Commun. Math. Phys. (2025) 406:270
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-025-05398-z

Communications in
Mathematical Physics

A String-Like Realization of Hyperbolic Kac–Moody Algebras

Saverio Capolongo[1], **Axel Kleinschmidt**[1,2], **Hannes Malcha**[1], **Hermann Nicolai**[1]

[1] Max Planck Institute for Gravitational Physics (Albert Einstein Institute), 14476, Potsdam, Germany.
[2] Faculty of Physics, University of Vienna, Boltzmanngasse 5, 1090 Vienna, Austria.
E-mail: axel.kleinschmidt@aei.mpg.de

Received: 20 December 2024 / Accepted: 29 June 2025

Abstract: We propose a new approach to studying hyperbolic Kac–Moody algebras, focussing on the rank-3 algebra $\mathfrak{F}$ first investigated by Feingold and Frenkel. Our approach is based on the concrete realization of this Lie algebra in terms of a Hilbert space of transverse and longitudinal physical string states, which are expressed in a basis using DDF operators. When decomposed under its affine subalgebra $A_1^{(1)}$, the algebra $\mathfrak{F}$ decomposes into an infinite sum of affine representation spaces of $A_1^{(1)}$ for all levels $\ell \in \mathbb{Z}$. For $|\ell| > 1$ there appear in addition coset Virasoro representations for all minimal models of central charge $c < 1$, but the different level-ℓ sectors of $\mathfrak{F}$ do not form proper representations of these because they are incompletely realized in $\mathfrak{F}$. To get around this problem we propose to nevertheless exploit the coset Virasoro algebra for each level by identifying for each level a (for $|\ell| \geq 3$ infinite) set of 'Virasoro ground states' that are not necessarily elements of $\mathfrak{F}$ (in which case we refer to them as 'virtual'), but from which the level-ℓ sectors of $\mathfrak{F}$ can be fully generated by the joint action of affine and coset Virasoro raising operators. We conjecture (and present partial evidence) that the Virasoro ground states for $|\ell| \geq 3$ in turn can be generated from a *finite* set of 'maximal ground states' by the additional action of the 'spectator' coset Virasoro raising operators present for all levels $|\ell| > 2$. Our results hint at an intriguing but so far elusive secret behind Einstein's theory of gravity, with possibly important implications for quantum cosmology.

Contents

Published online: 03 October 2025

1. Introduction

Our central object of interest in this paper is the hyperbolic Kac–Moody algebra (KMA) $\mathfrak{F} \equiv \mathfrak{g}(A)$ associated with the indefinite Cartan matrix of rank three

$$(A_{ij}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}, \tag{1.1}$$

whose study was pioneered by Feingold and Frenkel [1] (see [2] for a general introduction to the theory of KMAs). The associated Dynkin diagram with our labelling of roots is shown in Fig. 1. The associated generators $\{e_i, f_i, h_i\}$ for $i \in \{-1, 0, 1\}$ obey the commutation relations ('Chevalley-Serre presentation')

$$\begin{aligned} [h_i, h_j] &= 0\,, \qquad [e_i, f_j] = \delta_{ij} h_i \\ [h_i, e_j] &= A_{ij}\, e_j\,, \quad [h_i, f_j] = -A_{ij}\, f_j\,, \\ \mathrm{ad}(e_i)^{1-A_{ij}}(e_j) &= \mathrm{ad}(f_i)^{1-A_{ij}}(f_j) = 0\,. \end{aligned} \tag{1.2}$$

With the $\mathfrak{sl}(2)$ building blocks consisting of the triples $\{e_i, f_i, h_i\}$ the Lie algebra $\mathfrak{g}(A)$ is then defined to be the Cartan subalgebra (CSA) spanned by the h_i plus the two free Lie algebras over the generators e_i and f_i, respectively, modulo the above relations [2].

The KMA based on the Cartan matrix (1.1), which is interchangeably designated as $\mathfrak{F}$ or $HA_1^{(1)}$, or AE_3 or A_1^{++} (the latter two designations being preferred in the physics literature), is the simplest hyperbolic KMA with a null root, and thus admits a distinguished

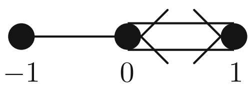

Fig. 1. Dynkin diagram of $\mathfrak{F}$ with labelling of nodes

affine subalgebra $A_1^{(1)} \equiv A_1^+$. Although not much is known about $\mathfrak{F}$, the following facts have been established [1]. The 'germ' of the algebra $\mathfrak{F}$ resides in the beginnings of its graded decomposition w.r.t. its distinguished affine subalgebra for levels $|\ell| \le 1$

$$\overline{\mathrm{V}} \oplus \mathfrak{F}^{(0)} \oplus \mathrm{V}\,, \tag{1.3}$$

where at the center we have the affine subalgebra $\mathfrak{F}^{(0)} \equiv A_1^{(1)} \subset \mathfrak{F}$. At level one, $\mathrm{V} \equiv \mathfrak{F}^{(1)} = L(\Lambda_0 + 2\boldsymbol{\delta})$ is the basic representation, while $\overline{\mathrm{V}} \equiv \mathfrak{F}^{(-1)}$ is the conjugate representation (see the following section for our definitions, conventions, and nomenclature). The algebra $\mathfrak{F}$ can then be generated by multiply commuting V and $\overline{\mathrm{V}}$. This task is, however, complicated enormously by the need to divide out ideals generated by the Serre relations (*i.e.* the last line in (1.2)). At level 2, this is still relatively simple, and we have [1]

$$\mathfrak{F}^{(2)} \cong \mathrm{V} \wedge \mathrm{V} \big/ \mathcal{J}_2 \equiv \mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(1)} \big/ \mathcal{J}_2\,, \tag{1.4}$$

where $\mathcal{J}_2$ is the ideal generated by the Serre relation involving the over-extended root with index -1 and $\mathcal{J}_2$ carries an action of the affine algebra $\mathfrak{F}^{(0)}$.[1] The above formula (1.3) is only the beginning of an infinite string of vector subspaces $\mathfrak{F}^{(\ell)}$ extending in both directions with $\ell \in \mathbb{Z}$, see (2.11) below, where each subspace $\mathfrak{F}^{(\ell)}$ consists of an infinite sum of affine representation spaces for $|\ell| > 1$. Consequently, the main obstacle towards a more 'global' understanding of $\mathfrak{F}$ is that the procedure of dividing out Serre relations gets more and more cumbersome with higher levels already for levels $\ell = 3$ and $\ell = 4$ [4,5]. For those levels, the complications are also evident from the formulas in [6] and our explicit results for $\ell = 3$ and $\ell = 4$.

As for products of affine representations, it has long been known that [7]

$$L(\Lambda_0 + 2\boldsymbol{\delta}) \wedge L(\Lambda_0 + 2\boldsymbol{\delta}) = \mathrm{Vir}(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(2\Lambda_1 + 3\boldsymbol{\delta})\,, \tag{1.5}$$

where $\mathrm{Vir}(\frac{1}{2}, \frac{1}{2})$ is the minimal representation of the coset Virasoro algebra with central charge $c = \frac{1}{2}$ and $h = \frac{1}{2}$; we recall that such a coset Virasoro algebra always accompanies the product of affine representations [8].[2] However, in the algebra $\mathfrak{F}$ the nice product structure of the r.h.s. is lost because one has to remove the top state associated with the Serre relation, and thus one whole affine representation $L(2\Lambda_1 + 3\boldsymbol{\delta})$, so that by (1.4) the level-2 sector of $\mathfrak{F}$ has the vector space structure [1]

$$\mathfrak{F}^{(2)} = \Big(\mathrm{Vir}(\tfrac{1}{2}, \tfrac{1}{2}) \ominus \mathbb{R}v_0\Big) \otimes L(2\Lambda_1 + 3\boldsymbol{\delta})\,, \tag{1.6}$$

where v_0 is the vacuum state of the $\mathrm{Vir}(\frac{1}{2}, \frac{1}{2})$ representation. Taking out the subspace $\mathbb{R}v_0$ leaves a 'hole' in the coset Virasoro representation space $\mathrm{Vir}(\frac{1}{2}, \frac{1}{2})$, as a result of which the level-2 sector of the KMA is *not* a representation of the coset Virasoro algebra anymore. Indeed, as we will show explicitly, the Virasoro algebra is no longer obeyed on the truncated representation space, a statement which extends to all levels of the Lie algebra $\mathfrak{F}$. We will exhibit a structure similar to (1.6) also for higher levels, where similar 'holes' will appear in the relevant coset Virasoro representations.

As a consequence, there is no 'easy' way to construct the algebra by simply multiplying affine representations as in (1.5), and to obtain the Lie algebra elements of a

---

1 An analogous decomposition for the maximal rank hyperbolic KMA E_{10} is given in [3].

2 Our conventions for tensor products of this type and the treatment of shifts by the null root $\boldsymbol{\delta}$ are explained below in section 4.2.

given level-ℓ sector by application of the affine and coset Virasoro raising generators to a given set of ground states that belong to $\mathfrak{F}$. In order to circumvent this difficulty, one main new tool employed in this work is to fill the 'holes' by introducing 'virtual states' which belong to the relevant tensor products (corresponding to the l.h.s. of (1.5)), but vanish as elements of the Lie algebra, in this way restoring the full coset Virasoro representation.

At least for levels $|\ell| \leq 2$ this trick enables us to generate the whole level-ℓ sector by acting with the coset Virasoro algebra and the affine algebra on a finite set of states that we will refer to as 'maximal ground states'. For levels $\ell > 2$ we encounter a vector space structure similar to (1.6) but with a 'pile-up' of coset Virasoro representations stemming from calculations similar to (1.5) (and more generally, (4.9)). This pile-up generates infinitely many copies of the finitely many maximal ground states. We call these copies 'Virasoro ground states'. The application of only affine and coset Virasoro raising generators does not allow us to generate these additional Virasoro ground states from the maximal ground states. Hence for level $\ell = 3$ we propose yet another set of operators that does exactly this. We conjecture that there exists a generalization of this operator for all $\ell > 3$. Together with the affine and coset Virasoro raising operators these operators would allow us to generate any level-ℓ sector $\mathfrak{F}^{(\ell)}$ of $\mathfrak{F}$ from the finite set of maximal ground states (which are essentially in one-to-one correspondence with the allowable weights at level ℓ (2.17)). An interactive visualization of the associated root systems is presented in [9].

A second new tool we rely on is the vertex operator formalism in the specific version developed in [10,11], which builds on the seminal work of [12–14]. In this formalism, the Lie algebra is realized as a subspace of a certain Hilbert space of physical string states, such that the elements of the Lie algebra are explicitly given in terms of DDF states built on certain tachyonic ground states, rather than in terms of multi-commutators (the Del Giudice, Di Vecchia, Fubini (DDF) formalism [15] is a well known and convenient tool to generate physical states in string theory). A key feature first pointed out in [10] is that for all levels $|\ell| > 1$, there also appear *longitudinal* DDF states in the algebra, in addition to the transversal DDF states familiar from the critical string. One main advantage of the vertex operator algebra formalism is that we do not have to worry about Jacobi identities and the Serre relations as these are automatically taken care of with the definition (3.4). That is, unlike in [1,3,6] there is no need to take out affine representations 'by hand', subtracting sub-representations and compensating for over-subtractions. Here, we will give explicit expressions for the maximal ground states for $\ell \leq 4$ in terms of the DDF basis. In this way, we seek to develop a perspective on hyperbolic KMAs different from the one usually taken in the mathematics literature, with the aim of gaining a more 'global' understanding of its structure, as well as a more concrete realization of the algebra itself (as opposed to merely counting root multiplicities).

The Virasoro ground states are here determined by imposing the conditions (4.8) on a given ansatz in terms of DDF states. With increasing level this method becomes more and more unwieldy (for instance, at level $\ell = 4$ the 'deepest' such state is so far inaccessible by our methods). Therefore it would be desirable to determine the maximal ground states by independent and more efficient means. If this can be done, we would have an efficient tool to explore higher levels. On top of unbounded pile-up of coset Virasoro representations described above, there is the added difficulty that in the final product for general level ℓ certain subspaces of affine representations must be taken out, in analogy with (1.6). The real complication is therefore not so much with products of affine representations but with the 'holes' in the coset Virasoro representations, which become more and more difficult to deal with as the level is increased. This proliferation of complications is reminiscent of the fractal structure of a Mandelbrot set, although we know of no Lie algebra analog of the self-similarity features.

To conclude this introduction we wish to underline the potential relevance of the KMA $\mathfrak{F}$ for physics. We claim that this algebra hides a deeply buried secret about Einstein's theory! To explain this point, observe that from the Cartan matrix (1.1) we see that $\mathfrak{F}$ possesses two distinguished rank-two subalgebras, both of which appear in the dimensional reduction of Einstein's theory to lower dimensions. Namely, the upper 2-by-2 submatrix corresponding to a $\mathfrak{sl}(3)$ subalgebra is associated with the Matzner-Misner SL(3) (actually GL(3)) group obtained by reducing Einstein gravity from four to one dimension. On the other hand, the lower 2-by-2 submatrix is associated with an $A_1^{(1)}$ affine symmetry, which is just the Lie algebra underlying the Geroch group of general relativity, obtained by reducing Einstein's theory to two dimensions [16,17]. The lowermost diagonal entry corresponds to the Ehlers $\mathfrak{sl}(2)$ symmetry obtained by dualizing the Kaluza-Klein vector in three dimensions. The Geroch algebra and the Matzner-Misner $\mathfrak{sl}(3)$ intersect in the middle entry corresponding to the Matzner-Misner SL(2), which likewise has been known for a long time in general relativity.

All this suggests that one might try to find a concrete physical realization of $\mathfrak{F}$ by simply combining the Matzner-Misner SL(3) and the Geroch symmetry [18]. However, it turns out that a simple dimensional reduction to one dimension cannot accomplish this because to realize the Geroch group, we need *two* coordinates for the duality transformations ([18] tried to circumvent this problem by means of a null reduction, but again finds that the bulk of $\mathfrak{F}$ is realized only trivially; see also [19] for a recent related investigation). The conclusion is that we cannot find a non-trivial realization of $\mathfrak{F}$ by sticking with Einstein's theory and standard notions of space-time based field theory, but need an extension from which standard general relativity 'emerges' only in a specific limit. Hints of such a theory have emerged from the study of cosmological billiards [20]. In particular, the celebrated BKL analysis [21] of cosmological singularities can be rephrased in terms of a cosmological billiard that takes place in the Weyl chamber of the Weyl group of $\mathfrak{F}$ (it is a main result of [1] that the even part of the Weyl group for $\mathfrak{F}$ is the modular group $PSL_2(\mathbb{Z})$).

In view of the compelling links with Einstein gravity on the one hand [20] and the horrendous complexity of $\mathfrak{F}$ on the other, one may also ask about possible implications for Big Bang cosmology. From a physics perspective, the pile-up of truncated Virasoro modules with increasing level may indicate that more and more degrees of freedom 'open up' in the approach towards the cosmological singularity. It is for this reason that [22] conjectured the emergence of a mathematically well-defined notion of non-computability towards the singularity which may thwart attempts at mathematically understanding the beginning of time, unless a more 'global' description of $\mathfrak{F}$ can be found. At the very least this shows that the restriction to finitely many degrees of freedom that underlies most investigations in quantum cosmology (*e.g.* by means of a mini-superspace approximation, where keeping only diagonal metric degrees of freedom would correspond to restricting $\mathfrak{F}$ to its CSA) may be far too naïve to understand the quantum origin of our universe.

2. Basic Facts About $\mathfrak{F}$

For (1.1) we denote the simple roots by $\{\mathbf{r}_{-1}, \mathbf{r}_0, \mathbf{r}_1\}$, such that their inner products yield

$$A_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j \tag{2.1}$$

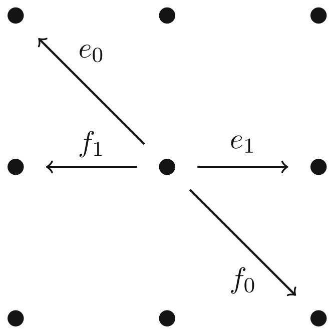

Fig. 2. The action of the Chevalley-Serre generators e_i and f_i depicted in the affine root sublattice Q'

with the over-extended root $\mathbf{r}_{-1}$. The affine null root is $\boldsymbol{\delta} = \mathbf{r}_0 + \mathbf{r}_1$, so that $\mathbf{r}_{-1} \cdot \boldsymbol{\delta} = -1$. The one-dimensional Ehlers $\mathfrak{sl}(2)$ root sublattice of the $\mathfrak{F}$ root lattice is simply $\mathbb{Z}\mathbf{r}_1$, consisting of the elements $\pm n\mathbf{r}_1$ of length $2n^2$. There are two important regular rank-two subalgebras, namely $A_2 \equiv \mathfrak{sl}(3)$ with simple roots $\{\mathbf{r}_{-1}, \mathbf{r}_0\}$, and the affine $A_1^{(1)}$ with simple roots $\{\mathbf{r}_0, \mathbf{r}_1\}$.

The three simple roots of $\mathfrak{F}$ are associated with the generators $\{e_i, f_i, h_i\}$ for $i \in \{-1, 0, 1\}$ which satisfy (1.2). With the root lattice $Q = \mathbb{Z}_{-1}\mathbf{r}_{-1} \oplus \mathbb{Z}_0\mathbf{r}_0 \oplus \mathbb{Z}_1\mathbf{r}_1$ we write the roots as

$$(a_{-1}, a_0, a_1) \equiv \mathbf{r} = a_{-1}\mathbf{r}_{-1} + a_0\mathbf{r}_0 + a_1\mathbf{r}_1 \,. \tag{2.2}$$

We can alternatively parametrize them in terms of the null root as

$$\mathbf{r} = a_{-1}\mathbf{r}_{-1} + k_{-1}\boldsymbol{\delta} + a_1\mathbf{r}_1 \equiv (a_{-1}, k_{-1}, a_1 + k_{-1}) \,. \tag{2.3}$$

For this combination to be a root we must have

$$\mathbf{r}^2 = 2a_{-1}(a_{-1} - k_{-1}) + 2a_1^2 \le 2 \,. \tag{2.4}$$

The level ℓ of a root $\mathbf{r}$ is defined by

$$\ell := \mathbf{r} \cdot \boldsymbol{\delta} = -a_{-1} \tag{2.5}$$

and thus counts the number of occurrences of e_{-1} or f_{-1} in a multi-commutator. Here we adopt the conventions of [3,11], so *positive level* is associated with *negative roots*.

The affine algebra $A_1^{(1)} \subset \mathfrak{F}$ identified above will play a central role in the remainder, as we will focus exclusively on the decomposition of $\mathfrak{F}$ w.r.t. this affine subalgebra, as in [1]. On the affine root sublattice $Q' = \mathbb{Z}_0\mathbf{r}_0 \oplus \mathbb{Z}_1\mathbf{r}_1$, the Chevalley-Serre generators induce the transformations shown in Fig. 2.

The generators h_0 and h_1 do not induce transformations in the root lattice but give rise to eigenvalue equations. A more explicit description of $A_1^{(1)}$ is afforded by

$$A_1^{(1)} = \mathrm{span}_{\mathbb{R}} \left\{ E_m, F_m, H_m, K, \mathfrak{d} \,|m \in \mathbb{Z} \right\}, \tag{2.6}$$

with the standard commutation relations

$$\begin{aligned} [H_m, E_n] &= 2E_{m+n} \,, & [H_m, F_n] &= -2F_{m+n} \,, \\ [E_m, F_n] &= H_{m+n} + mK\delta_{m+n,0} \,, & [H_m, H_n] &= 2mK\delta_{m+n,0} \,, \\ [E_m, E_n] &= [F_m, F_n] = 0 \end{aligned} \tag{2.7}$$

and where

$$\begin{aligned} E_0 &= e_1\,, & F_0 &= f_1\,, & H_0 &= h_1\,, \\ F_1 &= e_0\,, & E_{-1} &= f_0\,, & K - H_0 &= h_0\,. \end{aligned} \tag{2.8}$$

Furthermore, we define

$$K \;=\; h_0 + h_1 \quad , \qquad \mathfrak{d} = h_{-1} + 2h_0 + 2h_1\,, \tag{2.9}$$

where K commutes with all affine generators; its eigenvalue on an arbitrary element of $\mathfrak{F}$ is the level ℓ associated to that element and for this reason we will call K the 'level counting operator'. $\mathfrak{d}$ is the 'depth counting operator' with

$$\big[\mathfrak{d}\,,\,T_m\big] \;=\; -mT_m \quad \text{for all} \quad T_m \in \big\{E_m, F_m, H_m\big\} \tag{2.10}$$

and records the coefficient of $-\boldsymbol{\delta}$ for any given root (2.3) of $\mathfrak{F}$. Thus the depth *increases by one*, if the root is shifted by $-\boldsymbol{\delta}$ in our conventions.[3] Together with h_1, the operators K and $\mathfrak{d}$ span the CSA of $\mathfrak{F}$.

After these preparations the algebra can be decomposed into eigenspaces of K

$$\mathfrak{F} \;=\; \bigoplus_{\ell\in\mathbb{Z}} \mathfrak{F}^{(\ell)}\,. \tag{2.11}$$

We will refer to this decomposition as the 'level decomposition' of $\mathfrak{F}$. The level-ℓ subspace $\mathfrak{F}^{(\ell)}$, on which $K = \ell$, is thus the linear span of all multi-commutators $[f_{i_1}, \ldots, [f_{i_{n-1}}, f_{i_n}]\ldots]$ with ℓ generators f_{-1}. Negative levels are similarly associated with multi-commutators $[e_{i_1}, \ldots, [e_{i_{n-1}}, e_{i_n}]\ldots]$ with $|\ell|$ generators e_{-1} in the multi-commutator. Hence, each subspace $\mathfrak{F}^{(\ell)}$ decomposes into (generally infinitely many) irreducible representations of $A_1^{(1)}$. K is the central charge which commutes with all elements of $A_1^{(1)}$, but not of $\mathfrak{F}$. Furthermore, the subspace $\mathfrak{F}^{(-\ell)}$ is conjugate to $\mathfrak{F}^{(\ell)}$ and thus does not need to be studied separately. Below, we will therefore restrict attention to positive levels, *i.e. highest weight* representations of $A_1^{(1)}$, hence multi-commutators of $\{f_i\}$.

The following theorem is of central importance (for readers' convenience we include a short proof of this Theorem in Appendix A, see also Theorem 1 in [10]).

Theorem 1 (Feingold-Frenkel [1]).
Any level-ℓ element of $\mathfrak{F}$ can be obtained as a linear combination of commutators of level-one and level-$(\ell-1)$ elements, that is

$$\mathfrak{F}^{(\ell)} \;=\; \Big[\mathfrak{F}^{(1)},\, \mathfrak{F}^{(\ell-1)}\Big]\,. \tag{2.12}$$

Thus, one can proceed to explore the algebra level by level, moving up in level by one step at a time. The commutator of two elements thus provides a map from the tensor product $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)}$ to $\mathfrak{F}^{(\ell)}$ which we denote by $\mathcal{I}^{(\ell)}$, so that

$$\mathcal{I}^{(\ell)} \;:\; \mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)} \;\to\; \mathfrak{F}^{(\ell)} \quad , \qquad \mathcal{I}^{(\ell)}\big(u \otimes v\big) := [u, v] \qquad \big(u \in \mathfrak{F}^{(1)}\,,\; v \in \mathfrak{F}^{(\ell-1)}\big)\,. \tag{2.13}$$

3 The Cartan generator $\mathfrak{d}$ is related to the standard derivation $d = h_{-1} + h_0 + h_1$ by $\mathfrak{d} = d + K$.

The map $\mathcal{I}^{(\ell)}$ is surjective by Theorem 1, but has a non-trivial kernel, as a result of which we have the vector space isomorphism

$$\mathfrak{F}^{(\ell)} \;\cong\; \left(\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)}\right) \big/ \operatorname{Ker} \mathcal{I}^{(\ell)} \,. \tag{2.14}$$

While the tensor product of affine representations in the numerator can be evaluated by standard techniques, at least in principle, the main difficulty is in determining the kernels for all $\mathcal{I}^{(\ell)}$, which in particular include affine representations associated with the Serre relations. A main novelty of the present approach is that, once we have $\mathfrak{F}^{(1)}$ and $\mathfrak{F}^{(\ell-1)}$ in terms of DDF states, the commutator of any given pair of states can be directly evaluated by means of the universal formula (3.4), which at least in principle gives all elements of $\mathfrak{F}^{(\ell)}$, again in terms of DDF states, and in this way also furnishes information about the 'structure constants' of $\mathfrak{F}$. Let us stress that the quotient in (2.14) must be distinguished from the division of the free Lie algebra by the 'Serre ideal' $\oplus_{|\ell|\geq 2}\mathcal{J}_\ell$ that is employed in more standard approaches, *cf.* [6] and section 3 in [3]. Note also that our $\mathcal{I}^{(\ell)}$ is not the same as I_ℓ in [6], the latter being defined for the free Lie algebra.

We will also need the (hyperbolic) fundamental weights $\{\Lambda_j\}$, which are defined by

$$\mathbf{r}_i \cdot \Lambda_j \;=\; \delta_{ij} \,. \tag{2.15}$$

For the algebra $\mathfrak{F}$ they are given by

$$\Lambda_{-1} \;=\; -\boldsymbol{\delta} \;, \quad \Lambda_0 \;=\; -\mathbf{r}_{-1} - 2\boldsymbol{\delta} \;, \quad \Lambda_1 \;=\; -\mathbf{r}_{-1} - 2\boldsymbol{\delta} + \frac{1}{2}\mathbf{r}_1 \,. \tag{2.16}$$

From (2.16) it follows immediately that the highest affine weights which can appear at level ℓ are

$$\Lambda \;=\; p_0\Lambda_0 + 2p_1\Lambda_1 + p_{-1}\Lambda_{-1} \quad \text{with} \quad p_0 + 2p_1 \;=\; \ell \quad \text{and} \quad p_0\,,\; p_1 \in \mathbb{Z}_{\geq 0}\,,\; p_{-1} \in \mathbb{C} \tag{2.17}$$

($2p_1$ because $\mathbf{r}_1$ can only appear with integer coefficients as Λ must be an element of the root lattice of $\mathfrak{F}$). For given ℓ, the number p_{-1} will always be an integer; in fact, for level $|\ell| \geq 2$ representations $L(\Lambda)$ with infinitely many different values p_{-1} will occur for given p_0 and p_1. In the analysis of $\mathfrak{F}$ we will encounter almost all irreducible highest-weight representations of the affine algebra for Λ of the form (2.17). They are uniquely characterized by providing the highest weight from (2.17) and denoted by $L(\Lambda)$. The coefficients p_0 and p_1 are constrained to be non-negative integers, but the coefficient p_{-1} of $\Lambda_{-1} = -\boldsymbol{\delta}$ is arbitrary and the corresponding representation spaces differ only by the $\mathfrak{d}$ eigenvalues of the highest weight vectors.

For each level, there is an associated Sugawara realization of the Virasoro algebra, with generators ${}^{[\ell]}\mathcal{L}_m^{\text{sug}}$ $(m \in \mathbb{Z})$

$${}^{[\ell]}\mathcal{L}_m^{\text{sug}} := \frac{1}{2(\ell+2)} \sum_{k\in\mathbb{Z}} {}^{*}_{*} \left[\frac{1}{2} H_k H_{m-k} + E_k F_{m-k} + F_k E_{m-k}\right] {}^{*}_{*} \,. \tag{2.18}$$

The normal ordering for the affine generators in (2.18) is defined by (with a sum A over the $\mathfrak{sl}(2)$ generators paired by the Killing form)

$${}^{*}_{*} T_m^A T_n^A {}^{*}_{*} := \begin{cases} T_m^A T_n^A & m < 0\,, \\ T_n^A T_m^A & m \geq 0\,. \end{cases} \tag{2.19}$$

On each $\mathfrak{F}^{(\ell)}$ we have

$$\left[{}^{[\ell]}\mathcal{L}_m^{\text{sug}}, T_n\right] = -nT_{m+n} \tag{2.20}$$

for any affine generator $T_n \in \{E_n, F_n, H_n\}$ acting on the level-ℓ representations in $\mathfrak{F}^{(\ell)}$. The corresponding (Sugawara) central charge at level-ℓ is [23]

$$c_\ell^{\text{sug}} = \frac{3\ell}{\ell+2}\,. \tag{2.21}$$

Even more important for us is the fact that for each level there is a **coset Virasoro algebra** with generators ${}^{[\ell]}\mathfrak{L}_m^{\text{coset}}$ which commutes with the affine generators [8]. The action of ${}^{[\ell]}\mathfrak{L}_m^{\text{coset}}$ does not affect the affine representation but shifts the associated affine weight diagrams by $m\boldsymbol{\delta}$. The key point here is that the action ${}^{[\ell]}\mathfrak{L}_m^{\text{coset}}$ is defined only on the tensor product $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)}$, but not directly on $\mathfrak{F}^{(\ell)}$ where its implementation would lead to inconsistencies, for which we will give some examples below. Equivalently, there is no consistent action of the coset Virasoro algebra on the kernel Ker $\mathcal{I}^{(\ell)}$. In general, we therefore do not have a proper representation of the level-ℓ coset Virasoro algebras on the level-ℓ sectors of the Lie algebra.

More specifically, consider a level-one element $u \in \mathfrak{F}^{(1)}$ and a level-$(\ell-1)$ element $v \in \mathfrak{F}^{(\ell-1)}$ (for $\ell > 1$) and their tensor product $w = u \otimes v$. The action of the affine generators $T_m \in \{E_m, F_m, H_m\}$ on w obeys the usual distributive law

$$T_m w \equiv T_m\,(u \otimes v) = (T_m u) \otimes v + u \otimes (T_m v) \tag{2.22}$$

and remains valid in this form if the tensor product is replaced by a commutator. The action of the coset Virasoro element on tensor products is defined in terms of the Sugawara generators (2.18) by

$${}^{[\ell]}\mathfrak{L}_m^{\text{coset}} w \equiv {}^{[\ell]}\mathfrak{L}_m^{\text{coset}}(u \otimes v) := \left({}^{[1]}\mathcal{L}_m^{\text{sug}} u\right) \otimes v + u \otimes \left({}^{[\ell-1]}\mathcal{L}_m^{\text{sug}} v\right) - {}^{[\ell]}\mathcal{L}_m^{\text{sug}} w\,. \tag{2.23}$$

In general, level-ℓ elements are sums of such tensor products, in which case this formula applies summand by summand. When one replaces the tensor product by a Lie algebra commutator $[u, v]$ inconsistencies arise whenever this commutator vanishes although the tensor product does not. *For this reason formula* (2.23) *must not be used with Lie algebra commutators*, as this will lead to inconsistencies but only to elements of the tensor product $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)}$. Likewise applying this formula to a Lie algebra element that can be reached in two different ways by lower level commutators will lead to contradictory results. Below we will exhibit explicit examples of this phenomenon, and show how the coset Virasoro operator on the commutator fails already on level 2.

From the above theorem, it follows that the coset Virasoro central charge [8] associated with $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)}$, and hence also with $\mathfrak{F}^{(\ell)}$, is

$$c_\ell^{\text{coset}} = 1 + \frac{3(\ell-1)}{\ell+1} - \frac{3\ell}{\ell+2} = 1 - \frac{6}{(\ell+1)(\ell+2)}\,. \tag{2.24}$$

Consequently, all minimal Virasoro representations will occur in the analysis of $\mathfrak{F}$. Each level $\mathfrak{F}^{(\ell)}$ will thus decompose into sums of products of certain level-ℓ representations of the affine algebra and the associated truncated representations of the level-ℓ coset Virasoro algebra, furthermore adorned by an increasing tail of products of lower level coset Virasoro characters. The fact that the central charge (2.24) is bounded from above by 1 is not generally true, and in fact violated for higher rank hyperbolic algebras such as

E_{10}. For the minimal series the allowed ${}^{[\ell]}\mathfrak{L}_0^{\text{coset}}$ eigenvalues at level ℓ are then contained in the following list [8] (see also [24])

$$h_{r,s}^{(\ell)} = \frac{[(\ell+2)r-(\ell+1)s]^2-1}{4(\ell+1)(\ell+2)} . \qquad (r=1,\ldots,\ell\,;\ s=1,\ldots,r) \tag{2.25}$$

These are the values that can be assumed by the virtual ground states, but are shifted by integers in the coset Virasoro descendant states. There is no such restriction on the eigenvalues h to a discrete set for $c > 1$ [24]. In fact, in the ultimate analysis of $\mathfrak{F}$ there will appear such representations galore once one tries to simplify the 'porous' coset Virasoro representations by reducing products.

3. DDF Construction

Following [10,11] our main tool to analyze the algebra is to represent it in terms of a certain subspace of a Hilbert space of physical string states. More specifically, we will be dealing with a subcritical compactified bosonic string whose target space-time dimension is $d = 3$, equal to the rank of $\mathfrak{F}$, and whose momenta lie on the Lorentzian root lattice Q which is the $\mathbb{Z}$-linear span of the three simple roots $\{\mathbf{r}_{-1}, \mathbf{r}_0, \mathbf{r}_1\}$. Because $d = 3 < 26$ there will also appear longitudinal states in addition to the transversal states [10], and these will show up for all levels $|\ell| \geq 2$. For the details of this construction we refer to [10], and here only summarize some salient points.

3.1. Lie algebra of physical states. As usual, the string Fock space that we will associate to the Lie algebra comes equipped with elementary Virasoro operators

$$\mathrm{L}_m := \frac{1}{2}\sum_{n\in\mathbb{Z}} {:}\alpha_{m-n}^{\mu}\alpha_{n\mu}{:} \tag{3.1}$$

with the usual string oscillators α_m^μ for $\mu = 0, 1, 2$ and $m \in \mathbb{Z}$. We define the space $\mathfrak{P}_n$ by

$$\varphi \in \mathfrak{P}_n \quad \Leftrightarrow \quad \mathrm{L}_m\varphi = 0 \quad (m \geq 1) \quad \text{and} \quad (\mathrm{L}_0 - n)\varphi = 0 \tag{3.2}$$

and physical states belong to $\mathfrak{P}_n$ for $n = 1$. As shown in the theory of vertex operator algebras [12,14] (see also [10] for an introduction) the following quotient space is then a Lie algebra

$$\mathfrak{H} := \mathfrak{P}_1/\mathrm{L}_{-1}\mathfrak{P}_0 \,, \tag{3.3}$$

where, as explained at length in these references, the commutator between any two elements $\varphi, \psi \in \mathfrak{H}$ is defined via the state-operator correspondence through the formula

$$[\varphi\,,\ \psi] := \oint \frac{\mathrm{d}z}{2\pi i}\,\mathcal{V}(\varphi; z)\,\psi \tag{3.4}$$

and where $\mathcal{V}(\varphi; z)$ is the vertex operator associated to the state φ. As shown in [12,14] this definition satisfies all the requisite properties of a Lie bracket, to wit, antisymmetry and the Jacobi identity, *modulo* elements of $\mathrm{L}_{-1}\mathfrak{P}_0$. This is the reason for restricting $\mathfrak{P}_1$ to the quotient (3.3). The actual evaluation of (3.4) becomes more laborious with the excitation level, since for each state φ one first has to work out the associated vertex operator by use of standard formulas, and then re-express the result of the calculation

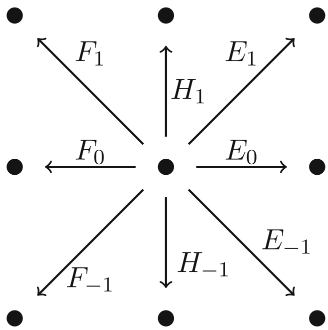

Fig. 3. The action of the affine generators E_m, F_m and H_m on the level-ℓ root sublattice Q'. H_0 gives rise to an eigenvalue equation. The positive Chevalley generators are identified as $e_1 \equiv E_0$ and $e_0 \equiv F_1$

in terms of DDF operators of the appropriate level. For more detailed explanations and simple examples see [10].

The affine subalgebra $\mathfrak{F}^{(0)}$ is a (tiny!) subspace of $\mathfrak{H}$. Adopting physicists' bra and ket notation, its Chevalley-Serre generators are associated with the following states in $\mathfrak{P}_1$

$$e_i := |\mathbf{r}_i\rangle \,, \quad f_i := -\,|{-\mathbf{r}_i}\rangle \,, \quad h_i := \mathbf{r}_{i\mu}\alpha^{\mu}_{-1}|0\rangle \qquad (i = 0, 1) \tag{3.5}$$

so e_i and f_i correspond to tachyonic states, while h_i correspond to photon states [10]. Using (3.4) it can be easily checked that the definitions (3.5) are such that (1.2) is satisfied. More generally, for the affine raising and lowering operators we have the operator-state correspondence

$$E_m \,\widehat{=}\, |\mathbf{r}_1 + m\boldsymbol{\delta}\rangle \,, \quad F_m \,\widehat{=}\, -\,|{-\mathbf{r}_1} + m\boldsymbol{\delta}\rangle \,, \quad H_m \,\widehat{=}\, \mathbf{r}_{1\mu}\alpha^{\mu}_{-1}|m\boldsymbol{\delta}\rangle \,. \tag{3.6}$$

With $\mathbf{r}_1 = \boldsymbol{\delta} - \mathbf{r}_0$ we can equivalently write these states in a Matzner-Misner type basis with $\mathbf{r}_1$ replaced by $\mathbf{r}_0$. The affine raising and lowering operators are thus all associated with tachyonic or photonic states. *The simple formulas* (3.5) *and* (3.6) *contain all that is required for the description of* $A_1^{(1)}$ *as a subspace of* $\mathfrak{H}$.

The affine generators induce the motions on the affine root sublattice (affine ladder diagram) Q' shown in Fig. 3.

At levels $|\ell| \geq 1$ there will be many more states, in particular those corresponding to higher excited string states. We will furthermore have many more tachyonic states. To describe them, we introduce for any root $\mathbf{r}$ its DDF decomposition by

$$\mathbf{r} = \mathbf{a} - n\mathbf{k} \quad \text{with} \quad \mathbf{a}^2 = 2 \,, \quad \mathbf{k}^2 = 0 \quad \text{and} \quad \mathbf{a}\cdot\mathbf{k} = 1 \,. \tag{3.7}$$

We refer to $\mathbf{a}$ as the tachyonic momentum associated to the root $\mathbf{r}$. In addition, we will need polarization vectors $\boldsymbol{\xi} \equiv \boldsymbol{\xi}(\mathbf{a})$ obeying

$$\boldsymbol{\xi}(\mathbf{a})\cdot\mathbf{a} = \boldsymbol{\xi}\cdot\mathbf{k} = 0 \tag{3.8}$$

and that we will normalize to 1. For a level-ℓ root $\mathbf{r} = -\ell\mathbf{r}_{-1} + \cdots$, the null vector $\mathbf{k} = \mathbf{k}(\ell) \equiv \mathbf{k}_\ell$ depends on the level, and we have

$$\mathbf{k}_\ell = \frac{1}{\ell}\boldsymbol{\delta} \,. \tag{3.9}$$

Importantly, *for* $\ell > 1$, $\mathbf{k}_\ell$ *does not belong to the root lattice*, but is nevertheless a necessary ingredient in the DDF construction. For level one we will usually drop the subscript, *i.e.* write $\mathbf{k} \equiv \mathbf{k}_1 = \boldsymbol{\delta}$. At level ℓ the relevant tachyonic momenta are

$$\mathbf{a}_n^{(\ell)} = -\ell \mathbf{r}_{-1} - \left(\ell + \frac{n^2-1}{\ell}\right)\boldsymbol{\delta} + n\mathbf{r}_1 \,, \qquad (n \in \mathbb{Z}) \tag{3.10}$$

with associated tachyon states $|\mathbf{a}_n^{(\ell)}\rangle$. For $\ell = 1$ all these momenta belong to the root lattice, while only for selected values of n if $|\ell| > 1$. The fact that they generally do not for $|\ell| > 1$ is an important feature of our construction of higher level states in $\mathfrak{F}$. On the space of tachyonic states we have the positive definite scalar product[4]

$$\langle \mathbf{a}_m^{(\ell)} \,|\, \mathbf{a}_n^{(\ell')} \rangle = \delta_{mn}\delta^{\ell\ell'} \,, \tag{3.11}$$

which eventually can be extended to the full Hilbert space by means of the DDF algebra (3.18). Denoting the 'spin' of a state by half its h_1 eigenvalue, the spin range of the $(2n+1)$-dimensonal (Ehlers) $\mathfrak{sl}(2)$ multiplet generated from $|\mathbf{a}_n^{(\ell)}\rangle$ is $-n \le s \le n$. The depth of $\mathbf{a}_n^{(\ell)}$ is $\mathfrak{d} = \ell + \frac{n^2-1}{\ell}$.

Furthermore, denoting the elementary Weyl reflections associated with $\mathbf{r}_0$ and $\mathbf{r}_1$, respectively, by w_0 and w_1, we have the following action of the affine Weyl group on $\mathbf{a}_n^{(\ell)}$ for $k \in \mathbb{Z}$

$$\begin{aligned} (w_1 w_0)^k (\mathbf{a}_n^{(\ell)}) &= \mathbf{a}_{n-k\ell}^{(\ell)} \,, \\ w_0 (w_1 w_0)^{k-1} (\mathbf{a}_n^{(\ell)}) &= \mathbf{a}_{-n+k\ell}^{(\ell)} \,. \end{aligned} \tag{3.12}$$

We recall that the affine Weyl group is the semi-direct product of an abelian translation group with a finite Weyl group [2]; in the present case $\mathbb{Z} \rtimes \mathbb{Z}_2$ and we take the translation group to be generated by $\mathfrak{t} \equiv w_1 w_0$. From (3.12), we see that the level-ℓ tachyonic momenta group into $\left[\frac{\ell}{2}\right] + 1$ distinct Weyl orbits. This is also the number of possible choices for Λ_0 and Λ_1 at level ℓ in (2.17). Hence for each admissible level-ℓ weight there is an appropriate tachyonic vacuum on which to build a DDF tower of string states, as explained below.

Physical states are built by acting on these tachyonic vacua with DDF operators [15]. Unlike in uncompactified string theory, for $\mathfrak{F}$ this is a *discrete* construction, in the sense that for any given level only a discrete subset of the momentum space continuum is admitted, of which the root lattice is a subset. This discrete set fills the continuum more and more densely as $\ell \to \infty$.

Because the string is subcritical, the DDF operators come in two varieties.

1) The *transversal* level-ℓ DDF operators ${}^{[\ell]}A_m$ (with $m \in \mathbb{Z}$) are defined by

$${}^{[\ell]}A_m := \oint \frac{dz}{2\pi i}\, \boldsymbol{\xi}_\mu \mathbf{P}^\mu(z) e^{im\mathbf{k}_\ell \cdot \mathbf{X}(z)} = \oint \frac{dz}{2\pi i}\, \boldsymbol{\xi}_\mu \mathbf{P}^\mu(z) e^{(m/\ell)\boldsymbol{\delta}\cdot \mathbf{X}(z)} \tag{3.13}$$

in terms of the usual string coordinate fields

$$\mathbf{X}^\mu(z) = q^\mu - ip^\mu \ln z + i\sum_{m\neq 0} \frac{1}{m}\alpha_m^\mu z^{-m} \,, \quad \mathbf{P}^\mu(z) = i\frac{d}{dz}\mathbf{X}^\mu(z) \,, \tag{3.14}$$

[4] It is the fundamental discreteness of our construction that allows such a discrete scalar product, whereas in the continuum we would have $\langle \mathbf{a}|\mathbf{a}'\rangle = \delta(\mathbf{a} - \mathbf{a}')$.

where again $\mu = 0, 1, 2$. By construction, the DDF operators are physical, to wit,

$$\left[\mathrm{L}_m \,,\, {}^{[\ell]}A_n\right] \,=\, 0\,. \tag{3.15}$$

The operator ${}^{[\ell]}A_m$ shifts the momentum of the state on which it acts by $m\mathbf{k}_\ell = (m/\ell)\boldsymbol{\delta}$. The transversal polarization vector $\boldsymbol{\xi}^\mu$ appearing in (3.13) is given by

$$\boldsymbol{\xi}\big(\mathbf{a}_n^{(\ell)}\big) \,=\, \frac{1}{\sqrt{2}}\big(\mathbf{r}_1 - 2n\mathbf{k}_\ell\big)\,. \tag{3.16}$$

Because for the non-zero modes in (3.13) a shift of the polarization vector by a multiple of $\boldsymbol{\delta}$ drops out as it is a total derivative according to

$$\boldsymbol{\delta}_\mu \mathbf{P}^\mu e^{m/\ell\boldsymbol{\delta}\cdot\mathbf{X}} = -\frac{i\ell}{m}\frac{d}{dz}e^{m/\ell\boldsymbol{\delta}\cdot\mathbf{X}} \quad \text{for all } m \neq 0. \tag{3.17}$$

For simplicity, we will therefore set $n = 0$ in (3.16) for *all* A_m including the zero mode A_0. Unlike for higher rank ($d > 3$) algebras there is only one transversal polarization, whence the DDF oscillators carry no transverse indices for $A_1^{(1)}$. Because of the orthogonality properties and $\boldsymbol{\delta}^2 = 0$ no normal ordering is required in (3.13).

On the level-ℓ subspace the DDF transversal operators obey the standard commutation relations

$$\left[{}^{[\ell]}A_m \,,\, {}^{[\ell]}A_n\right] \,=\, m\delta_{m+n,0} \tag{3.18}$$

because m is the eigenvalue of the operator $(m/\ell)K$ on $\mathfrak{g}^{(\ell)}$. On the tachyonic vacua they obey

$${}^{[\ell]}A_m|\mathbf{a}_n^{(\ell)}\rangle \,=\, 0 \qquad \text{for } m \geq 1\,. \tag{3.19}$$

The key property of the DDF operators is that their application to any physical state creates another physical state by (3.15). The same is true for all the 'composite operators' (affine generators and Sugawara operators) that we will construct from the transversal DDF operators.

2) The *longitudinal* level-ℓ DDF operators ${}^{[\ell]}B_m$ (with $m \in \mathbb{Z}$) are defined by [25]

$${}^{[\ell]}B_m(\mathbf{a}) := \oint \frac{dz}{2\pi i} : \left[-\mathbf{a}_\mu \mathbf{P}^\mu(z) + \frac{m}{2}(\mathbf{a}\cdot\mathbf{k}_\ell)\frac{\mathrm{d}}{\mathrm{d}z}\log\left(\frac{\mathbf{k}_{\ell\,\mu}}{\mathbf{a}\cdot\mathbf{k}_\ell}\mathbf{P}^\mu(z)\right)\right] e^{im\mathbf{k}_\ell\cdot\mathbf{X}(z)} :\,, \tag{3.20}$$

where we use ${}^{[\ell]}B_m$ rather than the more common notation ${}^{[\ell]}A_m^-$ for easier notational distinguishability between transversal and longitudinal DDF operators. Note also that we do *not* include the usual contribution quadratic in transversal DDF operators in this definition (this modification is often included to make transversal and longitudinal DDF operators commute, unlike (3.21)). The argument $\mathbf{a}$ is the tachyonic momentum of the state on which the longitudinal DDF operator ${}^{[\ell]}B_m$ acts. The definition implies [10]

$$\left[{}^{[\ell]}B_m(\mathbf{a})\,,\, {}^{[\ell]}A_n\right] \,=\, -n\,{}^{[\ell]}A_{m+n} + \frac{n}{\sqrt{2}\ell}\,(\mathbf{a}\cdot\mathbf{r}_1)\,\delta_{m+n,0}\,\boldsymbol{\delta}_\mu\alpha_0^\mu\,. \tag{3.21}$$

The longitudinal DDF operators are likewise physical, *viz.*

$$\left[\mathrm{L}_m \,,\, {}^{[\ell]}B_n\right] \,=\, 0\,. \tag{3.22}$$

Finally, the longitudinal DDF operators obey a Virasoro algebra of their own [10]

$$\left[{}^{[\ell]}B_m \,,\, {}^{[\ell]}B_n\right] \,=\, (m-n)\,{}^{[\ell]}B_{m+n} + 2(m^3 - m)\delta_{m+n,0} \tag{3.23}$$

with central charge 24. Let us emphasize that these operators are only well-defined on a subset of the full Hilbert space $\mathfrak{H}$, and only on states for which $\mathbf{a}\cdot\mathbf{k}\neq 0$.

In summary, the space of physical states at level ℓ is the linear span of states

$$\prod_{i=1}^{M} {}^{[\ell]}A_{-m_i} \prod_{j=1}^{N} {}^{[\ell]}B_{-n_j} \, |\mathbf{a}_n^{(\ell)}\rangle \qquad \text{for} \quad \begin{matrix} m_1 \geq \ldots \geq m_M \geq 1\,, \\ n_1 \geq \ldots \geq n_N \geq 2\,, \end{matrix} \qquad \text{and} \quad M, N \geq 0\,. \tag{3.24}$$

The restriction $2 \leq n_j$ is due to the fact that ${}^{[\ell]}B_{-1}$ creates null physical states because

$$\big(\ldots\big)\, {}^{[\ell]}B_{-1}|\mathbf{a}_n^{(\ell)}\rangle \;=\; \big(\ldots\big)\mathrm{L}_{-1}|\mathbf{a}_n^{(\ell)} - \mathbf{k}_\ell\rangle \;=\; \mathrm{L}_{-1}\big(\ldots\big)|\mathbf{a}_n^{(\ell)} - \mathbf{k}_\ell\rangle\,, \tag{3.25}$$

where $(\ldots)$ stands for any combination of DDF operators (which all commute with L_{-1}). These states must be omitted. The total momentum of the state (3.24) is $\mathbf{a}_n^{(\ell)} + (\sum_{i=1}^M m_i + \sum_{j=1}^N n_j)\mathbf{k}_\ell$, which in order to be an element of the root lattice Q must thus satisfy the condition

$$\sum_{i=1}^{M} m_i + \sum_{j=1}^{N} n_j \;=\; \ell P - 1\,. \qquad (P \in \mathbb{N}) \tag{3.26}$$

Importantly, *our Lie algebra formulas make sense only for states obeying this condition, in the sense that all elements of $\mathfrak{F}$ are associated with momenta subject to* (3.26). While the 'in-between' states with momenta not on the root lattice do exist as physical states, various operations, and in particular the definition of the commutator, fail for them, because appropriate cocycle factors cannot be consistently defined for fractional momenta (see Appendix C).

However, even restricting to admissible DDF states with momenta on the root lattice, we will see that the Lie algebra $\mathfrak{F}$ contains only a subset of these states. $\mathfrak{F} \subset \mathfrak{H}$ is thus a proper subspace of $\mathfrak{H}$. This can already be seen from the fact that the number of physical states (3.24) for a given root $\mathbf{r}$ is $p_2(1-\mathbf{r}^2/2) - p_2(-\mathbf{r}^2/2)$, where $\sum_{n\geq 0} p_d(n)q^n = \prod_{k>0}(1-q^k)^{-d}$ is the generating series of partitions of n in d dimensions. The known root multiplicities of $\mathfrak{F}$ are empirically bounded above by the smaller $p_1(1-\mathbf{r}^2/2)$ in accord with Frenkel's conjecture [13]. Our results shed no new light on this conjecture (which is known to be violated for E_{10} [3]). The main challenge is therefore to characterize how precisely $\mathfrak{F}$ is embedded in $\mathfrak{H}$. Below we will exhibit several examples where one can see which DDF states appear in $\mathfrak{F}$.

3.2. Level 1. As a 'warm-up' let us look at the level-one subspace $\mathfrak{F}^{(1)}$ where the longitudinal DDF operators do not yet appear, and where we have a fully explicit description of the so-called basic representation of $A_1^{(1)}$ in terms of the well-known Frenkel-Kac construction [26] (see also [27] and the introductory reviews [23,28]). We here rephrase these results in DDF language, as explained in [10]. By (3.18) the level-one transversal DDF operators are

$$A_m \;\equiv\; {}^{[1]}A_m \;=\; \oint \frac{dz}{2\pi i}\, \boldsymbol{\xi}_\mu \mathbf{P}^\mu(z) e^{m\boldsymbol{\delta}\cdot\mathbf{X}(z)}\,, \tag{3.27}$$

with the tachyonic ground states $|\mathbf{a}_n\rangle$ whose momenta are of the form

$$\mathbf{a}_n \;\equiv\; \mathbf{a}_n^{(1)} \;=\; -\mathbf{r}_{-1} - n^2\boldsymbol{\delta} + n\mathbf{r}_1\,. \qquad (n \in \mathbb{Z}) \tag{3.28}$$

They belong to a single Weyl orbit in the root lattice of $\mathfrak{F}$. With these definitions it is straightforward to write down the first few levels of the basic representation, and thus $\mathfrak{F}^{(1)}$, for depths $\mathfrak{d} \leq 4$ in terms of transversal DDF states

$$\begin{aligned}
\mathfrak{d} &= 0: & &|\mathbf{a}_0\rangle \\
\mathfrak{d} &= 1: & &A_{-1}|\mathbf{a}_0\rangle\,,\ |\mathbf{a}_{\pm 1}\rangle \\
\mathfrak{d} &= 2: & &A_{-2}|\mathbf{a}_0\rangle\,,\ A_{-1}^2|\mathbf{a}_0\rangle\,,\ A_{-1}|\mathbf{a}_{\pm 1}\rangle \\
\mathfrak{d} &= 3: & &A_{-3}|\mathbf{a}_0\rangle\,,\ A_{-2}A_{-1}|\mathbf{a}_0\rangle\,,\ A_{-1}^3|\mathbf{a}_0\rangle\,,\ A_{-2}|\mathbf{a}_{\pm 1}\rangle\,,\ A_{-1}^2|\mathbf{a}_{\pm 1}\rangle \\
\mathfrak{d} &= 4: & &A_{-4}|\mathbf{a}_0\rangle\,,\ A_{-3}A_{-1}|\mathbf{a}_0\rangle\,,\ A_{-2}A_{-2}|\mathbf{a}_0\rangle\,,\ A_{-2}A_{-1}^2|\mathbf{a}_0\rangle\,,\ A_{-1}^4|\mathbf{a}_0\rangle\,, \\
& & &A_{-3}|\mathbf{a}_{\pm 1}\rangle\,,\ A_{-2}A_{-1}|\mathbf{a}_{\pm 1}\rangle\,,\ A_{-1}^3|\mathbf{a}_{\pm 1}\rangle\,,\ |\mathbf{a}_{\pm 2}\rangle
\end{aligned} \tag{3.29}$$

(recall that the depth $\mathfrak{d}$ is the coefficient of $(-\boldsymbol{\delta})$ for the root corresponding to the given DDF state in the form (2.3)). Note that with our choice of polarization we have

$$A_0\,|\mathbf{a}_n\rangle \;=\; \sqrt{2}n\,|\mathbf{a}_n\rangle\,. \tag{3.30}$$

We refer to the uppermost state $|\mathbf{a}_0\rangle \;\equiv\; |-\mathbf{r}_{-1}\rangle \;=\; |\Lambda_0 + 2\boldsymbol{\delta}\rangle$ (corresponding to the Chevalley-Serre generator f_{-1}) as the 'ground state' of $\mathfrak{F}^{(1)}$. It is easily seen that the states (3.29) correspond to the $\mathfrak{sl}(2)$ representations

$$\mathbf{1}_0 \oplus \mathbf{3}_1 \oplus (\mathbf{1} \oplus \mathbf{3})_2 \oplus (\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3})_3 \oplus (\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{5})_4\,. \tag{3.31}$$

This list can be easily continued. As a special case of (3.11) we have the scalar product of the tachyonic states

$$\langle \mathbf{a}_m|\mathbf{a}_n\rangle \;=\; \delta_{mn}\,, \tag{3.32}$$

which can be extended to the full basic representation because we can evaluate all scalar products of DDF states by means of the commutation relations (3.18).

For a graphical representation of the level-one subspace $\mathfrak{F}^{(1)}$ see Fig. 6 in Appendix E. The level-one sector is a single affine representation, hence all its states can be reached by acting on the ground state with affine raising operators. The DDF states at different depths can also be connected via the level-one Sugawara generators [10]

$$^{[1]}\mathcal{L}_m^{\text{sug}} \;=\; \frac{1}{2}\sum_{n\in\mathbb{Z}} {}^*_*A_n A_{m-n}{}^*_* \quad\Rightarrow\quad \left[{}^{[1]}\mathcal{L}_m^{\text{sug}}, A_n\right] \;=\; -nA_{m+n} \tag{3.33}$$

where the normal ordering is defined as in (2.19). This simple quadratic formula works only for the basic representation, whereas at higher levels we encounter non-polynomial expressions in the DDF operators, see next section. With our choice of polarization the operator

$$^{[1]}\mathcal{L}_0^{\text{sug}} \;=\; \frac{1}{2}A_0^2 + A_{-1}A_1 + A_{-2}A_2 + \ldots \tag{3.34}$$

counts the depth of a given DDF state in (3.29), and can thus be identified with the operator $\mathfrak{d}$. The Sugawara operators provide a convenient tool, since $^{[\ell]}\mathcal{L}_m^{\text{sug}}$ while moving up in depth, does not change the $\mathfrak{sl}(2)$ representation content of a given state. Hence one can directly build infinite 'strings' of $\mathfrak{sl}(2)$ singlets, triplets, *etc.* by straightforward application of Sugawara operators $^{[\ell]}\mathcal{L}_{-1}^{\text{sug}}$, $^{[\ell]}\mathcal{L}_{-2}^{\text{sug}}, \ldots$ To move between the different $\mathfrak{sl}(2)$ representations contents we must use the affine generators defined below. It is also straightforward to re-express the list (3.29) in terms of multi-commutators, see *e.g.* [10].

We can also rephrase these results in terms of characters. Recall that $\mathfrak{F}^{(1)} = L(\Lambda_0{+}2\boldsymbol{\delta})$ and so Ch $\mathfrak{F}^{(1)} = q^{-2}\,\mathrm{Ch}\ L(\Lambda_0)$ with $q = e^{-\boldsymbol{\delta}}$, for which we have [1]

$$\mathrm{Ch}\ \mathfrak{F}^{(1)} = \frac{q^{-1}}{\varphi(q)}\,\Theta_{\Lambda_0} \quad , \qquad \Theta_{\Lambda_0} \equiv q^{-1}\sum_{n\in\mathbb{Z}} \exp\left(\mathrm{t}^n(\Lambda_0)\right). \tag{3.35}$$

where $\mathrm{t}{:=}w_1 w_0$ for the elementary affine translation. See Appendix B for a definition of the Euler function φ and the generalized theta function.

4. General Results for Higher Levels

4.1. Affine and Sugawara generators at arbitrary level. In order to implement the action of the affine and the Sugawara generators for arbitrary levels, we here specialize the formulas of [29] and [11] to the case of $\mathfrak{F}$ giving the affine step operators and the Sugawara operators explicitly in terms of transversal DDF operators for each level-ℓ subspace. In particular, these formulas generalize the well-known formula (3.33) at level-one to arbitrary levels. The essential new feature here (in comparison with the standard vertex operator construction) is that the exponents appearing in these expressions are expressed not in terms of standard string oscillators, but rather in terms of transversal DDF operators [10,11]; in the form given they are thus different for each level, and valid only on the respective level-ℓ subspace. The use of transversal DDF operators implies that both the affine and the Sugawara operators map physical states to physical states without changing the level, hence map elements of $\mathfrak{F}^{(\ell)}$ to other elements of $\mathfrak{F}^{(\ell)}$. This statement follows from the fact that $\mathfrak{F}^{(\ell)}$ is a representation of the affine subalgebra $A_1^{(1)}$ of fixed level, thus preserved by affine Lie algebra elements, and since the Sugawara generators (at that level) are bilinears in affine generators. We stress that $\mathfrak{F}^{(\ell)}$ is *not* a representation of any (coset) Virasoro algebra that commutes with the affine algebra.

We first give the formulas for the affine generators. The generators of the Heisenberg subalgebra are identified with a (for $\ell > 1$ proper) subset of the transversal DDF operators

$${}^{[\ell]}H_m := \oint \frac{dz}{2\pi i}\,\mathbf{r}_{1\,\mu}\,\mathbf{P}^\mu(z)e^{m\boldsymbol{\delta}\cdot\mathbf{X}(z)} = \sqrt{2}\,{}^{[\ell]}A_{\ell m}\,. \tag{4.1}$$

So it is only at level one that we have a one-to-one correspondence between the Heisenberg generators and the transversal DDF operators. This is the basic reason for the simplicity of formula (3.33) and its equivalence to a free field energy momentum tensor. For the affine raising and lowering operators we have the formulas

$$\begin{aligned}
{}^{[\ell]}E_m &:= \oint \frac{\mathrm{d}z}{2\pi i}\,z^{\ell m}\,\exp\left(+\sum_{n>0}\frac{\sqrt{2}}{n}\,{}^{[\ell]}A_{-n}z^n\right)\\
&\qquad\times e^{i\mathbf{r}_1\mathbf{Q}}z^{\mathbf{r}_1\mathbf{P}}\exp\left(-\sum_{n>0}\frac{\sqrt{2}}{n}\,{}^{[\ell]}A_n z^{-n}\right)c_{\mathbf{r}_1}\,,\\
{}^{[\ell]}F_m &:= -\oint \frac{\mathrm{d}z}{2\pi i}\,z^{\ell m}\,\exp\left(-\sum_{n>0}\frac{\sqrt{2}}{n}\,{}^{[\ell]}A_{-n}z^n\right)e^{-i\mathbf{r}_1\mathbf{Q}}z^{-\mathbf{r}_1\mathbf{P}}\\
&\qquad\times\exp\left(+\sum_{n>0}\frac{\sqrt{2}}{n}\,{}^{[\ell]}A_n z^{-n}\right)c_{-\mathbf{r}_1}\,,
\end{aligned}\tag{4.2}$$

where $c_{\pm\mathbf{r}_1}$ denotes the cocycle factor (see Appendix C), and where we have written out the normal ordering. The extra label on the affine generators is meant to indicate that these formulas are only valid on the corresponding level-ℓ subspace $\mathfrak{F}^{(\ell)}$ (as always we drop the label ℓ whenever statements are valid for all levels). As explained in [11] the zero mode contribution $\propto \mathbf{r}_1 \cdot \mathbf{Q}$ is *not* a shift in momentum space, but rather a Lorentz boost which rotates the tachyonic momenta (3.10) belonging to different depths into one another, as appropriate. One can now check that the operators (4.1) and (4.2) do satisfy the commutation relations (2.7) on the level-ℓ subspace.

All affine generators act on DDF states (3.24). The result of such an action can then be properly re-ordered by moving all annihilation operators to the right by means of (3.18) and (3.21), until they annihilate the tachyonic ground state by virtue of (3.19), to obtain a new DDF state. ${}^{[\ell]}A_0$ commutes with all other (transversal and longitudinal) DDF operators.

In the sequel we will refer to the Ehlers $\mathfrak{sl}(2)$ multiplet belonging to an affine highest weight state as an **affine ground state** or **affine ground state multiplet**. The members ψ of this $\mathfrak{sl}(2)$ multiplet obey

$$T_m \psi = 0 \quad \text{for all} \quad T_m \in \left\{ E_m, F_m, H_m \mid m \geq 1 \right\}. \tag{4.3}$$

For level one, there is only one affine ground state, the $\mathfrak{sl}(2)$ singlet $|-\mathbf{r}_{-1}\rangle$. By contrast, for $|\ell| \geq 2$ there will be infinitely many such affine ground states for all levels, of increasing depths. In all cases the action of the affine generators on such states is straightforward to evaluate by means of formulas (4.1) and (4.2), with the result again being a DDF state.

In addition to the affine generators, the Sugawara generators can also be represented in DDF form *for arbitrary levels* by plugging (4.1) and (4.2) into (2.18) [11,29]. This has been done explicitly in [11] for general hyperbolic Kac–Moody algebras. We specialize the result given in (3.15) of [11] to $\mathfrak{F}$ by setting $d = 3$ and $h^\vee = 2$ to arrive at

$$\begin{aligned}
{}^{[\ell]}\mathcal{L}_m^{\text{sug}}(\mathbf{a}) = {} & \frac{1}{2\ell} \sum_n {}^*_* \, {}^{[\ell]}A_{\ell n} \, {}^{[\ell]}A_{\ell(m-n)} {}^*_* \\
& + \frac{1}{\ell(\ell+2)} \sum_{n \neq 0 \bmod \ell} {}^*_* \, {}^{[\ell]}A_n \, {}^{[\ell]}A_{\ell m - n} {}^*_* + g_\ell(\mathbf{r}_1 \cdot \mathbf{a}) \delta_{m,0} \\
& - \frac{1}{2\ell(\ell+2)} \sum_{p=1}^{\ell-1} \frac{\zeta^{p\,\mathbf{r}_1 \cdot \mathbf{a}}}{|\zeta^p - 1|^2} \oint \frac{\mathrm{d}z}{2\pi i} \, z^{\ell m - 1} \\
& \qquad \times {}^*_* \left(\exp \left[\sum_{n \neq 0} \frac{\sqrt{2}}{n} \, {}^{[\ell]}A_n z^{-n} (1 - \zeta^{-pn}) \right] - 1 \right) {}^*_* \\
& - \frac{1}{2\ell(\ell+2)} \sum_{p=1}^{\ell-1} \frac{\zeta^{-p\,\mathbf{r}_1 \cdot \mathbf{a}}}{|\zeta^p - 1|^2} \oint \frac{\mathrm{d}z}{2\pi i} \, z^{\ell m - 1} \\
& \qquad \times {}^*_* \left(\exp \left[- \sum_{n \neq 0} \frac{\sqrt{2}}{n} \, {}^{[\ell]}A_n z^{-n} (1 - \zeta^{-pn}) \right] - 1 \right) {}^*_* ,
\end{aligned} \tag{4.4}$$

where again $\mathbf{a}$ is the tachyonic momentum of the state on which the Sugawara operator acts, and ζ is a primitive ℓ-th root of unity. Furthermore, we have introduced the periodic function

$$g_\ell(k) := \frac{\ell^2-1}{12\ell(\ell+2)} - \frac{f_\ell(k)}{\ell(\ell+2)} \tag{4.5}$$

with [29]

$$f_\ell(k) := \sum_{p=1}^{\ell-1} \frac{\zeta^{pk}}{|\zeta^p - 1|^2} = \frac{\ell^2-1}{12} - \frac{k(\ell-k)}{2} \quad \text{for} \quad 0 \le k \le \ell - 1\,. \tag{4.6}$$

This definition of $f_\ell(k)$ can be extended to all integer values of k by means of the relation (which actually follows from the definition (4.6)

$$f_\ell(k) = f_\ell(k+n\ell) = f_\ell(-k)\,, \tag{4.7}$$

thus ensuring symmetry and periodicity of $g_\ell(k)$. As before, the operator (4.4) is to be used and valid only on the level-ℓ subspace $\mathfrak{F}^{(\ell)}$. As we can see, (4.4), being non-polynomial, is no longer equivalent to a free field construction. Nevertheless, the action of the Sugawara generators is again straightforward to evaluate on any DDF state by means of (4.4).

4.2. Ground states. As we explained in section 2, the crucial ingredients in analyzing the higher level sectors of $\mathfrak{F}$ are the affine representation theory and the level-ℓ coset Virasoro algebra. Their combination motivates the following definition valid for all levels $|\ell| \ge 2$.

Definition 1. A **Virasoro ground state (multiplet)** $\Psi^{(\ell)} \in \mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)}$ at level $\ell \ge 2$ is an affine ground state multiplet, which in addition to (4.3) obeys the conditions

$${}^{[\ell]}\mathfrak{L}_m^{\text{coset}}\,\Psi^{(\ell)} = 0 \quad \text{for all} \quad m \ge 1\,. \tag{4.8}$$

These product states are non-vanishing elements of the tensor product, but may vanish as elements of the Lie algebra, that is, after the conversion of the tensor product into a Lie algebra commutator by means of the prescription (3.4). In that case we refer to $\Psi^{(\ell)}$ as a *virtual ground state*. By definition virtual ground states thus belong to the kernel of $\mathcal{I}^{(\ell)}$ defined in (2.13).

We can distinguish two level-ℓ Virasoro ground state multiplets of the same type by their depth and ${}^{[\ell]}\mathfrak{L}_0^{\text{coset}}$ eigenvalue. On level 2 there are finitely many Virasoro ground states and on levels $\ell \ge 3$ there are infinitely many. This motivates the additional definition.

Definition 2. Let $\Psi^{(\ell)} \in \mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)}$ be a Virasoro ground state at level $\ell \ge 2$ and depth $\mathfrak{d}$ with ${}^{[\ell]}\mathfrak{L}_0^{\text{coset}}$ eigenvalue $h_{r,s}^{(\ell)}$. Then $\Psi^{(\ell)}$ is said to be a **maximal (Virasoro) ground state (multiplet)** if there is no level-ℓ Virasoro ground state with the same ${}^{[\ell]}\mathfrak{L}_0^{\text{coset}}$ eigenvalue and depth $< \mathfrak{d}$.

The number of maximal ground states on any level $\ell > 2$ is finite, and essentially in one-to-one correspondence with the admissible level-ℓ weights (2.17). At level 2 all Virasoro ground states are maximal. Moreover, all maximal ground states at levels 2 and 3 are virtual. On level $\ell = 4$ not all maximal ground states are virtual.

The main observation is now that one can generate from the collection of all Virasoro ground states from Defintion 1 *all* states in $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)}$ by the combined action of the affine and coset Virasoro generators. Because $\mathcal{I}^{(\ell)}$ is surjective, we obtain in this way *all* elements of $\mathfrak{F}^{(\ell)}$ as DDF states after conversion of the tensor product to a Lie algebra commutator by means of (3.4), that is under the map $\mathcal{I}^{(\ell)}$. The main open problem is then the determination of $\operatorname{Ker}\mathcal{I}^{(\ell)}$. In section 6.2 we show at $\ell = 3$ that we can obtain all Virasoro ground states from the finite set of maximal ground states with yet another operator. We furthermore argue that there exists a generalization of this operator to all higher levels. Thus, it would be enough to know the finite set of maximal ground states.

In practical terms the evaluation of the tensor product $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)}$ boils down to the evaluation of a finite number of affine tensor products, with the previously generated Virasoro representations for $\mathfrak{F}^{(\ell-1)}$ as 'spectators'. Schematically, the general structure of such a tensor product is

$$L(\Lambda_0 + 2\boldsymbol{\delta}) \otimes L(\Lambda^{(\ell-1)}) = \bigoplus_a \mathrm{Vir}(c_\ell, h_a^{(\ell)}) \otimes L(\Lambda_a^{(\ell)} + n_a\boldsymbol{\delta}) \tag{4.9}$$

where $\Lambda^{(\ell-1)}$ is any admissible level-$(\ell-1)$ weight from (2.17), that is with $p_0 + 2p_1 = \ell - 1$. Likewise, $\Lambda_a^{(\ell)}$ is an admissible level-ℓ weight from (2.17), while the parameter $h_a^{(\ell)}$ is from the list (2.25) of allowed level-ℓ eigenvalues of ${}^{[\ell]}\mathfrak{L}_0^{\mathrm{coset}}$. In this schematic form, not all values of $h_a^{(\ell)} = h_{r,s}^{(\ell)}$ from (2.25) are meant to occur, and which ones do depends on the specific weights on the left-hand side. The formula (4.9) illustrates the *infinite reducibility* of such tensor products, with the coset Virasoro algebra as an additional ingredient to handle and distinguish an infinite number of identical copies of the same affine representation, with appropriately shifted (integer) coefficients p_{-1} in (2.17). This feature characterizes all tensor products for higher levels.

The coset Virasoro module $\mathrm{Vir}(c_\ell, h_a^{(\ell)})$ associated with the tensor product decomposition thus records the infinitely many isomorphic repetitions of the same affine module with highest weight $\Lambda^{(\ell)}$, that occur shifted by multiples of $\boldsymbol{\delta}$. Our convention here is that the coefficient n_a is such that $\Lambda_a^{(\ell)} + n_a\boldsymbol{\delta}$ is the first (highest) instance of the infinite repetition of affine highest weight states. In particular, there is always the case with $h_a^{(\ell)} = 0$ (corresponding to $r = s = 1$ in (2.25)) and $\Lambda_a^{(\ell)} + n_a\boldsymbol{\delta} = \Lambda_0 + 2\boldsymbol{\delta} + \Lambda^{(\ell-1)}$, corresponding to the tensor product of the two affine highest weight states on the left-hand side of (4.9). In Figs. 7, 8 and 9, we show the root diagrams of $\mathfrak{F}^{(\ell)}$ for $2 \le \ell \le 4$ and in those diagrams this highest weight vector corresponds to the right-most red diamond.

When we pass to the characters corresponding to the above product the notation is further refined by writing the character of a term on the r.h.s. of (4.9) as

$$\mathrm{Ch}\,\mathrm{Vir}\big(c_\ell, h_a^{(\ell)}\big)(q) \cdot \mathrm{Ch}\, L\Big(\Lambda_a + (n_a + h_a^{(\ell)})\boldsymbol{\delta}\Big)\,. \tag{4.10}$$

That is, we assign a fractional multiple of $-h_a^{(\ell)}\boldsymbol{\delta}$ of the null root to the Virasoro character, in accord with the fact that we define the (minimal) Virasoro characters by $\mathrm{Tr}(q^{L_0}) = q^h + \cdots$, see appendix B, and recall that $q = e^{-\boldsymbol{\delta}}$. This shift has to be compensated for in the affine character, which explains the extra shift shown in the formula. Matters are further complicated by the fact that each affine representation $L(\Lambda^{(\ell-1)})$ in $\mathfrak{F}^{(\ell-1)}$ comes with its own baggage of factors of 'porous' coset Virasoro representations from previous representation products, whose $\boldsymbol{\delta}$-shifts must also be taken into account in the final formulas. This accounts for increasingly more complicated patterns of fractional powers of q in the final character formulas.

5. Level 2

Generally speaking, the level-2 sector is spanned by all commutators of all level-one elements. It is thus contained in the antisymmetric product of two basic representations [1], which results in (1.5). The crucial point is that not all elements of this tensor product belong to the Lie algebra because some of them vanish on account of the Serre relations (and at higher levels also the Jacobi identities) once the tensor product is converted to a Lie algebra commutator. More precisely, we have [1]

$$\mathrm{Ker}\,\mathcal{I}^{(2)} = \mathcal{J}_2 = L\big(2\Lambda_1 + 3\boldsymbol{\delta}\big)\,. \tag{5.1}$$

5.1. Maximal ground states for $\ell = 2$. From (1.6) we can read off that the dominant state of the virtual maximal ground state multiplet has momentum $2\Lambda_1 + 3\boldsymbol{\delta} = -2\mathbf{r}_{-1} - \boldsymbol{\delta} + \mathbf{r}_1$. So it belongs to a triplet and sits at depth 1, as can also be seen from the position of the right-most red diamond in Fig. 7. This triplet belongs to a coset Virasoro representation with eigenvalue $h = \frac{1}{2}$. We shall see below that this is enough information to uniquely characterize them. Specifically, for level 2 there are no Virasoro ground state multiplets besides the maximal ground state multiplet. The multiplet is an $\mathfrak{sl}(2)$ triplet in $\mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(1)}$ which consists of the three product states built out of level-one DDF states

$$\Psi^{(2)}_{1,\mathbf{3},\pm 1} = |\mathbf{a}^{(1)}_{\pm 1}\rangle \wedge |\mathbf{a}^{(1)}_0\rangle \quad \text{and} \quad \Psi^{(2)}_{1,\mathbf{3},0} = -\frac{1}{\sqrt{2}}\,{}^{[1]}A_{-1}|\mathbf{a}^{(1)}_0\rangle \wedge |\mathbf{a}^{(1)}_0\rangle\,. \tag{5.2}$$

The labeling on Ψ is as follows:

- the first subscript gives the depth,
- the second the $\mathfrak{sl}(2)$ representation through its dimension,
- and the third entry is half the h_1 weight of the corresponding state in the given $\mathfrak{sl}(2)$ representation,
- while the superscript indicates the level.

These elements are perfectly well-defined (up to normalization) and non-vanishing as elements of the tensor product $\mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(1)}$, but when we convert the wedge products to actual commutators with (3.4) they vanish

$$\big[|\mathbf{a}^{(1)}_{\pm 1}\rangle\,,\,|\mathbf{a}^{(1)}_0\rangle\big] = \big[{}^{[1]}A_{-1}|\mathbf{a}^{(1)}_0\rangle\,,\,|\mathbf{a}^{(1)}_0\rangle\big] = 0 \tag{5.3}$$

because the associated momenta obey $(-2\mathbf{r}_{-1} - \boldsymbol{\delta} \pm \mathbf{r}_1)^2 = 6 > 2$ and $(-2\mathbf{r}_{-1} - \boldsymbol{\delta})^2 = 4 > 2$. In terms of multi-commutators of Chevalley-Serre generators, the first of these states corresponds to $[f_{-1}, [f_{-1}, f_0]]$ which vanishes by the Serre relation. However, here this vanishing does not need to be imposed 'by hand' but rather follows directly from the formula (3.4). The ${}^{[2]}\mathfrak{L}^{\text{coset}}_0$ eigenvalue of any of the triplet states (5.2) is $\frac{1}{2}$ in agreement with (2.25).

Acting with ${}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}$ on the virtual triplet states we obtain three descendant states in $\mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(1)}$

$$\begin{aligned} {}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\Psi^{(2)}_{1,\mathbf{3},\pm 1} &= \sqrt{2}|\mathbf{a}^{(1)}_{\pm 1}\rangle \wedge {}^{[1]}A_{-1}|\mathbf{a}^{(1)}_0\rangle \\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\Psi^{(2)}_{1,\mathbf{3},0} &= -\frac{1}{\sqrt{2}}\;{}^{[1]}A_{-2}|\mathbf{a}^{(1)}_0\rangle \wedge |\mathbf{a}^{(1)}_0\rangle\,, \end{aligned} \tag{5.4}$$

which now carry momenta $2\Lambda_1+2\boldsymbol{\delta} = -2\mathbf{r}_{-1}-2\boldsymbol{\delta}+\mathbf{r}_1, 2\Lambda_1+2\boldsymbol{\delta}-2\mathbf{r}_1 = -2\mathbf{r}_{-1}-2\boldsymbol{\delta}-\mathbf{r}_1$ and $2\Lambda_0 + 2\boldsymbol{\delta} = -2\mathbf{r}_{-1} - 2\boldsymbol{\delta}$ which square to 2, 2, and 0, respectively. Hence, the commutators no longer vanish but give honest non-vanishing elements of $\mathfrak{F}^{(2)}$. Explicitly, we obtain the following $\mathfrak{sl}(2)$ triplet states after the evaluation of the commutators by means of (3.4) (with the root labels in the left column)

$$\begin{aligned}
-(2,2,3):&\qquad \psi^{(2)}_{2,\mathbf{3},-1} = \sqrt{2}\Big[|\mathbf{a}^{(1)}_{-1}\rangle, {}^{[1]}A_{-1}|\mathbf{a}^{(1)}_0\rangle\Big] &&= 2\,|\mathbf{a}^{(2)}_{-1}\rangle\,,\\
-(2,2,2):&\qquad \psi^{(2)}_{2,\mathbf{3},0} = -\Big[|\mathbf{a}^{(1)}_{-1}\rangle, |\mathbf{a}^{(1)}_1\rangle\Big] + L_{-1}(\ldots) &&= -\sqrt{2}\,{}^{[2]}A_{-1}|\mathbf{a}^{(2)}_0\rangle\,,\\
-(2,2,1):&\qquad \psi^{(2)}_{2,\mathbf{3},1} = -\sqrt{2}\Big[|\mathbf{a}^{(1)}_{1}\rangle, {}^{[1]}A_{-1}|\mathbf{a}^{(1)}_0\rangle\Big] &&= 2\,|\mathbf{a}^{(2)}_{1}\rangle\,.
\end{aligned}\tag{5.5}$$

These three states form an affine ground state triplet at depth 2, hence we denote them by $\psi^{(\ell)}$. In the following, descendant states which are not affine ground states will be denoted by $\phi^{(\ell)}$ with appropriate indices, as in (5.5). In this way we distinguish the elements of $\mathfrak{F}^{(\ell)}$ from the (virtual) ground states $\Psi^{(\ell)}$ that live in $\mathfrak{F}^{(1)}\otimes\mathfrak{F}^{(\ell-1)}$. In general, there will always be $L_{-1}(\ldots)$ contributions to the DDF commutators, which must be dropped by (3.25) [10]. Notice that there are no cocycle factors appearing anywhere in this section with our conventions in Appendix C.

Let us also note that an analogous triplet exists *for all levels* with the three level-ℓ states

$$2\,\big|\mathbf{a}^{(\ell)}_{\pm 1}\big\rangle \quad\text{and}\quad -\sqrt{2}\,{}^{[\ell]}A_{-1}\big|\mathbf{a}^{(\ell)}_0\big\rangle \tag{5.6}$$

at depth ℓ, and a virtual triplet analogous to (5.2) at depth $(\ell-1)$. For the computation we evaluate the action of any coset Virasoro generator on a tensor product state by means of formula (2.23), where we use the standard expression for the Sugawara generators on the separate factors, and the original expression (2.18) on the third term in (2.23), a procedure that will also work at all higher levels. As we will see we can generate the full level-2 sector by repeated application of the affine and coset Virasoro raising operators to the virtual ground state multiplet.

At level $\ell=2$ and depth 3 we have a total of seven DDF states that form two triplets and a singlet. These states are (without the commutators that generate them)

$$\begin{aligned}
-(2,3,4):&\qquad \psi^{(2)}_{3,\mathbf{3},-1} = \Big[{}^{[2]}A_{-1}{}^{[2]}A_{-1} + \frac{3}{2}\,{}^{[2]}B_{-2}\Big]\,|\mathbf{a}^{(2)}_{-1}\rangle\,,\\
-(2,3,3):&\qquad \psi^{(2)}_{3,\mathbf{3},0} = \Big[-\frac{7}{6\sqrt{2}}\,{}^{[2]}A_{-3} + \frac{2\sqrt{2}}{3}\,{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1} - \frac{3}{2\sqrt{2}}\,{}^{[2]}A_{-1}{}^{[2]}B_{-2}\Big]\,|\mathbf{a}^{(2)}_{0}\rangle\,,\\
-(2,3,2):&\qquad \psi^{(2)}_{3,\mathbf{3},1} = \Big[{}^{[2]}A_{-1}{}^{[2]}A_{-1} + \frac{3}{2}\,{}^{[2]}B_{-2}\Big]\,|\mathbf{a}^{(2)}_{1}\rangle\,,\\
-(2,3,3):&\qquad \phi^{(2)}_{3,\mathbf{1},0} = -4\,{}^{[2]}A_{-2}{}^{[2]}A_{-1}\,|\mathbf{a}^{(2)}_{0}\rangle\,,\\
-(2,3,4):&\qquad \phi^{(2)}_{3,\mathbf{3},-1} = \sqrt{2}\,{}^{[2]}A_{-2}\,|\mathbf{a}^{(2)}_{1}\rangle\,,\\
-(2,3,3):&\qquad \phi^{(2)}_{3,\mathbf{3},0} = \Big[-\frac{\sqrt{2}}{3}\,{}^{[2]}A_{-3} - \frac{\sqrt{2}}{3}\,{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}\Big]\,|\mathbf{a}^{(2)}_{0}\rangle\,,\\
-(2,3,2):&\qquad \phi^{(2)}_{3,\mathbf{3},1} = -\sqrt{2}\,{}^{[2]}A_{-2}\,|\mathbf{a}^{(2)}_{-1}\rangle\,.
\end{aligned}\tag{5.7}$$

Table 1. Specific information about the $\mathfrak{sl}(2)$ multiplets at level-2 of $\mathfrak{F}^{(2)}$ up to depth 7

Level	Depth	$-(a_{-1}, a_0, a_1)$	dim $\mathfrak{sl}(2)$ rep.	Outer multiplicity
2	2	(2,2,3) (2,2,2) (2,2,1)	3	1
2	3	(2,3,4) (2,3,3) (2,3,2)	3	2
2	3	(2,3,3)	1	1
2	4	(2,4,6) …(2,4,2)	5	1
2	4	(2,4,5) (2,4,4) (2,4,3)	3	4
2	4	(2,4,4)	1	2
2	5	(2,5,7) …(2,5,3)	5	3
2	5	(2,5,6) (2,5,5) (2,5,4)	3	8
2	5	(2,5,5)	1	4
2	6	(2,6,9) …(2,6,3)	7	1
2	6	(2,6,8) …(2,6,4)	5	6
2	6	(2,6,7) (2,6,6) (2,6,5)	3	15
2	6	(2,6,6)	1	8
2	7	(2,7,10) …(2,7,4)	7	2
2	7	(2,7,9) …(2,7,5)	5	13
2	7	(2,7,8) (2,7,7) (2,7,6)	3	27
2	7	(2,7,7)	1	14

The blue states form an affine ground state triplet. The red state is the singlet, and the green states form the remaining triplet. The latter two are affine descendants of (5.5), as can also be recognized from the fact that there appear no longitudinal states. By contrast, the blue states are the result of the action of the coset Virasoro raising operators; the longitudinal states result from the evaluation of the commutators.

Continuing in this way, and starting from the dominant root of the maximal ground state triplet, we construct the level-2 part of the root lattice shown in Fig. 7. Expressing the first seven rows of Fig. 7 in a table, we obtain Table 1.

In Appendix D, we rewrite the states above in terms of multi-commutators of Chevalley-Serre generators.

5.2. Coset Virasoro action. As we already pointed out, there is no proper coset Virasoro representation on level-2 of $\mathfrak{F}$, because the Virasoro ground state triplet is absent from $\mathfrak{F}^{(2)}$. Acting with the coset Virasoro operators on the virtual triplet yields

$$\begin{aligned} {}^{[2]}\mathfrak{L}_1^{\text{coset}}\Psi^{(2)}_{1,\mathbf{3},w} &= 0\,, \\ {}^{[2]}\mathfrak{L}_0^{\text{coset}}\Psi^{(2)}_{1,\mathbf{3},w} &= \frac{1}{2}\Psi^{(2)}_{1,\mathbf{3},w}\,, \\ \mathcal{I}^{(2)}\left({}^{[2]}\mathfrak{L}_{-1}^{\text{coset}}\Psi^{(2)}_{1,\mathbf{3},w}\right) &= \psi^{(2)}_{2,\mathbf{3},w}\,, \\ \mathcal{I}^{(2)}\left({}^{[2]}\mathfrak{L}_{-2}^{\text{coset}}\Psi^{(2)}_{1,\mathbf{3},w}\right) &= \frac{3}{4}\psi^{(2)}_{3,\mathbf{3},w}\,. \end{aligned} \tag{5.8}$$

The coset Virasoro eigenvalue of the virtual Virasoro ground states $\Psi^{(2)}_{1,\mathbf{3},w}$ is exactly the one we expect from (1.6) and also $c_2 = \frac{1}{2}$. We see from the results above that we always obtain affine ground states when acting with ${}^{[2]}\mathfrak{L}_m^{\text{coset}}$ for $m \leq 0$ on $\Psi^{(2)}_{1,\mathbf{3},w}$. To show this we first use (2.20) which implies that the coset Virasoro operator commutes with the affine generators

$$\left[{}^{[\ell]}\mathfrak{L}_m^{\text{coset}}, T_n\right]\Phi = 0 \tag{5.9}$$

for any state $\Phi \in \mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(\ell-1)}$. Taking $\Phi = \Psi^{(2)}_{1,\mathbf{3},w}$, it is straightforward with the formulas (4.2) to check that

$$E_1\Psi^{(2)}_{1,\mathbf{3},-1} = H_1\Psi^{(2)}_{1,\mathbf{3},-1} = E_1\Psi^{(2)}_{1,\mathbf{3},0} = F_1\Psi^{(2)}_{1,\mathbf{3},0} = F_1\Psi^{(2)}_{1,\mathbf{3},1} = H_1\Psi^{(2)}_{1,\mathbf{3},1} = 0 \tag{5.10}$$

and

$$F_1\Psi^{(2)}_{1,\mathbf{3},-1} = H_1\Psi^{(2)}_{1,\mathbf{3},0} = E_1\Psi^{(2)}_{1,\mathbf{3},1} = |\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(1)}_0\rangle \,. \tag{5.11}$$

Moreover,

$$\begin{aligned} {}^{[2]}\mathfrak{L}^{\text{coset}}_n\left(|\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(1)}_0\rangle\right) &= {}^{[1]}\mathcal{L}^{\text{sug}}_n|\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(1)}_0\rangle + |\mathbf{a}^{(1)}_0\rangle \otimes {}^{[1]}\mathcal{L}^{\text{sug}}_n|\mathbf{a}^{(1)}_0\rangle \\ &\quad - {}^{[2]}\mathcal{L}^{\text{sug}}_n\left(|\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(1)}_0\rangle\right) \\ &= L_{-1}(\ldots) \end{aligned} \tag{5.12}$$

because the argument is symmetric. Hence, the virtual triplet generates all affine ground states on level-2 of $\mathfrak{F}$.

There are two ways to reach the affine ground states $\psi^{(2)}_{3,\mathbf{3},w}$ at depth 3 by applying either ${}^{[2]}\mathfrak{L}^{\text{coset}}_{-2}$ or $\left({}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\right)^2$ to $\Psi_{1,\mathbf{3},w}$ and they must be related. This is expressed by the relation

$$\left[{}^{[2]}\mathfrak{L}^{\text{coset}}_{-2} - \frac{3}{4}\left({}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\right)^2\right]\Psi^{(2)}_{1,\mathbf{3},w} = 0 \tag{5.13}$$

in agreement with the first null vector from Vir$(\frac{1}{2}, \frac{1}{2})$ [24]. For the next two coset Virasoro operators we find the relations

$$\left[\frac{3}{4}(4n_1+3n_2)\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-3} - n_2\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-2} - n_1\left({}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\right)^3\right]\Psi^{(2)}_{1,\mathbf{3},w} = 0 \tag{5.14}$$

and

$$\begin{aligned} &\left[(24m_1+18m_2+8m_3)\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-4} - \frac{4}{3}(12m_1+9m_2+4m_3)\left({}^{[2]}\mathfrak{L}^{\text{coset}}_{-2}\right)^2\right. \\ &\left.+ m_3\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-3} + m_2\left({}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\right)^2\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-2} + m_1\left({}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\right)^4\right]\Psi^{(2)}_{1,\mathbf{3},w} = 0 \end{aligned} \tag{5.15}$$

from which all null vectors can be read off by choosing the coefficients $m_i, n_i \in \mathbb{C}$ appropriately. Let us stress once more that the Virasoro null vectors are actually zero in our formalism. Altogether, the action of the coset Virasoro operator on the virtual maximal ground states thus yields two new multiplets at depth 5, in agreement with the general theory [24]. The dominant states in these multiplets are

$$\begin{aligned} &\mathcal{I}^{(2)}\left({}^{[2]}\mathfrak{L}^{\text{coset}}_{-4}\Psi^{(2)}_{1,\mathbf{3},1}\right) \\ &= \left[\frac{51}{80}\ {}^{[2]}A_{-5}{}^{[2]}A_{-1} + \frac{1}{144}\ {}^{[2]}A_{-3}{}^{[2]}A_{-3} + \frac{11}{36}\ {}^{[2]}A_{-3}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}\right. \\ &\quad - \frac{23}{360}\ {}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1} \end{aligned}$$

$$+\frac{3}{8}\,{}^{[2]}A_{-3}{}^{[2]}A_{-1}{}^{[2]}B_{-2}+\frac{1}{32}\,{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}B_{-2}{}^{[2]}B_{-2}$$
$$+\frac{1}{16}\,{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}B_{-2}+\frac{11}{192}\,{}^{[2]}B_{-2}{}^{[2]}B_{-2}{}^{[2]}B_{-2}\Big]|\mathbf{a}_1^{(2)}\rangle \quad (5.16)$$

and

$$\mathcal{I}^{(2)}\left({}^{[2]}\mathfrak{L}^{\text{coset}}_{-3}\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\Psi^{(2)}_{1,\mathbf{3},1}\right)$$
$$=\Big[\frac{73}{80}\,{}^{[2]}A_{-5}{}^{[2]}A_{-1}+\frac{11}{16}\,{}^{[2]}A_{-3}{}^{[2]}A_{-3}$$
$$-\frac{1}{12}\,{}^{[2]}A_{-3}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}+\frac{17}{120}\,{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}$$
$$+\frac{1}{8}\,{}^{[2]}A_{-3}{}^{[2]}A_{-1}{}^{[2]}B_{-2}+\frac{3}{32}\,{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}B_{-2}{}^{[2]}B_{-2}$$
$$-\frac{1}{16}\,{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}B_{-2}+\frac{13}{192}\,{}^{[2]}B_{-2}{}^{[2]}B_{-2}{}^{[2]}B_{-2}\Big]|\mathbf{a}_1^{(2)}\rangle$$
$$+\,{}^{[2]}H_{-1}\Big[\frac{5}{12}\,{}^{[2]}A_{-3}{}^{[2]}A_{-1}+\frac{1}{24}\,{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}A_{-1}$$
$$+\frac{1}{8}\,{}^{[2]}A_{-1}{}^{[2]}A_{-1}{}^{[2]}B_{-2}+\frac{15}{96}\,{}^{[2]}B_{-2}{}^{[2]}B_{-2}\Big]|\mathbf{a}_1^{(2)}\rangle\,. \quad (5.17)$$

Notice that the result of the second calculation is not a pure affine ground state. This is no contradiction to (5.9)–(5.12).

We have already mentioned that $\mathfrak{F}$ does not decompose into proper representations of the coset Virasoro algebra, as we will now make more explicit for $\mathfrak{F}^{(2)}$. Hence, we shall work on the tensor product space and exchange tensor products for commutators only after acting with the coset Virasoro operators. The tensor product space forms a representation of the coset Virasoro algebra

$$\left[{}^{[\ell]}\mathfrak{L}^{\text{coset}}_m,\ {}^{[\ell]}\mathfrak{L}^{\text{coset}}_n\right]=(m-n)\ {}^{[\ell]}\mathfrak{L}^{\text{coset}}_{m+n}+\frac{c^{\text{coset}}_\ell}{12}(m^3-m)\delta_{m+n,0}\,, \quad (5.18)$$

with central charge (2.24). The important point now is that on $\mathfrak{F}$ this algebra is *not* satisfied because we do not have a complete Virasoro representation. This can be seen if one tries to apply formula (2.23) in the form

$${}^{[\ell]}\mathfrak{L}^{\text{coset}}_m w\ \equiv\ {}^{[\ell]}\mathfrak{L}^{\text{coset}}_m([u\,,\,v])\ \stackrel{?}{=}\ \left[{}^{[1]}\mathcal{L}^{\text{sug}}_m u\,,\,v\right]+\left[u\,,\ {}^{[\ell-1]}\mathcal{L}^{\text{sug}}_m v\right]\ -\ {}^{[\ell]}\mathcal{L}^{\text{sug}}_m w\,. \quad (5.19)$$

Indeed, this erroneous application of the formula (2.23) leads to contradictions whenever the third term on the r.h.s. of (5.19) vanishes on account of the Serre relations. For example, we find for the dominant state of the maximal ground state triplets

$$\left[{}^{[2]}\mathfrak{L}^{\text{coset}}_1,\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}\right]\psi^{(2)}_{2,\mathbf{3},1}\ =\ {}^{[2]}\mathfrak{L}^{\text{coset}}_1\left(-\psi^{(2)}_{3,\mathbf{3},1}+\phi^{(2)}_{3,\mathbf{3},1}\right)-\ {}^{[2]}\mathfrak{L}^{\text{coset}}_{-1}0\ =\ 4\psi^{(2)}_{2,\mathbf{3},1}\,. \quad (5.20)$$

By contrast, direct application of the result of the Virasoro commutator leads to a different answer:

$$2\ {}^{[\ell]}\mathfrak{L}^{\text{coset}}_0\psi^{(2)}_{2,\mathbf{3},1}\ =\ 3\psi^{(2)}_{2,\mathbf{3},1} \quad (5.21)$$

in contradiction with the Virasoro algebra.

5.3. Affine characters at $\ell = 2$. Of course, our findings can be rephrased using characters, the tool mostly employed in the mathematical literature. Recall that level 2 of $\mathfrak{F}$ is formally given by (1.4) [1]. In terms of characters this translates into the formula

$$\text{Ch } \mathfrak{F}^{(2)} = q^{-7/2} \left(\chi_{2,1}^{4,3}(q) - q^{1/2}\right) \cdot \text{Ch } L(2\Lambda_1)\,, \tag{5.22}$$

where we use the notation

$$\text{Ch Vir}\big(c^{p,p'}, h_{r,s}^{p,p'}\big)(q) \equiv \chi_{r,s}^{p,p'}(q) = q^{h_{r,s}^{p,p'}} + \dots \tag{5.23}$$

for the Virasoro characters. The character of $L(2\Lambda_1)$ is given by [1]

$$\text{Ch } L(2\mathbf{\Lambda}_1) = \frac{q^{3/2}}{2}\frac{\varphi(q)}{\varphi(q^2)} \times \left[\left(\frac{1}{\varphi(\sqrt{q})} - \frac{1}{\varphi(-\sqrt{q})}\right)\Theta_{2\mathbf{\Lambda}_0} + \left(\frac{1}{\varphi(\sqrt{q})} + \frac{1}{\varphi(-\sqrt{q})}\right)\Theta_{2\mathbf{\Lambda}_1}\right]. \tag{5.24}$$

If we combine these equations with the definitions of $\varphi(q)$, Θ_Λ and $\chi_{2,1}^{4,3}(q)$ given in Appendix B, we obtain an expansion for Ch $\mathfrak{F}^{(2)}$ similar to (3.35).

At arbitrary level ℓ the character of $\mathfrak{F}^{(\ell)}$ can be computed recursively from the Weyl-Kac denominator formula [2] for $\mathfrak{F}$, or alternatively with the program `SimpLie` [30] which offers an implementation of the Peterson recursion formula.

6. Level 3

Ref. [6] gives the following formula in eqn.(24) for the isomorphism as affine modules for the level-3 sector of the algebra

$$\mathfrak{F}^{(3)} \simeq \mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(2)} / \wedge^3 \mathfrak{F}^{(1)}\,, \tag{6.1}$$

which is equivalent to the statement that

$$\text{Ker}\,\mathcal{I}^{(3)} = \wedge^3 \mathfrak{F}^{(1)} \tag{6.2}$$

and where $\wedge^3$ denotes the third exterior product of a representation. Indeed, this subtraction eliminates the terms which vanish on account of the Jacobi identity. In terms of characters this implies[5]

$$\text{Ch } \mathfrak{F}^{(3)} = \text{Ch } \mathfrak{F}^{(1)} \cdot \text{Ch } \mathfrak{F}^{(2)} - \text{Ch}\,(\wedge^3 \mathfrak{F}^{(1)})\,. \tag{6.3}$$

We have verified this equation up to depth 30 in an explicit calculation. In [6] the authors also give an expression for the character of (6.1), for which, however, we find a slightly different result, see below. To determine $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(2)}$ we first compute (for any s)

$$\begin{aligned} L(\Lambda_0 + 2\boldsymbol{\delta}) \otimes L(2\Lambda_1 + s\boldsymbol{\delta}) = \;& \text{Vir}(\tfrac{7}{10}, \tfrac{1}{10}) \otimes L(\Lambda_0 + 2\Lambda_1 + (s+2)\boldsymbol{\delta}) \\ & \oplus \text{Vir}(\tfrac{7}{10}, \tfrac{3}{2}) \otimes L(3\Lambda_0 + (s+1)\boldsymbol{\delta})\,. \end{aligned} \tag{6.4}$$

5 At higher level such a simple subtraction no longer works. For higher rank algebras such as E_{10}, an extra subtraction is required already at level $\ell = 3$.

As the prefactor in (1.6) encodes the infinite repetitions of representations of this type according to

$$\mathfrak{F}^{(2)} = \bigoplus_{n\geq 0} a_n L(2\Lambda_1 + (2+n)\boldsymbol{\delta}) \,, \tag{6.5}$$

the (outer) multiplicities a_n are the coefficients of the q-series

$$\chi^{4,3}_{2,1}(q) - q^{1/2} = q^{3/2} \sum_{n\geq 0} a_n q^n \,. \tag{6.6}$$

From this we deduce that, as a product of coset Virasoro and affine representations, we get

$$\begin{aligned} \mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(2)} \simeq \bigoplus_{n\geq 0} a_n \Big(& \mathrm{Vir}(\tfrac{7}{10}, \tfrac{1}{10}) \otimes L(\Lambda_0 + 2\Lambda_1 + (4+n)\boldsymbol{\delta}) \\ & \oplus \mathrm{Vir}(\tfrac{7}{10}, \tfrac{3}{2}) \otimes L(3\Lambda_0 + (3+n)\boldsymbol{\delta}) \Big) \,. \end{aligned} \tag{6.7}$$

This is the level-3 analogue of (1.5). Because there are now two affine representations in this decomposition, we expect two kinds of Virasoro ground state multiplets on level 3. Namely, singlets and triplets.

In view of (6.6) we can write the character of this tensor product also as

$$\begin{aligned} \mathrm{Ch}\,(\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(2)}) = & \big(\chi^{4,3}_{2,1}(q) - q^{1/2}\big) \\ & \Big(q^{-21/10} \chi^{5,4}_{3,3}(q)\, \mathrm{Ch}\, L(\Lambda_0 + 2\Lambda_1) + q^{-5/2} \chi^{5,4}_{3,1}(q)\, \mathrm{Ch}\, L(3\Lambda_0) \Big) \,. \end{aligned} \tag{6.8}$$

Even though tempting, we do not write the vector space as a tensor product of truncated Virasoro modules.

Moreover, compared to (1.5), the equation (6.8) exhibits a pile-up of Virasoro characters which will give rise to infinitely many Virasoro ground states. Recall that level 1 of $\mathfrak{F}$ is simply given by the affine module $L(\Lambda_0 + 2\boldsymbol{\delta})$. Hence, there is one affine ground state (namely $|\mathbf{a}_0^{(1)}\rangle$) from which we obtain all DDF states on level 1 by the action of the affine generators (4.1) and (4.2). Then level 2 of $\mathfrak{F}$ is given by the tensor product of an affine module and a (truncated) coset Virasoro representation (*cf.* (1.6)). Hence, there are now infinitely many affine ground states that arise from the action of the coset Virasoro raising operator on the Virasoro ground state multiplet (5.2). In particular, there is only one Virasoro ground state multiplet.

Similarly on level 3 there are now infinitely many Virasoro ground states which are generated from the action of the (truncated) coset Virasoro representation $\mathrm{Vir}(\frac{1}{2}, \frac{1}{2}) \ominus \mathbb{R}v_0$ on the two Virasoro ground states of

$$\mathrm{Vir}(\tfrac{7}{10}, \tfrac{1}{10}) \otimes L(\Lambda_0 + 2\Lambda_1 + 5\boldsymbol{\delta}) \oplus \mathrm{Vir}(\tfrac{7}{10}, \tfrac{3}{2}) \otimes L(3\Lambda_0 + 4\boldsymbol{\delta}) \,. \tag{6.9}$$

In the following, we disentangle this structure by first identifying the maximal ground states of $\mathfrak{F}^{(3)}$ and subsequently investigating how the action of $\mathrm{Vir}(\frac{1}{2}, \frac{1}{2}) \ominus \mathbb{R}v_0$ on these maximal ground states yields their infinite duplication.

The discussion of the different kinds of ground states is summarized in the Table 2.

Table 2. Number of each kind of ground state for the levels 1–3. The number of maximal ground states is finite for any level ℓ

	Affine	Virasoro	Maximal
Level 1	1	0	0
Level 2	∞	1	1
Level 3	∞	∞	2

6.1. Maximal ground states for $\ell = 3$. Equation (6.7) together with (6.31) below tells us that we have two virtual maximal ground state multiplets on level 3 of $\mathfrak{F}$. The first of which has dominant momentum $\Lambda_0 + 2\Lambda_1 + 4\boldsymbol{\delta} = -3\mathbf{r}_{-1} - 2\boldsymbol{\delta} + \mathbf{r}_1$.[6] Hence, this first multiplet is a triplet at depth 2. From the coset Virasoro prefactor we can read off its eigenvalue $\frac{1}{10}$. Similarly, we can determine that the other multiplet is a singlet at depth 3 with coset Virasoro eigenvalue $\frac{3}{2}$. The explicit expressions of the dominant states in these multiplets in terms of DDF operators as elements of $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(2)}$ are

$$\Psi^{(3)}_{2,\mathbf{3},1} = 2|\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(2)}_1\rangle\,, \tag{6.10}$$

$$\begin{aligned}\Psi^{(3)}_{3,\mathbf{1},0} &= 2|\mathbf{a}^{(1)}_{-1}\rangle \otimes |\mathbf{a}^{(2)}_1\rangle - 2\,{}^{[1]}\!A_{-1}|\mathbf{a}^{(1)}_0\rangle \otimes {}^{[2]}\!A_{-1}|\mathbf{a}^{(2)}_0\rangle + 2|\mathbf{a}^{(1)}_1\rangle \otimes |\mathbf{a}^{(2)}_{-1}\rangle \\ &\quad + |\mathbf{a}^{(1)}_0\rangle \otimes {}^{[2]}\!A_{-2}{}^{[2]}\!A_{-1}|\mathbf{a}^{(2)}_0\rangle\,,\end{aligned} \tag{6.11}$$

with associated squared momenta $(-3\mathbf{r}_{-1} - 2\boldsymbol{\delta} \pm \mathbf{r}_1)^2 = 8 > 2$ and $(-3\mathbf{r}_{-1} - 2\boldsymbol{\delta})^2 = 6 > 2$ for the triplet and $(-3\mathbf{r}_{-1} - 3\boldsymbol{\delta})^2 = 0$ for the singlet. While the triplet thus vanishes by the Serre relation after conversion of the tensor product into a commutator, the singlet state has allowed momentum. Hence, it must vanish for a different reason when replacing the tensor product with the commutator

$$\begin{aligned}&2\left[|\mathbf{a}^{(1)}_{-1}\rangle, |\mathbf{a}^{(2)}_1\rangle\right] - 2\left[{}^{[1]}\!A_{-1}|\mathbf{a}^{(1)}_0\rangle, {}^{[2]}\!A_{-1}|\mathbf{a}^{(2)}_0\rangle\right] \\ &+ 2\left[|\mathbf{a}^{(1)}_1\rangle, |\mathbf{a}^{(2)}_{-1}\rangle\right] + \left[|\mathbf{a}^{(1)}_0\rangle, {}^{[2]}\!A_{-2}{}^{[2]}\!A_{-1}|\mathbf{a}^{(2)}_0\rangle\right] \\ &= -\sqrt{2}\left[|\mathbf{a}^{(1)}_{-1}\rangle, \left[|\mathbf{a}^{(1)}_1\rangle, {}^{[1]}\!A_{-1}|\mathbf{a}^{(1)}_0\rangle\right]\right] - \sqrt{2}\left[{}^{[1]}\!A_{-1}|\mathbf{a}^{(1)}_0\rangle, \left[|\mathbf{a}^{(1)}_{-1}\rangle, |\mathbf{a}^{(1)}_1\rangle\right]\right] \\ &\quad + \sqrt{2}\left[|\mathbf{a}^{(1)}_1\rangle, \left[|\mathbf{a}^{(1)}_{-1}\rangle, {}^{[1]}\!A_{-1}|\mathbf{a}^{(1)}_0\rangle\right]\right] + \left[|\mathbf{a}^{(1)}_0\rangle, \left[{}^{[1]}\!A_{-2}{}^{[1]}\!A_{-1}|\mathbf{a}^{(1)}_0\rangle, |\mathbf{a}^{(1)}_0\rangle\right]\right].\end{aligned} \tag{6.12}$$

The first three terms on the right-hand side vanish with the Jacobi identity. The last term vanishes by itself, as it can be shown to be a null state. Thus, we see that the states (6.11) and (6.11) are both in the kernel of $\mathcal{I}^{(3)}$. It is straightforward to check the affine and coset Virasoro vacuum conditions. We can summarise the action of the coset Virasoro on the virtual states as follows.

$$\begin{aligned}
{}^{[3]}\mathfrak{L}^{\text{coset}}_1 \Psi^{(3)}_{2,\mathbf{3},w} &= 0\,, & {}^{[3]}\mathfrak{L}^{\text{coset}}_1 \Psi^{(3)}_{3,\mathbf{1},0} &= 0\,, \\
{}^{[3]}\mathfrak{L}^{\text{coset}}_0 \Psi^{(3)}_{2,\mathbf{3},w} &= \frac{1}{10}\Psi^{(3)}_{2,\mathbf{3},w}\,, & {}^{[3]}\mathfrak{L}^{\text{coset}}_0 \Psi^{(3)}_{3,\mathbf{1},0} &= \frac{3}{2}\Psi^{(3)}_{3,\mathbf{1},0}\,, \\
\mathcal{I}^{(3)}\left({}^{[3]}\mathfrak{L}^{\text{coset}}_{-1} \Psi^{(3)}_{2,\mathbf{3},w}\right) &= \psi^{(3)}_{3,\mathbf{3},w}\,, & \mathcal{I}^{(3)}\left({}^{[3]}\mathfrak{L}^{\text{coset}}_{-1} \Psi^{(3)}_{3,\mathbf{1},0}\right) &= \psi^{(3)}_{4,\mathbf{1},0}\,, \\
\mathcal{I}^{(3)}\left({}^{[3]}\mathfrak{L}^{\text{coset}}_{-2} \Psi^{(3)}_{2,\mathbf{3},w}\right) &= \psi^{(3)}_{4,\mathbf{3},w}\,, & \mathcal{I}^{(3)}\left({}^{[3]}\mathfrak{L}^{\text{coset}}_{-2} \Psi^{(3)}_{3,\mathbf{1},0}\right) &= \psi^{(3)}_{5,\mathbf{1},0}\,.
\end{aligned} \tag{6.13}$$

[6] There is a difference of $+1\boldsymbol{\delta}$ here compared to the argument of $L(\Lambda_0 + 2\Lambda_1 + 5\boldsymbol{\delta})$ because the q-series of $\mathrm{Vir}(\frac{1}{2}, \frac{1}{2}) \ominus \mathbb{R}v_0$ is truncated and starts with $q^{3/2}$.

We use the notational convention explained below (5.5) for distinguishing Virasoro ground states from affine ground states. In particular, we see that the coset Virasoro eigenvalues agree with the general prediction, and without shifts in $\boldsymbol{\delta}$. The first descendant states, which are now elements of $\mathfrak{F}^{(3)}$, are given by

$$\psi^{(3)}_{3,\mathbf{3},\pm 1} = -2|\mathbf{a}^{(3)}_{\pm 1}\rangle\,, \quad \psi^{(3)}_{3,\mathbf{3},0} = \sqrt{2}\,{}^{[3]}A_{-1}|\mathbf{a}^{(3)}_0\rangle \tag{6.14}$$

and

$$\begin{aligned}\psi^{(3)}_{4,\mathbf{1},0} = \Big[&-\frac{4}{3}\,{}^{[3]}A_{-2}{}^{[3]}A_{-2} + 3\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1} - \frac{14}{3}\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}B_{-2} \\ &+ \frac{7}{18}\,{}^{[3]}B_{-2}{}^{[3]}B_{-2} - \frac{7}{18}\,{}^{[3]}B_{-4}\Big]|\mathbf{a}^{(3)}_0\rangle\,. \end{aligned}\tag{6.15}$$

The states resulting from the application of ${}^{[3]}\mathfrak{L}^{\text{coset}}_{-2}$ on the maximal ground states and subsequent evaluation of the commutator are

$$\begin{aligned}\psi^{(3)}_{4,\mathbf{3},-1} = \Big[&-\frac{25}{54}\,{}^{[3]}B_{-3} + \frac{1}{36}\,{}^{[3]}A_{-2}{}^{[3]}A_{-1} \\ &+ \frac{5\sqrt{2}}{36}\,{}^{[3]}A_{-1}{}^{[3]}B_{-2} - \frac{4\sqrt{2}}{27}\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}\Big]|\mathbf{a}^{(3)}_1\rangle \end{aligned}\tag{6.16}$$

and

$$\begin{aligned}\psi^{(3)}_{5,\mathbf{1},0} = \Big[&-\frac{385}{5832}\,{}^{[3]}B_{-7} - \frac{15089}{9720}\,{}^{[3]}A_{-5}{}^{[3]}A_{-2} - \frac{385}{5832}\,{}^{[3]}B_{-4}{}^{[3]}B_{-3} - \frac{871}{1944}\,{}^{[3]}A_{-4}{}^{[3]}A_{-1}{}^{[3]}B_{-2} \\ &- \frac{115}{486}\,{}^{[3]}A_{-2}{}^{[3]}A_{-2}{}^{[3]}B_{-3} + \frac{427}{1296}\,{}^{[3]}A_{-2}{}^{[3]}A_{-1}{}^{[3]}B_{-4} + \frac{385}{5832}\,{}^{[3]}B_{-3}{}^{[3]}B_{-2}{}^{[3]}B_{-2} \\ &- \frac{379}{648}\,{}^{[3]}A_{-4}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1} + \frac{95}{108}\,{}^{[3]}A_{-2}{}^{[3]}A_{-2}{}^{[3]}A_{-2}{}^{[3]}A_{-1} \\ &- \frac{427}{1296}\,{}^{[3]}A_{-2}{}^{[3]}A_{-1}{}^{[3]}B_{-2}{}^{[3]}B_{-2} - \frac{196}{243}\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}B_{-3}{}^{[3]}B_{-2} \\ &+ \frac{197}{216}\,{}^{[3]}A_{-2}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}B_{-2} + \frac{169}{324}\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}B_{-3} \\ &+ \frac{3}{10}\,{}^{[3]}A_{-2}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}\Big]|\mathbf{a}^{(3)}_0\rangle\,. \end{aligned}\tag{6.17}$$

All these states are affine ground states.

6.2. Virasoro ground states for $\ell = 3$. The characters of the two terms in

$$\mathrm{Vir}(\tfrac{7}{10}, \tfrac{1}{10}) \otimes L(\Lambda_0 + 2\Lambda_1 + 5\boldsymbol{\delta}) \oplus \mathrm{Vir}(\tfrac{7}{10}, \tfrac{3}{2}) \otimes L(3\Lambda_0 + 4\boldsymbol{\delta}) \tag{6.18}$$

are given by $q^{-21/10}\,\chi^{5,4}_{3,3}(q)\ \mathrm{ch}\,L(\boldsymbol{\Lambda}_0 + 2\boldsymbol{\Lambda}_1)$ and $q^{-5/2}\,\chi^{5,4}_{3,1}(q)\ \mathrm{ch}\,L(3\boldsymbol{\Lambda}_0)$ (see Appendix B for the relevant definitions). We can represent these characters by root systems shown in Figure 4.

We can obtain all (DDF) states in these two diagrams from the action of the affine generators E_m, F_m, H_m and the coset Virasoro operator ${}^{[3]}\mathfrak{L}^{\text{coset}}_m$ on the two maximal ground state multiplets (6.10) and (6.11). We illustrate this with the triplet (i.e. the left) character. Recall that (up to an at the moment unimportant $\boldsymbol{\delta}$ shift) its dominant maximal ground state (i.e. the top right state) is given by

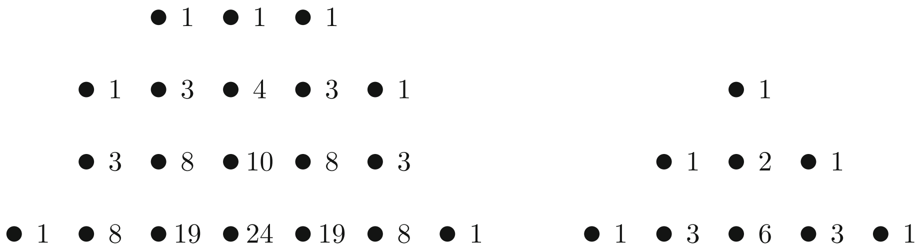

Fig. 4. Partial visualization of $q^{-21/10}\chi^{5,4}_{3,3}(q)$ ch $L(\mathbf{\Lambda}_0 + 2\mathbf{\Lambda}_1)$ (left) and $q^{-5/2}\chi^{5,4}_{3,1}(q)$ ch $L(3\mathbf{\Lambda}_0)$ (right). The top right root of the left character is $(-3, -4, -3)$ (depth 4) and the top root of the right character is $(-3, -5, -5)$ (depth 5). The numbers in the figure indicate the multiplicities

$$\Psi^{(3)}_{2,\mathbf{3},1} \;=\; 2|\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(2)}_1\rangle \,. \tag{6.19}$$

Acting with the affine generators and the coset Virasoro operator we obtain the three states right below this state

$$\begin{aligned}
\Phi^{(3)}_{3,\mathbf{3},1} &= \left(H_{-1} + \frac{1}{2}F_0E_{-1}\right)\Psi^{(3)}_{2,\mathbf{3},1} = \sqrt{2}\;{}^{[1]}\!A_{-1}|\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(2)}_1\rangle + 2\sqrt{2}|\mathbf{a}^{(1)}_0\rangle \otimes {}^{[2]}\!A_{-2}|\mathbf{a}^{(2)}_1\rangle \\
&\quad - \sqrt{2}|\mathbf{a}^{(1)}_1\rangle \otimes {}^{[2]}\!A_{-1}|\mathbf{a}^{(2)}_0\rangle\,, \\
\Phi^{(3)}_{3,\mathbf{5},1} &= \frac{1}{2}F_0E_{-1}\Psi^{(3)}_{2,\mathbf{3},1} = -\sqrt{2}\;{}^{[1]}\!A_{-1}|\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(2)}_1\rangle - \sqrt{2}|\mathbf{a}^{(1)}_1\rangle \otimes {}^{[2]}\!A_{-1}|\mathbf{a}^{(2)}_0\rangle\,, \\
\hat{\Phi}^{(3)}_{3,\mathbf{3},1} &= \mathfrak{L}^{\text{coset}}_{-1}\Psi^{(3)}_{2,\mathbf{3},1} = -\frac{2\sqrt{2}}{5}\;{}^{[1]}\!A_{-1}|\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(2)}_1\rangle + \frac{\sqrt{2}}{5}|\mathbf{a}^{(1)}_0\rangle \otimes {}^{[2]}\!A_{-2}|\mathbf{a}^{(2)}_1\rangle \\
&\quad + \frac{2\sqrt{2}}{5}|\mathbf{a}^{(1)}_1\rangle \otimes {}^{[2]}\!A_{-1}|\mathbf{a}^{(2)}_0\rangle\,.
\end{aligned} \tag{6.20}$$

It can be checked that these three states are linearly independent and that any other action of the affine generators on $\Psi^{(3)}_{2,\mathbf{3},1}$ yields a state in the linear span of these three states.

To go from the characters described by Fig. 4 to the character of $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(2)}$ we can use (6.8), according to which we must multiply the character of (6.18) by the q-series (*cf.* (5.22))

$$\begin{aligned}
&q^{-7/2}\left(\chi^{4,3}_{2,1}(q) - q^{1/2}\right) \\
&\quad = q^{-2}\left(1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + \dots\right)\,.
\end{aligned} \tag{6.21}$$

In terms of Fig. 4, the multiplication with mq^n amounts to making m copies of Fig. 4 and shifting them by n rows into the $-y$ direction. After the multiplication with (6.21) the two root systems in Fig. 4 become those shown in Figure 5.

Once again we study the left diagram. Below the maximal ground state $\Psi^{(3)}_{2,\mathbf{3},1}$ (sitting in the top right position) there are now four instead of the three states that we had before. The fourth state is a Virasoro ground state. It cannot be reached by the action of E_m, F_m, H_m or ${}^{[3]}\mathfrak{L}^{\text{coset}}_m$ on $\Psi^{(3)}_{2,\mathbf{3},1}$. Instead it arises from the prefactor associated with Vir$(\frac{1}{2}, \frac{1}{2})$ in front of Vir$(\frac{7}{10}, \frac{1}{10}) \otimes L(\mathbf{\Lambda}_0 + 2\mathbf{\Lambda}_1 + 5\boldsymbol{\delta})$. Because (6.21) never terminates, there are infinitely many such states. The action of E_m, F_m, H_m and ${}^{[3]}\mathfrak{L}^{\text{coset}}_m$ on each of these Virasoro ground states generates one Virasoro and infinitely many affine multiplets.

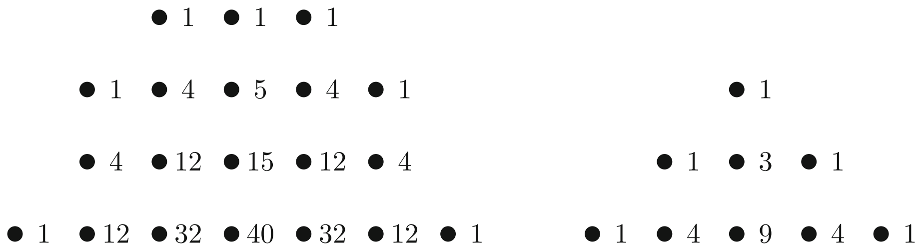

Fig. 5. Partial visualization of $q^{-28/5}(\chi^{4,3}_{2,1}(q) - q^{1/2})\,\chi^{5,4}_{3,3}(q)\,\mathrm{ch}\,L(\mathbf{\Lambda}_0 + 2\mathbf{\Lambda}_1)$ (left) and $q^{-6}(\chi^{4,3}_{2,1}(q) - q^{1/2})\,\chi^{5,4}_{3,1}(q)\,\mathrm{ch}\,L(3\mathbf{\Lambda}_0)$ (right). The top right root of the left character is $(-3, -2, -1)$ and the top root of the right character is $(-3, -3, -3)$. The numbers in the figure indicate the multiplicities

In the following, we explain how to derive all Virasoro ground states for $\ell = 3$ from the maximal ground states (6.10) and (6.11). Starting from the maximal ground state $\Psi^{(3)}_{2,\mathbf{3},1}$ we introduce the triple tensor product state

$$\Psi^{\otimes(3)}_{2,\mathbf{3},1} = \frac{1}{2\sqrt{2}}|\mathbf{a}^{(1)}_0\rangle \otimes \left({}^{[1]}A_{-1}|\mathbf{a}^{(1)}_0\rangle \wedge |\mathbf{a}^{(1)}_1\rangle + {}^{[1]}A_{-1}|\mathbf{a}^{(1)}_1\rangle \wedge |\mathbf{a}^{(1)}_0\rangle\right) . \tag{6.22}$$

This state is such that

$$(1 \otimes \mathcal{I}^{(2)})\Psi^{\otimes(3)}_{2,\mathbf{3},1} = \Psi^{(3)}_{2,\mathbf{3},1} \tag{6.23}$$

and that $\Psi^{\otimes(3)}_{2,\mathbf{3},1}$ is the affine ground state of a triplet multiplet. This uniquely fixes $\Psi^{\otimes}_{2,\mathbf{3},1}$. Subsequently, we obtain the fourth state at $(-3, -3, -2)$ in the left root lattice of Fig. 5 via

$$\Psi^{(3)}_{3,\mathbf{3},1} = \left(1\otimes\mathcal{I}^{(2)}\circ\ {}^{[2]}\mathfrak{L}^{\mathrm{coset}}_{-1}\right)\Psi^{\otimes(3)}_{2,\mathbf{3},1} = 2|\mathbf{a}^{(1)}_0\rangle\otimes{}^{[2]}A_{-1}{}^{[2]}A_{-1}|\mathbf{a}^{(2)}_1\rangle+3|\mathbf{a}^{(1)}_0\rangle\otimes{}^{[2]}B_{-2}|\mathbf{a}^{(2)}_1\rangle . \tag{6.24}$$

Besides being a Virasoro ground state, $\Psi^{(3)}_{3,\mathbf{3},1}$ is distinguished from $\hat{\Phi}^{(3)}_{3,\mathbf{3},1}$ by the fact that it has coset Virasoro eigenvalue $\frac{1}{10}$, while $\hat{\Psi}^{(3)}_{3,\mathbf{3},1}$ has coset Virasoro eigenvalue $\frac{1}{10} + 1 = \frac{11}{10}$. Moreover, let us note that under the action of $\mathcal{I}^{(3)}$ both $\hat{\Phi}_{3,\mathbf{3},1}$ and $\Psi_{3,\mathbf{3},1}$ get mapped to the same DDF state, namely

$$\mathcal{I}^{(3)}\left(\hat{\Phi}_{3,\mathbf{3},1}\right) = \frac{1}{8}\mathcal{I}^{(3)}\left(\Psi_{3,\mathbf{3},1}\right) = -2|\mathbf{a}^{(3)}_1\rangle . \tag{6.25}$$

However, we do not expect this to happen for all Virasoro ground states and their descendants.

Repeating this analysis for the singlet multiplet we find the following uniquely determined affine singlet ground state (again a triple tensor product)

$$\begin{aligned}\Psi^{\otimes(3)}_{3,\mathbf{1},0} =& \frac{1}{2\sqrt{2}}|\mathbf{a}^{(1)}_{-1}\rangle \otimes \left({}^{[1]}A_{-1}|\mathbf{a}^{(1)}_1\rangle \wedge |\mathbf{a}^{(1)}_0\rangle + {}^{[1]}A_{-1}|\mathbf{a}^{(1)}_0\rangle \wedge |\mathbf{a}^{(1)}_1\rangle\right)\\ &- \frac{1}{4\sqrt{2}}\,{}^{[1]}A_{-1}|\mathbf{a}^{(1)}_0\rangle \otimes \left(\sqrt{2}\,{}^{[1]}A_{-2}|\mathbf{a}^{(1)}_0\rangle \wedge |\mathbf{a}^{(1)}_0\rangle + 2\,|\mathbf{a}^{(1)}_{-1}\rangle \wedge |\mathbf{a}^{(1)}_1\rangle\right)\\ &- \frac{1}{2\sqrt{2}}|\mathbf{a}^{(1)}_1\rangle \otimes \left({}^{[1]}A_{-1}|\mathbf{a}^{(1)}_{-1}\rangle \wedge |\mathbf{a}^{(1)}_0\rangle + {}^{[1]}A_{-1}|\mathbf{a}^{(1)}_0\rangle \wedge |\mathbf{a}^{(1)}_{-1}\rangle\right)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{8\sqrt{2}} |\mathbf{a}_0^{(1)}\rangle \otimes \Big(\sqrt{2}\, {}^{[1]}A_{-2} |\mathbf{a}_0^{(1)}\rangle \wedge {}^{[1]}A_{-1} |\mathbf{a}_0^{(1)}\rangle \\
&+ \sqrt{2}\, {}^{[1]}A_{-2} {}^{[1]}A_{-1} |\mathbf{a}_0^{(1)}\rangle \wedge |\mathbf{a}_0^{(1)}\rangle \\
&\quad - 2\, {}^{[1]}A_{-1} |\mathbf{a}_1^{(1)}\rangle \wedge |\mathbf{a}_{-1}^{(1)}\rangle + 2\, {}^{[1]}A_{-1} |\mathbf{a}_{-1}^{(1)}\rangle \wedge |\mathbf{a}_1^{(1)}\rangle \Big)
\end{aligned} \tag{6.26}
$$

which satisfies

$$
(1 \otimes \mathcal{I}^{(2)}) \Psi^{\otimes (3)}_{3,\mathbf{1},0} = \Psi^{(3)}_{3,\mathbf{1},0} \,. \tag{6.27}
$$

Again, we find the additional state below the top state in the singlet root system of Fig. 5 compared to Fig. 4.

Since the q series of the prefactor can be generated using the coset Virasoro generators describing level $\ell = 2$ we obtain that the infinite set of Virasoro ground states in $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(2)}$ is given by

$$
\Big(1 \otimes \mathcal{I}^{(2)} \circ \;{}^{[2]}\mathfrak{L}^{\text{coset}}_{-n_1} \cdots \;{}^{[2]}\mathfrak{L}^{\text{coset}}_{-n_N} \Big) \quad \text{with} \quad n_1 \geq \ldots \geq n_N \quad \text{and} \quad N \geq 0 \,. \tag{6.28}
$$

We conjecture that a construction similar to (6.28) exists for all levels $\ell \geq 3$ and that for every level $\ell \geq 3$ the infinite set of Virasoro ground states can be obtained by the action of similar operators on the finite set of maximal ground states. For this conjecture we rely on Theorem 1 and first write everything potential state as a tensor product state and then place coset Virasoro generators followed by the maps $\mathcal{I}^{(k)}$ successively. We leave the derivation of this operator and the further discussion of our conjecture for future work.

6.3. Affine characters at $\ell = 3$. To get the character of equation (6.7), we need the character of the level-3 modules

$$
\operatorname{Ch} L(i\mathbf{\Lambda}_0 + 2j\mathbf{\Lambda}_1) = q^3 \sum_{\substack{(m,n)\in\{\mathbb{Z}_{\geq 0},\mathbb{Z}\} \\ m+2n=3}} C^{m,2n}_{i,2j}\, \Theta_{m\mathbf{\Lambda}_0 + 2n\mathbf{\Lambda}_1} \tag{6.29}
$$

with the string functions [6]

$$
\begin{aligned}
C^{1,2}_{1,2} = C^{5,-2}_{1,2} &= \frac{q^{-2/5}}{\varphi(q)}\, \chi^{6,5}_{3,3}(q)\,, \\
C^{3,0}_{1,2} &= \frac{q^{-2/5}}{\varphi(q)} \Big(\chi^{6,5}_{2,1}(q) + \chi^{6,5}_{3,1}(q) \Big)\,, \\
C^{1,2}_{3,0} = C^{5,-2}_{3,0} &= \frac{1}{\varphi(q)}\, \chi^{6,5}_{4,3}(q)\,, \\
C^{3,0}_{3,0} &= \frac{1}{\varphi(q)} \Big(\chi^{6,5}_{1,1}(q) + \chi^{6,5}_{4,1}(q) \Big)\,,
\end{aligned} \tag{6.30}
$$

where the Virasoro characters $\chi^{p,p'}_{r,s}(q)$ are defined in (B.14). Equation (6.29) follows immediately from (B.7) and (B.8). The fact that the string functions can be expressed in terms of (coset) Virasoro characters for the central charges shown follows from the general property [2, Prop. 12.12] that they stem from a coset construction of $\mathfrak{sl}(2)/\mathfrak{gl}(1)$ at level ℓ, where the $\mathfrak{gl}(1)$ is due to the Cartan subalgebra. The corresponding central charge is in general $c = \frac{3\ell}{\ell+2} - 1$, which for $\ell = 3$ yields $c = \frac{4}{5} = c^{6,5}$.

For the character of the last term in (6.1) we find

$$\begin{aligned}
\mathrm{Ch}\,(\wedge^3\mathfrak{F}^{(1)}) &= \frac{1}{6}\Big((\mathrm{Ch}\,\mathfrak{F}^{(1)})^3 - 3(\mathrm{Ch}\,\mathfrak{F}^{(1)})(\mathrm{Ch}\,\mathfrak{F}^{(1)})^{*2} + 2(\mathrm{Ch}\,\mathfrak{F}^{(1)})^{*3}\Big)\\
&= \frac{q^{-6}}{3}\Big[-\chi^{5,4}_{1,1}(q)\chi^{4,3}_{1,1}(q) + 2\chi^{5,4}_{3,1}(q)\chi^{4,3}_{2,1}(q) + q^{-1/15}\frac{\varphi(q)}{\varphi(q^3)}\chi^{6,5}_{3,3}(q)\Big]\mathrm{Ch}\ L(3\Lambda_0)\\
&\quad + \frac{q^{-28/5}}{3}\Big[-\chi^{5,4}_{3,2}(q)\chi^{4,3}_{1,1}(q) + 2\chi^{5,4}_{3,3}(q)\chi^{4,3}_{2,1}(q) - q^{-1/15}\frac{\varphi(q)}{\varphi(q^3)}\chi^{6,5}_{4,3}(q)\Big]\mathrm{Ch}\ L(\Lambda_0+2\Lambda_1)\,,
\end{aligned} \tag{6.31}$$

where the (Adams) operation $(\mathrm{Ch}\,)^{*n}$ on a formal character $\mathrm{Ch}\ = \sum_\lambda m_\lambda e^\lambda$ is $(\mathrm{Ch}\,)^{*n} = \sum_\lambda m_\lambda e^{n\lambda}$.

One important step in the derivation of (6.31) is to write $\Theta_{3\Lambda_0}$ in terms of affine and Virasoro characters in the spirit of §2 of [6], leading to

$$\Theta_{3\Lambda_0} = \varphi(q)\Big[q^{-1/15}\,\chi^{6,5}_{3,3}(q)\,\mathrm{Ch}\ L(3\Lambda_0) - q^{1/3}\,\chi^{6,5}_{4,3}(q)\,\mathrm{Ch}\ L(\Lambda_0+2\Lambda_1)\Big]\,. \tag{6.32}$$

This together with the identity

$$(\mathrm{Ch}\,\mathfrak{F}^{(1)})^{*3} \;=\; \frac{q^{-3}}{\varphi(q^3)}\,\Theta_{3\Lambda_0} \tag{6.33}$$

leads to (6.31). Combining these equations with formula (6.3) and the known formulas for $\mathrm{Ch}\,\mathfrak{F}^{(1)}$ and $\mathrm{Ch}\,\mathfrak{F}^{(2)}$ gives the character of $\mathfrak{F}^{(3)}$

$$\begin{aligned}
\mathrm{Ch}\ \mathfrak{F}^{(3)} = q^{-6}\Big[&\frac{1}{3}\chi^{5,4}_{1,1}(q)\chi^{4,3}_{1,1}(q) + \chi^{5,4}_{3,1}(q)\left(\frac{1}{3}\chi^{4,3}_{2,1}(q) - q^{1/2}\right)\\
&- \frac{q^{-1/15}}{3}\frac{\varphi(q)}{\varphi(q^3)}\chi^{6,5}_{3,3}(q)\Big]\mathrm{Ch}\ L(3\Lambda_0)\\
+ q^{-28/5}\Big[&\frac{1}{3}\chi^{5,4}_{3,2}(q)\chi^{4,3}_{1,1}(q) + \chi^{5,4}_{3,3}(q)\left(\frac{1}{3}\chi^{4,3}_{2,1}(q) - q^{1/2}\right)\\
&+ \frac{q^{-1/15}}{3}\frac{\varphi(q)}{\varphi(q^3)}\chi^{6,5}_{4,3}(q)\Big]\mathrm{Ch}\ L(\Lambda_0+2\Lambda_1)\,.
\end{aligned} \tag{6.34}$$

As a further confirmation of these formulae, we have checked that the root multiplicities between the l.h.s. and the r.h.s. of (6.1) match up to depth 30. From (6.34) we can now in principle derive a closed form expression for the level-3 root multiplicities by substituting the formulas (6.30) for the affine characters. We refrain from writing out the final result, as it is “explicit, but ugly” [6], and as such not very illuminating. A further check would be to compare these results with those of [4,5], a test which we leave to future work.

7. Level 4

At level four our formula (2.14) reads

$$\mathfrak{F}^{(4)} \;\cong\; \big(\mathfrak{F}^{(1)}\otimes\mathfrak{F}^{(3)}\big)\,\big/\,\mathrm{Ker}\,\mathcal{I}^{(4)}\,. \tag{7.1}$$

and we thus need to determine $\mathrm{Ker}\,\mathcal{I}^{(4)}$. From Ref. [6] and their equations (10), (13) and (22) we extract the following formula:

$$\mathrm{Ker}\,\mathcal{I}^{(4)} \;=\; \mathfrak{W}^{(4)} \;-\; \big(\mathfrak{F}^{(1)}\otimes\mathcal{J}_3\big) \;\cap\; \mathfrak{W}^{(4)} \tag{7.2}$$

where

$$\mathcal{J}_3 \cong \mathfrak{F}^{(1)} \otimes \mathcal{J}_2 \equiv \mathfrak{F}^{(1)} \otimes \left(\mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(1)} - \mathfrak{F}^{(2)}\right) \tag{7.3}$$

and

$$\mathfrak{W}^{(4)} = \text{Weyl}\left(\mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(1)}\right)^2 := \text{Sym}\left(\mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(1)}\right)^2 - \wedge^4 \mathfrak{F}^{(1)} \tag{7.4}$$

projects out the Weyl tensor tableau. In principle, we should be able to find expressions for the characters associated with Young-tableau-type expressions or products of such Young-tableau-type expressions, but that is not so easy for the intersection term on the r.h.s. of (7.2), as would be required for the derivation of an explicit formula for the level-4 root multiplicities [6]. Whereas the determination of such intersections is straightforward for finite dimensional Lie groups and their representations, this is not the case here, as one also needs to match the Virasoro eigenvalues. We therefore leave the verification of (7.2) to future work. A more general conjecture would be that for all ℓ, the kernel of $\mathcal{I}^{(\ell)}$ can be expressed by such Young-tableau-like combinations of lower $\mathfrak{F}^{(\ell)}$, in analogy with (6.1), but in addition we expect there to be intersection terms which hamper more explicit calculations, because it is not clear how the permutation group acts on them [6].

For the product $\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(3)}$, the other ingredient of (2.14), we can find an explicit expression for the character by multiplication of the level 1 and level 3 modules. In view of (6.34), there are two products to be considered, namely

$$\begin{aligned} &L(\Lambda_0 + 2\boldsymbol{\delta}) \otimes L(3\Lambda_0 + 3\boldsymbol{\delta}) \\ &\cong \text{Vir}(\tfrac{4}{5}, 0) \otimes L(4\Lambda_0 + 5\boldsymbol{\delta}) \oplus \text{Vir}(\tfrac{4}{5}, \tfrac{2}{3}) \otimes L(2\Lambda_0 + 2\Lambda_1 + 4\boldsymbol{\delta}) \\ &\quad \oplus \text{Vir}(\tfrac{4}{5}, 3) \otimes L(4\Lambda_1 + \boldsymbol{\delta}) \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} &L(\Lambda_0 + 2\boldsymbol{\delta}) \otimes L(\Lambda_0 + 2\Lambda_1 + 4\boldsymbol{\delta}) \\ &\cong \text{Vir}(\tfrac{4}{5}, \tfrac{7}{5}) \otimes L(4\Lambda_0 + 5\boldsymbol{\delta}) \oplus \text{Vir}(\tfrac{4}{5}, \tfrac{1}{15}) \otimes L(2\Lambda_0 + 2\Lambda_1 + 6\boldsymbol{\delta}) \\ &\quad \oplus \text{Vir}(\tfrac{4}{5}, \tfrac{2}{5}) \otimes L(4\Lambda_1 + 5\boldsymbol{\delta}) , \end{aligned} \tag{7.6}$$

where again the $\boldsymbol{\delta}$ shifts are in accord with our convention stated in section 4.2. Combining this with the level 3 Virasoro prefactors from (6.34)

$$\mathcal{V}^{\text{singlet}} = q^{-6}\left[\frac{1}{3}\chi^{5,4}_{1,1}(q)\chi^{4,3}_{1,1}(q) + \chi^{5,4}_{3,1}(q)\left(\frac{1}{3}\chi^{4,3}_{2,1}(q) - q^{1/2}\right) - \frac{q^{-1/15}}{3}\frac{\varphi(q)}{\varphi(q^3)}\chi^{6,5}_{3,3}(q)\right] \tag{7.7}$$

and

$$\mathcal{V}^{\text{triplet}} = q^{-28/5}\left[\frac{1}{3}\chi^{5,4}_{3,2}(q)\chi^{4,3}_{1,1}(q) + \chi^{5,4}_{3,3}(q)\left(\frac{1}{3}\chi^{4,3}_{2,1}(q) - q^{1/2}\right) + \frac{q^{-1/15}}{3}\frac{\varphi(q)}{\varphi(q^3)}\chi^{6,5}_{4,3}(q)\right] \tag{7.8}$$

we obtain the character

$$\begin{aligned} \text{Ch}\,(\mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(3)}) &= \mathcal{V}^{\text{singlet}} \cdot \text{Ch}\,(V(\Lambda_0 + 2\boldsymbol{\delta}) \otimes V(3\Lambda_0 + 3\boldsymbol{\delta})) \\ &\quad + \mathcal{V}^{\text{triplet}} \cdot \text{Ch}\,(V(\Lambda_0 + 2\boldsymbol{\delta}) \otimes V(2\Lambda_0 + \Lambda_1 + 4\boldsymbol{\delta})) . \end{aligned} \tag{7.9}$$

Closed form expressions for the characters of the level 4 modules $L(4\Lambda_0)$, $L(2\Lambda_0{+}2\Lambda_1)$ and $L(4\Lambda_1)$ can be deduced for example from the results in [31,32] for the string functions in terms of coset Virasoro modules (of central charge $\frac{3\ell}{\ell+2} - 1$).

From the tensor products, we can read off the ground state structure at level 4. To each summand on the r.h.s. of (7.5) and (7.6) there is a maximal ground state, thus

- There are two singlets that both sit at depth 4. They have $\mathfrak{L}_0^{\text{coset}}$ eigenvalues 0 and $\frac{7}{5}$, respectively;
- There are two triplets that sit at depths 5 and 3 with eigenvalues $\frac{2}{3}$ and $\frac{1}{15}$, respectively;
- There are two fiveplets that sit at depths 4 and 8 with eigenvalues $\frac{2}{5}$ and 3, respectively.

We next investigate what they look like as tensor products of DDF states and investigate whether they are virtual or not. The tensor product form is obtained by making a general ansatz in terms of DDF states and then imposing the conditions (4.8). This we have succeeded in doing for all ground states but the final fiveplet at depth 7 which is a deeply nested commutator of DDF states.

The dominant state of the triplet at depth 3 is given by

$$\Psi^{(4)}_{3,\mathbf{3},1} = -2|\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(3)}_1\rangle\,. \tag{7.10}$$

The two singlets at depth 4 are

$$\begin{aligned}\Psi^{(4)}_{4,\mathbf{1},0} &= |\mathbf{a}^{(1)}_0\rangle \otimes \Big[14\,{}^{[3]}A_{-3}{}^{[3]}A_{-1} + 3\,{}^{[3]}A_{-2}{}^{[3]}A_{-2} + 2\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}\Big]|\mathbf{a}^{(3)}_0\rangle \\ &\quad + 42|\mathbf{a}^{(1)}_{-1}\rangle \otimes |\mathbf{a}^{(3)}_1\rangle - 42\,{}^{[3]}A_{-1}|\mathbf{a}^{(1)}_0\rangle \otimes {}^{[3]}A_{-1}|\mathbf{a}^{(3)}_0\rangle + 42|\mathbf{a}^{(1)}_1\rangle \otimes |\mathbf{a}^{(3)}_{-1}\rangle \end{aligned} \tag{7.11}$$

and

$$\begin{aligned}\bar{\Psi}^{(4)}_{4,\mathbf{1},0} &= |\mathbf{a}^{(1)}_0\rangle \otimes \Big[-\frac{7}{54}{}^{[3]}B_{-4} - \frac{4}{9}{}^{[3]}A_{-2}{}^{[3]}A_{-2} + \frac{7}{54}{}^{[3]}B_{-2}{}^{[3]}B_{-2} \\ &\quad - \frac{14}{9}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}B_{-2} + {}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}\Big]|\mathbf{a}^{(3)}_0\rangle\,. \end{aligned} \tag{7.12}$$

Moreover, the dominant state of the fiveplet at depth 4 is

$$\Psi^{(4)}_{4,\mathbf{5},2} = 24|\mathbf{a}^{(1)}_0\rangle \otimes |\mathbf{a}^{(3)}_2\rangle + 24|\mathbf{a}^{(1)}_1\rangle \otimes |\mathbf{a}^{(3)}_1\rangle\,. \tag{7.13}$$

All these states are virtual, i.e. they are in the kernel of the map $\mathcal{I}^{(4)}$. For the triplet and the fiveplet this is easy to see as their dominant states have momentum squared > 2. The two singlets have allowed momenta but their associated DDF states still vanish due to the Serre relations and the Jacobi identity.

Subsequently, we investigate the triplet at depth 5. Its dominant state is given by

$$\begin{aligned}\Psi^{(4)}_{5,\mathbf{3},-1} &= |\mathbf{a}^{(1)}_0\rangle \\ &\otimes \Big[-\frac{22}{9}\,{}^{[3]}A_{-5}{}^{[3]}A_{-1} + \frac{55}{36}\,{}^{[3]}A_{-4}{}^{[3]}A_{-2} - \frac{77}{18\sqrt{2}}\,{}^{[3]}A_{-4}{}^{[3]}B_{-2} \\ &\quad - \frac{7}{18\sqrt{2}}\,{}^{[3]}A_{-2}{}^{[3]}B_{-4} + \frac{11}{18\sqrt{2}}\,{}^{[3]}A_{-4}{}^{[3]}A_{-1}{}^{[3]}A_{-1} \\ &\quad + \frac{7}{36\sqrt{2}}\,{}^{[3]}A_{-2}{}^{[3]}A_{-2}{}^{[3]}A_{-2} - \frac{77}{36}\,{}^{[3]}A_{-2}{}^{[3]}A_{-2}{}^{[3]}B_{-2} \\ &\quad + \frac{7}{18\sqrt{2}}\,{}^{[3]}A_{-2}{}^{[3]}B_{-2}{}^{[3]}B_{-2} - \frac{7}{18}\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}B_{-4} \\ &\quad + \frac{73}{36}\,{}^{[3]}A_{-2}{}^{[3]}A_{-2}{}^{[3]}A_{-1}{}^{[3]}A_{-1} + \frac{35}{9\sqrt{2}}\,{}^{[3]}A_{-2}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}B_{-2} \end{aligned}$$

$$
\begin{aligned}
&+\frac{7}{18}\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}B_{-2}{}^{[3]}B_{-2}-\frac{23}{18\sqrt{2}}\,{}^{[3]}A_{-2}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}\\
&-\frac{7}{18}\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}B_{-2} \qquad (7.14)\\
&-\frac{1}{18}\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}\Bigg]|\mathbf{a}_1^{(3)}\rangle\\
&+|\mathbf{a}_1^{(1)}\rangle\\
&\otimes\left[\frac{7}{6}\,{}^{[3]}B_{-4}+4{}^{[3]}A_{-2}{}^{[3]}A_{-2}-\frac{7}{6}\,{}^{[3]}B_{-2}{}^{[3]}B_{-2}+14{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}B_{-2}\right.\\
&\left.-9\,{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}{}^{[3]}A_{-1}\right]|\mathbf{a}_0^{(3)}\rangle\,.
\end{aligned}
$$

This is the first maximal ground state which is not virtual. Instead we find

$$
\begin{aligned}
\mathcal{I}^{(4)}\left(\Psi^{(4)}_{5,\mathbf{3},-1}\right) = \Bigg[&-\frac{45}{32}\,{}^{[4]}B_{-4}-\frac{576}{32}\,{}^{[4]}A_{-3}{}^{[4]}A_{-1}-\frac{201}{32}\,{}^{[4]}A_{-2}{}^{[4]}A_{-2}\\
&-\frac{783}{32\sqrt{2}}\,{}^{[4]}A_{-2}{}^{[4]}B_{-2}-\frac{45}{8\sqrt{2}}\,{}^{[4]}A_{-1}{}^{[3]}B_{-3}+\frac{975}{256}\,{}^{[4]}B_{-2}{}^{[4]}B_{-2}\\
&+\frac{717}{16\sqrt{2}}\,{}^{[4]}A_{-2}{}^{[4]}A_{-1}{}^{[4]}A_{-1}-\frac{771}{64}\,{}^{[4]}A_{-1}{}^{[4]}A_{-1}{}^{[4]}B_{-2}\\
&-\frac{915}{64}\,{}^{[4]}A_{-1}{}^{[4]}A_{-1}{}^{[4]}A_{-1}{}^{[4]}A_{-1}\Bigg]|\mathbf{a}_1^{(4)}\rangle\,.
\end{aligned}
\tag{7.15}
$$

The sixth missing maximal fiveplet at depth 8 is currently out of reach for our computational machinery.

The action of the coset Virasoro operator on the (virtual) maximal ground states confirms the expected eigenvalues and gives rise to the descendant DDF states. For the triplet at depth 3 we have

$$
\begin{aligned}
{}^{[4]}\mathfrak{L}_1^{\text{coset}}\Psi^{(4)}_{3,\mathbf{3},w} &= 0\,,\\
{}^{[4]}\mathfrak{L}_0^{\text{coset}}\Psi^{(4)}_{3,\mathbf{3},w} &= \frac{1}{15}\Psi^{(4)}_{3,\mathbf{3},w}\,,\\
\mathcal{I}^{(4)}\left({}^{[4]}\mathfrak{L}_{-1}^{\text{coset}}\Psi^{(4)}_{3,\mathbf{3},w}\right) &= \psi^{(4)}_{4,\mathbf{3},w}
\end{aligned}
\tag{7.16}
$$

with the affine ground states (as a special case of (5.6))

$$
\psi^{(4)}_{4,\mathbf{3},\pm1} = \frac{8}{5}|\mathbf{a}^{(4)}_{\pm1}\rangle\,,\quad \psi^{(4)}_{4,\mathbf{3},0} = -\frac{4}{5}\sqrt{2}\,{}^{[4]}A_{-1}|\mathbf{a}_0^{(4)}\rangle\,.
\tag{7.17}
$$

Similarly, we obtain for the singlet maximal ground states at depth 4

$$
\begin{aligned}
{}^{[4]}\mathfrak{L}_1^{\text{coset}}\Psi^{(4)}_{4,\mathbf{1},0} &= 0\,, & {}^{[4]}\mathfrak{L}_1^{\text{coset}}\bar{\Psi}^{(4)}_{4,\mathbf{1},0} &= 0\,,\\
{}^{[4]}\mathfrak{L}_0^{\text{coset}}\Psi^{(4)}_{4,\mathbf{1},0} &= \frac{7}{5}\Psi^{(4)}_{4,\mathbf{1},0}\,, & {}^{[4]}\mathfrak{L}_0^{\text{coset}}\bar{\Psi}^{(4)}_{4,\mathbf{1},0} &= 0\,,\\
\mathcal{I}^{(4)}\left({}^{[4]}\mathfrak{L}_{-1}^{\text{coset}}\Psi^{(4)}_{4,\mathbf{1},0}\right) &= \psi^{(4)}_{5,\mathbf{1},0}\,, & \mathcal{I}^{(4)}\left({}^{[4]}\mathfrak{L}_{-1}^{\text{coset}}\bar{\Psi}^{(4)}_{4,\mathbf{1},0}\right) &= 0\,,
\end{aligned}
\tag{7.18}
$$

where

$$\psi^{(4)}_{5,\mathbf{1},0} = \Big[-\frac{189}{40}\,{}^{[4]}B_{-5} - 21\,{}^{[4]}A_{-3}{}^{[4]}A_{-2} + \frac{189}{40}\,{}^{[4]}B_{-3}{}^{[4]}B_{-2} - \frac{189}{5}\,{}^{[4]}A_{-2}{}^{[4]}A_{-1}{}^{[4]}B_{-2} - \frac{189}{5}\,{}^{[4]}A_{-1}{}^{[4]}A_{-1}{}^{[4]}B_{-3} + \frac{336}{5}\,{}^{[4]}A_{-2}{}^{[4]}A_{-1}{}^{[4]}A_{-1}{}^{[4]}A_{-1} \Big] |\mathbf{a}^{(4)}_0\rangle\,. \tag{7.19}$$

We observe that for the second singlet we actually have ${}^{[4]}\mathfrak{L}^{\text{coset}}_{-1}\Psi^{(4)}_{4,\mathbf{5},w}$ without the application of $\mathcal{I}^{(4)}$ since this state is actually a null state in the Virasoro module $\text{Vir}(\frac{4}{5}, 0)$.

For the fiveplet we have

$$\begin{aligned} {}^{[4]}\mathfrak{L}^{\text{coset}}_{1}\Psi^{(4)}_{4,\mathbf{5},w} &= 0 \\ {}^{[4]}\mathfrak{L}^{\text{coset}}_{0}\Psi^{(4)}_{4,\mathbf{5},w} &= \frac{2}{5}\Psi^{(4)}_{4,\mathbf{5},w}\,, \\ \mathcal{I}^{(4)}\left({}^{[4]}\mathfrak{L}^{\text{coset}}_{-1}\Psi^{(4)}_{4,\mathbf{5},w}\right) &= 0\,. \end{aligned} \tag{7.20}$$

Finally, we also confirm the eigenvalue of the triplet at depth 5 via

$$\begin{aligned} {}^{[4]}\mathfrak{L}^{\text{coset}}_{1}\Psi^{(4)}_{5,\mathbf{3},w} &= 0\,, \\ {}^{[4]}\mathfrak{L}^{\text{coset}}_{0}\Psi^{(4)}_{5,\mathbf{3},w} &= \frac{2}{3}\Psi^{(4)}_{5,\mathbf{3},w}\,. \end{aligned} \tag{7.21}$$

8. Isotropy and Anisotropy of Root Multiplicities

It has been observed from the available data for root multiplicities that they can be anisotropic. By isotropy we mean the the root multiplicities depend only on the norm $\mathbf{r}^2$ of the root, and not on its orientation. From [1] it follows that isotropy is respected at level-2.

In this section, we explain why the character of $\mathfrak{F}$ is isotropic only up to level 3 and give a general criterion for determining whether a level is isotropic or not. This criterion is based on the fact that roots at a given level belong to affine representations whose characters in turn are expressed in terms of generalized theta functions and string functions. The generalized theta functions relate elements that are related by (even) Weyl transformations and therefore all roots appearing in one theta function have the same multiplicity and norm.

The string functions correspond to adding multiples of $\boldsymbol{\delta}$ to the elements from the Θ-function. These elements all have the same multiplicity. Let $\mathbf{a}$ be one of these elements on level ℓ, meaning that $\mathbf{a} = -\ell\mathbf{r}_{-1} - d\boldsymbol{\delta} + n\mathbf{r}_1$, where d is the depth of the root. The roots generated by the string function are of the form $\mathbf{a}+m\boldsymbol{\delta}$ and have norm $(\mathbf{a}+m\boldsymbol{\delta})^2 = \mathbf{a}^2 - 2m\ell$ and so get more and more imaginary as one descends the string. Since $\boldsymbol{\delta}$ is invariant under the affine Weyl group, the roots that are obtained by descending m steps from one top element have the same multiplicities as those obtained from another top element of the same Θ-function and that is why the string functions and the Θ-function factorize in the character.

In general, the character of $\mathfrak{F}$ at a given level is a sum of Θ-functions multiplied by their string functions.[7] The question of isotropy or not of a given level can thus be determined, as a sufficient condition, if the individual Θ functions appearing in the character of that level generate disjoint sets of norms of roots. If they do, anisotropy is ruled out. We do not know whether the converse is also true, but believe so based on available data.

Let us go through this analysis for the lowest levels. At level 1, we can express the character schematically as

$$\text{Ch } \mathfrak{F}^{(1)} = f_1(q)\,\Theta_{\mathbf{\Lambda}_0}\,, \tag{8.1}$$

with a q-series $f_1(q)$. Hence, all root strings are connected to each other through Weyl reflections. Since a Weyl reflection preserves the norm, the roots of $\mathfrak{F}$ on level one are isotropic.

On levels $\ell = 2$ and $\ell = 3$ we can express the character schematically as

$$\text{Ch } \mathfrak{F}^{(\ell)} = f_{\ell,1}(q)\,\Theta_{\ell,1} + f_{\ell,2}(q)\,\Theta_{\ell,2}\,, \tag{8.2}$$

with some q-series $f_{\ell,i}$ and $\Theta_{2,1} = \Theta_{2\mathbf{\Lambda}_1}$, $\Theta_{2,2} = \Theta_{2\mathbf{\Lambda}_0}$ and $\Theta_{3,1} = \Theta_{\mathbf{\Lambda}_0+2\mathbf{\Lambda}_0} + \Theta_{5\mathbf{\Lambda}_0-2\mathbf{\Lambda}_0}$, $\Theta_{3,2} = \Theta_{3\mathbf{\Lambda}_0}$. The norms of the roots in the q-series $f_{\ell,1}(q)$ are $\{2-2m\cdot\ell \mid m \in \mathbb{Z}_{\geq 0}\}$. The norms of the roots in the q-series $f_{\ell,2}(q)$ are $\{-2m\cdot\ell \mid m \in \mathbb{Z}_{\geq 0}\}$. These two sets of norms are disjoint and thus two roots of the same norm belong to the same Weyl orbit and hence must have the same multiplicity. Thus the roots of $\mathfrak{F}$ on level 2 and 3 are isotropic.

On level 4 we do not have isotropy anymore. We can express the character schematically as

$$\begin{aligned}\text{Ch } \mathfrak{F}^{(4)} = {} & f_{4,1}(q)\,\Theta_{4\mathbf{\Lambda}_0} + f_{4,2}(q)\,\left(\Theta_{2\mathbf{\Lambda}_0+2\mathbf{\Lambda}_1} + \Theta_{6\mathbf{\Lambda}_0-2\mathbf{\Lambda}_1}\right)\\ & + f_{4,3}(q)\,\left(\Theta_{4\mathbf{\Lambda}_1} + \Theta_{8\mathbf{\Lambda}_0-4\mathbf{\Lambda}_1}\right)\,.\end{aligned} \tag{8.3}$$

The roots in the corresponding q-series $f_{4,i}(q)$ have norms

$$\{-2\cdot 4m \mid m \in \mathbb{Z}_{\geq 0}\}\,, \quad \{2-2\cdot 4m \mid m \in \mathbb{Z}_{\geq 0}\} \quad \text{and} \quad \{-2\cdot 4m \mid m \in \mathbb{Z}_{\geq 0}\}\,. \tag{8.4}$$

Thus the roots in the singlet and 5-plet q-series have the same norms. However, the q-series themselves do not agree. We find the first mismatch for the roots $(-4, -12, -12)$ and $(-4, -13, -11)$ which both have norm -64 but multiplicity 10107 respectively 10108.

For the general case we make the following conjecture. The roots on a given level ℓ are isotropic if and only if ℓ is prime or $\ell = 2p$ with $p > 2$ and p prime. Since multiplicities are Weyl-invariant, we only need to consider level-ℓ highest weights and their associated norms. The set of highest weights at level ℓ, up to possible shifts by $m\boldsymbol{\delta}$, is given by

$$\{(\ell-2n)\Lambda_0 + 2n\Lambda_1\} \quad \text{with} \quad (\ell-2n) \geq 0 \quad \text{and} \quad n \in \mathbb{Z}_{\geq 0}\,. \tag{8.5}$$

For each element in the set with $n \geq 0$ there is one q-series. The roots in these q-series have norms

$$\{-2\ell^2 + 2n^2 - 2\ell m \mid m \in \mathbb{Z}\}\,. \tag{8.6}$$

---

[7] We here use the term string function more generally to mean the full q-series that is produced from the affine character together with the Virasoro characters.

For two norms $-2\ell^2 + 2n_1^2 - 2\ell m_1$ and $-2\ell^2 + 2n_2^2 - 2\ell m_2$ from two of these sets (i.e. with $n_1 \neq n_2$) to agree, we would need

$$m_1 = m_2 + \frac{n_1^2 - n_2^2}{\ell} \,. \tag{8.7}$$

Clearly this equation has no solution in $\mathbb{Z}$ if ℓ is prime or $\ell = 2p$ with $p > 2$ and p prime. Hence the condition is sufficient. We checked that the condition is also necessary by testing it up to level 100 and roots of height ≤ 300 explicitly.

9. Outlook

In this paper we have initiated a new approach to studying the hyperbolic KMA $\mathfrak{F}$. This approach relies on the methods developed in [10, 11], and we have shown that these methods have the potential to reach beyond the low level sectors of $\mathfrak{F}$ studied so far. At the very least, they offer a much more concrete realization of $\mathfrak{F}$. In contradistinction to more conventional approaches relying on the division by ‘Serre ideals’ here the main open problem, besides working out products of affine representations, is in determining and understanding the kernel of the map $\mathcal{I}^{(\ell)}$ introduced in (2.13). A further complication as one moves up in level ℓ, is that each lower level factor comes with its own coset Virasoro representations, so that the products of such representations increase without bound as $\ell \to \infty$. We have presented partial evidence that, beyond the affine and coset Virasoro operators, there exist operators also for the ‘spectator Virasoro representations’. These should eventually enable us to generate the full level-ℓ sectors from a finite set of maximal ground states for each level. Importantly, our approach is not so much aimed at obtaining multiplicity formulas (a main focus in the mathematical literature), but rather at understanding the Lie algebra structure itself. Since $\mathfrak{F}$ is a subalgebra of the vertex operator algebra which is realized here with DDF operators, we also have a very concrete realization of the Lie bracket of $\mathfrak{F}$ with (3.4) that can be used for calculation.

It remains to be see whether or not our approach can give insight into all-level properties of $\mathfrak{F}$. Nevertheless, implications for fundamental physics (which we mentioned only briefly in the introduction) are of considerable interest, especially with regard to quantum cosmology and the physics of the Big Bang. Their exploration will be a fascinating topic for further study.

Acknowledgement We would like to thank Denis Bernard, Ilka Brunner, Stefan Fredenhagen, and especially Alex Feingold for illuminating discussions. HN thanks Luis Alvarez-Gaumé and the Simons Center for Geometry and Physics for hospitality and support during the final redaction of this paper. SC would like to thank the Humboldt-Universität of Berlin and the Erasmus+ program, respectively for hospitality and support during part of the work. AK, HM and HN would also like to thank the Mathematisches Forschungsinstitut Oberwolfach, where these results were first presented, for hospitality. This work was supported in part by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 740209).

Funding Open Access funding enabled and organized by Projekt DEAL. Partial financial support was received from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 740209).

Data Availability The authors declare that the data relevant to this paper are available within the paper and can also be reproduced using the software [9].

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

A. Proof of Theorem 1

In this appendix we provide a proof of Theorem 1. We start by noting the standard fact that $\mathfrak{F}$ has a basis of in terms of standard multicommutators, *i.e.*

$$\mathfrak{F}^{(\ell)} = \text{span}\{f_{i_1\dots i_n} \mid \text{with } \ell \text{ generators } f_{-1} \text{ and } n \geq \ell\}\,, \tag{A.1}$$

which implies that the inclusion $[\mathfrak{F}^{(1)}, \mathfrak{F}^{(\ell-1)}] \subseteq \mathfrak{F}^{(\ell)}$ is obvious. We use the notation of appendix D for multi-commutators and the set $\{f_{i_1\dots i_n}\}$ whose span is taken is not a basis.

For the reverse inclusion $\mathfrak{F}^{(\ell)} \subseteq [\mathfrak{F}^{(1)}, \mathfrak{F}^{(\ell-1)}]$ let $f_{i_1\dots i_n} \in \mathfrak{F}^{(\ell)}$. We proceed by induction on the first place k for which $i_k = -1$. Therefore assume that $i_k = -1$ and $-1 \notin \{i_1, \dots, i_{k-1}\}$ and we will show that we can write $f_{i_1\dots i_n}$ as a linear combination of commutators of level 1 and level $\ell - 1$ elements.

Base Case: $k = 1$. For $k = 1$ we simply have

$$f_{i_1\dots i_n} = \left[f_{-1}, f_{i_2\dots i_n}\right]. \tag{A.2}$$

Clearly $f_{-1} \in \mathfrak{F}^{(1)}$ and $f_{i_2\dots i_n} \in \mathfrak{F}^{(\ell-1)}$ because $f_{i_2\dots i_n}$ contains $\ell - 1$ elements f_{-1}.

Induction Step: $k \to k+1$. Consider the multi-commutator

$$f_{i_1\dots i_n} = f_{i_1\dots i_k,-1,i_{k+1},\dots i_n} = \left[f_{i_1}, f_{i_2,\dots,i_k,-1,i_{k+1},\dots i_n}\right]. \tag{A.3}$$

Since $f_{i_1} \neq f_{-1}$ we can write $f_{i_2,\dots,i_k,-1,i_{k+1},\dots i_n}$ by the induction hypothesis as a linear combination of multi-commutators of level 1 and level $\ell - 1$. Schematically we have a finite sum

$$f_{i_2,\dots,i_k,-1,i_{k+1},\dots i_n} = \sum_a \left[x_a^{(1)}, x_a^{(\ell-1)}\right], \tag{A.4}$$

where $x_a^{(1)} \in \mathfrak{F}^{(1)}$ and $x_a^{(\ell-1)} \in \mathfrak{F}^{(\ell-1)}$. We then use the Jacobi identity to get

$$\begin{aligned} f_{i_1,i_2,\ldots,i_k,-1,i_{k+1},\ldots i_n} &= \left[f_{i_1}, \sum_a \left[x_a^{(1)}, x_a^{(\ell-1)} \right] \right] \\ &= \sum_a \left[\left[f_{i_1}, x_a^{(1)} \right], x_a^{(\ell-1)} \right] + \sum_a \left[x_a^{(1)}, \left[f_{i_1}, x_a^{(\ell-1)} \right] \right], \end{aligned} \tag{A.5}$$

which, since $f_{i_1} \neq f_{-1}$ does not change the level, is clearly a linear combination of commutators in $[\mathfrak{F}^{(1)}, \mathfrak{F}^{(\ell-1)}]$ as claimed. □

B. Character Formulas

Here, we collect all our conventions and important formulae for computing the character of $\mathfrak{F}^{(\ell)}$.

Let $P = \mathbb{C}\Lambda_{-1} \oplus \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1$ be the weight lattice. Moreover let $P(\Lambda)$ be the set of all weights in the integrable highest-weight module $L(\mathbf{\Lambda})$ and P^ℓ the set of all weights of level ℓ. Then the formal character of $L(\mathbf{\Lambda})$ is given by

$$\mathrm{Ch}\ L(\mathbf{\Lambda}) = \sum_{\boldsymbol{\lambda} \in P(\Lambda)} \mathrm{mult}(\boldsymbol{\lambda})\, e^{\boldsymbol{\lambda}}, \tag{B.1}$$

where $\mathrm{mult}(\boldsymbol{\lambda})$ is the multiplicity of $\boldsymbol{\lambda} \in P(\Lambda)$. The formal characters $\mathrm{Ch}\ \mathfrak{F}^{(\ell)}$ are defined analogously, except that they are (in general infinite) sums of affine characters of type (B.1). The characters (B.1) are functions of three variables in general, and we will make particular use of the variable $q = e^{\Lambda_{-1}} = e^{-\boldsymbol{\delta}}$. All characters can be determined from the Weyl-Kac character formula [2]

$$\mathrm{Ch}\ L(\Lambda) = R^{-1} \sum_{w \in \mathcal{W}} e^{w(\boldsymbol{\rho}+\Lambda)-\boldsymbol{\rho}}, \tag{B.2}$$

where the sum is over the Weyl group $\mathcal{W} = \mathbb{Z} \rtimes \mathbb{Z}_2 \equiv \mathfrak{T} \rtimes \mathbb{Z}_2$, with the Weyl vector

$$\boldsymbol{\rho} = \Lambda_{-1} + \Lambda_0 + \Lambda_1 = -2\mathbf{r}_{-1} - 5\boldsymbol{\delta} + \frac{1}{2}\mathbf{r}_1, \tag{B.3}$$

and the denominator for affine $\mathfrak{sl}_2$ is

$$R := \prod_{n \geq 0} \left(1 - e^{-\mathbf{r}_0 - n\boldsymbol{\delta}}\right) \prod_{n > 0} \left(1 - e^{-n\boldsymbol{\delta}}\right) \prod_{n \geq 0} \left(1 - e^{-\mathbf{r}_1 - n\boldsymbol{\delta}}\right). \tag{B.4}$$

The abelian translation subgroup of the Weyl group is $\mathfrak{T} = \{\mathfrak{t}^n \mid n \in \mathbb{Z}\}$ where $\mathfrak{t} = w_1 w_0$ in terms of simple reflections and $\mathfrak{t}^n$ acts on a weight $\boldsymbol{\lambda} = p_{-1}\Lambda_{-1} + p_0\Lambda_0 + p_1\Lambda_1 \in P$ by

$$\mathfrak{t}^n(\boldsymbol{\lambda}) = \boldsymbol{\lambda} + \left[p_0 n^2 + p_1 n(n-1)\right]\Lambda_{-1} + 2n(p_0 + p_1)\Lambda_0 - 2n(p_0 + p_1)\Lambda_1. \tag{B.5}$$

In particular, the coefficient p_{-1} of $\Lambda_{-1} = -\boldsymbol{\delta}$ does not enter in the orbit under translations. The so-called generalized theta function of a weight $\boldsymbol{\lambda} \in P^\ell$ is defined by

$$\Theta_{\boldsymbol{\lambda}} = e^{-\frac{|\boldsymbol{\lambda}|^2}{2\ell}\boldsymbol{\delta}} \sum_{\mathfrak{t} \in \mathfrak{T}} e^{\mathfrak{t}(\boldsymbol{\lambda})} = e^{-\frac{|\boldsymbol{\lambda}|^2}{2\ell}\boldsymbol{\delta}} \sum_{n \in \mathbb{Z}} e^{\mathfrak{t}^n(\boldsymbol{\lambda})}. \tag{B.6}$$

We note that the definition is invariant under shifts of $\boldsymbol{\lambda}$ by multiples of $\boldsymbol{\delta}$, i.e. $\Theta_{\boldsymbol{\lambda}+s\boldsymbol{\delta}} = \Theta_{\boldsymbol{\lambda}}$ for any $s \in \mathbb{C}$. The full Weyl group is given by the semi-direct product $W = \{1, w_1\} \ltimes \mathfrak{T}$, where

$$w_1(p_{-1}\Lambda_{-1} + p_0\Lambda_0 + p_1\Lambda_1) = p_{-1}\Lambda_{-1} + (p_0 + 2p_1)\Lambda_0 - p_1\Lambda_1 \tag{B.7}$$

and $w_1 \mathfrak{t} = \mathfrak{t}^{-1} w_1$.

Since any Weyl transformation preserves the multiplicity of a root, we may organize the character along orbits of the translation group using the generalized theta function (B.6). Since the orbits for weights shifted by $\boldsymbol{\lambda}$ look similar, we additionally need to keep track of the multiplicities of weights along a string of weights shifted by $\boldsymbol{\lambda}$. To this end we define the set of *maximal* weights $\max(\Lambda)$ associated with a highest weight representation $L(\Lambda)$ to be the set of $\boldsymbol{\lambda}$ such that $\boldsymbol{\lambda} + \boldsymbol{\delta} \notin P(\Lambda)$. Then we can write the character of $L(\Lambda)$ for Λ a highest weight of level ℓ as

$$\mathrm{Ch}\ L(\Lambda) = \sum_{\substack{\boldsymbol{\lambda}\in\max(\Lambda) \\ \boldsymbol{\lambda} \bmod \mathfrak{T}}} C_\Lambda^{\boldsymbol{\lambda}}\, \Theta_{\boldsymbol{\lambda}}\,, \tag{B.8}$$

where $C_\Lambda^{\boldsymbol{\lambda}}$ is a so-called string function. It describes the multiplicities of a string of weights extending in the $-\boldsymbol{\delta}$ direction. Note that with the definition (B.6), the string function

$$C_\Lambda^{\boldsymbol{\lambda}} = e^{\frac{|\Lambda|^2}{2\ell}\boldsymbol{\delta}} + \ldots = q^{-\frac{|\Lambda|^2}{2\ell}} + \ldots \tag{B.9}$$

We also make use of the inverse of the generating series of partitions with $\varphi(q) = \prod_{n\geq 1}^{\infty}(1-q^n)$, such that

$$\frac{1}{\varphi(q)} = \sum_{n\geq 0} p(n) q^n \tag{B.10}$$

with $p(n)$ the classical number of partitions of n into positive integers (with repetitions).

As an example, we give the character of the basic representation on level $\ell = 1$ in this form:

$$\mathrm{Ch}\ L(\Lambda_0 + n\boldsymbol{\delta}) = q^{-n}\,\mathrm{Ch}\ L(\Lambda_0) = q^{-n}\, C_{\Lambda_0}^{\Lambda_0}\Theta_{\Lambda_0} = \frac{q^{1-n}}{\varphi(q)}\,\Theta_{\Lambda_0} \tag{B.11}$$

since there is only one maximal weight up to the action by $\mathfrak{T}$ and the multiplicities in the basic module are simply given by the classical integer partitions [2].

In the main text, we have already introduced the Sugawara central charge and allowed eigenvalues of unitary minimal models. They are a special case of the (p, p') minimal model with central charge and allowed eigenvalues (with $p > p'$ without loss of generality)

$$\begin{aligned} c^{p,p'} &= 1 - 6\frac{(p-p')^2}{pp'}\,, \\ h_{r,s}^{p,p'} &= \frac{(pr - p's)^2 - (p-p')^2}{4pp'}\,, \end{aligned} \tag{B.12}$$

with $1 \le r \le p'-1$ and $1 \le s \le p-1$ as well as $p's \le pr$. These are only unitary for $|p-p'|=1$ and then give the same formulas as (2.24) and (2.25), but with a different parametrization where

$$\ell = \frac{1}{2}\frac{p+p'}{p-p'} - \frac{3}{2} . \tag{B.13}$$

Finally, we define the Virasoro character of the minimal model $\mathrm{Vir}(c^{p,p'}, h^{p,p'}_{r,s})$ by [24]

$$\chi^{p,p'}_{r,s}(q) = \frac{1}{\varphi(q)}\left[q^{h^{p,p'}_{r,s}} + \sum_{k=1}^{\infty}(-1)^k\left(q^{h_1(k)} + q^{h_2(k)}\right)\right] = q^{h^{p,p'}_{r,s}} + \dots . \tag{B.14}$$

with

$$h_1(k) = h^{p,p'}_{r+kp',(-1)^k s+(1-(-1)^k)p/2} , \qquad h_2(k) = h^{p,p'}_{r,kp+(-1)^k s+(1-(-1)^k)p/2} . \tag{B.15}$$

C. Cocycle Factors

In this appendix, we discuss the so-called cocycle factors and fix the related convention used throughout this paper. Cocycle factors were first introduced in [26,27,33] to ensure the anti-symmetry of the commutator (3.4). For this work we adopt the cocycle conventions of [2]. We begin by repeating some important notation.

Recall that any DDF state is of the form

$$\varphi_{\mathbf{r}} = \prod_{i=1}^{M} {}^{[\ell]}A_{-m_i} \prod_{j=1}^{N} {}^{[\ell]}B_{-n_j} |\mathbf{a}_n^{(l)}\rangle \qquad M, N \ge 0 \tag{C.1}$$

with total momentum

$$\mathbf{r} = \mathbf{a}_n^{(l)} + \left(\sum_{i=1}^{M} m_i + \sum_{j=1}^{N} n_j\right)\mathbf{k}_l . \tag{C.2}$$

If the DDF state $\varphi_{\mathbf{r}}$ is an element of the Lie algebra $\mathfrak{F}$ then the total momentum $\mathbf{r}$ is an element of the root lattice Q, i.e.

$$\mathbf{r} = a_{-1}\mathbf{r}_{-1} + k_{-1}\boldsymbol{\delta} + a_1\mathbf{r}_1 \quad \text{with} \quad a_{-1}, k_{-1}, a_1 \in \mathbb{Z} . \tag{C.3}$$

In [10] it is explained in detail how the commutator of any two DDF states is defined via the state-operator correspondence. However, it is well known that this definition does not make the commutator anti-symmetric for all DDF states. Hence [26,27,33] introduced the cocycle factors $c_{\mathbf{r}}$ which commute with all DDF states $\varphi_{\mathbf{s}}$ ($\mathbf{r}, \mathbf{s} \in Q$) and satisfy

$$c_{\mathbf{r}} c_{\mathbf{s}} = \epsilon(\mathbf{r}, \mathbf{s}) c_{\mathbf{r+s}} , \tag{C.4}$$

where $\epsilon : Q \times Q \mapsto \{\pm 1\}$ is a 2-cocycle. We then define $\varphi{:=}\varphi_{\mathbf{r}} c_{\mathbf{r}}$ and $\psi{:=}\psi_{\mathbf{s}} c_{\mathbf{s}}$ such that the commutator (3.4) becomes

$$\begin{aligned}[\varphi, \psi] &= \oint \frac{\mathrm{d}z}{2\pi i}\, \mathcal{V}(\varphi_r c_{\mathbf{r}}; z)\, \psi_s c_{\mathbf{s}} = \oint \frac{\mathrm{d}z}{2\pi i}\, \mathcal{V}(\varphi_r; z)\, c_{\mathbf{r}} \psi_s c_{\mathbf{s}} \\ &= \epsilon(\mathbf{r}, \mathbf{s}) \oint \frac{\mathrm{d}z}{2\pi i}\, \mathcal{V}(\varphi_r; z)\, \psi_s c_{\mathbf{r+s}} ,\end{aligned} \tag{C.5}$$

where $\oint \frac{dz}{2\pi i}\, \mathcal{V}(\varphi_r; z)\, \psi_s$ is a new DDF state with total momentum $\mathbf{r}+\mathbf{s}$. This commutator is now anti-symmetric for all DDF states [26,27,33].

For the 2-cocycle ϵ we impose the following conditions [2]

$$\begin{aligned} \epsilon(\mathbf{r}+\mathbf{r}', \mathbf{s}) &= \epsilon(\mathbf{r}, \mathbf{s})\epsilon(\mathbf{r}', \mathbf{s}), \\ \epsilon(\mathbf{r}, \mathbf{s}+\mathbf{s}') &= \epsilon(\mathbf{r}, \mathbf{s})\epsilon(\mathbf{r}, \mathbf{s}') \end{aligned} \tag{C.6}$$

and

$$\epsilon(\mathbf{r}, \mathbf{r}) = (-1)^{\frac{1}{2}(\mathbf{r}\cdot\mathbf{r})} \tag{C.7}$$

for all $\mathbf{r}, \mathbf{s} \in Q$. Moreover, we normalize the cocycle factor to

$$\epsilon(0, 0) = 1. \tag{C.8}$$

By replacing $\mathbf{r}$ with $\mathbf{r}+\mathbf{s}$ in (C.7) we obtain

$$\epsilon(\mathbf{r}, \mathbf{s})\epsilon(\mathbf{s}, \mathbf{r}) = (-1)^{\mathbf{r}\cdot\mathbf{s}}. \tag{C.9}$$

The bi-multiplicativity (C.6) and the normalization (C.8) imply

$$\epsilon(m\mathbf{r}, n\mathbf{r}) = \epsilon(\mathbf{r}, \mathbf{r})^{mn} = (-1)^{\frac{1}{2}mn(\mathbf{r}\cdot\mathbf{r})}. \qquad (m, n \in \mathbb{Z}) \tag{C.10}$$

These conditions are sufficient to specify

$$\begin{aligned} \epsilon(\mathbf{r}_{-1}, \mathbf{r}_{-1}) &= -1, \\ \epsilon(\boldsymbol{\delta}, \boldsymbol{\delta}) &= 1, \\ \epsilon(\mathbf{r}_1, \mathbf{r}_1) &= -1 \end{aligned} \tag{C.11}$$

and

$$\begin{aligned} \epsilon(\mathbf{r}_{-1}, \boldsymbol{\delta}) &= -\epsilon(\boldsymbol{\delta}, \mathbf{r}_{-1}), \\ \epsilon(\mathbf{r}_{-1}, \mathbf{r}_1) &= \epsilon(\mathbf{r}_1, \mathbf{r}_{-1}), \\ \epsilon(\boldsymbol{\delta}, \mathbf{r}_1) &= \epsilon(\mathbf{r}_1, \boldsymbol{\delta}). \end{aligned} \tag{C.12}$$

Subsequently, we choose

$$\epsilon(\mathbf{r}_{-1}, \boldsymbol{\delta}) = -1 \quad \text{and} \quad \epsilon(\mathbf{r}_{-1}, \mathbf{r}_1) = \epsilon(\boldsymbol{\delta}, \mathbf{r}_1) = 1 \tag{C.13}$$

in agreement with the conventions given in [2]. Fixing also these last three cocycle factors explicitly has the advantage that now the cocycle factor of any two momenta $\mathbf{r}, \mathbf{s} \in Q$ is completely determined which considerably simplifies the results of all computations in the main text involving commutators and affine generators.

The affine generators ${}^{[\ell]}E_m$ and ${}^{[\ell]}F_m$ defined in (4.2) also come with a cocycle factor because they change the total momentum of the state on which they act. Let ${}^{[\ell]}E_m \equiv {}^{[\ell]}\hat{E}_m c_{\mathbf{r}_1}$ and ${}^{[\ell]}F_m \equiv {}^{[\ell]}\hat{F}_m c_{-\mathbf{r}_1}$, so we find

$$\begin{aligned} {}^{[\ell]}E_m\varphi &= {}^{[\ell]}\hat{E}_m c_{\mathbf{r}_1}\varphi_{\mathbf{s}}c_{\mathbf{s}} = \epsilon(\mathbf{r}_1, \mathbf{s})\ {}^{[\ell]}\hat{E}_m\varphi_{\mathbf{s}}c_{\mathbf{s}+\mathbf{r}_1}, \\ {}^{[\ell]}F_m\varphi &= {}^{[\ell]}\hat{F}_m c_{-\mathbf{r}_1}\varphi_{\mathbf{s}}c_{\mathbf{s}} = \epsilon(-\mathbf{r}_1, \mathbf{s})\ {}^{[\ell]}\hat{F}_m\varphi_{\mathbf{s}}c_{\mathbf{s}-\mathbf{r}_1}. \end{aligned} \tag{C.14}$$

Finally, we remark that the cocycle factors do not commute with the affine generators ${}^{[\ell]}\hat{E}_m$ and ${}^{[\ell]}\hat{F}_m$ since $c_{\mathbf{r}}e^{i\mathbf{s}\cdot\mathbf{Q}} = \epsilon(\mathbf{r}, \mathbf{s})e^{i\mathbf{s}\cdot\mathbf{Q}}c_{\mathbf{r}}$.

D. Translation of DDF States to Multi-commutators at Level 2

For completeness we spell out the relation between these DDF states (5.7) and multi-commutators of the Chevalley-Serre generators more explicitly. To this end we introduce the shorthand notation for words $w = lw'$ where l is a letter in the non-commutative alphabet $\{-1, 0, 1\}$ recursively by

$$f_w = f_{lw'} := [f_l, f_{w'}] \tag{D.1}$$

with $f_w = f_{lw'} = f_l$ for words w of length one (i.e. $w' = \emptyset$).

For low depths and level 2 the translation from DDF states to multi-commutators can be done relatively easily. For the above three ground states at depth 2, we find

$$\begin{aligned} \psi^{(2)}_{2,\mathbf{3},-1} &= \frac{1}{2}\, f_{-1,-1,0,1,0}\,, \\ \psi^{(2)}_{2,\mathbf{3},0} &= -\frac{1}{4}\, f_{-1,-1,0,1,0,1}\,, \\ \psi^{(2)}_{2,\mathbf{3},1} &= -\frac{1}{4}\, f_{-1,-1,1,0,1,0,1}\,. \end{aligned} \tag{D.2}$$

with the notation (D.1). At depth 3 the relation between the seven states and the seven linearly independent multi-commutators is a little more complicated. The multi-commutators are

$$\begin{aligned} &-(2,3,4): && \chi^{(2)}_{3,6} = f_{-1,-1,1,0,1,0,1,0,1} && \chi^{(2)}_{3,7} = f_{-1,1,0,-1,1,0,1,0,1} \\ &-(2,3,3): && \chi^{(2)}_{3,3} = f_{-1,-1,0,1,0,1,0,1} && \chi^{(2)}_{3,4} = f_{-1,0,-1,1,0,1,0,1} \\ & && \chi^{(2)}_{3,5} = f_{-1,1,0,-1,0,1,0,1} && \\ &-(2,3,2): && \chi^{(2)}_{3,1} = f_{-1,-1,0,0,1,0,1} && \chi^{(2)}_{3,2} = f_{-1,0,-1,0,1,0,1}\,. \end{aligned} \tag{D.3}$$

Then we find that

$$\begin{pmatrix} \psi^{(2)}_{3,\mathbf{3},-1} \\ \psi^{(2)}_{3,\mathbf{3},0} \\ \psi^{(2)}_{3,\mathbf{3},1} \\ \phi^{(2)}_{3,\mathbf{1},0} \\ \phi^{(2)}_{3,\mathbf{3},-1} \\ \phi^{(2)}_{3,\mathbf{3},0} \\ \phi^{(2)}_{3,\mathbf{3},1} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \chi^{(2)}_{\mathbf{3},1} \\ \chi^{(2)}_{\mathbf{3},2} \\ \chi^{(2)}_{\mathbf{3},3} \\ \chi^{(2)}_{\mathbf{3},4} \\ \chi^{(2)}_{\mathbf{3},5} \\ \chi^{(2)}_{\mathbf{3},6} \\ \chi^{(2)}_{\mathbf{3},7} \end{pmatrix}. \tag{D.4}$$

It is not hard to see that this transformation is invertible. Similar relations exist for higher depths.

E. Root Systems

We provide figures of the root system of $\mathfrak{F}$ at levels $\ell = 1, 2, 3, 4$. For an interactive 2D and 3D version of these root systems see [9].

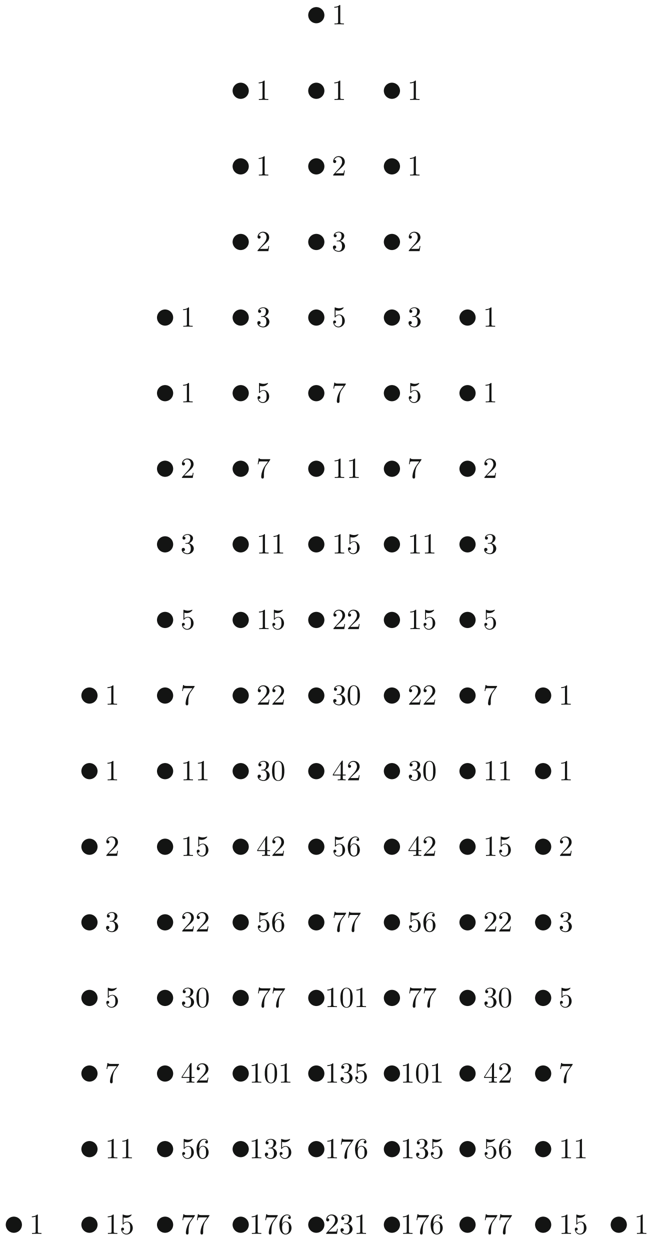

Fig. 6. The partial level-1 root system associated to $\mathfrak{F}^{(1)}$. The numbers specify the number of DDF states at each point and coincide with the multiplicities of the associated roots. They are in one-to-one correspondence with (3.35) and (3.31). The uppermost root is $(-1, 0, 0)$ the triplet below is $(-1, -1, -2)$, $(-1, -1, -1)$ and $(-1, -1, 0)$ from the left to right. The corresponding DDF states are $|\mathbf{a}_0\rangle$ and $|\mathbf{a}_{-1}\rangle$, $A_{-1}|\mathbf{a}_0\rangle$, $|\mathbf{a}_{+1}\rangle$. As explained in the main text the first number indicates the level, the second number indicates the depth and the third number is the sum of the depth and the weight. Thus, the states in the center column are $(-1, -\mathfrak{d}, -\mathfrak{d})$. A state n points left of the center column is $(-1, -\mathfrak{d}, -\mathfrak{d} - n)$. This correspondence holds at all levels

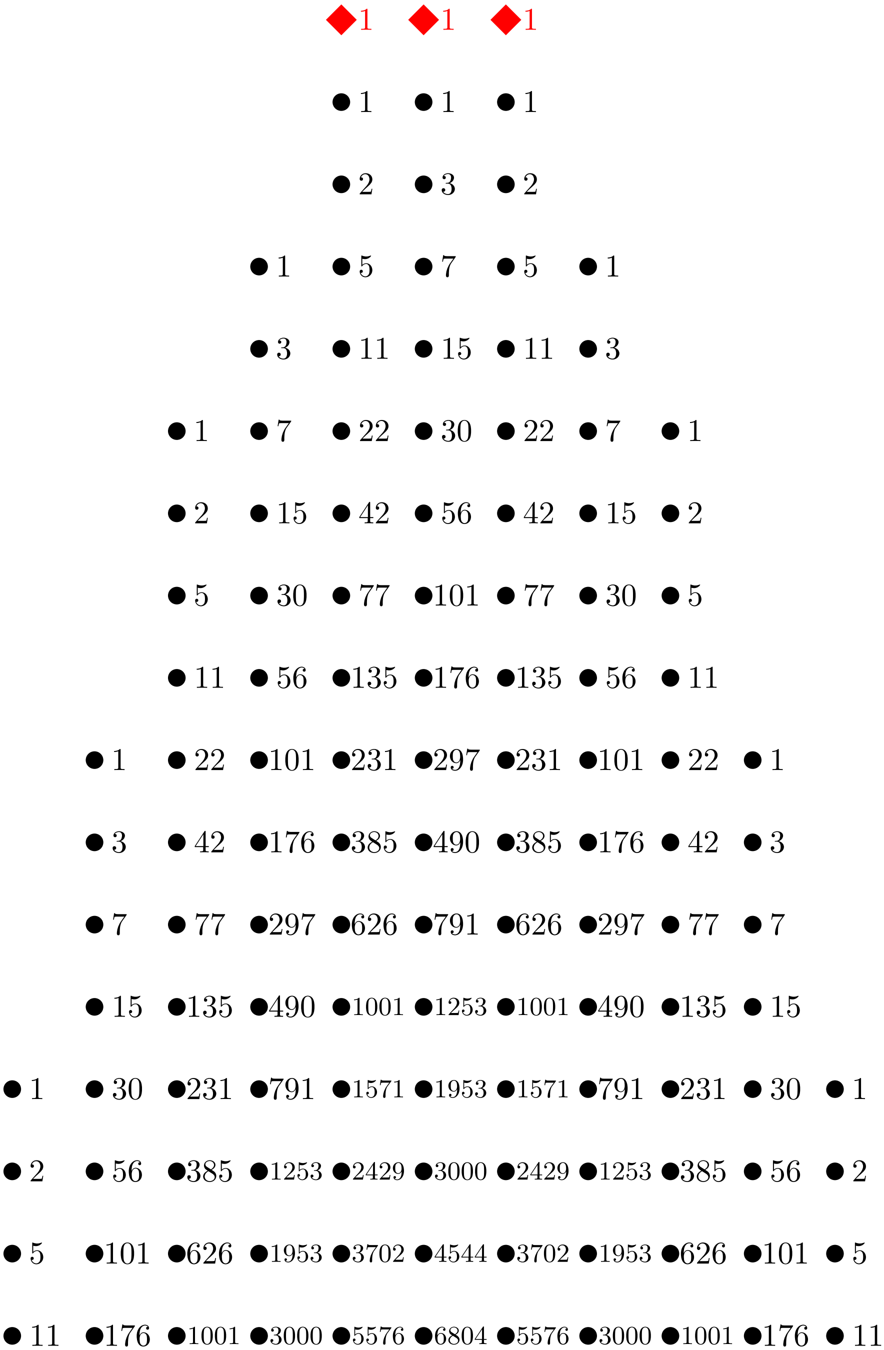

Fig. 7. The partial level-2 root system associated to $\mathfrak{F}^{(2)}$. The numbers specify the number of states at each point, and are equal to the multiplicities of the associated roots. The red diamonds denote the virtual triplet (5.2) at depth one that is not part of $\mathfrak{F}^{(2)}$ but only $\mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(1)}$. The top left root is $(-2, -1, -2)$. Not drawn in this picture (nor included in the multiplicities) is the affine tower erected over the red diamonds as none of its states belong to $\mathfrak{F}^{(2)}$

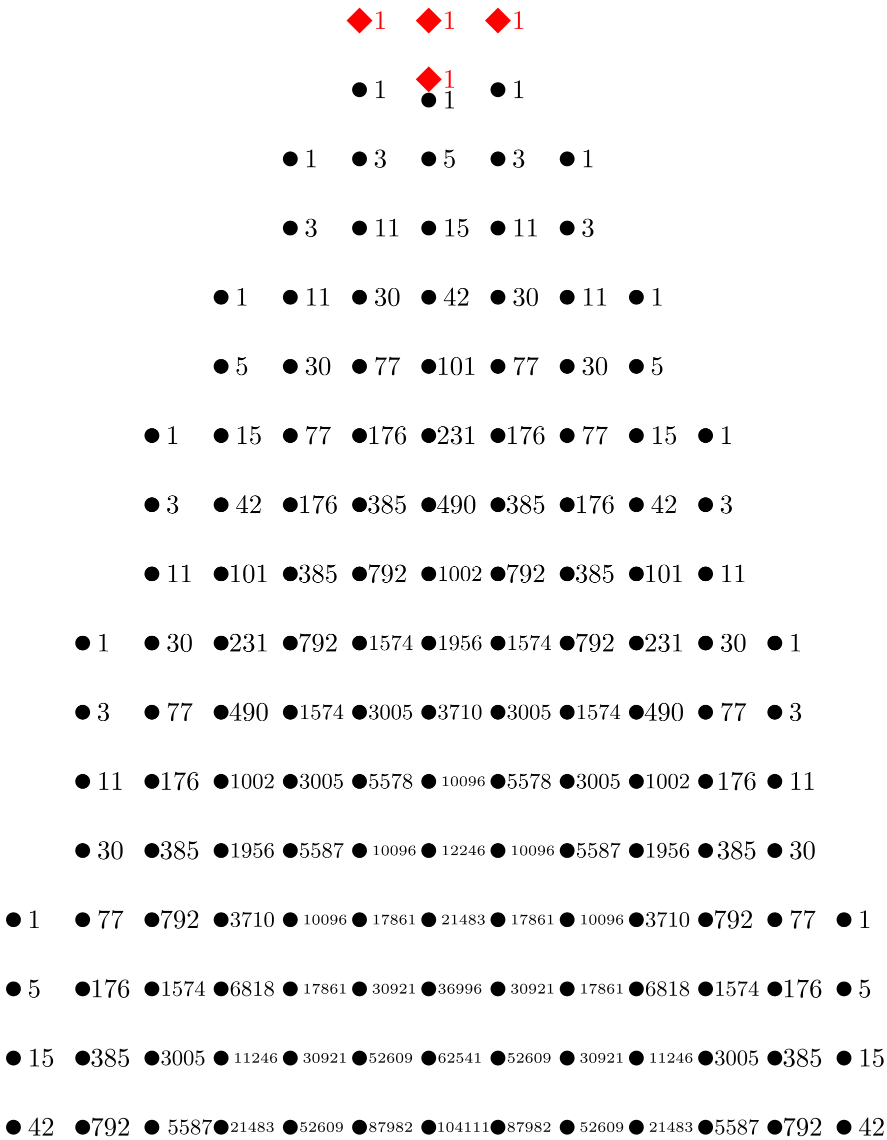

Fig. 8. The partial level-3 root system associated to $\mathfrak{F}^{(3)}$. The numbers specify the number of states at each point, and are equal to the multiplicities of the associated roots. The red diamonds denote the virtual singlet (6.11) at depth three and virtual triplet (6.10) at depth two that are not part of $\mathfrak{F}^{(3)}$. At depth 3 the virtual singlet sits among the regular states. Again we have not depicted the affine towers built on the virtual red diamond states

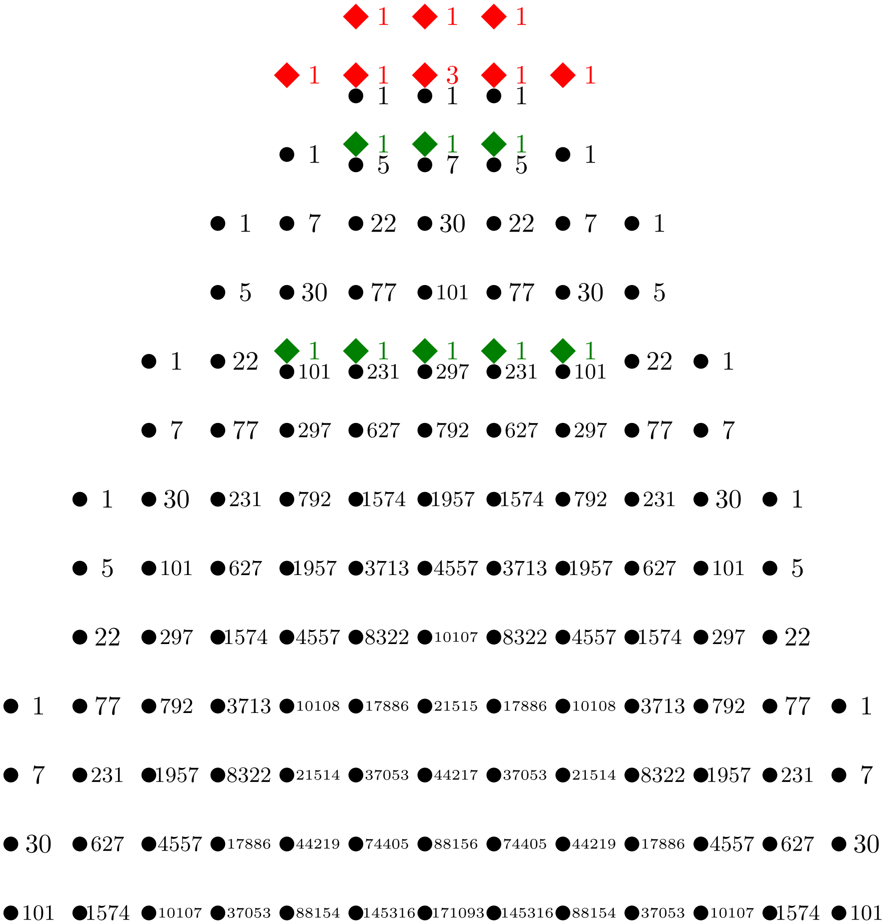

Fig. 9. The partial level-4 root system associated to the level-4 part $\mathfrak{F}^{(4)}$ indicated in black, again with root multiplicities indicated. The highest root on this level is $(-4, -4, -3)$ and is the top right black dot. The red diamonds indicate the virtual maximal ground states described in (7.5) and (7.6) and on which affine and coset Virasoro modules can be built that contain all level-4 elements of the hyperbolic algebra. Similarly the green diamonds denote the maximal ground states which also belong to $\mathfrak{F}^{(4)}$. For the fiveplet at depth 8 we do not yet have an expression in terms of DDF states. Hence we cannot yet tell whether it is virtual or not

References

1. Feingold, A.J., Frenkel, I.B.: A hyperbolic Kac-Moody algebra and the theory of Siegel modular forms of genus 2. Math. Ann. **263**, 87–144 (1983). https://doi.org/10.1007/BF01457086
2. Kac, V.G.: Infinite-Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990). https://doi.org/10.1017/CBO9780511626234
3. Kac, V.G., Moody, R.V., Wakimoto, M.: On E_{10}. In: Bleuler, K., Werner, M. (eds.) Differential Geometrical Methods in Theoretical Physics. Springer, Dordrecht (1988). https://doi.org/10.1007/978-94-015-7809-7
4. Kang, S.J.: Root multiplicities of the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$. J. Algebra **160**, 492–523 (1993). https://doi.org/10.1006/jabr.1993.1198
5. Kang, S.J.: Root multiplicities of Kac-Moody algebras. Duke Math. J. **74**, 635–666 (1994). https://doi.org/10.1215/S0012-7094-94-07423-1

6. Bauer, M., Bernard, D.: On root multiplicities of some hyperbolic Kac-Moody algebras. Lett. Math. Phys. **42**, 153–166 (1997). https://doi.org/10.1023/A:1007317602691. arXiv:hep-th/9612210
7. Kac, V.G., Wakimoto, M.: Unitarizable Highest Weight Representations of the Virasoro, Neveu-Schwarz and Ramond Algebras. Lecture Notes in Physics, vol. 261, pp. 345–372 (1986). https://math.mit.edu/~kac/not-easily-available/unitarizable.pdf
8. Goddard, P., Kent, A., Olive, D.I.: Virasoro algebras and coset space models. Phys. Lett. B **152**, 88–92 (1985). https://doi.org/10.1016/0370-2693(85)91145-1
9. Malcha, H.: VisualLie (2024). https://hmalcha.github.io/VisualLie/
10. Gebert, R.W., Nicolai, H.: On E(10) and the DDF construction. Commun. Math. Phys. **172**, 571–622 (1995). https://doi.org/10.1007/BF02101809. arXiv:hep-th/9406175
11. Gebert, R.W., Nicolai, H.: An affine string vertex operator construction at arbitrary level. J. Math. Phys. **38**, 4435–4450 (1997). https://doi.org/10.1063/1.532135. arXiv:hep-th/9608014
12. Borcherds, R.E.: Vertex algebras, Kac-Moody algebras, and the Monster. Proc. Natl. Acad. Sci. **83**(10), 3068–3071 (1986). https://doi.org/10.1073/pnas.83.10.3068
13. Frenkel, I.B.: Representations of Kac-Moody algebras and dual resonance models. In: Applications of Group Theory in Physics and Mathematical Physics (Chicago, 1982). Lectures in Appliation Mathematics, vol. 21, pp. 325–353. American Mathematical Society, Providence (1985)
14. Frenkel, I.B., Lepowsky, J., Meurman, A.: Vertex Operator Algebras and the Monster, Pure and Applied Mathematics, vol. 134. Academic Press, San Diego (1988)
15. Del Giudice, E., Di Vecchia, P., Fubini, S.: General properties of the dual resonance model. Ann. Phys. **70**, 378–398 (1972). https://doi.org/10.1016/0003-4916(72)90272-2
16. Julia, B.: Infinite Lie algebras in physics. In: Johns Hopkins Workshop on Current Problems in Particle Physics: Unified theories and Beyond, Johns Hopkins University, Baltimore (1981). https://lib-extopc.kek.jp/preprints/PDF/1981/8112/8112035.pdf
17. Breitenlohner, P., Maison, D.: On the Geroch group. Ann. Inst. H. Poincaré. Phys. Théor. **46**, 215 (1987). https://lib-extopc.kek.jp/preprints/PDF/1987/8702/8702388.pdf
18. Nicolai, H.: A hyperbolic Lie algebra from supergravity. Phys. Lett. B **276**, 333–340 (1992). https://doi.org/10.1016/0370-2693(92)90328-2
19. Penna, R.F.: The Geroch Group in one dimension. arXiv:2112.05661 [hep-th]
20. Damour, T., Henneaux, M., Nicolai, H.: Cosmological Billiards. Class. Quant. Grav. **20**, R145–R200 (2003). https://doi.org/10.1007/0-387-24992-3_5
21. Belinskii, V.A., Khalatnikov, I.M., Lifshitz, E.M.: Oscillatory approach to a singular point in the relativistic cosmology. Adv. Phys. **19**, 525 (1970)
22. Nicolai, H.: Complexity and the Big Bang. Class. Quant. Grav. **38**(18), 187001 (2021). https://doi.org/10.1088/1361-6382/ac1b07. arXiv:2104.09626 [gr-qc]
23. Goddard, P., Olive, D.I.: Kac-Moody and Virasoro algebras in relation to quantum physics. Int. J. Mod. Phys. A **1**, 303 (1986). https://doi.org/10.1142/S0217751X86000149
24. Di Francesco, P., Mathieu, P., Senechal, D.: Conformal Field Theory. Springer, Berlin (1997). https://doi.org/10.1007/978-1-4612-2256-9
25. Brower, R.C.: Spectrum generating algebra and no ghost theorem for the dual model. Phys. Rev. D **6**, 1655–1662 (1972). https://doi.org/10.1103/PhysRevD.6.1655
26. Frenkel, I.B., Kac, V.G.: Basic representations of affine Lie algebras and dual resonance models. Invent. Math. **62**, 23–66 (1980). https://doi.org/10.1007/BF01391662
27. Segal, G.: Unitarity representations of some infinite dimensional groups. Commun. Math. Phys. **80**, 301–342 (1981). https://doi.org/10.1007/BF01208274
28. Goddard, P., Olive, D.I.: In: Lepowsky, J., Mandelstam, S., Singer, I.M. (eds.) Vertex Operators in Mathematics and Physics. Springer (1983). https://doi.org/10.1007/978-1-4613-9550-8
29. Gebert, R.W., Koepsell, K., Nicolai, H.: The Sugawara generators at arbitrary level. Commun. Math. Phys. **184**, 119–141 (1997). https://doi.org/10.1007/s002200050055
30. Nutma, T.: SimpLie (2015). https://github.com/teake/simplie
31. Kac, V.G., Peterson, D.H.: Infinite-dimensional Lie algebras, theta functions and modular forms. Adv. Math. **53**, 125–264 (1984). https://doi.org/10.1016/0001-8708(84)90032-X
32. Mortenson, E.T.: On Hecke-type double-sums and general string functions for the affine Lie algebra $A_1^{(1)}$. Ramanujan J. **63**(3), 553–582 (2024). https://doi.org/10.1007/s11139-023-00737-x. arXiv:2110.02615 [math.NT]
33. Goddard, P., Olive, D.I.: Kac-Moody and Virasoro algebras: a preprint volume for physicists. Adv. Ser. Math. Phys. (1988). https://doi.org/10.1142/0485

Communicated by Y. Kawahigashi