'-\boldsymbol{q})-i(\boldsymbol{k}\times\boldsymbol{\sigma})\cdot(\boldsymbol{q}'-\boldsymbol{q})}{2M}\Big] \\
&\quad +\frac{\delta_{aa'}}{f_\pi^2\sqrt{2\omega_q}}\sqrt{2\omega_{q'}}\Big[2c_1m_\pi^2+(c_2+c_3)\omega_q\omega_{q'}-c_3\boldsymbol{q}\cdot\boldsymbol{q}'\Big] \\
&\quad -\frac{c_4}{2f_\pi^2}\frac{\epsilon_{aa'b}\tau_b}{\sqrt{2\omega_q}\sqrt{2\omega_{q'}}}(\boldsymbol{q}\times\boldsymbol{q}')\cdot\boldsymbol{\sigma}\ . \qquad \text{(C.26)}
\end{aligned}
$$

From the Lagrangian Eq. (2.66) we have also the contact terms

$$
{}^{PC}M^{00}_{\alpha_1'\alpha_2',\alpha_1\alpha_2} = \frac{1}{2}[C_S + C_T\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]\ . \qquad \text{(C.27)}
$$

C.3 PV vertex functions

The PV πNN vertices are due to interaction terms proportional to the LEC's h^1_π in Eq. (B.17), h^0_V, h^1_V and h^2_V in Eq. (B.18), h^1_2, h^1_3, and h^1_{12} in Eq. (B.19), $\tilde{h}^0_1$, $\tilde{h}^1_1$ and $\tilde{h}^2_1$ in Eq. (B.20). Up to order Q^3 we have

$$
\begin{aligned}
{}^{PV}M^{\pi NN,01}_{\alpha'\alpha,\boldsymbol{q}\,a} &= -\frac{h^1_\pi}{\sqrt{2}}\frac{\epsilon_{3ab}\tau_b}{\sqrt{2\omega_q}}\Big[1-\frac{2K^2+i(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}}{4M^2}\Big] \\
&\quad -\frac{i}{f_\pi}\Big(\frac{h^0_V}{2}\tau_a + h^1_V\,\delta_{a,3}+\frac{2}{3}\,h^2_V\,\mathcal{I}^{ab}\,\tau_b\Big) \\
&\quad \times\frac{1}{\sqrt{2\omega_q}}\Big[\omega_q\Big(1+\frac{-k^2/2+i(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}}{4M^2}\Big)-\frac{\boldsymbol{q}\cdot\boldsymbol{K}}{M}\Big] \\
&\quad +\frac{8}{f_\pi^2}\frac{\epsilon_{3ab}\tau_b}{\sqrt{2\omega_q}}\Big[(h^1_2-h^1_3)\,m_\pi^2-2h^1_{12}\,\omega_q^2\Big] \\
&\quad +\frac{16}{f_\pi^2}\frac{\epsilon_{3ab}\tau_b}{\sqrt{2\omega_q}}\omega_q\Big(2(\boldsymbol{q}\cdot\boldsymbol{K})-\frac{\boldsymbol{q}\cdot\boldsymbol{k}}{2}\Big)+\frac{i}{f_\pi^3}\Big(\tilde{h}^0_1\tau_a+2\tilde{h}^1_1\,\delta_{a,3} \\
&\quad +\frac{4}{3}\,\tilde{h}^2_1\,\mathcal{I}^{ab}\,\tau_b\Big)\frac{4m_\pi^2\omega_q}{\sqrt{2\omega_q}}\ , \qquad \text{(C.28)}
\end{aligned}
$$

$$
\begin{aligned}
{}^{PV}M^{\pi NN,10}_{\alpha'\alpha,\boldsymbol{q}\,a} &= -\frac{h^1_\pi}{\sqrt{2}}\frac{\epsilon_{3ab}\tau_b}{\sqrt{2\omega_q}}\Big[1-\frac{2K^2+i(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}}{4M^2}\Big] \\
&\quad +\frac{i}{f_\pi}\Big(\frac{h^0_V}{2}\tau_a + h^1_V\,\delta_{a,3}+\frac{2}{3}\,h^2_V\,\mathcal{I}^{ab}\,\tau_b\Big) \\
&\quad \times\frac{1}{\sqrt{2\omega_q}}\Big[\omega_q\Big(1+\frac{-k^2/2+i(\boldsymbol{k}\times\boldsymbol{K})\cdot\boldsymbol{\sigma}}{4M^2}\Big)-\frac{\boldsymbol{q}\cdot\boldsymbol{K}}{M}\Big] \\
&\quad +\frac{8}{f_\pi^2}\frac{\epsilon_{3ab}\tau_b}{\sqrt{2\omega_q}}\Big[(h^1_2-h^1_3)\,m_\pi^2-2h^1_{12}\,\omega_k^2\Big]
\end{aligned}
$$

$$+\frac{16}{f_\pi^2}\frac{\epsilon_{3ab}\tau_b}{\sqrt{2\omega_q}}\omega_q\Big(2(\boldsymbol{q}\cdot\boldsymbol{K})-\frac{\boldsymbol{q}\cdot\boldsymbol{k}}{2}\Big)-\frac{i}{f_\pi^3}\Big(\tilde{h}_1^0\tau_a+2\tilde{h}_1^1\,\delta_{a,3}$$
$$+\frac{4}{3}\,\tilde{h}_1^2\,\mathcal{I}^{ab}\,\tau_b\Big)\frac{4m_\pi^2\omega_q}{\sqrt{2\omega_q}}\ . \tag{C.29}$$

The PV $\pi\pi NN$ vertices follow from the interaction terms proportional to h_A^1 and h_A^2 in Eq. (B.21), and are given by

$$\begin{aligned}
{}^{PV}M^{\pi\pi NN,02}_{\alpha'\alpha,\boldsymbol{q}'a'\,\boldsymbol{q}\,a} &= \frac{i}{2f_\pi^2}\frac{1}{\sqrt{2\omega_q}\sqrt{2\omega_{q'}}}\Big[-h_A^1\,\epsilon_{3aa'}(\boldsymbol{q}-\boldsymbol{q}')\cdot\boldsymbol{\sigma}\\
&-\frac{1}{3}h_A^2\,\epsilon_{aa'b}\,I^b\,\tau_b\,(\boldsymbol{q}-\boldsymbol{q}')\cdot\boldsymbol{\sigma}\\
&+\frac{1}{3}h_A^2\,\epsilon_{aa'b}\,\tau_b\,(I^a\,\boldsymbol{q}-I^{a'}\,\boldsymbol{q}')\cdot\boldsymbol{\sigma}\Big]\ , &(C.30)\\
{}^{PV}M^{\pi\pi NN,11}_{\alpha'\alpha,\boldsymbol{q}'a'\,\boldsymbol{q}\,a} &= \frac{i}{f_\pi^2}\frac{1}{\sqrt{2\omega_q}\sqrt{2\omega_{q'}}}\Big[-h_A^1\,\epsilon_{3aa'}(\boldsymbol{q}+\boldsymbol{q}')\cdot\boldsymbol{\sigma}\\
&-\frac{1}{3}h_A^2\,\epsilon_{aa'b}\,I^b\,\tau_b\,(\boldsymbol{q}+\boldsymbol{q}')\cdot\boldsymbol{\sigma}\\
&+\frac{1}{3}h_A^2\,\epsilon_{aa'b}\,\tau_b\,(I^a\,\boldsymbol{q}+I^{a'}\,\boldsymbol{q}')\cdot\boldsymbol{\sigma}\Big]\ , &(C.31)\\
{}^{PV}M^{\pi\pi NN,20}_{\alpha'\alpha,\boldsymbol{q}'a'\,\boldsymbol{q}\,a} &= \frac{i}{2f_\pi^2}\frac{1}{\sqrt{2\omega_q}\sqrt{2\omega_{q'}}}\Big[-h_A^1\,\epsilon_{3aa'}\,(\boldsymbol{q}'-\boldsymbol{q})\cdot\boldsymbol{\sigma}\\
&-\frac{1}{3}h_A^2\,\epsilon_{aa'b}\,I^b\tau_b\,(\boldsymbol{q}'-\boldsymbol{q})\cdot\boldsymbol{\sigma}\\
&+\frac{1}{3}h_A^2\,\epsilon_{aa'b}\,\tau_b\,(I^{a'}\,\boldsymbol{q}'-I^a\,\boldsymbol{q})\cdot\boldsymbol{\sigma}\Big]\ , &(C.32)
\end{aligned}$$

where the factor I^a has been defined as $I^a=(-1,-1,2)$.

The contact vertex function is given in Appendix A.4.

C.4 TV vertex functions

The TV πNN vertices up to order Q^2 coming from interactions terms proportional g_0^θ and g_1^θ read

$$\begin{aligned}
{}^{TV}M^{\pi NN,01}_{\alpha'\alpha,\boldsymbol{q}\,a} &= \frac{g_0^\theta\tau_a+g_1^\theta\delta_{a,3}}{f_\pi\sqrt{2\omega_{\boldsymbol{q}}}}\ , &(C.33)\\
{}^{TV}M^{\pi NN,01}_{\alpha'\alpha,\boldsymbol{q}\,a} &= {}^{TV}M^{\pi NN,10}_{\alpha'\alpha,\boldsymbol{q}\,a}\ . &(C.34)
\end{aligned}$$

The other vertex functions come from the 3π Lagrangian term proportional to Δ^θ and read

$$ {}^{TV}M^{3\pi,03}_{\boldsymbol{p}b\,\boldsymbol{q}'a'\,\boldsymbol{q}\,a} = \frac{\Delta^\theta}{f_\pi}\frac{1}{\sqrt{8\omega_{\boldsymbol{q}}\omega_{\boldsymbol{q}'}\omega_{\boldsymbol{p}}}}\left[\frac{\delta_{a,a'}\delta_{b,3}+\delta_{a,b}\delta_{a',3}+\delta_{b,a'}\delta_{a,3}}{3}\right] , \tag{C.35} $$

$$ {}^{TV}M^{3\pi,12}_{\boldsymbol{p}b\,\boldsymbol{q}'a'\,\boldsymbol{q}\,a} = \frac{\Delta^\theta}{f_\pi}\frac{1}{\sqrt{8\omega_{\boldsymbol{q}}\omega_{\boldsymbol{q}'}\omega_{\boldsymbol{p}}}}\left[\delta_{a,a'}\delta_{b,3}+\delta_{a,b}\delta_{a',3}+\delta_{b,a'}\delta_{a,3}\right] , \tag{C.36} $$

$$ {}^{TV}M^{3\pi,21}_{\boldsymbol{p}b\,\boldsymbol{q}'a'\,\boldsymbol{q}\,a} = \frac{\Delta^\theta}{f_\pi}\frac{1}{\sqrt{8\omega_{\boldsymbol{q}}\omega_{\boldsymbol{q}'}\omega_{\boldsymbol{p}}}}\left[\delta_{a,a'}\delta_{b,3}+\delta_{a,b}\delta_{a',3}+\delta_{b,a'}\delta_{a,3}\right] , \tag{C.37} $$

$$ {}^{TV}M^{3\pi,30}_{\boldsymbol{p}b\,\boldsymbol{q}'a'\,\boldsymbol{q}\,a} = \frac{\Delta^\theta}{f_\pi}\frac{1}{\sqrt{8\omega_{\boldsymbol{q}}\omega_{\boldsymbol{q}'}\omega_{\boldsymbol{p}}}}\left[\frac{\delta_{a,a'}\delta_{b,3}+\delta_{a,b}\delta_{a',3}+\delta_{b,a'}\delta_{a,3}}{3}\right] . \tag{C.38} $$

From the contact terms we obtain the following vertex function

$$ {}^{TV}M^{00}_{\alpha_1'\alpha_1\alpha_2'\alpha_2} = \frac{\overline{C_1}^\theta}{2\Lambda_\chi^2 f_\pi}\, i(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{k} + \frac{\overline{C_2}^\theta}{2\Lambda_\chi^2 f_\pi}\,(\vec{\tau}_1\cdot\vec{\tau}_2) i(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{k} \tag{C.39} $$

Appendix D

Regularization

In this Appendix we will discuss the method used to regularize the integrals that come from the loops. We will use the dimensional regularization (DR) method to control the logarithmic divergences. In case of the linear divergences we will compare the result of DR with the use of a cut-off.

D.1 Useful relations

We will make use of the following identity (the "Feynman trick")

$$\frac{1}{AB} = \int_0^1 \mathrm{d}y \, \frac{1}{[yA + (1-y)B]^2} \ , \tag{D.1}$$

furthermore in order to simplify some energy factors we will use the following reppresentations [61]:

$$\frac{1}{\omega_+ + \omega_-} = \frac{2}{\pi} \int_0^\infty \mathrm{d}\beta \, \frac{\beta^2}{(\omega_+^2 + \beta^2)(\omega_-^2 + \beta^2)} \ , \tag{D.2}$$

$$\frac{1}{\omega_+ \, \omega_- \, (\omega_+ + \omega_-)} = \frac{2}{\pi} \int_0^\infty \mathrm{d}\beta \, \frac{1}{(\omega_+^2 + \beta^2)(\omega_-^2 + \beta^2)} \ . \tag{D.3}$$

When we use the DR, it is better to "rescale" all the dimensional quantities with an energy scale μ. Therefore we define $q = \tilde{q}\mu$, $m = \tilde{m}\mu$, etc., where the "tilde" quantities are adimensional. We can now perform the integrations in d dimension

$$\int \frac{d^3\tilde{q}}{(2\pi)^3} \to \int \frac{d^d\tilde{q}}{(2\pi)^d} \equiv \int_{\tilde{\boldsymbol{q}}} \ , \tag{D.4}$$

and then use the following integrals:

$$\int_{\tilde{q}} \frac{1}{\left((\tilde{q})^2 + A\right)^\alpha} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)} A^{-(\alpha-d/2)} , \tag{D.5}$$

$$\int_{\tilde{q}} \frac{(\tilde{q})^2}{\left((\tilde{q})^2 + A\right)^\alpha} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(\alpha - d/2 - 1)}{\Gamma(\alpha)} A^{-(\alpha-d/2-1)} , \tag{D.6}$$

$$\int_{\tilde{q}} \frac{(\tilde{q})^4}{\left((\tilde{q})^2 + A\right)^\alpha} = \frac{1}{(4\pi)^{d/2}} \frac{d\,(d+2)}{4} \frac{\Gamma(\alpha - d/2 - 2)}{\Gamma(\alpha)} A^{-(\alpha-d/2-2)} , \tag{D.7}$$

where $\Gamma(z)$ is defined by $z\,\Gamma(z) = \Gamma(z+1)$, with the following trend for $z \to 0$

$$\Gamma(z) = \frac{1}{z} - \gamma + \left(\frac{\gamma^2}{2} + \frac{\pi^2}{12}\right) z + O(z^2) , \tag{D.8}$$

and $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

Lastly we will need the following integrals

$$\int_{-1}^{1} \mathrm{d}x \quad ln(a+x) = -2 + ln\frac{a+1}{a-1} + ln(a^2-1), \tag{D.9}$$

$$\int_{-1}^{1} \mathrm{d}x \quad x^2 ln(a+x) = \frac{1}{9}\left[-2 - 6a^2 + 3a^3 ln\frac{a+1}{a-1} + 3ln(a^2-1),\right] \tag{D.10}$$

$$\int_{-1}^{1} \mathrm{d}x \quad \frac{1}{\sqrt{a-x^2}} = 2 \arctan\left(\frac{1}{\sqrt{a-1}}\right), \tag{D.11}$$

$$\int_{-1}^{1} \mathrm{d}x \quad \sqrt{a-x^2} = \sqrt{a-1} + a \arctan\left(\frac{1}{\sqrt{a-1}}\right), \tag{D.12}$$

$$\int_{-1}^{1} \mathrm{d}x \quad \frac{x^2}{\sqrt{a-x^2}} = -\sqrt{a-1} + a \arctan\left(\frac{1}{\sqrt{a-1}}\right). \tag{D.13}$$

We also define the following quantities:

$$L(k) = \frac{1}{2}\frac{s}{k} \ln\frac{s+k}{s-k} , \qquad H(k) = \frac{m_\pi^2}{s^2} L(k) , \qquad A(k) = \frac{1}{2k} \arctan\left(\frac{k}{2m_\pi^2}\right) , \tag{D.14}$$

where $s = \sqrt{4\,m_\pi^2 + k^2}$.

D.2 Integrals with logarithmic divergences

In Eqs. (4.24) and (4.37) all the integrals have the following function to integrate

$$f(\omega_+, \omega_-) \equiv \frac{\omega_+^2 + \omega_+\,\omega_- + \omega_-^2}{\omega_+^3\,\omega_-^3(\omega_+ + \omega_-)} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,m_\pi^2} \frac{1}{\omega_+\,\omega_-\,(\omega_+ + \omega_-)} , \tag{D.15}$$

so we need to consider the following integral

$$I^{(0)}(k) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\omega_+ \, \omega_- \, (\omega_+ + \omega_-)} \,, \tag{D.16}$$

and then we take the derivative of the result respect to m_π^2. Using the transformation $q \to \tilde{q}\mu$ and then going to d dimension we have

$$I^{(0)}(k) \to \frac{2}{\pi} \int \frac{d^d\tilde{q}}{(2\pi)^d} \int_0^\infty \mathrm{d}\beta \, \frac{1}{\Big((\tilde{\omega}_+)^2 + \beta^2\Big)\Big((\tilde{\omega}_-)^2 + \beta^2\Big)} \,, \tag{D.17}$$

where $\tilde{\omega}_\pm = \sqrt{(\tilde{\boldsymbol{q}} \pm \tilde{\boldsymbol{k}})^2 + 4\,(\tilde{m}_\pi)^2}$. Using the "Feynman trick" (D.1) with $A = (\tilde{\omega}_+)^2 + \beta^2$ e $B = (\tilde{\omega}_-)^2 + \beta^2$, we obtain

$$\begin{aligned} I^{(0)}(k) &= \frac{2}{\pi} \int_{\tilde{\boldsymbol{q}}} \int_0^1 \mathrm{d}y \int_0^\infty \mathrm{d}\beta \left[\Big[\tilde{\boldsymbol{q}} + (2\,y - 1)\,\tilde{\boldsymbol{k}}\Big]^2 + 4\,[(\tilde{m}_\pi)^2 - y\,(y-1)\,k^2] + \beta^2 \right]^{-2} \\ &= \frac{1}{2} \int_{\tilde{\boldsymbol{q}}} \int_0^1 \mathrm{d}y \left[(\tilde{q})^2 + 4\,[(\tilde{m})^2 - y\,(y-1)\,(\tilde{k})^2] \right]^{-3/2} \,, \end{aligned} \tag{D.18}$$

where in the second line we have changed the integration variable $\tilde{\boldsymbol{q}} \to \tilde{\boldsymbol{q}} + (2\,y - 1)\,\tilde{\boldsymbol{k}}$. Exploiting Eq. (D.5) with $\alpha = 3/2$, and $A = 4\left[(\tilde{m}_\pi)^2 - y\,(y-1)\,(\tilde{k})^2\right]$, and using the following asymptotic behaviors for $\epsilon^+ \to 0$

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + O(\epsilon) \,, \tag{D.19}$$

$$\left(\frac{A}{4\,\pi}\right)^{-\epsilon/2} = 1 - \frac{\epsilon}{2} \ln \frac{A}{4\,\pi} + O(\epsilon^2) \,, \tag{D.20}$$

we obtain, neglecting $\mathcal{O}(\epsilon)$ terms,

$$I^{(0)}(k) = \frac{1}{8\,\pi^2}\left(\ln\pi + \frac{2}{\epsilon} - \gamma\right) - \frac{1}{8\,\pi^2} \int_0^1 \mathrm{d}y \ln\left[(\tilde{m}_\pi)^2 - y\,(y-1)\,(\tilde{k})^2\right] \,. \tag{D.21}$$

At the end, setting $y \to (x+1)/2$ and using the integral in Eq. (D.9), we obtain

$$I^{(0)}(k) = -\frac{1}{8\,\pi^2}\left(\frac{s}{k}\ln\frac{s+k}{s-k} - \frac{2}{\epsilon} + \gamma - \ln\pi + \ln\frac{m_\pi^2}{\mu^2} - 2\right) \,. \tag{D.22}$$

where we have expressed the results in terms of the dimensional quantities. In the following, we define

$$d_\epsilon = \frac{2}{\epsilon} - \gamma + \ln\pi - \ln\frac{m_\pi^2}{\mu^2} \,, \tag{D.23}$$

which contains the diergent part, so

$$I^{(0)}(k) = -\frac{1}{4\,\pi^2}\Big(L(k) - d_\epsilon + 2\Big)\ . \tag{D.24}$$

In the same way we calculate the integrals

$$I^{(2)}(k) = \int_{\mathbf{q}} \frac{q^2}{\omega_+\,\omega_-\,(\omega_+ + \omega_-)}\,, \tag{D.25}$$

$$I^{(2)}_{ij}(k) = \int_{\mathbf{q}} \frac{q_i\,q_j}{\omega_+\,\omega_-\,(\omega_+ + \omega_-)}\,, \tag{D.26}$$

obtaining

$$I^{(2)}(k) = \frac{1}{24\,\pi^2}\left[4S^2 L(k) + 2\,k^2\left(-d_\epsilon - \frac{5}{3}\right) + 18\,m_\pi^2\left(-d_\epsilon - \frac{11}{9}\right)\right], \tag{D.27}$$

$$\begin{aligned} I^{(2)}_{ij}(k) &= \frac{1}{24\,\pi^2}\,\delta_{ij}\left[2s^2 L(k) + k^2\,(-d_\epsilon - 2) + 6\,m_\pi^2\left(-d_\epsilon - \frac{5}{3}\right)\right] \\ &- \frac{1}{24\,\pi^2}\,\frac{k_i k_j}{k^2}\left[2s^2 L(k) + k^2\left(-d_\epsilon - \frac{8}{3}\right) - 8\,m_\pi^2\right]\ . \end{aligned} \tag{D.28}$$

Performing the derivative we obtain the integral of the integrals containing the quantity f in Eq. (D.15):

$$J^{(0)}(k) = \int_{\boldsymbol{q}} f(\omega_+,\omega_-) = \frac{1}{4\,\pi^2}\,\frac{H(k)}{m_\pi^2}\ , \tag{D.29}$$

$$J^{(2)}(k) = \int_{\boldsymbol{q}} q^2 f(\omega_+,\omega_-) = -\frac{1}{8\,\pi^2}\left[4L(k) + 3\left(-d_\epsilon - \frac{2}{3}\right)\right]\ , \tag{D.30}$$

$$\begin{aligned} J^{(2)}_{ij}(k) &= \int_{\boldsymbol{q}} q_i q_j f(\omega_+,\omega_-) = -\frac{1}{8\,\pi^2}\delta_{ij}\left[2L(k) + \left(-d_\epsilon - \frac{4}{3}\right)\right] \\ &\quad + \frac{1}{8\,\pi^2}\,\frac{k_i k_j}{k^2}\,(2L(k) - 2)\ . \end{aligned} \tag{D.31}$$

D.3 Integrals with linear divergences

From Sections 4.3.3 and 4.3.4, it results that we need to calculate the integrals that contains the following function

$$g(\omega_+,\omega_-) = \frac{1}{\omega_+^2\,\omega_-^2}\ . \tag{D.32}$$

Let us start, for example, with the integral

$$K^{(0)}(k) = \int \frac{d^3q}{(2\pi)^3}\,\frac{1}{\omega_+^2\,\omega_-^2} \tag{D.33}$$

using the formula in Eq. (D.1) we get

$$K^{(0)}(k) = \int_{\boldsymbol{q}} \int_0^1 \mathrm{d}y \frac{1}{(y\,\omega_+^2 + (1-y)\,\omega_-^2)} \tag{D.34}$$

where we don't need to go in d dimensions and using the tilde quantities since the integral does not have divergences. Thus we get, changing $\boldsymbol{q} \to \boldsymbol{q} + (2\,y - 1)\,\boldsymbol{k}$,

$$\begin{aligned} K^{(0)}(k) &= \int_{\boldsymbol{q}} \int_0^1 \mathrm{d}y \frac{1}{(q^2 + A)^2} \\ &= \frac{1}{8\pi} \int_0^1 \mathrm{d}y \frac{1}{\sqrt{4(m_\pi^2 - y\,(y-1)k^2)}} \ , \end{aligned} \tag{D.35}$$

where $A = 4\,[(m)^2 - y\,(y-1)\,(k)^2]$ and in the last step we use Eq. (D.5) with $d = 3$. Then performing the transformation $y \to (x+1)/2$, we obtain

$$K^{(0)}(k) = \frac{1}{8\pi\,k} \arctan\big(\frac{k}{2m_\pi^2}\big) \ , \tag{D.36}$$

and recalling the definition of the function $A(k)$ given in Eq. (D.14), and so we can rewrite this result as

$$K^{(0)}(k) = \frac{1}{4\pi} A(k) \ . \tag{D.37}$$

Let us now consider the integrals

$$K^{(2)}(k) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{q^2}{\omega_+^2\,\omega_-^2} \tag{D.38}$$

$$K_{ij}^{(2)}(k) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{q_i\,q_j}{\omega_+^2\,\omega_-^2}. \tag{D.39}$$

In these cases the integrals are divergent for $q \to \infty$. Using however DR as before, we obtain

$$K^{(2)}(k) = -\frac{s^2\,A(k)}{4\pi} - \frac{m_\pi}{2\pi} \tag{D.40}$$

$$K_{ij}^{(2)}(k) = \Big(-\frac{s^2 A(k)}{8\pi} - \frac{m_\pi}{8\pi}\Big)\delta_{ij} + \Big(\frac{s^2 A(k)}{8\pi} - \frac{m_\pi}{8\pi}\Big)\frac{k_i\,k_j}{k^2}, \tag{D.41}$$

which do not contain any divergent factor d_ϵ. In fact the DR does not match the linear divergences. In order to clarify what is happening, let us evaluate these integrals using a simple cut off. For example, the integral $K_{ij}^{(2)}$ is given in Eq. (D.39), using the Feynman trick and performing the change of variable $\boldsymbol{q} \to \boldsymbol{q} + (2\,y-1)\,\boldsymbol{k}$, becomes

$$K^{(2)}_{ij}(k) = \int_{\boldsymbol{q}} \int_0^1 \mathrm{d}y\, \frac{q_i q_j + (2y-1)\,k_i k_j}{(q^2+A)^2}. \tag{D.42}$$

Only the integration over $q = |\boldsymbol{q}|$ is divergent so we can write,

$$K^{(2)}_{ij}(k) = \int_{\boldsymbol{q}} \int_0^1 \mathrm{d}y\, \frac{q_i q_j}{(q^2+A)^2} \longrightarrow \frac{\delta_{ij}}{6\pi} \int_0^1 \mathrm{d}y \int_0^\Lambda \mathrm{d}q\, \frac{q^4}{(q^2+A)^2}, \tag{D.43}$$

which gives

$$K^{(2)}_{ij}(k) = \frac{\delta_{ij}}{6\pi} \int_0^1 \mathrm{d}y \Big(\frac{A\Lambda}{2(A+\Lambda)^2} - \frac{3}{2}\sqrt{A}\arctan\big(\frac{\Lambda}{\sqrt{A}}\big) + \Lambda \Big). \tag{D.44}$$

Then developing the terms with Λ in Laurent series $\Lambda/A \gg 1$ we obtain

$$K^{(2)}_{ij}(k) = \frac{\delta_{ij}}{6\pi}\Big[\Lambda - \frac{3}{4}\pi \int_0^1 \mathrm{d}y\sqrt{A} + \mathcal{O}\Big(\int_0^1 \mathrm{d}y \frac{A}{\Lambda}\Big)\Big]. \tag{D.45}$$

Note that $\int_0^1 \mathrm{d}y \frac{A}{\Lambda} \sim \frac{k^2}{\Lambda}$ and so this term contributes only to the next order in ChPT as discussed in Section 4.4.2. Performing the integration in $\mathrm{d}y$ using Eq. (D.12) and summing the contribution of the non divergent part, we get

$$K^{(2)}_{ij}(k) = \frac{\Lambda}{12\pi^2}\delta_{ij} + \Big(-\frac{s^2 A(k)}{8\pi} - \frac{m_\pi}{8\pi}\Big)\delta_{ij} + \Big(\frac{s^2 A(k)}{8\pi} - \frac{m_\pi}{8\pi}\Big)\frac{k_i\,k_j}{k^2} + \mathcal{O}\big(\frac{k^2}{\Lambda}\big), \tag{D.46}$$

Note that we have obtained a term proportional to Λ which contains the full divergence of the integral. In the same way

$$K^{(2)}(k) = \frac{\Lambda}{4\pi^2} - \frac{s^2\,A(k)}{4\pi} - \frac{m_\pi}{2\pi} + \mathcal{O}\big(\frac{k^2}{\Lambda}\big). \tag{D.47}$$

The integrals given in Eq. (4.62) can be decomposed as follows

$$\begin{aligned} L^{(2)}_{ij}(k) &= \int_{\boldsymbol{q}} \frac{\omega_+^2 + \omega_-^2}{\omega_+^4\,\omega_-^4}(q^2-k^2) q_i q_j \\ &= \frac{1}{2}\int_{\boldsymbol{q}} q_i q_j \Big(\frac{1}{\omega_+^4} + \frac{1}{\omega_-^4} + \frac{2}{\omega_+^2\,\omega_-^2}\Big) \\ &\qquad -4(k^2+2m_\pi^2)\int_{\boldsymbol{q}} q_i q_j \frac{(q^2+s^2)}{\omega_+^4\,\omega_-^4}, \end{aligned} \tag{D.48}$$

where the first part contains all the divergences and can be calculated imposing a cut-off as discussed above while the second part is finite and can be calculated in an elementary way. A similar decomposition can be performed for the integral give in Eq. (4.62)

$$\begin{aligned} L^{(2)}(k) &= \int_{\boldsymbol{q}} \frac{\omega_+^2 + \omega_-^2}{\omega_+^4\,\omega_-^4}(q^2-k^2)^2 \\ &= \frac{1}{2}\int_{\boldsymbol{q}} \Big(\frac{(q^2-k^2)}{\omega_+^4} + \frac{(q^2-k^2)}{\omega_-^4} + \frac{2(q^2-k^2)}{\omega_+^2\,\omega_-^2}\Big) \\ &\quad -4(k^2+2m_\pi^2)\int_{\boldsymbol{q}} (q^2-k^2)\frac{(q^2+s^2)}{\omega_+^4\,\omega_-^4} \end{aligned} \tag{D.49}$$

where we have the same structure as before. After a length calculation, the final results are

$$\begin{aligned} L_{ij}^{(2)}(k) &= \Big(\frac{\Lambda}{4\pi^2} - \frac{3m_\pi}{8\pi} - \frac{1}{8\pi}\big(3 - \frac{4m_\pi^2}{s^2}\big)s^2 A(k)\Big)\delta_{ij} \\ &\quad + \frac{k_i k_j}{8\pi k^2}\big(3 - \frac{4m_\pi^2}{s^2}\big)s^2 A(k) + \mathcal{O}\big(\frac{k^2}{\Lambda}\big), \qquad \text{(D.50)} \\ L^{(2)}(k) &= \frac{3\Lambda}{4\pi^2} - \frac{11m_\pi}{8\pi} - \frac{1}{\pi}\big(1 - \frac{2m_\pi^2}{s^2}\big)s^2 A(k) + \frac{m_\pi^3}{2\pi s^2} + \mathcal{O}\big(\frac{k^2}{\Lambda}\big), \qquad \text{(D.51)} \end{aligned}$$

where we have used the results of the integral type,

$$\int_{-1}^{1} \mathrm{d}x\, \frac{x^n}{(a-x^2)^{\frac{m}{2}}}, \tag{D.52}$$

which is possible to find in [65] for different m and n.

Appendix E

Matrix elements of the PV and TV potentials

In this Appendix we will report the explicit analytic expression of the spin-isospin-angular matrix elements of the operators entering the PV and TV potentials. In the first section we will report some useful formulas and then in the second and third sections we will discuss the spin-angular and isospin matrix elements respectively.

E.1 Useful formulas

In this Appendix we will use the following notation

$$\Big[T_K\Psi_L\Big]_{JJ_z} = \sum_{\kappa M}(K\kappa, LM|JJ_z)T_{K\kappa}\Psi_{LM} \ , \tag{E.1}$$

where $T_{K\kappa}$ is a generic spherical tensor operator of rank K and component z given by κ, while Ψ_{LM} rappresents a state of angular momentum L, M. $(K\kappa, LM|JJ_z)$ is a Clebsh-Gordan coefficient.

Coupling of two spherical harmonics

$$\Big[Y_{\ell_1}Y_{\ell_2}\Big]_{LM} = \frac{B^{\ell}_{\ell_1\ell_2}}{\sqrt{4\pi}}Y_{LM}(\hat{\boldsymbol{r}}) \ , \qquad B^{L}_{\ell_1\ell_2} = \hat{\ell}_1\hat{\ell}_2(-)^{\ell_1+\ell_2}\begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix}, \tag{E.2}$$

where $\hat{\ell} = \sqrt{2\ell+1}$.

Coupling of 3 angular momenta

$$|(j_1j_2)_{j_{12}}j_3JM\rangle = \sum_{j_{23}} T^{j_1j_2j_3}_{j_{12}j_{23}J}|j_1(j_2j_3)_{j_{23}}JM\rangle \tag{E.3}$$

where

$$T^{j_1 j_2 j_3}_{j_{12} j_{23} J} = (-)^{j_1+j_2+j_3+J} \hat{j}_{12} \hat{j}_{23} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{array} \right\} . \quad \text{(E.4)}$$

Coupling of 4 angular momenta

$$|(j_1 j_2)_{j_{12}} (j_3 j_4)_{j_{34}} JM\rangle = \sum_{j_{13} j_{24}} N^{j_1 j_2 j_3 j_4}_{j_{12} j_{34} j_{13} j_{23} J} |(j_1 j_3)_{j_{13}} (j_2 j_4)_{j_{24}} JM\rangle \quad \text{(E.5)}$$

where

$$N^{j_1 j_2 j_3 j_4}_{j_{12} j_{34} j_{13} j_{23} J} = \hat{j}_{13} \hat{j}_{24} \hat{j}_{12} \hat{j}_{34} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{array} \right\} . \quad \text{(E.6)}$$

Spherical Tensors The Cartesian operators $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{r}}$ can be written like spherical tensor of rank 1:

$$\begin{aligned} \sigma_{10} &= \sigma_z \\ \sigma_{1-1} &= \frac{1}{\sqrt{2}} (\sigma_x - i\sigma_y) \\ \sigma_{1+1} &= -\frac{1}{\sqrt{2}} (\sigma_x + i\sigma_y) \ , \end{aligned}$$

where with the notation $\boldsymbol{\sigma}_{1\mu}$ we indicate the rank 1 tensor and component z given by $\mu = \pm 1, 0$. Furthermore

$$\begin{aligned} \frac{z}{r} &= \sqrt{\frac{4\pi}{3}} Y_{10} \ , \\ \frac{y}{r} &= \sqrt{\frac{4\pi}{3}} i \frac{Y_{1-1} + Y_{1+1}}{\sqrt{2}} \ , \\ \frac{x}{r} &= \sqrt{\frac{4\pi}{3}} \frac{Y_{1-1} - Y_{1+1}}{\sqrt{2}} \ . \end{aligned}$$

We can rewrite the scalar and vector products in terms of these spherical components

$$(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}) = -\sqrt{4\pi} \left[\sigma_1 Y_1\right]_{00} \ , \qquad (\boldsymbol{\sigma} \times \hat{\boldsymbol{r}})_{1\mu} = i\sqrt{\frac{8\pi}{3}} \left[\sigma_1 Y_1\right]_{1\mu} \ , \quad \text{(E.7)}$$

$$(\boldsymbol{\sigma}(1) \times \boldsymbol{\sigma}(2))_{1\mu} = -i\sqrt{2} \left[\sigma_1(1) \sigma_1(2)\right]_{1\mu} \ , \quad \text{(E.8)}$$

where $\boldsymbol{\sigma}(i)$ indicates Pauli's matrices acting on the spin state of the i-th particle. Other useful formula are (remembering that $\chi_{SS_z} = |s_1 s_2\rangle_{SS_z}$):

$$\begin{aligned} \Big[\sigma_1|\frac{1}{2}\rangle\Big]_{\frac{1}{2}m} &= -\sqrt{3}|\frac{1}{2}m\rangle \ , && \text{(E.9)} \\ \Big[\sigma_1\chi_S\Big]_{S'} &= \sqrt{3}T^{1\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}SS'}\chi_{S'} \ , && \text{(E.10)} \\ \Big[\sigma_2\chi_S\Big]_{S'} &= (-)^{S+S'}\sqrt{3}T^{1\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}SS'}\chi_{S'} \ . && \text{(E.11)} \end{aligned}$$

where T are given in Eq. (E.4).

E.2 Spin-angular matrix elements

First let us compute, the effect of an operator $\mathcal{O}$ on a spin-angular state $\Big[Y_L(\hat{\boldsymbol{r}})\chi_S\Big]$, namely

$$\mathcal{O}\Big[Y_L(\hat{\boldsymbol{r}})\chi_S\Big]_{JJ_z} = \sum_{L'S'} O^J_{L'S',LS}\Big[Y_{L'}(\hat{\boldsymbol{r}})\chi_{S'}\Big]_{JJ_z} \ , \qquad \text{(E.12)}$$

where $\mathcal{O}$ is one of the following operators:

$$\begin{aligned} S^{\pm}_r &= (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}} \ , \quad S^{\pm}_p = (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot(-i\boldsymbol{\nabla}) \ , \quad S^{\times}_r = (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{r}} \ , \\ S_L &= \boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{r}}\boldsymbol{\sigma}_2\cdot\boldsymbol{L} + \boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{r}}\boldsymbol{\sigma}_1\cdot\boldsymbol{L} \ , \qquad \text{(E.13)} \end{aligned}$$

and the quantities $O^J_{L'S',LS}$ are

$$O^J_{L'S',LS} = \langle Y_{L'}\chi_{S'}|\mathcal{O}|Y_L\chi_S\rangle. \qquad \text{(E.14)}$$

Operators $S^{\pm}_r$ and $S^{\times}_r$. Using the Eq. (E.2) and Eq. (E.10)

$$\begin{aligned} S^{\pm}_r\left[Y_L\chi_S\right]_{JJ_z} &= \sum_{L'S'}(1 \pm (-)^{S+S'})\sqrt{3}N^{11LS}_{0jL'S'j}B^{L'}_{1L}T^{1\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}SS'}\left[Y_{L'}\chi_{S'}\right]_{JJ_z} \ , && \text{(E.15)} \\ S^{\times}_r\left[Y_L\chi_S\right]_{JJ_z} &= \sum_{L'S'} i3\sqrt{2}N^{11L_1S_1}_{0jL'S'j}B^{L'}_{1L}N^{11\frac{1}{2}\frac{1}{2}}_{1S\frac{1}{2}\frac{1}{2}S'}\left[Y_{L'}\chi_{S'}\right]_{JJ_z} \ , && \text{(E.16)} \end{aligned}$$

Operators $S^{\pm}_p$ e S_L. The gradient in spherical coordinates can be written as:

$$\boldsymbol{\nabla} = \hat{\boldsymbol{r}}\frac{\partial}{\partial \boldsymbol{r}} - \frac{i}{r}(\hat{r}\times\boldsymbol{L}) \ . \qquad \text{(E.17)}$$

Therefore

$$S^{\pm}_p = S^{\pm}_r\frac{\partial}{\partial r} + \frac{1}{r}S^{\pm}_L \ , \qquad S^{\pm}_L = -i(\boldsymbol{\sigma}_1 \pm \boldsymbol{\sigma}_2)\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{L}) \ . \qquad \text{(E.18)}$$

We get the matrix element of the operators $S_L^{\pm}$ considering that

$$\Big[\boldsymbol{L}Y_L(\hat{\boldsymbol{r}})\Big]_{L'M} = -\sqrt{L(L+1)}\delta_{LL'}Y_{L'M}(\hat{\boldsymbol{r}}) \ , \tag{E.19}$$

so

$$S_L^{\pm}\left[Y_L\chi_S\right]_{JJ_z} = \sum_{L'S'} \sqrt{6}\sqrt{L(L+1)}N^{11LS}_{0jL'S'j}T^{11L}_{1LL'}B^{L'}_{1L}T^{1\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}SS'}(1\pm(-)^{S+S'})|\left[Y_{L'}\chi_{S'}\right]_{JJ_z} \tag{E.20}$$

In the same way we find

$$S_L\left[Y_L\chi_S\right]_{JJ_z} = -3\sqrt{3}N^{1111}_{00ll0}N^{llLS}_{0jL'S'j}N^{11\frac{1}{2}\frac{1}{2}}_{lS\frac{1}{2}\frac{1}{2}S'}T^{11L}_{1LL'}\sqrt{L(L+1)}B_{1LL'}|\left[Y_{L'}\chi_{S'}\right]_{JJ_z} \tag{E.21}$$

E.3 Isospin matrix elements

In this case we need the matrix elements of the following operators

$$\vec{\tau}_1\cdot\vec{\tau}_2 \ , \qquad (\vec{\tau}_1\pm\vec{\tau}_2)_z \ , \qquad (\vec{\tau}_1\times\vec{\tau}_2)_z \ , \qquad \mathcal{I}_{ij}(\vec{\tau}_1)_i(\vec{\tau}_2)_j = 3(\vec{\tau}_1)_z(\vec{\tau}_2)_z - \vec{\tau}_1\cdot\vec{\tau}_2 \ . \tag{E.22}$$

In general the total isospin is not conserved (only its z components). So in this case it is convenient to decompose the isospin states ξ_{TT_z} in terms of the isospin state of the two nucleons, and then evaluate the sum over the z components. We have explicitly,

$$\langle\xi_{TT_z}|(\vec{\tau}_1\cdot\vec{\tau}_2)|\xi_{T'T_z}\rangle = (4T-3)\delta_{TT'} \tag{E.23}$$

$$\langle\xi_{TT_z}|(\vec{\tau}_1\pm\vec{\tau}_2)_z|\xi_{T'T_z}\rangle = \sum_{t_{1z}t_{2z}} C^{\frac{1}{2}\frac{1}{2}T'}_{t_{1z}t_{2z}T_z}C^{\frac{1}{2}\frac{1}{2}T}_{t_{1z}t_{2z}T_z}(t_{1z}\pm t_{2z}) \tag{E.24}$$

$$\langle\xi_{TT_z}|3(\vec{\tau}_1)_z(\vec{\tau}_2)_z-\vec{\tau}_1\cdot\vec{\tau}_2|\xi_{T'T_z}\rangle = \sum_{t_{1z}t_{2z}} C^{\frac{1}{2}\frac{1}{2}T'}_{t_{1z}t_{2z}T_z}C^{\frac{1}{2}\frac{1}{2}T}_{t_{1z}t_{2z}T_z}3t_{1z}t_{2z} -(4T-3)\delta_{TT'} \tag{E.25}$$

$$\langle\xi_{TT_z}|(\vec{\tau}_1\times\vec{\tau}_2)_z|\xi_{T'T_z}\rangle = \sum_{t_{1z}t_{2z}} C^{\frac{1}{2}\frac{1}{2}T'}_{t_{1z}t_{2z}T_z}C^{\frac{1}{2}\frac{1}{2}T}_{t_{1z}+1,t_{2z}-1,T_z}(1-t_{1z})(1+t_{2z}) + C^{\frac{1}{2}\frac{1}{2}T'}_{t_{1z}t_{2z}T_z}C^{\frac{1}{2}\frac{1}{2}T}_{t_{1z}+1,t_{2z}-1,T_z}(t_{1z}+1)(t_{2z}-1) \tag{E.26}$$

where $C^{\frac{1}{2}\frac{1}{2}T}_{t_{1z}t_{2z}T_z} = \langle\frac{1}{2}\frac{1}{2};t_{1z}t_{2z}|\frac{1}{2}\frac{1}{2};TT_z\rangle$.

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