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Article

Invariance of Elliptic Genus Under Wall-Crossing

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Wall-crossing formulas for various flavors of elliptic genus can be obtained using master spaces. We give a topological criterion which implies that such wall-crossing formulas are trivial. Applications are given for the following: GIT quotients, following Thaddeus; moduli of sheaves, following Mochizuki; and Donaldson–Thomas and Vafa–Witten theory, following Joyce and Tanaka–Thomas, respectively.

1 Introduction

1.1

Let X be a smooth proper scheme over $\mathbb{C}$. Recall the q-Pochhammer and (normalized) odd Jacobi theta functions

$$\phi(z) := \prod_{n>0}(1-q^n z), \qquad \vartheta(z) := (1-z^{-1})\phi(qz)\phi(qz^{-1}),$$

respectively. Both can be extended to functions of vector bundles on X as

$$\Phi(\mathcal{E}) := \prod_{\mathcal{L}\in\mathcal{E}} \phi(\mathcal{L}), \qquad \Theta(\mathcal{E}) := \prod_{\mathcal{L}\in\mathcal{E}} \vartheta(\mathcal{L}),$$

where the products range over the Chern roots $\mathcal{L}$ of $\mathcal{E}$. Then, following [19, 27], the *elliptic genus* of X is

$$E_{-y}(X) := \chi\left(X, \frac{\Theta(y\mathcal{T}_X)}{\Phi(\mathcal{T}_X)\Phi(\mathcal{T}_X^\vee)}\right), \tag{1}$$

where $\mathcal{T}_X$ is the tangent bundle and y is a formal variable. If G is a group of automorphisms of X, then (1) is naturally an element

$$E_{-y}(X) \in K_G(\mathrm{pt})[y^{\pm 1}][[q]],$$

where, for a scheme Z with G-action, $K_G(Z)$ denotes the G-equivariant K-theory of Z, that is, the Grothendieck group of G-equivariant coherent sheaves on Z.

Note that the $q=0$ specialization of $E_{-y}(X)$ is the Hirzebruch χ_{-y} genus of X.

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1.2

To give a sense of what elliptic genus looks like, consider the simple case where a torus $\mathrm{T} = (\mathbb{C}^\times)^r$ acts on X and the fixed point set X^{T} consists only of isolated points. Then, by torus-equivariant localization [43],

$$\mathrm{E}_{-y}(X) = \sum_{p \in X^{\mathrm{T}}} \prod_{w \in T_p X} \frac{\vartheta(yw)}{\vartheta(w)}, \tag{2}$$

where the product ranges over the T-weights of T_pX. Indeed, the definition (1) is purposefully designed so that (2) holds.

More generally, let $\mathcal{N}_{X^{\mathrm{T}} \subset X}$ be the normal bundle of $X^{\mathrm{T}} \subset X$. Then localization, combined with the multiplicativity of Θ and Φ, produces

$$\mathrm{E}_{-y}(X) = \chi\left(X^{\mathrm{T}}, \frac{\Theta(y\mathcal{N}_{X^{\mathrm{T}} \subset X})}{\Theta(\mathcal{N}_{X^{\mathrm{T}} \subset X})} \frac{\Theta(y\mathcal{T}_{X^{\mathrm{T}}})}{\Phi(\mathcal{T}_{X^{\mathrm{T}}})\Phi(\mathcal{T}_{X^{\mathrm{T}}}^\vee)}\right). \tag{3}$$

Note that only the first factor in the integrand has non-trivial T-dependence.

1.3

It turns out $\mathrm{E}_{-y}(X)$ has better properties when it is valued in equivariant *elliptic cohomology*, rather than equivariant K-theory as written in (1). For a torus $\mathrm{T} = (\mathbb{C}^\times)^r$, this means being a section of a line bundle on the elliptic cohomology scheme [17]

$$\mathbf{Ell}_{\mathrm{T}}(\mathrm{pt}) := \mathrm{T}/q^{\operatorname{cochar} \mathrm{T}} \cong (\mathbb{C}^\times/q^{\mathbb{Z}})^r,$$

rather than merely on the K-theory scheme $\operatorname{\mathbf{Spec}} K_{\mathrm{T}}(\mathrm{pt}) = \mathrm{T}$. Here $q = \exp(2\pi i\tau)$ where τ is the elliptic modulus, $\mathbb{C}^\times/q^{\mathbb{Z}}$ is the Tate elliptic curve, and $\operatorname{\mathbf{cochar}} \mathrm{T}$ and $\operatorname{\mathbf{char}} \mathrm{T}$ denote the cocharacter and character lattices of T.

We may view q either as a formal variable or as a complex number, so that $\mathbb{C}^\times/q^{\mathbb{Z}}$ is a family of complex elliptic curves over either $\mathbb{Z}[[q]]$ or the punctured disk $0 < |q| < 1$, respectively. Since both $\phi(z)$ and $\vartheta(z)$ are absolutely convergent, all convergence issues may be disregarded.

Properness of $\mathrm{T}/q^{\operatorname{cochar} \mathrm{T}}$, in contrast to T, provides strong control over $\mathrm{E}_{-y}(X)$ and related quantities.

1.4

Recall [7] that meromorphic sections of a degree d line bundle on $\mathbb{C}^\times/q^{\mathbb{Z}}$ correspond to meromorphic functions $f(s)$ on the pre-quotient $\mathbb{C}^\times$ satisfying the q-difference equation

$$f(qs) = cs^{-d} f(s), \quad c \in \mathbb{C}^\times. \tag{4}$$

For example, ϑ satisfies $\vartheta(qz) = -(qz)^{-1}\vartheta(z)$. Given a line bundle $\mathcal{L}$ on X^{T}, let $\mathbf{wt}_{\mathrm{T}}(\mathcal{L}) \in \operatorname{\mathbf{char}} \mathrm{T}$ correspond to its T-weight. Using (3) and (4), one can check that if

$$y^{-\sum_{\mathcal{L} \in \mathcal{N}_{X^{\mathrm{T}} \subset X}} \langle \sigma, \mathbf{wt}_{\mathrm{T}}(\mathcal{L}) \rangle} \text{ is constant on } X^{\mathrm{T}} \tag{5}$$

for any cocharacter $\sigma \in \operatorname{\mathbf{cochar}} \mathrm{T}$, then $\mathrm{E}_{-y}(X)$ becomes a genuinely elliptic object, that is, it becomes a meromorphic section of a (degree-0) line bundle on $\mathrm{T}/q^{\operatorname{cochar} \mathrm{T}}$.

The simplest way to satisfy (5) is to assume that $\det \mathcal{T}_X$ is an N-th power, so that $\sum_{\mathcal{L}} \mathbf{wt}_{\mathrm{T}}(\mathcal{L})$ is a multiple of N, and to specialize y to an N-th root of unity $\zeta_N \neq 1$. Thus, the following two cases will feature prominently in this paper:

(i) (X is spin) $y = -1$ and the canonical bundle $\mathcal{K}_X$ admits a square root; and
(ii) (X is Calabi–Yau) y is arbitrary and $\mathcal{K}_X = \mathcal{O}_X$.

The $y = -1$ specialization $\mathrm{E}_1(X)$ is particularly notable because it is the historically earlier notion of elliptic genus due to Landweber–Stong, Ochanine, and Witten [29, 45].

1.5

In this paper, we consider the following geometric setup for studying (equivariant) wall-crossing problems: T is a torus and M is a T-*equivariant master space*. This means that M is a smooth proper scheme with a $(\mathsf{T} \times \mathsf{S})$-action, where $\mathsf{S} := \mathbb{C}^\times$ with coordinate denoted s, such that the S-fixed locus is a disjoint union of the following T-invariant pieces:

(i) smooth divisors $\iota_\pm : Z_\pm \hookrightarrow M$ with normal bundles of S-weights $s^{\pm 1}$, respectively;
(ii) other proper component(s) $\iota_0 : Z_0 \hookrightarrow M$ whose normal sheaf $\mathcal{N}_{\iota_0}$ is locally free.

(More generally, M can be a smooth Deligne–Mumford stack satisfying the weaker properness condition of Remark 2.9. For elliptic genus, the S-fixed components $Z_\pm$ are still required to be schemes.) Generally, in applications, $Z_\pm$ will be two different stable loci in an ambient algebraic stack.

Many master spaces exist in the literature; examples include [15, 22, 35, 38, 41, 47]. In many of these, M is only *quasi-smooth* instead of smooth (see Remark 2.8).

1.6

Theorem (Main theorem). Suppose $\mathcal{N}_{\iota_0}\big|_{Z_0^{\mathsf{T}}} = \mathcal{E}_+ \oplus \mathcal{E}_-$ only has pieces of S-weight $s^{\pm 1}$, and

$$\operatorname{\mathbf{rank}} \mathcal{E}_+ \equiv \operatorname{\mathbf{rank}} \mathcal{E}_- \mod N \tag{6}$$

for some integer $N > 0$. Then, for any N-th root of unity $\zeta_N \neq 1$,

$$\mathsf{E}_{-\zeta_N}(Z_+) = \mathsf{E}_{-\zeta_N}(Z_-).$$

To be clear, Z_0^{T} may have many connected components. While $\operatorname{\mathbf{rank}} \mathcal{E}_\pm$ need not be constant on Z_0^{T}, (6) must hold on each connected component.

Note that if $\operatorname{\mathbf{rank}} \mathcal{E}_+ = \operatorname{\mathbf{rank}} \mathcal{E}_-$, that is, the theorem holds for all $N > 0$, then $\mathsf{E}_{-y}(Z_+) = \mathsf{E}_{-y}(Z_-)$ for general y because q-coefficients of E_{-y} are Laurent polynomials in y.

In practice, the condition that only pieces of S-weights $s^{\pm 1}$ appear is often satisfied automatically, but it may also be removed by replacing (6) with a more complicated condition.

1.7

The proof of Theorem 1.6, given in §2, is fairly elementary. A very general wall-crossing procedure, arising from $(\mathsf{T} \times \mathsf{S})$-equivariant localization on M, expresses the difference between $\mathsf{E}_{-y}(Z_+)$ and $\mathsf{E}_{-y}(Z_-)$ in terms of a contour integral

$$\oint_{|s| \approx 1} \frac{\Theta(y(\mathcal{E}_+ + \mathcal{E}_-))}{\Theta(\mathcal{E}_+ + \mathcal{E}_-)} \frac{ds}{s} \tag{7}$$

on S. The integrand is a function on S in general, but specializing $y = \zeta_N$ makes it $q^{\operatorname{cochar} \mathsf{S}}$-invariant, so the contour integral descends to the elliptic curve $\mathsf{S}/q^{\operatorname{cochar} \mathsf{S}}$. There, the contour encloses all the poles of the integrand, and therefore vanishes by Cauchy's residue theorem and properness of $\mathsf{S}/q^{\operatorname{cochar} \mathsf{S}}$.

1.8

Previous work studying elliptic genus under birational transformations focused on the Calabi–Yau setting of 1.4(ii), and the *non-equivariant* (i.e., T is trivial) elliptic genus

$$\mathsf{E}_{-y} \colon \mathbf{MSU}_* \otimes \mathbb{Q} \to \mathbf{Jac}$$

viewed as a homomorphism from the **SU**-bordism ring to Jacobi forms in (y, τ). A bordism argument by Totaro [44,§4] shows that E_{-y} is invariant under certain Calabi–Yau flops, using Krichever–Höhn's *elliptic rigidity* [19, 27]. More sophisticated work by Borisov and Libgober extends this to arbitrary crepant birational transformations [8].

It is possible that an equivariant version of such arguments can be used to prove Theorem 1.6, as explained below. But our approach outlined in §1.7 is more versatile; for instance, it applies equally well to virtual chiral elliptic genus (see §1.13).

1.9

In fact, a simple geometric formula for the contour integral (7) exists in general. In Appendix A, we explain how Jeffrey–Kirwan integration expresses contour integrals of this shape as equivariant Euler characteristics on GIT quotients—in our case (Proposition A.2), as

$$\chi\left(\mathbb{P}(V), \frac{\Theta(y\mathcal{O}(1)\otimes V_+ + y^{-1}\mathcal{O}(1)\otimes V_-)}{\Phi(\mathcal{O}(1)\otimes V)\Phi(\mathcal{O}(-1)\otimes V^\vee)}\right), \tag{8}$$

where $V := V_+ \oplus V_-$ and coordinates of $V_\pm$ correspond to Chern roots of $\mathcal{E}_\pm$. A slightly different application of Jeffrey–Kirwan integration expresses (7) as the change in E_{-y} across certain toric flips (Remark A.5). The invariance of $E_{-\zeta_N}$ under these toric flips is equivalent to our main Theorem 1.6.

Unfortunately for wall-crossing, in general (8) appears to be a non-trivial function of the Chern roots of $\mathcal{E}_\pm$, with no productive closed form.

1.10

We give two direct applications of Theorem 1.6: to the original Thaddeus master space, studying variation of GIT (§3), and to Mochizuki's *enhanced* master space, studying moduli of sheaves (§4). In each, most of the work is to identify sufficiently explicit and useful criteria such that the topological condition (6) holds. As such, the following two theorems describe certain "nice" cases which may be less general than permitted by their constituent pieces.

1.11

Theorem (Theorem 3.6, Theorem 3.7). Let $X /\!\!/_{\mathcal{L}_\pm} G$ be two smooth GIT quotients separated by a single, simple (§3.3) wall $\mathcal{L}_0$ in the space of GIT stability conditions, and let $X^{\mathrm{sst}}(\mathcal{L}) \subset X$ denote the $\mathcal{L}$-semistable locus. The natural maps

$$(X^{\mathrm{sst}}(\mathcal{L}_0) \setminus X^{\mathrm{sst}}(\mathcal{L}_\mp)) /\!\!/_{\mathcal{L}_\pm} G \to (X^{\mathrm{sst}}(\mathcal{L}_0) \setminus (X^{\mathrm{sst}}(\mathcal{L}_+) \cup X^{\mathrm{sst}}(\mathcal{L}_-))) /\!\!/_{\mathcal{L}_0} G$$

are always locally trivial fibrations by weighted projective spaces. If in fact they are locally trivial $\mathbb{P}^{N_\pm}$-fibrations with $N_+ - N_- \equiv 0 \bmod N$, then

$$E_{-\zeta_N}(X /\!\!/_{\mathcal{L}_+} G) = E_{-\zeta_N}(X /\!\!/_{\mathcal{L}_-} G).$$

1.12

Theorem (Theorem 4.6, Lemma 4.7, Theorem 4.10, Corollary 4.12). Let Y be a smooth projective variety with canonical bundle $\mathcal{K}_Y$. Under the assumptions of §4.1 and §4.2, consider two stable loci $\mathfrak{M}_\alpha^{\mathrm{sst}}(\pm) \subset \mathfrak{M}_\alpha^{\mathrm{sst}}(0)$ in a moduli stack of sheaves of class $\alpha \in H^*(Y)$ on Y, separated by a single wall at 0.

- (Spin) If $\mathcal{K}_Y$ admits a square root, then

$$E_1(\mathfrak{M}_\alpha^{\mathrm{sst}}(+)) = E_1(\mathfrak{M}_\alpha^{\mathrm{sst}}(-)).$$

- (Calabi–Yau) If $\mathcal{K}_Y^{\otimes k} = \mathcal{O}_Y$ for some integer k, and the wall is simple (see 4.4(i)), then

$$E_{-y}(\mathfrak{M}_\alpha^{\mathrm{sst}}(+)) = E_{-y}(\mathfrak{M}_\alpha^{\mathrm{sst}}(-)).$$

Unfortunately, in practice, the assumption 4.2(ii) that all semistable loci are smooth is too strong. For instance, it is satisfied for surfaces Y if Y is Fano (Remark 4.3) (i.e., a del Pezzo surface), but then only $Y = \mathbb{P}^2$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1$ have canonical bundles which are divisible (i.e. non-trivial powers of some line bundle).

In the Calabi–Yau case, the assumption that the wall is simple is an artefact of Mochizuki's setup, and is possibly unnecessary; see Remark 4.13.

1.13

In §5, we replace the smoothness condition on X or the master space M with the condition that they admit equivariantly-symmetric perfect obstruction theories. Then $\mathrm{E}_{-y}(X)$ may be replaced by the *virtual chiral elliptic genus*

$$\mathrm{E}_{-y}^{\mathrm{vir}/2}(X) := \chi\left(X, \frac{\mathcal{O}_X^{\mathrm{vir}} \otimes \mathcal{K}_{\mathrm{vir}}^{1/2}}{\Phi(\mathcal{T}_X^{\mathrm{vir}})\Phi((\mathcal{T}_X^{\mathrm{vir}})^\vee)}\right)$$

following ideas of [14,§8], and there is a virtual version (Theorem 5.6) of our main theorem. The twist by a square root of the virtual canonical $\mathcal{K}_{\mathrm{vir}} := \mathbf{det}(\mathcal{T}_X^{\mathrm{vir}})$ is crucial, as first observed in [36].

Symmetric perfect obstruction theories are a hallmark of Donaldson–Thomas-type theories on Calabi–Yau 3-folds, to which we apply Theorem 5.6 as follows.

1.14

Let Y be a quasi-projective Calabi–Yau 3-fold, acted on by a torus with proper fixed loci, such that the (trivial) canonical bundle has non-trivial weight y. For the Donaldson–Thomas (DT) moduli stack $\mathfrak{N}_\alpha$, and a stability condition σ with no strictly σ-semistable objects, we define the *elliptic DT invariant* as the virtual chiral elliptic genus

$$\mathrm{DT}_{-y}^{\mathbf{Ell}/2}(\alpha;\sigma) := \mathrm{E}_{-y}^{\mathrm{vir}/2}(\mathfrak{N}_\alpha^{\mathrm{sst}}(\sigma))$$

of the σ-stable locus in $\mathfrak{N}_\alpha$, whenever it has proper torus-fixed loci (Definition 6.3). Note that the $q = 0$ specialization is what is usually referred to as a K-theoretic DT invariant.

A special case is *Vafa–Witten (VW) theory* [40], where $Y = \mathbf{tot}(\mathcal{K}_S)$ is an equivariant local surface (§6.10) for a smooth projective surface S acted on by a torus T. Up to some modifications to $\mathfrak{N}_\alpha$, the elliptic DT invariant becomes the *elliptic VW invariant* $\mathrm{VW}_{-y}^{\mathbf{Ell}/2}(\alpha;\sigma)$.

1.15

Theorem (Theorem 6.5, Remark 6.6, Lemma 6.10, Lemma 4.7, Corollary 6.12, Lemma 6.13). Consider two stable loci $\mathfrak{N}_\alpha^{\mathrm{sst}}(\pm) \subset \mathfrak{N}_\alpha^{\mathrm{sst}}(0)$ in the VW moduli stack of class $\alpha \in H^*(S)$, separated by a single wall at 0 (see 4.1(i)).

(i) (Spin) If $\mathcal{K}_S$ admits a square root, and $\mathcal{K}_S|_{S^T}$ has non-trivial T-weight on each component, then

$$\mathrm{VW}_1^{\mathbf{Ell}/2}(\alpha;+) = \mathrm{VW}_1^{\mathbf{Ell}/2}(\alpha;-).$$

(ii) (Calabi–Yau) If $\mathcal{K}_S^{\otimes k} = \mathcal{O}_S$ for some integer k, and the wall is simple (see 4.4(i)), then

$$\mathrm{VW}_{-y}^{\mathbf{Ell}/2}(\alpha;+) = \mathrm{VW}_{-y}^{\mathbf{Ell}/2}(\alpha;-).$$

In contrast to Theorem 1.12, the smoothness assumption 4.2(ii) is no longer required. For instance, S can be a Hirzebruch surface of even degree, satisfying the spin condition, or an Enriques surface, satisfying the Calabi–Yau condition for $k = 2$.

As in §1.12, in the Calabi–Yau case it may be unnecessary to require the wall to be simple.

1.16 Notation

All schemes are Noetherian and over $\mathbb{C}$. Given a torus $T = (\mathbb{C}^\times)^r$, its character and cocharacter lattices are $\mathbf{char}(T) := \mathbf{Hom}(T, \mathbb{C}^\times)$ and $\mathbf{cochar}(T) := \mathbf{Hom}(\mathbb{C}^\times, T)$ respectively, and:

- $K_T(\mathrm{pt}) = \mathbb{Z}[t^\omega : \omega \in \mathbf{char}\, T]$ where $t \in T$ denotes the coordinate;
- $K_T(\mathrm{pt})_{\mathrm{loc}} := K_T(\mathrm{pt})[(1 - t^\omega)^{-1} : 0 \neq \omega \in \mathbf{char}\, T]$; and
- $K_T(X)_{\mathrm{loc}} := K_T(X) \otimes_{K_T(\mathrm{pt})} K_T(\mathrm{pt})_{\mathrm{loc}}$ is the localized T-equivariant K-group of X.

The monomials $t^\omega \in K_T(\mathrm{pt})$ are referred to as T-*weights*.

In K-theory, $\chi(X, -) := \sum_i (-1)^i H^i(X, -)$ is the Euler characteristic, $\wedge_z^\bullet := \sum_i z^i \wedge^i$ is the exterior algebra, and all functors are derived, for example, $\mathbf{Ext}_X(-,-) := \sum_i (-1)^i \, \mathbf{Ext}_X^i(-,-)$.

Given a closed embedding $\iota\colon Z \hookrightarrow Z'$, let $\mathcal{N}_\iota$ or $\mathcal{N}_{Z\subset Z'}$ denote its normal sheaf.
Finally, $\ln(z) := \log(z)/2\pi i$ so that $\exp(2\pi i \ln z) = z$.

2 Wall-Crossing and Proof of the Main Theorem

2.1

Let M be the T-equivariant master space from §1.5, with action by $\mathsf{T} \times \mathsf{S}$. Let s denote the coordinate on $\mathsf{S} = \mathbb{C}^\times$.

In this section, we explain the general strategy for obtaining wall-crossing formulas from M, and prove the main Theorem 1.6. It has a long history with many interesting applications to cohomological and K-theoretic invariants, particularly when M is allowed to be quasi-smooth instead of smooth. But it has not yet been systematically applied to elliptic genus.

2.2

Wall-crossing formulas arise from the $(\mathsf{T} \times \mathsf{S})$-equivariant localization formula on M. Explicitly, if $\mathcal{F}$ is a coherent sheaf on M, then

$$\begin{aligned}\chi(M,\mathcal{F}) = \chi\left(Z_-^{\mathsf{T}}, \frac{1}{\wedge_{-1}^\bullet(\mathcal{N}^\vee_{Z_-^{\mathsf{T}}\subset Z_-})}\frac{\mathcal{F}|_{Z_-^{\mathsf{T}}}}{1 - s\mathcal{L}_-^\vee}\right) + \chi\left(Z_+^{\mathsf{T}}, \frac{1}{\wedge_{-1}^\bullet(\mathcal{N}^\vee_{Z_+^{\mathsf{T}}\subset Z_+})}\frac{\mathcal{F}|_{Z_+^{\mathsf{T}}}}{1 - s^{-1}\mathcal{L}_+^\vee}\right)\\ + \chi\left(Z_0^{\mathsf{T}}, \frac{1}{\wedge_{-1}^\bullet(\mathcal{N}^\vee_{Z_0^{\mathsf{T}}\subset Z_0})}\frac{\mathcal{F}|_{Z_0^{\mathsf{T}}}}{\wedge_{-1}^\bullet(\mathcal{N}^\vee_{\iota_0})}\right),\end{aligned} \tag{9}$$

where $\mathcal{N}_{\iota_\pm} =: s^{\pm}\mathcal{L}_\pm$ are the normal line bundles from 1.5(i), and several obvious pullbacks have been omitted. Each integrand is written so that only the second factor is a non-trivial (rational) function of s.

The left-hand side $\chi(M,\mathcal{F})$ is an element in the *non-localized* K-group $K_{\mathsf{T}\times\mathsf{S}}(\mathrm{pt})$ since M is proper and $\mathcal{F}$ is coherent, but each individual term in the right hand side of (9) lives in the localized $K_{\mathsf{T}\times\mathsf{S}}(\mathrm{pt})_{\mathrm{loc}}$.

2.3

The idea is to apply an operation res on rational functions of s such that:

(i) $\mathrm{res}\colon K_{\mathsf{T}\times\mathsf{S}}(\mathrm{pt}) \mapsto 0$;
(ii) for any T-equivariant line bundle $\mathcal{L}$ on a Deligne–Mumford stack with trivial T-action,

$$\mathrm{res}\,\frac{1}{1 - s\mathcal{L}} = 1.$$

Property (i) ensures the left-hand side of (9) will vanish, while property (ii) ensures the first two terms on the right-hand side of (9) will become

$$\chi\left(Z_\pm^{\mathsf{T}}, \frac{\mathcal{F}|_{s^{\mp1}=\mathcal{L}_\pm}}{\wedge_{-1}^\bullet(\mathcal{N}_{Z_\pm^{\mathsf{T}}\subset Z_\pm})}\right) = \chi\left(Z_\pm, \mathcal{F}|_{s^{\mp1}=\mathcal{L}_\pm}\right),$$

where the equality is T-equivariant localization on $Z_\pm$.

2.4

As the notation suggests, res is given by the *K-theoretic residue map*

$$\begin{aligned}\mathrm{res}\colon K_{\mathsf{T}\times\mathsf{S}}(\mathrm{pt})_{\mathrm{loc}} &\to K_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}\\ f &\mapsto \frac{1}{2\pi i}\oint_{|s|\approx 1} f\,\frac{ds}{s}\end{aligned} \tag{10}$$

where the contour encloses precisely the poles of the form $s^k = w$ for all $0 \neq k \in \mathbb{Z}$ and T-weights w. (The notation $|s| \approx 1$ is solely a suggestive name for the contour.) Equivalently, res computes the sum of residues at all such poles in s in the function f/s.

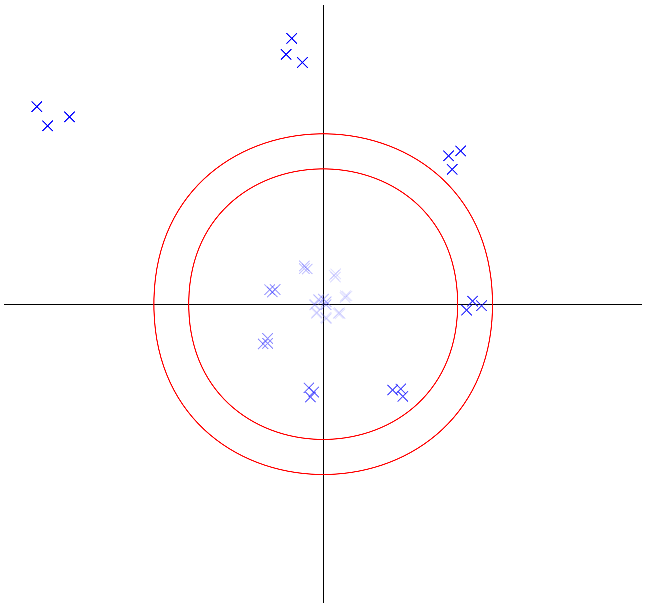

Fig. 1. The contour $|s| \approx 1$ on S. It can be represented by the annulus drawn in red. Crosses indicate poles of a sample elliptic function.

For any Deligne–Mumford stack Z with $(\mathsf{T} \times \mathsf{S})$-action, we continue to use res to denote the map

$$\mathrm{res}\colon K_{\mathsf{T}\times\mathsf{S}}(Z)_{\mathrm{loc}} \to K_{\mathsf{T}}(Z^{\mathsf{S}})_{\mathrm{loc}}$$

induced by tensor product with $K(Z^{\mathsf{T}\times\mathsf{S}})$. It is not difficult to check that res satisfies properties 2.3(i) and 2.3(ii), and in fact is uniquely characterized by them. For instance, Laurent polynomials in s have poles only at $s = 0$ and $s = \infty$, and neither point is enclosed by the contour $|s| \approx 1$.

2.5

More generally, res is well-defined for any meromorphic function on S. In particular we will apply it to elliptic functions, i.e. meromorphic functions invariant under $s \mapsto qs$, and related objects. Analytically, we treat the T-weights w as complex numbers much closer to 1 than q is, i.e. $|1 - w| \ll |1 - q|$ for all T-weights w. Thus, by definition, the contour excludes poles of the form $s^k q^n = w$ for $n \neq 0$. Figure 1 contains an illustration.

2.6

Example. Let t be any T-weight. Then $\vartheta(yst)/\vartheta(st)$ has a simple pole at $s = t^{-1}$, which is enclosed by $|s| \approx 1$, and it has no other poles enclosed by the contour. Hence,

$$\mathrm{res}\,\frac{\vartheta(yst)}{\vartheta(st)} = \lim_{s\to t^{-1}}(1 - st)\frac{\vartheta(yst)}{(1-(st)^{-1})\phi(st)\phi((st)^{-1})} = -\frac{\vartheta(y)}{\phi(1)^2}.$$

This still holds when s is replaced by s^{-1}, without the minus sign on the right-hand side.

2.7

For elliptic genus, take $\mathcal{F} = \Theta(y\mathcal{T}_M)/\Phi(\mathcal{T}_M)\Phi(\mathcal{T}_M^\vee)$ in (9) and apply res. Recall that in K-theory, there are splittings such as

$$\mathcal{T}_M\big|_{Z_\pm^{\mathsf{T}}} = s^{\pm}\mathcal{L}_\pm\big|_{Z_\pm^{\mathsf{T}}} + \mathcal{N}_{Z_\pm^{\mathsf{T}}\subset Z_\pm} + \mathcal{T}_{Z_\pm^{\mathsf{T}}},$$

and Θ and Φ are multiplicative. Combined with the calculation in Example 2.6, we get

$$0 = \frac{\vartheta(y)}{\phi(1)^2}\left(\mathrm{E}_{-y}(Z_-) - \mathrm{E}_{-y}(Z_+)\right) + \chi\left(Z_0^{\mathrm{T}}, \cdots \otimes \operatorname{res} \frac{\Theta(y\mathcal{N}_{\iota_0})}{\Theta(\mathcal{N}_{\iota_0})}\right), \tag{11}$$

a wall-crossing formula relating $\mathrm{E}_{-y}(Z_-)$ and $\mathrm{E}_{-y}(Z_+)$. In the remaining integrand, the terms represented by $\cdots$ are irrelevant because our goal is to study the case where the residue vanishes, and therefore $\mathrm{E}_{-y}(Z_-) = \mathrm{E}_{-y}(Z_+)$.

The assumption $y \neq 1$ of the main Theorem 1.6 is important for this step (and only this step), in order for the right-hand side of (11) to be non-trivial.

2.8

Remark. Smoothness of M is crucial for (11) to hold. It is natural to ask whether we can allow M to be only *quasi*- or *virtually* smooth, meaning that M may not be smooth but instead has a perfect obstruction theory [6]. The naive answer is no (cf. §5), for the following reason.

For quasi-smooth schemes, the elliptic genus E_{-y} should be upgraded to the *virtual elliptic genus* $\mathrm{E}_{-y}^{\mathrm{vir}}$ by replacing all instances of $\mathcal{T}$ in (1) with the virtual tangent bundle $\mathcal{T}^{\mathrm{vir}}$ [13]. Virtual localization [16] provides a virtual version of (9). However, the residue arguments fail to work for $\mathrm{E}_{-y}^{\mathrm{vir}}$: the left-hand side of (11) is no longer zero because the term

$$\Theta(y\mathcal{T}_M^{\mathrm{vir}}) \in K_{\mathrm{T}\times\mathrm{S}}(M)_{\mathrm{loc}}$$

in the integrand $\mathcal{F} = \Theta(y\mathcal{T}_M^{\mathrm{vir}})/\Phi(\mathcal{T}_M^{\mathrm{vir}})\Phi((\mathcal{T}_M^{\mathrm{vir}})^\vee)$ may now have poles at $|s| \approx 1$. Put differently, $\chi(M, \mathcal{F})$ is now an element of $K_{\mathrm{T}\times\mathrm{S}}(\mathrm{pt})_{\mathrm{loc}}$, not $K_{\mathrm{T}\times\mathrm{S}}(\mathrm{pt})$, and there is no good way to control its poles in s. So virtual elliptic genus is not amenable to our wall-crossing setup.

2.9

Remark. Properness of M is not crucial for (11) to hold. We used properness to conclude that $\operatorname{res} \chi(M, \mathcal{F}) = 0$, but it can be replaced by the weaker property that

- for any $(\mathrm{T}\times\mathrm{S})$-weight w with non-trivial S-component, the fixed locus M^{T_w} is proper, where $\mathrm{T}_w \subset \mathbf{ker}(w)$ is the maximal torus.

A priori, $\chi(M, \mathcal{F}) \in K_{\mathrm{T}\times\mathrm{S}}(\mathrm{pt})_{\mathrm{loc}}$, but by applying the pole cancellation Lemma 2.10 below, it has no poles at $w = 1$ for any w with non-trivial S-component. In other words,

$$\chi(M, \mathcal{F}) \in K_{\mathrm{T}}(\mathrm{pt})_{\mathrm{loc}} \otimes_{\mathbb{Z}} K_{\mathrm{S}}(\mathrm{pt}).$$

This is enough to imply $\operatorname{res} \chi(M, \mathcal{F}) = 0$.

2.10

Lemma (Pole cancellation). Let M be a scheme acted on by a torus T. Let w be a T-weight and $\mathrm{T}_w \subset \mathbf{ker}(w)$ be the maximal torus. If the T_w-fixed locus of M is proper, then

$$\chi(M, \mathcal{F})\big|_{w=1} \in K_{\mathrm{T}_w}(\mathrm{pt})_{\mathrm{loc}}$$

is well defined. In particular, $\chi(M, \mathcal{F})$ has no pole at $w = 1$.

Proof. This is a geometric observation from [5, Proposition 3.2], see also [32, Lemma 5.5]. Properness of M^{T_w} means that T_w-equivariant localization can be used to compute the right-hand side of

$$\chi(M, \mathcal{F})\big|_{w=1} = \chi\left(M, \mathcal{F}\big|_{w=1}\right).$$

The result is an element of $K_{\mathrm{T}_w}(\mathrm{pt})_{\mathrm{loc}}$. ■

2.11

Remark. The abstract study of residue maps, meaning module homomorphisms satisfying 2.3(i) and 2.3(i), was first explicitly done in [34]. There, (10) appeared in the slightly different form

$$f \mapsto -(\mathbf{Res}_{s=0} + \mathbf{Res}_{s=\infty})\left(f\,\frac{ds}{s}\right) = \sum_{s_0 \in \mathbb{C}^\times} \mathbf{Res}_{s=s_0}\left(f\,\frac{ds}{s}\right),$$

where **Res** denotes the usual notion of the residue of a differential 1-form and the equality is the residue theorem. For $f \in K_{T\times S}(\mathrm{pt})_{\mathrm{loc}}$, which have poles in s only in the set $S := \{0, \infty\} \sqcup \{|s| \approx 1\}$, we may replace $s_0 \in \mathbb{C}^\times$ with $s_0 \in \{|s| \approx 1\}$. Thus, for such f, Metzler's residue map is equivalent to ours. However, this is no longer true for elliptic functions f, which, in addition to poles in S, have poles at every q-shift of S (as in Figure 1).

2.12

Using the wall-crossing formula (11), the main Theorem 1.6 is reduced to the following.

Proposition. Let $\mathcal{E}_\pm$ be equivariant vector bundles of S-weight $s^{\pm 1}$, respectively, on a scheme with trivial $(T \times S)$-action. If

$$\mathbf{rank}\,\mathcal{E}_+ \equiv \mathbf{rank}\,\mathcal{E}_- \mod N$$

for some integer $N > 0$, then, for any N-th root of unity ζ_N,

$$\left(\mathrm{res}\,\frac{\Theta(y(\mathcal{E}_+ + \mathcal{E}_-))}{\Theta(\mathcal{E}_+ + \mathcal{E}_-)}\right)\Bigg|_{y=\zeta_N} = 0. \tag{12}$$

In §5.8, we will explain why this theorem continues to hold even if $\mathcal{E}_\pm$ are virtual bundles.

2.13

Proof. of Proposition 2.12 To be very explicit, the left-hand side of (12), before specializing to $y = \zeta_N$, is

$$\oint_{|s|\approx 1} \prod_i \frac{\vartheta(ysa:i,\tau)}{\vartheta(sa:i,\tau)} \prod_j \frac{\vartheta(ys^{-1}b_j,\tau)}{\vartheta(s^{-1}b_j,\tau)}\,\frac{ds}{s} \tag{13}$$

for variables $\{a_i\}_i$ and $\{b_j\}_j$ which will eventually be specialized to the T-equivariant Chern roots of $\mathcal{E}_+$ and $\mathcal{E}_-$, respectively. These Chern roots have the form $w \otimes \mathcal{L}$, where w is a T-weight and $\mathcal{L}$ is a non-equivariant line bundle. In particular, $\mathcal{L}$ is unipotent:

$$(1-\mathcal{L})^{\otimes M} = 0, \qquad \forall M \gg 0.$$

Analytically, we can therefore treat the variable corresponding to the Chern root the same way as the T-weight w. ∎

2.14

Let $I(s)$ denote the integrand in (13), so that $I(s)$ is a meromorphic 1-form on S once the T-weights are fixed to be some appropriately-generic elements in S. Using the basic q-difference equation $\vartheta(qs) = -(qs)^{-1}\vartheta(s)$, one checks easily that

$$I(qs) = y^{\mathbf{rank}\,\mathcal{E}_- - \mathbf{rank}\,\mathcal{E}_+} I(s).$$

Hence, specializing $y = \zeta_N$, both the integrand $I(s)$ and the contour integral (13) descend to the elliptic curve $E_S := S/q^{\mathrm{cochar}\,S}$. But on E_S, all poles of $I(s)$ are enclosed by the contour, and therefore the contour integral is zero by Cauchy's residue theorem [18, Chapter VII] and properness of E_S.

This concludes the proof of the main Theorem 1.6 as well.

3 Example: GIT Quotients

3.1

Let X be a smooth projective variety with the action of a reductive group G. Pick two ample linearizations $\mathcal{L}_\pm$ which lie in adjacent chambers in the space of ample linearizations of this G-action ([12,§3.3], [41,§2]). Explicitly, let

$$\mathcal{L}_t := \mathcal{L}_+^{(1+t)/2} \otimes \mathcal{L}_-^{(1-t)/2}, \qquad t \in [-1, 1],$$

and assume, without loss of generality, that $\mathcal{L}_0$ is the only point that is "on the wall", that is, there are strictly $\mathcal{L}_t$-semistable points only for $t = 0$. Let $X^{sst}(t)$ denote the $\mathcal{L}_t$-semistable locus in X, and write

$$X /\!\!/_t\, G := X^{sst}(t)/G$$

for the GIT quotient, for short. We assume $X /\!\!/_\pm G$ are smooth, and refer to this setup as *variation of GIT* (*VGIT*).

If X is acted on by a torus T commuting with G, then clearly everything above can be made T-equivariant.

3.2

Thaddeus [41,§3] constructs a master space for VGIT as follows: for the natural G-action on $\mathcal{L}_+ \oplus \mathcal{L}_-$, it is the projective GIT quotient

$$M := \mathbb{P}(\mathcal{L}_+ \oplus \mathcal{L}_-) /\!\!/_{\mathcal{O}(1)} G,$$

where the ample line bundle $\mathcal{O}(1)$ has the canonical linearization. The torus $S := \mathbb{C}^\times$, with coordinate s, acts on M by scaling the $\mathcal{L}_+$ factor with weight s. Let

$$\iota_\pm : X \cong \mathbb{P}(\mathcal{L}_\pm) \hookrightarrow \mathbb{P}(\mathcal{L}_+ \oplus \mathcal{L}_-)$$

be the inclusion of the 0 and ∞ sections, and $\mathbb{P}^\circ(\mathcal{L}_+ \oplus \mathcal{L}_-) := \mathbb{P}(\mathcal{L}_+ \oplus \mathcal{L}_-) \setminus (\mathbf{im}(\iota_+) \cup \mathbf{im}(\iota_-))$.

3.3

Throughout this section, we assume that $\mathcal{L}_0$ is a *simple wall*, meaning that for all x in

$$X^0 := X^{sst}(0) \setminus (X^{sst}(+) \cup X^{sst}(-)) :$$

(i) the stabilizer G_x of the G-action on x is $\mathbb{C}^\times$;
(ii) G_x acts on the fiber $(\mathcal{L}_+ \otimes \mathcal{L}_-^\vee)|_x$ by scaling with weight one.

Lemma. Under these assumptions, M is a T-equivariant master space in the sense of §1.5.

Proof sketch. We must check M is a smooth scheme. The semistable locus in $\mathbb{P}(\mathcal{L}_+ \oplus \mathcal{L}_-)$ decomposes as

$$\iota_+(X^{sst}(+)) \cup \iota_-(X^{sst}(-)) \cup \bigcup_{t \in [-1,1]} \pi^{-1}(X^{sst}(t)) \cap \mathbb{P}^\circ(\mathcal{L}_+ \oplus \mathcal{L}_-) \tag{14}$$

where $\pi : \mathbb{P}(\mathcal{L}_+ \oplus \mathcal{L}_-) \to X$ is the projection. Points in the image of the first two terms in M must be smooth because $X /\!\!/_\pm G$ are smooth. The assumptions imply the G-action on the third term is free. See [41,§4] for details. ∎

Note that $(\mathcal{L}_+ \otimes \mathcal{L}_-^\vee)|_x \setminus \{0\}$ is the fiber of the projection $\mathbb{P}^\circ(\mathcal{L}_+ \oplus \mathcal{L}_-) \to X$.

3.4

Remark (Toric varieties). When G is a torus, the assumptions of §3.3 automatically hold, possibly after replacing $\mathcal{L}_+$ and $\mathcal{L}_-$ by some positive powers. In particular, if $X = \mathbb{C}^r$ and $G \subset (\mathbb{C}^\times)^r$ is a sub-torus of the maximal torus of $\mathbf{GL}(r)$, then $X /\!\!/_\pm G$ are toric varieties and VGIT reduces to the combinatorics of lattice polyhedra.

Recall that, conversely, all toric varieties without torus factor have the form $\mathbb{C}^r /\!\!/_\theta G$, where $G \subset (\mathbb{C}^\times)^r$ is an algebraic subgroup and θ is a G-weight.

3.5

In the decomposition (14), clearly the first two terms $\iota_\pm(X^{\mathrm{sst}}(\pm))$ are S-fixed and induce inclusions which we also denote

$$\iota_\pm \colon X /\!\!/_\pm G \hookrightarrow M,$$

whose normal bundles are $\mathcal{L}_\pm$. In the third term, a point becomes S-fixed in M if and only if it has positive-dimensional stabilizer under the $(G \times S)$-action. By hypothesis, such points must belong to $\pi^{-1}(X^0)$, so the remaining S-fixed locus is exactly

$$\iota_0 \colon (\pi^{-1}(X^0) \cap \mathbb{P}^\circ(\mathcal{L}_+ \oplus \mathcal{L}_-)) /\!\!/_{\mathcal{O}(1)} G \hookrightarrow M$$

and the stabilizer G_x of points in X^0 is identified with S. Thus, $\mathcal{N}_{\iota_0}$ is identified with $\mathcal{N}_{X^0 \subset X}$.

3.6

Theorem. Under the assumptions of §3.3, if $\mathcal{N}_{X^0 \subset X}|_{(X^0)^T} = \mathcal{E}_+ \oplus \mathcal{E}_-$ only has pieces of S-weight $s^{\pm 1}$, and

$$\mathbf{rank}\, \mathcal{E}_+ \equiv \mathbf{rank}\, \mathcal{E}_- \bmod N$$

for some integer $N > 0$, then $E_{-\zeta_N}(X /\!\!/_+ G) = E_{-\zeta_N}(X /\!\!/_- G)$.

This is a restatement of the main Theorem 1.6 in the current setting, using the description of S-fixed loci in §3.5. It remains to describe the S-weight pieces of $\mathcal{N}_{X^0 \subset X}$ in some useful way, which Thaddeus provides and we summarize as Theorem 3.7 below.

3.7

Theorem ([41, Proposition 4.6, Theorem 4.7]). Let $X^\pm := X^{\mathrm{sst}}(0) \setminus X^{\mathrm{sst}}(\mp)$. Under the assumptions of §3.3,

$$X^\pm /\!\!/_\pm G \to X^0 /\!\!/_0 G$$

are locally trivial fibrations whose fibers are weighted projective spaces $\mathbb{P}(|w_i^\pm|)$, where $\{w_i^\pm\}_i \in \mathbb{Z}$ are (the exponents of) the positive and negative S-weights, respectively, of $\mathcal{N}_{X^0 \subset X}$.

Proof sketch. This claim may be checked affine-locally. For example, the Bialynicki-Birula decomposition theorem shows that the positive and negative S-weight parts of $\mathcal{N}_{X^0 \subset X}$ are equal to $\mathcal{N}_{X^0 \subset X^\pm}$, respectively. ■

3.8

Example (Blow-ups). Let $Y := X /\!\!/_- G$ and suppose that $X /\!\!/_+ G = \mathbf{Bl}_p Y$ is the blow-up of a point $p \in Y$. (Conversely, a large class of blow-ups of points in GIT quotients can be realized as VGIT [20].) Let $N := \mathbf{dim}\, Y - 1$ and assume that the wall is simple. Then $X^\pm /\!\!/_\pm G \to X^0 /\!\!/_0 G$ are the exceptional loci, which are $\mathbb{P}^N$ and $\mathbb{P}^0$-fibrations respectively, so Theorem 3.6 says

$$E_{-y}(Y) = E_{-y}(\mathbf{Bl}_p Y) \quad \text{if } y = \zeta_N. \tag{15}$$

Note that if Y is toric and p is a torus-fixed point, then, using torus-equivariant localization as in (2), the equality of elliptic genera is equivalent to the identity

$$\prod_{i=1}^{N+1} \frac{\vartheta(yx_i)}{\vartheta(x_i)} = \sum_{i=1}^{N+1} \frac{\vartheta(yx_i)}{\vartheta(x_i)} \prod_{j\neq i} \frac{\vartheta(yx_j/x_i)}{\vartheta(x_j/x_i)} \quad \text{if } y = \zeta_N. \tag{16}$$

Here, x_i are weights of T_pY. In the blow-up, p is replaced by $N+1 = \mathbf{dim}\, Y$ different torus-fixed points, corresponding to the terms in the sum on the right hand side.

3.9

Remark. In fact, some toric computation can strengthen the results in Example 3.8. In particular we claim that (15) (as a claim about a class of varieties Y), and therefore (16), is an "if and only if". First, by explicit computation,

$$E_{-y}(\mathbb{P}^{N+1}) = \chi\left(\mathbb{P}^{N+1}, \wedge^{\bullet}_{-y}(\mathcal{T}^{\vee}_{\mathbb{P}^{N+1}})\right) + \cdots = \frac{1-y^{N+2}}{1-y} + \cdots \tag{17}$$

where $\cdots$ denotes terms involving q. Second, if $\pi : A \to B$ is a $\mathbb{P}^n$-fibration, then

$$E_{-y}(A) = \chi\left(B, \frac{\Theta(y\mathcal{T}_B)}{\Phi(\mathcal{T}_B)\Phi(\mathcal{T}_B^{\vee})} \otimes \pi_* \frac{\Theta(y\mathcal{T}_\pi)}{\Phi(\mathcal{T}_\pi)\Phi(\mathcal{T}_\pi^{\vee})}\right) = E_{-y}(\mathbb{P}^n)E_{-y}(B).$$

The first equality is the projection formula for π, and the second equality is because all relative cohomology bundles $R^i\pi_*(\mathcal{T}_\pi^j)$ are canonically trivialized by powers of the hyperplane class. Hence, for a torus-fixed point $p \in \mathbb{P}^{N+1}$, the $\mathbb{P}^1$-fibration $\pi : \mathbf{Bl}_p\, \mathbb{P}^{N+1} \to \mathbb{P}^N$ implies

$$E_{-y}(\mathbf{Bl}_p\, \mathbb{P}^{N+1}) = E_{-y}(\mathbb{P}^1)E_{-y}(\mathbb{P}^N) = \frac{1-y^2}{1-y}\frac{1-y^{N+1}}{1-y} + \cdots .$$

Comparing with (17), the q-constant terms match if and only if $y = \zeta_N$. So the hypothesis in (15) is really necessary.

4 Example: Moduli of Sheaves

4.1

Let Y be a smooth projective variety acted on by a torus T, and $\mathscr{A} \subset D^b\mathscr{C}(Y)$ be an abelian subcategory of its bounded derived category of coherent sheaves. (For the purposes of this section, with some care it is also possible to consider an exact subcategory $\mathscr{B} \subset \mathscr{A}$ which may not be abelian but is closed under isomorphisms and direct sum; see, e.g., the setup of [22,§5.1].) We refer to $\mathscr{A}$ as a *moduli of sheaves* even though the objects involved may in general be complexes of sheaves.

Given $\alpha \in H^*(Y)$, a typical wall-crossing problem in $\mathscr{A}$ involves a continuous family $\{\sigma_\xi\}_{\xi\in[-1,1]}$ of stability conditions such that (we use Joyce's notion of stability condition: functions σ from non-zero classes α into some totally-ordered set, such that if $\alpha = \beta + \gamma$ for $\alpha, \beta, \gamma \neq 0$, then either $\sigma(\beta) > \sigma(\alpha) > \sigma(\gamma)$ or $\sigma(\beta) = \sigma(\alpha) = \sigma(\gamma)$ or $\sigma(\beta) < \sigma(\alpha) < \sigma(\gamma)$):

(i) for $\xi \neq 0$, there are no strictly σ_ξ-semistable objects $E \in \mathscr{A}$ with $\mathbf{ch}(E) = \alpha$;
(ii) for any ξ, there exist algebraic moduli stacks (acted on by T)

$$\mathfrak{M}^{\mathrm{sst}}_\beta(\xi) := \{E \in \mathscr{A} : E \text{ is } \sigma_\xi\text{-semistable and } \mathbf{ch}(E) = \beta\}$$

for all relevant β, which includes α and the classes appearing in (19).

The goal is to relate elliptic genus of $\mathfrak{M}^{\mathrm{sst}}_\alpha(+)$ and $\mathfrak{M}^{\mathrm{sst}}_\alpha(-)$, where $\pm$ means ± 1. Continuity implies that all $\{\sigma_\xi\}_{\xi\in(0,1]}$ are equivalent, and similarly for $\{\sigma_\xi\}_{\xi\in[-1,0)}$.

4.2

To consider enumerative invariants, one usually makes a properness assumption:

(i) the moduli stacks $\mathfrak{M}_\alpha^{\mathrm{sst}}(\pm)$ are proper algebraic spaces. (For this paper, the words "algebraic space" and "scheme" are basically interchangeable. In practice $\mathfrak{M}_\alpha^{\mathrm{sst}}(\pm)$ are often projective schemes.)

To be precise, all objects in $\mathscr{A}$ have at least a $\mathbb{C}^\times$'s worth of scalar automorphisms, and $\mathfrak{M}_\alpha^{\mathrm{sst}}(\xi)$ denotes the *rigidified* moduli stacks where this $\mathbb{C}^\times$ has been removed [1, Appendix A]. So, objects in $\mathfrak{M}_\alpha^{\mathrm{sst}}(\pm)$ have no non-trivial automorphisms and $\mathfrak{M}_\alpha^{\mathrm{sst}}(\pm)$ are automatically algebraic spaces.

Our main Theorem 1.6 requires the master space to be smooth; see Remark 2.8. To satisfy this, it is typically enough to assume that:

(ii) for any ξ, in particular $\xi = 0$, the moduli stack $\mathfrak{M}_\alpha^{\mathrm{sst}}(\xi)$ is smooth.

In particular, $\mathfrak{M}_\alpha^{\mathrm{sst}}(\pm)$ are smooth and proper, so their elliptic genera $\mathrm{E}_y(\mathfrak{M}_\alpha^{\mathrm{sst}}(\pm))$ exist.

4.3

Remark. Conditions 4.1(i), 4.1(ii), and 4.2(i) are relatively weak for most wall-crossing problems of interest. For instance, if for any ξ,

$$\mathbf{Ext}_Y^{<0}(E,E) = 0, \quad \forall [E] \in \mathfrak{M}_\beta^{\mathrm{sst}}(\xi),$$

then 4.1(ii) holds [31]. There also exists machinery [3] for verifying 4.2(i) in great generality, or, more concretely, one can often use Langton's strategy for semistable reduction [30].

On the other hand, condition 4.2(ii) is very strong — it is basically the requirement that for any ξ, including $\xi = 0$ where there exist strictly semistable objects,

$$\mathbf{Ext}_Y^{\geq 1}(E,E) = 0, \quad \forall [E] \in \mathfrak{M}_\alpha^{\mathrm{sst}}(\xi). \tag{18}$$

This is automatic if $\dim Y \leq 1$. If $\dim Y = 2$, this is equivalent to the vanishing of $\mathbf{Ext}_Y^2(E,E) = \mathbf{Hom}_Y(E, E \otimes \mathcal{K}_Y)^\vee$ which, for example, holds if $\mathcal{K}_Y$ is anti-ample by a standard degree argument for semistable objects [21, Proposition 1.2.6], cf. [22, Definition 7.47]. If $\dim Y \geq 3$, this is typically hopeless.

4.4

We begin with the case of a *simple wall* (cf. §3.3), namely:

(i) all strictly semistable $[E] \in \mathfrak{M}_\alpha^{\mathrm{sst}}(0)$ split as $E = E_1 \oplus E_2$ where E_1, E_2 are both σ_0-stable.

In other words, the automorphism group of objects in $\mathfrak{M}_\alpha^{\mathrm{sst}}(0)$ is at worst $\mathbb{C}^\times$, given by scaling E_1. Then, the strategy behind the Thaddeus master space can be applied directly to obtain a master space; this is done explicitly in [23,§4], or can also be recovered implicitly from the more general wall-crossing machinery of Mochizuki [35,§1.3, §1.6.1] or of Joyce [22]. (Both [35] and [22] are written in a vastly more general setting where the assumptions 4.1(i), 4.2(i) and 4.2(ii) may not hold. They must pass to auxiliary moduli stacks, impose a quasi-smoothness assumption and work with virtual cycles, like in §5 and §6. In our simpler setting, these complications may be ignored.)

In the notation of §1.5, the complicated locus Z_0 in the master space is

$$Z_0 = \bigsqcup_{\substack{\alpha_1+\alpha_2=\alpha \\ \sigma_0(\alpha_1)=\sigma_0(\alpha_2)}} \left\{ [E_1 \oplus E_2] \in \mathfrak{M}_\alpha^{\mathrm{sst}}(0) : [E_i] \in \mathfrak{M}_{\alpha_i}^{\mathrm{sst}}(0) \text{ stable} \right\} \hookrightarrow \mathfrak{M}_\alpha^{\mathrm{sst}}(0). \tag{19}$$

4.5

Remark. If α is rank-2 and torsion-free, then all walls are simple because the rank can only decompose as $2 = 1 + 1$. For rank greater than two, typically non-simple walls exist.

4.6

Theorem (Simple wall). Assume 4.4(i). If, for all $[E_1 \oplus E_2] \in Z_0^T$,

$$\dim \mathrm{Ext}_Y(E_1, E_2) \equiv \dim \mathrm{Ext}_Y(E_2, E_1) \bmod N$$

for some integer $N > 0$, then $\mathrm{E}_{-\zeta_N}(\mathfrak{M}_\alpha^{\mathrm{sst}}(+)) = \mathrm{E}_{-\zeta_N}(\mathfrak{M}_\alpha^{\mathrm{sst}}(-))$.

Lemma 4.7 below gives some situations in which the dimension condition is satisfied.

Proof. The S-action scales E_1 with weight s, so the normal bundle is given by

$$\mathcal{N}_{\iota_0}\Big|_{[E_1\oplus E_2]} = -s^{-1}\,\mathbf{Ext}_Y(E_1, E_2) - s\,\mathbf{Ext}_Y(E_2, E_1). \tag{20}$$

We conclude by a direct application of Theorem 1.6. ■

4.7

Lemma. Let S be a smooth projective surface with canonical bundle $\mathcal{K}_S$, and take $E, F \in \mathscr{A}$.

(i) If $\mathcal{K}_S$ admits a square root, then $\dim \mathbf{Ext}_S(E, F) \equiv \dim \mathbf{Ext}_S(F, E) \bmod 2$.
(ii) If $\mathcal{K}_S^{\otimes k} = \mathcal{O}_S$ for some integer k, then $\dim \mathbf{Ext}_S(E, F) = \dim \mathbf{Ext}_S(F, E)$.

Proof. By Serre duality, we are comparing $\dim \mathbf{Ext}_S(E, F)$ and $\dim \mathbf{Ext}_S(E, F \otimes \mathcal{K}_S)$. Using bilinearity of $\mathbf{Ext}_S$, assume that $E, F \in \mathscr{C}(S)$. Also, assume E is locally free; if not, take a locally free resolution and consider each term of the resolution. So, without loss of generality, we are comparing $\dim \chi(S, F)$ and $\dim \chi(S, F \otimes \mathcal{K}_S)$. Hirzebruch–Riemann–Roch says

$$\dim \chi(S, F) = \mathrm{rank}(\mathcal{F})\chi(S) - \frac{1}{2}c_1(F)K + \mathrm{ch}_2(F),$$

where $K := c_1(\mathcal{K}_S)$ is the canonical divisor. Then,

$$\dim \chi(S, F) - \dim \chi(S, F \otimes \mathcal{K}_S) = \frac{1 - \mathrm{rank}(F)}{2}K^2 - c_1(F)K.$$

For (i), $K = 2D$ for some D, so this is divisible by 2. For (ii), $K = 0$ so this is zero. ■

4.8

Remark. From Remark 4.3, if we only consider the case of Fano surfaces Y, then either $Y = \mathbb{P}^1 \times \mathbb{P}^1$ or Y is the blow-up of $\mathbb{P}^2$ at ≤ 8 points. In the latter case, the canonical divisor

$$K_Y = \pi^* K_{\mathbb{P}^2} + \sum_i E_i$$

is either $\pi^* K_{\mathbb{P}^2} = -3H$ or a primitive vector. Here E_i denote the exceptional divisors and H is the hyperplane class. So the hypotheses of Lemma 4.7 are only satisfied for Fano surfaces Y if $Y = \mathbb{P}^1 \times \mathbb{P}^1$, where $\mathcal{K}_Y = \mathcal{O}_{\mathbb{P}^1}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2)$ admits a square root. Nonetheless, for more general surfaces Y, the results of this section will be useful in §6.

4.9

Generally, walls in $\mathscr{A}$ are not simple, meaning that condition 4.4(i) does not hold. In [35] and [22], this issue is solved by lifting the wall-crossing problem to the auxiliary abelian category

$$\widetilde{\mathscr{A}}^{\mathrm{Fr}} := \{(E, V^\bullet) : E \in \mathscr{A} \text{ and } V^\bullet \text{ is a full flag in } \mathbf{Fr}(E)\}$$

associated to a *framing functor*, which is an exact functor

$$\mathbf{Fr}\colon \mathscr{A}' \to \mathscr{V} \tag{21}$$

on a full exact subcategory $\mathscr{A}' \subset \mathscr{A}$ containing all objects of interest, such that

$$\mathbf{Hom}(E, E) \to \mathbf{Hom}(\mathbf{Fr}(E), \mathbf{Fr}(E))$$

is injective for all $E \in \mathscr{A}'$. (The injectivity condition is explicitly stated by Joyce [22, Assumption 5.1(g)(iii)] but not mentioned by Mochizuki, who only uses $\mathbf{Fr}(E) := H^0(E \otimes \mathcal{O}_Y(m))$.) For example, if $\mathscr{A} \subset \mathscr{C}(Y)$, a common choice is $\mathbf{Fr}(E) := H^0(E \otimes \mathcal{O}_Y(m))$ for $m \gg 0$. See §4.11 for our notion of a full flag.

The additional data of the flag "resolves" the non-simple wall in $\mathscr{A}$ into multiple simple walls in $\mathscr{A}^{\mathbf{Fr}}$. Each such simple wall can be crossed like in §4.4; the so-called *enhanced master space*, for the auxiliary stacks, sits inside a flag variety fibration over the original master space and is therefore still smooth. The complicated locus Z_0^{T} in the master space now involves splittings

$$(E, V^\bullet) = (E_1 \oplus E_2, V_1^\bullet \oplus V_2^\bullet), \quad \sigma_0(E_1) = \sigma_0(E_2) \tag{22}$$

where $V_i^\bullet$ is a full flag in $\mathbf{Fr}(E_i)$ and each $(E_i, V_i^\bullet)$ is σ_0-stable.

4.10

Theorem. If, for all splittings (22) appearing in all auxiliary wall-crossings,

$$\dim \mathrm{Ext}_Q(V_1^\bullet, V_2^\bullet) - \dim \mathrm{Ext}_Y(E_1, E_2) \equiv \dim \mathrm{Ext}_Q(V_2^\bullet, V_1^\bullet) - \dim \mathrm{Ext}_Y(E_2, E_1) \bmod N$$

for some integer $N > 0$, then $\mathsf{E}_{-\zeta_N}(\mathfrak{M}_\alpha^{\mathrm{sst}}(+)) = \mathsf{E}_{-\zeta_N}(\mathfrak{M}_\alpha^{\mathrm{sst}}(-))$.

Note that this is not as widely applicable as Theorem 4.6, because we get very little control over the Ext_Q terms; see Lemma 4.11 below.

Proof. The normal bundle is now, cf. (20),

$$\begin{aligned}\mathcal{N}_{\iota_0}\Big|_{[(E_1\oplus E_2, V_1^\bullet \oplus V_2^\bullet)]} &= \mathsf{s}^{-1}\left(\mathrm{Ext}_Q(V_1^\bullet, V_2^\bullet) - \mathrm{Ext}_Y(E_1, E_2)\right) \\ &\quad + \mathsf{s}\left(\mathrm{Ext}_Q(V_2^\bullet, V_1^\bullet) - \mathrm{Ext}_Y(E_2, E_1)\right).\end{aligned} \tag{23}$$

Here, viewing flags as representations of a type A quiver Q, the quiver part of the deformation theory is given by the standard formula

$$\mathrm{Ext}_Q(V_1^\bullet, V_2^\bullet) := \sum_i \left(\mathrm{Hom}(V_1^i, V_2^{i+1}) - \mathrm{Hom}(V_1^i, V_2^i)\right). \tag{24}$$

We conclude by a direct application of Theorem 1.6. ■

4.11

Let W be a vector space. For us, $V^\bullet$ being a *full flag* of length K in W means that

$$\dim V^k \le \dim V^{k+1} \le \dim V^k + 1$$

for $0 \le k \le K$, with the convention that $V^0 = 0$ and $V^{K+1} = W$. We write $\dim V^\bullet := \dim W$.

Lemma. Let $\mathrm{Split}(V^\bullet; d_1, d_2)$ be the set of splittings $V^\bullet = V_1^\bullet \oplus V_2^\bullet$ of a full flag into two smaller full flags with $\dim V_i^\bullet = d_i$. Then

$$\left\{\dim \mathrm{Ext}_Q(V_1^\bullet, V_2^\bullet) - \dim \mathrm{Ext}_Q(V_2^\bullet, V_1^\bullet)\right\} = \{-d_1d_2, -d_1d_2 + 2, \ldots, d_1d_2 - 2, d_1d_2\},$$

where the left-hand side ranges over all splittings in $\mathrm{Split}(V^\bullet; d_1, d_2)$.

Proof. Note that if $V_i^k = V_i^{k+1}$ for $i = 1, 2$ and some k, then by (24) we can remove the k-th step from both flags without affecting $\dim \mathrm{Ext}_Q(V_1^\bullet, V_2^\bullet)$ or $\dim \mathrm{Ext}_Q(V_2^\bullet, V_1^\bullet)$. So, without loss of generality, $V^\bullet$ has the shortest possible length $d := \mathbf{dim}\, V^\bullet$. Then it is convenient to use the bijection

$$\mathrm{Split}(V^\bullet; d_1, d_2) \xrightarrow{\sim} \{I \subset \{1, 2, \ldots, d\} : |I| = d_1\}$$
$$(V_1^\bullet, V_2^\bullet) \mapsto \{i : V_1^i \neq V_1^{i+1}\}.$$

Let $v_i^k := \mathbf{dim}\, V_i^k$, and write the quantity of interest as $\mathbf{ext}_Q^1(I, +) - \mathbf{ext}_Q^1(I, -)$ where

$$\mathbf{ext}_Q^1(I, +) := \sum_i v_1^i v_2^{i+1}, \quad \mathbf{ext}_Q^1(I, -) := \sum_i v_2^i v_1^{i+1}.$$

It is straightforward that, for $I_{\max} := \{1, 2, \ldots, d_1\}$ and $I_{\min} = \{d_1 + 1, d_1 + 2, \ldots, d\}$,

$$\mathbf{ext}_Q^1(I_{\max}, +) = d_1 d_2, \quad \mathbf{ext}_Q^1(I_{\max}, -) = 0$$
$$\mathbf{ext}_Q^1(I_{\min}, +) = 0, \quad \mathbf{ext}_Q^1(I_{\min}, -) = d_1 d_2.$$

These result in the maximum and minimum values $\pm d_1 d_2$. For the intermediate values, suppose I and I' differ by replacing an element k by $k+1$. This amounts to $(v_1')^{k+1} = v_1^{k+1} - 1$, and consequently $(v_2')^{k+1} = v_2^{k+1} + 1$, while all other dimensions remain unchanged, so

$$\mathbf{ext}_Q^1(I', +) = \mathbf{ext}_Q^1(I, +) + v_1^k - v_2^{k+2}, \quad \mathbf{ext}_Q^1(I', -) = \mathbf{ext}_Q^1(I, -) + v_1^{k+2} - v_2^k.$$

Since $|I| = |I'| = d_1$, it must be that $v_i^{k+2} - v_i^k = 1$ for $i = 1, 2$. ■

4.12

Corollary. If Y is a smooth projective surface whose canonical bundle admits a square root, then the hypothesis of Theorem 4.10 is satisfied for $N = 2$.

Lemma 4.11 shows that there is no way to generalize this to $N > 2$, in contrast to, for example, Theorem 4.6 in the setting of Lemma 4.7(ii).

Proof. By Lemma 4.11, it suffices to ensure that $\mathbf{dim}(V_1^\bullet)\,\mathbf{dim}(V_2^\bullet) = \mathbf{dim}\,\mathbf{Fr}(E_1)\,\mathbf{dim}\,\mathbf{Fr}(E_2)$ is always even. This can be done by replacing the framing functor $\mathbf{Fr}$ in the wall-crossing machinery with $\mathbf{Fr}^{\oplus 2}$. ■

4.13

Remark. We expect that the $N = 2$ restriction of Corollary 4.12 is an artefact of the choice of auxiliary category $\tilde{\mathscr{A}}^{\mathrm{Fr}}$, rather than an intrinsic limitation. Namely, for any given $N \geq 2$, one may speculate that there exist auxiliary categories $\tilde{\mathscr{A}}_N$ which work equally well for wall-crossing, for which the contribution from the "auxiliary" part of the obstruction theory, that is, the dimensions in Lemma 4.11, are all $0 \bmod N$ instead of merely $0 \bmod 2$.

For example, the Calabi–Yau case of Theorem 1.12 should hold *without* the assumption that the wall is simple. Evidence for this includes the fact that if Y is a K3 or abelian surface, α is primitive (and not too small), and $\{\sigma_\xi\}_\xi$ is a general family of Gieseker stability conditions, then $\mathfrak{M}_\alpha^{\mathrm{sst}}(+)$ is deformation-equivalent to $\mathfrak{M}_\alpha^{\mathrm{sst}}(-)$ [46, Theorems 0.1 and 8.1] and therefore their elliptic genera are equal.

5 The Virtual Chiral Version

5.1

Let X be a proper scheme, and, instead of assuming X is smooth, assume the weaker condition:

(i) X has a $\tilde{\mathsf{T}}$-equivariant perfect obstruction theory [6], obtained from a perfect complex $\mathbb{E} \in D^b\mathscr{C}_{\mathsf{T}}(X)$ which satisfies

$$\mathbb{E} = y \otimes \mathbb{E}^\vee[1] \tag{25}$$

for some non-trivial $\tilde{\mathsf{T}}$-weight y; we say $\mathbb{E}$ is *equivariantly symmetric*.

(One typically assumes that $\mathbb{E}$ admits a two-term resolution by vector bundles. However, recent technical advances [4,§5] suggest that this condition is unnecessary, so we do not worry about it.) Then X has a virtual structure sheaf $\mathcal{O}_X^{\mathrm{vir}}$ and virtual tangent bundle $\mathcal{T}_X^{\mathrm{vir}}$ [11, 26], both elements of $K_{\widetilde{T}}(X)$. Let $\mathcal{K}_{\mathrm{vir}} := \mathbf{det}(\mathcal{T}_X^{\mathrm{vir}})^\vee$ be the virtual canonical.

The notation $\widetilde{T}$, instead of T, is a reminder that y is now an equivariant weight as opposed to a formal variable. For instance, if X is smooth with T-action, then let $\mathbb{C}^\times$ scale fibers of the cotangent bundle $Y := T^*X$ with weight y and view X as its zero section. Then X satisfies 5.1(i) with $\widetilde{T} := T \times \mathbb{C}^\times$ and $\mathbb{E} = (\mathcal{T}_Y^\vee - y\mathcal{T}_Y)|_X$ in K-theory.

5.2

Whenever a square root $\mathcal{K}_{\mathrm{vir}}^{1/2}$ exists, following [36,§3.1] let

$$\widehat{\mathcal{O}}_X^{\mathrm{vir}} := \mathcal{O}_X^{\mathrm{vir}} \otimes \mathcal{K}_{\mathrm{vir}}^{1/2} \in K_{\widetilde{T}}(X)$$

After possibly passing to a double cover of $\widetilde{T}$ so that $y^{1/2}$ exists, the *virtual chiral elliptic genus* of X is

$$E_{-y}^{\mathrm{vir}/2}(X) := \chi\left(X, \frac{\widehat{\mathcal{O}}_X^{\mathrm{vir}}}{\Phi(\mathcal{T}_X^{\mathrm{vir}})\Phi((\mathcal{T}_X^{\mathrm{vir}})^\vee)}\right) \in K_{\widetilde{T}}(\mathrm{pt})[[q]];$$

cf. [14]. The expression $E_{-y_0}^{\mathrm{vir}/2}(X)$, for $y_0 \in \mathbb{C}^\times$, means the specialization of $E_{-y}^{\mathrm{vir}/2}(X)$ to $y = y_0$. Since X is proper, the only poles in y are at 0 and ∞, so this specialization is always well defined.

The deformation invariance of virtual cycles and the class $\mathcal{T}_X^{\mathrm{vir}}$ immediately implies the deformation invariance of $E_{-y}^{\mathrm{vir}/2}(X)$; see, for example, [13,§3.5].

5.3

Remark. In [14,§8.1], virtual chiral elliptic genus was defined without requiring the equivariant symmetry (25), and without inserting $\mathcal{K}_{\mathrm{vir}}^{1/2}$. In that generality, it has no hope of being a truly elliptic class (see §1.3) and will not have any of the nice wall-crossing properties considered in this paper.

5.4

Remark 5.1. The perfect obstruction theory in condition 5.1(i) is really only used to construct $\mathcal{O}_X^{\mathrm{vir}}$ and $\mathcal{T}_X^{\mathrm{vir}}$. There are many weaker, more local notions that also suffice, with obvious analogues of the equivariant symmetry (25):

(i) weak perfect obstruction theories (equivalent to complex Kuranishi structures) [37];
(ii) almost perfect obstruction theories [24];
(iii) semi-perfect obstruction theories [10].

These are ordered such that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii), and it is known that virtual localization holds at least for (ii) [25]. The content of this section therefore holds at the level of (ii).

5.5

For wall-crossing with virtual chiral elliptic genus, we assume the $\widetilde{T}$-equivariant master space M is a proper scheme (more generally M can be a Deligne–Mumford stack satisfying the weaker properness condition of Remark 2.9) satisfying 5.1(i) for an action of $\widetilde{T} \times S$ where $S = \mathbb{C}^\times$, and the S-fixed locus is a disjoint union of the following $\widetilde{T}$-invariant pieces (cf. §1.5):

(i) $\iota_\pm : Z_\pm \hookrightarrow M$ with $\mathcal{N}_{\iota_\pm}^{\mathrm{vir}} = \mathcal{L}_\pm - y^{-1}\mathcal{L}^\vee$ for line bundles $\mathcal{L}_\pm$ of S-weights $s^{\pm 1}$;
(ii) other proper component(s) $\iota_0 : Z_0 \hookrightarrow M$ with

$$\mathcal{N}_{\iota_0}^{\mathrm{vir}} = \mathcal{N}_0^{\mathrm{vir}/2} - y^{-1}(\mathcal{N}_0^{\mathrm{vir}/2})^\vee$$

for some virtual bundle $\mathcal{N}_0^{\mathrm{vir}/2}$.

Here $\mathcal{N}_f^{\mathrm{vir}}$ denotes the *virtual* normal bundle of the closed embedding f, namely the S-moving part of the restriction $f^*\mathcal{T}_M^{\mathrm{vir}}$.

To emphasize, unlike in the non-virtual setting, M does not need to be actually smooth (cf. Remark 2.8).

5.6

Theorem. (Virtual analogue of Theorem 1.6) Suppose $\mathcal{N}_0^{\mathrm{vir}/2}|_{Z_0^T} = \mathcal{E}_+ \oplus \mathcal{E}_-$ only has pieces of S-weight $s^{\pm 1}$, and

$$\operatorname{rank} \mathcal{E}_+ \equiv \operatorname{rank} \mathcal{E}_- \bmod N \tag{26}$$

for some integer $N > 0$. Then, for any N-th root of unity $\zeta_N \neq 1$,

$$E_{-\zeta_N}^{\mathrm{vir}/2}(Z_+) = E_{-\zeta_N}^{\mathrm{vir}/2}(Z_-).$$

Here $\mathcal{E}_\pm$ may be *virtual* vector bundles, i.e. of the form $\mathcal{E}_\pm^1 - \mathcal{E}_\pm^2$ for genuine vector bundles $\mathcal{E}_\pm^1$ and $\mathcal{E}_\pm^2$.

5.7

Proof of Theorem 5.6. Since all steps are analogous to the proof of Theorem 1.6, we only indicate what needs to be modified.

All wall-crossing considerations from §2 continue to hold, with the following minor adjustments:

- in §2.2, the *virtual* localization formula [16] is used to obtain (9);
- in §2.7, the integrand is $\mathcal{F} = \widehat{\mathcal{O}}_M^{\mathrm{vir}} / \Phi(\mathcal{T}_M^{\mathrm{vir}})\Phi((\mathcal{T}_M^{\mathrm{vir}})^\vee)$;
- in §2.7, the resulting wall-crossing formula (11) is

$$0 = \frac{y^{\frac{1}{2}}\vartheta(y)}{\phi(1)^2}\left(E_{-y}^{\mathrm{vir}/2}(Z_-) - E_{-y}^{\mathrm{vir}/2}(Z_+)\right) + \chi\left(Z_0^T, \cdots \otimes \operatorname{res} \frac{(\det \mathcal{N}_{\iota_0}^{\mathrm{vir}})^{-\frac{1}{2}}}{\Theta(\mathcal{N}_{\iota_0}^{\mathrm{vir}})}\right).$$

To compare with (11) more closely, note that

$$\frac{(\det \mathcal{N}_{\iota_0}^{\mathrm{vir}})^{-\frac{1}{2}}}{\Theta(\mathcal{N}_{\iota_0}^{\mathrm{vir}})} = y^{\frac{1}{2}\operatorname{rank} \mathcal{N}_0^{\mathrm{vir}/2}} \frac{\Theta(y\mathcal{N}_0^{\mathrm{vir}/2})}{\Theta(\mathcal{N}_0^{\mathrm{vir}/2})}.$$

■

5.8

It remains to explain why Theorem 2.12 continues to hold for when $\mathcal{E}_\pm$ are allowed to be virtual vector bundles. If $\{a_i\}_i$ and $\{b_j^{-1}\}$ are the Chern roots of $\mathcal{E}_+^1$ and $\mathcal{E}_-^1$, respectively, like in §2.13, and $\{a'_k\}_k$ and $\{(b'_l)^{-1}\}_l$ are the Chern roots of $\mathcal{E}_+^2$ and $\mathcal{E}_-^2$, respectively, then the contour integral of interest is

$$\oint_{|s|\approx 1} \prod_i \frac{\vartheta(ysa:i,\tau)}{\vartheta(sa:i,\tau)} \prod_j \frac{\vartheta(ys^{-1}b_j^{-1},\tau)}{\vartheta(s^{-1}b_j^{-1},\tau)} \prod_k \frac{\vartheta(sa:k',\tau)}{\vartheta(ysa:k',\tau)} \prod_l \frac{\vartheta(s^{-1}(b'_l)^{-1},\tau)}{\vartheta(ys^{-1}(b'_l)^{-1},\tau)} \frac{ds}{s}.$$

Upon the change of variables $a''_k := ya'_k$ and $b''_l := yb'_l$, the result is exactly the original contour integral (13) using the variables $\{a_i\}_i \cup \{b''_l\}_l$ and $\{(a''_k)^{-1}\}_k \cup \{b_j^{-1}\}_j$. Analytically, upon specializing all variables to complex numbers, note that $|a'_k| = |a''_k|$ and $|b'_l| = |b''_l|$ since y is eventually specialized to a root of unity, so, the new variables $\{a''_k\}_k$ and $\{b''_l\}_l$ may be treated the same way as the old variables $\{a_i\}_i$ and $\{b_j\}_j$. Hence, the remainder of the proof of Theorem 2.12 can proceed in exactly the same way.

This concludes the proof of Theorem 5.6.

6 Example: Donaldson–Thomas Theory

6.1

Let Y be a quasi-projective Calabi–Yau 3-fold acted on by a torus $\widetilde{T}$ such that

$$\mathcal{K}_Y = y \otimes \mathcal{O}_Y$$

for a $\widetilde{T}$-weight y, and the $\widetilde{T}$-fixed locus in Y is proper. For instance, Y could be toric and $\widetilde{T} = (\mathbb{C}^\times)^3$ the standard torus, or, more generally, Y can be a local curve or surface.

We work in the setup of §4.1, denoting the stability condition by σ, but using the moduli substack

$$\mathfrak{N}_\alpha^{\mathrm{sst}}(\sigma) := \{\det E = \mathcal{L}\} \subset \mathfrak{M}_\alpha^{\mathrm{sst}}(\sigma) \tag{27}$$

of fixed-determinant objects. Here the line bundle $\mathcal{L}$ must be chosen such that $c_1(\mathcal{L})$ agrees with the $H^2(Y)$ component of $\alpha \in H^*(Y)$. Such moduli stacks $\mathfrak{N}$ are part of the *Donaldson–Thomas (DT) theory* of Y, whose characterizing property is the existence of an equivariantly-symmetric perfect obstruction theory given at the point $[E]$ by

$$\mathbb{E}^\vee[-1]\Big|_{[E]} = \mathbf{Ext}_Y(E, E)_0, \tag{28}$$

where the subscript 0 denotes trace-less part [42, Theorem 3.30]. We will assume that $\mathbf{rank}\,\alpha > 0$ until §6.10.

6.2

Let σ be a stability condition with no strictly semistable objects in class α. The moduli stack $\mathfrak{N}_\alpha^{\mathrm{sst}}(\sigma)$ is an algebraic space (see §4.2), but is generally not proper since Y is not proper. So, throughout this section, we assume the following analogue of 4.2(i):

(i) the T_y-fixed locus in $\mathfrak{N}_\alpha^{\mathrm{sst}}(\sigma)$ is proper, where $T_y \subset \mathbf{ker}(y)$ is the maximal torus.

Since $T_y \subset \widetilde{T}$, this implies:

(ii) the $\widetilde{T}$-fixed locus in $\mathfrak{N}_\alpha^{\mathrm{sst}}(\sigma)$ is proper.

But 6.2(i) is a stronger assumption, and we really need its full strength to study wall-crossing for elliptic DT invariants (Definition 6.3).

In practice, properness of the T_y-fixed locus in Y is usually enough to imply 6.2(i). This is a much more manageable condition. For instance, if Y is toric, it is equivalent to the condition that non-compact edges in its toric 1-skeleton cannot have $\widetilde{T}$-weight y^k for any $k \in \mathbb{Z}$ (but weights $y^k w$ for non-trivial w are allowed).

6.3

Definition. Let $\alpha \in H^*(Y)$ and σ be a stability condition with no strictly semistable objects in $\mathscr{A}$ of class α. Assume 6.2(ii). Then the *elliptic DT invariant* is

$$\mathrm{DT}_{-y}^{\mathbf{Ell}/2}(\alpha;\sigma) := \mathrm{E}_{-y}^{\mathrm{vir}/2}(\mathfrak{N}_\alpha^{\mathrm{sst}}(\sigma)) \in K_{\widetilde{T}}(\mathrm{pt})_{\mathrm{loc}}[[q]], \tag{29}$$

where the virtual chiral elliptic genus is defined by $\widetilde{T}$-equivariant localization, that is, as the right-hand side of (3). Consequently the result lives in the *localized* K-group, and in particular it may have non-trivial poles in $\{|y| = 1\}$. But if we further assume 6.2(i), then no such poles in y exist by Lemma 2.10, and y may be specialized to any root of unity.

6.4

Remark. In (27), $\mathfrak{N}_\alpha^{\mathrm{sst}}(\sigma)$ has obstruction theory given by the traceless $\mathbf{Ext}_Y(E, E)_0$ while $\mathfrak{M}_\alpha^{\mathrm{sst}}(\sigma)$ has obstruction theory given by $\mathbf{Ext}_Y(E, E)$. If $H^1(\mathcal{O}_Y) \neq 0$, the latter contains trivial summands $H^1(\mathcal{O}_Y)$ and its dual $H^2(\mathcal{O}_Y) = y^{-1}H^1(\mathcal{O}_Y)^\vee$, and therefore all enumerative invariants of $\mathfrak{M}_\alpha^{\mathrm{sst}}(\sigma)$ vanish. This is why we generally work with $\mathfrak{N}_\alpha^{\mathrm{sst}}(\sigma)$.

However, the discrepancy between $\mathbf{Ext}$ and traceless $\mathbf{Ext}$ does not affect the "off-diagonal" terms $\mathbf{Ext}_Y(E_1, E_2)$ and $\mathbf{Ext}_Y(E_2, E_1)$ appearing in arguments below. This is true of many reductions one may want to perform on the obstruction theory, for example, [40,§1.6] in Vafa–Witten theory.

6.5

Theorem. Assume 6.2(i) and use the notation of §4.

(i) (Simple wall) Assume 4.4(i). If, for all $[E_1 \oplus E_2] \in Z_0^{\widetilde{T}}$,

$$\dim \mathrm{Ext}_Y(E_1, E_2) \equiv 0 \bmod N$$

for some integer $N > 0$, then $\mathrm{DT}^{\mathrm{Ell}/2}_{-\zeta_N}(\alpha; +) = \mathrm{DT}^{\mathrm{Ell}/2}_{-\zeta_N}(\alpha; -)$.

(2) (General wall) If, for all splittings (22) appearing in all auxiliary wall-crossings,

$$\dim \mathrm{Ext}_Y(E_1, E_2) \equiv \dim \mathrm{Ext}_Q(V_1^\bullet, V_2^\bullet) - \dim \mathrm{Ext}_Q(V_2^\bullet, V_1^\bullet) \bmod N$$

for some integer $N > 0$, then $\mathrm{DT}^{\mathrm{Ell}/2}_{-\zeta_N}(\alpha; +) = \mathrm{DT}^{\mathrm{Ell}/2}_{-\zeta_N}(\alpha; -)$.

This is the direct analogue of Theorems 4.6 and 4.10, but without requiring the strong assumption 4.2(ii) on the smoothness of the moduli spaces.

6.6

Remark. Suppose that the numerical conditions in Theorem 6.5 hold for all $N > 0$, that is, they are equalities instead of congruences mod N. Then one obtains equalities

$$\mathrm{DT}^{\mathrm{Ell}/2}_{-y}(\alpha; +) = \mathrm{DT}^{\mathrm{Ell}/2}_{-y}(\alpha; -)$$

under only the weaker assumption 6.2(ii) which ensures both sides are well-defined. This is because both sides only have finitely many poles in $\{|y| = 1\}$, so their $y = \zeta_N$ specializations are well defined for all $N \gg 0$. Coefficients of the q-series on both sides are rational functions of y, which are therefore equal if and only if they are equal at $y = \zeta_N$ for all $N \gg 0$.

6.7

Proof of Theorem 6.5. In [22,§10.6], Joyce constructs a master space M, roughly the moduli space of triples $(E, \mathbf{V}, \boldsymbol{\rho})$ where $[E] \in \mathfrak{N}_\alpha^{\mathrm{sst}}(0)$ and $(\mathbf{V}, \boldsymbol{\rho})$ is a representation of a certain quiver. There is a forgetful morphism

$$\pi : M \to \mathfrak{N}_\alpha^{\mathrm{sst}}(0)$$

which is smooth as a morphism of algebraic stacks. Then *symmetrized pullback* ([32,§2], [28,§2]) along π of the equivariantly-symmetric obstruction theory on $\mathfrak{N}_\alpha^{\mathrm{sst}}(0)$ (not necessarily perfect!) results in a equivariantly-symmetric *almost perfect* obstruction theory on M. By Remark 5.1, this suffices for wall-crossing.

Although the almost perfect obstruction theory on M can only be compared étale-locally to the (pullback along π of the) original obstruction theory on $\mathfrak{N}_\alpha^{\mathrm{sst}}(0)$, there is a well-defined global virtual tangent bundle on M, satisfying

$$\mathcal{T}_M^{\mathrm{vir}} = \pi^* \mathcal{T}^{\mathrm{vir}}_{\mathfrak{N}_\alpha^{\mathrm{sst}}(0)} + (\mathcal{T}_\pi - y^{-1}\mathcal{T}_\pi^\vee) \in K_{\widetilde{T}}(M), \tag{30}$$

where $\mathcal{T}_\pi$ is the relative tangent complex of π. ∎

6.8

We will apply Theorem 5.6. The analogue of (20),

$$\mathcal{N}_{\iota_0}^{\mathrm{vir}}\Big|_{[E_1 \oplus E_2]} = -s^{-1}\,\mathrm{Ext}_Y(E_1, E_2) - s\,\mathrm{Ext}_Y(E_2, E_1)$$

continues to hold. By Serre duality, in the setting of a simple wall,

$$\mathcal{N}_0^{\mathrm{vir}/2}\Big|_{[E_1 \oplus E_2]} = -s^{-1}\,\mathrm{Ext}_Y(E_1, E_2).$$

In the setting of a non-simple wall where the wall-crossing problem has been lifted to the auxiliary abelian category $\mathscr{A}^{\mathrm{Fr}}$, like in §4.9,

$$\mathcal{N}_0^{\mathrm{vir}/2}\Big|_{[E_1\oplus E_2, V_1^\bullet\oplus V_2^\bullet)]} = -s^{-1}\,\mathbf{Ext}_Y(E_1,E_2) + \big(s^{-1}\,\mathbf{Ext}_Q(V_1^\bullet,V_2^\bullet) + s\,\mathbf{Ext}_Q(V_2^\bullet,V_1^\bullet)\big)$$

using (30). This explains the numerical conditions in Theorem 6.5.

6.9

Finally, care is required when applying Theorem 5.6, because the master space M is not proper. We will use the argument in Remark 2.9 to work around this issue, by checking that all $\widetilde{T}_w$-fixed loci of M are proper, for maximal tori

$$\widetilde{T}_w \subset \ker(w) \subset \widetilde{T}\times S$$

where w is a $(\widetilde{T}\times S)$-weight with non-trivial S-component. This is the same argument as in [32, Lemma 5.7], which we summarize for the sake of completeness. Take any $\widetilde{T}$-equivariant compactification $\overline{Y}$ of Y. Let $\overline{\mathfrak{N}}$ denote the moduli stack $\mathfrak{N}$ but for objects on $\overline{Y}$, and similarly let $\overline{M}$ denote the master space for $\overline{\mathfrak{N}}$. It is known that $\overline{M}$ is proper. Consider the inclusions

$$M^{\widetilde{T}_w} \subset \overline{M}^{\widetilde{T}_w} \subset \overline{M}.$$

The second inclusion is clearly closed. The first inclusion is also closed: on triples $(E,\mathbf{V},\boldsymbol{\rho})$ parameterized by $\overline{M}$, only $\widetilde{T}\subset\widetilde{T}_w$ acts on E, and so

$$M^{\widetilde{T}_w} = \{(E,\mathbf{V},\boldsymbol{\rho}) : \mathbf{supp}\,E \subset Y^{\widetilde{T}} \subset \overline{Y}^{\widetilde{T}}\},$$

where $\mathbf{supp}$ means set-theoretic support. In other words, the $\widetilde{T}_w$-fixed locus in M is a collection of certain $\widetilde{T}_w$-fixed components of $\overline{M}$. Closed subsets of proper spaces are proper.

Applying Theorem 5.6 concludes the proof.

6.10

We give one explicit situation, *Vafa–Witten (VW) theory* [40], in which the divisibility of $\mathbf{Ext}_Y(E_1,E_2)$, required by Theorem 6.5, can be controlled. VW theory is a form of DT theory when $Y = \mathbf{tot}(\mathcal{K}_S)$ is an *equivariant local surface*, meaning:

- S is a smooth projective surface acted on by a torus T;
- $\widetilde{T} := T\times\mathbb{C}^\times$ where $\mathbb{C}^\times$ acts by scaling the fibers of $\pi: Y\to S$ with weight y^{-1}.

Let $\mathfrak{M}$ be the moduli stack of *compactly-supported* coherent sheaves on Y. By the spectral construction [40,§2], a point $[\mathcal{E}]\in\mathfrak{M}$ is equivalent to a pair $(\overline{\mathcal{E}},\phi)$ where

$$\overline{\mathcal{E}} = \pi_*\mathcal{E}\in\mathscr{C}(S),\quad \phi\in\mathbf{Hom}_S(\overline{\mathcal{E}},\overline{\mathcal{E}}\otimes\mathcal{K}_S).$$

Lemma ([40, Proposition 2.14]) . For $\mathcal{E}_1,\mathcal{E}_2\in\mathfrak{M}$,

$$\mathbf{Ext}_Y(\mathcal{E}_1,\mathcal{E}_2) = \mathbf{Ext}_S(\overline{\mathcal{E}}_1,\overline{\mathcal{E}}_2) - y^{-1}\,\mathbf{Ext}_S(\overline{\mathcal{E}}_2,\overline{\mathcal{E}}_1)^\vee.$$

6.11

The VW moduli stack comes in two flavors. Fix a class $\alpha = (r,c_1,c_2)\in\mathbb{Z}_{>0}\oplus H^2(S)\oplus H^4(S)$.

- (U) If $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$, then define the moduli substack

$$\mathfrak{N}_\alpha := \{\mathbf{ch}(\overline{\mathcal{E}}) = \alpha\}\subset\mathfrak{M}.$$

- (**SU**) Otherwise, pick $\mathcal{L} \in \mathrm{Pic}(S)$ with $c_1(\mathcal{L}) = c_1$ and define the moduli substack

$$\mathfrak{N}_\alpha := \{\det \overline{\mathcal{E}} = \mathcal{L},\ \operatorname{tr}\phi = 0,\ \operatorname{ch}(\overline{\mathcal{E}}) = \alpha\} \subset \mathfrak{M}.$$

Then $\mathfrak{N}_\alpha$ has equivariantly-symmetric perfect obstruction theory given by $\mathrm{Ext}_Y(\mathcal{E}, \mathcal{E})$ or some reduction of it [40, Corollary 2.26, Theorem 5.46]. Assuming 6.2(ii), Definition 6.3 produces *elliptic VW invariants*

$$\mathrm{VW}_{-y}^{\mathrm{Ell}/2}(\alpha; \sigma) \in K_{\tilde{T}}(\mathrm{pt})_{\mathrm{loc}}[[q]].$$

Typically, σ is Gieseker stability with respect to a choice of ample line bundle, but more general σ are permitted as long as the properness assumption 6.2(i) holds.

6.12

Corollary. If S is a smooth projective surface whose canonical bundle admits a square root, then the hypothesis of Theorem 6.5 is satisfied for $N = 2$.

This is the direct analogue of Corollary 4.12.

Proof. By Lemma 6.10 and Lemma 4.7(i), $\dim \mathrm{Ext}_Y(\mathcal{E}_1, \mathcal{E}_2) \equiv 0 \bmod 2$. By the same argument as in the proof of Corollary 4.12, the framing functor may be doubled so that $\dim \mathrm{Ext}_Q(V_1^\bullet, V_2^\bullet) - \dim \mathrm{Ext}_Q(V_2^\bullet, V_1^\bullet) \equiv 0 \bmod 2$ as well. ■

6.13

Finally, in the VW setting, we can simplify the properness assumption 6.2(i).

Lemma. Suppose that $\mathcal{K}_S|_{S^T}$ has non-trivial T-weight on each component. Then assumption 6.2(i) is satisfied.

Proof. Since $T_y = T$, by hypothesis the T_y-fixed locus of Y lies within S and is therefore proper. Then the T_y-fixed locus of $\mathfrak{N}_\alpha^{\mathrm{sst}}(\sigma)$ is also proper by the same argument as in §6.9: it is a closed subspace in the analogous moduli space for any choice of compactification $\overline{Y}$, which is proper for standard reasons [21]. ■

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A Appendix: A Geometric Formula for the Wall-Crossing Term

A.1

In this appendix, we study the general contour integral

$$C_n(\boldsymbol{a}, \tau; y) := \oint_{|s|\approx 1} s^n \prod_{i=1}^{r_+} \frac{\vartheta(ysa : i; \tau)}{\vartheta(sa : i; \tau)} \prod_{j=1}^{r_-} \frac{\vartheta(y^{-1}sb : j; \tau)}{\vartheta(sb : j; \tau)} \frac{ds}{s},$$

where the $\boldsymbol{a} := (a_1, \ldots, a_{r_+}, b_1, \ldots, b_{r_-})$ are viewed as fixed, generic elements of S with $|a_i|, |b_j| \gg 1$. Using that $\vartheta(z^{-1}; \tau) = -z\vartheta(z; \tau)$, clearly $C_0(\boldsymbol{a}, \tau; y)$ is the contour integral (13) up to an overall factor y^{r_-}.

The main results are a geometric formula (Proposition A.2) for $C_n(\boldsymbol{a}, \tau; y)$, and some mild control over where $C_n(\boldsymbol{a}, \tau; y)$ has poles, as a meromorphic function of $\boldsymbol{a}$ and τ. Neither result is used in any other part of the paper and may be safely skipped on a first reading. While the main Theorem 1.6 of this paper concerns the vanishing of $C_0(\boldsymbol{a}, \tau; \zeta_N)$, we hope the results in this appendix will help control the wall-crossing term in the future study of more general, non-trivial wall-crossings.

We assume throughout this section that $r_+ + r_- > 0$.

A.2

Proposition. Let $A := (\mathbb{C}^\times)^{r_+ + r_-}$ with coordinates identified with $\boldsymbol{a}$. Then

$$C_n(\boldsymbol{a}, \tau; y) = \chi\left(\mathbb{P}(V), \mathcal{O}(n) \otimes \frac{\Theta(y\mathcal{O}(1) \otimes V_+ + y^{-1}\mathcal{O}(1) \otimes V_-)}{\Phi(\mathcal{O}(1) \otimes V)\Phi(\mathcal{O}(-1) \otimes V^\vee)}\right) \in K_A(\mathrm{pt})[y^{\pm 1}][[q]] \tag{A.1}$$

is an A-equivariant Euler characteristic, where $V := V_+ \oplus V_- := \mathbb{C}^{r_+} \oplus \mathbb{C}^{r_-}$ and $A \subset \mathbf{GL}(V)$ acts as the maximal torus.

Note that $\mathbb{P}(V)$ is proper, and each coefficient of y and q is an element of $K_A(\mathbb{P}(V))$, so indeed the result is valued in the Laurent polynomial ring $K_A(\mathrm{pt})$ instead of its localization $K_A(\mathrm{pt})_{\mathrm{loc}}$.

A.3

Remark. While this paper mostly focuses on the case where $C_0(\boldsymbol{a}, \tau; y)$ vanishes for some specialization of y, it is possible that this explicit formula may be useful for the wall-crossing in §2. However, it is clear from this formula that $C_0(\boldsymbol{a}, \tau; y)$ will depend non-trivially on the coordinates $a_i \in A$. In wall-crossing, these correspond to the Chern roots of $\mathcal{N}_{\iota_0}$, and typically one has very little control over these Chern roots.

A.4

Proof of Proposition A.2. This is a standard application of the Jeffrey–Kirwan residue formula for integrals over GIT quotients; see [2, Appendix A], or the more general [39, Proposition 2.4] (written in cohomology, not K-theory).

To summarize the basic idea in our setting, let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}(V) = (V \setminus 0)/S$ which is induced by restriction from an S-equivariant coherent sheaf $\widetilde{\mathcal{F}}$ on V. Concretely, $\mathcal{F}$ is a coefficient of y and q in the integrand of (A.1). Then,

$$\chi((V \setminus 0)/S, \mathcal{F}(m)) = \chi(V \setminus 0, \widetilde{\mathcal{F}} \otimes s^m)^S \cong \chi(V, \widetilde{\mathcal{F}} \otimes s^m)^S.$$

Analytically, $\chi(V, \widetilde{\mathcal{F}} \otimes s^m)$ converges to a rational function on $A \times S$ for $|a_i| \gg 1$, with no poles on the maximal compact subgroup $\{|s| = 1\} \subset S$, so

$$\begin{aligned}\chi(V, \widetilde{\mathcal{F}} \otimes s^m)^S &= \int_{|s|=1} \chi(V, \widetilde{\mathcal{F}} \otimes s^m)\, \frac{ds}{s} \\ &= \int_{|s|=1} \frac{\widetilde{\mathcal{F}}|_0 \otimes s^m}{\prod_i (1 - a_i^{-1}s^{-1}) \prod_j (1 - b_j^{-1}s^{-1})}\, \frac{ds}{s}\end{aligned}$$

where the second equality is $(A \times S)$-equivariant localization on V. Since $\widetilde{\mathcal{F}}|_0$ has bounded degree in s, for $m \gg 0$ there is no pole at $s = 0$ and therefore $\{|s| = 1\}$ and $\{|s| \approx 1\}$ enclose the same poles. We obtain the desired formula upon specializing $m = 0$. This is valid because

$$\chi((V \setminus 0)/S, \mathcal{F}(m)) \in \mathbb{Q}[m, x_1^{\pm m}, x_2^{\pm m}, \ldots],$$

where the x_i may be roots of unity or $(A \times S)$-weights, and for elements of such a ring, if an equality holds for all $m \gg 0$, then in fact it holds for all $m \in \mathbb{Z}$. ■

A.5

Remark. By the same reasoning as in the proof of Proposition A.2, it turns out that the vanishing of $C_0(\boldsymbol{a}, \tau; \zeta_N)$ (Proposition 2.12) is equivalent to the invariance of elliptic genus under certain toric flips. Namely, the contour integral in (13) may also be expressed as

$$E_{-y}(\mathbf{tot}(\mathcal{O}_{\mathbb{P}(V_+)}(-1) \otimes V_-^\vee)) - E_{-y}(\mathbf{tot}(\mathcal{O}_{\mathbb{P}(V_-)}(-1) \otimes V_+^\vee)), \tag{A.2}$$

where $\mathbf{tot}$ denotes total space. These non-compact toric geometries $Y_\pm := \mathbf{tot}(\mathcal{O}_{\mathbb{P}(V_\pm)}(-1) \otimes V_\mp^\vee)$ are related by the birational transformation

$$\begin{array}{ccc} Y_+ & & Y_- \\ & \searrow^{f_+} \quad {}^{f_-}\swarrow & \\ & Y_0 & \end{array} \tag{A.3}$$

where $f_\pm$ contracts the zero section $\mathbb{P}(V_\pm)$. If $\pi_\pm : Y_\pm \to \mathbb{P}(V_\pm)$ denotes the projection, then the canonical bundles are

$$\begin{aligned} \mathcal{K}_{Y_\pm} &= \pi_\pm^* \left(\mathcal{K}_{\mathbb{P}(V_\pm)} \otimes \mathbf{det}(\mathcal{O}_{\mathbb{P}(V_\pm)}(1) \otimes V_\mp)\right) \\ &= \pi_\pm^* \mathcal{O}_{\mathbb{P}(V_\pm)}(\mathbf{dim}\, V_\pm - \mathbf{dim}\, V_\mp) \otimes \mathbf{det}(V_\pm)\, \mathbf{det}(V_\mp), \end{aligned}$$

so (A.3) is a flip in general and a flop if and only if $\mathbf{dim}\, V_+ = \mathbf{dim}\, V_-$. For instance, the classical Atiyah flop is modeled by the case $\mathbf{dim}\, V_+ = \mathbf{dim}\, V_- = 2$.

In particular, if $\mathbf{dim}\, V_- = 0$, then the vanishing of (A.2) becomes $E_{-\zeta_N}(\mathbb{P}^{N-1}) = 0$.

A.6

We turn to studying poles of $C_n(\boldsymbol{a}, \tau; y)$ as a meromorphic function on $A \times \mathbb{H}$, where $\mathbb{H} \ni \tau$ is the upper half plane. Recall that $\vartheta(z; \tau)$ is a holomorphic function of τ. By construction, poles of $C_n(\boldsymbol{a}, \tau; y)$ can only occur at

$$\mathrm{Poles}(q^m) := \bigcup_\mu \{q^m \boldsymbol{a}^\mu = 1\} \subset A \times \mathbb{H}$$

for various $m \in \mathbb{Z}$, where μ ranges over finitely many non-trivial A-characters. (More precisely, by A-equivariant localization applied to (A.1), $\boldsymbol{a}^\mu$ ranges over all A-weights of the normal bundle of $\mathbb{P}(V)^A \subset \mathbb{P}(V)$.) Proposition A.2 allows us to be more precise about where poles actually occur.

A.7

Proposition A.1. $C_n(\boldsymbol{a}, \tau; y)$ is holomorphic on $\mathrm{Poles}(q^0)$.

Proof. This is the same argument as in [9, Proposition 5.1] or [33, Lemma 1.3], so we give only a sketch. The idea is to obtain more mileage from Proposition A.2 by computing (A.1) via A-equivariant localization on $\mathbb{P}(V)$. The resulting expression, in the domain

$$\bigcap_\mu \{|q| < |\boldsymbol{a}^\mu| < |q|^{-1}\}, \quad 0 < |q| < 1, \tag{A.4}$$

has an expansion of the form $\sum_{j \geq 0} c_j(\boldsymbol{a}) q^j$ where the $c_j(\boldsymbol{a})$ are rational functions of $\boldsymbol{a}$ which only have poles on $\mathrm{Poles}(A; q^0)$, and the possible locations of such poles are independent of j. On the other hand, from properness, we also know that the $c_j(\boldsymbol{a})$ are Laurent polynomials of bounded degree (polynomial in j), and in particular have no poles on $\mathrm{Poles}(q^0)$. This is enough to conclude the desired holomorphicity since (A.4) contains $\mathrm{Poles}(q^0)$. ∎

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