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Aspects of Supersymmetric Conformal Field Theories in Various Dimensions

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Aspects of Supersymmetric Conformal Field Theories in Various Dimensions

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Physics

by

Emily M. Nardoni

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Professor Patrick Diamond
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Professor Justin Roberts

2018

The dissertation of Emily M. Nardoni is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

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DEDICATION

To my family.

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Chapter 3 is, in part, a reprint of material from published work done in collaboration with Ken Intriligator, as it appears in the Journal of High Energy Physics, as well as on the arXiv. K. Intriligator and E. Nardoni, "Deformations of $W_{A,D,E}$ SCFTs," *JHEP* **09** (2016) 043, arXiv:1604.04294. The dissertation author was the primary investigator and author of this material.

Chapter 4 is, in part, a reprint of material from published work done in collaboration with Kazunobu Maruyoshi and Jaewon Song, as it appears on the arXiv. K Maruyoshi, E. Nardoni and J. Song, "Landscape of Simple Superconformal Field Theories in 4d," arXiv:1806.08353. The dissertation author was the primary investigator and author of this material.

Chapters 5 and 7 are, in part, a reprint of material from published work done in collaboration with Ibrahima Bah, as it appears on the arXiv. I. Bah and E. Nardoni, "Structure of Anomalies of 4d SCFTs from M5-branes, and Anomaly Inflow," arXiv:1803:00136. The dissertation author was the primary investigator and author of this material.

Chapter 6 is, in part, a reprint of material from published work as it appears in the Journal of High Energy Physics, as well as on the arXiv. E. Nardoni, "4d SCFTs from Negative-Degree Line Bundles," *JHEP* **08** (2018) 199, arXiv:1611.01229. The dissertation author was the investigator and author of this material.

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PUBLICATIONS

Kenneth Intriligator and Emily Nardoni, "Deformations of $W_{A,D,E}$ SCFTs," *JHEP* **09** (2016) 043, arXiv:1604.04294.

Emily Nardoni, "4d SCFTs from Negative-Degree Line Bundles," *JHEP* **08** (2018) 199, arXiv:1611.01229.

Ibrahima Bah and Emily Nardoni, "Structure of Anomalies of 4d SCFTs from M5-branes, and Anomaly Inflow," arXiv:1803:00136.

Kazunobu Maruyoshi, Emily Nardoni and Jaewon Song, "Landscape of Simple Superconformal Field Theories in 4d," arXiv:1806.08353.

ABSTRACT OF THE DISSERTATION

Aspects of Supersymmetric Conformal Field Theories in Various Dimensions

by

Emily M. Nardoni

Doctor of Philosophy in Physics

University of California San Diego, 2018

Professor Kenneth Intriligator, Chair

In this dissertation we study properties of superconformal field theories (SCFTs) that arise from a variety of constructions. We begin with an extended review of various techniques in supersymmetry that are relevant throughout the work. In Chapter 3, we discuss aspects of theories with superpotentials given by Arnold's A, D, E singularities, particularly the novelties that arise when the fields are matrices. We focus on four-dimensional $\mathcal{N} = 1$ variants of supersymmetric QCD, with $U(N_c)$ or $SU(N_c)$ gauge group, N_f fundamental flavors, and adjoint matter fields X and Y appearing in $W_{A,D,E}(X,Y)$ superpotentials. We explore these issues by considering various deformations of the $W_{A,D,E}$ superpotentials, and the resulting RG flows and IR theories. In Chapter 4, we examine the infrared fixed points of four-dimensional $\mathcal{N} = 1$ supersymmetric

$SU(2)$ gauge theory coupled to an adjoint and two fundamental chiral multiplets. We focus on a particular RG flow that leads to the $\mathcal{N}=2$ Argyres-Douglas theory H_0, and a further deformation to an $\mathcal{N}=1$ SCFT with low a central charge. Then for the latter half of the dissertation we turn our attention to 4d SCFTs that arise from compactifications of M5-branes. In Chapter 6, we field-theoretically construct 4d $\mathcal{N}=1$ quantum field theories by compactifying the 6d (2,0) theories on a Riemann surface with genus g and n punctures, where the normal bundle decomposes into a sum of two line bundles with possibly negative degrees p and q. In Chapter 7, we study the 't Hooft anomalies of the SCFTs that arise from these compactifications. In general there are two independent contributions to the anomalies: there is a bulk term obtained by integrating the anomaly polynomial of the world-volume theory on the M5-branes over the Riemann surface, and there is a set of contributions due to local data at the punctures. Using anomaly inflow in M-theory, we describe how this general structure arises for cases when the four-dimensional theories preserve $\mathcal{N}=2$ supersymmetry, and derive terms that account for the local data at the punctures.

Chapter 1

Introduction

1.1 Outlook

The framework of Quantum Field Theory (QFT) encompasses the mathematical structure of a wide variety of physical systems, including systems of particle physics, general relativity, statistical physics, and the early universe. QFT unifies Einstein's theory of relativity with quantum mechanics, permitting us to describe the physics of the very small and the very fast in one unified framework. In QFT the fundamental object is a *field* that is valued at every point in spacetime, and particles are excitations of these fundamental fields. One reason we must switch to thinking about fields rather than particles is that in relativistic processes, particles can be created or destroyed. When particle number is not conserved, we need a theory that can describe more than single particle dynamics. Another fundamental reason for a quantum theory of fields is that we need to construct laws of nature that are local, which requires taking into account the fact that information travels at a finite speed. QFT provides a natural framework for doing this.

Symmetry is the main organizing principle and tool for studying QFT. A symmetry of a physical system is a transformation of the system that does not change the results of any possible experiments. An eloquent way to state this is that *physics should not depend on the physicist*.[1] The more symmetry we have at our disposal, the better our handle on the properties of a theory. In QFT, fields are classified by how they transform under the symmetries of the physical system under consideration.

A related and important fact is that symmetries imply conservation laws. In particular, Noether's theorem tells us that continuous symmetries have corresponding conserved quantities. For example, a reasonable symmetry one might impose is the following: Say I do an experiment standing at a particular position in space, and then take a step to the right by a meter. If there is no other difference between where I started and where I ended (e.g., there is no wind at either location, and so on), then the results of doing the same experiment at the second location should be exactly the same as at the first location. In other words, we expect that the laws of physics should be invariant under spatial translations. In this case, momentum conservation is the corresponding conservation law due to Noether's theorem as applied to the symmetry of spatial translations. Similarly, requiring that the laws of physics are invariant under translations in time yields the law of energy conservation.

An important set of QFTs are those with Poincaré symmetry. A key consequence of relativity is that time and space should be put on equal footing, into the more general concept of *spacetime*. Poincaré invariance is the requirement that physics is invariant under isometries (distance-preserving maps) of Minkowski spacetime—the flat spacetime relevant to special relativity. This is a requirement for relativistic quantum field theories: that the laws of physics should take the same form in all inertial frames of reference. Again, physics should not depend on the physicist. For example, in a relativistic framework energy and momentum conservation are derived together from general spacetime translation invariance (a subset of Poincaré invariance), and can be seen as packaged into one conserved object known as the energy-momentum tensor.

An idea which has proven itself useful time and time again in the study of QFT is

[1] As stated by Anthony Zee in the excellent *Quantum Field Theory in a Nutshell.*

to study theories with extra symmetries as a testing ground for ideas in more general QFTs. With this in mind, we will focus our attention in this dissertation mainly on QFTs with two additional symmetries beyond Poincaré invariance: conformal symmetry, and supersymmetry. The Haag-Lopuszanski-Sohnius theorem [1] states that supersymmetry is the only extension of Poincaré symmetry as a spacetime symmetry in a consistent QFT.[2] Supersymmetry involves the addition of fermionic (anticommuting) generators, the *supercharges*, to the Poincaré algebra. In theories with supersymmetry, fermions and bosons are related by a symmetry transformation, and representations of the supersymmetry algebra—usually called supersymmetry *multiplets*—have equal numbers of fermions and bosons. The addition of supersymmetry often allows us to solve aspects of these theories exactly.

To explain the utility of conformal symmetry, we need to introduce the concept of the renormalization group. The *renormalization group* (RG) is a framework that tells you how a theory looks at different distance scales. Couplings in a QFT determine the strength of the force of an interaction between fields, and the RG equations tell you how the couplings change, or *run*, as a function of the distance scale. This framework gives a precise way for how high energy degrees of freedom can be accounted for by an effective theory at lower energies.

Here it is worth taking an aside to explain what we mean by scale, which requires a diversion into dimensional analysis. The fundamental dimensions we use to measure physical quantities are length, time, and mass. For example, in S.I. units length is measured in units of meters, time in seconds, and mass in kilograms. In nature there are three fundamental dimensionful constants: the speed of light c with dimensions of length/time, which according to Einstein's theory of relativity must be measured to be the same in any inertial frame of reference; Planck's (reduced) constant $\hbar$ with dimensions of mass$\cdot$ length^2 / time, which controls the scale at which quantum effects become important; and Newton's gravitational constant G with dimensions of length^3 / (mass $\cdot$ time^2), which tells us about the scale of gravitational effects. High energy

[2] More specifically as a symmetry of the S-matrix, assuming (among other things) analyticity of the S-matrix.

physicists commonly use natural units, in which we redefine $c = \hbar = 1$. Then, for instance, velocities will be given as numbers in units of the speed of light. This restricts the units of length equal to the units of time, equal to the units of inverse mass, equal to the units of inverse energy. For example, in natural units the mass m of a particle is equal to its rest energy $E = mc^2$, as well as its inverse Compton wavelength $mc/\hbar$. So, in natural units we have some choices as to which units we care to keep track of. Particle physicists typically choose to use units of energy, with length and time both given in units of inverse energy. So truly, when we refer to large energy scales we are referring as well to small length scales, and vice versa. *End aside.*

Back to the renormalization group. Generically, at very long distance scales quantum field theories become scale invariant—they reach a point at which the physics looks the same even if we change the length scale, known as a *fixed point* of the renormalization group. At such a fixed point in the space of couplings, the couplings no longer run with scale. In general, this scale invariance at a fixed point is enhanced to conformal invariance.[3] Conformal symmetry is the largest possible non-supersymmetric spacetime symmetry of an interacting field theory compatible with Poincaré invariance. A very useful picture of QFT is as a flow under the renormalization group between conformal field theories (CFTs) in various limits. Our perspective is that by studying CFT, we can start to map out the space of more general QFTs. Said another way: by focusing on CFTs, we essentially aim to map out the end points of RG flows between more general theories.

There are various possibilities for how QFTs can behave at different energy scales. One possibility is that at long distances, the strength of the coupling decreases. Such a theory is said to be *infrared (IR) free*. This is, for example, the case for quantum electrodynamics (QED). In QED, vacuum polarization renormalizes the electric charge $e(r)$—the coupling constant for QED, which as we stress is really best not thought of as a constant—to smaller values at bigger distances. At distances r greater than the inverse mass of the electron, the coupling settles onto a constant. In this limit, the potential energy between two separated static test sources goes like $V(r) \sim e^2/r$,

[3]For unitary QFTs in $d = 2$ this is proven [2], in $d = 4$ this is argued but not rigorously proven.

which is the Coulomb potential we learn about in high school. At the other end of the scale, for small enough distances the coupling diverges in what is known as a Landau pole, at which point the theory needs modification.

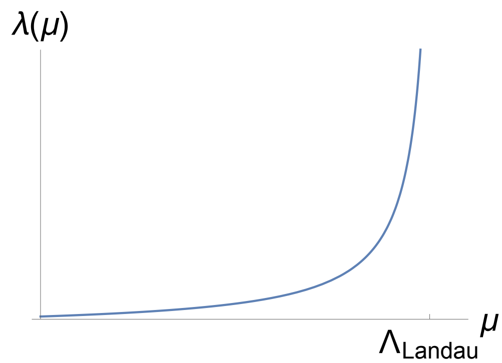


Figure 1.1: The coupling λ as a function of energy scale μ plotted for an infrared free theory that breaks down at $\mu \sim \Lambda_{\text{Landau}}$.

Another possibility is that at short distances, the strength of the coupling decreases. Such a theory is called *asymptotically free*, or *ultraviolet (UV) free*. This is the case, for instance, for quantum chromodynamics (QCD)—the theory of the strong interactions that act on quarks and gluons. At long distances / low energies the quarks are strongly coupled, and we only can observe composite objects with no color charge. This is called *confinement*. But at high energies, the theory consists of weakly coupled quarks and gluons.

There are various possibilities for the strongly coupled regime in an asymptotically free theory (the small μ part of Figures 1.2 and 1.3). In the case of QCD, it is thought that as the coupling keeps getting stronger in the infrared, the theory will dynamically generate a mass scale Λ_{QCD} by the strong interactions. So, for energies much less than $\Lambda_{\text{QCD}} \sim 300$ MeV the theory is strongly coupled (confines), and for energies much bigger than Λ_{QCD} the theory is weakly coupled.

Another interesting possibility is that in the strongly coupled regime, the coupling will flow to a fixed point where it no longer changes with scale. At large distances, the physics is completely independent of the scale. Interestingly, an asymptotically free gauge theory with enough matter

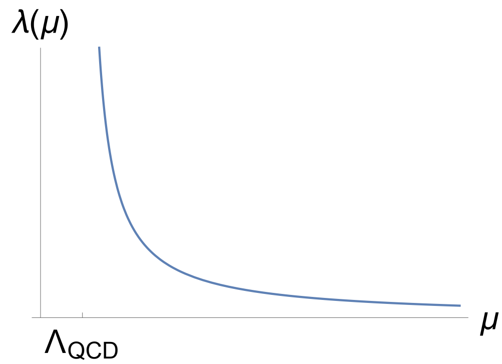


Figure 1.2: The coupling λ as a function of energy scale μ for an asymptotically free theory that develops a dynamical scale at $\mu \sim \Lambda_{\text{QCD}}$.

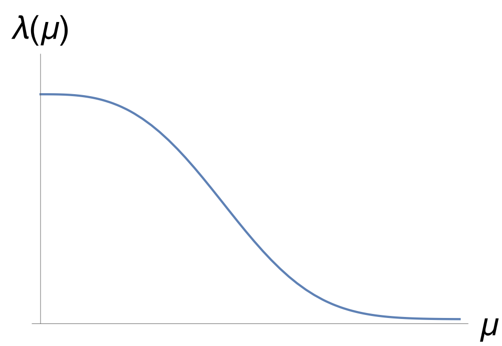


Figure 1.3: The coupling λ as a function of energy scale μ for an asymptotically free theory that develops an interacting fixed point at low energies.

content might flow to an interacting CFT at low energies. There exist many examples of nontrivial, interacting CFTs in various dimensions—especially using supersymmetry!

It is natural at this point to mention that the idea of the renormalization group leads naturally to the idea of *duality*. As we just saw in some examples, the UV and IR physics of a quantum field theory might look very different. Two physical systems that look different at short distances might behave the same way at long distances. Two such theories are said to be infrared dual to one another, or to lie in the same universality class. Interestingly, a single CFT fixed point generically describes the long distance physics of many different physical systems. In this way, understanding CFTs can teach us about the universal features of a wide variety of quantum field theories.

Most of the QFTs that we will discuss in this thesis have both conformal invariance and supersymmetry, i.e. are superconformal quantum field theories (SCFTs). When an interacting supersymmetric fixed point exists, we can use superconformal symmetry to derive some exact results about the theory. Broadly speaking, one can view this thesis as a study of special classes of SCFTs in various dimensions, with the broader goal of understanding general phenomena in quantum field theory.

1.2 Outline

The rest of this dissertation is organized as follows. In Chapter 2 we give an extended introduction to some of the technical aspects of SCFTs that we will utilize throughout. We first review properties of supersymmetric gauge theories, and then review some useful properties superconformal field theories, mainly focusing on the four-dimensional case.

In Chapter 3 we discuss aspects of theories with superpotentials given by Arnold's A,D,E singularities, particularly the novelties that arise when the fields are matrices. We focus on 4d $\mathcal{N}=1$ variants of susy QCD, with $U(N_c)$ or $SU(N_c)$ gauge group, N_f fundamental flavors, and adjoint matter fields X and Y appearing in $W_{A,D,E}(X,Y)$ superpotentials. The 4d $W_{A,D,E}$ SQCD-type theories RG flow to superconformal field theories, and there are proposed duals in the literature for the W_{A_k}, W_{D_k}, and W_{E_7} cases. The $W_{D_{\text{even}}}$ and W_{E_7} duals rely on a conjectural, quantum truncation of the chiral ring. We explore these issues by considering various deformations of the $W_{A,D,E}$ superpotentials, and the resulting RG flows and IR theories. Rather than finding supporting evidence for the quantum truncation and $W_{D_{\text{even}}}$ and W_{E_7} duals, we note some challenging evidence to the contrary.

In Chapter 4 we explore the infrared fixed points of four-dimensional $\mathcal{N}=1$ supersymmetric $SU(2)$ gauge theory coupled to an adjoint and two fundamental chiral multiplets under all possible relevant deformations and F-term couplings to gauge-singlet chiral multiplets. We find 35 fixed points, including the $\mathcal{N}=2$ Argyres-Douglas theories H_0 and H_1. The theory with minimal central charge a is identical to the mass-deformed H_0 theory, and the one with minimal c has the smallest a among the theories with $U(1)$ flavor symmetry. We examine the RG flow to the mass-deformed H_0 theory.

In the latter half of the dissertation we turn our attention to a class of 4d SCFTs that arise from compactifications of M5-branes. In Chapter 5 we give an introduction to the 4d theories of Class $\mathscr{S}$, which are constructed by compactifying the 6d (2,0) theories on a Riemann surface with genus g and n punctures.

In Chapter 6, we field-theoretically construct 4d $\mathscr{N}=1$ quantum field theories of Class $\mathscr{S}$, where the normal bundle decomposes into a sum of two line bundles with possibly negative degrees p and q. Previously the only available field-theoretic constructions required the line bundle degrees to be nonnegative, although supergravity solutions were constructed in the literature for the zero-puncture case for all p and q. Here, we provide field-theoretic constructions and computations of the central charges of 4d $\mathscr{N}=1$ SCFTs that are the IR limit of M5-branes wrapping a surface with general p or q negative, for general genus g and number of maximal punctures n.

In Chapter 7, we study the 't Hooft anomalies of the SCFTs that arise from these compactifications. In general there are two independent contributions to the anomalies: there is a bulk term obtained by integrating the anomaly polynomial of the world-volume theory on the M5-branes over the Riemann surface, and there is a set of contributions due to local data at the punctures. Using anomaly inflow in M-theory, we describe how this general structure arises for cases when the four-dimensional theories preserve $\mathscr{N}=2$ supersymmetry, and derive terms that account for the local data at the punctures.

Chapter 2

Technical Introduction

Here we collect some of the main facts and methods that will be useful to us in our study of superconformal field theories. This material can be found in a myriad of textbooks and reviews, some of the most useful of which (in the author's opinion) we will mention when they are relevant. We make no attempt to be comprehensive, and instead utilize this chapter as a depository for a variety of useful facts. Our focus will be largely on QFTs in four dimensions, with some additional comments on other dimensions.

2.1 Supersymmetry: The Basics

We begin with a brief review of the supersymmetry algebra and its irreducible representations. Useful references for this material are the classic textbooks [3, 4].

2.1.1 The supersymmetry algebra

The four-dimensional supersymmetry algebra is an extension of the Poincaré algebra of spacetime symmetries by $\mathcal{N}$ anti-commuting generators,

$$\{Q^A_\alpha, Q^{\dagger B}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^{AB}, \qquad A,B = 1,\ldots,\mathcal{N}. \tag{2.1}$$

The Q^A_α are complex anti-commuting spinors with spinor indices α, $\dot{\alpha} = 1,2$, and in particular transform nontrivially under the Lorentz group. The index μ runs over four-dimensional spacetime. $\sigma^\mu_{\alpha\dot{\alpha}} = (1, \sigma^i)$ for σ^i the usual Pauli matrices. The Poincaré algebra is a subalgebra of the supersymmetry algebra; the other nonzero commutators between the supercharges and the Poincaré generators involve the Lorentz boosts $M_{\mu\nu}$. We refer the reader to the textbooks for a discussion of the full algebra.

The $\mathscr{N} = 1$ supersymmetry algebra possesses an internal global $U(1)$ symmetry known as an R-symmetry. This can be seen from the fact that when $\mathscr{N} = 1$ the supersymmetry algebra (2.1) is invariant under multiplication of the Q's by a phase. Denoting the generator of the $U(1)$ R-symmetry by R, we have that

$$[R, Q_\alpha] = -Q_\alpha, \qquad [R, Q^\dagger_{\dot{\alpha}}] = Q^\dagger_{\dot{\alpha}}, \tag{2.2}$$

such that the Q's have R-charge -1, and the $Q^\dagger$'s have R-charge $+1$. *n.b.* that the R-symmetry is generally not part of the supersymmetry algebra, although the algebra can include the R-symmetry as an extension (which is the case, as we will see, for superconformal algebras).

A theory with $\mathscr{N} > 1$ is said to have extended supersymmetry.[1] In four dimensions the smallest spinor representation (either a Weyl or Majorana spinor) has four real degrees of freedom, such that the actual number of supercharges is $N_Q = 4\mathscr{N}$. For example, the 4d $\mathscr{N} = 1$ algebra has four supercharges. A 4d theory with $\mathscr{N}$ supersymmetries generally has a corresponding global R-symmetry of $U(\mathscr{N})$, corresponding to the rotation of the $Q's$ by a $U(\mathscr{N})$ matrix.

In d spacetime dimensions, the supercharges are promoted to spinors of $SO(d-1,1)$. The general d-dimensional algebra has the same structure as in four dimensions, with the Pauli matrices σ^μ promoted to Dirac matrices Γ^μ that satisfy the Clifford algebra in d dimensions, $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ for η the spacetime metric. For example, in $d = 6$ the smallest spinor representation has 8 real degrees of freedom, so a 6d QFT with $\mathscr{N}$-extended supersymmetry has $N_Q = 8\mathscr{N}$

[1] In this case the supersymmetry algebra can contain central charges, corresponding to additional terms on the right-hand-side of (2.1). We will not discuss this possibility further here.

supercharges.

2.1.2 Irreducible representations of supersymmetry in 4d

In general, one characterizes the irreducible representations (irreps) of supersymmetry on asymptotic single particle states via Casimir operators—operators that commute with all the supersymmetry generators. One such Casimir operator is $P^2 = P^\mu P_\mu$, which is also used to construct representations of the Poincaré algebra. Because P^2 is a Casimir, particles in the same irrep have the same mass. For example, for a particle of mass m one can go to the rest frame $P_\mu = (m, \vec{0})$, from which it follows that $P^2 = m^2$.

The single particle states of a supersymmetric theory fall into irreps of the supersymmetry algebra called *supermultiplets*. Supermultiplets are in general reducible representations of the Poincaré algebra, and therefore correspond to a collection of particles in the usual sense. Particles in the same supermultiplet must have equal masses and lie in the same representation of the gauge group, so must have the same electric charges, weak isospin, and color degrees of freedom. Each supermultiplet contains an equal number of fermionic and bosonic degrees of freedom. Note that since the R-symmetry does not commute with the supersymmetry generators, component fields of a supermultiplet do not all carry the same R charge.

First consider the case of massive particles, where $P^2 = m^2$. The nonzero anticommutators simplify to $\{Q^A_\alpha, Q^{\dagger B}_{\dot{\alpha}}\} = 2m\delta_{\alpha\dot{\alpha}}\delta^{AB}$, which is precisely the Clifford algebra. Denote the lowest weight state—the Clifford vacuum of spin s—by $|\Omega_s\rangle$. This is annihilated by Q^A_α, which acts as a lowering operator. $Q^{\dagger B}_{\dot{\alpha}}$ acts as a raising operator. Then, generally there are $2\mathcal{N}$ creation and annihilation operators.

Next consider the case of massless particles. We can pick a frame in which $P_\mu = (E, 0, 0, E)$, such that $P^2 = 0$. The anticommutators simplify in this case to $\{Q^A_1, Q^{\dagger B}_1\} = 4E\delta^{AB}$, which is the Clifford algebra with only $\mathcal{N}$ raising operators. To construct representations we choose a Clifford vacuum $|\Omega_h\rangle$ of fixed helicity h, and construct representations by acting with the raising operators.

It will be useful to enumerate a selection of massless multiplets in four dimensions. Denote the gauge group by G. Let $a^\dagger$ denote a raising operator. Then, the state $a^\dagger|\Omega_h\rangle$ has helicity $h+1/2$. As we mentioned, there are $\mathcal{N}$ such creation operators for massless states. An irrep will in general have a total of $2^{\mathcal{N}}$ states. However, we might need to add the CPT conjugate to construct a full CPT eigenstate such that the multiplet will actually include $2\cdot 2^{\mathcal{N}}$ states.

We begin with the case of $\mathcal{N}=1$, where we denote the R-symmetry generator of the $U(1)_R$ as $R_{\mathcal{N}=1}$. We normalize such that the R-charge of the gluino λ_α is 1, and that of the gluon is 0. Note that in general the fermion component ψ of a chiral superfield Φ has $R[\psi]=R[\phi]-1$, for ϕ the bosonic component (since it is obtained by the action with a Q). The components of a massless $\mathcal{N}=1$ vector multiplet V are given in Table 2.1, and the components of a massless $\mathcal{N}=1$ chiral multiplet Φ in a representation r of the gauge group are given in Table 2.2.

Table 2.1: $\mathcal{N}=1$ vector multiplet V. This multiplet consists of two massless susy irreps $(|\Omega\rangle, a^\dagger|\Omega\rangle)$ paired to make a CPT eigenstate, for a total of $2\times 2^1=4$ states.

		G	$R_{\mathcal{N}=1}$	states
Weyl fermion	λ_α	adj	1	$\{|\Omega_{1/2}\rangle, a^\dagger|\Omega_{-1}\rangle\}$
massless spin 1	A_μ	adj	0	$\{a^\dagger|\Omega_{1/2}\rangle, |\Omega_{-1}\rangle\}$

Table 2.2: $\mathcal{N}=1$ chiral multiplet Φ. This multiplet also has $2\times 2^1=4$ states.

		G	$R_{\mathcal{N}=1}$	states
complex scalar	Q	r	$R(Q)$	$\{|\Omega_0\rangle, a^\dagger|\Omega_{-1/2}\rangle\}$
Weyl fermion	ψ_α	r	$R(Q)-1$	$\{a^\dagger|\Omega_0\rangle, |\Omega_{-1/2}\rangle\}$

An $\mathcal{N}=2$ theory has an R-symmetry $U(2)_R\simeq U(1)_R\times SU(2)_R$, with generators that we denote by $R_{\mathcal{N}=2}$ and $I^a,\ a=1,2,3$ respectively. We use a basis for the Cartan subalgebra of the R-symmetry labeled by $\left(R_{\mathcal{N}=2}, I^3\right)$. We can fix an $\mathcal{N}=1$ subalgebra in the $\mathcal{N}=2$ algebra, such that the $\mathcal{N}=1$ R-symmetry is given by

$$R_{\mathcal{N}=1}=\frac{1}{3}R_{\mathcal{N}=2}+\frac{4}{3}I^3. \tag{2.3}$$

With this choice, the linear combination

$$J = R_{\mathcal{N}=2} - 2I^3 \tag{2.4}$$

commutes with the $\mathcal{N}=1$ subalgebra, and is a flavor symmetry from the $\mathcal{N}=1$ point of view. The components of a massless $\mathcal{N}=2$ vector multiplet are given in Table 2.3, and the components of a massless $\mathcal{N}=2$ hypermultiplet in a representation r of the gauge group are given in Table 2.4.

Table 2.3: ($\mathcal{N}=2$ vector multiplet) = $V \oplus \Phi$. This has $2\times 2^2 = 8$ states. Note that $R_{\mathcal{N}=1}(\lambda'_\alpha) = R_{\mathcal{N}=1}(\phi) - 1$, since these come from the $\mathcal{N}=1$ chiral multiplet. λ_α and λ'_α form an $SU(2)_R$ doublet.

		G	$R_{\mathcal{N}=2}$	I_3	$R_{\mathcal{N}=1}$	states
Weyl fermion	λ_α	adj	1	1/2	1	$\{a_1^\dagger\|\Omega_0\rangle, a_2^\dagger\|\Omega_{-1}\rangle\}$
Weyl fermion	λ'_α	adj	1	-1/2	$-1/3$	$\{a_2^\dagger\|\Omega_0\rangle, a_1^\dagger\|\Omega_{-1}\rangle\}$
vector field	A_μ	adj	0	0	0	$\{a_1^\dagger a_2^\dagger\|\Omega_0\rangle, \|\Omega_{-1}\}$
complex scalar	ϕ	adj	2	0	$2/3$	$\{\|\Omega_0\rangle, a_1^\dagger a_2^\dagger\|\Omega_{-1}\rangle\}$

Table 2.4: ($\mathcal{N}=2$ hypermultiplet) = $\Phi \oplus \bar{\Phi}$. This has $2\times 2^2 = 8$ states. Note that $R_{\mathcal{N}=1}(\psi_\alpha) = R_{\mathcal{N}=1}(Q) - 1$, since these come from the $\mathcal{N}=1$ chiral multiplet. A and $\widetilde{Q}^\dagger$ form an $SU(2)_R$ doublet.

		G	$R_{\mathcal{N}=2}$	I_3	$R_{\mathcal{N}=1}$	states
Weyl fermion	ψ_α	r	-1	0	$-1/3$	$\{a_1^\dagger a_2^\dagger\|\Omega_{-1/2}\rangle, \|\Omega_{-1/2}\rangle\}$
Weyl fermion	$\widetilde{\psi}^\dagger_{\dot\alpha}$	$\bar{r}$	1	0	$1/3$	$\{a_1^\dagger a_2^\dagger\|\Omega_{-1/2}\rangle, \|\Omega_{-1/2}\rangle\}$
complex scalar	Q	r	0	$1/2$	$2/3$	$\{a_1\|\Omega_{-1/2}\rangle, a_2\|\Omega_{-1/2}\rangle\}$
complex scalar	$\widetilde{Q}^\dagger$	$\bar{r}$	0	$-1/2$	$-2/3$	$\{a_1\|\Omega_{-1/2}\rangle, a_2\|\Omega_{-1/2}\rangle\}$

An $\mathcal{N}=4$ theory has an R-symmetry[2] $SU(4)_R \simeq SO(6)_R$. We list the components of an $\mathcal{N}=4$ vector multiplet in Table 2.5.

[2] The R-symmetry is $SU(\mathcal{N})$ and not $U(\mathcal{N})$ in this case because for $\mathcal{N}=4$, a $U(1)$ decouples and becomes an outer automorphism. This can be seen at the level of the commutation relations, since $[Q^B_\alpha, R^A_A] = 0$.

Table 2.5: $\mathcal{N}=4$ vector multiplet = $\Phi\oplus\Phi\oplus\bar{\Phi}\oplus V$. This has 2^4 states, and is self-conjugate. The scalars ϕ^I are in the rank 2 antisymmetric (**6**) representation of $SU(4)_R$. The four $a_i^\dagger|\Omega_{-1}\rangle$ states with helicity $-1/2$ form the **4**, and the other four of helicity $1/2$ form the $\bar{\mathbf{4}}$.

		$R_{\mathcal{N}=1}$	states	
Weyl fermions	λ_α	1	$8\ \{a_i^\dagger\|\Omega_{-1}\rangle, a_i^\dagger a_j^\dagger a_k^\dagger\|\Omega_{-1}\rangle\}$	$\mathbf{4},\bar{\mathbf{4}}$ of $SU(4)_R$
	ψ_α	$-1/3$		
	ψ_α	$-1/3$		
	$\tilde{\psi}_\alpha$	$-1/3$		
real scalars	ϕ^1	2/3	$6\ \{a_i^\dagger a_j^\dagger\|\Omega_{-1}\rangle\}$	**6** of $SU(4)_R$
	ϕ^2	2/3		
	ϕ^3	2/3		
	ϕ^4	2/3		
	ϕ^5	2/3		
	ϕ^6	2/3		
vector field	A_μ	0	$2\ \{\|\Omega_{-1}\rangle, a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger\|\Omega_{-1}\rangle\}$	

2.2 Some Properties of Supersymmetric Gauge Theories

2.2.1 $\mathcal{N}=1$ supersymmetric actions and the power of holomorphy

Here we will be rather schematic, just pointing out some particular features of $\mathcal{N}=1$ supersymmetric actions to emphasize. Consider a 4d $\mathcal{N}=1$ theory of massless chiral and vector multiplets. It is useful to add an auxiliary field to each: add a complex field F to the chiral multiplet, and a real field D to the vector multiplet, to furnish superfields with components

$$\Phi^i: \quad Q^i,\ \psi^i_\alpha,\ F^i \tag{2.5}$$

$$V^a: \quad A^a_\mu,\ \lambda^a_\alpha,\ D^a \tag{2.6}$$

The F component of a chiral superfield and D component of a vector superfield transform by a total derivative under an $\mathcal{N}=1$ supersymmetry transformation.

We can write a supersymmetry and gauge invariant action for chiral superfields as an integral over superspace,

$$S=\int d^4x\, d^4\theta\ K(\Phi^\dagger, e^{gT^aV^a}\Phi)+\int d^4x\, d^2\theta\ W(\Phi)+h.c. \tag{2.7}$$

The superpotential $W(\Phi)$ is a holomorphic function of the chiral superfields and has R-charge 2.

K is the Kähler potential, a vector superfield that yields the kinetic terms upon expansion into components. Terms proportional to $\int d^4\theta$ are known as D-terms, and terms proportional to $\int d^2\theta$ are known as F-terms. Here g is the gauge coupling.

We can write the $\mathcal{N} = 1$ super Yang-Mills action in terms of the field strength chiral superfield W_α. W_α is constructed out of the vector superfield V^a, with components

$$W^a_\alpha: \quad \lambda^a_\alpha,\ F^a_{\mu\nu},\ D^a. \tag{2.8}$$

The gauge invariant supersymmetric action for pure super Yang-Mills is

$$S_{SYM} = \frac{1}{8\pi}\mathrm{Im}\left[\tau\int d^4x\, d^2\theta\ \mathrm{Tr}W^\alpha W_\alpha\right]. \tag{2.9}$$

Here, W^2_α is the supersymmetric completion of $F^2 + iF\widetilde{F}$. τ is the complex holomorphic gauge coupling,

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{2.10}$$

It is nontrivial that supersymmetry combines the gauge coupling g and the θ-angle[3] into a single holomorphic quantity.

D-terms receive both perturbative and nonperturbative quantum corrections. The quantum corrections to F-terms, however, are highly constrained because they must maintain holomorphy in the chiral superfields. Because of this, the superpotential is not renormalized to any order in perturbation theory. This fact is known as a nonrenormalization theorem. Of course, the more supersymmetry a theory as, the more nonrenormalization theorems apply.

A nice related trick if I want to write an effective theory below some scale is to think of the UV couplings as the lowest components of background chiral superfields. Then, the low energy effective superpotential is constrained by holomorphy in the UV coupling constants. The use of holomorphy to obtain exact superpotentials was first exploited in [5]—for a nice review and more references, see [6].

[3] *n.b.* that θ in (2.10) is not the same as the superspace coordinate in (2.9).

2.2.2 NSVZ β-function

Recall that the one-loop renormalization of the gauge coupling g in a general Lagrangian field theory with gauge group G is

$$\beta(g) \equiv \mu \frac{dg(\mu)}{d\mu} = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C(\text{adj}) - \frac{2}{3}\sum_f C(r_f) - \frac{1}{3}\sum_b C(r_b)\right]. \tag{2.11}$$

Here r_f is the gauge group representation of the fermions, and r_b is the representation of the complex scalar bosons. The quantity in parenthesis is the one-loop coefficient, which we will refer to as b_0. $C(r)$ is the quadratic Casimir in the representation r, which satisfies

$$\text{Tr}_r T^a T^b = C(r)\delta^{ab}. \tag{2.12}$$

$C(\text{adj})$ is equal to the dual Coxeter number of the group; e.g. for $G = SU(N)$, $C(\text{adj}) = N$.

For a theory with $\mathcal{N} = 1$ supersymmetry, the vector multiplet contributes an additional $-2/3C(\text{adj})$ to the one loop β-function, and a chiral multiplet in a representation r_Φ has one Weyl fermion and one complex boson. Then, the RHS of (2.11) reduces to

$$\beta(g)_{\mathcal{N}=1} = -\frac{g^3}{(4\pi)^2}\left[3C(\text{adj}) - C(R_\Phi)\right]. \tag{2.13}$$

Since τ in (2.10) is a holomorphic quantity, its running under the renormalization group must preserve holomorphy. Then we have that the one-loop running coupling is

$$2\pi i \frac{d\tau}{d\ln\mu} = -b_0 = \frac{16\pi^2}{g^3}\frac{dg}{d\ln\mu} \quad \Rightarrow \quad \tau_{1-\text{loop}} = \frac{b_0}{2\pi i}\ln\left(\frac{\Lambda}{\mu}\right) \tag{2.14}$$

with Λ the complex dynamically generated holomorphic scale of the theory, and b_0 the one-loop β-function coefficient. Because this must be holomorphic, the β-function for τ is one-loop exact—it is only corrected nonperturbatively by n-instanton corrections. In particular, the combination

$$\Lambda^{b_0} = \mu^{b_0} e^{2\pi i \tau(\mu)} \tag{2.15}$$

is not corrected at any order in perturbation theory.

The exact NSVZ (for Novikov, Shifman, Vainshtein, and Zakharov) β-function [7] is given as

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{1}{1 - \frac{g^2 C(\text{adj})}{8\pi^2}}\right)\left(3C(\text{adj}) - \sum_j C(r_j)(1-\gamma_j)\right). \tag{2.16}$$

γ_j is the anomalous dimension of a matter field in a representation r_j of the gauge group. Importantly, the one- and two-loop β-function coefficients are scheme independent.

2.2.3 Moduli space of vacua

The scalar potential V in a supersymmetric theory takes the form

$$V = |F|^2 + \frac{1}{2} D^a D^a, \tag{2.17}$$

where

$$F = \frac{\partial W}{\partial \Phi}. \tag{2.18}$$

A supersymmetric vacuum is a zero of the scalar potential V, and vice versa; a zero of the scalar potential is also a supersymmetric vacuum. Therefore, supersymmetric vacua are the set of scalar field vacuum expectation values which simultaneously solve the F- and D-terms. Note that all expectation values of Φ for which $\partial W / \partial \Phi = 0$ correspond to supersymmetric, global minima of the potential.

Classical supersymmetric gauge theories often have a classical moduli space of degenerate vacua. The classical moduli space of a theory is given by the space of all scalar vacuum expectation values satisfying the D-term equations, modulo gauge equivalence and the classical F-terms.

The moduli space can always be given a gauge-invariant description in terms of the space of expectation values of gauge-invariant polynomials X_r in the fields, subject to any classical relations. This is because setting the potential to zero and modding out by the gauge group is equivalent to modding out by the complexified gauge group—holomorphy of the superpotential promotes a global symmetry group of the theory to a complexified symmetry group of the superpotential.[4] The gauge invariant polynomials correspond to matter fields left massless after the Higgs mechanism, and are classical moduli, $W(X_r) = 0$ [8]. Note that vacua with different expectation values of the fields are physically inequivalent; in particular, the masses of the vector bosons depend on the $\langle X_r \rangle$. The classical degeneracy can be lifted in the quantum theory by a

[4] Although, note that the Kähler potential is only invariant under the real symmetry group.

dynamically generated effective superpotential, $W_{\text{eff}}(X_r)$.

2.2.4 A short introduction to anomalies in QFT

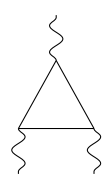
Classically, invariance under a continuous global symmetry group G implies the existence of conserved currents. If the symmetry is anomalous, then there are quantum corrections that make the currents no longer conserved. Then, the quantum effective action varies as

$$\delta S_{\text{eff}} = \int \lambda^a D^\mu j^a_\mu. \tag{2.19}$$

When G is a gauge symmetry, the anomaly indicates a fundamental inconsistency in the theory. Such an anomaly is often called an ABJ (for Adler-Bell-Jackiw) anomaly. For G a global symmetry, anomalies do not indicate any inconsistency, but rather often have interesting physical consequences. These are called *'t Hooft anomalies*.

The anomaly is related to an $(n+1)$-gon diagram with external insertions of the symmetry current. A well-known example is the chiral anomaly in 4d $SU(N)$ gauge theory. In this case, $\delta A_\mu = D_\mu \lambda$, and the nonconservation of the current j^μ is evident in the $(n+1)$-gon Feynman diagram (a triangle with three gauge currents) proportional to $\text{Tr}F^{n+1} = \text{Tr}F^3$:

$d = 4$:



$\propto \text{Tr}F^3$

One interesting physical consequence of 't Hooft anomalies is 't Hooft anomaly matching [9]. The argument due to 't Hooft goes as follows: Consider an asymptotically free, anomaly-free gauge theory with G global symmetry and chiral fermions. In the asymptotically free regime compute the triangle anomaly for three G currents. Assuming this is nonzero, denote this by A^{UV}.

Now gauge G, and add some massless gauge-singlet spectator fields with only G gauge couplings such that their G anomaly cancels A^{UV}: $A^S = -A^{UV}$. Since we've assumed that the original gauge theory confines, we can study the IR effective theory of the massless excitations below the strong coupling scale. At an IR scale less than the scale of strong interactions, if G is not spontaneously broken (so that we can assume there are no massless Nambu-Goldstone bosons) its anomaly must still vanish. This is due to the fact that we started with a consistent theory to begin with, by assumption. Therefore, the G anomaly at this scale must cancel $A^{IR} = -A^{S}$. The IR anomaly corresponds to a set of massless bound states which have the same anomaly as the original fields. Therefore, $A^{IR} = A^{UV}$. Now we can take the limit that the gauge coupling goes to zero to decouple the spectators. The result that $A^{IR} = A^{UV}$ still holds, and so is true for the original theory with G global symmetry and no spectators.

For a nice review of anomalies in gauge theories, see [10, 11]. We will have quite a bit more to say about anomalies in Chapter 7.

2.3 Superconformal Field Theories

For a supersymmetric theory with conformal symmetry, the supersymmetry algebra is extended to the superconformal algebra. Importantly, the superconformal algebra includes the R-symmetry as a bosonic subalgebra.

2.3.1 A word on conformal symmetry

To orient ourselves, we first discuss some aspects of conformal field theories (without supersymmetry) in various dimensions. The conformal algebra in d dimensions is $\mathfrak{so}(d,2)$. It is generated by Lorentz rotations / boosts, translations, special conformal transformations, and dilatations. Local operators in a unitary CFT must organize into unitary irreducible representations, commonly called conformal multiplets, of the conformal algebra. States are labeled by their scaling dimension Δ and their $SO(d)$ weights. The structure of the multiplet is completely labeled by the conformal primary operator, corresponding to the lowest weight state.

Unitarity require that all states have positive norm, which yields bounds on allowed CFT representations. Unitarity bounds take the form $\Delta \geq f(j_i)$ for j_i the $\lfloor d/2 \rfloor$ weights of $SO(d)$, and where (Δ, j_i) are the quantum numbers of the primary [12, 13]. When the inequality is saturated, some states—called null states—have zero norm, such that the multiplet is "short". For example, a free scalar field is at the bottom of a short multiplet. In that case, Δ saturates an inequality, with

$$\Delta(\mathscr{O}) = (d-2)/2, \qquad \mathscr{O} = \text{gauge-invariant, spin 0}. \tag{2.20}$$

In $d = 4$, $SO(4) \simeq SU(2) \times SU(2)$, and so representations are labeled by two half-integer spins j_1, j_2. In this case, that the unitarity bounds are given by (schematically)

$$\Delta \geq f(j_1) + f(j_2). \tag{2.21}$$

See [12] for the derivation and description of these bounds.

2.3.2 A tour of SCFTs in various dimensions

With the addition of supersymmetry, conformal symmetry is enhanced to superconformal symmetry. Superconformal algebras exist in dimensions $d \leq 6$ [14]. The supersymmetry generators transform as spinors of $SO(d,2)$. Superconformal algebras with $N_Q > 16$ supercharges in $d = 4, 6$ do not admit a stress tensor multiplet, and so for unitary SCFTs we restrict to $N_Q \leq 16$ in these dimensions. In $d = 3$ SCFTs with $N_Q > 16$ exist, but are necessarily free [15]. The case $d = 5$ is special because there is a unique superconformal algebra, $\mathscr{N} = 1$ with $N_Q = 8$.

The bosonic subalgebra of the superconformal algebra in d dimensions is $\mathfrak{so}(d,2) \times \mathfrak{R}$, where $\mathfrak{R}$ is the R-symmetry algebra. We are interested in representations that are unitary irreducible representations of $\mathfrak{so}(d) \times \mathfrak{so}(2) \times \mathfrak{R}$, which is the maximal compact subalgebra. These are completely specified by the lowest (or depending on your convention, highest) weights. As in the non-supersymmetric case, each unitary irrep of the superconformal group contains a unique operator of lowest scaling dimension, known as a superconformal primary, and the multiplet is completely specified by the quantum numbers of the primaries. Superconformal multiplets for $d > 2$ are comprehensively enumerated in [15].

2.3.3 Facts about fixed points

Here, we review a myriad of useful facts about SCFTs which will come up repeatedly throughout the rest of this dissertation.

Conformal field theories in even spacetime dimensions have Weyl anomalies. The conformal anomaly of the trace of the stress tensor $T_{\mu\nu}$ on a curved background is given by (schematically)

$$\langle T^{\mu}_{\mu} \rangle \sim a(\text{Euler}) + \sum_i c_i I_i. \tag{2.22}$$

Here, (Euler) refers to the d-dimensional Euler density, and I_i the local Weyl invariants. The dimensionless coefficients a, c_i are known as the central charges of the CFT. In 2d there are no I_i and a is commonly called c, which corresponds to the Virasoro central charge; in four dimensions there is one I_1; and in six dimensions there are two.

As we review below, in superconformal theories the a central charge is related to the 't Hooft anomalies for the superconformal R-symmetry. This follows from the fact that $T_{\mu\nu}$ is in the same multiplet as the R-symmetry current.

4d $\mathcal{N} = 1$

The conformal anomaly of the trace of the four-dimensional energy momentum tensor on a curved background in four dimensions is

$$\langle T^{\mu}_{\mu} \rangle = -\frac{1}{16\pi^2}\left[a(\text{Euler}) - c(\text{Weyl})^2\right], \tag{2.23}$$

where

$$(\text{Weyl})^2 = (R_{\mu\nu\rho\sigma})^2 - 2(R_{\mu\nu})^2 + \frac{1}{3}R^2, \quad (\text{Euler}) = (R_{\mu\nu\rho\sigma})^2 - 4(R_{\mu\nu})^2 + R^2. \tag{2.24}$$

For an $\mathcal{N} = 1$ superconformal theory, the superconformal algebra places the R-symmetry current in the same multiplet as the stress tensor and supersymmetry currents. From this it follows that the central charges a and c are related to the 't Hooft anomalies of the superconformal $U(1)_R$ symmetry as [16]

$$a = \frac{3}{32}\left(3\mathrm{Tr}R^3_{\mathcal{N}=1} - \mathrm{Tr}R_{\mathcal{N}=1}\right), \quad c = \frac{1}{32}\left(9\mathrm{Tr}R^3_{\mathcal{N}=1} - 5\mathrm{Tr}R_{\mathcal{N}=1}\right). \tag{2.25}$$

Further, all 3-point functions among elements of this supermultiplet are determined by a and c.

The anomaly-free condition for the R-symmetry is closely tied with the condition for a fixed point. In particular, the requirement that the $U(1)_R$ symmetry be free from ABJ anomalies is precisely the condition that the NSVZ exact β-function vanish. Explicitly, the R-symmetry current R_μ (the lowest component of the supercurrent superfield containing the stress tensor and supersymmetry currents) satisfies

$$\partial_\mu R^\mu = \frac{1}{48\pi^3}\left(3C(\mathrm{adj}) - \sum_j C(r_j)(1-\gamma_j)\right) F^a_{\mu\nu}\widetilde{F}^a_{\mu\nu}. \tag{2.26}$$

Comparing with (2.16), the expression in parenthesis is precisely the numerator of the NSVZ β-function.

The 't Hooft anomalies for the superconformal $U(1)_R$ symmetry for a vector and chiral multiplet are

$$V: \quad \mathrm{Tr}R_{\mathcal{N}=1} = \mathrm{Tr}R^3_{\mathcal{N}=1} = |G| \tag{2.27}$$

$$\Phi: \quad \mathrm{Tr}R_{\mathcal{N}=1} = (R(Q)-1)|r|, \quad \mathrm{Tr}R^3_{\mathcal{N}=1} = (R(Q)-1)^3|r| \tag{2.28}$$

where r is the representation of Φ. These are easily computed by adding the contributions of the fermions in Tables 2.1 and 2.2. Then, for a theory with $n_v^{(1)}$ $\mathcal{N}=1$ vector multiplets and $n_\phi^{(1)}$ $\mathcal{N}=1$ chiral multiplets, the a central charge is

$$a = \frac{3}{32}\left[2n_v^{(1)} + n_\phi^{(1)}(R(Q)-1)(3(R(Q)-1)^2-1)\right]. \tag{2.29}$$

If the theory has a flavor symmetry G with generators T^a, the flavor central charge k_G is defined

$$k_G\delta^{ab} = -3\mathrm{Tr}R_{\mathcal{N}=1}T^aT^b. \tag{2.30}$$

For scalar chiral primary operators $\mathcal{O}$, the R-charge and dimension are proportional to one another:

$$\Delta(\mathcal{O}) = \frac{3}{2}R(\mathcal{O}) \geq 1. \tag{2.31}$$

The inequality comes from the unitarity bound $\Delta_{\mathcal{O}} \geq (d-2)/2$ given in (2.20).

4d $\mathcal{N}=2$

An $\mathcal{N}=2$ SCFT has an R-symmetry $U(1)_R \times SU(2)_R$, with generators $R_{\mathcal{N}=2}$ and I^a $(a=1,2,3)$ respectively. We use a basis for the Cartan subalgebra of the R-symmetry labeled by $(R_{\mathcal{N}=2}, I^3)$. The R-charge assignment for free $\mathcal{N}=2$ vector multiplets and hypermultiplets is given in Tables 2.3 and 2.4, which we repeat in a convenient form in Table 2.6.

Table 2.6: R-charge assignments for $\mathcal{N}=2$ multiplets.

$R_{\mathcal{N}=2} \setminus I^3$	$\frac{1}{2}$	0	$-\frac{1}{2}$
0		A_μ	
1	λ_α		λ'_α
2		ϕ	

$R_{\mathcal{N}=2} \setminus I^3$	$\frac{1}{2}$	0	$-\frac{1}{2}$
-1		ψ_α	
0	Q		$\widetilde{Q}^\dagger$
1		$\widetilde{\psi}^\dagger_{\dot\alpha}$	

With this charge assignment, the nonzero anomaly coefficients for an $\mathcal{N}=2$ vector multiplet and hypermultiplet are

$$\begin{aligned}
\mathcal{N}=2 \text{ vector}: \quad & \mathrm{Tr}R_{\mathcal{N}=2} = \mathrm{Tr}R^3_{\mathcal{N}=2} = (1+1)|G| = 2|G|, \\
& \mathrm{Tr}R_{\mathcal{N}=2}I_3^2 = (1(1/2)^2 + 1(-1/2)^2)|G| = 1/2|G|, \\
\mathcal{N}=2 \text{ hyper}: \quad & \mathrm{Tr}R_{\mathcal{N}=2} = \mathrm{Tr}R^3_{\mathcal{N}=2} = (-1-1)|r| = -2|r|, \\
& \mathrm{Tr}R_{\mathcal{N}=2}I_3^2 = (-1(0)^2 - 1(0)^2)|r| = 0.
\end{aligned} \tag{2.32}$$

(Note that due to the dagger, $\widetilde{\psi}^\dagger_{\dot\alpha}$ contributes -1 rather than 1). Then, for a theory with n_v vectors and n_h hypers we have

$$\mathrm{Tr}R_{\mathcal{N}=2} = \mathrm{Tr}R^3_{\mathcal{N}=2} = 2(n_v - n_h), \qquad \mathrm{Tr}R_{\mathcal{N}=2}I_3^2 = \frac{n_v}{2}. \tag{2.33}$$

The central charges a and c are related to the anomaly coefficients as [17]

$$\mathrm{Tr}R^3_{\mathcal{N}=2} = \mathrm{Tr}R_{\mathcal{N}=2} = 48(a-c), \qquad \mathrm{Tr}R_{\mathcal{N}=2}I^aI^b = 2\delta^{ab}(2a-c). \tag{2.34}$$

The flavor central charge k_G for a global symmetry G with generators T^a is

$$k_G\delta^{ab} = -2\mathrm{Tr}R_{\mathcal{N}=2}T^aT^b. \tag{2.35}$$

Using n_v and n_h to represent the number of free vector multiplets and free hypermultiplets, the central charges of an $\mathcal{N}=2$ superconformal theory can be written

$$a = \frac{1}{24}\left(n_h + 5n_v\right), \qquad c = \frac{1}{12}\left(n_h + 2n_v\right). \tag{2.36}$$

As we discussed previously, we can fix an $\mathcal{N}=1$ subalgebra in the $\mathcal{N}=2$ algebra, such that the $\mathcal{N}=1$ R-symmetry is given by (2.3). With this choice, the linear combination $J = R_{\mathcal{N}=2} - 2I_3$ commutes with this $\mathcal{N}=1$ subalgebra, and thus is a flavor symmetry from the $\mathcal{N}=1$ point of view. (2.3) is the unique $\mathcal{N}=1$ R-symmetry that has the properties of a superconformal $U(1)_R$ when the theory has enhanced $\mathcal{N}=2$ supersymmetry.

For scalar chiral primary operators $\mathcal{O}$, using (2.3) and (2.31) (and $I^3=0$) we can relate the dimension and R-charge of $\mathcal{O}$ as

$$\Delta(\mathcal{O}) = \frac{1}{2}R_{\mathcal{N}=2} \geq 1. \tag{2.37}$$

The inequality is saturated only for free fields.

4d $\mathcal{N}=4$

An $\mathcal{N}=4$ SCFT has R-symmetry $SU(4)_R \simeq SO(6)_R$. The 't Hooft anomalies for an $\mathcal{N}=4$ massless vector multiplet can be written

$$\mathcal{N}=4 \text{ vector}: \quad \mathrm{Tr}R_{\mathcal{N}=1} = 0, \qquad \mathrm{Tr}R^3_{\mathcal{N}=1} = 8/9(N^2-1), \tag{2.38}$$

in terms of the $\mathcal{N}=1$ subalgebra (2.3). The N^2-1 factor comes from the dimension of the adjoint. Then, we see that

$$a = c = \frac{N^2-1}{4}. \tag{2.39}$$

2.3.4 The a-theorem, and a-maximization

In two dimensions the RG flow is a gradient flow, meaning that the a central charge satisfies a gradient condition along the flow. This is Zamolodchikov's a-theorem [18], usually called the c-theorem since in 2d there is only one central charge.

In four dimensions, the a-theorem states that the endpoints of all unitary RG flows must satisfy

$$a_{UV} > a_{IR}. \tag{2.40}$$

The four-dimensional a-theorem was conjectured in [19], analyzed in [20], and proven in [21, 22]. Note that in 4d $c > 0$, since it appears as the coefficient of a 2-point function, but that c does not satisfy a similar a-theorem. Also, note that $a \geq 0$ in both 2d and 4d, with $a = 0$ if and only if the theory has no local degrees of freedom [18, 23].

The intuition of there being a quantity that monotonically decreases along nontrivial RG flows comes from the idea that along a flow, relevant deformations lift some of the massless degrees of freedom. Indeed, for a free conformal theory, a and c are given by the free field content, and therefore decrease as more fields are integrated out at lower energies.

The a central charge of a 4d $\mathcal{N} = 1$ SCFT is completely determined by the $U(1)_R$ 't Hooft anomalies as in (2.25). In cases where the $U(1)_R$ can mix with additional $U(1)$ global flavor symmetries, the exact superconformal R-symmetry is determined by a-maximization [24], which requires locally maximizing (2.25) over all possible $U(1)_R$ symmetries. In particular, one can parameterize the most general possible R-symmetry as

$$R_{\text{trial}} = R_0 + \sum_i \varepsilon_i F_i \tag{2.41}$$

for F_i the generators of other $U(1)$'s in the theory. The result of a-maximization is that the values of the ε_i such that R_{trial} at the fixed point is the correct $U(1)_R$ are those that locally maximize

$$a_{\text{trial}} = \frac{3}{32}\left(3\text{Tr}R_{\text{trial}}^3 - \text{Tr}R_{\text{trial}}\right). \tag{2.42}$$

This follows from showing that [24]

$$\frac{\partial a_{\text{trial}}}{\partial \varepsilon_i} = \frac{3}{32}(9\text{Tr}R_{\text{trial}}^2 F_i - \text{Tr}F_i) = 0, \qquad \frac{\partial^2 a_{\text{trial}}}{\partial \varepsilon_i \varepsilon_j} = \frac{27}{16}\text{Tr}R_{\text{trial}} F_i F_j < 0. \tag{2.43}$$

Note that any flavor symmetry that satisfies $\text{Tr}F_i = 0$ (such as non-Abelian flavor symmetries) does not mix with the superconformal $U(1)_R$, since by (2.43) R commutes with such an F_i.

Cases with accidental symmetries or irrelevant interactions require special care: one then maximizes (2.25) over R-symmetries that are not obvious from the original description.

One situation where such enhanced symmetries are evident is when a gauge-invariant operator saturates, or seemingly violates, an SCFT unitarity bound. For instance, for scalar chiral primary operators $\mathscr{O}$ we must satisfy (2.31). Apparent violations instead actually saturate the inequality, with an accidental symmetry $U(1)_{\mathscr{O}}$ which only acts on the IR-free-field composite operator. Then, this accidental $U(1)$ mixes with the $U(1)_R$ in (2.41). See [25] for how a-maximization is modified in such cases.

2.3.5 Chiral ring

Chiral operators are operators that are annihilated by the supercharges of one chirality. A chiral superfield has a chiral operator as its lowest component. As we've already reviewed, in theories with four supercharges chiral primary operators have dimension proportional to their $U(1)_R$ charge, which is hence additive. The product of two chiral operators is again a chiral operator. Then, their OPEs have a ring structure, known as the *chiral ring*. In the ring, chiral operators are considered modulo operators of the form $\{Q^\dagger_{\dot\alpha}, \dots\}$—i.e. two chiral operators are considered equivalent if they differ by a $Q^\dagger_{\dot\alpha}$ exact term.

In terms of a microscopic Lagrangian description, the chiral ring consists of gauge-invariant composites formed from the microscopic chiral superfields. Superpotentials lead to chiral ring relations, since $\partial_\Phi W$ is not a primary, and is thus set to zero in the ring. Ring relations can also come from the finiteness of the chiral operators as matrices in a representation of the gauge group.

2.3.6 4d $\mathscr{N} = 1$ superconformal index

The Witten index [26] is defined as $\mathrm{Tr}(-1)^F$, where the Witten operator $(-1)^F$ distinguishes bosons from fermions: $(-1)^F|\text{boson}\rangle = +1|\text{boson}\rangle$, and $(-1)^F|\text{fermion}\rangle = -1|\text{fermion}\rangle$.

The idea of an index for a superconformal theory is essentially as a Witten index in radial quantization. Consider a supercharge Q with $\{Q, Q^\dagger\} = 2\Delta$ for Δ some conserved charge. Then, one can generally define an index

$$I[\mu_i] = \mathrm{Tr}(-1)^F e^{-\beta\Delta} e^{-\mu_i q_i} \tag{2.44}$$

where the q_i are charges that commute with the supercharges $Q, Q^\dagger$. The trace is over states of the theory quantized on $S^3 \times \mathbb{R}$. The necessity of the compact manifold S^3 is due to the fact that one can rarely compute the Witten index in flat space, since supersymmetric theories have a moduli space of vacua. The index (2.44) will only receive contributions from $\Delta = 0$ since states with $\Delta > 0$ come in boson-fermion pairs. As an alternative formulation, one can define a superconformal index as a supersymmetric partition function on $S^3 \times S^1$.

The superconformal index for an $\mathcal{N} = 1$ theory was first defined by Römelsberger, and can be written [27, 28]

$$I(p,q) = \mathrm{Tr}(-1)^F p^{j_1+j_2+R/2} q^{j_2-j_1+R/2} \tag{2.45}$$

where (j_1, j_2) are the spins of the Lorentz group $SO(4) \simeq SU(2) \times SU(2)$, and R is the $U(1)_R$ charge. When the theory has a global symmetry with Cartan generator f, we include a fugacity y such that there is a term y^f included in the product.

Equivalently, it is useful to use

$$p = tx, \qquad q = \frac{t}{x}, \tag{2.46}$$

to rewrite

$$I(t,x) = \mathrm{Tr}(-1)^F t^{R+2j_2} x^{2j_1}. \tag{2.47}$$

Note that this convention differs from some conventions—including the one used in Chapter 4 of this thesis—by a rescaling $t \to t^3$. As an aside, one can similarly define the $\mathcal{N} = 2$ index as

$$I(p,q,t)_{\mathcal{N}=2} = \mathrm{Tr}(-1)^F p^{j_1+j_2+R_{\mathcal{N}=2}} q^{j_2-j_1+R_{\mathcal{N}=2}} t^{I^3-R_{\mathcal{N}=2}} \tag{2.48}$$

with I^3 and $R_{\mathcal{N}=2}$ the Cartan generators. For the rest of this subsection we will continue our focus on the $\mathcal{N} = 1$ case.

The index can be determined first on single particle states as

$$i(t,x,h,g) = \frac{2t^2 - t(x+x^{-1})}{(1-tx)(1-tx^{-1})}\chi_{\text{adj}}(g) + \sum_i \frac{t^{R_i}\chi_{r_{F,i}}(h)\chi_{r_{G,i}}(g) - t^{2-R_i}\chi_{\bar{r}_{F,i}}(h)\chi_{\bar{r}_{G,i}}(g)}{(1-tx)(1-tx^{-1})}. \quad (2.49)$$

Here G is the gauge group, and F is a flavor symmetry group. The first term represents the contribution for gauge fields from the vector multiplet, and the second set of terms sums the contribution of chiral matter fields in representations r_i of the corresponding groups. In the sum, the first term of the numerator represents the contribution of a chiral scalar with R-charge R_i, while the second represents the contribution of the fermionic descendent of its anti-chiral partner. This expression depends on the symmetry group elements $g \in G$, and $h \in F$. From (2.49), we can determine the index for all gauge singlet operators via the plethystic exponential

$$I(t,x,h) = \int_G d\mu(g) \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i(t^n, x^n, h^n, g^n)\right) \quad (2.50)$$

where $d\mu(g)$ is the G-invariant measure. This formulation of the $\mathcal{N}=1$ index is nicely summarized in [29].

The index has an expansion of the form

$$I(t,x,h) = \sum_i n_i \frac{t^{\alpha_i}\chi_{j_i}(x)\chi_{r_{F,i}}(h)}{(1-tx)(1-tx^{-1})}. \quad (2.51)$$

Here the n_i are integer coefficients, and the χ_j are $SU(2)$ characters for the spin j representation. For example, for low spin j we have

$$\chi_1(z) = 1, \qquad \chi_2(z) = z + z^{-1}, \qquad \chi_3(z) = 1 + z^2 + z^{-2}. \quad (2.52)$$

To see the operator spectrum from the index, it is convenient to consider the reduced index (called the corrected index in [30])

$$I_r = (1-tx)(1-tx^{-1})(I(t,x,h) - 1). \quad (2.53)$$

This is an infinite series in t from which one can read off the quantum numbers of operators in the theory, up to the ambiguity of recombination of short multiplets into long multiplets.

Part of the utility of the index comes from the fact that the superconformal index is constant under continuous variations of the theory that preserve superconformal invariance. For example, the index is invariant under renormalization group flow. One can compute the index in the UV, RG flow to the IR, and if superconformal symmetry is preserved then compute the infrared index, taking into account the fact that the $U(1)_R$ charge preserved at the IR fixed point might need modification. One then has a recipe to evaluate the superconformal index at a low energy fixed point given the index in the UV.

Chapter 3

Deformations of $W_{A,D,E}$ SCFTs

We discuss aspects of theories with superpotentials given by Arnold's A, D, E singularities, particularly the novelties that arise when the fields are matrices. We focus on 4d $\mathcal{N} = 1$ variants of susy QCD, with $U(N_c)$ or $SU(N_c)$ gauge group, N_f fundamental flavors, and adjoint matter fields X and Y appearing in $W_{A,D,E}(X,Y)$ superpotentials. Many of our considerations also apply in other possible contexts for matrix-variable $W_{A,D,E}$. The 4d $W_{A,D,E}$ SQCD-type theories RG flow to superconformal field theories, and there are proposed duals in the literature for the W_{A_k}, W_{D_k}, and W_{E_7} cases. As we review, the $W_{D_{\text{even}}}$ and W_{E_7} duals rely on a conjectural, quantum truncation of the chiral ring. We explore these issues by considering various deformations of the $W_{A,D,E}$ superpotentials, and the resulting RG flows and IR theories. Rather than finding supporting evidence for the quantum truncation and $W_{D_{\text{even}}}$ and W_{E_7} duals, we note some challenging evidence to the contrary.

3.1 Introduction

The simply-laced Lie groups, A_k, D_k, and E_6, E_7, and E_8 ("ADE") relate to, and classify, far-flung things in physical mathematics. The Platonic solids are classified by the discrete subgroups $\Gamma_G \subset SU(2)$—cyclic, dihedral, tetrahedral, octahedral, and icosahedral—which connect to the ADE Lie algebras via the McKay correspondence[1]. Another connection is in Arnold's simple surface singularities, which follow an ADE classification [31]:

$$W_{A_k} = X^{k+1}, \qquad W_{D_{k+2}} = X^{k+1} + XY^2, \tag{3.1}$$

$$W_{E_6} = Y^3 + X^4, \quad W_{E_7} = Y^3 + YX^3, \quad W_{E_8} = Y^3 + X^5. \tag{3.2}$$

These have resolutions, via lower order deformations, associated with the corresponding ADE Cartan, with the adjacency of the singularities that of the ADE Cartan matrix.

In two dimensions, the ADE groups arise in the classification of minimal models and their partition functions [32]. The 2d $\mathcal{N} = 2$ minimal models with $\widehat{c} < 1$ are given by Landau-Ginzburg theories with the $W_{G=A,D,E}$ superpotentials (3.2) [33, 34, 35]. The chiral ring of the W_G 2d $\mathcal{N} = 2$ SCFT is related to the ADE group's Cartan, with $r_G = \mathrm{rank}(G)$ chiral primary operators. Deforming the theory by adding these chiral ring elements to the superpotential, $W \to W + \Delta W$, the deformation parameters can be associated with expectation values in the adjoint of G. The deformation leads to multiple vacua, where the ADE group breaks into a subgroup. This breaking pattern is in accord with adjoint Higgsing, preserving the rank r_G and corresponding to deleting a node from the extended Dynkin diagram, e.g.

$$D_{k_1+k_2+2} \to D_{k_1+2} + A_{k_2}, \qquad E_7 \to E_6 + A_1, \qquad E_6 \to D_5 + A_1. \tag{3.3}$$

The generic deformation gives $G \to r_G\, A_1$, giving $\mathrm{Tr}(-1)^F = r_G$ susy vacua. The solitons of the integrable ΔW deformations also exhibit the ADE structure, e.g. [36].

A related connection with ADE groups is via local Calabi-Yau geometries: when the defining hypersurface has a singularity (3.2), there are (collapsed) cycles corresponding to the ADE Dynkin diagram nodes, with intersections given by the group's Cartan matrix. String theory on these backgrounds can yield the corresponding ADE gauge groups in spacetime [37]. In this context, the geometric resolutions of the local singularities corresponding to ΔW deformations lead to adjoint Higgsing of the corresponding group.

[1]The irreducible representations R_i of Γ_G correspond to the nodes of the extended Dynkin diagram for G, with $R_F = \sum_j a_{ij} R_j$ for R_F the fundamental of $SU(2)$ and $C_{ij} = 2\delta_{ij} - a_{ij}$ the ADE Cartan matrix.

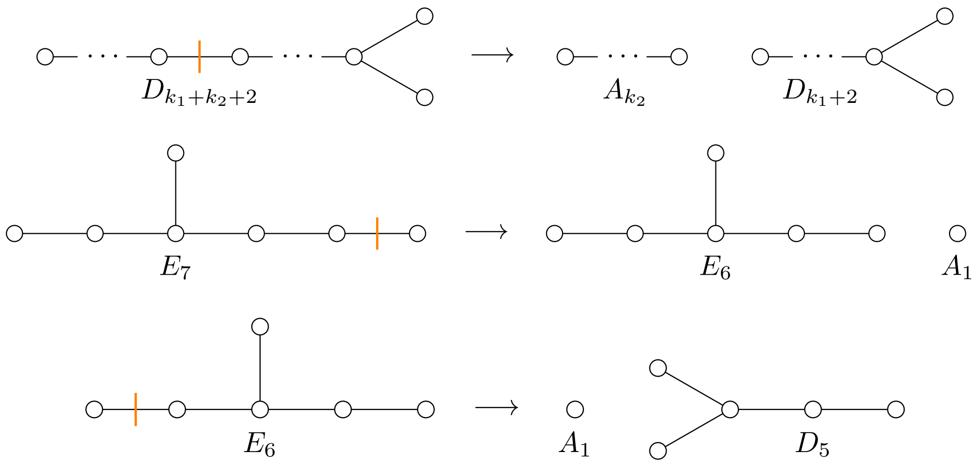


Figure 3.1: Deforming the $W_{A,D,E}$ 2d $\mathcal{N}=2$ SCFTs corresponds to adjoint Higgsing of the ADE group, hence cutting the Dynkin diagrams, as illustrated here for the flows in (3.3). This gives the vacua associated with 1d representations of the F-terms.

3.1.1 The chiral ring of $W_{A,D,E}(X,Y)$ for matrix fields X and Y

We are interested in an ADE classification that arises in the context of a family of 4d $\mathcal{N}=1$ SCFTs [38]. Before delving into specifics, we highlight a difference in comparison with (3.2): now X and Y are matrices, with

$$W_{A_k} = \mathrm{Tr}(X^{k+1}+Y^2), \qquad W_{D_{k+2}} = \mathrm{Tr}(X^{k+1}+XY^2), \tag{3.4}$$

$$W_{E_6} = \mathrm{Tr}(Y^3+X^4), \quad W_{E_7} = \mathrm{Tr}(Y^3+YX^3), \quad W_{E_8} = \mathrm{Tr}(Y^3+X^5). \tag{3.5}$$

The fact that matrices allow non-zero, nilpotent solutions to the equations of motion, and can have $[X,Y]\neq 0$, makes for important differences—even classically.

Recall that in theories with four supercharges, chiral primary operators have dimension proportional to their $U(1)_R$ charge, which is hence additive, and their OPEs yield the chiral ring. In terms of a microscopic, Lagrangian description, the chiral ring consists of gauge-invariant composites formed from the microscopic chiral superfields. Superpotentials lead to chiral ring relations, since $\partial_X W \sim \overline{Q}^2 \partial_X K$ is not a primary, and is thus set to zero in the ring; for instance the LG theories (3.2) then have $r_{G=A,D,E}$ elements.

We are here interested in aspects of the chiral rings for theories with matrix X and Y

superpotentials (3.5), and their $W \to W + \Delta W$ deformations. Our focus is on the application to 4d $\mathcal{N} = 1$ QFTs and renormalization group (RG) flows, but much of the analysis also applies to other possible contexts—for instance, in 2d or 3d— where one could also consider theories with the superpotentials (3.5) with matrix fields.

If the fields X and Y are $N_c \times N_c$ matrices, the superpotentials (3.5) have a $GL(N_c, C)$ symmetry under which X and Y transform in the adjoint representation: $X \to M^{-1}XM$, $Y \to M^{-1}YM$ for $M \in GL(N_c, C)$. Then, we can decide whether or not to gauge a subgroup of this symmetry, say $SU(N_c)$ or $U(N_c)$. If we do not gauge, (3.5) will leave unlifted a large space of flat directions. For instance, consider the matrix variable A_k superpotential in (3.5), whose F-term chiral ring relations, $\partial_X W = \partial_Y W = 0$, are given by

$$W_{A_k}: \qquad F_X \sim \partial_X W \sim X^k = 0, \qquad F_Y \sim \partial_Y W \sim Y = 0. \tag{3.6}$$

Y is massive and could be integrated out, setting $Y = 0$; we merely included it here to make the ADE cases in (3.5) more uniform. For $k = 1$ and any N_c, X is also massive, and there is a unique supersymmetric vacuum at $X = Y = 0$. For $k > 1$ and $N_c = 1$, (3.6) gives isolated vacua at $X = 0$, and resolving the singularity by lower order ΔW shows that there are $\mathrm{Tr}(-1)^F = r_G = k$ such vacua. For both $k > 1$ and $N_c > 1$, on the other hand, $X^k = 0$ has a non-compact moduli space of flat direction solutions with nilpotent X; for example, X could contain a block $v(\sigma_1 + i\sigma_2)$ for arbitrary complex v.

In our context, $SU(N_c)$ or $U(N_c)$ is gauged, and the nilpotent matrix solutions of (3.6) are lifted by the gauge D-term potential: supersymmetric vacua must have

$$V_D = 0: \quad [X, X^\dagger] + [Y, Y^\dagger] + \text{other matter field contributions} = 0. \tag{3.7}$$

The "other matter field contributions" are for example the contributions from N_f fundamentals and anti-fundamentals Q, $\widetilde{Q}$ in variants of SQCD, which we need not consider for the moment; i.e. we consider the theory at $Q = \widetilde{Q} = 0$. For the A_k case, (3.7) gives $[X, X^\dagger] = 0$, implying X and $X^\dagger$ can be simultaneously diagonalized; then nilpotent solutions are eliminated, and (3.6) implies that the vacua are all at $X = 0$.

The D and E cases, with $N_c > 1$, have more matrix-related novelties since generally $[X,Y] \neq 0$. For the D-series, the F-terms in the undeformed case are

$$W_{D_{k+2}}: \qquad F_X \sim X^k + Y^2 = 0, \qquad F_Y \sim \{X,Y\} = 0. \tag{3.8}$$

The 1d representations are the same as in the $N_c = 1$ case, giving $r_{D_{k+2}} = k+2$ chiral ring elements. For matrices X and Y, the chiral-ring relations (3.8) lead to a qualitative difference between k odd and k even. For k odd, (3.8) imply that $Y^3 \sim YX^k \sim -YX^k = 0$, and thus there are $3k$ independent chiral ring elements formed from X and Y, given by

$$k \text{ odd}: \qquad \Theta_{\ell j} = X^{\ell-1}Y^{j-1}, \qquad \ell = 1, \ldots, k; \;\; j = 1,2,3. \tag{3.9}$$

For k even, $Y^{m\geq 3} \neq 0$ in the ring, so there are chiral ring elements with allowed values of j that do not truncate, i.e. they do not have a maximum value independent of N_c.

Likewise, for W_{E_6} the chiral ring relations

$$W_{E_6}: \qquad F_X \sim X^3 = 0, \qquad F_Y \sim Y^2 = 0, \tag{3.10}$$

allow for $r_{E_6} = 6$ chiral ring elements with 1d representations, $\{1, X, Y, X^2, XY, X^2Y\}$. For $N_c > 1$, one can form, for example, $\text{Tr}(XY)^\ell$ with arbitrary ℓ as independent chiral ring elements, so the ring does not truncate. Similarly, for W_{E_7}, the chiral ring relations

$$W_{E_7}: \qquad F_X \sim X^2Y + XYX + YX^2 = 0, \qquad F_Y \sim Y^2 + X^3 \sim 0, \tag{3.11}$$

lead to $r_{E_7} = 7$ chiral ring elements when $N_c = 1$, while for $N_c > 1$ the classical chiral ring is not truncated. For W_{E_8}, the chiral ring relations

$$W_{E_8}: \qquad F_X \sim X^4 = 0, \qquad F_Y \sim Y^2 = 0, \tag{3.12}$$

lead to $r_{E_8} = 8$ chiral ring elements for 1d representations ($X^{\ell-1}Y^{j-1}$ for $\ell = 1, \ldots, 4$ and $j = 1,2$), but the classical chiral ring does not truncate for matrix representations.

3.1.2 $W_{A,D,E}$ in 4d SQCD with fundamental plus adjoint matter

We consider ADE superpotentials in the context of 4d $\mathcal{N} = 1$ SCFTs, with gauge group $SU(N_c)$ or $U(N_c)$, X and Y adjoint chiral superfields, and N_f (anti)fundamental flavors Q (and

$\tilde{Q}$). The possible interacting SCFTs were classified in [38] as

$$W_{\widehat{O}} = 0, \qquad W_{\widehat{A}} = \mathrm{Tr}Y^2, \qquad W_{\widehat{D}} = \mathrm{Tr}XY^2, \qquad W_{\widehat{E}} = \mathrm{Tr}Y^3 \tag{3.13}$$

along with (3.5). The reappearance of Arnold's ADE classification in this context [38] was unexpected. Some interesting ideas and conjectures for a geometric explanation of the $W_{A,D,E}$ in this context appeared in [39], in connection with matrix models and the construction of [40]. We will not further explore these interesting ideas here.

The IR phase of the theory depends on N_f and N_c. It is convenient to consider these theories in the Veneziano limit of large N_c and N_f, with the ratio

$$x = N_c/N_f \tag{3.14}$$

held fixed; the IR phase then only depends on x. The $\widehat{O}$ theory is (or is not) asymptotically free for $x > 1$ (or for $x \leq 1$), and RG flows to an interacting (or free electric) theory. Larger x values means that the theory is more asymptotically free, and hence the original "electric" description is more strongly coupled in the IR. The asymptotically free theories are expected[2] to be in the interacting SCFT conformal phase for all $N_f < 2N_c$ (i.e. $x > \frac{1}{2}$) for the $\widehat{A}$ cases, and for all $N_f < N_c$ (i.e. $x > 1$) for the $\widehat{O}$, $\widehat{D}$ and $\widehat{E}$ cases. For the $W_{A,D,E}$ theories (3.5), on the other hand, there are more possible IR phases.

In the W_{A_1} case, the adjoints are massive and can be integrated out. The resulting IR theory is SQCD, which has the duality [42], with "magnetic" gauge group $SU(N_f - N_c)$. The dual reveals the bottom of the conformal window, and the existence of the IR-free magnetic phase for $\frac{2}{3} \leq x \leq 1$; for $x > 1$, the theory generates a dynamical superpotential [43]. The $W_{A_{k>1}}$ theories were considered in [44, 45], where a duality was proposed and checked. Following [46, 47] we write the W_{A_k} duality in a way that will generalize to some cases:

$$\text{(some cases)} \quad W_G : \qquad SU(N_c) \leftrightarrow SU(\alpha_G N_f - N_c), \qquad \text{with} \quad \alpha_{A_k} = k. \tag{3.15}$$

Superpotential deformations of W_{A_k} were considered in [48], where the fact that $\alpha_{A_k} = k$ was

[2] This can be seen e.g. for the $\widehat{A}$ theories with $N_f > 0$ as in [41]: a superpotential deformation leads to $\mathscr{N} = 2$ SQCD, and all the mutually non-local, massless monopole and dyon points in the moduli space collapse to the origin in the original theory. This has no free-field interpretation.

shown to tie in with the fact that upon a generic ΔW deformation, Arnold's A_k singularity is resolved as

$$A_k \to kA_1, \tag{3.16}$$

since the low-energy theory in each of the k vacua has the $SU(n_i) \leftrightarrow SU(N_f - n_i)$ duality of [42]. The IR phases and relevance of the W_{A_k} theories were clarified in [25] using a-maximization [24], including accounting for accidental symmetries.

A duality of the form (3.15) for the case of two adjoint chiral superfields X and Y, with $W_{D_{k+2}}$ as in (3.5), was proposed in [49], with

$$\alpha_{D_{k+2}} = 3k. \tag{3.17}$$

The IR phases and relevance of the superpotential terms were clarified in [38], where it was also noted how the $\alpha_{D_{k+2}}$ value (3.17) can be understood / derived from ΔW deformations; this will be discussed much further, and clarified, in the present chapter.

More recently, a duality for the case of W_{E_7} was proposed in [46], with

$$\alpha_{E_7} = 30. \tag{3.18}$$

The value (3.18) was moreover shown in [47] to be compatible with the superconformal index in the Veneziano limit[3], and it was argued [46, 47] that the W_{E_6} and W_{E_8} theories cannot have duals of the simple form (3.15); it is not yet know if these theories have duals. A motivating goal of our work was to obtain some additional insight into the value (3.18), and its connection with the flows in Fig. 3.2.

3.1.3 $W_{A,D,E} + \Delta W$ RG flows

Possible flows between these fixed points are illustrated in Figure 3.2, taken from [38]. We here emphasize that this figure is somewhat incomplete: the ΔW superpotential deformations

[3] The exact matching of the electric and magnetic indices beyond this limit requires mathematical identities which have only been demonstrated explicitly for the W_{A_1} SQCD duality case [29, 50]; the needed identities are conjectural for the $A_{k>1}$, D_{k+2}, and E_7 dualities.

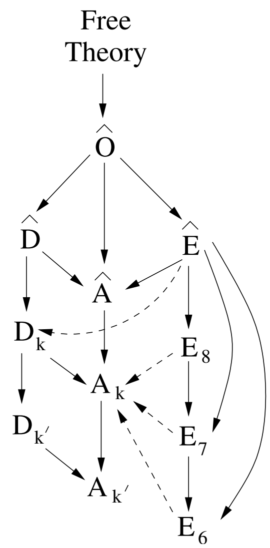


Figure 3.2: Flows among the fixed points of SQCD with two adjoints.

give additional vacua, with a richer IR structure than is indicated in the figure. Indeed, even the 1d ($N_c = 1$) representations of the chiral ring of the deformed $W_{A,D,E} + \Delta W$ superpotentials give rank $r_{G=A,D,E}$ vacua, as in the examples (3.3). The two-matrix D and E cases with $N_c > 1$ give additional vacua. Incidentally, much as in 2d, there are BPS solitons—here domain walls—interpolating between the vacua; we will not discuss them further here.

To illustrate the multiple vacua and possibility for additional vacua, consider $W_{D_{k+2}}$ with $U(N_c)$ gauge group for $N_c > 1$. The generic ΔW deformation gives[4] [51, 38]

$$D_{k+2} \to (k+2)A_1^{1d} + \left\lfloor \frac{1}{2}(k-1) \right\rfloor A_1^{2d}. \tag{3.19}$$

The $1d$ and $2d$ labels refer to the dimension of the representation of the (deformed) chiral ring. The higher-dimensional representations of the chiral ring are the new elements of the matrix-variable superpotentials. The gauge group is then broken as [38]

$$U(N_c) \to \prod_{i=1}^{k+2} U(n_i) \prod_{j=1}^{\lfloor \frac{1}{2}(k-1) \rfloor} U(n_j^{2d}) \qquad \text{with} \qquad \sum_{i=1}^{k+2} n_i + \sum_{j=1}^{\lfloor \frac{1}{2}(k-1) \rfloor} 2n_j^{2d} = N_c. \tag{3.20}$$

For k odd, the low-energy theory is SQCD for each factor, with N_f flavors for the $U(n_i)$ groups

[4]We use the standard notation for the floor and ceiling functions, $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively. So, for k odd, $\lfloor \frac{1}{2}(k-1) \rfloor = \lceil \frac{1}{2}(k-1) \rceil = \frac{1}{2}(k-1)$; for k even, $\lfloor \frac{1}{2}(k-1) \rfloor = \frac{1}{2}(k-2)$ and $\lceil \frac{1}{2}(k-1) \rceil = \frac{k}{2}$.

and $2N_f$ flavors for the $U(n_j^{2d})$ groups, and then the duality of [42] in each factor fits with the value (3.17) [38].

We will discuss even vs. odd D_{k+2} and the duality of [49] in much more detail in what follows. We will also report on our attempt to understand the duality [47]—and the value (3.18)—by considering various ΔW deformations, similar to (3.19) and (3.20).

3.1.4 $W_{A,D,E}$ flat direction flows

The $W_{A,D,E}$ theories can also be deformed by moving away from the origin, on the moduli space of supersymmetric vacua. There are fundamental matter flat directions associated with expectation values for the Q and $\widetilde{Q}$ matter fields (e.g. $\langle Q_{N_f}\rangle = \langle \widetilde{Q}_{N_f}\rangle \neq 0$), and adjoint flat directions associated with expectation values $\langle X\rangle$ and/or $\langle Y\rangle$, as well as mixed directions where both fundamentals and adjoints receive expectation values. We will here primarily focus on the purely adjoint flat directions.

For X and Y adjoints of $SU(N_c)$ gauge group, there are certain flat directions which exist for special values of N_c that do not exist for the $U(N_c)$ case. For example, for W_{A_k} there are flat directions when $N_c = kn$ for integer n; along such flat directions,

$$SU(kn) \to U(n)^k/U(1), \tag{3.21}$$

where in the low-energy theory each $U(n)$ factor is a decoupled copy of SQCD with N_f flavors. As we will review in Section 3.3.4, this gives another check of $\alpha_{A_k} = k$ in the duality (3.15). We will discuss similar checks of $\alpha_{D_{k+2}} = 3k$, for the case of k odd. As we will emphasize, the D_{even} case is quite different from D_{odd}; similar series of flat directions for D_{even} and E_7 have a more subtle story.

For the cases where the classical chiral ring does not truncate—namely, $W_{D_{k+2}}$ for k even and W_{E_7}—we show that there are classically unlifted flat directions given by matrix solutions to the F- and D-terms of the undeformed theories. We argue that these flat directions are not lifted or removed by any dynamics, and they thus present a possible challenge for the proposed duals for these theories.

3.1.5 Outline

The outline of the rest of this chapter is as follows. In Section 3.2 we review some technical details, including a review of the known and conjectured dualities for the 4d $W_{A,D,E}$ SCFTs, and a discussion of their moduli spaces of vacua—especially with respect to higher-dimensional vacua. In Section 3.3 we review some aspects of the $W_{\hat{A}}$ and W_{A_k} theories to set the stage for subsequent analysis.

In Section 3.4, we consider the $W_{D_{k+2}}$ theories. First, we study a matrix-related classical moduli space of supersymmetric vacua present for the D_{even} theory, which poses a puzzle for duality for D_{even}, and argue that these flat directions are not lifted by quantum effects. We demonstrate that these flat directions seem to violate the a-theorem, and discuss possible resolutions to this puzzle. We then study $SU(N_c)$-specific flat directions of the D_{k+2} theories, reviewing that such flat directions provide a nontrivial check of the D_{odd} duality, and then showing that they lead to puzzles for the conjectured D_{even} duality. Next, we study RG flows from the $W_{D_{k+2}}$ SCFTs via relevant superpotential deformations, again finding nontrivial checks of duality for D_{odd}, and more hurdles for D_{even}. We conclude Section 3.4 with comments on hints as to how these puzzles might be resolved.

In Section 3.5, we similarly analyze the W_{E_7} SCFT. We study matrix-related flat directions and $SU(N_c)$-specific flat directions of the E_7 theory, which turn out to be analogous to the puzzling D_{even} flat directions. We then study some ΔW RG flows from the W_{E_7} SCFT, noting some features in the resulting higher-dimensional vacuum structure that are new to the E-series. Finally, we conclude in Section 3.6 with comments on future directions, and some discussion of how the present work might be applied to the W_{E_6} and W_{E_8} SCFTs. In an appendix, we explore additional E-series RG flows.

3.2 Technical Review

3.2.1 The a-theorem, and a-maximization

As we reviewed in Chapter 2, the 4d a-theorem [19, 52, 21] implies that the endpoints of all RG flows must satisfy

$$a_{UV} > a_{IR}. \tag{3.22}$$

In superconformal theories, a is related to the 't Hooft anomalies for the superconformal $U(1)_R$ symmetry as [16] (we rescale to a convenient normalization):

$$a(R) = 3\mathrm{Tr}R^3 - \mathrm{Tr}R. \tag{3.23}$$

In cases where $U(1)_R$ can mix with $U(1)_F$ global flavor symmetries, the exact superconformal R-symmetry is determined by a-maximization [24], by locally maximizing (3.23) over all possible $U(1)_R$ symmetries. Cases with accidental symmetries or irrelevant interactions require special care: one then maximizes (3.23) over R-symmetries that are not obvious from the original description. One such situation is when a gauge-invariant operator saturates, or seemingly violates, an SCFT unitarity bound, e.g. for scalar chiral primary operators $\mathscr{O}$:

$$\Delta(\mathscr{O}) = \frac{3}{2}R(\mathscr{O}) \geq 1. \tag{3.24}$$

The inequality is saturated for free chiral superfields, and apparent violations instead actually saturate the inequality, with an accidental symmetry $U(1)_{\mathscr{O}}$ which only acts on the IR-free-field composite operator. See [25] for how a-maximization is modified in such cases, and its application to the $\widehat{A}$ SCFTs. See [38] for additional applications to the other theories in Fig. 3.2, and additional discussion.

The a-theorem (3.22) requires, for example, that a decreases when a fundamental flavor is given a mass and integrated out,

$$a_{SCFT}(N_c, N_f) > a_{SCFT}(N_c, N_f - 1), \tag{3.25}$$

where SCFT refers to any of the SCFTs in Fig. 3.2. In the Veneziano limit, (3.25) for this RG flow requires (recall $x \equiv N_c/N_f$) [38]

$$\frac{d}{dx}\left(x^{-2}a(x)/N_f^2\right) < 0. \tag{3.26}$$

Upon computing $a(x)$ for the SCFTs in Fig 3.2, it is verified that $x^{-2}a(x)/N_f^2$ is indeed monotonically decreasing for small x, but then flattens out when x is sufficiently large, e.g. at $x \approx 13.8$ for the W_{E_6} SCFTs [38]. The a-theorem implies that some new dynamical effect must kick in for x at or before the problematic range where (3.26) is violated.

One such effect, for sufficiently large x, is that a dynamical superpotential could be generated, and the theory is no longer conformal; this is referred to as the stability bound. For W_{A_k} theories, the stability bound is $x < x_{\text{stability}} = k$ [43, 44, 48]. Another effect, which can occur for $x < x_{\text{stability}}$, is that the theory could develop non-obvious accidental symmetries. In cases with known duals, such accidental symmetries can be evident in the dual description, where it is seen that some superpotential terms—or the dual gauge interaction—become irrelevant when x_{elec} is too large (x_{mag} is too small). It is satisfying that the a-theorem condition (3.26) is indeed satisfied in the W_{A_k} theories [25] and the $W_{D_{k+2}}$ theories [38] upon taking such accidental symmetries into account.

3.2.2 Duality for the 4d SCFTs

Recall that the chiral ring consists of gauge-invariant composites, e.g. meson, baryon, and glueball operators, formed from the microscopic chiral superfields: here X and Y, the fundamentals and anti-fundamentals Q, $\widetilde{Q}$, and the gauge field strength fermionic chiral superfields W_α, subject to classical and quantum relations. Such theories, with adjoint(s) X (or X and Y, or similarly, other two-index representations, e.g. in the examples in [53]) only have a known dual if the chiral ring of products of the adjoint(s) *truncates*. Here, *truncate* means that the number of independent elements in the ring is independent[5] of N_c. An example of an untruncated case is the

[5] There is a classical chiral ring relation that the adjoint-valued operator X^{N_c} can be expressed in terms of products of lower powers $X^{\ell<N_c}$ and the $u_j \equiv \text{Tr}X^j$. To see this, write the characteristic polynomial $P(x,u_j) \equiv \det(x-X) = x^{N_c} - x^{N_c-1}u_1 + \ldots$, and note that $P(x,u_j)|_{x=X} = 0$. Thus one can write any gauge invariant $\text{Tr}X^\ell = \mathscr{P}_\ell(u_1,\ldots,u_{N_c})$

$\widehat{A}$ theory, for which a basis of adjoint-valued products is given by $\Theta_j(X) = X^{j-1}$, for $j = 1, \ldots, N_c$; such theories do not have a known dual. A truncated case is W_{A_k}, where $\Theta_j(X) = X^{j-1}$, for $j = 1, \ldots, k$.

More generally, suppose that a truncated case has a basis of elements $\Theta_j(X,Y)$, with $j = 1, \ldots, \alpha$; these are holomorphic products without traces, so gauge-invariant chiral ring elements are formed by taking traces or contracting with Q and $\tilde{Q}$. One can form dressed quarks $Q_{(j)} \equiv \Theta_j(X,Y)Q$, which can then be used to construct gauge-invariant operators, such as the αN_f^2 mesonic operators

$$M_j = \widetilde{Q}\Theta_j Q, \qquad j = 1, \ldots, \alpha. \tag{3.27}$$

(We suppress flavor indices: each $M_j = (M_j)_{f,\widetilde{f}}$ is in the (N_f, N_f) of $SU(N_f)_L \times SU(N_f)_R$). For $SU(N_c)$ there are also baryonic operators, built out of the dressed quarks:

$$B^{(l_1,\ldots,l_\alpha)} = Q_{(1)}^{l_1} \cdots Q_{(\alpha)}^{l_\alpha}, \quad \sum_{j=1}^{\alpha} l_j = N_c. \tag{3.28}$$

As shown in [46], the many constraints on any possible dual—including matching of the chiral operators, invariance under the same global symmetries, 't Hooft anomaly matching, and matching of the superconformal index [47]—essentially determines the dual (assuming it is of a similar form) to have gauge group $SU(\tilde{N}_c)$, with $\tilde{N}_c = \alpha N_f - N_c$, again with N_f flavors q and $\widetilde{q}$ in the (anti)fundamental of the gauge group, and adjoint fields we denote by $\hat{X}, \hat{Y}$. The ratio (3.14) of the dual theory is

$$\hat{x} \equiv \tilde{N}_c / N_f = \alpha - x. \tag{3.29}$$

The electric mesons (3.27) map to elementary operators of the dual theory, which couple in W_{dual} to a corresponding mesonic composite operator in the magnetic theory. Magnetic baryons map to

for some polynomial $\mathscr{P}_\ell$. As shown in [54], such relations can be modified by instantons for sufficiently large ℓ. See e.g. [54, 55], and references therein, for examples of chiral ring relations involving the adjoint-valued gaugino and gauge field chiral superfield W_α, including the glueball operator $S \sim \mathrm{Tr} W_\alpha W^\alpha$ and generalizations. Relations involving W_α and S will not be discussed in this current work.

electric baryons as

$$B^{(l_1,\ldots,l_\alpha)} \leftrightarrow \hat{B}^{(\hat{l}_1,\ldots,\hat{l}_\alpha)}, \quad \hat{l}_j = \alpha N_f - l_j. \tag{3.30}$$

The truncation of the ring to α generators is a necessary ingredient for these classes of conjectured dualities. The chiral ring of the electric theory truncates classically in the A_k and D_{odd} cases, and has been conjectured to truncate quantum mechanically in the D_{even} [49] and E_7 [46] cases.

The $W_{A,D,E}$ theories are understood in terms of the RG flows in Fig. 3.2, starting from the top, $W=0$ theories. If $x>1$ the gauge coupling is asymptotically free, so it is a relevant deformation of the UV-free fixed point, driving the RG flow of the top arrow in Fig. 3.2 into the $\widehat{O}$ SCFT. Deforming by $W_{\widehat{A}}$, $W_{\widehat{D}}$, or $W_{\widehat{E}}$ gives flows, as in the figure, that are also all relevant for $x>1$ (the $\widehat{A}$ case can be defined down to $x>\frac{1}{2}$). Generally, as long as the gauge coupling is asymptotically free, its negative contribution to anomalous dimensions drives the cubic superpotential terms to be relevant. The $\widehat{A}\to A_k$, and $\widehat{D}\to D_{k+2}$, and $\widehat{E}\to E_r$ flows with non-cubic terms in $W(X,Y)$ only occur if $x>x_{\text{min}}$, such that the negative anomalous dimension from the gauge interactions is large enough to drive the $W(X,Y)$ terms relevant; the values of x_{min} were obtained using a-maximization for W_{A_k} in [25] and in [38] for the other $W_{G=A,D,E}$ theories. Duality, if it is known and applicable, clarifies the IR phase structure of the theories for $x>x_{\text{min}}$, where the magnetic dual becomes more weakly coupled. The fixed point theories whose duals are known or conjectured all have a similar phase structure [25, 38, 46]:

Table 3.1: Conjectured phases of the$W_{A,D,E}$ SCFTs.

$x \leq 1$	free electric
$1 < x \leq x_{\text{min}}$	$(\widehat{A},\widehat{D},\widehat{E})$ electric
$x_{\text{min}} < x < \alpha - \hat{x}_{\text{min}}$	(A_k, D_{k+2}, E_r) conformal window
$\alpha - \hat{x}_{\text{min}} \leq x < \alpha - 1$	$(\widehat{A},\widehat{D},\widehat{E})$ magnetic
$\alpha - 1 \leq x \leq \alpha$	free magnetic
$\alpha < x$	no vacuum

3.2.3 Moduli spaces of vacua of the theories

Recall that, as we reviewed in Chapter 2, 4d $\mathscr{N}=1$ theories with $W=0$ have a classical moduli space of vacua $\mathscr{M}_{cl}$, given by expectation values of the microscopic matter fields, subject to the D-term conditions (3.7) and modulo gauge equivalence. Alternatively, $\mathscr{M}_{cl}$ is given by expectation values of gauge invariant composite, chiral superfield operators, modulo classical chiral ring relations (see for instance [8]). When $W \neq 0$, one also imposes the F-term chiral ring relations. The quantum moduli space $\mathscr{M}_{qu}$ can be (fully or partially) lifted if W_{dyn} is generated, or deformed for a specific N_f as in [56] or variants[6]; the constraints of symmetries and holomorphy often exactly determine the form of such effects, and with sufficient matter (e.g. sufficiently small x) this implies that $W_{dyn}=0$ and $\mathscr{M}_{cl} \cong \mathscr{M}_{qu}$.

We will here focus on vacua with $Q=\tilde{Q}=0$, with non-zero expectation values for the adjoints, X and Y; such vacua preserve the $SU(N_f)_L \times SU(N_f)_R$ global flavor symmetry. The $N_c \times N_c$ matrices X and Y are decomposed into multiple copies of a set of basic, irreducible solutions of the D- and F-flatness conditions. We refer to such a basic vacuum solution representation as being d-dimensional if X and Y are represented as $d \times d$ matrices, which cannot be decomposed into smaller matrices.

For the $\widehat{A}$ and A_k theories and their ΔW deformations, we can set $Y=0$ and the D-terms give $[X, X^\dagger]=0$. Thus, X and $X^\dagger$ can be simultaneously diagonalized by an appropriate gauge choice, and all vacuum solutions are $d=1$ dimensional, represented by eigenvalues on the diagonal of X. More generally, vacua with $[X,Y]=0$ allow for simultaneously diagonalizing X, $X^\dagger$, Y, and $Y^\dagger$, so the representations are $d=1$ dimensional. For cases other than $\widehat{A}$ and A_k in Fig. 3.2, there are generally also $d>1$ dimensional vacua, where $[X,Y]\neq 0$. In such cases, we cannot in general fully diagonalize neither X nor Y. We can use the gauge freedom to e.g. diagonalize the real part of X (or Y), and then impose the D-term to get an adjoint-worth of constraints on the

[6]There are exotic examples of classical flat directions that are lifted by, for example, confinement (see e.g. [57]); this can only occur if a gauge group remains unbroken and strong there.

remaining three real adjoints. We indeed find examples of vacua where neither X nor Y can be fully diagonalized.

The independent representations for X and Y vacuum solutions can be characterized by the independent solutions for the Casimir[7] products of X and Y. For example, if the F-terms imply that $[X^3,Y]=0,\ [Y^2,X]=0$ then we use the eigenvalues $X^3=x^3\mathbf{1}_d,\ Y^2=y^2\mathbf{1}_d$ to label the vacua. In some cases we find there are no such Casimirs (other than the zero F-terms themselves); then different X and Y eigenvalues give different vacuum solutions. In general, a $d>1$ dimensional representation is not reducible if: $[X,Y]\neq 0$, the eigenvectors of X and Y collectively span at least a d-dimensional space, and X and Y do not share an eigenvector corresponding to a zero eigenvalue.

Consider a general $W_{A,D,E}$ theory, deformed by a generic ΔW. Let i run over the vacuum solutions, and d_i be their dimension. There are always precisely $r_G\equiv \mathrm{rank}(G)$ different $d_i=1$ dimensional (diagonalized) vacuum solutions for X and Y, as with the original, $N_c=1$ Landau-Ginzburg theories (3.2). For the D and E cases, with $N_c>1$, there are $d_i>1$ dimensional vacuum solutions. In all cases, the full $N_c\times N_c$ matrix expectation values of X and Y decompose into blocks, with n_i copies of the i'th representation, such that

$$N_c=\sum_i n_i d_i. \tag{3.31}$$

The vacua are given by all such partitions of N_c into the n_i, subject to quantum stability constraints (to be discussed). The non-zero X and Y Higgs $U(N_c)$ or $SU(N_c)$, with the unbroken gauge group depending on the n_i.

It turns out that if there are n copies of a d-dimensional vacuum, there will be an unbroken $U(n)_D\subset U(N_c)$, where $U(n)_D$ can be regarded as coming from breaking a $U(dn)\subset U(N_c)$ as $U(dn)\to U(n)^d\to U(n)_D$. The $U(n)^d$ factors each have N_f flavors, so the diagonally embedded $U(n)_D$ has dN_f flavors. If both adjoints receive a mass from the superpotential F-terms, the low-energy $U(n)_D$ will then be SQCD with dN_f flavors. This factor then has a dual gauge

[7] Casimir here means matrices commuting with X and Y, not the $U(N_c)$ or $SU(N_c)$ Casimir traces.

group $U(dN_f-n)_D$, with dN_f flavors (with $SU(N_f)_{L,R}$ enhanced to $SU(dN_f)_{L,R}$ as an accidental symmetry in the IR limit). By the dual analog of the electric Higgsing, this low-energy $U(dN_f-n)_D$ can be embedded in a $U(d^2N_f-dn)$ with N_f flavors. For example, consider the case of n copies of a 2d vacuum, with $\langle X\rangle$ breaking $U(2n)\to U(n)\times U(n)$, and then $\langle Y\rangle$ in the bifundamental breaking to $U(n)_D$. Duality maps this process as follows:

$$\begin{array}{ccccc} U(2n) & \to & U(n)\times U(n) & \to & U(n)_D \\ \downarrow & & & & \downarrow \\ U(4N_f-2n) & \to & U(2N_f-n)\times U(2N_f-n) & \to & U(2N_f-n)_D \end{array} \tag{3.32}$$

The low-energy theory for such a vacuum is denoted as A_1^{2d} if all the adjoints are massive, where the $2d$ superscript indicates that it comes from a 2d representation, and thus has $2N_f$ (or more generally, dN_f) flavors. Applying such considerations for all d_i vacua in (3.31) suggests that the dual theory has α given by

$$\alpha \stackrel{?}{=} \sum_i d_i^2. \tag{3.33}$$

This relation indeed works for the A_k and the D_{odd} theories, but not for D_{even} or $E_{6,7,8}$.

For $W_{D_{k+2}}$, the generic deformation has $k+2$ 1d vacuum solutions, and $\lfloor\frac{1}{2}(k-1)\rfloor$ 2d representations. If there are n_i copies of the i'th 1d solution, and n_j^{2d} copies of the j'th 2d solution, then $U(N_c)$ is broken as in (3.20). For odd k, (3.33) indeed gives $\alpha=3k$.

3.3 Example and Review: $\widehat{A}$ and A_k One-Adjoint Cases

3.3.1 $\widehat{A}\to A_k$ flow and A_k duality

Consider $SU(N_c)$ SQCD with N_f chiral superfields $Q(\tilde{Q})$ in the (anti)fundamental of the gauge group, and adjoint chiral superfields X and Y with superpotential

$$W_{A_k}=\frac{t_k}{k+1}\text{Tr}X^{k+1}+\frac{m_Y}{2}\text{Tr}Y^2. \tag{3.34}$$

The Y field is massive and can be integrated out; this is the $\widehat{O} \to \widehat{A}$ RG flow in Fig. 3.2. The t_k coupling, if relevant, drives the $\widehat{A} \to A_k$ RG flow in Fig. 3.2; if irrelevant, the IR theory is instead an $\widehat{A}$ SCFT. For $k=1$, $t_k = m_X$ is an X mass term and is always relevant; then both X and Y can be integrated out and the IR A_1 theory is ordinary SQCD. For $k=2$, t_k is marginally relevant as long as the matter content is within the asymptotically free range, thanks to the gauge coupling. For $k>2$, the t_k coupling is relevant only if $x > x_k^{\min}$ [25].

The chiral ring of the A_k theory truncates classically, and we may write the k generators

$$\Theta_j = X^{j-1}, \quad j = 1, \ldots, k. \tag{3.35}$$

There are then kN_f^2 meson operators (3.27), with $\alpha_{A_k} = k$, and baryonic operators (3.28).

The $\widehat{A}$ theory ($t_k = 0$) does not have a known dual description. The magnetic description of the A_k SCFT [44, 45, 48] has gauge group $SU(\tilde{N}_c)$ with $\tilde{N}_c = kN_f - N_c$, so $\hat{x} \equiv \tilde{N}_c/N_f = k - x$. The dual has N_f (anti)fundamentals $q(\tilde{q})$, adjoints $\hat{X}, \hat{Y}$, and k gauge singlets M_j transforming in the bifundamental of the $SU(N_f) \times SU(N_f)$, with superpotential

$$W_{A_k}^{mag} = \frac{\hat{t}_k}{k+1} \mathrm{Tr} \hat{X}^{k+1} + \frac{\hat{m}_Y}{2} \mathrm{Tr} \hat{Y}^2 + \frac{t_k}{\mu^2} \sum_{j=1}^{k} M_j \tilde{q} \hat{X}^{k-j} q. \tag{3.36}$$

We can rescale X and $\hat{X}$ to set $t_k = \hat{t}_k = 1$, and μ is a scale that appears in the scale matching of the electric and magnetic theories. The kN_f^2 mesonic gauge invariant operators (3.27) of the electric theory map to elementary gauge-singlets M_j in the dual. The other gauge-invariant, composite operators in the chiral ring of the electric theory—i.e. the generalized baryons (3.28), operators $\mathrm{Tr}X^{j-1}$, and glueball-type operators composed from W_α—all map directly to the corresponding composite gauge-invariant chiral operators in the magnetic dual theory. Both theories have the same anomaly free global symmetries, $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$, and the 't Hooft anomalies properly match [44, 45, 48].

The $\widehat{A}$ theories have a quantum moduli space of vacua, $W_{dyn} = 0$, for all N_f and N_c. The A_k theories, however, generate $W_{dyn} \neq 0$ if $kN_f < N_c$. For example, SQCD (W_{A_1}) for $N_f < N_c$ has $W_{dyn} \neq 0$ [43], giving a $\widetilde{Q}Q \to \infty$ runaway instability for massless flavors or $\mathrm{Tr}(-1)^F = N_c$

gapped susy vacua for massive flavors. We are here interested in cases with massless flavors and $W_{dyn}(M_j) = 0$, so we restrict to $kN_f > N_c$, i.e. $x < x_{\text{stability}} = k$; this is the vacuum stability bound [44, 45]. For $kN_f < N_c$, the quantum theory A_k has a moduli space of vacua, where the M_j mesons have expectation values. The classical constraints on this moduli space, e.g. $\text{rank}(M_k) \leq N_c$, are recovered in the magnetic dual description from its stability bound, $\hat{x} < k$, since the M_k expectation value gives masses via (3.36) to the dual quarks q, $\widetilde{q}$.

3.3.2 $W_{A_k} + \Delta W$ deformations and $A_k \to A_{k'<k}$ RG flows

The A_k theories of different k are connected by RG flows upon resolving the A_k singularity (3.34) by lower order ΔW deformations. The generic deformation, for instance by a mass term $\Delta W = \frac{1}{2} m_X \text{Tr} X^2$, leads to an RG flow with k vacuum solutions for $\langle X \rangle$, with X massive in each, hence k copies of SQCD in the IR—i.e. $A_k \to kA_1$.

We now consider a partial resolution, by tuning the superpotential couplings such that some of the eigenvalues coincide. We first consider the $U(N_c)$ case, which is simpler because we don't have to worry about imposing the tracelessness of X. Consider the deformation

$$W_{elec} = W_{A_k} + \Delta W, \qquad \Delta W = \sum_{i=k'}^{k-1} \frac{t_i}{i+1} \text{Tr} X^{i+1}. \tag{3.37}$$

(The t_{k-1} deformation is trivial in the chiral ring, and it can be shifted away by shifting X, at the expense of inducing lower order terms. Such items affect the RG flow, so we keep t_{k-1} non-zero here.) The F-terms of (3.37) have a discrete set of solutions for the eigenvalues of X, with one solution at $X = 0$ and $(k - k')$ solutions at non-zero values of $\langle X \rangle$.

The vacua are given by all possible partitions of N_c into the possible vacuum eigenvalues; in such a vacuum, the electric gauge group is broken as

$$U(N_c)_{A_k} \to U(n_0)_{A_{k'}} \times \prod_{i=1}^{k-k'} U(n_i)_{A_1}, \qquad N_c = n_0 + \sum_{i=1}^{k-k'} n_i. \tag{3.38}$$

The subscripts denote the low-energy theory, obtained by expanding (3.37) around the corresponding vacuum, $X = \langle X \rangle + \delta X$. The vacua at $\langle X \rangle = 0$ have the most relevant term in (3.37) given by $W_{low} \sim \mathrm{Tr}(\delta X)^{k'+1} = W_{A_{k'}}$. The vacua at $\langle X \rangle \neq 0$ have a mass term for the low-energy adjoint, $W_{low} \sim \mathrm{Tr}(\delta X)^2 = W_{A_1}$. We write this breaking pattern as

$$A_k \to A_{k'} + (k-k')A_1. \tag{3.39}$$

By further tuning the t_i parameters in the deformation (3.37), we could cause some or all of the $(k-k')$ SQCD vacua to coincide, e.g leading to

$$A_k \to A_{k'} + A_{k-k'}: \qquad \text{i.e.} \qquad U(N_c)_{A_k} \to U(n_0)_{A_{k'}} \times U(N_c - n_0)_{A_{k-k'}}. \tag{3.40}$$

Quantum mechanically, the vacuum stability condition—needed to have $W_{dyn} = 0$—requires each $U(n)_{A_k}$ vacuum in (3.38) to have $kN_f > n$ [44, 45, 48].

In the magnetic dual, we deform by the dual analog of the perturbations in (3.37). The vacuum solutions of the deformed electric and dual theories, $W'_{elec}(X) = 0$ and $W'_{mag}(\hat{X}) = 0$, thus appropriately match, so if the electric breaking pattern is as in (3.39) or (3.40), it will have the corresponding pattern in the magnetic dual. Each vacuum gauge group in the low-energy theories maps under duality as [44, 45, 48]

$$U(n)_{A_k} \leftrightarrow U(kN_f - n)_{\widetilde{A_k}} \tag{3.41}$$

and the stability bound in the electric theory ensures that $kN_f - n > 0$. The theories on the UV and IR sides of (3.38) thus map in the dual as

$$U(kN_f - N_c)_{\widetilde{A_k}} \to U(k'N_f - n_0)_{\widetilde{A_{k'}}} \times \prod_{i=1}^{k-k'} U(N_f - n_i)_{\widetilde{A_1}}. \tag{3.42}$$

For the case in (3.40) the map is

$$U(kN_f - N_c)_{\widetilde{A_k}} \to U(k'N_f - n_0)_{\widetilde{A_{k'}}} \times U((k-k')N_f - N_c + n_0)_{\widetilde{A_{k-k'}}}. \tag{3.43}$$

The two sides of the RG flow arrow in (3.42) properly fit together as a dual description of the flow associated with the ΔW deformation, since $(k'N_f - n_0) + \sum_{i=1}^{k-k'}(N_f - n_i) = kN_f - \sum_{i=0}^{k-k'} n_i = kN_f - N_c$. This demonstrates that the value $\alpha_{A_k} = k$ (see Section 3.2.2) ties in with the fact that the A_k deformation breaking patterns (e.g. as in (3.39)) have matching sum on the two sides. This matching gives a check on the duality [48]—a perspective which we utilize throughout the present work.

As an aside, we note that the a-theorem (3.22) applies for any choice of the IR vacuum; i.e. for any fixed choice of how to distribute the N_c eigenvalues of X among solutions to $W'(X) = 0$ (subject to the stability bounds). Regarding a as counting a suitably defined "number of degrees of freedom" of the QFT, one might wonder if a hypothetical stronger statement holds: if a_{UV} is also larger than the sum $\sum_i a_{IR,i}$ over all IR vacua. These examples demonstrate that the hypothetical stronger statement is false. There are so many vacua from the many partitions of N_c that it is straightforward to explicitly verify that $\sum_i a_{IR,i}$ can be larger than a_{UV}.

3.3.3 Comments on $SU(N_c)$ vs $U(N_c)$ RG flows

It is standard that the local[8] dynamics of 4d $U(N_c)$ and $SU(N_c)$ are the same: the overall $U(1)$ factor in $U(N_c)$ is IR-free anyway in 4d (although that is not the case in 3d and lower). The original dualities of [42, 44, 45, 48, 49] etc. were written in terms of $SU(N_c)$, with $U(1)_B$ as a global symmetry. Since $U(1)_B$ is anomaly free, one can gauge it on both sides of the duality, leading to $U(N_c) \to U(\alpha N_f - N_c)$ dualities. For the theories with adjoint matter, the $U(N_c)$ version of the theories are simpler, in that we do not need to impose the tracelessness of the adjoints. The adjoints X of the $SU(N_c)$ vs $U(N_c)$ theories are related by $X_{U(N_c)} = X_{SU(N_c)} + X_0 \mathbf{1}_{N_c}$, where $\mathrm{Tr} X_{SU(N_c)} = 0$ and X_0 is an $SU(N_c)$ singlet. In the purely $SU(N_c)$ theory, it is standard to eliminate X_0 by including a Lagrange multiplier λ_x: $W_{A_k} = \mathrm{Tr} X^{k+1}/(k+1) - \lambda_x \mathrm{Tr} X$. Then λ_x pairs up with X_0, giving it a mass, and the vacua have $X_0 = 0$. The $W_X = 0$ chiral ring relation here gives $X^k = \lambda_x \mathbf{1}_{N_c}$.

Upon deforming $W_{A,D,E} \to W_{A,D,E} + \Delta W$, the $\mathrm{Tr} X_{SU(N_c)} = \mathrm{Tr} Y_{SU(N_c)} = 0$ constraints complicate the $SU(N_c)$ theories compared with $U(N_c)$. This is particularly the case if we are interested in ΔW flows as in Fig. 3.2 which have some X and Y dynamics remaining in the IR, rather than flowing all the way down to just decoupled copies of SQCD. We can enforce

[8] Of course the global dynamics and observables distinguish the different center of $U(N_c)$ vs $SU(N_c)$.

$\mathrm{Tr}X_{SU(N_c)} = \mathrm{Tr}Y_{SU(N_c)} = 0$ via Lagrange multipliers, which shifts the eigenvalues of X and Y along the flow away from the preferred $U(N_c)$ origin at $X = Y = 0$. Such a shift will induce the more general, relevant ΔW deformations which were tuned to zero for the $U(N_c)$ case, unless the reintroduced ΔW terms are subtracted off by a tuned choice of coefficients in the initial ΔW. We will see that there are subtleties—especially for the D and E cases—from the $d > 1$ dimensional vacuum representations.

Consider for example the flow $A_3 \to A_2 + A_1$. For $U(N_c)$, we get the enhanced A_2 in the IR (vs the generic $3A_1$) by taking $k' = 2$ in (3.37):

$$W = \frac{1}{4}\mathrm{Tr}X^4 + \frac{t_2}{3}\mathrm{Tr}X^3 + \frac{1}{2}\mathrm{Tr}Y^2. \tag{3.44}$$

For the $SU(N_c)$ version of this flow, we add the Lagrange multiplier λ_x to eliminate X_0, shifting the X eigenvalues. But simply doing this shift in (3.44) would induce the $\mathrm{Tr}X^2$ term, giving instead $A_3 \to 3A_1$. To get $A_3 \to A_2 + A_1$, we must add to (3.44) the remaining $t_{m<2}$ terms in (3.37),

$$W = \frac{1}{4}\mathrm{Tr}X^4 + \frac{t_2}{3}\mathrm{Tr}X^3 + \frac{1}{2}\mathrm{Tr}Y^2 + \frac{t_1}{2}\mathrm{Tr}X^2 - \lambda_x \mathrm{Tr}X - \lambda_y \mathrm{Tr}Y, \tag{3.45}$$

with t_1 tuned in terms of the multiplicities n_0, n_1 of eigenvalues in the A_2 and A_1 solutions. For fixed t_1, vacua with other partitions $N_c = n_0' + n_1'$ will instead have $3A_1$ in the IR.

It is not immediately apparent if this procedure works in the D and E cases to shift higher-dimensional representations in just the right way to be able to map any $U(N_c)$ deformation into a corresponding $SU(N_c)$ one. The chiral ring algebra that determines how one labels the higher-dimensional vacua is sensitive to additional deformation terms in both X and Y, with $[X,Y] \neq 0$. While such a shift maps between the 1d $U(N_c)$ and $SU(N_c)$ solutions, the higher-dimensional solutions can differ; indeed, we will see examples of this later on. Additional subtleties arise when there are multiple ways to perform the shift between the 1d solutions of $SU(N_c)$ and $U(N_c)$. We find cases in the D- and E-series where different deformation shifts agree for the 1d solutions but result in different Casimirs along the flow, thus affecting the labeling of higher-dimensional vacua. We will explore these issues with examples in Sections 3.4.5 and 3.5.4.

3.3.4 $SU(N_c)$ flat direction deformations

The ADE SCFTs, for $SU(N_c)$ gauge group and special values of N_c, have flat directions that are not present for $U(N_c)$. These are discussed for the A_k case in [48]. Adding a Lagrange multiplier term $\lambda_x \mathrm{Tr}X$ to (3.34), there is a flat direction of supersymmetric vacua when $N_c = km$ for integer m, labeled by arbitrary complex λ_x:

$$\langle X \rangle = \lambda_x^{1/k} \begin{pmatrix} \omega \mathbf{1}_m & & & \\ & \omega^2 \mathbf{1}_m & & \\ & & \ddots & \\ & & & \omega^k \mathbf{1}_m \end{pmatrix}, \tag{3.46}$$

where $\omega = e^{2\pi i/k}$ is a k'th root of unity and the off-diagonals are zero. This flat direction breaks $SU(N_c) \to SU(m)^k \times U(1)^{k-1}$. In each vacuum the adjoints are massive, so in the IR we end up with k copies of SQCD. The magnetic A_k theory has an analogous flat direction, along which the low-energy theory matches to that of the k copies of SQCD via Seiberg duality:

$$\begin{array}{ccc} SU(km) & \stackrel{\lambda_x \neq 0}{\longrightarrow} & SU(m)^k \times U(1)^{k-1} \\ \downarrow & & \downarrow \\ SU(k(N_f - m)) & \to & SU(N_f - m)^k \times U(1)^{k-1} \end{array} \tag{3.47}$$

This gives yet another check that the A_k duality has $\tilde{N}_c = \alpha N_f - N_c$, with $\alpha = k$.

3.4 The $W_{D_{k+2}}$ Fixed Points and Flows

The $W_{D_{k+2}}$ SCFTs are the IR endpoints of the RG flow from the $\hat{D}$ SCFT, and correspond to the superpotential (with Y normalized to set the coefficient of the first term to 1)

$$W_{D_{k+2}} = \mathrm{Tr}XY^2 + \frac{t_k}{k+1}\mathrm{Tr}X^{k+1}. \tag{3.48}$$

Such theories were first studied in [49]. The $\mathrm{Tr}XY^2$ term in (3.48) is always relevant and drives the RG flow $\widehat{O} \to \widehat{D}$, while the second term in (3.48) gives the $\widehat{D} \to D_{k+2}$ RG flow. For $k = 1$, $W_{D_3} \cong W_{A_3}$, since then (3.48) contains the (relevant) X-mass term $\mathrm{Tr}X^2$, and integrating out X yields $W_{low} \sim \mathrm{Tr}Y^4 \sim W_{A_3}$. For $k = 2$, the superpotential (3.48) is cubic, and hence relevant as long as the gauge group is asymptotically free, i.e. $x > 1$. For $k > 2$, the $\widehat{D} \to D_{k+2}$ flow associated with the coupling t_k is relevant only if $x > x^{\min}_{D_{k+2}}$, where $x^{\min}_{D_{k+2}}$ was determined via a-maximization in [38],

$$x^{\min}_{D_{k+2}} \begin{cases} = & \frac{1}{3\sqrt{2}}\sqrt{10-34k+19k^2} & k<5 \\ < & \frac{9}{8}(k+1) & k \text{ large} \end{cases}. \tag{3.49}$$

For relevant t_k, we can normalize X to set $t_k = 1$ at the IR D_{k+2} SCFT. For $x < x^{\min}_{D_{k+2}}$, $t_k \to 0$ in the IR and the theory stays at the $\widehat{D}$ SCFT. We will here assume that $x > x^{\min}_{D_{k+2}}$.

The F-terms of the undeformed D_{k+2} superpotential (3.48) are given by

$$Y^2 + t_k X^k = 0, \tag{3.50}$$

$$\{X, Y\} = 0. \tag{3.51}$$

For k odd, it follows from (3.50) and (3.51) (as explained after (3.8)) that the chiral ring classically truncates to the $3k$ generators (3.9). As in the A_k case, there is a stability bound: we must require $x < x_{\text{stability}}$ in order to avoid W_{dyn}, which would lead to a runaway potential for the generalized mesons. For $x < x_{\text{stability}}$, there is instead a moduli space of supersymmetric vacua with $W_{dyn} = 0$. As we will review (at least for odd k) $x_{\text{stability}} = 3k$, which is related to the fact that the chiral ring has $3k$ elements.

3.4.1 Previously proposed dualities for $W_{D_{k+2}}$

A dual description of the D_{k+2} theories was proposed in [49], and many of the usual, non-trivial checks were verified—for instance matching of the global symmetries, 't Hooft anomaly matching, and mapping of the chiral ring operators. As reviewed in Section 3.2.2, the conjectured duals have gauge group $SU(\alpha_{D_{k+2}} N_f - N_c)$ with $\alpha_{D_{k+2}} = 3k$ (3.17), and matter content consisting of N_f (anti)fundamentals $q(\tilde{q})$, adjoints $\hat{X}, \hat{Y}$, and $3k$ gauge singlet mesons $M_{\ell j}$ which map to the composite meson operators of the electric theory as

$$M_{\ell j} = \tilde{Q} X^{\ell-1} Y^{j-1} Q, \quad \ell = 1, ..., k; \;\; j = 1, 2, 3. \tag{3.52}$$

The dual theory has superpotential

$$W_{D_{k+2}}^{mag} = \mathrm{Tr}\hat{X}\hat{Y}^2 + \frac{1}{k+1}\mathrm{Tr}\hat{X}^{k+1} + \frac{1}{\mu^4}\sum_{\ell=1}^{k}\sum_{j=1}^{3} M_{\ell j}\tilde{q}\hat{X}^{k-\ell}\hat{Y}^{3-j}q. \tag{3.53}$$

A detailed calculation, via a-maximization, is needed to determine the $\hat{x}_{min}$ (3.29) values for the various non-cubic terms in (3.53) to be relevant rather than irrelevant [38].

The above dual, with $\alpha_{D_{k+2}} = 3k$ mesonic operators (3.52), requires the chiral ring truncation (3.9), which is only evident from the classical F-terms for k odd. It was conjectured in [49] that quantum effects make the even k theories similar to odd k, with a quantum truncation of the chiral ring, in order for the duality to hold for both even and odd k. It is as-yet unknown if and how such a quantum truncation occurs for the even k case, and thus the status of the duality remains uncertain for even k. The fact that e.g. the 't Hooft anomaly matching checks work irrespective of whether k is even or odd can be viewed as evidence that the duality also applies for D_{even}, or perhaps just a coincidence following merely from the fact that these checks are meaningful for odd k.

In addition to the usual checks of duality, the proposed chiral ring truncation and duality for D_{even} were used in [49] to predict a duality for an $SU(N_c) \times SU(N_c')$ quiver gauge theory with (anti)fundamentals and an adjoint for each node, and (anti)bifundamentals between. This latter duality was later re-derived, and confirmed, by considering deformations of the more solid, odd k D_{k+2} theories [58]. But it was also noted in [58] that the D_{k+2} duality implies some other dualities that are clearly only applicable for k odd, with fields appearing in the superpotentials with powers like $X^{(k+1)/2}$. The fractional power for k even suggests an incomplete description, which is missing some additional degrees of freedom. The status of the D_{even} duality thus remained (and it still remains) inconclusive.

A powerful, more recent check of dualities is to verify that the superconformal indices of the electric and magnetic theories match; see e.g. [29, 50]. In [47], the superconformal indices for the electric and magnetic dual D_{k+2} theories are verified to indeed match in the Veneziano limit for both even and odd k. The matching beyond the Veneziano limit provides a physical basis for a conjectural mathematical identity. It was moreover noted in [47] that the conjectural quantum truncation of the k even chiral ring should be verifiable via the the index, by expanding it to the appropriate order in the fugacities and checking if the contributions from operators that are eliminated by the quantum constraints are indeed cancelled by those of other operators. It was noted, however, that this check is complicated by the fact that there are many possible contributing operators, so it was not yet completed.

One of the original arguments for the D_{even} quantum truncation is based on the fact that one can RG flow from $D_{\text{odd}} \to D_{\text{even}}$ via appropriate ΔW deformation, e.g. $D_{k+2} \to D_{k+1} + A_1$. Another, similar argument [38] uses the connection between the stability bound and the chiral ring truncation. The duality suggests that the original electric theory has an instability, e.g. via $W_{dyn} \neq 0$ leading to a runaway vacuum instability, when $3kN_f - N_c < 0$, i.e. for $x > 3k$, and we expect RG flows to reduce the stability bound in the IR. Flowing, for instance, from $D_{k+2} \to D_{k+1} + A_1$ for k odd, the UV D_{k+2} theory has a truncated chiral ring and stability bound, which suggests that the IR (even) D_{k+1} theory should also have a stability bound, and hence chiral ring truncation. We will analyze such RG flows in detail here, and show that there are subtleties.

In summary, the evidence that the duality holds for D_{odd} is compelling, while the evidence for D_{even} is mixed, with aspects that are not understood. Our analysis here fails to find evidence for the quantum truncation of the chiral ring for D_{even}, and instead points out additional hurdles for the conjectured duality.

3.4.2 Matrix-related flat directions at the origin

A 2d line of flat directions for D_{even}

We consider the moduli space of vacuum solutions of (3.50) and (3.51), and the D-term constraints (3.7), taking $X, Y \neq 0$ with $Q = \tilde{Q} = 0$. The 1d versions of these equations, where we replace the matrices with 1d eigenvalue variables $X \to x$, $Y \to y$, are only solved at $x = y = 0$, corresponding to the D_{k+2} singularity at the origin of the moduli space of the undeformed $W_{D_{k+2}}$ theory. Now consider $d > 1$ dimensional representations of the solutions of (3.50)-(3.51) and (3.7). The second F-term shows that $[X^2, Y] = 0$, so X^2 is a Casimir. Likewise, it follows from (3.50) that $[Y^2, X] = 0$, so Y^2 is also a Casimir; the representation must have $X^2 = x^2 \mathbf{1}_d$, and $Y^2 = y^2 \mathbf{1}_d$. For D_{odd}, (3.51) would then imply that X is also a Casimir, so there can not be a non-trivial $d > 1$ dimensional representation. For D_{even}, on the other hand, the F-terms, D-terms, and Casimir conditions are solved by the 2-dimensional solutions

$$k \text{ even}: \quad X = x\sigma_3, \quad Y = y\sigma_1, \qquad y^2 + t_k x^k = 0. \tag{3.54}$$

This gives a moduli space of supersymmetric vacua, passing through the origin. Modding out by gauge transformations, which take $x \to -x$ and $y \to -y$, the moduli space can be labeled by x^2 and y^2 satisfying (3.54), which allows for an additional $\mathbf{Z}_{k/2}$ phase for x^2. Since X and Y in (3.54) are traceless, this flat direction is present for either $SU(N_c)$ or $U(N_c)$.

More generally, D_{even} has vacua with multiple copies of the 2d vacuum solution (3.54), with the remaining eigenvalues of X and Y at the origin. There can be $\lfloor N_c/2 \rfloor$ copies of the 2d representation, giving a moduli space of supersymmetric vacua labelled by x_i^2 and y_i^2 satisfying (3.54), for $i = 1, \ldots, \lfloor N_c/2 \rfloor$. The $SU(N_f)_L \times SU(N_f)_R$ global symmetries are unbroken along this subspace, so it can be distinguished from the mesonic or baryonic directions where the Q_f or $\tilde{Q}_{\tilde{f}}$ have expectation value. The classically unbroken gauge symmetry is enhanced when various x_i are either zero or equal to each other. Consider, for example, $N_c = 2n$, with all n of the x_i

non-zero and equal. In this direction of the moduli space, by a similarity transformation we have

$$\begin{aligned}\langle X\rangle &= x\mathbf{1}_n\otimes\sigma_3 \stackrel{B\langle X\rangle B^{-1}}{\longrightarrow} x\sigma_3\otimes\mathbf{1}_n\\ \langle Y\rangle &= y\mathbf{1}_n\otimes\sigma_1 \stackrel{B\langle Y\rangle B^{-1}}{\longrightarrow} y\sigma_1\otimes\mathbf{1}_n.\end{aligned} \qquad (3.55)$$

Consider the Higgsing in stages: first, $\langle X\rangle$ breaks $U(2n)\to U(n)\times U(n)$, and then $\langle Y\rangle$ breaks $U(n)\times U(n)\to U(n)_D$, the diagonally embedded subgroup (for simplicity, we write the gauge groups as $U(m)$, and corresponding expressions apply if we work in terms of $SU(m)$ groups). This breaking pattern leaves five uneaten $U(n)_D$ adjoints from X and Y, four of which get a mass from the $W_{D_{k+2}}$ superpotential (3.48). The low-energy $U(n)_D$ along this moduli space has a massless adjoint matter field and $W_{low}=0$; i.e. it is a $U(n)_D\,\widehat{A}$ theory. Giving general expectation values to the adjoint matter field of the low-energy $\widehat{A}$ theory corresponds to unequal expectation values of the x_i in the n copies of the 2d vacuum (3.54), leading to the more generic breaking pattern $U(2n)\to U(n)_D\to U(1)^n$. Note also that the low-energy $U(n)_D\,\widehat{A}$ theory, along the moduli space (3.55) has $N_f^{low}=2N_f$ flavors, since the fundamentals decompose as $\mathbf{2n}\to(\mathbf{n},\mathbf{1})+(\mathbf{1},\mathbf{n})\to 2\cdot\mathbf{n}$; the enhanced flavor symmetry arises as an accidental symmetry. In summary, there is a (classical) flat direction

$$D_{k+2=\text{even}}\to\widehat{A},\qquad \text{with}\quad U(N_c)\to U(\lfloor N_c/2\rfloor)_D\quad \text{and}\quad N_f^{low}=2N_f, \qquad (3.56)$$

so $x^{low}=N_c^{low}/N_f^{low}=(N_c/2)/(2N_f)=x/4$.

We have not found a mechanism for this classical moduli space to be lifted by a dynamical superpotential or removed by quantum effects. The low-energy $U(\lfloor N_c/2\rfloor)_{\widehat{A}}$ theory with $2N_f$ flavors clearly has $W_{dyn}=0$, and unmodified quantum moduli space. The original theory can have additional effects e.g. from instantons in the broken part of the group (see [59, 60] for discussion and examples), from the last step of the breaking $U(N_c)\to U(\lfloor N_c/2\rfloor)^2\to U(\lfloor N_c/2\rfloor)_D$ in (3.56). Indeed, for x above the stability bound, there can be a W_{dyn} which leads to runaway expectation values for the mesonic operators. But holomorphy, the $U(1)_R$ symmetry, and the condition that W_{dyn} must lead to a potential that, by asymptotic freedom, goes to zero far from the origin of the moduli space, precludes any W_{dyn} that only lifts the 2d flat directions (3.54) without generating a runaway W_{dyn} for the mesonic operators. As usual, the low-energy theory along the flat direction is less asymptotically free than the theory at the origin, and the theory is more weakly coupled for vacua farther from the origin on the moduli space. The original D_{even} theory at the origin is asymptotically free for $N_f < N_c$, while the low-energy $U(\lfloor N_c/2 \rfloor)_{\widehat{A}}$ theory far along the flat direction is IR-free if $N_f > (N_c/2)$, i.e. if $x < 2$. In that case, the IR spectrum consists of the IR-free $U(\lfloor N_c/2 \rfloor)_{\widehat{A}}$ gauge fields and matter.

We now consider if this D_{even} flat direction is compatible with the conjectural, dual $U(3kN_f - N_c)_{D_{\text{even}}}$ theory. That theory has an analogous moduli space of vacua where the dual adjoints $\hat{X}, \hat{Y}$ satisfy F-term equations analogous to (3.50)-(3.51), with copies of the 2d representation (3.54). Chiral ring elements like $\text{Tr}X^n$ should indeed map to similar elements in the dual, e.g. $\text{Tr}X^n \leftrightarrow \text{Tr}\widehat{X}^n$. The moduli space of eigenvalues of the 2d representation is $\frac{1}{2}(3kN_f - N_c)$-dimensional, along which the gauge group is broken to $U(\lfloor \frac{1}{2}(3kN_f - N_c) \rfloor)_{\widehat{A}}$. The dimensions of the two moduli spaces differ, which is a contradiction with the conjectural dual unless some quantum effect eliminates the difference (as indeed happens with the mesonic directions of the moduli space, where the classical constraints on the rank of the meson matrices arise from quantum effects in the dual). In addition to the moduli spaces differing, the low-energy theories on the flat directions of the two conjectured duals, i.e. $U(\lfloor N_c/2 \rfloor)_{\widehat{A}}$ and $U(\lfloor 3kN_f/2 - N_c/2 \rfloor)$, are not in any clear way dual to each other; there is no known dual for the $W_{\widehat{A}}$ SCFTs.

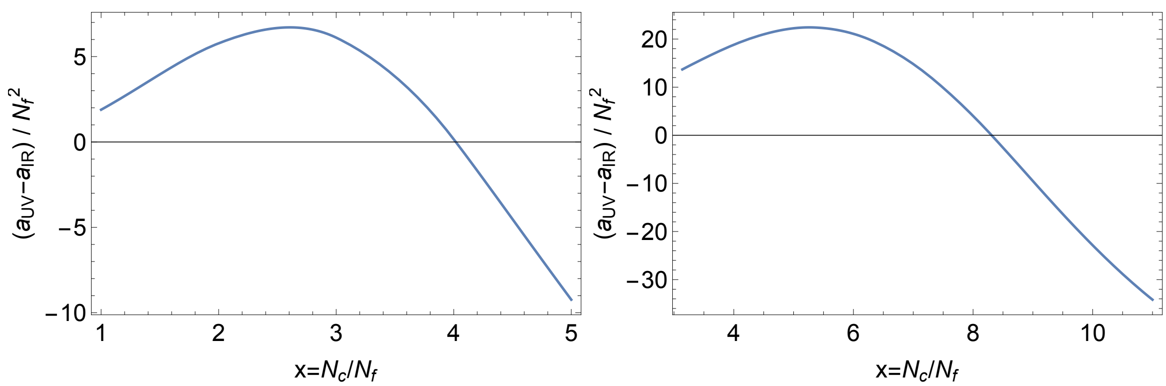
As in the electric theory, we do not yet see a mechanism for quantum effects to modify the classical dimensions of these moduli spaces. Note that the low-energy $U(\lfloor 3kN_f/2 - N_c/2 \rfloor)_{\widehat{A}}$ theory is IR-free if $x > 3k - 2$, which is non-overlapping with the range $x < 2$ where the corresponding electric theory is IR-free; this at least avoids an immediate, sharp contradiction with the duality, since two theories cannot have a different IR-free spectrum in the same region of the moduli space. As a concrete example, consider the case $k = 2$, i.e. W_{D_4}, and take N_c even. The electric W_{D_4} superpotential (3.48) is relevant as long as the gauge group is asymptotically free, for $x > 1$. The stability bound suggested by the conjectural $U(6N_f - N_c)$ dual is $x < 6$. The electric theory has the flat direction to the low-energy $U(N_c/2)_{\widehat{A}}$ theory, which is IR-free if $x < 2$. The dual theory has a flat direction to a low-energy $U(3N_f - N_c/2)_{\widehat{A}}$ theory with $2N_f$ flavors, which is IR-free if $x > 4$.

This D_{even} flat direction is related to the fact that the chiral ring of the D_{even} theory does not classically truncate; one can think of it as coming from the massless degrees of freedom present in the non-truncated ring. Its existence provides us with a new way to rephrase the puzzle of how the truncation occurs: does some quantum effect lift this flat direction? If not, the flat direction seems inconsistent with duality.

A puzzle for the $W_{D_{\text{even}}}$ flat directions (3.54): apparent a-theorem violations

The supersymmetric flat direction discussed in the previous subsection has another puzzle, independent of the conjectured duality: it leads to naive violations of the a-theorem (3.22) for sufficiently large x. The exact a_{SCFT} is evaluated by using the relation (3.23) between a and the 't Hooft anomalies for the superconformal $U(1)_R$ symmetry, along with a-maximization (when needed) and accounting for all accidental symmetries. The values of a_{SCFT} for the $W_{D_{k+2}}$ theories were analyzed in [38], following the W_{A_k} analysis in [25] with regard to the crucial role of including the effect of accidental symmetries in a-maximization. One type of accidental symmetry, when gauge invariant chiral operators hit the unitarity bound and decouple, is readily apparent in the electric theory. Dualities reveal other types of accidental symmetries, e.g. those where the analog of the t_k coupling in (3.48) for the magnetic dual is irrelevant, or where the magnetic gauge coupling is irrelevant (the free-magnetic phase); such accidental symmetries are—as far as we know—unseen without knowing the dual.

We consider Δa for the RG flow associated with the flat direction in (3.56). We compute $a_{UV}(x)$ corresponding to the D_{k+2} theory with gauge group $SU(N_c)$ and N_f flavors as in [38], and $a_{IR}(x)$ corresponding to an $\widehat{A}$ theory with gauge group $SU(N_c/2)$ and $2N_f$ flavors as in [25], including as there the effects of all mesons hitting the unitarity bound and becoming IR-free. We plot the results for the cases $k = 2$ and $k = 4$, working in the Veneziano limit of large N_c and N_f, with x fixed. ($U(N_c)$ vs $SU(N_c)$ is a subleading difference in this limit.)



(a) $W_{D_4} \to \widehat{A}$ ($k = 2$).

(b) $W_{D_6} \to \widehat{A}$ ($k = 4$).

Figure 3.3: $(a_{UV} - a_{IR})/N_f^2$ plotted for x in the conformal window. The W_{D_4} theory is IR-free for $x \leq 1$. The W_{D_6} theory requires $x > 3.14$ for the $\mathrm{Tr}X^5$ term in W_{D_6} to be relevant, while the corresponding term in the Brodie dual is relevant if $x < 8.93$. The $\widehat{A}$ theory is asymptotically free in both plotted domains.

As we can see in Figures 3.3a and 3.3b, both the $k = 2$ and the $k = 4$ flat direction RG flows seem to violate the a-theorem for sufficiently large x. For the W_{D_4} case, the conformal window where both the electric and magnetic theories are asymptotically free is $1 < x < 5$, and the cubic t_k term in (3.48), or its magnetic analog, is relevant in this entire x range. As seen in Fig. 3.3a, the a-theorem is seemingly violated for $x \geq 4$, within the conformal window. For W_{D_6}, the situation is plotted in Fig. 3.3b: the flat direction seemingly violates the a-theorem for $x \geq 8.31$. This is within the expected W_{D_6} conformal window (i.e. below $3k - \hat{x}_{\min} \approx 8.93$ beyond which duality suggests that the theory is instead in the $\widehat{D}_{mag}$ phase, and also below $x = 11$, where duality suggests the IR-free magnetic phase).

Of course, we do not believe that there will be violations of the a-theorem, so the puzzle of these apparent violations must somehow be resolved. We also note that the apparent violations first occur for x still below the values where mesons involving Y^3 would first hit the unitarity bound (this occurs first at $x = 5$ for $k = 2$, and at $x = 9.33$ for $k = 4$). Thus, the calculation of a_{UV}

is not affected by the issue of whether or not such mesons should be included—we've removed them in the plots above, which would be correct if Brodie duality is correct for D_{even} and the quantum truncation indeed occurs.

We see two possible resolutions to the puzzle of the apparent a-theorem violations. 1) These classical flat directions are somehow lifted by quantum effects, in a way that we do not yet understand. 2) Some additional degrees of freedom make the calculation of a wrong, e.g. giving a larger value for a_{UV} for the $W_{D_{\text{even}}}$ theory. We do not yet know the resolution.

Option 1) could also resolve the conflict with Brodie-duality, discussed in the previous subsection. As we discussed there, asymptotic freedom, along with holomorphy and the R-symmetry, suggests that $W_{exact} = 0$, but perhaps another mechanism could remove the flat directions—at least for x large enough to be in the problematic range. The existence of the classical flat direction fits with the classically untruncated chiral ring, and it sharpens the issue of if, and how, the chiral ring for the D_{even} theory is quantumly truncated.

Additional evidence that the $W_{D_{\text{even}}} \to \widehat{A}$ flat directions aren't lifted

We here present additional arguments against any quantum barrier to the $W_{D_{\text{even}}} \to W_{\widehat{A}}$ flat directions. The idea is to explore more of the full moduli space of supersymmetric vacua, going along Q-flat directions, until the low-energy theory is IR-free.

Consider an even D_{k+2} theory at the origin, with $N_f < N_c$ such that the theory is asymptotically free. Going along a Q-flat direction by giving a vev to a flavor, $\langle Q_f \rangle = (v, 0, ..., 0) = \langle \tilde{Q}_f \rangle$, gives a low-energy theory that is less asymptotically free. The gauge group is Higgsed $SU(N_c) \to SU(N_c - 1)$, under which the adjoints decompose $X \to \check{X} + F_x + \tilde{F}_x + s_x$ for $\check{X}$ an adjoint and s_x a singlet (and likewise for Y). Then, the number of light flavors in the low-energy theory is $N_f - 1 + 2 = N_f + 1$, where the -1 is for the eaten flavor and the $+2$ is from additional light flavors, $F_{x,y}$. Expanding the superpotential under this decomposition gives, for instance for W_{D_4}, an IR superpotential of the form

$$W_{D_4} = \mathrm{Tr}\bigg(\frac{1}{3}\check{X}^3 + \check{X}\check{Y}^2 + \check{X}\tilde{F}_x F_x + \check{X}\tilde{F}_y F_y + \check{Y}\tilde{F}_y F_x + \check{Y}\tilde{F}_x F_y + s_x F_x \tilde{F}_x + s_x F_y \tilde{F}_y + s_y F_x \tilde{F}_y + s_y F_y \tilde{F}_x + s_x^3 + s_x s_y^2\bigg). \tag{3.57}$$

Along the above flat direction, the 1-loop beta function coefficient changes by $b_1 = N_c - N_f \to (N_c - 1) - (N_f + 1) = b_1 - 2$ so, as usual, the low-energy theory is less asymptotically free. We iterate this procedure, giving expectation values to n flavors of Q and $\tilde{Q}$, and thus reducing $N_c \to N_c - n$, with $N_f \to N_f + n$ and $b_1 \to b_1 - 2n$, until the low-energy theory is no longer asymptotically free, i.e. $n > (N_c - N_f)/2$. Then X decomposes as

$$X \longrightarrow \left(\begin{array}{ccc|c} s_x^1 & & & F_x^1 \\ & \ddots & & \vdots \\ & & s_x^n & F_x^n \\ \hline \tilde{F}_x^1 & \dots & \tilde{F}_x^n & \check{X} \end{array}\right) \tag{3.58}$$

with $\check{X}$ adjoints of an unbroken $SU(N_c - n)$, and similarly for Y.

At this point, we can take $\check{X}$ and $\check{Y}$ in the low-energy $SU(N_c - n)$ theory to have an expectation value with $m \leq (N_c - n)/2$ copies of the 2d vev (3.54), resulting in the $\widehat{A}$ flat direction where $SU(N_c - n) \to SU(m)_D \times SU(N_c - n - 2m)$. By choice of n, the intermediate $SU(N_c - n)$ theory is already IR-free, and so the $\check{X}$ and $\check{Y}$ expectation values make the low-energy theory even more weakly coupled; thus, the terms in W_{low} (e.g. in (3.58)) involving the singlets and fundamentals are irrelevant and can be ignored. The number of flavors of the low-energy $SU(m)_D$ theory is $2(N_f + n - r)$, where $N_f + n$ flavors came from the n iterations of Q-Higgsing, $r \leq n$ is the number of the $F_{x,y}$ flavors that receive a mass from $\langle \check{X} \rangle, \langle \check{Y} \rangle$ in the superpotential, and the 2 comes from Higgsing $SU(2m) \to SU(m) \times SU(m) \to SU(m)_D$. By taking m sufficiently small and n sufficiently large, the low-energy $SU(m)_D$ $W_{\widehat{A}}$ theory will have a 1-loop beta function of non-asymptotically free sign, so the theory will be IR-free and thus weakly coupled. Because every interaction is IR-free in this region of the moduli space, quantum effects from the intermediate or low-energy theory cannot lift or remove the $W_{D_{\mathrm{even}}} \to W_{\widehat{A}}$ flat direction. As remarked earlier,

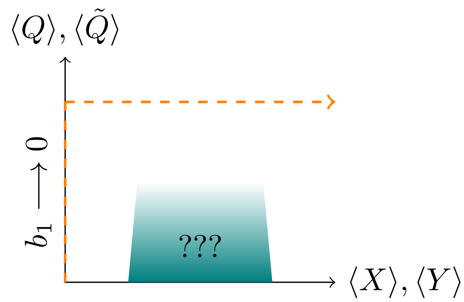


Figure 3.4: Using the Q-flat directions to bypass the strong coupling regime.

any possible effects from the Higgsed, original gauge theory at the origin (e.g. instantons in the broken part of the group) must moreover slope to zero for vacua farther from the origin on the classical moduli space (3.54).

In sum, as illustrated in Fig. 3.4, we use the Q-flat directions to bypass any hypothetical quantum barrier to the flat directions (3.54) by going to a region of moduli space where the theory is IR-free. This suggests that the $W_{D_{\text{even}}} \to W_{\widehat{A}}$ moduli space is indeed present in the full, quantum theory. As discussed in the previous subsection, there would then have to be some missing contribution to a for the D_{even} theory to avoid the apparent a-theorem violation along this moduli space for sufficiently large x.

3.4.3 $SU(N_c)$-specific (as opposed to $U(N_c)$) flat directions

For $SU(N_c)$, one includes Lagrange multipliers λ_x, λ_y to impose $\text{Tr}X = \text{Tr}Y = 0$:

$$W_{D_{k+2}} = \text{Tr}XY^2 + \frac{t_k}{k+1}\text{Tr}X^{k+1} - \lambda_x \text{Tr}X - \lambda_y \text{Tr}Y. \tag{3.59}$$

For D_{odd}, and $N_c = 2m + kn$ for m, n integers, there is a flat direction labeled by λ_x [49]

$$\langle X \rangle = \left(\frac{\lambda_x}{t_k}\right)^{\frac{1}{k}} \begin{pmatrix} \mathbf{0}_m & & & & \\ & \mathbf{0}_m & & & \\ & & \mathbf{1}_n & & \\ & & & \ddots & \\ & & & & \omega^{k-1}\mathbf{1}_n \end{pmatrix},$$

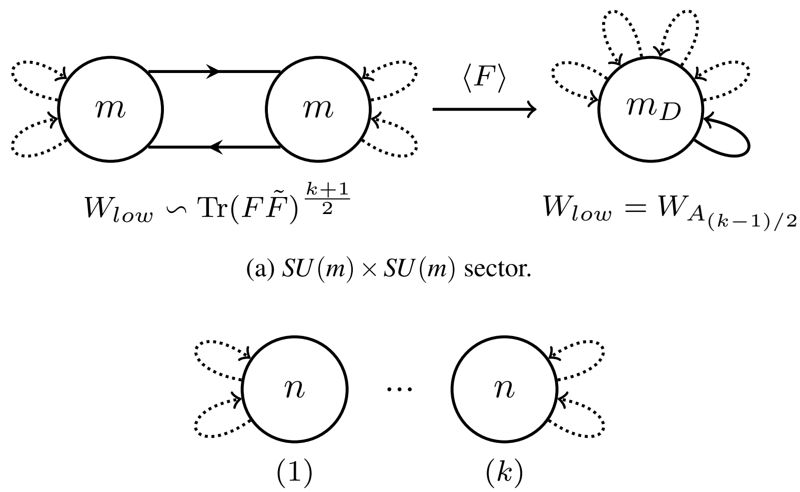
$$\langle Y \rangle = (\lambda_x)^{\frac{1}{2}} \begin{pmatrix} \mathbf{1}_m & & & & \\ & -\mathbf{1}_m & & & \\ & & \mathbf{0}_n & & \\ & & & \ddots & \\ & & & & \mathbf{0}_n \end{pmatrix}, \tag{3.60}$$

where $\omega = e^{2\pi i/k}$. The gauge group is Higgsed as $SU(2m+kn) \to SU(m)^2 \times SU(n)^k \times U(1)^{k+1}$. The $SU(n)^k$ theories are, in the IR, k decoupled copies of SQCD, each with N_f flavors. The low-energy $SU(m)^2$ sector includes SQCD, with N_f massless flavors, along with bifundamentals F and $\tilde{F}$ coming from the adjoint X of the original theory at the origin, with a low-energy superpotential $W_{low} \sim \mathrm{Tr}(F\tilde{F})^{(k+1)/2}$. All other components from X and Y are either eaten in the Higgsing, or get a mass from the superpotential (3.59) along the flat direction (3.60). This low-energy theory is depicted in Fig. 3.5, where as usual adjoints are arrows that start and end on the same node of the quiver diagram, and dotted adjoints depict those that get a mass term from the superpotential. Brodie duality along this flat direction is then compatible with a duality in [53] (see Section 8 there) for the $SU(m)^2$ factor, and with Seiberg duality for the $SU(n)$ factors:

$$\begin{array}{ccc} SU(2m+kn) & \overset{\langle X \rangle, \langle Y \rangle}{\longrightarrow} & SU(m) \times SU(m) \times SU(n)^k \times U(1)^{k+1} \\ \downarrow & & \downarrow \\ SU(3kN_f - (2m+kn)) & \longrightarrow & SU(kN_f - m)^2 \times SU(N_f - n)^k \times U(1)^{k+1} \end{array} \tag{3.61}$$

where horizontal arrows are the flat direction (3.60) and vertical arrows are the duality.

The low-energy $SU(m)^2$ theory has a further flat direction, where F has non-zero expectation value, breaking to $SU(m)_D$ [53]. The low-energy $SU(m)_D$ has an adjoint $\tilde{A}$, with superpotential $W \backsim \mathrm{Tr}\tilde{A}^{\frac{k+1}{2}}$ corresponding to an $A_{(k-1)/2}$ theory with $2N_f$ flavors. The duality of the low-energy $W_{A_{(k-1)/2}}$ theory along this flat direction then reduces to that of [44]. We summarize these flat directions in Figure 3.5. In sum, for D_{odd}, Brodie duality along the flat direction (3.60) is nicely consistent with other dualities.



(a) $SU(m) \times SU(m)$ sector.

(b) $SU(n)^k$ sector.

Figure 3.5: Flat directions for D_{odd}, $N_c = 2m + kn$, integrating out massive fields (denoted by dotted lines) and fields eaten by the Higgs mechanism (not shown).

We now consider the analogous flat directions (3.60) for the puzzling D_{even} cases, which again exist for $N_c = 2m + kn$ and are parameterized by arbitrary λ_x. As in the D_{odd} case, the gauge group is Higgsed $SU(2m+kn) \to SU(m)^2 \times SU(n)^k \times U(1)^{k+1}$, where the $SU(m)^2$ and $SU(n)^k$ decouple from each other at low energies. But the D_{even} case differs from D_{odd} in two respects. First, the $SU(m)^2$ sector has massless bifundamentals F and $\tilde{F}$, with $W_{low}(F\tilde{F}) = 0$. Similarly, the $SU(n)^k$ sector reduces at low-energy to $k/2$ decoupled copies of $SU(n)^2$ which each have, in addition to N_f flavors, massless bifundamentals with $W_{low}(F\tilde{F}) = 0$. For example, for $k = 2$,

$N_c = 2(m+n)$, and expanding (3.59) along the flat direction (3.60) gives

$$W_{SU(m)^2} \supset \frac{t_2}{3}\mathrm{Tr}A_{x,1}^3 + \mathrm{Tr}A_{x,1}A_{y,1}^2 + 2\lambda_x^{1/2}\mathrm{Tr}A_{x,1}A_{y,1} - \lambda_x \mathrm{Tr}A_{x,1} + t_2 \mathrm{Tr}A_{x,1}F_x\tilde{F}_x + (1 \to 2, A_y \to -A_y) \tag{3.62}$$

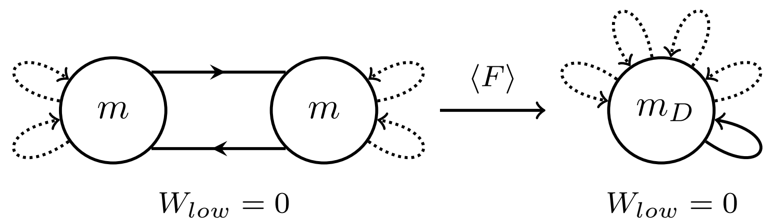
$$\begin{aligned} W_{SU(n)^2} \supset &\frac{t_2}{3}\mathrm{Tr}A_{x,3}^3 + \mathrm{Tr}A_{x,3}A_{y,3}^2 + \left(\frac{\lambda_x}{t_2}\right)^{1/k}\left(t_2\mathrm{Tr}A_{x,3}^2 + \mathrm{Tr}A_{y,3}^2\right) - \lambda_x\mathrm{Tr}A_{x,3} \\ &+ \frac{t_2}{3}\mathrm{Tr}A_{x,4}^3 + \mathrm{Tr}A_{x,4}A_{y,4}^2 - \left(\frac{\lambda_x}{t_2}\right)^{1/k}\left(t_2\mathrm{Tr}A_{x,4}^2 + \mathrm{Tr}A_{y,4}^2\right) - \lambda_x\mathrm{Tr}A_{x,4} \\ &+ \mathrm{Tr}(A_{x,3}+A_{x,4})F_y\tilde{F}_y. \end{aligned} \tag{3.63}$$

Subscripts x, y refer to which $SU(2m+2n)$ adjoint X, Y the field comes from, the $A_{1,2}$ are $SU(m)$ adjoints, and the $A_{3,4}$ are $SU(n)$ adjoints. Both of these IR superpotentials reduce to $W_{low}(F\tilde{F}) = 0$ upon integrating out the massive adjoints. The $SU(m) \times SU(m)$ theories with bifundamentals and $W_{low}(F\tilde{F}) = 0$ do not have a known dual. Indeed, they have a flat direction where F gets an expectation value and Higgses $SU(m) \times SU(m) \to SU(m)_D$, where the low-energy $SU(m)_D$ is an $\widehat{A}$ theory, with massless adjoint $\mathscr{X}$ (coming from $\tilde{F}$) and $2N_f$ fundamentals, with $W_{low}(\mathscr{X}) = 0$.

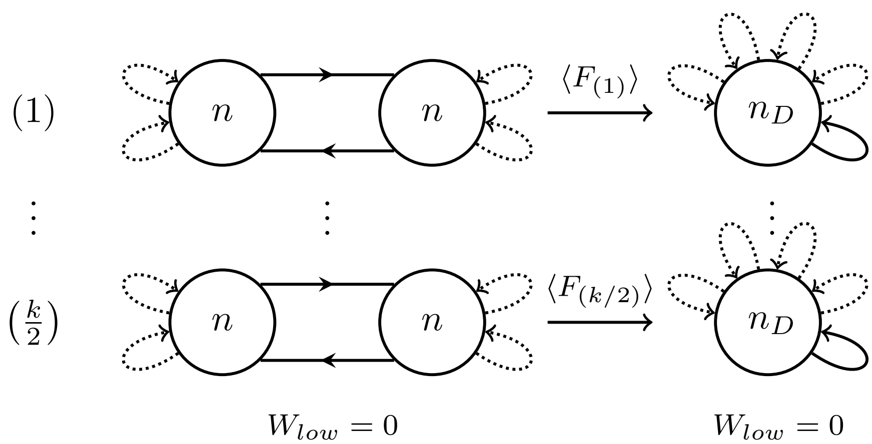
More generally, for even $k > 2$, since $\omega = e^{2\pi i/k}$ in (3.60), there will be $k/2$ massless bifundamental pairs. The low-energy $SU(n)^k$ theory then reduces to $k/2$ decoupled $SU(n)^2$ quiver gauge theories, where the i'th node couples to the $(k/2+i)$'th node via a pair of massless bifundamental fields. Each $SU(n)^2$ theory has a flat direction to an $SU(n)_D$ $\widehat{A}$ theory. The low-energy theories along these flat direction are as depicted in Figure 3.6.

The conclusion is that, for D_{even}, we end up with $(k/2+1)$ $\widehat{A}$ theories corresponding to nodes with $2N_f$ flavors[9]. The $\widehat{A}$ theories along the flat direction are puzzling, as in Section 3.4.2: we have not found a quantum mechanism for lifting these flat directions, and have not found how to make these flat directions compatible with Brodie's proposed duality.

[9]For $N_c = 2m$, there is a similar generalization of these flat directions parameterized by both λ_x and λ_y, with $\langle X \rangle \propto \langle Y \rangle \propto \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_m \end{pmatrix}$, which again leads to a low-energy $SU(m)_D$ $\widehat{A}$ theory with $2N_f$ flavors.



(a) $SU(m) \times SU(m)$ sector, Higgses to $W_{\widehat{A}}$



(b) $SU(n)^k$ sector, Higgses to $k/2$ decoupled $W_{\widehat{A}}$ theories.

Figure 3.6: Flat directions for D_{even}, with $N_c = 2m + kn$. Again we integrate out massive fields (denoted by dotted lines), and those eaten by the Higgs mechanism (not shown).

3.4.4 D_{k+2} RG flows from relevant ΔW deformations

In this subsection, we consider RG flows from the $W_{D_{k+2}}$ SCFTs upon deforming by relevant ΔW. As in the previous subsections, we find that cases involving only D_{odd} are nicely compatible with the duality of [49], while those involving D_{even} exhibit subtleties. For simplicity, we mostly consider $U(N_c)$, with brief discussion of the more complicated $SU(N_c)$ version in Section 3.4.5.

We begin with the class of ΔW deformation RG flows $D_{k+2} \to D_{k'+2}$, which is relevant for $k' < k$ (taking $x > x^{\min}_{D_{k+2}} > x^{\min}_{D_{k'+2}}$ as in (3.49)):

$$W = \text{Tr}XY^2 + \sum_{i=k'}^{k} \frac{t_i}{i+1} \text{Tr}X^{i+1}, \tag{3.64}$$

which yields the F-terms

$$Y^2 + \sum_{i=k'}^{k} t_i X^i = 0 \tag{3.65}$$

$$\{X, Y\} = 0. \tag{3.66}$$

The solution $X = Y = 0$ corresponds to the $D_{k'+2}$ theory at the origin. There are also $(k-k')$ 1d solutions with non-zero X-eigenvalue, corresponding to A_1's. The representation theory of (3.65)-(3.66) was discussed in [51, 38]. Taking X and Y to be matrices, it follows from (3.65)-(3.66) that X^2 and Y^2 are Casimirs (proportional to the unit matrix), so we may rewrite the first F-term as $(y^2 + Q_{\lfloor k/2 \rfloor}(x^2))\mathbf{1} + P_{\lfloor (k-1)/2 \rfloor}(x^2)X = 0$, where the subscripts on P and Q denote the degrees of the polynomials in x^2. There are 2d representations of the second F-term, taking $X = x\sigma_2$, $Y = y\sigma_1$; then a non-zero solution for X requires $P_{\lfloor (k-1)/2 \rfloor}(x^2) = 0$. Hence, there are $\lfloor (k-1)/2 \rfloor$ independent such solutions for x^2, and then y^2 is uniquely fixed[10]. If X and Y have n_j copies of such a vacuum, where $j = 1, \ldots, \lfloor (k-1)/2 \rfloor$ labels the value of x_j^2, then the non-zero X and Y values break $SU(2n_j) \to SU(n_j) \times SU(n_j) \to SU(n_j)_D$, where the low-energy $SU(n_j)_D$ theory has $2N_f$ flavors. Expanding $W(X,Y)$ in such vacua, the X and Y adjoints have mass terms and the low-energy theory is SQCD; we label such vacua as A_1^{2d}. In sum, the ΔW deformation (3.64) leads to vacua

$$D_{k+2} \;\longrightarrow\; D_{k'+2} + (k-k')A_1 + \left(\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lceil \frac{k'-1}{2} \right\rceil \right) A_1^{2d}. \tag{3.67}$$

The $N_c \times N_c$ matrices X and Y are decomposed into blocks, distributed among these vacua, with n_0 eigenvalues at the origin, n_i at the i'th A_1 node, and n_j^{2d} in the j'th A_1^{2d} node, with $N_c = n_0 + \sum_i n_i + 2\sum_j n_j^{2d}$. The gauge group is Higgsed in the electric and dual magnetic descriptions

[10] $x \to -x$ or $y \to -y$ is a gauge rotation so does not give additional vacua.

(for x in the conformal window) as:

$$\begin{array}{ccc} U(N_c) & \stackrel{\langle X\rangle,\langle Y\rangle}{\longrightarrow} & U(n_0)\prod_{i=1}^{k-k'}U(n_i)\prod_{j=1}^{\lfloor\frac{k-1}{2}\rfloor-\lceil\frac{k'-1}{2}\rceil}U(n_j^{2d}) \\ \downarrow & & \downarrow \\ U(3kN_f-N_c) & \longrightarrow & U(3k'N_f-n_0)\prod_{i=1}^{k-k'}U(N_f-n_i)\prod_{j=1}^{\lfloor\frac{k-1}{2}\rfloor-\lceil\frac{k'-1}{2}\rceil}U(2N_f-n_j^{2d}) \end{array} \tag{3.68}$$

The down arrows are Brodie duality for the D_{k+2} $U(N_c)$ theory in the UV, and Brodie or Seiberg duality for each approximately decoupled low-energy gauge group factor in the IR. Comparing the UV (LHS) and the IR (RHS) of the dual theories in the lower row of (3.68), the IR theory only properly matches the dual Higgsing pattern of the UV theory if $\tilde{N}_c = \tilde{n}_0 + \sum_i \tilde{n}_i + 2\sum_j \tilde{n}_j^{2d}$. This equality holds if and only if k and k' are both odd; this is a non-trivial check of Brodie duality for $D_{\text{odd}} \to D_{\text{odd'}}$. But if either k or k' is even, there is a mismatch of $2N_f$ between $\tilde{N}_c = 3kN_f - N_c$ on the LHS and its IR decomposition on the RHS of (3.68), and a mismatch of $4N_f$ if both k and k' are even.

We now consider the RG flow $D_{k+2} \to A_{k'}$, by adding $\Delta W = \frac{m_Y}{2}\text{Tr}Y^2$ to the superpotential in (3.64). There is then a low-energy $A_{k'}$ theory at the origin, $X = Y = 0$, along with $(k-k')$ A_1's corresponding to the 1d solutions of the vacuum equations with eigenvalues $y = 0$, $\sum_{i=0}^{k-k'} t_{i+k'}x^i = 0$, along with two more A_1 theories at $y = \pm\sqrt{-\sum_{i=k'}^{k} t_i x^i}$, $x = -\frac{m_Y}{2}$. As always, these 1d solutions of the F-term equations match the rank of the ADE group: $k+2$ in the UV matches the IR sum $k' + (k-k') + 2$. In addition, there are 2d representations of the D- and F-terms, with Casimirs $Y^2 = y^2\mathbf{1}$ and $\sum_{i=k'}^{k} t_i X^i = f(x)\mathbf{1}$. The 2d vacua may thus be parameterized as $X = -\frac{y}{2}\mathbf{1} + x_1\sigma_1$, $Y = y\sigma_3$, and the F-terms have $\lfloor (k-1)/2 \rfloor$ solutions for x_1, each of which determines $f(x_1)$ and specifies the 2d vacuum. In each such vacuum, the low-energy theory is SQCD (both X and Y have mass terms) with the X and Y expectation values breaking $SU(2n_j^{2d}) \to SU(n_j) \times SU(n_j) \to SU(n_j)_D$, with $2N_f$ flavors in the low-energy theory. In sum, the full (classical) structure of the vacua from such deformations is

$$D_{k+2} \longrightarrow A_{k'} + (k-k'+2)A_1 + \left\lfloor \frac{k-1}{2} \right\rfloor A_1^{2d}. \tag{3.69}$$

Taking $N_c = n_0 + \sum_{i=1}^{k-k'+2} + \sum_{j=1}^{\lfloor (k-1)/2 \rfloor} 2n_j^{2d}$, the deformation results in the following Higgsing in the electric and magnetic descriptions:

$$\begin{array}{ccc} U(N_c) & \overset{\langle X \rangle, \langle Y \rangle}{\longrightarrow} & U(n_0) \times \prod_{i=1}^{k-k'+2} U(n_i) \times \prod_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} U(n_j^{2d}) \\ \downarrow & & \downarrow \\ U(3kN_f - N_c) & \longrightarrow & U(k'N_f - n_0) \prod_{i=1}^{k-k'+2} U(N_f - n_i) \prod_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} U(2N_f - n_j^{2d}) \end{array} . \tag{3.70}$$

Again, the down arrows are duality in the UV theory on the LHS, and in each of the low-energy decoupled IR theories on the RHS. Again, for odd k the UV and the IR groups properly fit together, while for even k there is a mismatch in the dual gauge group of $2N_f$.

In summary, whenever the RG flows only involve D_{odd}, there is a successful, non-trivial check that the deformation maps properly between the UV and IR theories. On the other hand, whenever we flow to/from a D_{even} theory, there is a mismatch in the dual gauge groups pre and post deformation. An especially peculiar mismatch arises if we flow through an intermediate $D_{k'=\text{even}}$ theory, first deforming by $\sum_{i=k'}^{k-1} t_i X^{i+1}$ as in (3.67), and then deforming by $\frac{v}{2}\text{Tr}Y^2$ as in (3.69), which gives

$$D_{k+2} \longrightarrow A_{k'} + (k-k'+2)A_1 + \left(\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lceil \frac{k'-1}{2} \right\rceil + \left\lfloor \frac{k'-1}{2} \right\rfloor \right) A_1^{2d}. \tag{3.71}$$

For k' even, $\lfloor \frac{k'-1}{2} \rfloor - \lceil \frac{k'-1}{2} \rceil = -1$, and the number of 2d vacua in (3.71) differs from that in (3.69) for flowing directly with both $\sum_{i=k'}^{k-1} t_i X^{i+1}$ and $\frac{v}{2}\text{Tr}Y^2$ deformations together. Perhaps the conjectured quantum truncation of the chiral ring for D_{even} eliminates these puzzling mismatches in the higher dimensional representations for these flows, but we have not yet succeeded in showing how. We leave this as a challenge for future understanding.

3.4.5 The $SU(N_c)$ version of the RG flows

The above analysis was for $U(N_c)$. To adapt it for $SU(N_c)$, we write $X_{U(N_c)} = X_{SU(N_c)} + X_0 \mathbf{1}_{N_c}$, where $\mathrm{Tr} X_{SU(N_c)} = 0$, and likewise for Y, and can eliminate the unwanted X_0 and Y_0 fields via Lagrange multipliers, as in Section 3.3.3. The complication is that if we want to keep the enhanced $D_{k'+2}$ or $A_{k'}$ singularities as in (3.67) or (3.69), we need to add lower order ΔW terms, beyond those already present for the $U(N_c)$ version of the RG flows. These extra terms are needed in order to re-tune, to zero, the corresponding ΔW relevant deformations which would be generated by adding the Lagrange multiplier constraint terms, and which would generically further deform the RG flow to merely multiple A_1 vacua. For flows starting at D_{k+2} as in (3.48), the needed deformations are included in

$$\Delta W \subset \sum_{i=1}^{k-1} \frac{t_i}{i+1} \mathrm{Tr} X^{i+1} + \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{u_i}{i+1} \mathrm{Tr} X^{i+1} Y + \frac{m_Y}{2} \mathrm{Tr} Y^2 - \lambda_x \mathrm{Tr} X - \lambda_y \mathrm{Tr} Y. \tag{3.72}$$

For generic couplings in (3.72), the RG flow leads to vacua as

$$D_{k+2} \to (k+2) A_1 + \left\lfloor \frac{k-1}{2} \right\rfloor A_1^{2d}, \tag{3.73}$$

which is the same for $SU(N_c)$ and $U(N_c)$. One can now tune the couplings in (3.72) to enhance to an $A_{k'}$ or $D_{k'+2}$ singularity, and then the flow involves Higgsing as in e.g. (3.68), but with all $U(N)$ factors replaced with $SU(N)$. The tuning shifts of the couplings in (3.72) are complicated, and depend on how many eigenvalues n_0 are in the enhanced $D_{k'+2}$ or $A_{k'}$ vacua. We have verified that, despite these technical complications, the vacuum structure is qualitatively similar to that of the $U(N_c)$ case, replacing $U(n) \to SU(n)$ everywhere in Section 3.4.4.

Interestingly, there can be several options in performing the wanted shift, and these can result in different Casimirs along the flow. We illustrate this for the example $D_5 \to D_3$, and note that there are similar versions for other D flows. The first way to enhance to D_3 is via a tuned addition of the $\{m_Y, \lambda_x\}$ deformations to (3.64), where the needed shift of these couplings depends on the $\{t_3, t_2, t_1\}$ couplings in (3.64), as well as the multiplicities of the eigenvalues in the vacua. The Casimirs along the flow are then Y^2 and $t_3X^3+t_2X^2+t_1X$. Much as in (3.67), we indeed find one A_1^{2d} vacuum. Another option for $D_5 \to D_3 + \dots$ is to add only the $\frac{u_1}{2}\mathrm{Tr}X^2Y$ deformation in (3.72), with the other ΔW couplings set to zero. Then X^2 and Y^2 are Casimirs, but X and Y no longer anticommute as they did in the $U(N_c)$ case, and so a 2d solution is now of the form $X = x_1\sigma_1 - ix_3\sigma_3, Y = y_1\sigma_1 + iy_3\sigma_3$. We again find one 2d representation of the F- and D-terms, which reduces to the $U(N_c)$ 2d solution as $u_1 \to 0$. Different sets of lower order deformations in the chiral ring lead to different Casimirs along the flow, but nevertheless non-trivially give the same counting for the higher-dimensional vacua.

3.4.6 The $D_{\mathbf{odd}} \to D_{\mathbf{even}}$ RG flow and the hypothetical $D'_{\mathbf{even}}$ theory

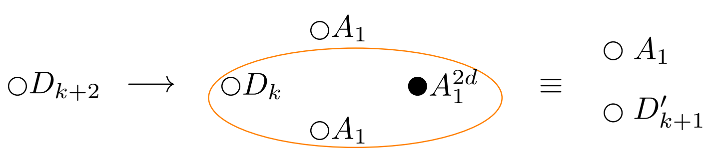
As discussed in the previous subsections, the D_{even} theories have some puzzles, whereas the D_{odd} theories appear to be under control. This suggests trying to understand the D_{even} theories via RG flows from the understood UV case: $D_{\text{odd}} \to D_{\text{even}}$. Indeed, the idea of embedding D_{even} in D_{odd} was the basis for the original conjecture [49] that quantum effects somehow make the troubling D_{even} theories similar to the nice D_{odd} theories. In this subsection, we examine the $D_{\text{odd}} \to D_{\text{even}}$ RG flow more carefully, and note that this flow has its own subtleties.

As seen in (3.67), the ΔW RG flow from $D_{k+2} \to D_{k'+2}$ comes with jumping number of A_1^{2d} representations, from the floor and ceiling functions, which is only straightforward for the $D_{\text{odd}} \to D_{\text{odd'}}$ cases. We here further discuss the relation and difference between $D_{\text{odd}} \to D_{\text{odd}}$ vs $D_{\text{odd}} \to D_{\text{even}}$. Consider starting from the D_{k+2} SCFT, with k odd, and deforming by ΔW. To simplify the discussion, we consider $U(N_c)$ (as opposed to $SU(N_c)$) and start with the ΔW deformation considered in (3.67) with $k' = k-2$: $D_{k+2} \to D_k + 2A_1 + A_1^{2d}$. The low-energy D_k theory is at $X = Y = 0$, the $2A_1$ theories are at X having eigenvalues $x_\pm$ with $Y = 0$, and the A_1^{2d} theory has (X,Y) values at (x_{2d}, y_{2d}) given by:

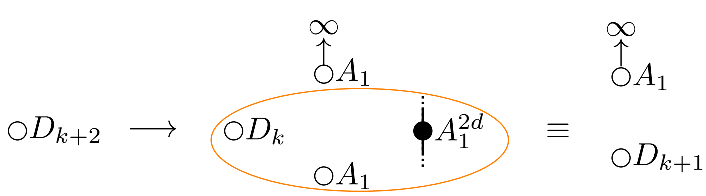
$$(x,y) = \begin{cases} (0,0) & \\ (x_\pm, 0) & t_k x_\pm^2 + t_{k-1} x_\pm + t_{k-2} = 0, \\ (x_{2d}, y_{2d}) & t_k (x_{2d})^2 + t_{k-2} = 0, \quad (y_{2d})^2 + t_{k-1}(x_{2d})^{k-1} = 0. \end{cases} \tag{3.74}$$

If we start at the D_{k+2} theory (as opposed to $\widehat{D}$), we can set $t_k = 1$, and t_{k-1} and t_{k-2} are the ΔW deformation parameters.

We now try to tune the superpotential couplings to collide the D_k singularity with an A_1 singularity, to get an enhanced D_{k+1} singularity. This can be accomplished by tuning $t_{k-2} \to 0$ in (3.74), which brings one of the A_1 singularities (x_+ or x_-) to the origin. Note that $t_{k-2} \to 0$ also brings x_{2d} and y_{2d} to the origin. We denote this enhancement as $D_k + A_1 + A_1^{2d} \to D'_{k+1}$, where the prime distinguishes the theory from the even D_{k+1} theory that one would obtain by flowing directly from the $\widehat{D}$ theory. We can formally obtain that latter theory, D_{k+1}, directly from the $\widehat{D}$ fixed point, by taking $t_k \to 0$ along with $t_{k-2} \to 0$ in (3.74); this brings one of the $x_\pm$ to the origin and the other to infinity, and then the last equation in (3.74) gives the line of A_1^{2d} solutions (3.54) where $D_{\text{even}} \to \widehat{A}$, since (3.74) is satisfied for all x_{2d} when $t_k = 0$. The two procedures are indicated in the Figure 3.7.



(a) The 2d vacuum and an A_1 both collapse to the origin.



(b) One A_1 goes to the origin while the other goes off to infinity, and the 2d vacuum becomes a degenerate line of 2d representations as in (3.54).

Figure 3.7: We enhance the k odd D_k singularity to a D_{k+1} singularity in one of two ways.

The two procedures suggest that perhaps there are actually two types of D_{even} theories. One is the D'_{even} theory of Figure 3.7a, which can actually be obtained from the RG flow

$D_{\text{odd}} \to D'_{\text{even}}$, and which therefore inherits the simpler properties of D_{odd}. The other is the mysterious D_{even} theory of Figure 3.7b, which actually is not obtained from RG flow from D_{odd}, but instead only from $\widehat{D} \to D_{\text{even}}$, since it requires $t_k = 0$ and the D_{odd} theory had $t_k = 1$. The latter, D_{even} theory has the puzzles, discussed in the previous subsections, associated with the $D_{\text{even}} \to \widehat{A}$ moduli space of vacua and the non-truncated chiral ring.

We have thus considered the possibility that Brodie duality actually only applies to the simpler D'_{even} theory, which inherits the truncated chiral ring from D_{odd}, and does not apply to the D_{even} theory. However, this scenario also has challenges. If we take seriously the idea that a D'_{k+1} (for k odd) theory is made by bringing together $D_k + A_1 + A_1^{2d}$, this seems to suggest that the chiral ring of the D'_{k+1} theory contains $(3k-1)N_f^2$ mesonic operators, where the $-N_f^2$ are those in the A_1 singularity, which decouples from D'_{k+1} in the IR. On the other hand, assuming that Brodie duality applies to D'_{k+1}, we would have expected $3(k-1)N_f^2$ mesonic operators. The D'_{k+1} theory has an extra $2N_f^2$ mesonic operators. Perhaps then, in collapsing the A_1 and A_1^{2d} theories to the D_k theory at the origin, a slightly modified version of Brodie duality applies, with $\alpha_{D'_{k+2}} = 3k+2$. We have also tried to cure the apparent a-theorem violations by adding the $2N_f^2$ mesons to the UV D_{even} theory. But the results did not look promising: the extra operators seem to become free at too large x to cure the apparent wrong sign of Δa. It is still possible that some modified version of Brodie duality resolves these puzzles, and we invite the interested reader to try.

3.5 The W_{E_7} Fixed Point and Flows

The W_{E_7} SCFT arises as the IR limit of a relevant superpotential deformation to the $\widehat{E}$ SCFT, with corresponding superpotential

$$W_{E_7} = \frac{1}{3}\text{Tr}Y^3 + s_1 \text{Tr}YX^3. \tag{3.75}$$

The $\text{Tr}YX^3$ term is a relevant deformation to the $\widehat{E}$ fixed point for $x > x_{E_7}^{\min} \approx 4.12$, where $x_{E_7}^{\min}$ was determined via a-maximization in [38]; here we will assume that $x > x_{E_7}^{\min}$.

The F-terms of the undeformed E_7 superpotential in (3.75) are given by

$$Y^2 + s_1 X^3 = 0, \tag{3.76}$$

$$X^2 Y + XYX + YX^2 = 0, \tag{3.77}$$

from which it follows that the chiral ring does not truncate classically. We may write the generators of the classical chiral ring in a basis

$$\begin{aligned}
\Theta_{(1,n)} &= X^n, \\
\Theta_{(2,n)} &= YX^n, \\
\Theta_{(3,n)} &= XYX^n, \\
\Theta_{(4,n)} &= YXYX^n; \quad n = 0, 1, ...
\end{aligned} \tag{3.78}$$

3.5.1 Previously proposed dualities for W_{E_7}

In [46], it was pointed out that for the W_{E_7} theories the condition (3.26) is violated for $x \gtrsim 27$, so some new dynamics is needed there, or at smaller x. The dual theory proposed in [46] resolves this apparent a-theorem violation, since it implies different IR phases for $x \gtrsim 26.11$ [46]. The duality of [46] requires that the chiral ring truncates, similar to the conjecture in [49] for D_{even}, as

$$YX^6 + bXYX^5 = 0 \quad \text{in the chiral ring} \tag{3.79}$$

for some constant b. It is not yet known if the proposed quantum constraint (3.79) is correct, or how it arises. Imposing (3.79), the chiral ring of the electric theory is truncated to 30 independent generators, listed for reference in Table 3.2. The resulting IR dual description of the E_7 fixed point has gauge group $SU(\alpha_{E_7} N_f - N_c)$ with $\alpha_{E_7} = 30$, and the usual duality map reviewed in Section 3.2.2. The dual theory has superpotential[11] [46]

$$W_{E_7}^{mag} \backsim \frac{1}{3}\text{Tr}\hat{Y}^3 + \hat{s}_1 \text{Tr}\hat{Y}\hat{X}^3 + \sum_{j=1}^{30} M_j \tilde{q} \Theta_{30-j}(\hat{X}, \hat{Y}) q. \tag{3.80}$$

[11] As in [46], we scale the factors of μ to unity.

In addition to the usual tests of duality—'t Hooft anomaly matching, that the charge assignment for the magnetic fields under the global symmetry is consistent with the duality map—it was verified in [47] that the superconformal index of the dual theories agrees, at least in the Veneziano limit (away from that limit, the duality and agreement of their superconformal indices suggests new mathematical identities).

As we discuss in the following subsections, we find similar puzzles for the E_7 theories as with the D_{even} theories. In the following, we mirror our analysis of the $W_{D_{k+2}}$ theories for W_{E_7}; as such, we will be brief when analysis or discussion is similar to what has already been discussed in Section 3.4. Much as we found for D_{even}, we fail to find evidence for this truncation, and point out additional hurdles for the conjectured duality.

Table 3.2: We list the 30 independent generators $\Theta_j,\ j = 1, \ldots, 30$ of the proposed E_7 chiral ring, where N is the polynomial degree.

j	N	Θ_j	j	N	Θ_j
1	1	1	16	11	YX^4
2	2	X	17	11	XYX^3
3	3	Y	18	12	X^6
4	4	X^2	19	12	$YXYX^2$
5	5	YX	20	13	YX^5
6	5	XY	21	13	XYX^4
7	6	X^3	22	14	X^7
8	7	YX^2	23	14	$YXYX^3$
9	7	XYX	24	15	YX^6
10	8	X^4	25	16	X^8
11	8	YXY	26	16	$YXYX^4$
12	9	YX^3	27	17	YX^7
13	9	XYX^2	28	18	X^9
14	10	X^5	29	19	YX^8
15	10	$YXYX$	30	21	YX^9

3.5.2 Matrix-related flat directions at the origin

We consider the moduli space of vacuum solutions of (3.76)-(3.77) with D-term constraints (3.7), setting $Q=\tilde{Q}=0$. The only 1d solution corresponds to the E_7 singularity at the origin. (3.76) shows that Y^2 and X^3 are Casimirs, yielding Casimir conditions $X^3=x^3\mathbf{1}_d$, and $Y^2=y^2\mathbf{1}_d$ for a d-dimensional representation. There is a line of $d=2$ solutions to these conditions analogous to (3.54),

$$X = x\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad Y = y\sigma_1$$

$$y^2 + s_1 x^3 = 0. \tag{3.81}$$

for $\omega = e^{2\pi i/3}$. As X and Y are not traceless, this flat direction is present for only $U(N_c)$[12].

In general, E_7 has vacua with multiple copies of the solution (3.81), with the remaining eigenvalues of X and Y at the origin, giving a moduli space of supersymmetric vacua labeled by y_i^2 and x_i^3 satisfying (3.81), for $i=1,\ldots,\lfloor N_c/2\rfloor$. These vacua Higgs the gauge group in a way that turns out to be analogous to the D_{even} case discussed in 3.4.2. In particular, for $N_c=2n$ with n copies of the 2d vacuum (3.54) and unequal expectation values of the y_i^2, x_i^3, the resulting breaking pattern is $U(2n)\to U(n)_D\to U(1)^n$. In summary, much as in (3.56), there is a (classical) flat direction:

$$E_7 \to \widehat{A}, \qquad \text{with} \quad U(N_c)\to U(\lfloor N_c/2\rfloor)_D \quad \text{and} \quad N_f^{low} = 2N_f, \tag{3.82}$$

so $x^{low} = N_c^{low}/N_f^{low} = (N_c/2)/(2N_f) = x/4$. If we assume that Kutasov-Lin's duality [46] holds, then we are led to a puzzle similar to that of the D_{even} theories: the moduli spaces of the electric theory and its dual differ, and the low-energy theories on the flat directions of the two conjectured duals, $SU(N_c/2)_{\widehat{A}}$ and $SU(15N_f - N_c/2)$, are not clearly related. This flat direction is related to the classical nontruncation of the E_7 chiral ring, and again provides us with a way to sharpen the

[12] For special cases of (3.81) there will be $SU(N_c)$ flat directions; for example, when there are equal multiplicities of X, ωX, and $\omega^2 X$ along the line given in (3.81). In that case, one could check the proposed $SU(N_c)$ duality along the corresponding flat directions.

puzzle of how the truncation occurs by asking what lifts the flat direction.

Independent of the conjectured duality [46], the deformation (3.82) seemingly violates the a-theorem (3.22) for sufficiently large x. As in Section 3.4.2, we compute $a_{UV}(x)$ for the W_{E_7} theory, with gauge group $U(N_c)$ and N_f flavors, as in [38]. Likewise, $a_{IR}(x)$ for the $\widehat{A}$ theory, with gauge group $U(N_c/2)$ and $2N_f$ flavors, is computed as in [25]. We include the effects of all mesons hitting the unitarity bound assuming that the chiral ring is quantumly truncated, such that all the operators listed in Table 3.2 are taken into account, and work in the Veneziano limit. We plot until the bottom of the conformal window—which occurs before the electric E_7 theory's stability bound, $x < 30$ as predicted by duality—such that we expect the a-theorem to hold in the whole range plotted.

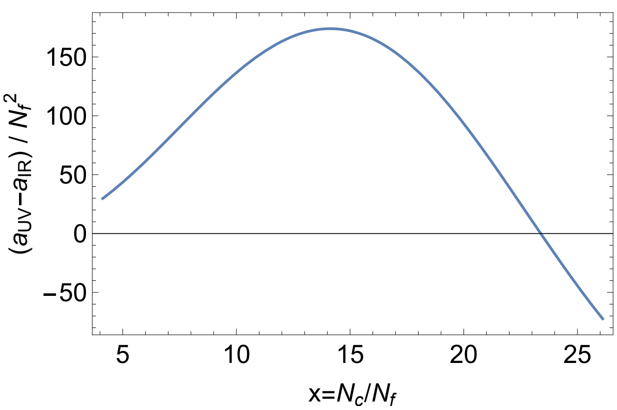


Figure 3.8: $(a_{UV} - a_{IR})/N_f^2$ for W_{E_7} in the UV and $\widehat{A}$ in the IR. The E_7 deformation term in the UV theory is relevant for $x \gtrsim 4.12$, while the corresponding term in Kutasov-Lin dual is relevant if $x \lesssim 26.11$. The $\hat{A}$ theory is UV-free in this whole range.

As seen Figure 3.8, this flat direction seems to violate the a-theorem in the conformal window for $x \gtrsim 23.39$. Unlike the D_{even} case, this violation occurs for x larger than the value where the mesons removed by the proposed quantum constraint 3.79 would hit the unitarity bound and become free; the first such meson that would be nonzero involves the operator YX^6, which would become free at $x = 21$. To understand the effect that these would-be mesons would have on the computation of a for this flat direction, we have performed the same check as in Figure 3.8, but without imposing the proposed constraint. It turns out that this is not enough; the effect of including these operators in the ring is only to push the range of the apparent a-theorem violation to $x \gtrsim 23.44$.

The apparent violation of the a-theorem for these flat directions must of course be somehow resolved. As in the discussion in Section 3.4.2, either these flat directions are lifted in a way we don't understand, or some additional degrees of freedom make the calculation of a incorrect—perhaps in the UV W_{E_7} theory. The arguments made in Section 3.4.2 would also apply here, and suggest that the former is not the solution. Since the calculation of a in Figure 3.8 already took into account the proposed W_{E_7} duality, we are left with a puzzle.

3.5.3 $SU(N_c)$-specific (as opposed to $U(N_c)$) flat directions

We now study $SU(N_c)$ flat directions of the W_{E_7} theory, imposing the tracelessness of the adjoints with Lagrange multipliers λ_x, λ_y:

$$W_{E_7} = \frac{1}{3}\mathrm{Tr}Y^3 + s_1 \mathrm{Tr}YX^3 - \lambda_x \mathrm{Tr}X - \lambda_y \mathrm{Tr}Y. \tag{3.83}$$

When $N_c = 2m + 3n$ for m, n integers, there is a flat direction labeled by λ_y,

$$\begin{aligned}\langle X \rangle &= \left(\frac{\lambda_y}{s_1}\right)^{\frac{1}{3}} \begin{pmatrix} \mathbf{0}_m & & & & \\ & \mathbf{0}_m & & & \\ & & \omega \mathbf{1}_n & & \\ & & & \omega^2 \mathbf{1}_n & \\ & & & & \omega^3 \mathbf{1}_n \end{pmatrix}, \\ \langle Y \rangle &= (\lambda_y)^{\frac{1}{2}} \begin{pmatrix} -\mathbf{1}_m & & & & \\ & \mathbf{1}_m & & & \\ & & \mathbf{0}_n & & \\ & & & \mathbf{0}_n & \\ & & & & \mathbf{0}_n \end{pmatrix},\end{aligned} \tag{3.84}$$

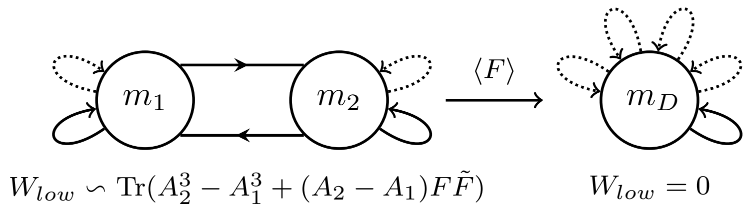
where $\omega = e^{2\pi i/3}$ and off-diagonals are zero. (3.84) is the special case of $k = 3$ in (3.60).

Along this flat direction, the gauge group is Higgsed $SU(2m+3n) \rightarrow SU(m)^2 \times SU(n)^3 \times U(1)^4$. The low-energy $SU(m)^2$ sector includes N_f massless flavors, along with bifundamentals $F, \tilde{F}$ and adjoints A_1, A_2 coming from the adjoint X of the original theory at the origin, with a low-energy superpotential that is cubic in the massless fields (written in Figure 3.9a). Thus, each $SU(m)$ node corresponds to a W_{A_2} theory plus extra flavors from the bifundamentals. The low-energy $SU(n)^3$ sector includes N_f massless flavors along with three pairs of bifundamentals F_{12}, F_{23}, F_{13}, and their conjugates, coming from the adjoint Y of the original theory at the origin. There is an IR superpotential for these fields $W_{low} \backsim \text{Tr}(F_{12}F_{23}\tilde{F}_{13} + \tilde{F}_{12}\tilde{F}_{23}F_{13})$, which corresponds to making a loop around the quiver diagram shown in Figure 3.9b. All other components from X and Y are either eaten in the Higgsing, or get a mass from the superpotential (3.83), such that the $SU(m)^2$ and $SU(n)^3$ sectors decouple from each other at low energies. These low-energy theories are summarized in the left-most quiver diagrams in Figure 3.9.

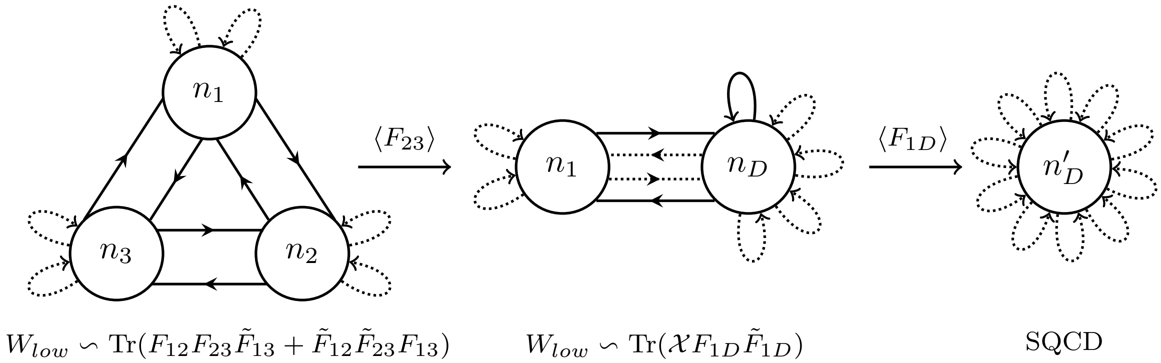
We can then go along a further flat direction of the low-energy $SU(m)^2$ theory, where we give an arbitrary vev to the massless F, such that $SU(m)_1 \times SU(m)_2$ breaks to the diagonal subgroup $SU(m)_D$. The low-energy $SU(m)_D$ has an adjoint that remains massless, and IR superpotential $W_{low} = 0$ from integrating out the massive fields, such that this node corresponds to an $\widehat{A}$ theory with $2N_f$ massless flavors. This IR theory is depicted on the RHS of Figure 3.9a.

The low-energy $SU(n)^3$ sector has a similar series of flat directions, where one of the massless bifundamentals has non-zero expectation value, depicted by the arrows in Figure 3.9b. For example, giving a vev first to F_{23} breaks $SU(n)^3 \rightarrow SU(n)_D \times SU(n)$, resulting in an IR theory with one massless adjoint $\mathscr{X}$ charged under $SU(n)_D$ and one massless bifundamental pair coming from $F_{13}, \tilde{F}_{12}$. Identifying the indices appropriately, these massless fields have an IR superpotential $W_{low} \backsim \text{Tr}(\mathscr{X} F_{1D}\tilde{F}_{1D})$. At this stage there is another flat direction where F_{1D} has a non-zero expectation value, Higgsing $SU(n) \times SU(n)_D \rightarrow SU(n)_D$, where the remaining node corresponds to SQCD with $3N_f$ massless flavors.

Again, it is not known if the $\widehat{A}$ theory in 3.9a at low-energies has a dual. On the other hand,



(a) $SU(m)_1 \times SU(m)_2$ sector, Higgses to $W_{\widehat{A}}$.



(b) $SU(n)_1 \times SU(n)_2 \times SU(n)_3$ sector, Higgses to SQCD.

Figure 3.9: Flat directions for E_7, $N_c = 2m + 3n$, integrating out massive fields (denoted by dotted lines) and fields eaten by the Higgs mechanism (not shown). The subscripts label the gauge groups and their matter.

the low-energy SQCD theory in 3.9b has a dual given by Seiberg duality. So we can consider, for example, $N_c = 3n$ ($m = 0$), which along the flat direction of 3.9b breaks to $SU(n)$ SQCD with $3N_f$ flavors, and thus has a Seiberg dual with gauge group $SU(n)_D \to SU(3N_f - n)_D$. The dual theory must also have moduli corresponding to the flat direction of the electric theory. Extrapolating back to the origin along these dual flat directions, to try to un-Higgs the $SU(3N_f - n)_D$ dual by reversing the process analogous to 3.9b, suggests a dual gauge group of $SU(9N_f - 3n)$ on the magnetic side at the origin. This disagrees with the dual gauge group of Kutasov-Lin, which maps $SU(3n) \to SU(30N_f - 3n)$. That latter theory has a flat direction, corresponding to 3.9b where instead $SU(30N_f - 3n) \to SU(10N_f - n)_D$.

To summarize, these flat directions pose puzzles for the proposed W_{E_7} duality, both in the $\widehat{A}$ theory of 3.9a, and in the SQCD theory of 3.9b. We have not found a quantum mechanism for lifting these flat directions.

3.5.4 Case studies in E_7 RG flows from ΔW deformations

In this subsection, we deform the W_{E_7} SCFT (3.75) by several examples of relevant ΔW for cases where the resulting IR theory is under better control in terms of duality. We study how the vacuum structure matches between the UV and IR electric and magnetic descriptions, focusing on the apparent puzzles of the UV E_7 theory.

These examples demonstrate several new features, as compared with the A_k and D_{k+2} series. One difference is that the deformed chiral ring admits $d > 2$ dimensional representations. Further, we explore cases in which enhancements of the singularities in the IR of an RG flow (via tuning couplings of the deformations) do not preserve the number of higher-dimensional vacua. Interestingly, for some RG flows the $SU(N_c)$ version of a flow with the same 1d vacuum structure as the corresponding $U(N_c)$ flow has a different set of higher-dimensional vacua. Furthermore, we explicitly construct some RG flows for which the ΔW deformations are not apparently relevant.

$E_7 \to A_2$: 3d vacua

We begin with the RG flow $E_7 \to A_2$ flow for gauge group $U(N_c)$, taking $x > x_{E_7}^{\min}$:

$$W = \frac{1}{3}\mathrm{Tr}Y^3 + s_1 \mathrm{Tr}YX^3 + \frac{t_1}{2}\mathrm{Tr}X^2, \tag{3.85}$$

which yields the F-terms

$$Y^2 + s_1 X^3 = 0 \tag{3.86}$$

$$s_1(YX^2 + XYX + X^2Y) + t_1 X = 0. \tag{3.87}$$

There are seven 1d solutions to (3.86)-(3.87): two coincident at $X = Y = 0$, corresponding to the A_2 theory, and five solutions with nonzero X and Y eigenvalues, corresponding to A_1 theories; as always, the 1d solutions correspond, as in Arnold's ADE singularity resolutions, to adjoint Higgsing of the $G = ADE$, preserving r_G. Taking X and Y to be matrices, it follows from (3.86)-(3.87) that $X^3 \backsim Y^2$ are Casimirs along the flow, so that we may write $X^3 = x^3\mathbf{1}_d$ and $Y^2 = y^2\mathbf{1}_d$ for a d-dimensional representation. There is a 2d as well as a 3d representation that solve the

F-terms, D-terms (3.7), and Casimir conditions,

$$X_{2d} = \frac{1}{2}\sqrt{\frac{|t_1|^2}{|s_1|}}(\sigma_1 + i\sigma_3), \quad Y_{2d} = \frac{1}{2}\sqrt{\frac{|s_1| t_1^{3/2}}{s_1^2 (t_1^*)^{1/2}}}(\sigma_1 - i\sigma_3) \tag{3.88}$$

$$X_{3d} = \sqrt{\frac{t_1}{2s_1}}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_{3d} = -\sqrt{\frac{t_1}{2s_1}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.89}$$

These solutions are nilpotent (recall the discussion in Section 3.2.2); these vacua are inherently nondiagonalizable, with the D-terms satisfied via $[X, X^\dagger] = -[Y, Y^\dagger]$. Expanding (3.85) in these vacua, the adjoints have mass terms, and so the low-energy theories are SQCD with extra massless flavors. The ΔW deformation in (3.85) thus gives the RG flow

$$E_7 \to A_2 + 5A_1 + A_1^{2d} + A_1^{3d} (+\ldots?). \tag{3.90}$$

The $(+\ldots?)$ indicate that there might be additional $d > 3$ dimensional vacuum solutions, beyond the ones that we found here[13]. In the following we will assume that there are no such additional vacua in (3.90), but we do not have a proof that this is the case.

If there are n_0 eigenvalues at the origin, n_i in the i'th A_1 node, n^{2d} in the A_1^{2d} node, and n^{3d} in the A_1^{3d} node, such that $N_c = n_0 + \sum_{i=1}^5 n_i + 2n^{2d} + 3n^{3d}$, then the gauge group is Higgsed in the electric and proposed magnetic descriptions (for x in the conformal window):

$$\begin{array}{ccc} U(N_c) & \xrightarrow{\langle X\rangle, \langle Y\rangle} & U(n_0)\prod_{i=1}^{5} U(n_i) \times U(n^{2d}) \times U(n^{3d}) \\ \downarrow & & \downarrow \\ U(30N_f - N_c) & \xrightarrow{\langle \hat{X}\rangle, \langle \hat{Y}\rangle} & U(2N_f - n_0)\prod_{i=1}^{5} U(N_f - n_i) \times U(2N_f - n^{2d}) \times U(3N_f - n^{3d}) \end{array} . \tag{3.91}$$

The down arrows are Kutasov-Lin duality for the E_7 $U(N_c)$ theory in the UV, and Kutasov

[13]We use the $SU(N)$ or $U(N)$ symmetry to gauge fix one real adjoint's worth of components in X, $X^\dagger$, Y, and $Y^\dagger$, and the remaining entries are constrained by the D- and F-terms, along with any Casimir conditions. We did not find an analytic way to construct, or exclude, higher-dimensional solutions beyond scanning computationally. Even gauge-fixing, scanning the solution space is harder for larger d, and so in (3.90) we only completed the scan for $d \leq 3$.

or Seiberg duality for the approximately decoupled low-energy gauge group factors in the IR. Comparing the UV and IR of the dual theories of the lower row of (3.91) as we did for the D-series, there is a mismatch in the dual gauge groups of $10N_f$. Indeed, it is immediately evident that (3.33) is not satisfied for $\alpha = 30$, since there are precisely 7 vacua with $d_i = 1$, and $23 \neq \sum d_i^2$ for integers $d_i > 1$. Something new is needed, beyond simply decoupled copies of SQCD in the various d_i-dimensional vacua.

To recover the $SU(N_c)$ version of this flow, we must deform the superpotential (3.85) by the operators $\mathrm{Tr}Y^2$, $\mathrm{Tr}XY$, $\mathrm{Tr}X$, $\mathrm{Tr}Y$, (the latter two with Lagrange multipliers) whose coefficients are shifted appropriately. The 2d representation for the deformed superpotential smoothly matches onto the $U(N_c)$ solution in (3.88) upon taking the coefficients of the lower order deformations to zero. The analogous check for the 3d representations in (3.89) turns out to be technically challenging, and while we expect that it also matches, such that the $SU(N_c)$ version of the flow will match onto (3.91), we have not verified this. (For reasons that will become apparent in Section 3.5.4, this can be a subtle issue in the E-series.)

$E_7 \to D_5$: Disappearing vacua?

We here consider the flow $E_7 \to D_5$ for $U(N_c)$ gauge group, which corresponds to the superpotential (normalizing the couplings in the UV E_7 theory to unity)

$$W = \mathrm{Tr}\frac{1}{3}Y^3 + \mathrm{Tr}YX^3 + t_1\mathrm{Tr}XY^2 + \frac{t_2}{4}\mathrm{Tr}X^4. \tag{3.92}$$

The F-terms of (3.92) are given by

$$Y^2 + X^3 + t_1\{X,Y\} = 0, \tag{3.93}$$

$$YX^2 + XYX + X^2Y + t_1Y^2 + t_2X^3 = 0. \tag{3.94}$$

The 1d vacuum structure along this flow consists of the D_5 theory at $X = Y = 0$, and 2 A_1's away from the origin. To study higher-dimensional vacua we note that there are no simple Casimirs of (3.93)-(3.94), except of course the F-terms themselves. There is a 2d solution to the F-terms (3.93)-(3.94) and D-terms (3.7) of the form $X = x_0\mathbf{1} + x_3\sigma_3$, $Y = y_0\mathbf{1} + y_3\sigma_3$, where

$\{x_0, x_3, y_0, y_3\}$ are determined functions of the couplings t_1 and t_2. Then, the RG flow leads to vacua

$$U(N_c), \quad t_i \text{ generic}: \quad E_7 \to D_5 + 2A_1 + A_1^{2d} \ (+\ldots?). \tag{3.95}$$

As in (3.90) and the associated footnote, there might additionally be $d > 3$ vacua, indicated here with $(+\ldots?)$. Performing the same check as in (3.91), assuming Kutasov-Lin duality for the UV $U(N_c)$ E_7 theory, there is a mismatch in the UV and IR dual gauge groups, this time of $15N_f$:

$$\begin{array}{ccc} U(N_c) & \overset{\langle X\rangle,\langle Y\rangle}{\longrightarrow} & U(n_0)\prod_{i=1}^{2}U(n_i)\times U(n^{2d}) \\ \downarrow & & \downarrow \\ U(30N_f - N_c) & \overset{\langle \hat{X}\rangle,\langle \hat{Y}\rangle}{\longrightarrow} & U(9N_f - n_0)\prod_{i=1}^{2}U(N_f - n_i)\times U(2N_f - n^{2d}) \end{array} . \tag{3.96}$$

The flow (3.95) is for generic deformations t_1, t_2 in (3.92). There are special values of the coupling t_2 for which the 2d representation "goes away" because X or Y becomes proportional to the identity, or proportional to each other—in either case, the solution is then accounted for by 1d vacua. This possibility does not occur for the D_{k+2} RG flows. The resulting flows are summarized by (for the rest of this subsection we refrain from putting the (+...?), but note that everywhere there is the possibility of $d > 2$ dimensional vacua):

$$U(N_c), \quad t_2 = t_1(7 \pm 2\sqrt{6}) \text{ or } 5t_1\frac{(-6 \mp \sqrt{6})}{(-6 \pm \sqrt{6})}: \quad E_7 \to D_5 + A_2 \tag{3.97}$$

$$U(N_c), \qquad t_2 = t_1 \text{ or } t_1(1 \pm \sqrt{6}): \qquad E_7 \to D_5 + 2A_1 \tag{3.98}$$

For the flow (3.97), the eigenvalues corresponding to the 1d and 2d A_1 singularities in (3.95) come together, enhancing to an A_2 singularity. Labeling the multiplicities of X and Y's eigenvalues as in (3.96), then for the enhancement (3.97) the eigenvalues rearrange such that the electric theory is Higgsed $U(N_c) \to U(n_0) \times U(n_1 + n_2 + 2n^{2d})$. For the case (3.98), the eigenvalues corresponding to the A_1^{2d} theory in (3.95) match onto copies of the eigenvalues corresponding to the 1d A_1 theories, such that in the IR the vacua are $D_5 + 2A_1$. In this case, the eigenvalues in the electric version of the flow rearrange such that $U(N_c) \to U(n_0) \times U(n_1 + n^{2d}) \times U(n_2 + n^{2d})$.

This feature that a 2d representation can "go away" is also present in the $SU(N_c)$ version of the flow (3.92). As was the case for the D-series flows discussed in Section 3.4.5, there are multiple sets of deformations ΔW that one can add to (3.92) to recover the same 1d vacuum structure as in (3.95) for $SU(N_c)$ gauge group[14]. For instance, one possibility is

$$\Delta W = v_1 \mathrm{Tr} X^2 Y + \frac{v_2}{3} \mathrm{Tr} X^3 + v_3 \mathrm{Tr} XY + v_4 \mathrm{Tr} X^2 + v_5 \mathrm{Tr} Y^2 - \lambda_x \mathrm{Tr} X - \lambda_y \mathrm{Tr} Y. \tag{3.99}$$

Surprisingly, there are three 2d vacua for the flow (3.92) plus (for instance) ((3.99)): one which matches continuously onto the $U(N_c)$ 2d vacuum when the couplings of the lower order deformations are taken to zero, and two which do not. The additional two 2d vacua have the property that X and Y become proportional to each other in the limit that the couplings of the additional deformations (e.g., the v_i and λ_x, λ_y in ((3.99))) vanish. In other words, these additional vacua vanish precisely when we cannot perform the shift of the $SU(N_c)$ flow to the preferred $U(N_c)$ origin—i.e., when we can only flow down to decoupled A_1 theories in the IR. In sum, the vacua of this flow are

$$SU(N_c), \quad t_i \text{ generic}: \quad E_7 \to D_5 + 2A_1 + 3A_1^{2d}. \tag{3.100}$$

However, as with our similar previous examples, using the known duals of the IR theories in (3.100) does not fit with the $\alpha_{E_7} = 30$ of Kutasov-Lin duality, essentially because (3.33) is not satisfied: here it is because $\alpha_{E_7} \neq \alpha_{D_5} + 2\alpha_{A_1} + 3 \times 2^2 \alpha_{A_1}$, i.e. $30 \neq 9 + 2 + 12$.

Analogously to the $U(N_c)$ flow (3.95), one of the 2d vacua in (3.100) reduces to 1d vacua in special cases. The difference here is that the other two 2d vacua in (3.100) remain:

$$SU(N_c), \quad t_2 = t_1(7 \pm 2\sqrt{6}): \quad E_7 \to D_5 + A_2 + 2A_1^{2d} \tag{3.101}$$

$$SU(N_c), \quad t_2 = t_1 \text{ or } t_1(1 \pm \sqrt{6}): \quad E_7 \to D_5 + 2A_1 + 2A_1^{2d} \tag{3.102}$$

This feature that the 2d vacua can "disappear" for particular values of the couplings is reminiscent of the wall crossing phenomena for BPS states. There are hints that this is a general phenomenon in the E-series. For instance, there is a similar effect in the $E_8 \to D_6$ flow, as we discuss in

[14] There are at least three possible sets of deformations, and we've explicitly checked that two of these (including ((3.99))) yield the same 2d vacuum structure.

Appendix A.2.3. It is presently unclear to us how to this phenomenon fits with proposed duals, and we leave such an exploration for future work.

$E_7 \to A_6$: A seemingly irrelevant deformation

As expected from Arnold's singularities and deformations, there are RG flows corresponding to adjoint Higgsing of $G = A, D, E$. For some of these Higgsing patterns, the corresponding ΔW deformation is not immediately apparent. A general treatment of how to deform and resolve the ADE singularities by giving expectations values to the Cartan elements is described in [61], and this formalism is applied in [62] to several of the resolutions of present interest to us. We adapted those constructions to obtain the deformations of this section.

We here consider the ΔW deformation which leads to the RG flow $E_7 \to A_6$. This is given by $\Delta W \sim \mathrm{Tr}X^7$. At first glance, this ΔW seems irrelevant at the W_{E_7} SCFT, since it scales with a higher $U(1)_R$ charge than the terms in (3.75), but we know that such a flow should be possible (for instance, we can cut the E_7 Dynkin diagram to recover the A_6 diagram, as demonstrated for other cases in Figure 3.1). The resolution to this puzzle is that only a special shift of the deformation couplings will recover the A_6 singularity in the IR—even for the $U(N_c)$ case. The clearest way to see the enhancement of the A_6 singularity is through a change of variables. Since the change of variables is already complicated in the $U(N_c)$ case, we will only consider this flow for $U(N_c)$ gauge group here. We analyze other E-series flows whose ΔW deformations seem irrelevant in this sense in Appendix A.2.

We start with W_{E_7} plus ΔW deformations,

$$W = \frac{1}{3}\mathrm{Tr}Y^3 + s_1\mathrm{Tr}YX^3 + t_1\mathrm{Tr}XY^2 + \frac{T_2}{2}\mathrm{Tr}Y^2 + T_3\mathrm{Tr}XY + \frac{T_4}{2}\mathrm{Tr}X^2. \tag{3.103}$$

It follows from the F-terms of (3.103) that there are seven 1d vacua in the IR, corresponding to seven A_1 theories (we will discuss higher-dimensional vacua below). It is useful to next linearly shift the fields $X \to X + n$, $Y \to Y + m$, where we choose m and n as functions of the couplings in (3.103) to cancel the linear terms in X and Y which result from the change of variables. Dropping constants, the superpotential can then be rewritten as

$$W = \frac{1}{3}\mathrm{Tr}Y^3 + s_1 \mathrm{Tr}YX^3 + t_1 \mathrm{Tr}XY^2 + t_2 YX^2 + \frac{t_3}{3}\mathrm{Tr}X^3 + \frac{t_4}{2}\mathrm{Tr}Y^2 + t_5 \mathrm{Tr}XY + \frac{t_6}{2}\mathrm{Tr}X^2, \quad (3.104)$$

where the t_i's are defined in terms of the couplings in (3.103) and m, n. We then implement the following change of variables for all $t_1 \neq 0$:

$$Y = U - \frac{3t_1}{7}X - \frac{7s_1}{t_1}X^2 - \frac{343s_1^2}{96t_1^3}X^3. \quad (3.105)$$

Such a change of variables is holomorphic, and has the property that the new field variable U is single-valued in terms of the variable being replaced (Y). (3.105) shifts around the R-charges of the fields, but causes no problems; in particular, the metric in the scalar potential acts to compensate and keep the actual vacua the same. Rewriting (3.104) in terms of U and X will result in many terms, including the terms $\mathrm{Tr}X^7$ and $\mathrm{Tr}U^2$ which we identify as corresponding to the A_6 theory and which are now apparently relevant from the perspective of the UV theory, plus eight even more relevant deformations.

So far, all we've accomplished is to rewrite the flow $E_7 \to 7A_1$ in a complicated way. At this point, however, one can show that there is a unique shift of the couplings $\{t_2, t_3, t_4, t_5, t_6\}$ in terms of t_1, s_1, such that all of the coefficients to terms more relevant than those which we will identify with the A_6 theory vanish. Implementing this shift, (3.104) becomes

$$\begin{aligned} W = {} & \frac{1}{3}\mathrm{Tr}U^3 - \frac{343s_1^2}{96t_1^3}\mathrm{Tr}U^2X^3 + \frac{117649s_1^4}{9216t_1^6}\mathrm{Tr}UX^6 - \frac{40353607s_1^6}{2654208t_1^9}\mathrm{Tr}X^9 - \frac{7s_1}{4t_1}\mathrm{Tr}U^2X^2 \\ & + \frac{2401s_1^3}{192t_1^4}\mathrm{Tr}UX^5 - \frac{823543s_1^5}{36864t_1^7}\mathrm{Tr}X^8 + \frac{4t_1}{7}\mathrm{Tr}U^2X - \frac{49s_1^2}{48t_1^2}\mathrm{Tr}UX^4 - \frac{16807s_1^4}{4608t_1^5}\mathrm{Tr}X^7 \\ & - \frac{48t_1^3}{343s_1}\mathrm{Tr}U^2. \end{aligned} \quad (3.106)$$

Studying the F-terms of this superpotential and expanding (3.106) in the vacua, there is one vacuum at the origin corresponding to the A_6 theory, and one away from the origin corresponding to an A_1 theory. Thus, we have recovered the desired flow.

We have also studied the 2d vacuum structure of this RG flow.[15] For generic values of the

[15] We have not as of writing attempted to find $d > 2$ dimensional vacua for this flow.

couplings in (3.104), there are nine 2d vacua which we can parameterize as $X = x_0\mathbf{1} + x_3\sigma_3$, $Y = y_0\mathbf{1} + y_3\sigma_3$, such that the generic ΔW deformations lead to the vacua

$$E_7 \to 7A_1 + 9A_1^{2d} \ (+\ldots?). \tag{3.107}$$

However, all of these 2d vacua "go away" in the enhancement to the A_6 theory, in the sense described in Section 3.5.4. In particular, of the 18 eigenvalue pairs corresponding to the A_1^{2d}'s in (3.107), 15 come to the origin to form the A_6 theory in the shift to (3.106), while the remaining 3 become copies of the shifted A_1 theory. Thus, the 1d and 2d vacuum structure of this flow appears to be

$$E_7 \to A_6 + A_1, \tag{3.108}$$

where the multiplicities of the eigenvalues corresponding to the 2d vacua of (3.107) have redistributed appropriately.

3.6 Conclusions, Future Directions, and Open Questions

3.6.1 Recap: puzzles and open questions for the $D_{\mathbf{even}}$ and E_7 theories

The ADE SCFTs have a rich structure of vacua, and deformations. The fact that the fields X and Y are matrices introduces many novelties, as we have here illustrated—but not yet fully understood. It is natural to expect that the higher-dimensional representations of the F- and D-terms have dimensions d_i given by some $G = A, D, E$ group theory quantities, e.g. the Dynkin indices n_i as with the McKay correspondence. But we find that $d_i \neq n_i$ in general, and we do not yet know how to analytically find the d_i and associated representations.

Our analysis of the E-series shows that even associating a fixed set of representations with the deformation flow can be subtle. For example, the case studies of Section 3.5.4 give the following puzzle: we can RG flow from the W_{E_7} SCFT via different ΔW deformations, to decoupled copies of SQCD (A_1) at low energies, and for different routes seemingly get different numbers of higher-dimensional representations in the IR. It will be interesting to understand how the proposed duality [46] fits in with this picture. The present work has raised several additional hurdles for the conjectured D_{even} and E_7 dualities, and it will be interesting to see how all of these puzzles are resolved.

3.6.2 Future directions: aspects of the W_{E_6} and W_{E_8} theories

The superpotentials that drive the RG flow from $\hat{O} \to \hat{E} \to E_{6,8}$ are (3.5):

$$W_{E_6} = \frac{1}{3}\text{Tr}Y^3 + \frac{s}{4}\text{Tr}X^4. \tag{3.109}$$

$$W_{E_8} = \frac{1}{3}\text{Tr}Y^3 + \frac{s}{5}\text{Tr}X^5. \tag{3.110}$$

The $\text{Tr}X^4$ and $\text{Tr}X^5$ terms are relevant for $x_{\text{min}}^{E_6} \approx 2.44$ and $x_{\text{min}}^{E_8} \approx 7.28$, respectively [38]. As reviewed in Section 3.1.1, the chiral rings of these theories do not classically truncate, and are especially rich since X and Y decouple in the F-terms (3.10) and (3.12). As shown in [46, 47], the $W_{E_{6,8}}$ theories cannot have a dual of the form reviewed following (3.15). It is unknown if there is a dual of some different form.

The a-theorem condition (3.26) is violated for sufficiently large x for both theories [38], showing that some new quantum effects must arise for large x. One possibility is that a W_{dyn} is generated, and the theory is no-longer conformal, for some $x > x_{\text{stability}}$. Another possibility is that there is some unknown dual description which becomes IR-free for large x. There are other reasons to expect that there might be some description of the IR physics of (3.109) and (3.110) in terms of dual variables: we can flow, for instance, $E_6 \to D_5$, and we expect that the stability bound is reduced $x_{E_6}^{\text{max}} > x_{D_5}^{\text{max}}$ along RG flow. It is also pointed out in [46, 47] that in E_6 the number of operators at a given value of R grows with R-charge, but somehow the theory must find a way to preserve unitarity.

We have studied a few aspects of the moduli space and ΔW deformations of the W_{E_6} and W_{E_8} SCFTs, looking for clues in formulating a dual description of the theories, but finding puzzles (similar to D_{even} and W_{E_7}). We here briefly report on some of our findings.

The undeformed W_{E_6} and W_{E_8} theories have a variety of flat directions similar to those discussed for the $W_{D_{\text{even}}}$ and W_{E_7} theories in Sections 3.4.2 and 3.5.2. In particular, both have 2d and 3d nilpotent flat directions (of course, a flat direction of E_6 is also a flat direction of E_8, since $X^3 = 0 \Rightarrow X^4 = 0$). The 2d vacuum solutions are of the form $X_{2d} = x(\sigma_3 + i\sigma_1)$, $Y_{2d} = -x(i\sigma_3 + \sigma_1)$ where arbitrary complex x labels the flat direction. There are several 3d flat directions of these theories, again labeled by x, for instance

$$X_{3d} = x\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_{3d} = x\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.111}$$

$$X_{3d'} = x\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_{3d'} = x\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.112}$$

As with the D_{even} and E_7 cases, these (classical) flat directions are surely related to the classical nontruncation of the ring. We expect, as with those cases, that some dynamics must alter these flat directions, at least for sufficiently large x, to avoid apparent violations of the a-theorem. It would be interesting to understand this further.

For $SU(N_c)$, as opposed to $U(N_c)$, upon imposing the tracelessness of the adjoints by adding Lagrange multiplier terms to (3.109) and (3.110), these theories have $SU(N_c)$ flat directions for particular values of N_c, similar to those discussed in Section 3.3.4, 3.4.3, and 3.5.3. The W_{E_6} theory has a flat direction for $N_c = 3m$ and/or $N_c = 2n$, while E_8 has a flat direction for $N_c = 2n$, for integer m and n. We expect low-energy $\widehat{A}$ theories along these classical flat directions; it would be interesting if one can obtain insights about the theory at the origin from these flat directions.

We now briefly comment on the RG flows from some ΔW deformations of the W_{E_6} and W_{E_8} SCFTs. Consider e.g. the flow $E_6 \to D_5$, obtained by adding $\Delta W = \mathrm{Tr} XY^2$ to (3.109). The

1d vacua correspond to the D_5 theory at the origin, and an A_1 theory away from the origin. The F-terms imply that $[Y^2, X] = 0$, and $[X^2, Y] = [X^3, Y] = 0$, so that $d > 1$ dimensional solutions to the F-terms must actually satisfy $X^2 = 0$. It is then straightforward to show that there are no 2d or 3d solutions that satisfy the F-terms and D-terms, so that the vacua along the flow are just the 1d vacua (up to possible $d > 3$ representations, again as in the discussion around (3.90))

$$E_6 \to D_5 + A_1 \qquad (+\ldots?). \tag{3.113}$$

While we do not yet know of a dual description of the W_{E_6} SCFT, in the IR of this flow Brodie duality and Seiberg duality map the low-energy gauge groups as

$$U(n_0) \times U(n_1) \stackrel{\text{duality}}{\longrightarrow} U(9N_f - n_0) \times U(N_f - n_1), \quad N_c = n_0 + n_1. \tag{3.114}$$

Perhaps understanding the IR limits of such flows will yield hints pointing towards a dual description of the W_{E_6},W_{E_8} theories. We invite the interested reader to try. Some additional comments on E-series flows are provided in Appendix A.2.

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Chapter 4

Landscape of Simple Superconformal Field Theories in 4d

We explore the infrared fixed points of four-dimensional $\mathcal{N} = 1$ supersymmetric $SU(2)$ gauge theory coupled to an adjoint and two fundamental chiral multiplets under all possible relevant deformations and F-term couplings to gauge-singlet chiral multiplets. We find 35 fixed points, including the $\mathcal{N} = 2$ Argyres-Douglas theories H_0 and H_1. The theory with minimal central charge a is identical to the mass-deformed H_0 theory, and the one with minimal c has the smallest a among the theories with $U(1)$ flavor symmetry. We also find a "next to minimal" $\mathcal{N} = 1$ SCFT with a chiral operator $\mathcal{O}$ with relation $\mathcal{O}^3 = 0$. In addition, we find 30 candidate fixed point theories possessing unphysical operators—including one with $(a, c) \simeq (0.20, 0.22)$—that need further investigation.

4.1 Introduction

Conformal field theory (CFT) is an important object in theoretical physics, which displays the physics of the low energy fixed point of some gauge theories and also of critical phenomena in condensed matter theories. One interesting question of CFT is to find the "minimal" interacting theory. In four dimensions, a measure of minimality is the a central charge, the coefficient to the Euler density term of the trace anomaly. This is because of the a-theorem [19, 21], $a_{UV} > a_{IR}$ for all unitary renormalization group (RG) flows. A related quantity is the c central charge, the coefficient to the two-point function of the stress-energy tensor.

In supersymmetric theories, these are tractable because of their relation to 't Hooft anomalies of the superconformal R-symmetry [16], which are in turn determined by the a-maximization technique [24]. The central charge c of any unitary interacting $\mathcal{N} = 2$ SCFT satisfies $c \geq \frac{11}{30}$ [63]. The theory that saturates the bound is the simplest Argyres-Douglas theory [64, 65], denoted as H_0 or (A_1, A_2) in the literature. H_0 also has the smallest known value of a for an interacting $\mathcal{N} = 2$ theory.

In $\mathcal{N} = 1$ theories, no analytic bound on the central charges is known so far. However, the numerical bootstrap program [66] suggests that the SCFT with the minimal central charge has a chiral operator $\mathcal{O}$ with chiral ring relation $\mathcal{O}^2 = 0$ [67, 68, 69], and a bound of $c \geq 1/9 \simeq 0.11$ [69]. The minimal theory thus far known in the literature has $a = \frac{263}{768} \simeq 0.34$ and $c = \frac{271}{768} \simeq 0.35$, and was constructed via a deformation of the H_0 theory [70, 71]. We will denote this theory as H_0^*. (See also a recent work on 3d $\mathcal{N} = 4$ theory [72].)

In the work [73], we initiate a classification of $\mathcal{N} = 1$ SCFT in four dimensions obtained from Lagrangian theories. We explore the space of RG flows and fixed points that originate from the simple starting point of supersymmetric $SU(2)$ gauge theory with one adjoint and a pair of fundamental chiral multiplets. From this minimal matter content, we consider all the possible relevant deformations, including deformations by coupling gauge-singlet chiral multiplets. Among the fixed points we obtain, two have enhanced $\mathcal{N} = 2$ supersymmetry: the Argyres-Douglas theories H_0 and H_1, as already found in [74, 75, 76]. The others are $\mathcal{N} = 1$ supersymmetric, including the H_0^* theory as a minimal theory in terms of a. We verify that these are "good" theories in the sense that there is no unitary-violating operator by utilizing the superconformal indices [28, 27].

In addition, in [73] we find a number of candidate fixed points which have an accidental global symmetry in the infrared and some unphysical operators, thus we refer to them as “bad” theories. Remarkably, these include theories with even smaller central charges than those of H_0^*. The minimal one, which we denote as $\mathscr{T}_M$, has $a \simeq 0.20$, and $c \simeq 0.22$. Although we are not able to conclude that these bad theories are really physical by the present techniques, we scope their properties.

In the present chapter, we review a particular subset of the flows in this landscape. We refer the reader to [73] for a description of the full landscape of fixed points and flows.

4.2 A Landscape of Simple SCFTs

We systematically enumerate a large set of superconformal fixed points via the following procedure:

1. Start with some fixed point theory $\mathscr{T}$.
2. Find the set of all the relevant chiral operators of $\mathscr{T}$, which we will call $\mathscr{R}_{\mathscr{T}}$. Let us also denote $\mathscr{S}_{\mathscr{T}} \subset \mathscr{R}_{\mathscr{T}}$ as the set of operators with R-charge less than $4/3$.
3. Consider the fixed points $\{\mathscr{T}_{\mathscr{O}}\}$ obtained by the deformation $\delta W = \mathscr{O}$ for all $\mathscr{O} \in \mathscr{R}_{\mathscr{T}}$.
4. Consider the fixed points $\{\mathscr{T}_{\overline{\mathscr{O}}}\}$ given by adding an additional gauge-singlet chiral field M and the superpotential coupling $\delta W = M\mathscr{O}$ for all $\mathscr{O} \in \mathscr{S}_{\mathscr{T}}$.
5. For each of the new fixed point theories obtained in previous steps, check if it has an operator $\mathscr{O}_d$ that decouples. Remove it by introducing a flip field X and a superpotential coupling $\delta W = X\mathscr{O}_d$.

6. For each new fixed point, repeat the entire procedure. Terminate if there is no new fixed point.

We employ the a-maximization procedure [24] and its modification [77] to compute the superconformal R-charges at each step. Beyond a-maximization, we check whether the theory passes basic tests as a viable unitary SCFT: one is the Hofman-Maldacena bounds for $\mathcal{N}=1$ SCFTs, $\frac{1}{2} \leq \frac{a}{c} \leq \frac{3}{2}$ [23]; the other one is the superconformal index. Some of the candidate fixed points have trivial index, or violate the unitarity constraints [78, 30].

We perform this procedure for one-adjoint SQCD with $SU(2)$ gauge group and with two fundamental chiral multiplets ($N_f = 1$). When there is no superpotential, this theory flows to an interacting SCFT $\hat{\mathscr{T}}$, as discussed in [41] (also see [79]), and a free chiral multiplet $\mathrm{Tr}\phi^2$. To pick up only the interacting piece, we add the additional singlet X and the superpotential $W_{\hat{\mathscr{T}}} = X\mathrm{Tr}\phi^2$.

Starting from $\hat{\mathscr{T}}$, we apply the deformation procedure, and find 35 non-trivial distinct fixed points. These theories pass every test we have checked, so we call them "good" theories.

There are an additional 30 distinct theories that pass almost all of our checks, except that there is a term in the index that signals a violation of unitarity. The existence of such a term implies that either the theory does not flow to an SCFT in the IR, or the answers we obtained were incorrect because we failed to take into account an accidental symmetry. In fact, these "bad" theories also have an accidental $U(1)$ symmetry which is not visible at the level of the superpotential, but is evident by the existence of the corresponding conserved current term present in the index. At present we do not know how to account for this accidental symmetry, and so cannot say for certain if these flows will lead to SCFTs or not.

Interestingly, 6 of these "bad" theories appear to have central charges lower than that of H_0^*. Denote the lowest one $\mathscr{T}_M$. This is a hint that there might be a minimal SCFT in this landscape.

We have plotted a, c for the "good" theories without this interesting complication in Figure

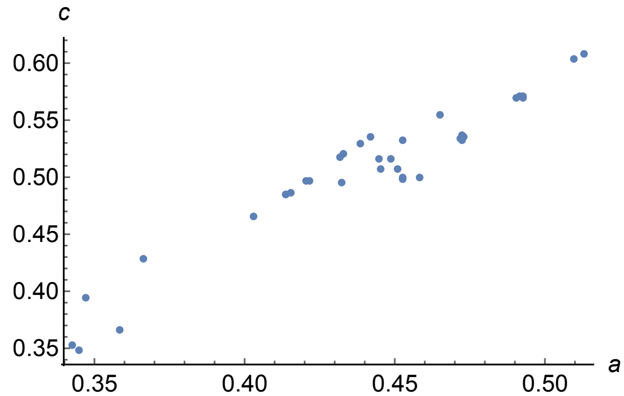

Figure 4.1: The central charges of the 35 "good" theories. The ratios a/c all lie within the range $(0.8246, 0.9895)$. The mean value of a/c is 0.8733 with standard deviation 0.03975.

4.1. We see that the distribution of a vs c are concentrated near the line of $a/c \sim 0.87$. Of the "good" theories, H_0^* has the smallest value of a. $\mathscr{T}_0$ has the smallest value of a among any theory with a $U(1)$ flavor symmetry. H_1^* has the smallest value of a among any theory with an $SU(2)$ flavor symmetry [1]. Below we examine each of these "minimal" theories in turn, as well as the lowest central charge theory $\mathscr{T}_M$, and the second-to-lowest a central charge "good" theory with no flavor symmetry, which we denote $\mathscr{T}_\mu$. We summarize the structure of RG flows among these special theories in Figure 4.2.

The superconformal indices of these theories can be computed using the Lagrangian description. We define the index as (see Chapter 2 for more details, and note the $t \to t^3$ notation change)

$$\mathscr{I}(t, y; x) = \mathrm{Tr}(-1)^F t^{3(r+2j_1)} y^{2j_2} x^f \ , \tag{4.1}$$

where (j_1, j_2) are the spins of the Lorentz group and r the $U(1)$ R-charge. When the theory has a global symmetry with Cartan generator f, we also include the fugacity x for it. For each of these special theories, we give the first few terms in the reduced superconformal index

$$\mathscr{I}_r(t, y) = (1 - t^3/y)(1 - t^3 y)(\mathscr{I}(t, y) - 1) \ , \tag{4.2}$$

which removes the conformal descendant contributions coming from spacetime derivatives. If

[1] There are two theories with 3 conserved currents with smaller a, but we do not find any evidence for the $SU(2)$ symmetry.

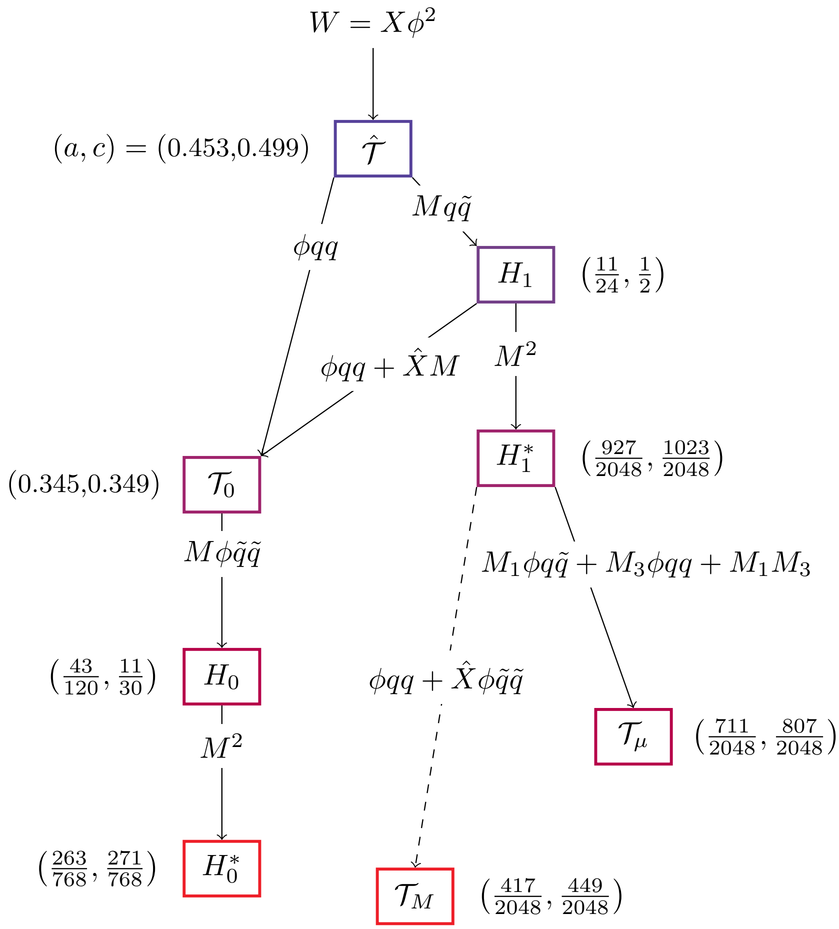

Figure 4.2: A subset of the fixed points that can be obtained from $SU(2)$ $N_f = 1$ adjoint SQCD with singlets. Note that the graph is not arranged vertically by decreasing a central charge, because the deformations we consider involve coupling in the singlet fields.

the reduced index contains a term $t^R\chi_j(y)$ with $R < 2+2j$ or a term $(-1)^{2j+1}t^R\chi_j(y)$ with $2+2j \leq R < 6+2j$, it violates the unitarity constraint [78, 30].

The coefficient of t^6y^0 allows us to read off the number of marginal operators minus the number of conserved currents [78]. The superpotential F-terms $\partial W/\partial\varphi = 0$ for the fields φ allow us to read off the classical chiral ring, and quantum modifications can be argued from the index. We will see that the chiral rings we study in this chapter are subject to the quantum corrections. The superconformal index turns out to be a useful tool to study the fully quantum corrected chiral rings of our models.

Table 4.1: The R-charges of the chiral multiplets at various fixed points. The $\mathscr{T}_\mu$ theory has 3 chiral multiplets labeled M, which we denote as $M_{1,2,3}$.

fields	$\mathscr{T}_0$	H_0^*	H_1^*	$\mathscr{T}_\mu$	$\mathscr{T}_M$
q	$\frac{543-\sqrt{1465}}{546} \simeq 0.924$	$11/12$	$1/2$	$1/4$	$7/8$
$\tilde{q}$	$\frac{75-\sqrt{1465}}{78} \simeq 0.471$	$5/12$	$1/2$	$3/4$	$1/8$
ϕ	$\frac{3+\sqrt{1465}}{273} \simeq 0.151$	$1/6$	$1/4$	$1/4$	$1/4$
M	$\cdot$	1	1	$(\frac{3}{4}, 1, \frac{5}{4})$	1
X	$\frac{2(270-\sqrt{1465})}{273} \simeq 1.70$	$5/3$	$3/2$	$3/2$	$3/2$
$\hat{X}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$3/2$

4.3 $\mathscr{T}_0$ —Minimal c, Minimal a with $U(1)$

Let us begin with the $\mathscr{T}_0$ SCFT which is obtained via a deformation of $\hat{\mathscr{T}}$,

$$W_{\mathscr{T}_0} = X\mathrm{Tr}\phi^2 + \mathrm{Tr}\phi qq, \tag{4.3}$$

and has irrational central charges

$$\begin{aligned} a_{\mathscr{T}_0} &= \frac{81108 + 1465\sqrt{1465}}{397488} \simeq 0.3451, \\ c_{\mathscr{T}_0} &= \frac{29088 + 1051\sqrt{1465}}{198744} \simeq 0.3488. \end{aligned} \tag{4.4}$$

The IR R-charges of the fields of the $\mathscr{T}_0$ and all other theories discussed below are given in Table 4.1. This theory has the second smallest value of a, and the smallest value of c among the 35 "good" fixed points we enumerate.[2]

The chiral ring of the theory can be easily studied: the F-term conditions from (4.3) are simply $\mathrm{Tr}\phi^2 = 0$, $q\phi = 0$ and $X\phi + q^2 = 0$. The first equation truncates the chiral ring by setting $\phi^2 = 0$. The second and third equations lead to the classical generators of the chiral ring: $\mathscr{O}' \equiv \mathrm{Tr}q\tilde{q}$, $\mathrm{Tr}\phi\tilde{q}\tilde{q}$ and X, with relation $\mathscr{O}'^2 \sim X\mathrm{Tr}\phi\tilde{q}\tilde{q}$.

This theory has an anomaly free $U(1)$ flavor symmetry that mixes with R. The reduced

[2] The theory with smaller c than H_0^* was also noticed by Sergio Benvenuti. We thank him for informing us on this.

index is given as

$$\begin{aligned}\mathscr{I}_r(t,y;x) = t^{3.28}x^{12} - t^{3.45}x^{-2}\chi_2(y) + t^{4.19}x^8 - t^6 \\ + t^{6.56}x^{24} + t^{7.46}x^{20} + t^{8.27}x^{-10} + \dots ,\end{aligned} \tag{4.5}$$

where we assigned the flavor charges for the fugacity x as $f_q = 1, f_{\tilde{q}} = 7, f_\phi = -2, f_X = 4$. Here and below $\chi_s(a)$ denotes the character for the $SU(2)$ flavor symmetry of dimension $s = 2j+1$. This index allows us to read off the quantum modified chiral ring: the terms $t^{3.28}x^{12}$ and $t^{4.19}x^8$ in the index come from the chiral operators $\mathrm{Tr}\phi\tilde{q}\tilde{q}$ and $\mathrm{Tr}q\tilde{q}$ respectively; the second term denotes the fermionic operator $\mathscr{O}_\alpha = \mathrm{Tr}\phi W_\alpha$. We see that the operator X (which would contribute $t^{5.10}x^4$ to the index if it exists) is absent from the chiral ring. We can read off the chiral ring relation $\mathscr{O}'^2 = \mathscr{O}_\alpha \cdot (\mathrm{Tr}\phi\tilde{q}\tilde{q}) = 0$ from the absence of the terms $t^{8.38}x^{16}$ and $-t^{6.73}\chi_2(y)x^{10}$.

4.4 H_0^*—Minimal a

The H_0 fixed point can be obtained from $\mathscr{T}_0$ by adding the $M\mathrm{Tr}\phi\tilde{q}\tilde{q}$ term. This superpotential is indeed a simplified version of the one considered in [74].

At the H_0 fixed point we further deform by a mass term M^2,

$$W_{H_0^*} = X\mathrm{Tr}\phi^2 + \mathrm{Tr}\phi qq + M\mathrm{Tr}\phi\tilde{q}\tilde{q} + M^2. \tag{4.6}$$

This flows to the H_0^* theory with the central charges

$$a_{H_0^*} = \frac{263}{768} \simeq 0.3424, \quad c_{H_0^*} = \frac{271}{768} \simeq 0.3529. \tag{4.7}$$

The H_0^* SCFT has been studied in [70, 71] as a deformation of the H_0 Argyres-Douglas theory. Utilizing the UV Lagrangian description presented here, we are able to confirm various predictions about H_0^*.

Classically, the F-terms of (4.6) imply that M, X, and $\mathscr{O}' \equiv \mathrm{Tr}q\tilde{q}$ generate the chiral ring, with relations $M^2 \sim 0$ and $\mathscr{O}'^2 \sim 0$. The superconformal index for the H_0^* theory can be computed to give a reduced index

$$\mathscr{I}_r(t,y) = t^3 - t^{\frac{7}{2}}\chi_2(y) + t^4 + t^7 + t^{\frac{17}{2}} + \dots \tag{4.8}$$

From this we see that the two generators M and $\mathcal{O}$ contribute the t^3 and t^4 respectively, while X is not a generator. We also find that the operator $\mathcal{O}_\alpha = \mathrm{Tr}(\phi W_\alpha)$ contributes to $t^{\frac{7}{2}}\chi_2(y)$. From the coefficients of t^6, t^7, t^8, we find $M^2 = M\mathcal{O}' = \mathcal{O}'^2 = 0$ in the chiral ring. The term t^7 comes from $(\mathcal{O}_\alpha)^2$. There is a relation for $\mathcal{O}_\alpha$ of the form $M\mathcal{O}_\alpha = \mathcal{O}'\mathcal{O}_\alpha = 0$ which can be read from the absence of the terms $-t^{\frac{13}{2}}\chi_2(y)$ and $-t^{\frac{15}{2}}\chi_2(y)$. These relations support the analysis of [70, 71].

4.5 H_1^*—Minimal a with $SU(2)$

The flow to H_1 in our setup is a simplified version of the flow considered in [75], and was also considered in [80]. From H_1 the H_1^* SCFT is then obtained via a mass deformation to the singlet,

$$W_{H_1^*} = X\mathrm{Tr}\phi^2 + M\mathrm{Tr}q\tilde{q} + M^2. \tag{4.9}$$

The central charges are

$$a_{H_1^*} = \frac{927}{2048} \simeq 0.4526, \quad c_{H_1^*} = \frac{1023}{2048} \simeq 0.4995. \tag{4.10}$$

Classically, the F-terms imply that the chiral ring is generated by M, X, $\mathcal{O}_2 \equiv \mathrm{Tr}\phi qq$, $\mathcal{O}_0 \equiv \mathrm{Tr}\phi q\tilde{q}$, $\mathcal{O}_{-2} \equiv \mathrm{Tr}\phi\tilde{q}\tilde{q}$, with relations $M^2 = M\mathcal{O}_i = X\mathcal{O}_i = 0$, and $\mathcal{O}_0^2 \sim \mathcal{O}_2\mathcal{O}_{-2}$. The last relation descends from that of the Higgs branch of the H_1 theory.

The reduced index is

$$\begin{aligned}\mathcal{I}_r(t,y;a) = t^3 + t^{\frac{15}{4}}(\chi_3(a) - \chi_2(y)) + t^{\frac{9}{2}} \\ - t^6\chi_3(a) + t^{\frac{15}{2}}(1+\chi_5(a)) + t^{\frac{33}{4}} + \dots\end{aligned} \tag{4.11}$$

We see the theory has the $SU(2)$ current from the $-t^6\chi_3(a)$ term, which is visible at the level of the superpotential. There are generators M, X and $\mathcal{O}_i$ satisfying the relations $M^2 = X^2 = 0$ and $\mathcal{O}_0^2 \sim \mathcal{O}_2\mathcal{O}_{-2}$. There are also fermionic operators $\mathcal{O}_\alpha = \mathrm{Tr}(\phi W_\alpha)$ with relations $M\mathcal{O}_\alpha = X\mathcal{O}_\alpha = 0$.

4.6 Discussion

One goal of this program is to search for and study minimal $\mathcal{N}=1$ SCFTs. One feature of the low-central charge SCFTs we have examined here is that there is a chiral operator satisfying a relation of the form $\mathcal{O}^n \sim 0$ for $n = 2,3$. Another feature is that the central charges of the SCFTs considered here lie in a narrow range of a/c. It would be interesting to pursue the reasons for this, and search for other $\mathcal{N}=1$ SCFTs with truncated chiral rings.

A common property of the RG flows in this landscape is that some operators that are irrelevant at high-energy can be relevant in the IR—such operators are called dangerously irrelevant. As such this is an interesting arena for studying RG flows along the lines of [81].

At present, the status of the "bad" theories is unclear, because it is not clear how to account for the accidental symmetry in the a-maximization procedure and thus check if the corrected theory would flow to an interacting SCFT. One way forward would be to identify the fermionic multiplet that contributes to the unitary-violating terms in the index and decouple it, as we naively did for the $\mathcal{T}_M$ theory. It would be interesting to resolve this question and understand how the accidental symmetry arises. This would settle whether one of these theories is indeed a new candidate minimal $\mathcal{N}=1$ theory, or strengthen the case for minimality of the H_0^* theory.

Acknowledgements

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Chapter 5

Interlude: An Introduction to the Theories of Class $\mathscr{S}$

In this chapter we review the construction of a class of four-dimensional SCFTs by compactification from six dimensions. These constructions will provide the arena for the remainder of this thesis.

5.1 Introduction

A large class of four-dimensional quantum field theories can be studied by compactifying six-dimensional $\mathscr{N} = (2,0)$ superconformal field theories over a punctured Riemann surface with a partial topological twist. The SCFTs that result from this procedure are known as theories of class $\mathscr{S}$ (for "six"). Generically, these theories are strongly coupled and do not have a known Lagrangian description, and yet many of their properties can be inferred by utilizing their origin in six dimensions and the compactification scheme.

These constructions have been revolutionary in that they provide a partial classification scheme for four-dimensional $\mathscr{N} = 2$ SCFT's, and bring to bear new geometric tools for studying them [82, 83, 84]. Soon after their introduction, it was demonstrated that these constructions can be generalized to study the space of four-dimensional $\mathcal{N}=1$ SCFT's [85, 86, 87, 88, 89, 78].

An important ingredient in the construction is a partial topological twist [90, 91], which is needed to preserve some supercharges in the compactification of the six-dimensional theory. Depending on the choice of twist, various amount of supersymmetry can be preserved in four dimensions. $\mathcal{N}=2$ theories of class $\mathcal{S}$ were first constructed and classified in [82, 84] (building on earlier work by [92]). A large class of $\mathcal{N}=1$ SCFTs and their dualities were studied via mass deformations of $\mathcal{N}=2$ theories in [85, 86, 87]. Later, it was demonstrated that $\mathcal{N}=1$ SCFTs could be directly constructed from compactifications of six-dimensional theories on a Riemann surface with a partial topological twist [89, 88]. We refer to these theories as BBBW theories, for the authors Bah, Beem, Bobev, Wecht.

A strong piece of evidence for the existence of these superconformal theories is the explicit construction of their large-N gravity duals. The gravity duals for the $\mathcal{N}=2$ theories corresponding to M5-branes wrapped on Riemann surfaces without punctures were constructed in [83], which are holographically dual to the Maldacena-Nuńez supergravity solutions [93]. The duals for the $\mathcal{N}=1$ theories were constructed in [89, 88] (without punctures) and in [94] (with punctures).

5.2 4d SCFTs from 6d

Generically, putting a QFT on a curved background breaks supersymmetry. A partial topological twist allows us to preserve some supersymmetry in the IR. In the twist, one turns on a background gauge field valued in the six-dimensional $SO(5)_R$ symmetry, and tunes it to cancel the background curvature on the Riemann surface. We identify an abelian subgroup of the six-dimensional R-symmetry as

$$U(1)_+ \times U(1)_- \subset SU(2)_+ \times SU(2)_- \subset SO(5)_R, \tag{5.1}$$

where the $U(1)_\pm$ are Cartans of the $SO(5)_R$. Then, embed the holonomy group of the Riemann surface $U(1)_h$ in the six-dimensional R-symmetry group by identifying the $U(1)_h$ generator R_h as a linear combination of the $U(1)_\pm$ generators $J_\pm$,

$$R_h = \frac{p_1}{p_1+p_2} J_+ + \frac{p_2}{p_1+p_2} J_- . \tag{5.2}$$

This fixes the parameters (p_1, p_2) in terms of the Euler characteristic χ of the surface as

$$p_1 + p_2 + \chi(\Sigma_{g,n}) = 0, \qquad \text{with} \qquad -\chi(\Sigma_{g,n}) = 2(g-1)+n. \tag{5.3}$$

This procedure in general preserves four supercharges in four dimensions, and breaks the bosonic symmetries of the six-dimensional theory as

$$SO(1,5) \times SO(5)_R \to SO(1,3) \times U(1)_+ \times U(1)_- . \tag{5.4}$$

When one of (p_1, p_2) is zero, eight supercharges will be preserved, and one of the $U(1)_\pm$ will be enhanced to $SU(2)_\pm$ to furnish the $\mathcal{N}=2$ R-symmetry of the four-dimensional theory.

The six-dimensional $(2,0)$ theories are labeled by a choice of gauge algebra $\mathfrak{g}$, which follows an ADE classification. The $\mathfrak{su}(N) = A_{N-1}$ and $\mathfrak{so}(2N) = D_N$ cases have a description in terms of M5-branes. In this dissertation we will focus on the A_{N-1} theories, in which case the six-dimensional theory arises as the effective world-volume theory of N coincident M5-branes. Then, the amount of supersymmetry that is preserved in the IR depends on the way the M5-branes are embedded in a Calabi-Yau threefold CY_3. From this perspective, the Riemann surface is described by a holomorphic curve $\mathscr{C}_{g,n}$ in CY_3. Generally, the Calabi-Yau threefold is a $U(2)$ bundle over $\mathscr{C}_{g,n}$, whose determinant line bundle is fixed to the canonical bundle of the surface. We choose to twist the Cartan of the $SU(2)$ bundle, such that the $U(2)$ bundle decomposes into a sum of two line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ with integer degrees (p_1, p_2) as[1]

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathscr{L}_1 \oplus \mathscr{L}_2 \\ & & \downarrow \\ & & \mathscr{C}_{g,n} \end{array} \tag{5.5}$$

[1]Often these parameters are called (p,q) in the literature, but here we reserve the (p,q) labels to specify the R-symmetry of locally $\mathcal{N}=2$ preserving punctures.

In this language, the topological twist involves embedding the holonomy group $U(1)_h$ of the Riemann surface in the $SO(5)$ structure group of the normal bundle to the M5-branes. The $U(1)_\pm$ global symmetries in (5.1) correspond to phase rotations of the two line bundles. Requiring that the first Chern class of the Calabi-Yau threefold vanish is equivalent to the condition (5.3). If one of the two line bundles is trivial, the threefold decomposes as $CY_3 = CY_2 \times \mathbb{C}$, and the background preserves eight supercharges.[2]

The four-dimensional theories of class $\mathscr{S}$ also have a description in terms of a generalized quiver gauge theory. To take the A_{N-1} case, one geometrically decomposes the curve $\mathscr{C}_{g,n}$ into 3-punctured spheres connected by tubes via pair-of-pants decompositions. S-duality relates different degeneration limits of the curve. The low energy effective description of N coincident M5-branes wrapping a 3-punctured sphere is known as the T_N theory, which is a strongly coupled $\mathscr{N} = 2$ SCFT with an $SU(N)^3$ global symmetry [82]. Gauging subgroups of these global symmetries via $\mathscr{N} = 1$ or $\mathscr{N} = 2$ vector multiplets (the tubes) corresponds geometrically to "gluing" punctures to form Riemann surfaces with general Euler characteristic. The classification of these four-dimensional building blocks, or "tinkertoys", has been carried out in [95, 96, 97, 98, 99]. As we discuss in some detail in the next chapter, similar field-theoretic constructions have recently been obtained for theories whose geometries have negative line bundle degrees p_1 and p_2 [100, 101, 102]. Such a field-theoretic approach can be useful in providing a different perspective on the properties of the theories of class $\mathscr{S}$.

Much of the richness of class $\mathscr{S}$ comes from the punctures on the Riemann surface. From the perspective of the parent six-dimensional $(2,0)$ theory, punctures are 1/2-BPS codimension-2 defects, specified by an embedding $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. Such embeddings are labeled by nilpotent orbits of the Lie algebra $\mathfrak{g}$ [103]. Regular defects correspond to singular boundary conditions to Hitchin's equation on the Riemann surface, which have been classified for A_{N-1} in [83], for D_N in [104], and have been discussed for other types of regular defects including twisted lines

[2]In the special case that the Riemann surface is a 2-torus ($g = 1$), $\mathscr{N} = 4$ supersymmetry can be preserved by fixing the normal bundle to the M5-brane world-volume to be trivial.

(possible when the ADE group admits an outer-automorphism) in [105, 103, 106, 107]. The generalization to $\mathscr{N} = 1$ Hitchin's equations was first discussed in [108]. In the present work, we will only discuss regular (also called "tame") defects, and omit discussion of irregular ("wild") punctures corresponding to higher order poles.[3]

When the (2,0) theories describe the effective world-volume theory of M5-branes, punctures correspond to points where the M5-branes branch out to infinity [109]. The punctures correspond to boundaries of the Riemann surface, and boundary conditions are needed for the M5-branes; this leads to global symmetries. In the Type IIA limit where we shrink the M-theory circle, the degrees of freedom at the puncture are associated to the intersection of D4/D6 branes [92].

At large N, one can look at AdS_5 dual solutions of M-theory corresponding to the near horizon limit of N M5-branes wrapping a Riemann surface [83, 94]. In these solutions, the new degrees of freedom are associated to additional M5-branes that are localized at the punctures (see [109]). These branes are extended along a direction normal to the Riemann surface, and end at monopole sources of a $U(1)$ connection of an S^1 bundle over the surface. This connection is associated to the topological twist in the field theory construction. A single M5-brane corresponds to a simple puncture, which can be analyzed in the probe approximation [83, 110]. In the full backreacted solution, the connection forms in the Ricci flat background pick up monopole sources.

To summarize, we require the following data in order to specify a theory of class $\mathscr{S}$: a choice of $\mathfrak{g}$ =ADE; the Euler characteristic χ of the Riemann surface; a choice of twist, i.e. the (p_1, p_2) that satisfy (5.3); and local data associated with the punctures. From this perspective, the class $\mathscr{S}$ construction allows us to organize a large space of four-dimensional SCFTs in a geometric way.

[3]The Argyres-Douglas theories discussed in the previous chapter in fact have a Class $\mathscr{S}$ description in terms of a sphere with one regular and one irregular puncture, but we will not further discuss this description here.

Acknowledgements

This chapter is, in part, a reprint of material from published work done in collaboration with Ibrahima Bah, as it appears on the arXiv. I. Bah and E. Nardoni, “Structure of Anomalies of 4d SCFTs from M5-branes, and Anomaly Inflow,” arXiv:1803:00136. The dissertation author was the primary investigator and author of this material.

Chapter 6

4d SCFTs from Negative Degree Line Bundles

We construct 4d $\mathcal{N} = 1$ quantum field theories by compactifying the (2,0) theories on a Riemann surface with genus g and n punctures, where the normal bundle decomposes into a sum of two line bundles with possibly negative degrees p and q.[1] Until recently, the only available field-theoretic constructions required the line bundle degrees to be nonnegative, although supergravity solutions were constructed in the literature for the zero-puncture case for all p and q. Here, we provide field-theoretic constructions and computations of the central charges of 4d $\mathcal{N} = 1$ SCFTs that are the IR limit of M5-branes wrapping a surface with general p or q negative, for general genus g and number of maximal punctures n.

6.1 Overview and Summary of Results

We here consider M5-branes wrapping a genus g Riemann surface with n maximal punctures $\mathscr{C}_{g,n}$, where the surface is embedded in a Calabi-Yau 3-fold alla (5.5). The BBBW

[1] Note that in this chapter, in order to match the original work we (regrettably) deviate from the notation in Chapter 5, where p and q were referred to as p_1 and q_1.

supergravity solutions are valid for all p and q; however, there is only an explicit field theory construction for the case of p and q nonnegative.[2] Our main goal at present is to understand such a construction for the case where one of the line bundle degrees is negative. Our construction requires a more general building block than the T_N theory. The necessary ingredient was provided in [100], which introduced a generalization of the T_N theory denoted $T_N^{(m)}$ for m a positive integer (and whose features we will review in Section 6.2.1). The field-theoretic constructions in [100] utilizing $T_N^{(m)}$ building blocks provided the first generalized quiver field theories with $p, q < 0$.

In the present work, we explicitly construct 4d $\mathscr{N} = 1$ field theories that result from compactifying the (2,0) theories on a surface with negative p or q, thereby providing field theoretic constructions for the duals of the BBBW gravity solutions obtained by gluing $T_N^{(m)}$ building blocks. We further generalize to the case of M5-branes wrapped on Riemann surfaces with maximal punctures, yielding formulae for the (trial) central charges of the resulting SCFTs that depend only on geometric data.

The organization of this chapter is as follows. In Section 6.2, we provide the field-theoretic construction of the 4d $\mathscr{N} = 1$ SCFTs that are dual to the BBBW gravity solutions for negative p or q. We begin by reviewing the definition of the $T_N^{(m)}$ theories as formulated in [100]. Then, we glue together $(2g-2)$ copies of the $T_N^{(m)}$ theories, yielding a genus $g > 1$ surface with no punctures and possibly negative p or q, thus providing an inherently field-theoretic construction of 4d $\mathscr{N} = 1$ field theories that arise as the IR limit of M5-branes wrapped on a surface with negative normal bundle degrees. We compute the central charges and operator dimensions for these theories, and find that they match precisely onto the BBBW formulae.

In Section 6.3, we study the genus zero case, which requires closing punctures on chains of $T_N^{(m)}$ theories. We consider the simplest case of a single $T_N^{(m)}$ theory whose $SU(N)$ flavor groups are Higgsed, which can yield theories with twist $|z| \geq 2$, and find that the trial central charges match onto the BBBW results. There will be corrections to these values from operators

[2] Similar constructions were recently considered in [101].

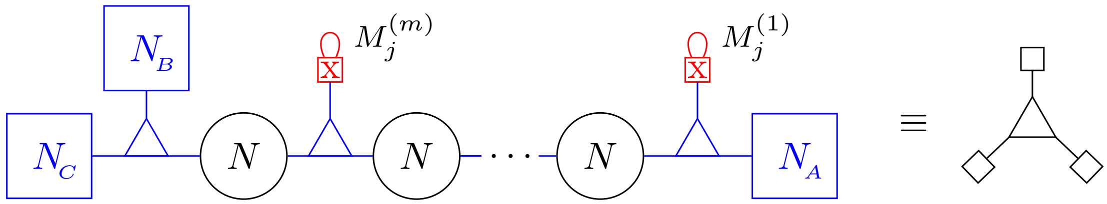

Figure 6.1: A UV generalized quiver description for the $T_N^{(m)}$ theories. The blue color of the $SU(N)_{A,B,C}$ flavor groups corresponds to punctures with sign $\sigma_{A,B,C} = +1$. The red boxes correspond to closed $\sigma = -1$ punctures. The singlets $M_j^{(i)}$ are the leftover components of the fluctuations of the $M^{(i)}$ about the vevs.

whose R-charges were shifted in the Higgsing procedure. We comment on these corrections, but leave their complete analysis to future work.

Having reviewed the machinery to close punctures in Section 6.3, in Section 6.4 we construct theories of genus g and n maximal punctures from $T_N^{(m_i)}$ building blocks. With these general constructions, we are able to compute the central charges for the torus as well. We conclude with a brief discussion of future directions.

6.2 Constructing the BBBW Duals from $T_N^{(m)}$ Building Blocks

6.2.1 $T_N^{(m)}$ review

The $T_N^{(m)}$ theories constructed in [100] are strongly coupled 4d $\mathscr{N} = 2$ SCFTs. They have a natural description as being of class $\mathscr{S}$, arising as the low-energy limit of wrapping N M5-branes (the (2,0) theories of type A_{N-1}) on a sphere with three punctures, $\mathscr{C}_{0,3}$. The sphere is embedded in a Calabi-Yau 3-fold, which decomposes into a sum of two line bundles as in (5.5). For our purposes, the novelty of this construction is that the degrees p and q of the line bundles are allowed to be negative, parameterized by a nonnegative integer m:

$$T_N^{(m)}: \quad p = m+1, \quad q = -m. \tag{6.1}$$

For $m = 0$, this construction reduces to the T_N theory.

These theories preserve a $U(1)_+ \times U(1)_-$ global symmetry which derives from the parent

$\mathscr{N} = (2,0)$ $SO(5)_R$ symmetry, as well as an $SU(N)^3$ global symmetry associated to the three punctures[3] which we denote $SU(N)_A \times SU(N)_B \times SU(N)_C$. Each puncture is labeled by a sign $\sigma_{A,B,C} = \pm 1$; in the notation of [100], +1 is blue-colored and -1 is red-colored, and in diagrams we'll take $+1$ to be unshaded and -1 to be shaded. Denoting the generators of $U(1)_+ \times U(1)_-$ by $J_\pm$, the exact superconformal R-symmetry is given by the linear combination (see Appendix B.1 for more on our conventions)

$$R_{\mathscr{N}=1}(\varepsilon) = \frac{1-\varepsilon}{2} J_+ + \frac{1+\varepsilon}{2} J_- . \tag{6.2}$$

The 't Hooft anomalies for the $T_N^{(m)}$ theories are given in Appendix B.3, in equation (B.12), and the chiral operators of the $T_N^{(m)}$ theories are listed in Table 6.1. The ϕ_i are adjoint chiral multiplets; the $\mu_j^{(i)}$ and $M_j^{(i)}$ are singlets; the μ_i are moment-map operators (i.e., chiral operators at the bottom of would-be $\mathscr{N} = 2$ current multiplets) of the m symmetry groups $SU(N)_i$ that are gauged in the construction of the theories; the $\mu_{A,B,C}$ are moment-map operators of the leftover $SU(N)_{A,B,C}$ flavor symmetries; and $Q(\widetilde{Q})$ are (anti)trifundamentals of the $SU(N)_A \times SU(N)_B \times SU(N)_C$ flavor symmetry. The singlets are coupled in a superpotential

$$W_{\text{singlets}} = \sum_{i=1}^{m} \sum_{j=1}^{N-1} \mu_j^{(i)} M_j^{(i)} \tag{6.3}$$

that arises from the construction of the $T_N^{(m)}$ theories by gluing $m+1$ copies of the T_N theory—see [100] for more details. We summarize the resulting UV generalized quiver description for the $T_N^{(m)}$ theories in Figure 6.1.

The superpotential (6.3) yields chiral ring relations for the chiral operators. For example, while naively one might worry that the singlets $\mu_j^{(i)}$ could violate the unitarity bound due to their negative J_- charge, the F-terms for the $M_j^{(i)}$ imply that the $\mu_j^{(i)}$ are in fact trivial in the ring. As checked in [100], none of the gauge-invariant chiral operators that are nontrivial in the ring decouple[4].

[3]This is taking the punctures to be maximal; to construct building blocks with generic three punctures whose flavor symmetries are non-maximal, one can use results in [111], [112], [95], [103].

[4]If the dimension of a chiral operator $\mathscr{O}$ appears to violate the unitarity bound $R(\mathscr{O}) < 2/3$ (the R-charge of a chiral operator is proportional to its dimension in theories with four supercharges), then $\mathscr{O}$ is in fact free, and

Table 6.1: Operators of the (unshaded) $T_N^{(m)}$ theories.

	$SU(N)_i$	$SU(N)_A$	$SU(N)_B$	$SU(N)_C$	(J_+,J_-)
Q		$\square$	$\square$	$\square$	$(N-1,0)$
$\widetilde{Q}$		$\bar{\square}$	$\bar{\square}$	$\bar{\square}$	$(N-1,0)$
$\phi_i\ (1\le i\le m)$	adj				$(0,2)$
$\mu_i\ (1\le i\le m)$	adj				$(2,0)$
μ_A		adj			$(2,0)$
μ_B			adj		$(2,0)$
μ_C				adj	$(2,0)$
$\mu_j^{(i)}\ (1\le j\le N-1)$					$(2,-2j)$
$M_j^{(i)}\ (1\le j\le N-1)$					$(0,2j+2)$

6.2.2 Gluing procedure

The gluing procedure corresponds to decomposing the geometry into pairs of pants, where each can be associated with $T_N^{(m)}$ theories, and gauging subgroups of the flavor symmetries associated to the punctures. We will label the ith block by $T_N^{(m_i)}$, where the m_i are in general different.

Before we do the general case, let us first illustrate the procedure of gluing two $T_N^{(m_i)}$, $i=1,2$ theories with either an $\mathscr{N}=1$ or $\mathscr{N}=2$ vector multiplet, as in [100]. Label the degrees of the blocks as

$$T_N^{(m_i)}:\quad (p_i,q_i)=\begin{cases}(m_i+1,\ -m_i) & \sigma_i=+1\\ (-m_i,\ m_i+1) & \sigma_i=-1\end{cases} \tag{6.4}$$

since p_i and q_i switch roles for an unshaded versus shaded block. We gauge an $SU(N)$ flavor symmetry of the two $T_N^{(m_i)}$ theories, leading to a superpotential for the moment-map operators of the gauged block,

$$W=\mathrm{Tr}\mu^+\mu^-. \tag{6.5}$$

For instance, μ^+ could be chosen to derive from μ_A in Table 6.1 for one block, and μ^- to come from μ_A for the other block. In order to write a superpotential of this form, when gluing with an

an accidental $U(1)$ symmetry acts on $\mathscr{O}$. One must account for the decoupling of these free operators, e.g. in computations of a and c [77].

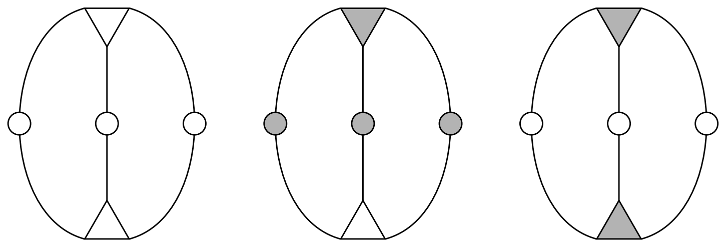

Figure 6.2: Options for gluing 2 $T_N^{(m_i)}$ blocks to form a genus 2 surface. Shaded triangles correspond to $T_N^{(m_i)}$ theories with $\sigma_i = -1$, while unshaded triangles have $\sigma_i = +1$. Shaded circular nodes correspond to $\mathscr{N} = 1$ vector multiplets, while unshaded circular nodes correspond to $\mathscr{N} = 2$ vector multiplets.

$\mathscr{N} = 1$ vector the (J_+, J_-) charge assignment of one of the $T_N^{(m_i)}$ blocks must be flipped such that μ^- has $J_\pm$ charges given by $(J_+, J_-) = (0, 2)$. In general, two blocks of the same color/shading should be glued by an $\mathscr{N} = 2$ vector, while two blocks of differing colors/shadings should be glued by an $\mathscr{N} = 1$ vector.

The result of this procedure is a four-punctured sphere $\mathscr{C}_{0,4}$, where the total degrees p and q of the embedding space satisfy $p + q = 2g - 2 + n = 2$.

6.2.3 Construction of $\mathscr{C}_{g>1,n=0}$ and computation of a and c

Here, we glue together $(2g-2)$ copies of the $T_N^{(m_i)}$ theories, $i = 1, ..., 2g-2$, yielding a genus $g > 1$ surface $\mathscr{C}_{g,0}^{(p,q)}$ with no punctures and possibly negative degrees p and q. (The sphere and torus cases are constructed separately in later sections.) We consider the general case of ℓ_1 (shaded) blocks with $\sigma_i = -1$, and ℓ_2 (unshaded) blocks with $\sigma_i = +1$, glued together with n_1 (shaded) $\mathscr{N} = 1$ vector multiplets and n_2 $\mathscr{N} = 2$ (unshaded) vector multiplets. Given the geometries we wish to construct, these parameters satisfy

$$\ell_1 + \ell_2 = 2(g-1), \qquad n_1 + n_2 = 3(g-1). \tag{6.6}$$

Label the degrees of the blocks as in (6.4). Then, the total degrees p and q of the space in which the genus g surface is embedded are given by

$$p = \sum_{\{\sigma_i=+1\}} m_i - \sum_{\{\sigma_i=-1\}} m_i + \ell_2, \qquad p+q = 2g-2. \tag{6.7}$$

The sum over $\{\sigma_i = +1\}$ runs over the ℓ_2 unshaded nodes, while the sum over $\sigma_i = -1$ runs over the ℓ_1 shaded nodes. As an example, the options for forming a genus 2 surface in this manner are shown in Figure 6.2.

We now compute the central charges a and c for these configurations. For a general 4d $\mathscr{N} = 1$ SCFT, the central charges a and c are determined by the 't Hooft anomalies [16],

$$a = \frac{3}{32}\left(3\text{Tr}R^3 - \text{Tr}R\right), \quad c = \frac{1}{32}\left(9\text{Tr}R^3 - 5\text{Tr}R\right). \tag{6.8}$$

For quivers made from $T_N^{(m)}$ building blocks, in the absence of accidental symmetries, the $\mathscr{N} = 1$ superconformal R-symmetry $R = R(\varepsilon)$ takes the form (see Appendix B.1 for conventions)

$$R(\varepsilon) = \frac{1}{2}(1-\varepsilon)J_+ + \frac{1}{2}(1+\varepsilon)J_-. \tag{6.9}$$

Then, the exact superconformal R-symmetry at an IR fixed point is determined by a-maximization with respect to ε [24].

The contributions to a and c of the various components of our constructions can be computed using the 't Hooft anomalies given in Appendix B.3, substituted into equations (6.8) and (6.9). The contribution of the i'th $T_N^{(m_i)}$ block is given by [100]

$$\begin{aligned} a_{T_N^{(m_i)}}(\varepsilon) =& \frac{3}{64}(N-1)(1-\varepsilon)\left(3N^2(1+\varepsilon)^2 - 3N(2\varepsilon^2+\varepsilon+1) - 2(3\varepsilon^2+3\varepsilon+2)\right) \\ & - m_i \frac{3}{32}\varepsilon\left(3N^3(\varepsilon^2-1) - 3\varepsilon^2 + 2N + 1\right) \\ \equiv& \mathscr{A}_0(\varepsilon) + m_i \mathscr{A}_1(\varepsilon). \end{aligned} \tag{6.10}$$

For convenience, we've defined $\mathscr{A}_0(\varepsilon)$ as the piece of $a_{T_N^{(m_i)}}(\varepsilon)$ that's independent of m_i, and $\mathscr{A}_1(\varepsilon)$ as the piece proportional to m_i. Our convention is that these formulae as written correspond to an unshaded ($\sigma_i = +1$) block, while taking $\varepsilon \to -\varepsilon$ (equivalently, swapping J_+ and J_-) yields the formulae for a shaded block. The contributions of an $\mathscr{N} = 2$ and $\mathscr{N} = 1$ vector multiplet are

$$a_{\mathcal{N}=2}(\epsilon) = \frac{3}{32}(N^2-1)\epsilon(3\epsilon^2-1) + \frac{6}{32}(N^2-1) \tag{6.11}$$

$$a_{\mathcal{N}=1} = \frac{6}{32}(N^2-1). \tag{6.12}$$

Here, the convention for $a_{\mathcal{N}=2}(\epsilon)$ is that as written we're gluing two unshaded flavor groups[5]; gluing two shaded flavor groups with an $\mathcal{N}=2$ vector corresponds to taking $\epsilon \to -\epsilon$. Then, $a(\epsilon)$ for this class of theories is given by a sum over these pieces,

$$\begin{aligned} a(\epsilon) =& \ell_2\mathscr{A}_0(\epsilon) + \mathscr{A}_1(\epsilon) \sum_{\{\sigma_i=+1\}} m_i + \ell_1\mathscr{A}_0(-\epsilon) + \mathscr{A}_1(-\epsilon) \sum_{\{\sigma_i=-1\}} m_i \\ &+ \frac{3}{2}(\ell_2-\ell_1)a_{\mathcal{N}=2}(\epsilon) + 3\ell_1 a_{\mathcal{N}=1}. \end{aligned} \tag{6.13}$$

Maximizing with respect to ϵ yields

$$\epsilon = \frac{N+N^2-\sqrt{z^2+N(1+N)\big(N(1+N)+z^2(4+3N(1+N))\big)}}{3(1+N+N^2)z}, \tag{6.14}$$

where we've written the answer in terms of the twist parameter z,

$$z = \frac{p-q}{p+q}, \qquad p=(g-1)(1+z), \qquad q=(g-1)(1-z). \tag{6.15}$$

ϵ in (6.14) matches the value computed in [89], as expected. The argument of the square root is always positive for the valid ranges of the parameters, $N \geq 2,\ g \geq 2$. ϵ is singular only for $q=p=g-1$, and re-maximizing with respect to ϵ for this special point yields $\epsilon=0$.

Substituting ϵ into (6.13), and performing the similar computation for c, we find

[5] Our conventions appropriately account for this, e.g. by not including an absolute value in the definition of n_2 in (6.6).

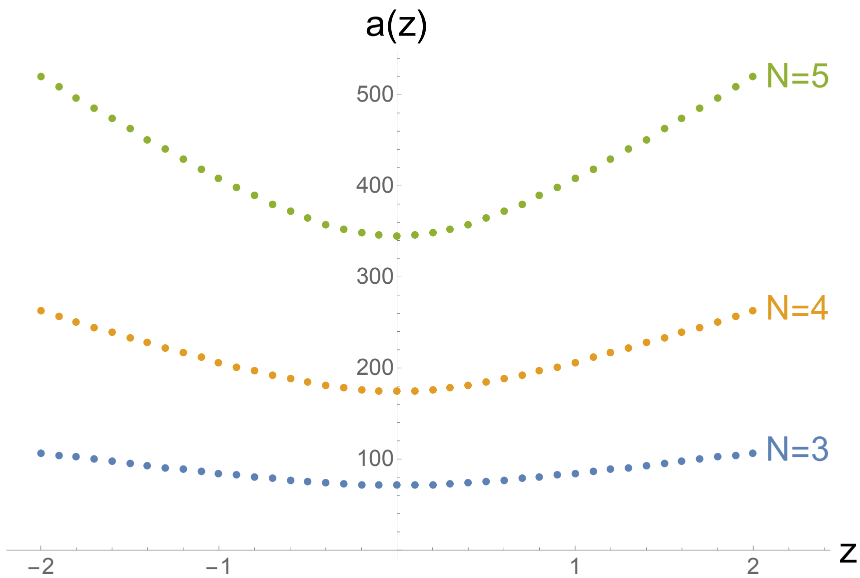

Figure 6.3: a as a function of the twist z for quivers constructed from $2g-2$ glued $T_N^{(m_i)}$ blocks, plotted for $g=11$ and various N.

$$a = \frac{(N-1)(g-1)}{48(1+N+N^2)z^2}\Big[-N^3(1+N)^3+3z^3(1+N+N^2)(3+N(1+N)(7+3N(1+N))) + \big(z^2+N(1+N)(N(1+N)+(4+3N(1+N))z^2)\big)^{3/2}\Big]. \tag{6.16}$$

$$c = \frac{(N-1)(g-1)}{48(1+N+N^2)^2z^2}\Big[-N^3(1+N)^3+z^2(1+N+N^2)(6+N(1+N)(17+9N(1+N))) + (N^2(1+N)^2+z^2(1+N+N^2)(2+3N(1+N)) \cdot\sqrt{z^2+N(1+N)\big(N(1+N)+z^2(4+3N(1+N))\big)}.\Big] \tag{6.17}$$

We plot a as a function of z for various values of N in Figure 6.6. These results precisely match the values of a and c that were computed by BBBW in [89] via integrating the anomaly eight-form of the M5-brane theory over $\mathscr{C}_{g,0}$. We emphasize that we've found this same result with a field theoretic construction. It is worth highlighting that it is nontrivial that the dependence on the m_i in (6.13) cancels to yield central charges (6.16) and (6.17) that depend only on the topological data z and g (and choice of N).

6.2.4 Operator dimensions and large-N

With ε fixed in (6.14), the dimensions of chiral operators can be determined by $\Delta(\mathscr{O}) = \frac{3}{2}R(\mathscr{O})$, using $R(\varepsilon)$ defined in (6.2). The dimensions of operators are then given by

$$\begin{aligned}
&\Delta[\mu] = \tfrac{3}{2}(1-\varepsilon), && \Delta[\phi] = \tfrac{3}{2}(1+\varepsilon) \\
&\Delta[Q] = \tfrac{3}{4}(N-1)(1-\varepsilon), && \Delta[u_k] = \tfrac{3}{2}(1+\varepsilon)k, \;\; k = 3,\ldots,N \\
&\Delta[M_j] = \tfrac{3}{2}(1+\varepsilon)(1+j), \;\; j = 1,\ldots,N-1.
\end{aligned} \tag{6.18}$$

Again, we use a convention where taking $\varepsilon \to -\varepsilon$ yields the R-charge of the operator corresponding to a shaded node; e.g. $\Delta[\phi](\varepsilon)$ corresponds to the adjoint chiral field in the $\mathscr{N} = 2$ vector multiplet of an unshaded node, while $\Delta[\phi](-\varepsilon)$ corresponds to the adjoint chiral of a shaded node. The μ are the various moment map operators of the $T_N^{(m_i)}$ blocks, detailed in Table 6.1.

We can construct gauge-invariant operators out of Q and $\widetilde{Q}$ that correspond to M2-brane operators wrapping the surface $\mathscr{C}_{g,0}$, as described in [89]. Schematically, these are

$$\mathscr{O}_{M2} = \prod_{i=1}^{2g-2} Q_i, \qquad \widetilde{\mathscr{O}}_{M2} = \prod_{i=1}^{2g-2} \widetilde{Q}_i \tag{6.19}$$

From (6.18), the dimensions of these operators are

$$\Delta[\mathscr{O}_{M2}] = \Delta[\widetilde{\mathscr{O}}_{M2}] = \frac{3}{4}(N-1)\left[(2g-2) + \varepsilon(\ell_1 - \ell_2)\right] \tag{6.20}$$

for ε given in (6.14), and where $0 \leq |\ell_1 - \ell_2| \leq 2g-2$. None of these operators decouple.

$|\varepsilon|$ in (6.14) ranges from 0 at $z = 0$, to $\frac{1}{\sqrt{3}}$ at large z and N. In particular, the new range of ε accessible for negative p and q versus the previously studied case of $p, q \geq 0$ [89] is the range $\frac{1}{3} \leq |\varepsilon| \leq \frac{1}{\sqrt{3}}$. In this range of ε, no operators violate the unitarity bound; thus, assuming no accidental IR symmetries, a and c are given by (6.16) and (6.17). We note that a and c given in (6.16), (6.17) are always positive and nonimaginary, and always (for $g > 1$) satisfy the Hofman-Maldacena bounds for $\mathscr{N} = 1$ SCFTs [23],

$$\frac{1}{2} \leq \frac{a}{c} \leq \frac{3}{2}. \tag{6.21}$$

At large N, the leading-order term of $a = c$ in (6.16) scales as N^3, and is given by

$$a_{\text{large-}N} = \frac{(1-g)(1-9z^2-(1+3z^2)^{3/2})}{48z^2}N^3. \tag{6.22}$$

This reproduces equation (2.22) in [89].

6.3 $\mathscr{C}_{g=0,n=0}$ from the Higgsed $T_N^{(m)}$

6.3.1 Constructing Higgsed $T_N^{(m)}$ theories

In this section, we review the procedure of closing the three maximal punctures of the $T_N^{(m)}$ theory. The Higgsing procedure we review below was detailed in [100], and also utilized in [75] (where the $|z| = 2$ case was first studied) and [101].

First, switch the color of the punctures to be opposite the color of the background $T_N^{(m)}$ (i.e. flip the colors of the $SU(N)_{A,B,C}$ flavor groups in Figure 6.1 from blue to red), and couple in three extra chiral fields $M_{A,B,C}$ that transform in the adjoint of the $SU(N)_{A,B,C}$ flavor groups, respectively. The superpotential contains terms that couple these adjoints to the moment map operators $\mu_{A,B,C}$,

$$W \supset \mu_A M_A + \mu_B M_B + \mu_C M_C. \tag{6.23}$$

Next, Higgs each of the flavor groups $SU(N)_{A,B,C}$ via a nilpotent vev[6]

$$\langle M_A \rangle = \langle M_B \rangle = \langle M_C \rangle = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}. \tag{6.24}$$

[6]In general, one could consider a nilpotent vev corresponding to an $SU(2)$ embedding $\rho : SU(2) \to SU(N)$ labeled by a partition of N, with the residual flavor symmetry given by the commutant of the embedding—see [112] for more details. Here, we consider only the principal embedding.

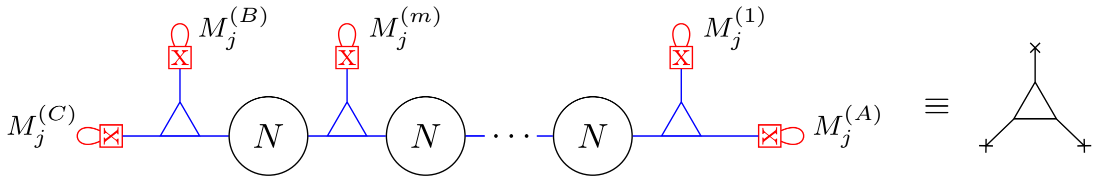

Figure 6.4: UV quiver for the $T_N^{(m)}$ theory with $\sigma_i = +1$, Higgsing the $SU(N)$ flavor nodes. The rightmost figure depicts a shorthand we use throughout, for reference.

A vev of this form corresponds to the principle embedding of $SU(2)$ into $SU(N)$, completely breaking each $SU(N)_{A,B,C}$ flavor group. The adjoint representation of $SU(N)$ decomposes into a sum of spin-j irreducible representations of $SU(2)$, such that the components of $M_{A,B,C}$ corresponding to fluctuations about the vev are labeled by the spin j and the σ_3-eigenvalue $m = -j, \ldots, j$ (e.g. see relevant discussion in [112] and [100]). The only components that don't decouple are those with $m = -j$, which we'll denote as $M_j^{(A,B,C)}$ below. After decoupling operators, the remaining superpotential is

$$W \supset \sum_{j=1}^{N-1} \left(M_j^{(A)} \mu_j^{(A)} + M_j^{(B)} \mu_j^{(B)} + M_j^{(C)} \mu_j^{(C)} \right). \tag{6.25}$$

The UV quiver is depicted in 6.4.

The Higgsing shifts the R-charges

$$J_+ \to J_+, \quad J_- \to J_- - \sum_{i=A,B,C} 2m^{(i)} \tag{6.26}$$

for $m^{(i)}$ the weights of the $SU(2)$ representations. The resulting R-charges of operators are given in Table 6.2. Note that the trifundamental Q's have decomposed into N^3 singlets, with R-charges shifted due to (6.26).

Geometrically, closing the punctures reduces the degrees of the normal bundle; starting with $p = m+1$, $q = -m$ as in (6.1), we flow to a theory with

$$\begin{gathered} p = m+1, \qquad q = -m-3, \qquad m \geq 0 \\ \Rightarrow z = -m-2. \end{gathered} \tag{6.27}$$

Note that this construction only yields 4d theories with $|z| \geq 2$, since starting with a $T_N^{(m)}$ theory with $\sigma = \pm 1$ results in $z = \mp(m+2)$ for $m \geq 0$.

Table 6.2: Operators in the Higgsed T_N theory.

		$SU(N)$	(J_+, J_-)
$Q^{(s)(t)(u)}, \widetilde{Q}^{(s)(t)(u)}$	$-\frac{(N-1)}{2} \leq \{s,t,u\} \leq \frac{N-1}{2}$		$\big(N-1, -2(s+t+u)\big)$
ϕ_i	$(i \leq 1 \leq m)$	adj	$(0,2)$
μ_i	$(i \leq 1 \leq m)$	adj	$(2,0)$
$\mu_j^{(A,B,C)}$	$(1 \leq j \leq N-1)$		$(2,-2j)$
$\mu_j^{(i)}$	$(1 \leq j \leq N-1)$		$(2,-2j)$
$M_j^{(i)}$	$(1 \leq j \leq N-1)$		$(0,2j+2)$
$M_j^{(A,B,C)}$	$(1 \leq j \leq N-1)$		$(0,2j+2)$

6.3.2 Computation of a_{trial} and c_{trial}

Next, we compute a and c for the Higgsed $T_N^{(m)}$ theories, assuming a flow to an IR fixed point. The central charges can be computed from the 't Hooft anomalies for the $T_N^{(m)}$ theory given in Appendix B.3, adding in the contribution from Higgsing the $SU(N)_{A,B,C}$ symmetries of the three punctures given in (B.11). The contribution to a from closing a single puncture can be expressed as

$$a_{\langle M\rangle}(\varepsilon) = \frac{3}{64}\left(2\varepsilon - 6\varepsilon^3 + 3N^3(\varepsilon-1)(\varepsilon+1)^2 + N(1+(2-3\varepsilon)\varepsilon) + N^2(2-\varepsilon+3\varepsilon^3)\right). \quad (6.28)$$

With these ingredients, we find that ε is given by

$$\varepsilon = \frac{N+N^2+\sqrt{z^2+N(1+N)(N(1+N)+z^2(4+3N(1+N)))}}{3(1+N+N^2)z} \quad (6.29)$$

and the central charges a and c are given by

$$a_{trial} = \frac{1}{48(1+N+N^2)^2z^2}\Big[(1+N)^3 - z^2(1+N+N^2)(9+3N(1+N)(7+3N(1+N)))$$
$$\cdot\left(z^2+N(1+N)(N(1+N)+z^2(4+3N(1+N)))\right)^{3/2}\Big],$$
$$c_{trial} = \frac{(N-1)}{48(1+N+N^2)^2z^2}\Big[N^3(1+N)^3 + \left(z^2+N(1+N)(N(1+N)+z^2(4+3N(1+N)))\right)^{3/2}$$
$$-z^2(1+N+N^2)\Big[6+N(1+N)(17+9N(1+N))$$
$$-\sqrt{z^2+N(1+N)(N(1+N)+z^2(4+3N(1+N)))}\Big]\Big]. \tag{6.30}$$

These match the BBBW results, given in (B.4) of Appendix B.2 with $\kappa = 1$ and $g = 0$. However, as we discuss in the next section, this is not the whole story, and there will be field theory corrections from operators in the theory hitting the unitarity bound. For this reason, we explicitly include the label a_{trial}, c_{trial}.

6.3.3 Comments on ruling out $g = 0$ SCFTs

For the Higgsed $T_N^{(m)}$ theory, the chiral operators are summarized in Table 6.2. $|\varepsilon|$ runs from $\frac{1}{3}\sqrt{\frac{19}{7}}$ for $N = 2$ and large-z, to $\frac{1}{6}(1+\sqrt{13})$ for $z = 2$ and large-N. For instance, the singlets $Q, \widetilde{Q}$ have dimension

$$\Delta[Q^{(s)(t)(u)}] = \Delta[\widetilde{Q}^{(s)(t)(u)}] = \frac{1}{2}\left((1-\varepsilon)(N-1) - 2(s+t+u)(1+\varepsilon)\right),$$
$$-\frac{N-1}{2} \leq s,t,u \leq \frac{N-1}{2}. \tag{6.31}$$

We generically find that some operators decouple, and at finite-N there will be field theory corrections to the central charges (6.30), and thus corrections to the BBBW results. We will not discuss these corrections in general, and instead briefly point out some features in the $|z| = 2, 3$ cases.

Our construction does not reach $|z| = 0, 1$. The case of $|z| = 2$, i.e. $m = 0$, is studied field-theoretically in [75], where they find that the central charges violate the Hofman-Maldacena bounds due to accidental symmetries from the decoupled operators along the flow to the IR SCFT.

For $z=3$, i.e. $m=1$, and $N>2$, the operator $\mathrm{Tr}\mu^2$ decouples. $Q, \widetilde{Q}$ operators with

$$|s+t+u| \geq \frac{-21+N(4N(2N-3)-11)+(N-1)\sqrt{9+4N(1+N)(9+7N(1+N))}}{2(9+10N(1+N)+\sqrt{9+4N(1+N)(9+7N(1+N))})} \quad (6.32)$$

violate the unitarity bound. For instance, for $N=2$ and $z=3$, the three operators $Q^{(s)(t)(u)}$ with $s+t+u=1/2$, and one with $3/2$ would have R-charges that unitarity. In general, the decoupling of these operators could lead to violations of the Hofman-Maldacena bounds, but we do not pursue this direction here[7].

6.4 General $\mathscr{C}_{g,n}$ from $T_N^{(m_i)}$ Building Blocks

6.4.1 Computing a_{trial} and c_{trial} for $g \neq 1$

Now we consider the most general case of constructing a genus g surface with n maximal punctures from $T_N^{(m_i)}$ building blocks, and computing a_{trial} and c_{trial}. One useful way to arrange this computation is to glue in chains of $T_N^{(m_i)}$ theories to the $n=0$ cases we constructed in Section 6.2.3. The result will be a genus g surface where the number of punctures depends on how many chains we add. Then, we can close arbitrarily many of these punctures via the Higgsing procedure discussed in Section 6.3.

Let us begin with the class of theories we considered in Section 6.2.3: start with ℓ_1 $T_N^{(m_i)}(\sigma_i=-1)$ blocks, and ℓ_2 $T_N^{(m_i)}(\sigma_i=+1)$ blocks, glued with n_1 $\mathcal{N}=1$ vectors, and n_2 $\mathcal{N}=2$ vectors, where $\ell_1+\ell_2=2g-2$ and $n_1+n_2=3(g-1)$.

Next, glue in some number of additional $T_N^{(m_i)}$ blocks such that we do not change the genus of the surface. In particular, introduce ℓ_1' $T_N^{(m_i)}(\sigma_i=-1)$ blocks and ℓ_2' $T_N^{(m_i)}(\sigma_i=+1)$ blocks. This will require that we introduce $\ell_1'+\ell_2'$ vectors to glue in these chains, where the number of additional $\mathcal{N}=2$ vectors n_2' is given by $\ell_2'-\ell_1'$. These $\ell_1'+\ell_2'$ blocks also introduce

[7] A discussion of the chiral operators in the ring of these theories on a sphere and their decoupling is given in [101], as well as some discussion on which geometries do not flow to SCFTs in the IR.

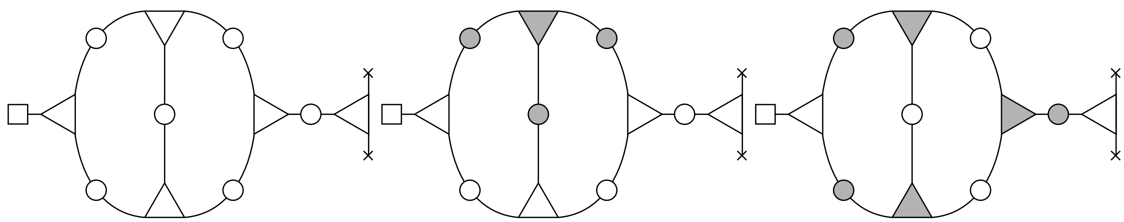

Figure 6.5: A genus 2, single-puncture example of a possible generalization of the diagrams in Figure 6.2. In our notation, these all have $n_{tot}=1$, $n_{dif}=-1$, and $\ell_1+\ell_2=2, n_1+n_2=\ell_1'+\ell_2'=n_1'+n_2'=3, h_2=2$. All three diagrams have the same IR central charges.

$\ell_1'+\ell_2'$ punctures to the surface, of various colors. As in Section 6.3, we can close each of these punctures by flipping their color, coupling in an adjoint chiral field, and letting the adjoint take a nilpotent expectation value that breaks the $SU(N)$ flavor group of the puncture. In particular, let us close h_1 of the $\sigma_i=-1$ punctures, and h_2 of the $\sigma_i=+1$ punctures. This will leave us with $n^{(-)}=\ell_1'-h_1$ minus punctures, and $n^{(+)}=\ell_2'-h_2$ plus punctures. An example of this construction applied to a genus 2 surface that results in one plus puncture is given in Figure 6.5.

The total degrees p and q for the surface will be given by

$$\begin{aligned} p &= \sum_{\{\sigma_i=+1\}} m_i - \sum_{\{\sigma_i=-1\}} m_i + (\ell_2+\ell_2') - h_1, \\ q &= -\sum_{\{\sigma_i=+1\}} m_i + \sum_{\{\sigma_i=-1\}} m_i + (\ell_1+\ell_1') - h_2, \\ p+q &= 2g-2+n^{(-)}+n^{(+)} = -\chi. \end{aligned} \tag{6.33}$$

The sum over $\{\sigma_i=+1\}$ runs over the $\ell_2+\ell_2'$ plus nodes, while the sum over $\{\sigma_i=-1\}$ runs over the $\ell_1+\ell_1'$ minus nodes. Below, we write out answers in terms of $z=(p-q)/(p+q)$, the combinations

$$n_{tot} \equiv n^{(-)}+n^{(+)}, \quad n_{dif} \equiv n^{(-)}-n^{(+)}, \tag{6.34}$$

and the Euler characteristic of the surface, $\chi=-2g+2-n_{tot}$. Summing the contributions to the trial central charges, we find $a(\epsilon)$ is given by

$$\begin{aligned} a(\epsilon) = -\frac{3}{64}(N-1)\Big[&(1+N)\left(2n_{tot}+n_{dif}\epsilon(1-3\epsilon^2)\right) \\ &+\chi\left(2+3(1-\epsilon^2)N(1+N)-z\epsilon\left(1+3N(1+N)+3\epsilon^2(1+N+N^2)\right)\right)\Big], \end{aligned} \tag{6.35}$$

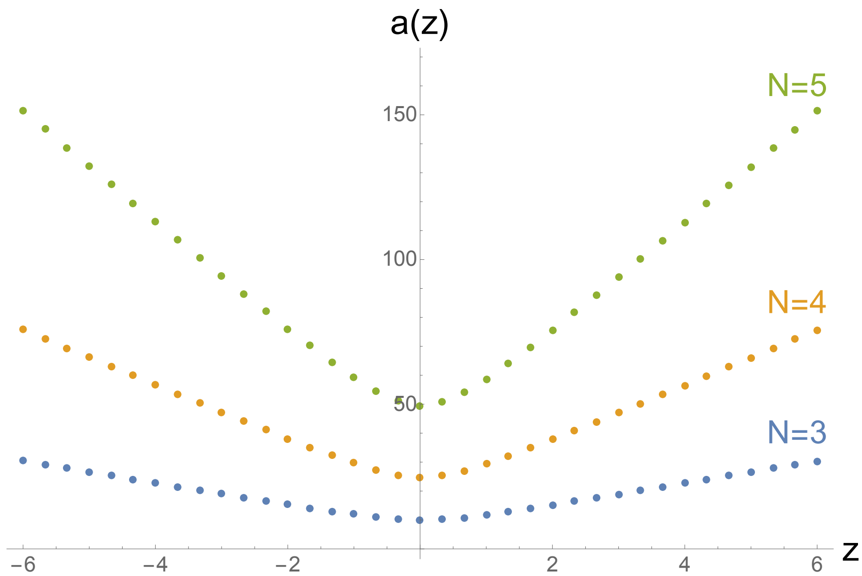

Figure 6.6: a as a function of the twist z for the $g=2, n_{tot}=1$ theories represented in Figure 6.5, plotted for $N=3,4,5$.

and $c(\varepsilon)$ is given by

$$c(\varepsilon) = -\frac{1}{64}(N-1)\Big[-4-(1+N)(9N-5\varepsilon+9n_{dif}\varepsilon^3)+n_{tot}\left(1-N(5+9N)\right) \\ +\chi\left(1-9\varepsilon^2N(1+N)+z\varepsilon\left(-5+9\varepsilon^2-9N(1+N)(1-\varepsilon^2)\right)\right)\Big]. \tag{6.36}$$

Maximizing with respect to ε, we find that ε is given by

$$\varepsilon = \frac{1}{3\left(\chi z(1+N+N^2)-n_{dif}(1+N)\right)} \cdot \\ \Bigg[\chi N(1+N)+\Big(n_{dif}(1+N)\left((1+N)(n_{dif}-4Nz\chi)-2\chi z\right) \\ +\chi^2\left(N^2(1+N)^2+z^2(1+N(1+N)(4+3N(1+N)))\right)\Big)^{1/2}\Bigg]. \tag{6.37}$$

For $n_{tot}=n_{dif}$=0, these formulae reproduce (6.16) and (6.17); for $g=n_{tot}=n_{dif}=0$ they reproduce (6.30).

At large-N, ε computed in (6.37) matches the BBBW result listed in (B.6). Then, the leading order piece of a and c in a large-N expansion is given by

$$a_{\text{large-}N} = \frac{N^3(2g-2+n_{tot})\left(9z^2-1+(1+3z^2)^{3/2}\right)}{96z^2}. \tag{6.38}$$

This is simply our result (6.22) with $(2g-2) \to (2g-2+n_{tot})$. Thus at large N, the central charges depend on the Riemann surface only through the Euler characteristic $\chi = -2g+2-n_{tot}$.

We highlight that these answers depend only on topological data: the total number of punctures n_{tot}, the difference between the number of plus and minus punctures n_{dif}, the twist z defined in terms of the normal line bundle degrees p and q, and the genus g of the surface (or equivalently, the Euler characteristic). It is satisfying that the computation organized such that the other parameters dropped out, leaving the (relatively) nice expressions (6.35)-(6.37).

6.4.2 Comments on operators

Deferring a detailed analysis of the chiral operators of this class of theories to the future, we point out one interesting feature in the chain operators $\mathcal{O}_{M2}$, $\widetilde{\mathcal{O}}_{M2}$. For illustration, consider the theories in Figure 6.5. There will be N^2 fundamentals (and anti-fundamental) operators of the form $(QQQQQ)_\ell^{(s)(t)}$, where $-\frac{(N-1)}{2} \leq s,t \leq \frac{(N-1)}{2}$ label the N^2 operators and yield a shift in the J_- charges of these operators as in Table 6.2, and ℓ runs from $1,\ldots,N$ is an index for the $SU(N)$ flavor symmetry of the remaining puncture. If we close the remaining puncture, then there will be N^3 degenerate chain operators that should correspond to M2-branes wrapping the surface.

More generally, for a class of theories constructed from gluing h_1 Higgsed shaded flavor groups and h_2 Higgsed unshaded flavor groups such that all the punctures are closed, there will be gauge-invariant chain operators

$$\mathcal{O}^{(s_1)\ldots(s_{h_1})(t_1)\ldots(t_{h_2})} = \prod_{i=1}^{2g-2+h_1+h_2} (Q_i)^{(s_1)\ldots(s_{h_1})(t_1)\ldots(t_{h_2})}, \quad -\frac{(N-1)}{2} \leq s_i,t_i \leq \frac{N-1}{2} \qquad (6.39)$$

as well as the corresponding operators constructed from the $\widetilde{Q}$'s. The R-charges of these operators will be given by

$$R\left(\mathscr{O}^{(s_1)\ldots(s_{h_1})(t_1)\ldots(t_{h_2})}\right) = \frac{1}{2}\Bigg(\left((N-1)(h_2+\ell_2)-2h_1\sum_{i=1}^{h_1}s_i\right)(1-\varepsilon) + \left((N-1)(2g-2+h_1-\ell_1)-2h_2\sum_{i=1}^{h_2}t_i\right)(1+\varepsilon)\Bigg). \tag{6.40}$$

Thus, this field-theoretic analysis suggests a degeneracy of possible M2-brane operators[8].

6.4.3 Computing a_{trial} and c_{trial} for the torus

Up to this point, we've considered $g \neq 1$. The case of M5-branes compactified on the 2-torus is special because the torus admits a flat metric, implying that the maximal amount of supersymmetry can be preserved by fixing the normal bundle to the M5-brane worldvolume to be trivial. The singular behavior at $g = 1$ in the computation of a and c is related to the fact that the M5-brane tension causes the volume of the torus to shrink. In our constructions, this means that we should get a 4d $\mathscr{N} = 4$ field theory in the IR when $g = 1$ and $z = 0$.

One can formulate a nonsingular construction that preserves only $\mathscr{N} = 1$ supersymmetry by taking the torus to have line bundles of equal and opposite degrees fibered over it, i.e. taking $p = -q$. Letting $g = 1$ and $p = -q$ in (6.33) means that we should require the total number of punctures be zero. In the construction detailed in Section 6.4.1, the simplest generalized quiver that this could correspond to is closing one of the punctures on a $T_N^{(m)}$ block, and then gluing the other two.

Redoing the computation of Section 6.4.1 for the torus, we find

$$g = 1: \quad \varepsilon = -\frac{1}{3}\sqrt{\frac{1+3N(1+N)}{1+N+N^2}}, \tag{6.41}$$

and

$$g = 1: \quad a = \frac{p(N-1)(1+3N(1+N))^{3/2}}{48\sqrt{1+N+N^2}}. \tag{6.42}$$

Indeed, (6.42) matches the BBBW result that we've written in (B.7), where our definition of p

[8]This degeneracy of operators was noted independently in [101].

matches their $|z|$.

6.5 Future Directions

There are many directions one can think about based on the present work, some of which we will list below.

First, we note that the authors of [101] study the chiral rings of these theories (much as [113] and [114] studied the chiral rings of the T_N theory in detail). It would be further interesting to study the moduli space of vacua of the 4d $\mathcal{N} = 1$ SCFTs realized by our constructions.

In the present work we don't discuss possible confinement of the gauge theories that result from the gluing procedure. However, as shown in [100], when two $T_N^{(m)}$ blocks with the same m and opposite shading are glued with an $\mathcal{N} = 1$ vector multiplet, the gauge node confines. It would be interesting to study the structure of confinement for our general constructions, as well as to understand how various duality maps of $T_N^{(m)}$ theories (discussed in [100], and [101]) act on our constructions.

It would be interesting to obtain a field-theoretic construction of the $N = 2,\ g = n = 0$ theories with $|z| = 0, 1$. While these theories do not have a conformal phase, it could be useful to study the IR dynamics from the field theory side[9].

Also, it is shown in [110] that the BBBW solutions with rational central charges allow probe M5-branes to break into multiple M5-branes at special points; in particular, when a and c at large-N (given in (6.22)) are rational. It would be interesting to understand field-theoretically what happens at these special points in our constructions.

[9] The author is grateful to Ibrahima Bah for pointing out this possibility.

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Chapter 7

Structure of Anomalies of 4d SCFTs from M5-branes, and Anomaly Inflow

We study the 't Hooft anomalies of four-dimensional superconformal field theories that arise from M5-branes wrapped on a punctured Riemann surface. In general there are two independent contributions to the anomalies. There is a bulk term obtained by integrating the anomaly polynomial of the world-volume theory on the M5-branes over the Riemann surface; this contribution knows about the punctures only through its dependence on the Euler characteristic of the surface. The second set of contributions comes from local data at the punctures; these terms are independent from the bulk data of the surface. Using anomaly inflow in M-theory, we describe the general structure of the anomalies for cases when the four-dimensional theories preserve $\mathcal{N}=2$ supersymmetry. In particular, we show how to account for the local data from the punctures. We additionally discuss the anomalies corresponding to (p,q) punctures in $\mathcal{N}=1$ theories.

7.1 Introduction

't Hooft anomalies provide a robust measure of the degrees of freedom in quantum field theories. In general, the anomalies for a given theory in even spacetime dimensions d can be encoded in a $(d+2)$-form polynomial known as the anomaly polynomial, which depends on the various curvature forms associated to the dynamical or background gauge and gravity fields. If the gauge or gravity field is dynamical, its anomalies should vanish or else the theory is inconsistent. Otherwise, the anomaly doesn't lead to an inconsistency, but often has interesting physical consequences. We refer to anomalies in background gauge or gravity fields as 't Hooft anomalies. For a review, see [115, 10].

In the last ten years, there has been a proliferation of new classes of four-dimensional Superconformal Field Theories (SCFT's), dubbed class $\mathscr{S}$, that are inherently strongly coupled and admit no known Lagrangian description. These theories can emerge from the low energy limit of six-dimensional $\mathscr{N}=(2,0)$ SCFT's wrapped on a punctured Riemann surface, which in certain cases describe the low energy dynamics of M5-branes. These constructions have been revolutionary in that they provide a partial classification scheme for four-dimensional $\mathscr{N}=2$ SCFT's, and bring to bear new geometric tools for studying them [82, 83, 84]. Soon after their introduction, it was demonstrated that these constructions can be generalized to study the space of four-dimensional $\mathscr{N}=1$ SCFT's [85, 86, 87, 88, 89, 78].

The basic set-up of class $\mathscr{S}$ theories was reviewed in Chapter 5. An important ingredient in the construction is a partial topological twist [90, 91], which is needed to preserve some supercharges in the compactification of the six-dimensional theory. Depending on the choice of twist, various amount of supersymmetry can be preserved in four dimensions.

Anomalies are particularly important observables for the theories of class $\mathscr{S}$, as they provide a measure of various degrees of freedom in these inherently strongly coupled field theories. The anomalies for $\mathscr{N}=2$ class $\mathscr{S}$ theories can be obtained, in some cases, by using

S-duality [82, 83, 95] and anomaly matching on the moduli space [103, 116]. In the special case of $\mathscr{N}=1$ theories we can use Seiberg duality as well as anomaly matching on the moduli space to obtain them [111, 117, 112].

In the cases where they are known, the anomaly polynomials of class $\mathscr{S}$ theories have two contributions which are independent and must be stated separately. The first is the contribution from the bulk Riemann surface, which we denote $I_6(\Sigma_{g,n})$. This depends only on the genus g and number of punctures n through the Euler characteristic $2g-2+n$, and on the anomaly polynomial of the UV six-dimensional theory. The second set of contributions come from new degrees of freedom localized at the punctures; these are related to the consistent boundary conditions for the six-dimensional theory at these locations. A contribution of a puncture to the anomalies of the four-dimensional theory is denoted as $I_6(P)$. The total six-form anomaly polynomial $I_6^{\mathscr{S}}$ for the class $\mathscr{S}$ theory takes the form

$$I_6^{\mathscr{S}} = I_6(\Sigma_{g,n}) + \sum_{i=1}^{n} I_6(P_i). \tag{7.1}$$

Since the theories of class $\mathscr{S}$ are defined by the compactification of a six-dimensional theory, there should exist a prescription for directly computing their anomaly polynomials from the geometric construction. Indeed, in the case of theories obtained by compactifying on a smooth Riemann surface without punctures, integrating the anomaly polynomial I_8 of the six-dimensional theory over the surface can yield the polynomial of the four-dimensional theory [118, 89][1], i.e.

$$I_6^{\mathscr{S}} = \int_{\Sigma_g} I_8. \tag{7.2}$$

This prescription requires shifting the curvature of the background R-symmetry gauge field with the curvature form of the Riemann surface, implementing the topological twist. The integration of the eight-form polynomial over the surface picks out the terms that are linear in the surface's curvature form, and therefore proportional to its volume form.

In the presence of punctures this prescription fails; we cannot obtain the full anomaly

[1]This procedure fails when there are accidental symmetries. This problem is most commonly encountered when compactifying on a Riemann surface with vanishing Euler characteristic—see for example [119].

polynomial of class $\mathscr{S}$ theories by simply shifting the background curvature and integrating. There can be additional terms in the anomaly polynomial of the six-dimensional theory, and the integration over the Riemann surface cannot account for the additional data localized at the punctures.

Our primary goal in this chapter is to develop new tools for computing the anomaly polynomial from first principles—i.e. from the six-dimensional theory, the compactification scheme, and from the punctured Riemann surface. We will argue that this general form (7.1) follows from anomaly inflow in M-theory on the M5-branes wrapping the punctured Riemann surface.

Our strategy is motivated by the holographic duals of class $\mathscr{S}$ theories from punctured Riemann surfaces [83, 94] (see [110] for probe analysis). In the gravity duals, the topological twists are manifested by non-trivial S^1-bundles over the Riemann surface. The connections on these bundles are related to the shifts of the background R-symmetry in the twist, and their curvatures F are proportional to the volume form of the Riemann surface. In the presence of a puncture, F picks up monopole sources that encode the new degrees of freedom associated to the puncture. These monopoles are end points of additional M5-branes localized at the puncture and extended along a direction normal to the surface. This gives a strong hint that in computing $I_6^{\mathscr{S}}$, we need to enrich the shifting prescription of the background gauge field of the R-symmetry to account for these sources. Moreover, in integrating the eight-form anomaly polynomial, there is an additional interval that is normal to the branes along which the monopole sources sit. We indeed recover all of these features in the anomaly inflow analysis.

We can summarize our main result from the inflow analysis for the case of $\mathscr{N}=2$ SCFTs as follows. For simplicity, in this discussion we will restrict to the case of N M5-branes wrapping a single-punctured surface $\Sigma_{g,1}$. If there are n punctures, we repeat the same step procedure for each of them.

- Given $\Sigma_{g,1}$, add an interval $[\mu]$ with coordinate μ in the range $[0,1]$ to obtain the space

$[\mu] \times \Sigma_{g,1}$.

- Along $[\mu]$ we add monopole sources for F localized at the puncture on the surface, and at $\mu = \mu^a$ (the a labels the different sources). In the region near the monopole, there is a $U(1)_a$ symmetry induced on the brane that ends on it. These $U(1)_a$'s are related to the local data at the puncture. The curvatures for the background gauge fields for these $U(1)_a$'s, which we denote F^a, can and do appear in the anomaly polynomial.

- The general solution for the curvature in the presence of the sources, $\widetilde{F}$, is constructed. It has a term proportional to the volume form of the Riemann surface, new terms that are proportional to the volume forms of spheres S^2_a surrounding the a'th monopole in the space $[\mu] \times \Sigma_{g,1}$, and terms that are proportional to the F^a's.

- We argue that $I_6^{\mathscr{S}}$ can be obtained by integrating a nine-form, I_9, on the space $[\mu] \times \Sigma_{g,1}$. The form is given schematically as
$$I_9 = d\left[\mu^3 I_8(\widetilde{F}) + N^3 I_8^p\right], \tag{7.3}$$
where $I_8(\widetilde{F})$ is the anomaly eight-form of the world-volume theory on the flat M5-branes with the R-symmetry curvature form shifted by $\widetilde{F}$. The form I_8^p depends on the local puncture data. Its dependence on the curvature forms is fixed up to some coefficients that we will study and determine in [120].

- The integral of I_9 over $[\mu] \times \Sigma_{g,1}$ is a sum of boundary terms. The one at $\mu = 1$ reproduces exactly the bulk term, $I_6(\Sigma_{g,1})$ in (7.1)—the form I_8^p does not contribute here. There are also terms that come from the internal boundaries, S^2_a, which depend only on the charge of the monopole and on the μ^a. These terms account for the puncture data. The μ^a and monopole charges must be quantized; we study them and their quantization conditions further in [120].

The structure of the rest of this chapter is as follows. In Section 7.2, we describe general features of the anomalies of of four-dimensional field theories preserving $\mathscr{N}=2$ or $\mathscr{N}=1$ supersymmetry from M5-branes wrapped on Riemann surfaces. Readers familiar with the class $\mathscr{S}$ construction and anomalies can skip to Section 7.3.

In Section 7.3 we take a field-theory detour. Focusing on the case where the four-dimensional field theory preserves $\mathscr{N}=1$ supersymmetry, we derive the anomalies corresponding to a large class of locally $\mathscr{N}=2$ preserving punctures in geometries in which the bulk preserves $\mathscr{N}=1$. We additionally discuss an illuminating way of parameterizing the anomaly coefficients we obtain in terms of an $\mathscr{N}=1$ generalization of an effective number of vector multiplets and hypermultiplets. This section and the inflow computation that follows may be read independently of one another.

In the remainder of the chapter, we turn to a computation of the class $\mathscr{S}$ anomalies by anomaly inflow in 11d supergravity in the presence of M5-branes, focusing on the case where the four-dimensional theory preserves $\mathscr{N}=2$. We begin with a review of inflow for flat M5-branes in Section 7.4 (originally discussed in [121, 122, 123]). In Section 7.5 we compute the anomaly eight-form of the M5-branes in the curved background, which we argue contains boundary terms when the Riemann surface has punctures. Integrating the eight-form over the surface, we are able to derive directly from first principles the structure of the class $\mathscr{S}$ anomalies laid out in (7.1). The local puncture contributions to the anomalies of the four-dimensional theories come from the new boundary terms in the integration.

7.2 Structure of Class $\mathscr{S}$ Anomalies

This section serves as an extended introduction to the four-dimensional theories obtained by compactifying the six-dimensional (2,0) theories on a Riemann surface, setting notation and focusing attention on the main points of interest in the rest of the chapter. The experienced reader can skip to Section 7.3.

7.2.1 Anomalies of the $(2,0)$ theories

The six-dimensional $\mathscr{N} = (2,0)$ theories are labeled by an ADE Lie algebra: $A_{N-1} = \mathfrak{su}(N), D_N = \mathfrak{so}(2N)$, or $\mathfrak{e}_{6,7,8}$. The six-dimensional (2,0) superconformal algebra is $\mathfrak{osp}(4|8)$. Its bosonic subgroup is $SO(2,6) \times SO(5)_R$, corresponding respectively to the conformal group and R-symmetry group. These theories arise from decoupling limits of string theory constructions [124, 125, 126].

The focus of the present work is the anomalies of the four-dimensional class $\mathscr{S}$ theories, which can be understood by tracking the anomaly polynomial of their parent six-dimensional theories in the compactification on the surface. The six-dimensional (2,0) theories cannot be written down in terms of the usual path integral of local fields, which makes understanding their properties a challenge. However, as anomalies are inherently topological quantities, they are accessible even for these mysterious theories.

The interacting A_{N-1} theory is the effective world-volume theory of N coincident M5-branes, and the D_N-type theories are realized on the world-volume of N coincident M5-branes at an $\mathbb{R}^5/\mathbb{Z}_2$ orbifold fixed point. In these cases, the derivation of six-dimensional $(2,0)$ anomalies can be understood in terms of inflow for M5-branes in 11d supergravity. As the M5-brane world-volume is six-dimensional, the anomalies will involve eight-dimensional characteristic classes, packaged in an eight-form anomaly polynomial which encodes anomalous diffeomorphisms of the world-volume of the M5-branes and their normal bundle. The idea of the inflow analysis is that in the presence of the M5-branes, the total anomaly from zero modes on the world-volume and inflow from the bulk should vanish in order for the theory to be consistent. Using these methods (which we review in Section 7.4), the anomaly eight-form for a single M5-brane is derived as [123, 121]

$$I_8[1] = \frac{1}{48}\left[p_2(NW) - p_2(TW) + \frac{1}{4}\left(p_1(TW) - p_1(NW)\right)^2\right]. \tag{7.4}$$

NW and TW are the normal bundle and tangent bundle to the M5-brane world-volume W,

Table 7.1: Rank, dimension, and Coxeter numbers for the simply-laced Lie groups. Note the useful group theory identity $d_G = r_G(h_G + 1)$.

G	r_G	d_G	h_G
A_{N-1}	$N-1$	N^2-1	N
D_N	N	$N(2N-1)$	$2N-2$
E_6	6	78	12
E_7	7	133	18
E_8	8	248	30

respectively, and p_k are the Pontryagin classes, reviewed in Appendix C.1. (7.4) is also the anomaly polynomial for a single, free $(2,0)$ tensor multiplet. The tensor multiplet is the only (2,0) superconformal multiplet that describes free fields, containing a self-dual three-form, as well as Weyl fermions in the spinor representation of $SO(5)_R$, and real scalars in the fundamental of $SO(5)_R$.

For a general six-dimensional (2,0) theory of type $\mathfrak{g}$ = ADE, the eight-form anomaly polynomial takes the form

$$I_8[\mathfrak{g}] = r_G I_8[1] + \frac{d_G h_G}{24} p_2(NW). \tag{7.5}$$

The values of r_G, d_G, h_G for the ADE groups are listed for reference in Table 7.1. Here, the normal bundle NW can be thought of as an $SO(5)$ bundle coupled to the six-dimensional R-symmetry.

This result was obtained for A_{N-1} in [122] via inflow with multiple M5-branes, and conjectured for all $\mathfrak{g}$ = ADE in [127] using purely field-theoretic reasoning. It was verified for D_N in [128] with an inflow analysis, and verified for all $\mathfrak{g}$ =ADE in [129] via anomaly matching on the tensor branch. An exact calculation of the a-anomaly for (2,0) theories via a similar field-theoretic derivation was given in [130]. The famous N^3 scaling at large N was first noticed in the context of black hole calculations of the thermal free energy [131], and was computed for the central charges via AdS/CFT [132].

7.2.2 Structure of class $\mathscr{S}$ anomalies

As we emphasized in the introduction, the anomalies of class $\mathscr{S}$ have two contributions which are independent and must be stated separately:

$$I_6^{\mathscr{S}} = I_6(\Sigma_{g,n}) + \sum_{i=1}^{n} I_6(P_i). \tag{7.6}$$

Here, we'll give a more complete discussion of this point.

The bulk piece $I_6(\Sigma_{g,n})$ is always obtained by integrating the eight-form anomaly polynomial (7.5) over the Riemann surface with a given Euler characteristic $\chi = -2g+2-n$, and with the appropriate topological twist (5.3), as in (7.2) [89, 88, 118]. This piece will be proportional to χ, since the terms in $I_8[\mathfrak{g}]$ that survive the integral are linear in the curvature two-form on the Riemann surface.

The second class of terms are due to the punctures. Deriving these contributions from a six-dimensional perspective is more subtle. These pieces depend on local data which add degrees of freedom to the theory, leading to global symmetries. In this note we'll be interested in the anomalies of a class of punctures dubbed regular punctures, which we review below. For more details on the anomalies of regular punctures, see Appendix C.2.

A regular puncture is labeled by an embedding $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. For $\mathfrak{g} = A_{N-1}$, the choice of embedding is 1-to-1 with a partition of N, and is therefore labeled by a Young diagram Y. For a Young diagram with n_i columns of height h_i, the field theory will have an unbroken flavor symmetry G associated to the puncture. G corresponds to the commutant of the embedding ρ, given as

$$G = S\left[\prod_i U(n_i)\right]. \tag{7.7}$$

The case of the maximal flavor symmetry $G = SU(N)$ is known as a maximal (or full) puncture, and the case of the minimal flavor symmetry $G = U(1)$ is known as a minimal (or simple) puncture.

The form of $I_6(P_i)$ can be derived from string dualities utilizing the generalized quiver descriptions of the four-dimensional theories [103]. One can derive the contributions to the anomaly polynomial from non-maximal punctures by Higgsing the associated flavor symmetry and keeping track of the multiplets which decouple [116]. In Section 7.5, we will show that the form of the $I_6(P_i)$ follows directly from inflow of M5-branes in the presence of punctures on the Riemann surface, and demonstrate that in fact the puncture anomalies can also be derived directly by integrating $I_8[\mathfrak{g}]$, in a way that we will make precise.

The additive structure of the anomalies (7.6) is motivated by the TQFT structure of the class $\mathscr{S}$ theories. Both the $\mathcal{N}=1$ and $\mathcal{N}=2$ class $\mathscr{S}$ SCFTs admit a formulation in the language of a 2d topological quantum field theory [133, 134, 135, 136, 78]. The superconformal index is then computed as an n-point correlation function of the TQFT living on the n-punctured Riemann surface, with punctures corresponding to operator insertions. Thus, the theories are organized topologically by specifying bulk information and local puncture information.

One should note, however, that even though the anomaly polynomial has a simple additive structure, quantities of interest such as the central charges are still nontrivial and don't follow immediately from topological arguments. For instance, in the $\mathcal{N}=1$ case there is an additional $U(1)$ flavor symmetry that mixes with the $U(1)_R$ symmetry. Given an R-symmetry, the anomaly polynomial encodes all the mixed anomalies with the global symmetries. However, a-maximization is required to specify the exact superconformal R-symmetry.

Anomalies for $\mathcal{N}=2$ SCFTs

Here, we review the anomalies for the four-dimensional $\mathcal{N}=2$ SCFTs which we will match onto in an inflow computation in Section 7.5.

The anomaly polynomial of a four-dimensional $\mathcal{N}=2$ superconformal theory with a flavor symmetry G and $SU(2)_R \times U(1)_R$ symmetry has the form[2]

[2]More generally, the term $k_G c_1(F_1) c_2(F_G)$ should be written in terms of the instanton number $n(F_G)$, normalized such that for $SU(N)$ $n(F_{SU(N)}) = c_2(F_{SU(N)})$—e.g. see [116].

$$I_6 = (n_v - n_h)\left(\frac{c_1(F_1)^3}{3} - \frac{c_1(F_1)p_1(T^4)}{12}\right) - n_v c_1(F_1)c_2(F_2) + k_G c_1(F_1)c_2(F_G). \tag{7.8}$$

This expression follows from the definition of the anomaly polynomial for four-dimensional Weyl fermions, as reviewed in Appendix C.1, and the $\mathcal{N}=2$ superconformal algebra [17]. In (7.8), F_1 (F_2) is the field strength for the background gauge field of the $U(1)_R$ ($SU(2)_R$) bundle, and F_G is the field strength of the flavor symmetry bundle. The flavor central charge k_G is defined in the Introduction. More generally, additional flavor symmetries would contribute additional terms in (7.8).

The parameters n_v and n_h are related to the central charges of the SCFT as $a = \frac{1}{24}(5n_v + n_h)$, and $c = \frac{1}{12}(2n_v + n_h)$. If the theory is free, then n_v and n_h denote the number of vector multiplets and hypermultiplets respectively; otherwise, we regard n_v and n_h as an effective number of vector and hypermultiplets. Even for interacting field theories, this notation serves as a useful bookkeeping device.

The R-symmetry of the $\mathcal{N}=2$ theories is identified as $SU(2)_+ \times U(1)_-$ when $p_1 = 0$, and as $U(1)_+ \times SU(2)_-$ when p_2 is zero. Denoting the generators of the $U(1)_\pm$ as $J_\pm$ and the $SU(2)_R \times U(1)_R$ generators by I^a and $R_{\mathcal{N}=2}$ respectively, this corresponds to the identification

$$\begin{aligned} p_1 = 0: &\qquad J_+ = 2I^3, \qquad J_- = R_{\mathcal{N}=2} \\ p_2 = 0: &\qquad J_+ = R_{\mathcal{N}=2}, \qquad J_- = 2I^3. \end{aligned} \tag{7.9}$$

As summarized in (7.6), the theories of class $\mathcal{S}$ have two contributions to their anomalies: contributions from the bulk, and local contributions from the punctures. For the $\mathcal{N}=2$ theories, as suggested by [83] it is convenient to write these in terms of an effective number of vector and hypermultiplets (n_v, n_h) as

$$n_v = n_v(\Sigma_{g,n}) + \sum_{i=1}^{n} n_v(P_i), \qquad n_h = n_h(\Sigma_{g,n}) + \sum_{i=1}^{n} n_h(P_i). \tag{7.10}$$

These terms were computed explicitly in [83, 95], with the help of a result in [137]. The bulk terms are given by[3]

$$n_v(\Sigma_{g,n}) = -\frac{\chi}{2}\left(r_G + \frac{4}{3}d_G h_G\right), \qquad n_h(\Sigma_{g,n}) = -\frac{\chi}{2}\left(\frac{4}{3}d_G h_G\right). \tag{7.11}$$

The puncture contributions $n_{v,h}(P_i)$ for the A_{N-1} case are reviewed in Appendix C.2. As explained there, these terms depend on the details of the Young diagrams corresponding to the punctures.

Together, the bulk contribution (7.11), and the puncture contributions (C.16) and (C.17) determine the full anomaly polynomial of the four-dimensional $\mathcal{N}=2$ class $\mathcal{S}$ SCFTs. Plugging into (7.8), this gives

$$I_6(\Sigma_{g,n}) = -\frac{\chi}{2}\left[\left(\frac{\left(c_1^+\right)^3}{3} - \frac{c_1^+ p_1(T^4)}{12}\right) r_G - c_1^+ c_2^- \left(r_G + \frac{4}{3}d_G h_G\right)\right], \tag{7.12}$$

$$I_6(P_i) = \left(n_v(P_i) - n_h(P_i)\right)\left(\frac{(c_1^+)^3}{3} - \frac{c_1^+ p_1(T^4)}{12}\right) - n_v(P_i) c_1^+ c_2^- + k_{G_i} c_1^+ c_2(F_{G_i}). \tag{7.13}$$

Here, we've chosen $p_2 = 0$ as our $\mathcal{N}=2$ limit, with $c_1^+ \equiv c_1(U(1)_+)$ and $c_2^- \equiv c_2(SU(2)_-)$. The terms proportional to $(c_1^+)^3$ and $c_1^+ c_2^-$ are 't Hooft anomalies for the background R-symmetry. The $c_1^+ p_1(T^4)$ pieces encode the mixed gauge-gravity anomalies. The last piece in $I_6(P_i)$ couples the global symmetry G_i preserved by the puncture with the $U(1)_R$ symmetry, and will have a separate term for each factor in the puncture flavor symmetry (7.7). The anomaly coefficients $\mathrm{Tr}R_{\mathcal{N}=2} = \mathrm{Tr}R^3_{\mathcal{N}=2}$ and $\mathrm{Tr}R_{\mathcal{N}=2} I_3^2$ are readily determined from (7.13) using (2.33) and (2.35).

General structure of $\mathcal{N}=1$ class $\mathcal{S}$ anomalies

The $\mathcal{N}=1$ theories of class $\mathcal{S}$ preserve a $U(1)_+ \times U(1)_-$ global symmetry which derives from the $\mathcal{N}=(2,0)$ $SO(5)_R$ symmetry as in (5.1). A combination of the $U(1)_\pm$ generators $J_\pm$ corresponds to a flavor symmetry $\mathcal{F}$, and we can pick an R-symmetry R_0, given as

$$R_0 = \frac{1}{2}(J^+ + J^-), \qquad \mathcal{F} = \frac{1}{2}(J^+ - J^-). \tag{7.14}$$

The exact superconformal R-symmetry $R_{\mathcal{N}=1}$ is

[3]Note that in much of the literature, the term proportional to the n in $\chi = -(2g-2+n)$ is instead grouped with the puncture contribution to the anomalies. The grouping we use here emphasizes the fact that the whole term proportional to χ comes from global considerations. E.g. regardless of the types of punctures, this term only depends on their total number.

$$R_{\mathcal{N}=1}(\varepsilon) = R_0 + \varepsilon\mathscr{F} = \frac{1}{2}(J_+ + J_-) - \frac{1}{2}\varepsilon(J_+ - J_-), \tag{7.15}$$

where ε is determined by a-maximization [24]. When $p_1 = 0$, ε is fixed to be $\frac{1}{3}$, and the generators $J_\pm$ are identified as in (7.9). In this case, we identify an $\mathcal{N}=1$ subalgebra in $\mathcal{N}=2$ as

$$R_{\mathcal{N}=1} = \frac{1}{3}R_{\mathcal{N}=2} + \frac{4}{3}I^3 = \frac{1}{3}J_+ + \frac{2}{3}J_-, \tag{7.16}$$

and $U(1)_-$ is enhanced to $SU(2)_-$. Similarly, when $p_2 = 0$, $\varepsilon = -\frac{1}{3}$ and $U(1)_+$ is enhanced to $SU(2)_+$.

The 't Hooft anomalies for the four-dimensional class $\mathscr{S}$ theories are encoded in a six-form anomaly polynomial. It follows from [16] and the definition of the anomaly polynomial discussed in Appendix C.1 that the anomaly polynomial for a four-dimensional theory with a $U(1)_+ \times U(1)_-$ global symmetry takes the form

$$I_6^{\mathscr{S}} = \frac{1}{6}\mathrm{Tr}\left[J_+c_1^+ + J_-c_1^-\right]^3 - \frac{1}{24}\mathrm{Tr}\left[J_+c_1^+ + J_-c_1^-\right]p_1(T^4). \tag{7.17}$$

Here, $c_1^\pm \equiv c_1(U(1)_\pm)$ are the first Chern classes of the $U(1)_\pm$ bundles. There could be additional flavor symmetries, which will mix with the R-symmetry and give additional terms in (7.17).

As discussed in Section 7.2.2, the anomaly polynomials for the $\mathcal{N}=1$ class $\mathscr{S}$ theories will decompose into background contributions from the bulk which can be computed directly by integrating $I_8[\mathfrak{g}]$ for the six-dimensional theory over the Riemann surface, and local contributions from the punctures. The bulk contribution to $I_6^{\mathscr{S}}$ is

$$\begin{aligned} I_6(\Sigma_{g,n}) = &-\frac{\chi(1+z)}{2}\left\{\left(\frac{\left(c_1^+\right)^3}{6} - \frac{c_1^+ p_1(T^4)}{24}\right)r_G - \frac{c_1^+\left(c_1^-\right)^2}{2}\left(r_G + \frac{4}{3}d_G h_G\right)\right\} \\ &-\frac{\chi(1-z)}{2}\left\{\left(\frac{\left(c_1^-\right)^3}{6} - \frac{c_1^- p_1(T^4)}{24}\right)r_G - \frac{c_1^-\left(c_1^+\right)^2}{2}\left(r_G + \frac{4}{3}d_G h_G\right)\right\}. \end{aligned} \tag{7.18}$$

We've written the answer in terms of the twist parameter z, defined

$$z = \frac{p_1 - p_2}{p_1 + p_2}, \qquad p_1 + p_2 = 2g - 2 + n = -\chi(\Sigma_{g,n}). \tag{7.19}$$

This result for the bulk anomalies follows from the analysis in [89]. In the next section, we'll give a discussion of the contributions of punctures to the anomalies of the $\mathcal{N}=1$ class $\mathscr{S}$ theories.

7.3 Anomalies of (p,q) Punctures in Class $\mathscr{S}$

In this section, we study the anomalies of a large class of allowed punctures in the $\mathcal{N}=1$ class $\mathscr{S}$ SCFTs which carry a (p,q) We present new results for the anomalies of (p,q) punctures. This section can be read independently of sections 7.4 and 7.5.

7.3.1 Anomalies of (p,q) punctures

When the bulk preserves $\mathcal{N}=1$, there are punctures that can locally preserve $\mathcal{N}=2$ supersymmetry. In this case the local degrees of freedom preserve $\mathcal{N}=2$ supersymmetry, and therefore there is a local $\mathcal{N}=2$ R-symmetry action. This action is identified with the background $J_\pm$ symmetries in a nontrivial way, with different choices labeled by (p,q). For a given background with fixed $J_\pm$, there is an infinite family of inequivalent (p,q)-labeled punctures. The existence of these (p,q) punctures has been demonstrated in the gravity duals [94], with the (p,q) restricted to co-prime integers, but as of yet they have not been understood in general from a field theory perspective.

We identify the generators of the $SU(2)_R\times U(1)_R$ symmetry locally near a (p,q) puncture as

$$R_{\mathcal{N}=2}=\frac{p}{p-q}J_+-\frac{q}{p-q}J_-,\qquad 2I_3=\frac{q}{q-p}J_+-\frac{p}{q-p}J_-. \tag{7.20}$$

Once $R_{\mathcal{N}=2}$ is fixed as a general linear combination of $J_\pm$, we can fix I_3 by identifying the flavor symmetry $(R_{\mathcal{N}=2}-2I_3)$ with the combination (J_+-J_-). Then, we can refine the statement of Chapter 5 of what local data is required to specify a $\mathcal{N}=1$ theory of class $\mathscr{S}$. When the Riemann surface has punctures that locally preserve $\mathcal{N}=2$, one must specify:

- A choice of embedding $\rho:\mathfrak{su}(2)\to\mathfrak{g}=$ADE, determining the flavor symmetry at the puncture, and
- A choice of (p,q), determining the R-symmetry locally at the puncture as (7.20).

The (p,q) punctures are a generalization of the notion of "colored" punctures that appear in field theory, e.g. in [117, 111, 112, 138]. Punctures in theories in which the bulk spacetime preserves $\mathscr{N}=1$ supersymmetry have an additional $\mathbb{Z}_2$-valued label $\sigma=\pm 1$ called the "color", which corresponds to the fact that we can choose one of the two normal directions to the M5-branes at the location of the puncture. In the gravity dual, the puncture corresponds to D4 branes ending on D6 branes, and the choice of σ corresponds to the choice of a plane transverse to the D4's along which the D6's are extended. In the more general framework of (p,q) punctures, these choices correspond to

$$\begin{aligned} \sigma=+1 \qquad &\leftrightarrow \qquad (p,q)=(p,0) \\ \sigma=-1 \qquad &\leftrightarrow \qquad (p,q)=(0,q). \end{aligned} \tag{7.21}$$

For $\sigma=+1$ the geometry locally preserves a $U(1)_+\times SU(2)_-$ bundle, while for $\sigma=-1$ a $U(1)_-\times SU(2)_+$ bundle is preserved. The overall normalization in (7.20) was fixed by matching onto these two limiting cases.

For general (p,q) punctures, the anomaly coefficients can be computed with the local twist (7.20). We'll express the answer for the anomaly coefficients in terms of a local twist parameter $\hat{z}$, defined analogously to (7.19) as

$$\hat{z}=\frac{p-q}{p+q}. \tag{7.22}$$

When $q=0$, $\hat{z}=1$, and when $p=0$, $\hat{z}=-1$, such that $\hat{z}$ reduces to the $\sigma=\pm 1$ label in these limits. The result is that a puncture corresponding to a flavor symmetry G with a (p,q) twist yields the following contribution to the anomaly polynomial of the four-dimensional theory:

$$\begin{aligned} I_6(P_i,\hat{z}) = (1+\hat{z})&\left[a_+^{(1)}\left(c_1^+\right)^3 - a_+^{(2)}c_1^+ p_1(T^4) - a_+^{(3)}c_1^+\left(c_1^-\right)^2 + \frac{k_G}{3}c_1^+ c_2(F_G)\right] \\ +(1-\hat{z})&\left[a_-^{(1)}\left(c_1^-\right)^3 - a_-^{(2)}c_1^- p_1(T^4) - a_-^{(3)}c_1^-\left(c_1^+\right)^2 + \frac{k_G}{3}c_1^- c_2(F_G)\right]. \end{aligned} \tag{7.23}$$

The coefficients $a_+^{(i)}$ are given by

$$
\begin{aligned}
a_+^{(1)} &= -\frac{1}{24}\left(n_h(P_i)(1+\hat{z})^2 + 2n_v(P_i)(1-4\hat{z}+\hat{z}^2)\right) \\
a_+^{(2)} &= -\frac{1}{24}(n_v(P_i)-n_h(P_i)), \qquad a_+^{(3)} = \frac{1}{8}\left(n_h(P_i)(1-\hat{z})^2 + 2n_v(P_i)(1+\hat{z}^2)\right),
\end{aligned} \tag{7.24}
$$

and $a_-^{(i)}(\hat{z}) = a_+^{(i)}(-\hat{z})$. The effective number of vector multiplets $n_v(P_i)$ and hypermultiplets $n_h(P_i)$ in the A_{N-1} case are given in (C.16) and (C.17). The flavor central charge terms are as given in (7.23) for our current definition of (p,q) punctures, but one can imagine a case where the $c_2(F_G)$ also splits. The rules for this splitting are not clear, and will not be further discussed here.

$I_6(P_i,\hat{z})$ reduces to the answer already known for punctures with $\hat{z}=\pm 1$. For example, the contribution of a maximal puncture with $\hat{z}=-1$ in the A_{N-1} case reduces to

$$
I_6(P_{\max},1) = -\frac{(N^2-1)}{2}\left(\frac{\left(c_1^+\right)^3}{3} - \frac{c_1^+ p_1(T^4)}{12} - c_1^+\left(c_1^-\right)^2\right) + 2Nc_1^+ c_2(F_{SU(N)}), \tag{7.25}
$$

which matches $n_v(P_{\max}) = -d_G/2$ and $n_h(P_{\max}) = 0$.

7.3.2 Effective n_v, n_h for $\mathscr{N}=1$ theories

It was conjectured in [112] that even when the bulk doesn't preserve $\mathscr{N}=2$ supersymmetry, an $\mathscr{N}=1$ analogue of n_v and n_h can be defined. In this section, we check this proposal for $\mathscr{N}=1$ class $\mathscr{S}$ theories with general (p,q) punctures. To do so, we'll use the convenient basis of $(R_0,\mathscr{F})$ defined in (7.14) for the four-dimensional global symmetries. When the $\mathscr{N}=1$ theory is derived from an $\mathscr{N}=2$ theory, R_0 and $\mathscr{F}$ are related to the $\mathscr{N}=2$ R-symmetry generators as $R_0 = R_{\mathscr{N}=2}/2 + I_3$, $\mathscr{F} = -R_{\mathscr{N}=2}/2 + I_3$. It will be further useful to express results in terms of the twist parameters z and $\hat{z}$, as defined in (7.19) and (7.22).

The proposal of [112] is that the anomaly coefficients for the $\mathscr{N}=1$ class $\mathscr{S}$ theories can be written in the form

$$
\begin{aligned}
\mathrm{Tr}R_0 = n_v - n_h, &\qquad \mathrm{Tr}R_0^3 = n_v - \frac{n_h}{4}, \\
\mathrm{Tr}\mathscr{F} = -(\hat{n}_v - \hat{n}_h), &\qquad \mathrm{Tr}\mathscr{F}^3 = -\hat{n}_v + \frac{\hat{n}_h}{4}, \\
\mathrm{Tr}R_0\mathscr{F}^2 = -\frac{n_h}{4}, &\qquad \mathrm{Tr}R_0^2\mathscr{F} = \frac{\hat{n}_h}{4}.
\end{aligned} \tag{7.26}
$$

These parameters are defined such that when it is possible to identify the $\mathcal{N}=1$ subalgebra as part of an $\mathcal{N}=2$ algebra, n_v and n_h are precisely the effective number of free vector and hypermultiplets of the $\mathcal{N}=2$ theory. The $\hat{n}_v$ and $\hat{n}_h$ parameters are defined analogously for the anomalies involving an odd power of the flavor symmetry $\mathscr{F}$, and should be loosely interpreted as the $\mathcal{N}=1$ version of an effective number vector and hypermultiplets.

We find that (7.26) is true only for $\hat{z}=\pm 1$ punctures, and requires some modification for more general (p,q) punctures. Writing the anomalies in terms of $(n_v, \hat{n}_v, n_h, \hat{n}_h)$, we find that the $\mathcal{N}=1$ class $\mathscr{S}$ anomaly coefficients take the form

$$\begin{aligned}
\mathrm{Tr} R_0 &= n_v - n_h, & \mathrm{Tr} F &= -\left(\hat{n}_v - \hat{n}_h\right), \\
\mathrm{Tr} R_0^3 &= -\frac{n_h}{4} + n_v + \frac{3}{2}\sum_i \delta_-(P_i), & \mathrm{Tr} F^3 &= \frac{\hat{n}_h}{4} - \hat{n}_v + \frac{1}{2}\sum_i \hat{z}_i \delta_+(P_i), \\
\mathrm{Tr} R_0 F^2 &= -\frac{n_h}{4}, & \mathrm{Tr} F R_0^2 &= \frac{\hat{n}_h}{4}.
\end{aligned} \tag{7.27}$$

Relative to the $\mathcal{N}=2$ class $\mathscr{S}$ theories, the n_v and n_h parameters have additional local terms. They break into separate bulk and local contributions as

$$n_{v,h} = n_{v,h}(\Sigma_{g,n}) + \sum_{i=1}^{n} \left[n_{v,h}(P_i) + \delta_-(P_i) \right], \tag{7.28}$$

$$\hat{n}_{v,h} = -z n_{v,h}(\Sigma_{g,n}) - \sum_{i=1}^{n} \hat{z}_i \left[n_{v,h}(P_i) + \delta_+(P_i) \right]. \tag{7.29}$$

The bulk pieces $n_{v,h}(\Sigma_{g,n})$ are the same as in the $\mathcal{N}=2$ case, which we repeat here for clarity:

$$n_v(\Sigma_{g,n}) = -\frac{\chi}{2}\left(r_G + \frac{4}{3} d_G h_G \right), \qquad n_h(\Sigma_{g,n}) = -\frac{\chi}{2}\left(\frac{4}{3} d_G h_G \right). \tag{7.30}$$

The local pieces $n_{v,h}(P_i)$ are reviewed in Appendix C.2 for the A_{N-1} case, and depend on the data of the Young diagram associated to the flavor symmetry at the puncture. In effect, the parameters $\hat{n}_{v,h}$ special to the $\mathcal{N}=1$ theories are a twisted version of the $\mathcal{N}=2$ parameters.

The deviation from the conjecture (7.26) lies in the $\delta_\pm(P_i)$ terms, which written in terms of the (p,q) parameters are given as

$$\delta_\pm(P_i) \equiv \frac{2pq}{(p+q)^2} \left(n_h(P_i) \pm 2 n_v(P_i) \right). \tag{7.31}$$

When either p or q is zero, (7.27) matches onto (7.26), and we recover the known answer for the effective $(n_v, n_h, \hat{n}_v, \hat{n}_h)$. Otherwise, these represent new contributions to the anomalies. For nonzero p and q, they contribute extra effective vector multiplets and hypermultiplets to the theory that depend on local puncture data, due to their appearance in (7.28) and (7.29). Additionally, they contribute new terms to the cubic anomalies, such that (7.27) deviates from (7.26). This result can be stated as the fact that we require more than four parameters to label the anomalies of theories with (p,q) punctures.

Discussion

For the moment, let's get some intuition as to the meaning of the $(n_v, n_h, \hat{n}_v, \hat{n}_h)$ parameters for the cases where $\delta_\pm(P_i) = 0$. Consider two class $\mathscr{S}$ theories that each have an $SU(N)$ flavor symmetry. By gauging a diagonal subgroup of the $SU(N) \times SU(N)$ symmetries with either an $\mathcal{N} = 1$ or $\mathcal{N} = 2$ vector multiplet, we glue the two punctures associated with the flavor symmetries. Then, we can isolate the contribution of the $\mathcal{N} = 1$ or $\mathcal{N} = 2$ vector multiplet to the anomalies as [112]

$$\begin{aligned} \mathcal{N} = 1 \text{ vector}: \quad & n_v = N^2 - 1, \quad n_h = 0, \quad \hat{n}_v = 0, \qquad \hat{n}_h = 0 \\ \mathcal{N} = 2 \text{ vector}: \quad & n_v = N^2 - 1, \quad n_h = 0, \quad \hat{n}_v = N^2 - 1, \quad \hat{n}_h = 0. \end{aligned} \tag{7.32}$$

These precisely correspond to subtracting the contributions of two maximal punctures of different colors (for $\mathcal{N} = 1$ gluing) or of the same color (for $\mathcal{N} = 2$ gluing).

Another simple example is to consider the A_{N-1} (2,0) theory compactified on a sphere with two maximal punctures and one minimal puncture. This is the T_N theory with one puncture partially closed, and corresponds to the theory of N^2 free hypermultiplets $H^i = (Q^i, \tilde{Q}^i),\ i = 1, \ldots, N$ in the bifundamental representation of the $SU(N) \times SU(N)$ flavor symmetry. For instance, with $N = 2$ the theory is Lagrangian, and one can explicitly check that the matter content is four $\mathcal{N} = 2$ hypermultiplets, or eight $\mathcal{N} = 1$ chiral multiplets. The contribution of these hypermultiplets to the anomaly is

$$(N, \bar{N}) \text{ hypermultiplets}: \quad n_v = 0, \quad n_h = N^2, \quad \hat{n}_v = 0, \quad \hat{n}_h = N^2, \tag{7.33}$$

as expected. In both of these cases, the $\hat{n}_{v,h}$ parameters have a natural interpretation in terms of splitting the $\mathscr{N}=2$ multiplets into $\mathscr{N}=1$ components.

As a final example, consider the case when the Riemann surface has $n^{(+)}$ maximal punctures with $\hat{z}=1$, and $n^{(-)}$ maximal punctures with $\hat{z}=-1$. Denote the total number of punctures as $n_{tot}=n^{(+)}+n^{(-)}$, and let $n_{dif}=n^{(-)}-n^{(+)}$. Then, $(n_v,\hat{n}_v,n_h,\hat{n}_h)$ reduce to the known results (see e.g. [102])

$$\begin{aligned} n_v &= -\frac{\chi}{2}(r_G+\frac{4}{3}d_Gh_G)-\frac{d_G}{2}n_{tot}, \qquad & n_h &= -\frac{2\chi}{3}d_Gh_G, \\ \hat{n}_v &= \frac{z\chi}{2}(r_G+\frac{4}{3}d_Gh_G)-\frac{d_G}{2}n_{dif}, \qquad & \hat{n}_h &= \frac{2z\chi}{3}d_Gh_G. \end{aligned} \tag{7.34}$$

In all of these cases, there is an interpretation of the $(n_v,n_h,\hat{n}_v,\hat{n}_h)$ parameters in the generalized quiver description of the $\mathscr{N}=1$ theory. The generalized quiver description is also useful in understanding colored punctures with non-maximal flavor symmetry by Higgsing an operator in the adjoint of the flavor symmetry group with a nilpotent vev, as discussed in the context of $\mathscr{N}=2$ theories in [139], and in the context of $\mathscr{N}=1$ class $\mathscr{S}$ theories in [112]. For general (p,q) punctures with $\delta_\pm\neq 0$, however, we do not currently have a field theory interpretation in terms of a generalized quiver. The fact that the anomalies for the (p,q) punctures take the form (7.27) implies that there is no straightforward field-theoretic interpretation of gluing (p,q) punctures. It would be further interesting to understand the operation of closing maximal punctures via nilpotent Higgsing from the perspective of the anomaly polynomial for the $\mathscr{N}=1$ class $\mathscr{S}$ theories, as was discussed for the $\mathscr{N}=2$ theories in [116]. The additional $U(1)$ symmetry in the $\mathscr{N}=1$ case that mixes with the R-symmetry naively complicates the problem. We leave these interesting questions to upcoming work.

7.4 Inflow for Flat M5-branes: A Review

The anomalies of the (2,0) theories of type A_{N-1} and D_N can be obtained by inflow in 11d supergravity in the presence of M5-branes. The eight-form anomaly polynomial (7.5) encodes anomalous diffeomorphisms of the six-dimensional world-volume of the M5-branes and their normal bundle. Here, we will restrict our attention to the A_{N-1} case, for which the four-dimensional class $\mathscr{S}$ theories have a description as the low energy limit of N coincident M5-branes wrapped on a punctured Riemann surface. Our goal will be to describe the inflow procedure for this class of theories, and in particular derive new terms in the anomaly eight-form for the M5-branes when the Riemann surface has punctures. Before we get there, we will take some time to review the standard inflow mechanism for M5-branes. In Section 7.5 we will extend this analysis to the main problem of interest.

7.4.1 Anomaly inflow

A QFT that admits chiral fields coupled to gauge or gravity fields may have anomalies. In even spacetime dimensions d, consistent anomalies are encoded in a $(d+2)$-form I_{d+2} known as the anomaly polynomial. I_{d+2} is a polynomial in the dynamical or background gauge and gravity fields[4], and is related to the anomalous variation of the quantum effective action as

$$\delta S_{\text{eff}} = 2\pi \int_{M_d} I_d^{(1)}. \tag{7.35}$$

Here, $I_d^{(1)}$ is a d-form obtained from I_{d+2} via the descent procedure [141, 115, 142],

$$I_{d+2} = dI_{d+1}^{(0)}, \qquad \delta I_{d+1}^{(0)} = dI_d^{(1)}. \tag{7.36}$$

δ indicates the gauge variation, and the superscripts indicate the order of the quantity in the gauge variation parameter.

In string theory, gauge theories can be obtained by considering the decoupling limit of extended objects—such as branes—in a gravitational background. Gauge transformations and/or diffeomorphisms restricted on the branes induce global symmetries. If effective degrees of freedom of the world-volume theory on the branes are chiral (possible when the world-volume is even-dimensional), then the induced global symmetries can be anomalous. Since

[4] The anomaly polynomial can also involve differential forms on the space of couplings of the theory, as was recently pointed out in the context of class $\mathscr{S}$ theories in [140].

diffeomorphisms in the full gravitational theory must be preserved, the action of the gravitational theory in the presence of the brane sources must be anomalous in order to cancel the anomalies of the world-volume theory.

In inflow, the anomaly is canceled by a term in the bulk effective action whose variation is localized on the brane [143, 144]. Such a coupling implies a source in the equations of motion, modifying the Bianchi identity for the $(D-p-1)$-form field strength $dH_{D-p-1} = \delta_{D-p}$ (for D-dimensional spacetime). The anomalous variation of the effective action can be written in terms of the descent of a $(p+3)$-form anomaly polynomial I_{p+3} as in (7.35), where the integral will be over the $(p+1)$-dimensional world-volume. This procedure was first explained in [144], while a detailed study of the role played by consistent versus covariant anomalies appeared in [145]. An extension to Green-Schwarz anomaly cancellation appeared in [146]. Such anomalous terms in the presence of Dp-branes were understood in [147, 148, 149]. For a review of D-brane and I-brane (intersecting D-brane) inflow, including an extended discussion on regularizing the delta function sources in this context, see [150].

In the context of M-theory, the (5+1)-dimensional M5-brane carries a chiral tensor multiplet, which has a one-loop anomaly; this is canceled by inflow from the bulk. The origin of the anomaly in M-theory comes from topological terms in the supergravity action, which have an anomalous variation in the presence of the M5-branes. Because the M5-brane acts as a magnetic source for the C_3 potential of M-theory, inflow can be understood as a result of the modified Bianchi identity (schematically) $dG_4 = \delta_5$. For a nice review of anomaly cancellation in M-theory, see [151].

7.4.2 M5-brane inflow

Now, we review the inflow analysis for flat M5-branes in 11d supergravity. Anomaly inflow for a single flat M5-brane was first discussed in [152], and the computation was done in [123] and [121]. Inflow for N flat M5-branes was computed in [122]. We will use the details and notation reviewed in this section as a jumping off point in the computation in Section 7.5.

The eleven-dimensional supergravity action is given by

$$S = \frac{1}{2\kappa_{11}^2}\int \sqrt{-g}\left(R - \frac{1}{2}|G_4|^2\right) - \frac{1}{12\kappa_{11}^2}\int C_3 \wedge G_4 \wedge G_4 - \mu_{M_2}\int C_3 \wedge I_8^{\text{inf}}. \tag{7.37}$$

C_3 is the three-form gauge field, $G_4 = dC_3$ the four-form field strength, and μ_{M_2} the M2-brane tension. The integrals are over eleven-dimensional spacetime, M_{11}.

The couplings satisfy

$$\frac{1}{2\kappa_{11}^2} = \frac{2\pi}{(2\pi\ell_p)^9}, \qquad \mu_{M_2} = \frac{2\pi}{(2\pi\ell_p)^3}, \qquad \mu_{M_5} = \frac{2\pi}{(2\pi\ell_p)^6}, \tag{7.38}$$

and we fix $2\pi\ell_p = 1$ such that $\mu_{M_2} = \mu_{M_5} = 2\pi$. The eight-form I_8^{inf} is a polynomial function of the spacetime curvature R on M_{11},

$$I_8^{\text{inf}} = -\frac{1}{48}\left(p_2(R) - \frac{1}{4}\left(p_1(R)\right)^2\right), \tag{7.39}$$

with conventions for the Pontryagin classes given in Appendix C.1.

Diffeomorphisms in the bulk are anomalous in the presence of M5-branes. For an M5-brane with six-dimensional world-volume W_6, the tangent bundle to M_{11} splits as

$$TM_{11}|_{W_6} = TW_6 \oplus NW_6, \tag{7.40}$$

with TW_6 an $SO(1,5)$ bundle, and NW_6 an $SO(5)$ bundle. Diffeomorphisms of M_{11} that map $W_6 \to W_6$ induce $SO(1,5)$ diffeomorphisms of the world-volume (gravitational anomalies) and $SO(5)$ gauge transformations of the normal bundle (gauge anomalies).

The M5-branes magnetically source the four-form flux G_4, modifying the Bianchi identity for G_4 as[5]

$$dG_4 = N\delta_5, \qquad \delta_5 = \delta(y^1)\ldots\delta(y^5)dy^1 \wedge \cdots \wedge dy^5. \tag{7.41}$$

Here, the y^a coordinates parameterize the transverse space to the M5-branes, which sit at $y^a = 0$. Terms in the bulk action (7.37) are singular due to (7.41), leading to inflow towards the world-volume that should be canceled by anomalies carried by degrees of freedom on the M5-branes.

[5]The source appears with units given as $dG_4 = 2\kappa_{11}^2\mu_{M5}N\delta_5$; in units where $2\pi\ell_p = 1$, $2\kappa_{11}^2\mu_{M5} = 1$.

A proper treatment requires that we smooth out the delta functions at the positions of the M5-branes [121]. We will need to replace the delta functions with bump functions, and impose regularity and gauge invariance of the field strength. This will imply a particular form of the gauge transformation of C_3. For this discussion we restrict to the case of a single brane.

To implement the smoothing of the source, parameterize the transverse directions to the M5-brane by an S^4 whose volume form is

$$dV_5 = \left(\frac{1}{4!}\varepsilon_{abcde} d\hat{y}^a \wedge d\hat{y}^b \wedge d\hat{y}^c \wedge d\hat{y}^d \hat{y}^e\right) \wedge r^4 dr \equiv d\Omega_4 \wedge r^4\, dr. \tag{7.42}$$

The $\hat{y}^a$ are isotropic coordinates on the S^4 fibers of the sphere bundle over the M5-brane world-volume. We smear the charge over the radial direction with a smooth function $\rho(r)$ that satisfies $\rho(0) = -1$ and $\rho(r \to \infty) = 0$, such that the Bianchi identity (7.41) is

$$dG_4 = d\rho(r) \wedge e_4. \tag{7.43}$$

The four-form e_4 is a closed, global angular form that is gauge invariant under $SO(5)$ transformations of the normal bundle and restricts to $d\Omega_4$ when the $SO(5)$ connection is taken to be trivial. Denoting the $SO(5)$ gauge field as F^{ab}, e_4 takes the form

$$e_4 = \frac{1}{V_4}\left(D\Omega_4 - \frac{2}{4!}\varepsilon_{abcde} F^{ab} \wedge D\hat{y}^c \wedge D\hat{y}^d \hat{y}^e + \frac{1}{4!}\varepsilon_{abcde} F^{ab} \wedge F^{cd} \hat{y}^e\right). \tag{7.44}$$

Here $D\Omega_4$ refers to $d\Omega_4$ with ordinary derivatives replaced with covariant derivatives, and V_4 refers to the area of the S^4, $V_4 = 8\pi^2/3$. This is normalized such that integrating e_4 over the S^4 gives unity[6]. Note that $de_4 = 0$ ensures that $d^2 G_4 = 0$.

(7.43) requires that the relationship between G_4 and C_3 is modified to

$$G_4 = dC_3 - d\rho \wedge e_3^{(0)}. \tag{7.45}$$

Here, $e_3^{(0)}$ is a two-form related to e_4 by the standard descent procedure,

$$e_4 = de_3^{(0)}, \quad \delta e_3^{(0)} = e_2^{(1)}. \tag{7.46}$$

Requiring gauge invariance of G_4 then implies that C_3 has an anomalous variation,

$$\delta C_3 = -d\rho \wedge e_2^{(1)}. \tag{7.47}$$

[6]Our conventions in this section follow [121], except that their normalization is such that $\int_{S_4} e_4^{\text{them}} = 2$.

In order to compute the variation of the action in the presence of the M5-branes, G_4 and C_3 need to replaced with quantities that are smooth and non-singular in the neighborhood of the branes. It follows from (7.47) that the correct replacement is

$$C_3 \;\to\; C_3 - \rho e_3^{(0)}, \qquad G_4 \;\to\; d\left(C_3 - \rho e_3^{(0)}\right). \tag{7.48}$$

Now we have the pieces to compute the variation of the bulk action and the anomaly due to the brane source. There are two terms in the bulk action that can lead to an anomaly: the linear coupling $C_3 \wedge I_8^{\text{inf}}$ (Green-Schwarz), and the $C_3 \wedge G_4 \wedge G_4$ (Chern-Simons) terms. From the decomposition of the tangent bundle (7.40), it follows that I_8^{inf} can be written as

$$I_8^{\text{inf}} = -\frac{1}{48}\left[p_2(TW_6) + p_2(NW_6) - \frac{1}{4}\left(p_1(TW_6) - p_1(NW_6)\right)^2\right]. \tag{7.49}$$

The variation of I_8^{inf} is given by the descent formalism as

$$I_8^{\text{inf}} = dI_7^{\text{inf}(0)}, \qquad \delta I_7^{\text{inf}(0)} = dI_6^{\text{inf}(1)}. \tag{7.50}$$

We will need to regulate the integrals by removing a neighborhood of radius ε around the M5-brane. Denote by $D_\varepsilon(W_6)$ the total space of the disc bundle with base W_6 and with fibers the discs of radius ε. First one computes the variation outside the disc with the shifted non-singular forms (7.48). Then, take the size of the disc to zero. The total space of the S^4 sphere bundle over W_6 which forms the boundary of $M_{11}/D_\varepsilon(W_6)$ will be denoted as $S_\varepsilon(W_6)$.

Then, the variation of the linear term leads to

$$\frac{\delta S_L}{2\pi} = -\lim_{\varepsilon\to 0}\int_{M_{11}/D_\varepsilon(W_6)} d\rho \wedge e_4 \wedge I_6^{\text{inf}(1)} = \int_{W_6} I_6^{\text{inf}(1)}. \tag{7.51}$$

The other source of anomalies in the bulk action is the Chern-Simons term, improved to take into account (7.48) as

$$\frac{S'_{CS}}{2\pi} = -\frac{1}{6}\lim_{\varepsilon\to 0}\int_{M_{11}/D_\varepsilon(W_6)} \left(C_3 - \rho e_3^{(0)}\right) \wedge (G_4 - \rho e_4) \wedge (G_4 - \rho e_4). \tag{7.52}$$

Its variation leads to

$$\frac{\delta S'_{CS}}{2\pi} = \frac{1}{6}\int_{S_\varepsilon(W_6)} e_2^{(1)} \wedge e_4 \wedge e_4 = \frac{1}{24}\int_{W_6} [p_2(NW_6)]^{(1)} \equiv \int_{W_6} I_6^{CS(1)}, \tag{7.53}$$

where the second equality is due to a result of Bott-Catteneo [153], and $[p_2(NW_6)]^{(1)}$ refers to the six-form related to $p_2(NW_6)$ by descent. In both (7.51) and (7.53) we have integrated by parts, and we've dropped the boundary terms in writing the final answers. A difference between this case and the case when the internal space has punctures will be that the punctures lead to boundaries, and the boundary terms in the integration by parts will need to be evaluated. We'll return to this point in the next section.

Combining this contribution with the contribution from the $C_3 \wedge I_8^{\text{inf}}$ term, the anomaly eight-form for a single M5-brane is then

$$I_8[1] = I_8^{\text{inf}} + I_8^{CS} = \frac{1}{48}\left[p_2(NW_6) - p_2(TW_6) + \frac{1}{4}\left(p_1(TW_6) - p_1(NW_6)\right)^2\right]. \tag{7.54}$$

This is precisely the result we quoted in (7.4). For N M5-branes, the Green-Schwarz term is linear in C_3 and thus also linear in N, and the Chern-Simons term is cubic in C_3 and thus also cubic in N (we take $\rho(r=0) = -N$). Anomaly cancellation then requires that $I_8[N]$ for N M5-branes is given by

$$I_8[N] = I_8^{CS}[N] + I_8^{\text{inf}}[N] = (N^3 - N)\frac{p_2(NW_6)}{24} + NI_8[1]. \tag{7.55}$$

To obtain the anomaly polynomial for the six-dimensional A_{N-1} theories, we must also subtract off an overall $U(1)$ corresponding to the center of mass motion of the branes, which amounts to subtracting $I_8[1]$ from (7.55).

7.5 Class $\mathscr{S}$ Anomalies from Inflow

We finally turn to the main problem of interest: anomaly inflow for cases where the M5-branes wrap a holomorphic curve $\mathscr{C}_{g,n}$.

As we emphasized in the previous section, anomaly inflow in a gravitational theory can be understood as accounting for sources of connection forms in the variation of the action. In the presence of punctures, the total space will have internal boundaries. The logic we will employ is as follows. We can account for boundary conditions for the M5-branes by considering additional sources at the boundaries. These sources model the branching off of the M5-branes at the punctures. This is consistent with the M-theory description of punctures as transverse M5-branes that intersect the Riemann surface at a point. Accounting for these sources, the inflow procedure yields additional contributions to the anomalies of the world-volume theory.

7.5.1 Inflow for curved M5-branes

As we reviewed in Section 7.4.2, in the presence of an M5-brane the tangent bundle to the full eleven-dimensional spacetime splits into the tangent bundle and normal bundle to the world-volume, as (7.40). When the M5-branes wrap a holomorphic curve $\mathscr{C}_{g,n}$, the tangent bundle over the branes further splits as

$$TW_6 = TM^{1,3} \oplus T\mathscr{C}_{g,n}. \tag{7.56}$$

Since the Riemann surface is embedded in a CY_3 that is a sum of two line bundles, the $SO(5)$ normal bundle over the branes reduces to a sum of two $SO(2)$ bundles,

$$NW_6 = SO(2)_+ \oplus SO(2)_-. \tag{7.57}$$

The structure group of the normal bundle restricted to the flat four-dimensional spacetime is covered by $U(1)_+ \times U(1)_-$, which correspond to the global symmetries in the field theory in (5.4). We will reduce the curvature as $SO(2) \times SO(2) \subset SO(5)$ (or $SO(2) \times SO(3) \subset SO(5)$ for the $\mathscr{N} = 2$ preserving case), and then use relations between the Pontryagin classes of real bundles and the Chern roots of their complexified covers—see Appendix C.1 for relevant formulae.

The curvature for the normal bundle NW_6 has two contributions: one from the Riemann surface, and the other from the four-dimensional spacetime $M^{1,3}$. Then, the roots of the normal bundle, which we'll denote as $n_\pm$, can be written as

$$n_\pm = \hat{t}_\pm + 2c_1\left(U(1)_\pm\right), \tag{7.58}$$

where $c_1\left(U(1)_\pm\right)$ is the first Chern class of the $U(1)_\pm$ symmetries of class $\mathscr{S}$, and $\hat{t}_\pm$ is the contribution of the curvature of the $SO(2)_\pm$ bundles over the Riemann surface. The Calabi-Yau condition, or the topological twist, restricts the $\hat{t}_\pm$ as

$$\hat{t}_+ + \hat{t}_- + \hat{t} = 0, \qquad \int_{\Sigma_{g,n}} \hat{t} = \chi\left(\Sigma_{g,n}\right). \tag{7.59}$$

In these expressions, $\hat{t}$ is the curvature of the tangent bundle of the Riemann surface. It will be useful to introduce the connection one-forms $A_\pm$ and their curvatures, $F_\pm$, as

$$\hat{t}_\pm = \frac{1}{2\pi} dA_\pm \equiv \frac{1}{2\pi} F_\pm. \tag{7.60}$$

If one of the contributions $\hat{t}_\pm$ is trivial, the compactification preserves eight supercharges and the four-dimensional quantum field theory preserves $\mathscr{N}=2$ supersymmetry. Without loss of generality, we choose our $\mathscr{N}=2$ limit to be $\hat{t}_- = 0$, in which case the $U(1)_-$ symmetry enhances to an $SU(2)_-$ R-symmetry. In this limit, we have the following parametrization:

$$\hat{t}_+ = \frac{dA}{2\pi} \equiv \frac{F}{2\pi}, \qquad n_+ = \frac{F}{2\pi} + 2c_1^+, \qquad \hat{t}_- = 0, \qquad n_-^2 = -4c_2^-. \tag{7.61}$$

Here we have dropped the $(+)$ subscript on F since $\hat{t}_-$ is trivial, and we are utilizing a shorthand notation

$$c_1^+ = c_1(U(1)_+), \qquad c_2^- = c_2(SU(2)_-). \tag{7.62}$$

In this chapter, we aim to simply discuss how to account for punctures in the inflow computation above. For that it is sufficient to restrict to systems with eight supercharges. The inflow analysis for systems with four supercharges can be discussed in the same way, however the reduction is reasonably more involved. Further details and the analysis for systems with four supercharges will appear elsewhere [120].

Angular forms

Now, we explain how to construct the angular form e_4 that appears in the Bianchi identity (7.43) to reflect the restricted $U(1)\times SU(2)$ isometry manifest on the four-sphere transverse to the M5-branes.

When the M5-branes are curved, the normal bundle is reduced. The magnetic source for

G_4 must be suitably modified to reflect this. The source can be written alla (7.43) as

$$dG_4 = d\rho(r) \wedge d\Omega_4(\widetilde{S}^4) \qquad (7.63)$$

where $d\rho(r)$ is the smoothing of $\delta^5(r)r^4dr$. Here, the angular form $d\Omega_4(\widetilde{S}^4)$ is for a four-sphere $\widetilde{S}^4$ that is not maximally symmetric.

This volume form depends on the normal bundle. If the twist preserves eight supercharges, the normal bundle of the branes has a $U(1) \times SU(2)$ structure group and therefore only a $U(1) \times SU(2)$ isometry is manifest on the four-sphere. The connection of the $U(1)$ has a nontrivial component over the Riemann surface, while the connection of the $SU(2)$ over the surface is trivial in order to preserve the $SU(2)$ symmetry. A metric over the four-sphere can be chosen as

$$ds^2(\widetilde{S}^4) = \frac{d\mu^2}{1-\mu^2} + (1-\mu^2)d\phi^2 + \mu^2 ds^2(S^2_\Omega), \qquad (7.64)$$

with μ the interval $[0,1]$. The gauge invariant volume form is then

$$D\Omega_4 = \frac{1}{V_4}\mu^2 d\mu \wedge D\phi \wedge D\Omega_2, \qquad D\phi \equiv d\phi - A_\phi - A, \qquad (7.65)$$

where V_4 is the area of the four-sphere, $V_4 = 8\pi^2/3$, and $D\Omega_2$ is the gauge-invariant volume form of the round two-sphere S^2_Ω given as

$$D\Omega_2 = \frac{1}{2}\varepsilon_{abc} D\hat{y}^a \wedge D\hat{y}^b \hat{y}^c, \qquad D\hat{y}^a = d\hat{y}^a - A^{ab}\hat{y}^b, \qquad \sum_{a=1}^{3} (\hat{y}^a)^2 = 1. \qquad (7.66)$$

A^{ab} is the connection for an $SO(3)$ bundle over the branes, with corresponding field strength

$$F^{ab} = dA^{ab} - A^{ac} \wedge A^{cb}. \qquad (7.67)$$

The connection A_ϕ is the contribution over the flat four-dimensional space, and A is the contribution over the Riemann surface. The corresponding curvatures are

$$dA = F, \qquad dA_\phi = F_\phi. \qquad (7.68)$$

While the angular form (7.65) is gauge invariant, it is not closed. The most general closed and gauge invariant angular form can be written as

$$E_4 = \frac{1}{3V_4} d\left[\mu D\phi \wedge \left(\mu^2 D\Omega_2 - h(\mu) F_2^\Omega\right) + \left(a_\phi A_\phi + a_s A\right) \wedge e_2^\Omega\right] \qquad (7.69)$$

where we have introduced an arbitrary function $h(\mu)$ and arbitrary constants (a_ϕ, a_s). The $SO(3)$

forms are given as

$$F_2^\Omega = \frac{1}{2}\varepsilon_{abc}F^{ab}\hat{y}^c, \qquad e_2^\Omega = D\Omega_2 - F_2^\Omega, \qquad d(D\Omega_2) = F_3^\Omega = \frac{1}{2}\varepsilon_{abc}F^{ab} \wedge D\hat{y}^c. \tag{7.70}$$

One choice of $h(\mu)$ and the a's corresponds to taking the $SO(5)$ gauge invariant angular form in (7.44) and reducing it so that only an $SO(2)\times SO(3)$ is manifest. This choice corresponds to $h(\mu) = 1$ and $a_\phi = a_s = 0$.

Overview of the computation

Before going forward with the details, we pause to summarize the steps necessary to carry out the inflow analysis in the presence of punctures. The details of the computation will follow in the rest of this section.

1. At the locations of the punctures, the connection one-form A over the Riemann surface is not defined. Motivated by the work in gravity [83, 94], we allow for explicit sources for the connection localized at the punctures. The symmetries of the sources allow us to account for the local puncture data.

2. Sources for the curvatures $F = dA$ induce sources for the four-form flux G_4. We explain how to account for these additional sources such that G_4 is non-singular at the locations of the punctures.

3. We compute the anomalies by varying the action. An important difference from the usual case reviewed in (7.51) and (7.53) is the fact that with punctures there are additional boundaries, and boundary terms in the integration by parts contribute to the variation. In fact, the boundary terms will entirely account for the new contributions from the punctures.

4. Integrating the anomaly polynomial for the world-volume theory over the Riemann surface, we compare with the known answer (7.13). The symmetries of step (1) manifest as global symmetries in the four-dimensional field theory.

A quick note on notation

The computation that follows requires the definition of various differential forms, and so before we dive in we'll take a moment to point out some features of our notation to facilitate ease of reading.

In general, a numerical subscript denotes the degree of the form, which we will often write explicitly for forms of degree greater than one. Then, the exterior derivative of a k-form is a $(k+1)$-form; e.g. df_k refers to a form of degree $k+1$. The only exception to this notation is when we write the angular two-form on a two-sphere as $d\Omega_2 \equiv \frac{1}{2}\varepsilon_{abc}d\hat{y}^a \wedge d\hat{y}^b \hat{y}^c$. While this expression utilizes Einstein summation notation, for clarity we will explicitly write any sums over indices that label branes or punctures.

We will use the letter e—either capital or lowercase, and possibly with identifying superscripts—to denote a gauge-invariant angular form. For instance, the angular form $d\Omega_2$ on the two-sphere will be promoted to the gauge-invariant e_2^{Ω}.

When an object satisfies the descent equations, we will use a superscript in parenthesis to denote the order of the object in the gauge variation parameter. For example, if a k-form f_k satisfies descent, we will write

$$f_k = df_{k-1}^{(0)}, \qquad \delta f_{k-1}^{(0)} = df_{k-2}^{(1)}. \tag{7.71}$$

Another notation we will frequently use is to put square brackets around a k-form, with the (0) or (1) superscript outside the square brackets. Such an object refers respectively to the $k-1$ or $k-2$ form related to the k-form by descent—for example,

$$[f_k]^{(1)} \equiv f_{k-2}^{(1)}. \tag{7.72}$$

This notation is useful when the main object of interest is the k-form. For further review on the descent procedure and our conventions regarding characteristic classes, refer to Appendix C.1.

7.5.2 Sources for connection forms

When the Riemann surface has punctures, the curvature form is not well-defined on them. The connection one-forms $A_\pm$ are singular at the punctures, and suitable boundary conditions are needed. We propose that in order to account for punctures in the inflow computation we should add magnetic sources for the curvature forms $F_\pm$, as

$$dF_\pm = 2\pi \sum_{\alpha=1}^{n} \delta(\vec{x}-\vec{x}_\alpha) df_\alpha^\pm \wedge dx^1 \wedge dx^2. \tag{7.73}$$

The (x^1, x^2) are coordinates on the Riemann surface and $\vec{x}_\alpha$ is the location of a puncture with label α. The functions $f_\alpha^\pm$ depend on the transverse coordinates and encode the boundary data for the connection one-forms. The allowed choices of $f_\alpha^\pm$ are constrained by supersymmetry. The supersymmetric analysis that constrains the f's will be presented elsewhere [120].

For the time being, we restrict to cases with one puncture, $\alpha = 1$, and reductions that preserve eight supercharges. For each brane that wraps the Riemann surface, we can turn on a source term. We write

$$dF = 2\pi\delta(\vec{x}-\vec{x}_1) \sum_a df^a(\mu) \wedge dx^1 \wedge dx^2, \qquad df^a = \hat{k}^a \delta(\mu - \mu^a)\, d\mu, \tag{7.74}$$

where a labels the different branes and the constant k^a is either zero or one. More general f^a could be obtained by smearing the delta function source.

Each source corresponds to a monopole located at $(\vec{x} = \vec{x}_1, \mu = \mu^a)$. In M-theory, this source is a co-dimension three object whose world-volume we denote as W_8. The tangent bundle of M-theory near the source decomposes as

$$TM_{11}|_{W_8} = TW_8 \oplus NW_8, \tag{7.75}$$

where TW_8 is the curvature bundle on the source and NW_8 is an $SO(3)$ normal bundle. The diffeomorphisms of M-theory induce an $SO(3)$ gauge symmetry on the world-volume of the source. The background geometry where the source lives splits the μ direction from the (x_1, x_2) directions, and therefore only a $U(1)$ subgroup of this $SO(3)$ gauge symmetry group is preserved, which we will call $U(1)_a$. Near the source we can pick coordinates (R_a, τ_a, φ^a) where R_a

is the overall radial coordinate, τ_a is an interval $[-1,1]$, and φ^a parameterizes the S^1 whose diffeomorphisms induce the $U(1)_a$ gauge symmetry. The explicit coordinate transformation is given in Appendix C.3.

This description of the sources is consistent with the picture in the holographic duals [83, 94]. In these solutions, there are additional M5-branes that are localized at the punctures. These branes are extended along a direction normal to the Riemann surface and end at monopole sources of a $U(1)$ connection of an S^1 bundle over the surface, which here corresponds to the A connection. The location of the monopole sources along the μ interval are denoted here as μ^a[7]. The global symmetry that the four-dimensional field theory sees will be related to the precise values of the μ^a.

Analogously to (7.43), we must smear the charge at each monopole over the radial direction. The gauge invariant and closed source for F can be written as

$$dF = \sum_a d\rho_a(R_a) \wedge e_2^a, \tag{7.76}$$

where the gauge invariant and closed angular form near the source is given as

$$e_2^a = \frac{1}{2} d\left[\tau_a D\varphi^a + a_a A^a\right], \qquad D\varphi^a = d\varphi^a - A^a. \tag{7.77}$$

In particular, e_2^a is closed, with $e_2^a = de_1^{a(0)}$. The constant a_a here is arbitrary. The special case that e_2^a derives from the restriction of the full $SO(3)$-invariant angular form corresponds to $a_a = 0$. The one-form A^a is the connection of the $U(1)_a$ over the flat four-dimensional space with curvature $F^a = dA^a$[8]. To smooth out the source, we have excised a ball around the source of size ε and replaced the delta functions with bump forms $d\rho_a(R_a)$ that satisfy

$$\rho_a(R_a \to \infty) \ \to \ 0, \qquad \rho_a(\varepsilon) = k^a \in \mathbb{Z}. \tag{7.78}$$

In particular, note that the k^a are quantized (to be further discussed in a moment).

The Bianchi identity (7.76) for F is solved by

$$F = dA - \sum_a d\rho_a(R^a) \wedge e_1^{a(0)}. \tag{7.79}$$

[7]The μ interval is the y interval in the backreacted systems in [83].

[8]The contribution of A^a along the sphere is set to zero in order to preserve the $SU(2)$ symmetry.

The first term dA is the flux associated to the holonomy of the Riemann surface, which contributes to the Euler characteristic. In particular, the background curvature of the tangent bundle of the surface $\hat{t}$, discussed around (7.59), still satisfies

$$dA = -2\pi\hat{t}, \qquad \int_{\Sigma_{g,1}} dA = -2\pi\chi(\Sigma_{g,1}). \tag{7.80}$$

The second set of terms in (7.79) depend explicitly on data at the monopoles.

The gauge transformation induced by the $U(1)_a$'s leads to

$$\delta_a A^a = d\lambda^a, \qquad \delta\varphi^a = \lambda^a, \qquad \delta A = -\sum_a e_0^{a(1)} d\rho_a. \tag{7.81}$$

From the gauge transformation, we see how to shift A and F such that they are non-singular at the sources, as

$$A \rightarrow \widetilde{A} = A - \sum_a \rho_a e_1^{a(0)}, \qquad F \rightarrow \widetilde{F} = d\widetilde{A}. \tag{7.82}$$

$\widetilde{A}$ and $\widetilde{F}$ are the well-defined forms that will need to be used instead of A and F in computing the variation of the action.

It is further convenient to split $\widetilde{F}$ into two pieces, as

$$\widetilde{F} = F_0 - \frac{1}{2}\sum_a \rho_a(a_a - \tau_a)F^a, \quad F_0 = dA - \sum_a \left(d\rho_a \wedge e_1^{a(0)} + \frac{1}{2}\rho_a d\tau_a \wedge D\varphi^a\right). \tag{7.83}$$

F_0 is the gauge-invariant volume form of the Riemann surface. The F_a are the curvature forms for the $U(1)_a$ symmetries, whose coefficients in $\widetilde{F}$ depend on the interval τ_a and the smearing function $\rho_a(R_a)$ centered at each monopole.

The boundary conditions for the background curvature and connection near the puncture are

$$R_a \rightarrow 0: \qquad dA \rightarrow 0, \quad \text{and} \quad A \rightarrow c_a d\varphi^a, \tag{7.84}$$

for some constant c_a. As R^a goes to zero, the connection is flat. This can be seen by looking at the background metric near the puncture [94]. This choice of boundary conditions allows us to write several integral identities that will be useful to the computation of anomalies later in this section, which we give in Appendix C.3.

The curvature form $\widetilde{F}$ satisfies a quantization condition. In particular, the flux of $\widetilde{F}$ through the sphere S_a^2 surrounding a monopole at μ^a is quantized, as

$$\frac{1}{2\pi}\int_{S_a^2}\widetilde{F}=k^a\in\mathbb{Z}. \tag{7.85}$$

Punctures that preserve different amounts of flavor symmetry in the four-dimensional field theory will correspond to different M5-brane profiles [83, 110], and different choices of flux k^a.

Multiple punctures

When there are n punctures on the Riemann surface, there is a source for each brane at the location of each puncture that must be smoothed. Then, there will be a $U(1)$ gauge symmetry induced on each source world-volume, such that e_2^a also receives an index α labeling the puncture. These sources can be written as

$$dF=\sum_{\alpha=1}^{n}\sum_{a=1}^{N}d\rho_{a,\alpha}(R_{a,\alpha})\wedge e_2^{a,\alpha}. \tag{7.86}$$

The non-singular form $\widetilde{F}$ will then also receive a separate contribution from each puncture,

$$\widetilde{F}=F_0-\frac{1}{2}\sum_{\alpha}\sum_{a}\rho_{a,\alpha}(a_{a,\alpha}-\tau_{a,\alpha})F^{a,\alpha}, \tag{7.87}$$

where the gauge-invariant volume form of the Riemann surface is given by

$$F_0=dA-\sum_{\alpha}\sum_{a}\left(d\rho_{a,\alpha}\wedge e_1^{a,\alpha(0)}+\frac{1}{2}\rho_{a,\alpha}d\tau_{a,\alpha}\wedge D\varphi^{a,\alpha}\right). \tag{7.88}$$

The background curvature dA integrates to the Euler characteristic on the n-punctured Riemann surface, as

$$\int_{\Sigma_{g,n}}dA=-2\pi\chi(\Sigma_{g,n})=2\pi(2g-2+n). \tag{7.89}$$

Because each source is localized by the (smoothed) delta functions, it suffices to understand the one-puncture case in order to generalize to the n-puncture case. For ease of reading, we will continue the calculation for a single puncture.

Consistent sources for G_4

The sources for F induce sources for G_4. Since the gauge invariant angular form in (7.65) has an explicit dependence on F, G_4 cannot be closed in the presence of sources for F. It needs to be further improved.

To understand the sources induced for G_4, we temporarily turn off the connections on the four-dimensional space. Then, the closed magnetic sources for G_4 in the presence of N M5-branes are

$$dG_4 = \frac{1}{V_4} d\rho(r) \wedge \left(\mu^2 d\mu \wedge (d\phi - A) + \frac{1}{3} \left(a_s - \mu^3 \right) F \right) \wedge d\Omega_2 + \frac{1}{V_4} \sum_a K_3^a \wedge d\Omega_2, \qquad (7.90)$$

$$dK_3^a = \frac{1}{3} \left(a_s - \mu^3 \right) d\rho(r) \wedge d\rho_a(R^a) \wedge d\tau_a \wedge d\varphi^a. \qquad (7.91)$$

The K_3^a terms are needed to close the source term for G_4 in presence of the monopoles. We observe that consistency of the sources requires the M5-branes wrapped on the Riemann surface to branch off at the punctures. This is consistent with the probe analysis for punctures in holography [83, 110].

In the presence of the monopole sources for F, the most general closed, gauge invariant, and global source for G_4 is given as

$$dG_4 = d\rho(r) \wedge \widetilde{E}_4. \qquad (7.92)$$

Our convention is $\rho(0) = -N$ and $\rho(r \to \infty) = 0$. The angular four-form $\widetilde{E}_4$ is obtained by taking E_4 in (7.69) and replacing the connection A and the curvature F with the global and non-singular forms $\widetilde{A}$ and $\widetilde{F}$. This substitution will naturally include K_3^a.

Now, G_4 can be written as

$$G_4 = dC_3 - d\rho \wedge \widetilde{E}_3^0 \qquad (7.93)$$

where we have

$$\widetilde{E}_4 = d\widetilde{E}_3^{(0)}, \qquad \delta \widetilde{E}_3^{(0)} = d\widetilde{E}_2^{(1)}. \qquad (7.94)$$

Similar to E_4, in these forms we substitute (A, F) with $(\widetilde{A}, \widetilde{F})$. Since G_4 is gauge invariant and

$\widetilde{E}_3^{(0)}$ transforms non-trivially, we must have

$$\delta C_3 = -d\rho \wedge \widetilde{E}_2^{(1)}. \tag{7.95}$$

This suggests that we shift the potential and the flux in the action as

$$C_3 \to \widetilde{C}_3 = C_3 - \rho \widetilde{E}_3^{(0)}, \qquad G_4 \to \widetilde{G}_4 = d\widetilde{C}_3. \tag{7.96}$$

In addition to the condition that G_4 integrates over the S^4-bundle to N, we also have the quantization condition that

$$\int_{S^2_\Omega \times S^2_a} G_4 \equiv f_a(\mu^a) \in \mathbb{Z}. \tag{7.97}$$

Here $f_a(\mu^a)$ is some function of the monopole locations μ^a. This is then a quantization condition for the μ^a.

7.5.3 Variation of the action

The variation of the action has two terms, given as

$$\frac{\delta S}{2\pi} = -\frac{1}{6}\delta \int_{\widetilde{M}_{11}} \widetilde{C}_3 \wedge \widetilde{G}_4 \wedge \widetilde{G}_4 - \delta \int_{\widetilde{M}_{11}} \widetilde{C}_3 \wedge \widetilde{I}_8^{\text{inf}}. \tag{7.98}$$

We will find that we need to improve $I_8^{\text{inf}}(F_\phi, F)$ to $I_8^{\text{inf}}(F_\phi, \widetilde{F}) \equiv \widetilde{I}_8^{\text{inf}}$. We've excised small regions around the M5-branes ($r < \varepsilon$) and around the monopoles ($R^a < \varepsilon^a$) from M_{11} to obtain $\widetilde{M}_{11}$. The variation of the action is computed by integrating over $\widetilde{M}_{11}$, and then taking the ε's to zero. In the region near $r = \varepsilon$, we split the eleven-dimensional manifold as $\widetilde{M}_{11} = [r] \times M^{1,3} \times \widetilde{M}_6$, where the six-dimensional part is the total space of the S^4 bundle over the Riemann surface with balls surrounding the monopoles removed, as per the discussion around (7.85). The boundary of each ball is a sphere of radius ε^a, which we have denoted as S^2_a. In particular, $\widetilde{M}_6$ and its boundary split as

$$\widetilde{M}_6 = S^1_\phi \times S^2_\Omega \times [\mu] \times \Sigma_{g,1}, \qquad \partial \widetilde{M}_6 = \sum_a S^1_\phi \times S^2_\Omega \times S^2_a. \tag{7.99}$$

The manifold has a boundary component labeled by a for each brane.

In this section, we will compute the variation from the Chern-Simons and linear terms in turn, and evaluate their contributions to the anomalies. As we discussed in the introduction, the final answer can be repackaged as a nine-form (7.3), which integrated over $[\mu] \times \Sigma_{g,1}$ gives the anomalies of the four-dimensional theory. After computing the variation, we will comment on how this simple form comes out of the analysis.

General features

Before getting into the details, we summarize the main features of the computation of the variation. We find that the variation of the action splits into a bulk contribution to the integral and a boundary contribution that depends on local puncture data, as

$$\frac{\delta S}{2\pi} = -\int_{M^{1,3}\times\Sigma_{g,1}} \left[I_8^{\text{bulk}}\right]^{(1)} - \int_{M^{1,3}\times S_a^2} \left[I_8^{\text{bdy}}\right]^{(1)}. \tag{7.100}$$

Each of these pieces receives contributions from both the linear and Chern-Simons terms in (7.98). We can understand these two contributions to the anomaly independently.

The bulk contribution comes from an integral over the bulk spacetime in the variation of the action, of the form

$$\frac{\delta S}{2\pi} \supset \int_{M^{1,3}\times\widetilde{M}_6} A^{\text{bulk}} = -\int_{M^{1,3}\times\Sigma_{g,1}} \left[I_8^{\text{bulk}}\right]^{(1)}. \tag{7.101}$$

A^{bulk} is a ten-form computed from (7.98), whose explicit expression will be given later. From (7.101) we reconstruct the eight-form polynomial I_8^{bulk}, which corresponds to the anomaly for the M5-branes branes wrapped on the surface.

The statement of class $\mathscr{S}$ is that the bulk anomaly polynomial is derived by integrating the anomaly polynomial for the six-dimensional world-volume theory over the Riemann surface. Indeed, we find that integrating I_8^{bulk} over the Riemann surface matches onto the bulk contribution $I_6(\Sigma_{g,1})$ in (7.6). In particular, the integrand A^{bulk} is proportional to the gauge invariant volume form on the full space $M^{1,3}\times\widetilde{M}_6$, which is proportional to $d\mu\wedge dA$. By the relations in (C.22) and (C.23), the integral vanishes for any term with a ρ_a, such that without loss of generality we can fix $\widetilde{F} = dA$. Thus, the terms in the anomaly polynomial for the four-dimensional theory coming from I_8^{bulk} will have no dependence on the local puncture data, and will be proportional to the Euler characteristic of the surface.

The puncture contribution to the anomalies comes from a boundary term in the variation of the bulk action, which is non-vanishing because $\widetilde{M}_6$ has internal boundaries of the form $S^1_\phi \times S^2_\Omega \times S^2_a$. In particular, the variation has a contribution of the form

$$\frac{\delta S}{2\pi} \supset \int_{M^{1,3}\times\widetilde{M}_6} d\left(B^{\text{bdy}}\right) = \int_{M^{1,3}\times\partial\widetilde{M}_6} B^{\text{bdy}} = -\int_{M^{1,3}\times S^2_a}\left[I_8^{\text{bdy}}\right]^{(1)}. \tag{7.102}$$

The integrand B^{bdy} is a nine-form that will be explicitly given later. The eight-form I_8^{bdy} packages the anomalies from the new degrees of freedom that arise due to the punctures. These degrees of freedom live at the intersection of the eight-dimensional monopole source and the world-volume of the M5-branes. Then from (7.102), we can directly compute the contribution to the anomaly six-form of the four-dimensional theory as[9]

$$[I_6(P)]^{(1)} = -\sum_a \int_{S^1_\phi\times S^2_\Omega\times S^2_a} B^{\text{bdy}}. \tag{7.104}$$

Variation of Chern-Simons term

The variation of the Chern-Simons term reduces to

$$\frac{\delta S_{CS}}{2\pi} = \frac{1}{6}\int_{\widetilde{M}_{11}} d\left(\rho\widetilde{E}_2^{(1)}\right)\wedge d\left(\rho\widetilde{E}_3^{(0)}\right)\wedge d\left(\rho\widetilde{E}_3^{(0)}\right) \tag{7.105}$$

$$= \frac{1}{6}\int_{[r]} d\rho^3 \int_{M^{1,3}\times\widetilde{M}_6}\left[\widetilde{E}_2^{(1)}\wedge\widetilde{E}_4\wedge\widetilde{E}_4 - \frac{2}{3}d\left(\widetilde{E}_2^{(1)}\wedge\widetilde{E}_3^{(0)}\wedge\widetilde{E}_4\right)\right]. \tag{7.106}$$

In evaluating the variation, we dropped terms involving C_3. The integrand factorizes in its r

[9] There is a subtlety with regards to the order in which we perform the descent in these expressions. The anomalies for the four-dimensional theory are given by first reconstructing I_8, and then integrating over the Riemann surface. Thus, e.g. for the bulk term we should reconstruct I_8^{bulk} from $[I_8^{\text{bulk}}]^{(1)}$ in terms of natural six-dimensional quantities: the roots $n_\pm$. Then to derive the anomalies of the four-dimensional theory, we decompose $n_\pm$ over the $M^{1,3}\times\Sigma_{g,1}$ base, and integrate. However, if we wish to compute the four-dimensional bulk anomalies directly from

$$[I_6(\Sigma_{g,1})]^{(1)} = \int_{\widetilde{M}_6} A^{\text{bulk}}, \tag{7.103}$$

we must be careful with the order in which we decompose over the base versus apply the descent formalism. Varying with respect to $\widetilde{F}$ and F_ϕ separately will give a different answer than varying with respect to n_+. This ordering shows itself in extra constraints on the descent parameters for reducible terms in the anomaly.

dependence and therefore we can pull out the overall ρ dependence. Since ρ vanishes as $r \to \infty$, the only contribution comes from $r = \varepsilon$ where we have

$$\int_{[r]} d\rho^3 = -N^3. \tag{7.107}$$

We also needed to integrate by parts, which lead to a boundary term that—unlike the flat branes case—can be nonvanishing.

We group the variation as in (7.100). First we will discuss the bulk term, given by

$$\left[I_{CS,8}^{\text{bulk}}\right]^{(1)} = \frac{N^3}{6} \int_{S^1_\phi \times S^2_\Omega \times [\mu]} \widetilde{E}_2^{(1)} \wedge \widetilde{E}_4 \wedge \widetilde{E}_4. \tag{7.108}$$

In general, the anomaly will depend on the choice of the function $h(\mu)$ in (7.69). In this chapter we will not analyze the general case, and will instead fix them to match onto the reduction of the $SO(5)$ bundle to $SO(2) \times SO(3)$, with $h(\mu) = 1$ and $a_\phi = a_s = 0$.

Expanding the integrand of (7.108), the only terms that survive are proportional to the volume form on $M^{1,3} \times \widetilde{M}_6$, and therefore to $d\mu \wedge dA$. The coefficient is a polynomial in the ρ_a's. By the relations in (C.22) and (C.23), the integral vanishes for any term with a ρ_a, and therefore in evaluating the bulk terms we can fix $\widetilde{F} = dA$ without loss of generality. In other words, the bulk term does not see the monopoles and we can simply evaluate the integral with $\left(\widetilde{E}_2^{(1)}, \widetilde{E}_4\right) \to \left(E_2^{(1)}, E_4\right)$. In this case, we reconstruct $I_{CS,8}^{\text{bulk}}$ as

$$I_{CS,8}^{\text{bulk}} = \frac{N^3}{24} n_+^2 n_-^2 = \frac{N^3}{24} p_2(NW_6). \tag{7.109}$$

The bulk anomaly contribution to the anomaly polynomial of the four-dimensional theory can then be computed from (7.109) as

$$I_{CS,6}^{\text{bulk}} = \int_{\Sigma_{g,1}} I_{CS,8}^{\text{bulk}} = -\frac{2N^3}{3} \int \frac{dA}{2\pi} \wedge c_1^+ c_2^- = \frac{2N^3}{3} \chi(\Sigma_{g,1})\, c_1^+ c_2^-. \tag{7.110}$$

Recall that the relation of the roots of the normal bundle $n_\pm$ to the Chern roots of the $U(1)_+ \times SU(2)_-$ is given in (7.61).

The story will be similar for general $h(\mu)$ and a_ϕ, a_s, When we compute (7.108), we will find that the only terms that survive the integral over $[\mu] \times \Sigma_{g,1}$ will be proportional to dA, such that we can still replace $\widetilde{F} = dA$. Then, n_+ is independent of μ and can be pulled out of the integral. The effect will be to simply multiply the answer for $I_{CS,6}^{\text{bulk}}$ by a function of a_ϕ and a_s. These constants can be fixed by regularity and matching conditions. We do not consider this more general case here.

The boundary term is given by

$$\left[I_{CS,8}^{\text{bdy}}\right]^{(1)} = -\frac{N^3}{9}\sum_a \int_{S_1^\phi \times S_2^\Omega} \widetilde{E}_2^{(1)} \times \widetilde{E}_3^{(0)} \times \widetilde{E}_4. \tag{7.111}$$

We need explicit expressions for the $\widetilde{E}$'s in order to compute the boundary contributions, which requires a more detailed discussion of the angular forms than we will give in this chapter. Instead, here we simply highlight the form of the answer. For general choice of $h(\mu)$ and a_ϕ, a_s, (7.111) will allow us to reconstruct the eight-form

$$I_{CS,8}^{\text{bdy}} = N^3 \sum_a \hat{\ell}(\mu^a) n_+^2 n_-^2 \big|_{\rho^a = k^a} \tag{7.112}$$

for some function $\hat{\ell}(\mu^a)$ (which will also depend on the values of a_ϕ, a_s). Recall that the μ^a—which are quantized—are constants corresponding to the locations of the monopole sources. The roots $n_\pm$ are restricted to the spheres surrounding the sources.

In (7.112), the only terms in n_+ that can contribute are those proportional F_0—the gauge invariant volume form on the Riemann surface defined in (7.83). Then, without loss of generality we can replace $\widetilde{F} \to F_0$ in n_+. It follows from (C.25) that μ will be fixed to μ^a in each term of the sum over the branes (which we've already implemented in (7.112)). Integrating over the boundaries S_a^2, the contribution to the four-dimensional anomalies will take the form

$$I_{CS,6}^{\text{bdy}} = \int_{S_a^2} I_{CS,8}^{\text{bdy}} = N^3 \sum_a c_1^+ c_2^- \, \ell(\mu^a), \tag{7.113}$$

where the coefficient $\ell(\mu^a)$ depends on the μ^a. We will not comment further on the exact form of this coefficient.

Variation of linear term

Next we evaluate the variation of the linear term. For this, we need to first reduce I_8^{inf} in (7.39) and then restrict to the case with eight supercharges. Under the decomposition of the curvature bundle, we have

$$p_1(R) = p_1(T^4) + \hat{t}^2 + n_+^2 + n_-^2 \tag{7.114}$$

$$p_2(R) = p_2(T^4) + p_1(T^4)\left(\hat{t}^2 + n_+^2 + n_-^2\right) + \hat{t}^2(n_+^2 + n_-^2) + n_+^2 n_-^2. \tag{7.115}$$

The relation of the roots of the normal bundle, $n_\pm$, to the Chern roots of the $U(1)_+ \times SU(2)_-$ is given in (7.61). The curvature of the Riemann surface is given as $\hat{t}$, which satisfies (7.80).

In evaluating the linear term, we will need to replace F with its global extension $\widetilde{F}$ given in (7.82). We reduce the eight-form while only keeping terms that can be non-trivial in the action, leading to

$$\widetilde{I}_8^{\text{inf}} \supset \frac{1}{192}\left(\frac{\widetilde{F}}{2\pi} + 2c_1^+\right)^4 - \frac{1}{96}\left(p_1(T^4) - 4c_2^-\right) \wedge \left(\frac{\widetilde{F}}{2\pi} + 2c_1^+\right)^2. \tag{7.116}$$

The variation of the linear term goes as

$$-\delta \int \widetilde{C}_3 \wedge \widetilde{I}_8^{\text{inf}} = -\delta \int \widetilde{G}_4 \wedge \widetilde{I}_7^{\text{inf}(0)} = -\int \widetilde{G}_4 \wedge d\widetilde{I}_6^{\text{inf}(1)} = \int d\left(\rho \widetilde{E}_3^{(0)}\right) \wedge d\widetilde{I}_6^{\text{inf}(1)} \tag{7.117}$$

$$= \int_{[r]} d\rho \int_{M^{1,3} \times \widetilde{M}_6} \left[\widetilde{E}_4 \wedge \widetilde{I}_6^{\text{inf}(1)} - d\left(\widetilde{E}_3^{(0)} \wedge \widetilde{I}_6^{\text{inf}(1)}\right)\right]. \tag{7.118}$$

Analogously to the contribution from the Chern-Simons term, the variation of the linear term has a bulk contribution and a boundary contribution due to the monopoles. The bulk term is determined by

$$\left[I_{L,8}^{\text{bulk}}\right]^{(1)} = N \int_{S_\phi^1 \times S_\Omega^2 \times [\mu]} \widetilde{E}_4 \wedge \widetilde{I}_6^{\text{inf}(1)}. \tag{7.119}$$

In evaluating this bulk contribution we can drop all terms proportional to the ρ_a's, since the integral with the volume form on $\widetilde{M}_3$ vanishes with them. Again, we can simply replace $\widetilde{E}_4$ with E_4 and $\widetilde{F}$ with dA. Since the polynomial I_8^{inf} has no dependence on the angular coordinates, it follows that the only contribution from E_4 is actually the volume form of the transverse four-sphere. This expression is independent of any choice for $h(\mu)$ in (7.69). We can then reconstruct the bulk contribution simply as

$$I_{L,8}^{\text{bulk}} = N I_8^{\text{inf}}. \tag{7.120}$$

Integrating over the Riemann surface, we compute the contribution to the six-form anomaly polynomial of the class $\mathscr{S}$ theory as

$$I_{L,6}^{\text{bulk}} = \int_{\Sigma_{g,1}} I_{L,8}^{\text{bulk}} = -\frac{\chi(\Sigma_{g,1})}{2} N \left(\frac{(c_1^+)^3}{3} - \frac{c_1^+ p_1(T^4)}{12} + \frac{c_1^+ c_2^-}{3} \right). \tag{7.121}$$

The boundary contribution is determined by

$$\left[I_{L,8}^{\text{bdy}} \right]^{(1)} = N \sum_a \int_{S^1_\phi \times S^2_\Omega} \widetilde{E}_3^{(0)} \wedge \widetilde{I}_6^{\text{inf}(1)}. \tag{7.122}$$

In evaluating this boundary term we note that $\widetilde{I}_8^{\text{inf}}$ has no legs along the $S^1_\phi \times S^2_\Omega$ directions, and so the integral is non-vanishing only for terms in $\widetilde{E}_3^{(0)}$ that have legs along the circle and sphere directions. This expression is uniquely fixed by the volume of form of the transverse four-sphere in $d\widetilde{E}_3^{(0)} = \widetilde{E}_4$, as $\widetilde{E}_3^{(0)} \to \frac{\mu^3}{3V_4} d\phi \wedge d\Omega_2$. Thus, we have that

$$\left[I_{L,8}^{\text{bdy}} \right]^{(1)} = N \sum_a (\mu^a)^3 \widetilde{I}_6^{\text{inf}(1)}. \tag{7.123}$$

To compute the contribution to the four-dimensional anomalies, we need to expand $\widetilde{I}_6^{\text{inf}(1)}$, and keep terms proportional to F_0 since these are the only ones that will not integrate to zero. In this computation we will take $a_a = 0$ in (7.77)—i.e. we consider the case that e_2^a derives from the $SO(3)$-invariant angular form. We compute

$$\begin{aligned} \widetilde{I}_6^{\text{inf}(1)} \supset & \frac{1}{2} \frac{F_0}{2\pi} \Bigg(\left[\frac{(c_1^+)^3}{3} - \frac{c_1^+ p_1(T^4)}{12} + \frac{c_1^+ c_2^-}{3} \right]^{(1)} \\ & + \sum_a \rho_a \tau_a c_1^a \left[\frac{(c_1^+)^2}{2} + \frac{c_2^-}{6} \right]^{(1)} + \sum_{a,b} (\rho_a \tau_a c_1^a)(\rho_b \tau_b c_1^b) \left[\frac{c_1^+}{4} \right]^{(1)} \Bigg). \end{aligned} \tag{7.124}$$

We have written this expression in terms of the first Chern classes of the $U(1)_a$ symmetries,

$$F^a = 2(2\pi) c_1(U(1)_a) \equiv 2(2\pi) c_1^a. \tag{7.125}$$

Next, we integrate over the boundary spheres. From the integral identities given in (C.25), we have that

$$\int_{S_2^a} h(\mu) \frac{F_0}{2\pi} = -k^a h(\mu^a), \qquad \int_{S_2^a} h(\mu) \frac{F_0}{2\pi} (\rho_a \tau_a)(\rho_b \tau_b) = -\frac{1}{3} \delta_{ab} h(\mu^a)(k^a)^3. \tag{7.126}$$

Any terms that are odd in τ_a integrate to zero. The final answer for the boundary contribution of the linear term to the anomalies of the four-dimensional theory is

$$I_{L,6}^{\text{bdy}} = -\frac{N}{2} \sum_a (\mu^a)^3 \left\{ k^a \left(\frac{(c_1^+)^3}{3} - \frac{c_1^+ p_1(T^4)}{12} + \frac{c_1^+ c_2^-}{3} \right) + (k^a)^3 (c_1^a)^2 \frac{c_1^+}{12} \right\}. \tag{7.127}$$

This answer is expressed in terms of the quantized k^a fluxes of the sources, and the locations μ^a of the monopoles along the transverse direction to the surface.

7.5.4 Total 4d anomaly from inflow

We now have all the pieces to give the total anomaly polynomial for the class $\mathscr{S}$ theory on the branes. First, we note a simple way to reformulate the results.

We collect the terms that contribute to the anomalies of the world-volume theory on the branes as

$$I_8^{\text{bulk}} = N\widetilde{I}_8^{\text{inf}} + \frac{N^3}{24} p_2(NW_6), \qquad I_8^{\text{bdy}} = \sum_a \Big(N(\mu^a)^3 \widetilde{I}_8^{\text{inf}} + N^3 \hat{\ell}(\mu^a) p_2(NW_6) \Big), \qquad (7.128)$$

where recall that $\widetilde{I}_8^{\text{inf}}$ is $-1/48(p_2(R) - 1/4(p_1(R))^2)$, with $p_{1,2}(R)$ given as (7.114)-(7.115), and n_+ is a function of $\widetilde{F}$. The function $\hat{\ell}(\mu^a)$ (from (7.112)) is determined by the local data at the punctures, which we will not determine here. The terms linear in N came from the linear term in the action, and the terms cubic in N from the Chern-Simons term.

We showed that integrating I_8^{bulk} over the Riemann surface gives the contributions to the anomalies of the four-dimensional theory whose coefficients are proportional to the Euler characteristic of the surface, χ. These terms do not see the monopole sources. Integrating I_8^{bdy} over the boundaries S_a^2 gives the contributions to the anomalies of the four-dimensional theory whose coefficients depend on the local data of the punctures—in particular the locations of the monopoles μ^a and the fluxes through spheres surrounding the sources, k^a.

These results can be conveniently repackaged in terms of the integration of a nine-form I_9 over the space $[\mu] \times \Sigma_{g,1}$. Define

$$I_9 = d\left[\mu^3 I_8^{\text{bulk}} + N^3 f(\mu) p_2(NW_6)\right], \qquad (7.129)$$

where $f(\mu)$ is related to $\hat{\ell}(\mu)$ as $\hat{\ell}(\mu) = f(\mu) + \mu^3/24$. Then, the contributions to the class $\mathscr{S}$

anomalies, $I_6^{\mathscr{S}}$, are derived simply as[10]

$$\int_{[\mu]\times\Sigma_{g,1}} I_9 = \int_{\Sigma_{g,1}} I_8^{\text{bulk}} + \int_{S_a^2} I_8^{\text{bdy}} = I_6^{\mathscr{S}}. \tag{7.130}$$

The bulk contributions to the four-dimensional anomalies, $I_6(\Sigma_{g,1})$, come from the boundary terms in the integration evaluated at $\mu = 0$ and $\mu = 1$. The puncture contributions, $I_6(P)$, come from the integration over the boundary spheres S_a^2. From this perspective, the total anomaly polynomial for the class $\mathscr{S}$ theory naturally reproduces

$$I_6^{\mathscr{S}} = I_6(\Sigma_{g,1}) + I_6(P). \tag{7.131}$$

These expressions validate the general expectation of the structure of anomalies of class $\mathscr{S}$ theories as described in (7.6).

Class $\mathscr{S}$ anomaly

Now, we give some discussion of the contributions to the class $\mathscr{S}$ anomalies. The bulk terms were computed in (7.110) and (7.121), which together yield

$$I_6(\Sigma_{g,1}) = -\frac{\chi(\Sigma_{g,1})}{2}\left[N\left(\frac{(c_1^+)^3}{3} - \frac{c_1^+ p_1(T^4)}{12} + \frac{c_1^+ c_2^-}{3}\right) - \frac{4}{3}N^3 c_1^+ c_2^-\right]. \tag{7.132}$$

Indeed, $I_6(\Sigma_{g,1})$ is of the form described in (7.2) where we integrate the polynomial from the M5-branes given in (7.55). The difference between the result of this computation and the anomaly polynomial of the A_{N-1} $(2,0)$ theory is due to an overall free tensor multiplet that decouples from the dynamics of the M5-branes.

As we discussed, the terms that depend on the local puncture data can be given as

$$I_6(P) = \int_{S_a^2} I_8^{\text{bdy}}, \tag{7.133}$$

with I_8^{bdy} given in (7.128). An important feature of these terms is that there are independent contributions from each monopole source, and moreover, there are no mixed terms between sources due to the integrals over $\rho_a \rho_b$ vanishing for $a \neq b$.

We explicitly computed the contribution of the linear term as

[10]This construction requires the boundary condition that $f(1) = f(0)$.

$$I_6(P) \supset -\frac{N}{2}\sum_a (\mu^a)^3 k^a \left(\frac{(c_1^+)^3}{3} - \frac{c_1^+ p_1(T^4)}{12} + \frac{c_1^+ c_2^-}{3} \right) - \frac{N}{24}\sum_a (\mu^a)^3 (k^a)^3 c_1^+ (c_1^a)^2 . \quad (7.134)$$

This gives the general form of the answer that one obtains from inflow. This answer depends on the μ^a, which correspond to the locations of the monopoles along the μ direction, and the charges k^a of the monopoles. More work must be done to actually match onto the known answer for the Class $\mathscr{S}$ theory. In particular, we must address the following:

- How to fix the precise form of the angular form $\widetilde{E}_4$, as well as the free parameters a_ϕ, a_s. The possible choices are intimately related to regularity conditions on the flux in various limits.

- How the parameters k^a, μ^a encode the data of the punctures in field theory. In particular, for the case of regular punctures these should be associated to the data of the Young diagram that corresponds to the flavor symmetry preserved in the CFT. One can hope to extend this analysis to the case of irregular punctures.

- The relation of the $U(1)_a$ symmetries to the flavor symmetry that emerges at the puncture in the CFT.

- What are the decoupled modes of the system? In particular, the inflow result includes modes that decouple with respect to the low energy CFT.

These issues will be addressed in the upcoming [120].

7.6 Conclusions

7.6.1 Summary

In Section 7.3, we studied the anomalies of the $\mathscr{N} = 1$ four-dimensional theories that derive from M5-branes wrapped on a punctured Riemann surface. When the bulk preserves

$\mathscr{N} = 1$, the R-symmetry locally at a puncture can preserve $\mathscr{N} = 2$ supersymmetry with a local twist of the R-symmetry generators labeled by integers (p,q). We derived the anomalies of these (p,q)-labeled punctures, and discussed an illuminating parameterization of the anomalies in terms of an $\mathscr{N} = 1$ version of an effective number of vector and hypermultiplets.

In Sections 7.4 and 7.5, we turned to the problem of computing the anomalies of the $\mathscr{N} = 2$ theories of class $\mathscr{S}$ from inflow for M5-branes wrapped on a punctured Riemann surface. The punctures lead to boundaries on the internal space. In general, one expects the fluxes to have additional sources on the boundaries. In our analysis, we motivated the addition of monopole sources at the locations of the punctures for the connection form on the Riemann surface. These appear as delta functions on the right-hand-side of the Bianchi identity for the associated curvature F, which had to be appropriately smoothed.

The M5-branes magnetically source the M-theory flux G_4. The sources for the connection form on the surface induce additional sources for G_4. Compatibility conditions between these sources require additional M5-branes that intersect the original ones at the punctures and end on the monopole sources. This is consistent with the picture in AdS/CFT. When the branes are backreacted, there is an AdS_5 spacetime that emerges in the near-horizon limit of the branes. The connection forms in a Ricci flat background pick up such monopole sources in the work of [83, 94].

Our analysis captures the anomaly contributions from the additional degrees of freedom at the intersection of these sources. We describe a well-defined way to derive these anomaly contributions by integrating the eight-form anomaly polynomial of the world-volume theory over the boundaries, whose coefficients depend on local data at the puncture. In particular, this local data is captured by the flux through the boundary spheres surrounding the monopole sources, and the locations of the sources along an interval transverse to the branes.

7.6.2 Outlook

In this chapter we have focused on four-dimensional field theories that preserve $\mathcal{N} = 2$ supersymmetry. The generalization to the $\mathcal{N} = 1$ theories will follow the steps we've laid out here, but with some interesting additional complications. In particular, in the $\mathcal{N} = 1$ case the normal bundle to the M5-branes decomposes as $SO(2)_+ \times SO(2)_- \subset SO(5)$. The field strengths for each $SO(2)_\pm$ will have sources at the punctures, whose profile in the normal directions can be more involved. We present the analysis for the $\mathcal{N} = 1$ story in the upcoming [120].

The $\mathcal{N} = 1$ theories of Class $\mathcal{S}$ are even more rich than their $\mathcal{N} = 2$ counterparts. One feature is that different kinds of punctures can be present in the $\mathcal{N} = 1$ class $\mathcal{S}$ construction. One example are the class of (p,q)-labeled punctured we discussed in Section 7.3. It would be interesting to understand these anomalies from an inflow analysis. More generally, the landscape of $\mathcal{N} = 1$ preserving punctures in these geometries is much less understood than their $\mathcal{N} = 2$ counterparts, and would be interesting to study further.

In the region near the puncture, the M-theory system can be reduced to Type IIA string theory, and the degrees of freedom at the puncture are associated to the intersection of D4/D6 branes [92]. From this perspective, the contributions from the punctures should be related to I-branes, as discussed in [147]. Such intersections are also related to D6/D8 brane intersections, which appear in the classification of (1,0) theories [154]. It would be interesting to explore these connections in the future.

Throughout this chapter, we have only discussed theories which have their origin from the (2,0) theories in six dimensions. One could also consider starting from theories with less supersymmetry, such as six-dimensional (1,0) SCFTs (with a recently proposed classification in [155, 156, 157]). These are far more numerous, and their compactifications are less understood than their (2,0) counterparts. One large class of such constructions—dubbed class $\mathcal{S}_k$—involve N M5-branes on an A_{k-1} singularity of M-theory compactified on a punctured Riemann surface [158, 159, 160, 119]. It would be very interesting to extend the inflow analysis we considered here in the class $\mathscr{S}$ context to these theories.

As a specific example, 't Hooft anomalies for four-dimensional theories that result from compactifications of the six-dimensional E-string theory on a punctured Riemann surface were computed in [161]. There, the contributions of the punctures to the anomalies were obtained by studying boundary conditions of the E-string theory at the punctures, and adding up the anomalies from chiral fields living on the boundary. It would be interesting to understand these contributions from the perspective we've advocated here.

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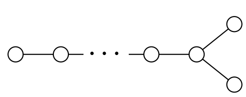
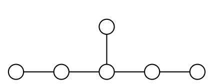
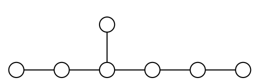
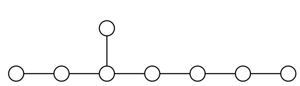
Appendix A

$W_{A,D,E}$ SCFTs

A.1 ADE Facts

Here we collect some facts about the ADE theories.

Table A.1: Some relevant data corresponding to the ADE Lie groups. h is the Coxeter number, and $\check{h}$ the dual Coxeter number.

G	Dynkin Diagram	$h(=\check{h})$	Exponents l_n♣	α♠
A_k $SU(k+1)$		$k+1$	$1,2,..,k$	k
D_{k+2} $SO(2k+4)$		$2(k+1)$	$1,3,5,\ldots,2k+1,k+1$	$3k$
E_6		12	$1,4,5,7,8,11$	?
E_7		18	$1,5,7,9,11,13,17$	30
E_8		30	$1,7,11,13,17,19,23,29$	?

♣l_n=(degree of invariant polynomials of the Lie algebra)-1. Eigenvalues of the adjacency matrix of the associated Dynkin diagram are $2\cos(\pi l_n/h)$. The l_n are 1-1 with the rank(G) chiral ring generators for the case when the fields are not matrix valued:

$$\begin{aligned} A_k: &\quad l_j = \frac{k+1}{2}R(\Theta_j)+1 \quad \text{for } \Theta_j = \{1,x,x^2,...,x^{k-1}\} \\ D_{k+2}: &\quad l_j = (k+1)R(\Theta_j)+1 \quad \text{for } \Theta_j = \{1,x,x^2,...,x^{k-1},y,y^2\} \\ E_7: &\quad l_j = 9R(\Theta_j)+1 \qquad \text{for } \Theta_j = \{1,x,x^2,x^3,x^4,y,xy\} \end{aligned} \tag{A.1}$$

♠number of generators of (supposedly truncated) chiral ring.

A.2 RG Flows Whose Deformations Seem Irrelevant

We briefly consider (as in Section 3.5.4) some cases where the ΔW's, corresponding to some ADE adjoint Higgsing pattern, are not immediately apparent. We focus on recovering the desired 1d vacuum structure for $U(N_c)$ flows, leaving a full analysis of the higher-dimensional structure for future work. The cases studied in Sections A.2.1 and A.2.2 are analogous to singularity resolutions studied in [62].

A.2.1 $E_6 \to A_5$

We start with the deformed E_6 superpotential,

$$W = \frac{1}{3}\mathrm{Tr}Y^3 + \frac{s}{4}\mathrm{Tr}X^4 + t_1\mathrm{Tr}YX^2 + t_2\mathrm{Tr}Y^2, \tag{A.2}$$

whose F-terms are

$$Y^2 + t_1X^2 + 2t_2Y = 0 \tag{A.3}$$

$$sX^3 + t_1\{X,Y\} = 0. \tag{A.4}$$

For 1d representations, $X = x\mathbf{1}, Y = y\mathbf{1}$, (A.3) and (A.4) yield vacua which correspond to the following IR theories (as usual, seen by expanding (A.2) in each vacuum):

$$(x,y) = \begin{cases} (0,0) & \leftrightarrow \quad A_3 \\ (0,-2t_2) & \leftrightarrow \quad A_1 \\ \left(\pm\sqrt{\frac{4t_1}{s}\left(t_2-\frac{t_1^2}{s}\right)}, -2\left(t_2-\frac{t_1^2}{s}\right)\right) & \leftrightarrow \quad 2A_1 \end{cases} \tag{A.5}$$

For the special value of $t_2 = \frac{t_1^2}{s}$, for nonzero s, the two eigenvalues on the last line of (A.5) collapse to the origin to enhance the A_3 singularity. This is more clearly seen by changing variables $Y = \frac{s}{t_1}(Z - X^2)$. Then, for the special value of $t_2 = \frac{t_1^2}{s}$, (A.2) rewritten in terms of the X, Z fields,

$$W = \frac{s^3}{t_1^3}\left(\frac{1}{3}\mathrm{Tr}Z^3 - \mathrm{Tr}Z^2X^2 + \mathrm{Tr}ZX^4\right) - \frac{s^3}{3t_1^3}\mathrm{Tr}X^6 + s\mathrm{Tr}Z^2. \tag{A.6}$$

The F-terms of (A.6) then yield the 1d vacua $A_5 + A_1$.

To find higher dimensional representations of vacua for this flow, we note that (A.3) and (A.4) implies $[X^2, Y] = 0$. Since $[Y^2, X] \backsim [X, Y]$, Y^2 is not a Casimir; instead we use $Y^2 + 2t_2Y = f(y)\mathbf{1}$. Parameterizing 2d solutions by $Y = y_0\mathbf{1} + y_1\sigma_1$ and $X = x\sigma_3$, Y's Casimir condition fixes y_0, so that the F-terms for 2d vacua simplify to

$$(-t_2^2 + y_1^2 + t_1x^2)\mathbf{1} = 0 \tag{A.7}$$

$$x(sx^2 - 2t_1t_2)\sigma_3 = 0. \tag{A.8}$$

(A.8) fixes the eigenvalue x, and the first fixes y_1, such that we indeed have a 2-dimensional vacuum (only one, as gauge symmetry relates $x \to -x$ and $y_1 \to -y_1$). This vacuum exists both for generic t_2, and for t_2 shifted to give the A_5 theory. In sum, the flow (A.2) has the following 1d and 2d vacua:

$$t_2 \text{ generic}: \quad E_6 \to A_3 + 3A_1 + A_1^{2d}\ (+\ldots?) \tag{A.9}$$

$$t_2 = \frac{t_1^2}{s}: \quad E_6 \to A_5 + A_1 + A_1^{2d}\ (+\ldots?). \tag{A.10}$$

A.2.2 $E_7 \to D_6$

Here, we start with the E_7 superpotential deformed by the D-series term $\text{Tr}XY^2$,

$$W_{E_7} + \Delta W = \frac{1}{3}\text{Tr}Y^3 + s\text{Tr}YX^3 + t\text{Tr}XY^2. \tag{A.11}$$

There are two sets of 1-dimensional vacuum solutions for X and Y, corresponding to the eigenvalues $(x = 0, y = 0)$, and $(x = \frac{5t^2}{9s}, y = -\frac{25t^3}{27s})$. Expanding near the origin appears to just give $W_{low} \backsim \text{Tr}XY^2 = W_{\widehat{D}}$. Consider though the following sequence of variable changes:

$$X = U - \frac{1}{3t}Y, \quad Y = \frac{s}{2t}(Z - U^2). \tag{A.12}$$

In terms of the U, Z fields, (A.11) becomes

$$\begin{aligned} W =& \frac{s^5}{108t^7}\left(-\frac{1}{4}\text{Tr}U^8 + \text{Tr}ZU^6 - \text{Tr}Z^2U^4 - \frac{1}{2}\text{Tr}(ZU^2)^2 + \text{Tr}Z^3U^2 - \frac{1}{4}\text{Tr}Z^4\right) \\ &+ \frac{s^4}{24t^5}\left(-\text{Tr}U^7 + 3\text{Tr}ZU^5 - 2\text{Tr}Z^2U^3 - \text{Tr}UZU^2Z + \text{Tr}Z^3U\right) \\ &+ \frac{s^3}{4t^3}\left(-\text{Tr}U^6 + 2\text{Tr}ZU^4 - \frac{2}{3}\text{Tr}Z^2U^2 - \frac{1}{3}\text{Tr}(UZ)^2\right) + \frac{s^2}{4t}\left(\text{Tr}UZ^2 - \text{Tr}U^5\right). \end{aligned} \tag{A.13}$$

We've organized the terms in (A.13) by increasing relevance from the perspective of the UV fixed point—the most relevant terms in the IR limit of the flow are those in the last parentheses, such that the D_6 theory resides at the origin. There is a 1d solution to the F-terms of (A.13) corresponding to an A_1 theory, such that for all $t \neq 0$ we recover the 1d vacua:

$$E_7 \to D_6 + A_1 \ (+\ldots?) \tag{A.14}$$

where here the (+...?) refers to the unexplored possibility of $d > 1$ dimensional vacuum solutions.

A.2.3 $E_8 \to D_7$

We start by deforming the E_8 theory with a D-series deformation and E_7 deformation,

$$W = \frac{1}{3}\text{Tr}Y^3 + \frac{s}{5}\text{Tr}X^5 + t_1\text{Tr}YX^3 + t_2\text{Tr}XY^2. \tag{A.15}$$

From the 1d F-terms of this superpotential, there is one eigenvalue pair at the origin and two away from the origin. As in the previous subsection, there is naively some ambiguity in identifying the solution at the origin, since each of $\text{Tr}YX^3$, $\text{Tr}X^5$, and $\text{Tr}XY^2$ appear to be marginal deformations of the UV theory, but the eigenvalue decomposition suggests that the theory at the origin corresponds to D_6. Then, the 1d vacua of (A.15) are D_6+2A_1.

There is a particular shift of the coefficients $t_2 = \frac{5t_1^2}{4s} \equiv t_*$ that brings one of the nonzero A_1 eigenvalue pairs to the origin. A change of variables clarifies what is happening: take $Y = U - \frac{2s}{5t_1}X^2$, such that (A.15) becomes

$$\begin{aligned} W = &\frac{1}{3}\mathrm{Tr}U^3 - \frac{2s}{5t_1}\mathrm{Tr}U^2X^2 + \frac{4s^2}{25t_1^2}\mathrm{Tr}UX^4 - \frac{8s^3}{375t_1^3}\mathrm{Tr}X^6 + \left(t_1 - \frac{4st_2}{5t_1}\right)\mathrm{Tr}UX^3 \\ &- \frac{s}{5t_1}\left(t_1 - \frac{4st_2}{5t_1}\right)\mathrm{Tr}X^5 + \frac{5t_1^2}{4s}\mathrm{Tr}U^2X. \end{aligned} \tag{A.16}$$

The 1d F-terms of (A.16) still yield one zero eigenvalue pair and two nonzero eigenvalue pairs, but if we now shift $t_2 = t_*$, then the D_6 theory at the origin is enhanced to a D_7 theory, while only one nonzero (1d) vacuum remains, in which both X and U receive masses. In sum, the shift $t_2 = t_*$ results in the 1d vacua $D_7 + A_1$.

We now study higher-dimensional representations of vacuum solutions to the F-terms of (A.15) and D-terms (3.7). For generic values of the couplings, there is a 2d vacuum (letting $s = 1$)

$$X = x_0\mathbf{1} + x_3\sigma_3, \quad Y = y_0\mathbf{1} + y_3\sigma_3,$$

$$x_0 = t_1\left(-\frac{9}{2}t_1^2 + 4t_2\right), \quad x_3 = \frac{1}{2}(9t_1^2 - 4t_2)^{1/2}(3t_1^2 - 2t_2), \tag{A.17}$$

$$y_0 = \frac{1}{2}t_1(-27t_1^4 + 45t_1^2t_2 - 20t_2^2), \quad y_3 = \frac{1}{2}(9t_1^2 - 4t_2)^{3/2}(t_1^2 - t_2). \tag{A.18}$$

Then, for generic values of t_1 and t_2, the 1d and 2d vacua of this flow are

$$E_8 \to D_6 + 2A_1 + A_1^{2d}\ (+\ldots?). \tag{A.19}$$

As is evident in (A.18), there exist special values of t_2 for which the 2d vacua “go away" in the sense of Section 3.5.4, e.g.

$$t_2 = \frac{5t_1^2}{4} \equiv t_* : \quad E_8 \to D_7 + A_1\ (+\ldots?) \tag{A.20}$$

$$t_2 = \frac{3t_1^2}{2} \text{ or } t_1^2 : \quad E_8 \to D_6 + 2A_1\ (+\ldots?) \tag{A.21}$$

$$t_2 = \frac{9t_1^2}{4} : \quad E_8 \to D_6 + A_2\ (+\ldots?) \tag{A.22}$$

In all cases above, the $(+\ldots?)$ refers to $d > 2$ dimensional vacua. The special case (A.20) corresponds precisely to the shift t_* already discussed, in which the D_6 singularity is enhanced to a D_7 singularity. In this case, one of the two eigenvalues corresponding to an A_1^{2d} in (A.18) goes the origin, and the other becomes a copy of the eigenvalues corresponding to the remaining A_1 theory. In (A.21), the eigenvalues corresponding to the A_1^{2d} theories in (A.18) become copies of the eigenvalues corresponding to the 1d A_1 theories. For the shift in (A.22), the two A_1 theories as well as the A_1^{2d} theory in (A.18) are enhanced to an A_2 theory.

Appendix B

Relevant Formulae for Gluing Negative-Degree Line Bundles

B.1 Conventions and Main T_N Formulae

The T_N theory is $\mathcal{N}=2$ supersymmetric with global symmetries $SU(2)_R \times U(1)_R \times SU(N)^3$. We use a basis for the Cartan subalgebra of the $\mathcal{N}=2$ R-symmetry $SU(2)_R \times U(1)_R$ labeled by $(I_3, R_{\mathcal{N}=2})$. The R-symmetry of an $\mathcal{N}=1$ subalgebra is given in (2.3). We can rewrite these in terms of the generators J_+, J_- of the $U(1)_+ \times U(1)_-$ symmetry preserved by the $\mathcal{N}=1$ theories of class $\mathscr{S}$, using $R_{\mathcal{N}=2} = J_-$, and $I_3 = \frac{1}{2}J_+$. With these conventions, for example, the adjoint field in the $\mathcal{N}=2$ vector multiplet has $(J_+, J_-) = (0,2)$.

The (J_+, J_-) charges of chiral operators of the T_N theory are The $\mu_{A,B,C}$ are moment-map

Table B.1: (J_+, J_-) charges of chiral operators of the T_N theory.

	(J_+, J_-)
u_k	$(0, 2k)$
$Q, \widetilde{Q}$	$(N-1, 0)$
$\mu_{A,B,C}$	$(2, 0)$

operators in the adjoint of (one of) the $SU(N)_A \times SU(N)_B \times SU(N)_C$ flavor symmetry groups, and the $Q(\widetilde{Q})$ transform in the trifundamental(anti-trifundamental) of the $SU(N)_A \times SU(N)_B \times SU(N)_C$ symmetry. The u_k are Coulomb branch operators of dimension k, with $k = 3, \ldots, N$.

The IR superconformal R-charge for operators of the T_N theory of color $\sigma_i = \pm 1$ is given by maximizing the following combination of R-charges with respect to ε:

$$\begin{aligned} R(\varepsilon) &= \left(\frac{1}{2}R_{\mathcal{N}=2} + I_3\right) + \sigma_i \varepsilon \left(\frac{1}{2}R_{\mathcal{N}=2} - I_3\right) \\ &= \frac{1}{2}(1-\sigma_i\varepsilon)J_+ + \frac{1}{2}(1+\sigma_i\varepsilon)J_-. \end{aligned} \tag{B.1}$$

B.2 Relevant BBBW Results

In [89], Bah, Beem, Bobev, and Wecht (BBBW) compute a and c of the IR $\mathcal{N}=1$ SCFTs obtained from compactifying the 6d (2,0) theories on a Riemann surface $\mathscr{C}_g$, where the surface is embedded in a Calabi-Yau three-fold that decomposes into a sum of line bundles as in (5.5). These are computed by integrating the anomaly eight-form of the M5-brane theory over the surface $\mathscr{C}_g$, and matching with the anomaly six-form, which is related to the anomalous divergence of the 4d $\mathcal{N}=1$ R-current by the descent procedure.

Due to the presence of an additional global symmetry $U(1)_{\mathscr{F}}$, the superconformal R-symmetry takes the form

$$R = K + \varepsilon \mathscr{F}, \tag{B.2}$$

where ε is a real number determined by a-maximization. For the (2,0) theory of type A_{N-1}, ε is found to be

$$\varepsilon = \frac{\eta + \kappa\zeta}{3(1+\eta)z}, \tag{B.3}$$

and the central charges a and c are found (for $g \neq 1$) to be

$$\begin{aligned} a &= (g-1)(N-1)\frac{\zeta^3 + \kappa\eta^3 - \kappa(1+\eta)(9+21\eta+9\eta^2)z^2}{48(1+\eta)^2 z^2}, \\ c &= (g-1)(N-1)\frac{\zeta^3 + \kappa\eta^3 - \kappa(1+\eta)(6-\kappa\zeta+17\eta+9\eta^2)z^2}{48(1+\eta)^2 z^2}. \end{aligned} \tag{B.4}$$

η and ζ are defined as

$$\eta = N(1+N), \quad \zeta = \sqrt{\eta^2 + (1+4\eta+3\eta^2)z^2}. \tag{B.5}$$

z is the twist parameter defined in terms of the degrees of the line bundles p and q as in (6.15), and $\kappa = 1$ for the sphere and $\kappa = -1$ for a hyperbolic Riemann surface. In the large N limit, these simplify to

$$\begin{aligned} \varepsilon_{\text{large-}N} &= \frac{1+\kappa\sqrt{1+3z^2}}{3z} \\ a_{\text{large-}N} = c_{\text{large-}N} &= (1-g)N^3\left(\frac{1-9z^2+\kappa(1+3z^2)^{3/2}}{48z^2}\right). \end{aligned} \tag{B.6}$$

The computation for $g = 1$ requires special care, as one can preserve $\mathcal{N} = 4$ supersymmetry in the IR by fixing the normal bundle to the M5-brane worldvolume theory to be trivial. However, taking $p = -q$ preserves only $\mathcal{N} = 1$ supersymmetry in the IR. Redoing the computation for this special value, BBBW find that for the A_{N-1} theory on the torus,

$$\begin{aligned} \varepsilon &= -\frac{1}{3}\sqrt{\frac{1+3\eta}{1+\eta}}, \\ a &= \frac{|z|}{48}\frac{(N-1)(1+3\eta)^{3/2}}{\sqrt{1+\eta}}, \quad c = \frac{|z|}{48}\frac{(N-1)(2+3\eta)\sqrt{1+3\eta}}{\sqrt{1+\eta}}, \end{aligned} \tag{B.7}$$

where at large-N,

$$a_{\text{large-}N} = c_{\text{large-}N} = \frac{\sqrt{3}}{16}|z|N^3. \tag{B.8}$$

B.3 't Hooft Anomalies for Gluing $T_N^{(m)}$ Building Blocks

The 't Hooft anomaly coefficients for a single T_N block are given by

$$\begin{array}{c|c} & T_N \text{ coefficients} \\ \hline J_+, J_+^3 & 0 \\ J_-, J_-^3 & (N-1)(-3N-2) \\ J_+^2 J_- & \frac{1}{3}(N-1)(4N^2-5N-6) \\ J_+ J_-^2 & 0 \\ J_+ SU(N)^2_{A,B,C} & 0 \\ J_- SU(N)^2_{A,B,C} & -N \end{array} \tag{B.9}$$

An $\mathcal{N}=2$ vector multiplet contains two fermions with $(J_+, J_-) = (1,1)$ and $(-1,1)$, so the only nonzero anomaly coefficients are

$$\mathcal{N}=2 \text{ vector}: \quad J_- = J_-^3 = J_+^2 J_- = 2(N^2-1). \tag{B.10}$$

Consider Higgsing an $SU(N)$ flavor group on a T_N block by giving a nilpotent vev to the adjoint chiral multiplet, $\langle M \rangle = \rho(\sigma_3)$, where the $SU(N)$ flavor corresponds to a maximal puncture whose color is opposite the background color. This can be computed[1] by shifting $J_- \to J_- - 2\rho(\sigma_3)$ and summing the contribution from the remaining $N-1$ singlets $M_j,\ j = 1, \ldots, N-1$ whose R-charges are shifted to $(J_+, J_-) = (0, 2+2j)$. This results in the following contribution to the

[1] The author is grateful to Prarit Agarwal for explaining this computation in more detail.

block being Higgsed:

	from Higgsing
J_+, J_+^3	$1-N$
J_-, J_-^3	N^2-1
$J_+^2 J_-$	N^2-1
$J_+ J_-^2$	$\frac{1}{3}(1-N)(4N^2+4N+3)$
$J_+ SU(N)^2_{A,B,C}$	0
$J_- SU(N)^2_{A,B,C}$	0

(B.11)

The 't Hooft anomaly coefficients for a single $T_N^{(m)}$ block are computed in [100] by summing the contributions of $(m+1)$ T_N blocks—m of which have a Higgsed flavor group—and m $\mathcal{N}=2$ vector multiplets, yielding

	$T_N^{(m)}$ coefficients
J_+, J_+^3	$m(1-N)$
J_-, J_-^3	$(N-1)(m-3N-2)$
$J_+^2 J_-$	$\frac{1}{3}(N-1)(4N^2-5N-6+m(4N^2+4N+3))$
$J_+ J_-^2$	$\frac{1}{3}m(3+N-4N^3)$
$J_+ SU(N)^2_{A,B,C}$	0
$J_- SU(N)^2_{A,B,C}$	$-N$

(B.12)

Taking $m=0$ reproduces the T_N 't Hooft anomalies.

Given these anomaly coefficients, we can compute the contribution to the central charges, using

$$\begin{aligned} a(\varepsilon) &= \frac{3}{32}\left(3\mathrm{Tr}R(\varepsilon)^3 - \mathrm{Tr}R(\varepsilon)\right) \\ &= \frac{3}{64}\left(\frac{3}{4}\left[(1-\varepsilon)^3 J_+^3 + (1+\varepsilon)^3 J_-^3 + 3(1-\varepsilon)^2(1+\varepsilon) J_+^2 J_- + 3(1-\varepsilon)(1+\varepsilon)^2 J_+ J_-^2\right]\right. \\ &\qquad \left. - (1-\varepsilon)J_+ - (1+\varepsilon)J_-\right). \end{aligned} \tag{B.13}$$

Appendix C

Class $\mathscr{S}$ Anomaly Conventions

C.1 Anomaly Polynomials and Characteristic Classes

As reviewed in the main text, anomalies are encoded in a $(d+2)$-form anomaly polynomial I_{d+2} that is related to the anomalous variation of the quantum effective action as

$$\delta S_{\text{eff}} = 2\pi \int_{M_d} I_d^{(1)}, \tag{C.1}$$

where

$$I_{d+2} = dI_{d+1}^{(0)}, \qquad \delta I_{d+1}^{(0)} = dI_d^{(1)}. \tag{C.2}$$

Anomalies for chiral fields in even $d = 2n$ dimensions are related to index theorems in two higher dimensions [141]. For example, the Atiyah-Singer index theorem for a chiral spin-1/2 fermion in $d+2$ dimensions relates the index density of the Dirac operator to characteristic classes of the curvatures, which in turn are related to the $(d+2)$-form anomaly polynomial as

$$I_{d+2} = \text{index}(i\not{D}) = \left[\hat{A}(R)\text{ch}(F)\right]_{d+2}. \tag{C.3}$$

The $(d+2)$ subscript in (C.3) instructs us to extract the $(d+2)$-form contribution in the expansion of the curvatures. $\text{ch}(F)$ is the Chern character, defined for a complex bundle in terms of the corresponding field strength F as

$$\text{ch}(F) = \text{Tr}_{\mathbf{r}} e^{iF/(2\pi)} = \dim(\mathbf{r}) + c_1(F) + \frac{1}{2}(c_1(F)^2 - 2c_2(F)) + \dots \tag{C.4}$$

The Chern classes c_k are $2k$-forms that are polynomials in F of degree k. For reference, the first two Chern classes take the form

$$c_1(F) = \frac{i}{2\pi}\text{Tr}F, \quad c_2(F) = \frac{1}{2(2\pi)^2}\left[\text{Tr}(F^2) - (\text{Tr}F)^2\right]. \tag{C.5}$$

Our notation is such that if the Chern roots of an $SU(N)$-bundle are given by λ_i, $c_2(F) = -\frac{1}{2}\sum_i \lambda_i^2$.

$\hat{A}(R)$ is the A-roof genus, a function of the curvature R of the spacetime tangent bundle with leading terms

$$\hat{A}(R) = 1 - \frac{1}{24}p_1(R) + \frac{7p_1(R)^2 - 4p_2(R)}{5760} + \dots \tag{C.6}$$

The p_k are the Pontryagin classes, $4k$-forms that are $2k$-order polynomials in R. For reference, the first two Pontryagin classes for a real vector bundle with curvature R are

$$p_1(R) = -\frac{1}{2}\frac{1}{(2\pi)^2}\text{Tr}(R^2), \tag{C.7}$$

$$p_2(R) = \frac{1}{8}\frac{1}{(2\pi)^4}\left[\left(\text{Tr}(R^2)\right)^2 - 2\text{Tr}(R^4)\right]. \tag{C.8}$$

For a real bundle with a complex cover, the Pontryagin classes can be related to the Chern classes. For an $SO(N)$ bundle E, $p_1(E)$ and $p_2(E)$ can be written in terms of the Chern roots λ_i as

$$p_1(E) = \sum_i \lambda_i^2, \qquad p_2(E) = \sum_{i<j} \lambda_i^2 \lambda_j^2. \tag{C.9}$$

Another useful set of identities relates the Pontryagin classes of a vector bundle which is the Whitney sum of two vector bundles, $E = E_1 \oplus E_2$, to the Pontryagin classes of the constituent $E_{1,2}$ as

$$p_1(E) = p_1(E_1) + p_1(E_2) \tag{C.10}$$

$$p_2(E) = p_2(E_1) + p_2(E_2) + p_1(E_1)p_1(E_2). \tag{C.11}$$

From (C.3), it follows that the six-form anomaly polynomial for one four-dimensional Weyl fermion with $U(1)$ charge q is

$$I_6 = \left[\hat{A}(T^4)\text{ch}(qF)\right]_6 = \frac{q^3}{6}c_1(F)^3 - \frac{q}{24}c_1(F)p_1(T^4). \tag{C.12}$$

Here, F is the field strength of the $U(1)$ bundle, and T^4 is the spacetime tangent bundle. More generally, a four-dimensional theory with a $U(1)$ R-symmetry and anomaly coefficients $\mathrm{Tr}R^3$ and $\mathrm{Tr}R$ (n.b. that R here does not refer to the curvature!) has the corresponding six-form anomaly polynomial:

$$I_6 = \frac{\mathrm{Tr}R^3}{6} c_1(F)^3 - \frac{\mathrm{Tr}R}{24} c_1(F) p_1(T^4). \tag{C.13}$$

F here is field strength of the $U(1)$ bundle coupled to the R-symmetry. I_6 is then related to the anomalous divergence of the R-symmetry current by the descent procedure.

C.2 Anomalies for Regular $\mathcal{N}=2$ Punctures

A regular $\mathcal{N}=2$ puncture is labeled by an embedding $\rho:\ \mathfrak{su}(2) \to \mathfrak{g}$. For $\mathfrak{g} = A_{N-1}$, the choice of ρ is 1-to-1 with a partition of N, i.e. a Young diagram Y with N boxes. In this appendix, we review the contributions of punctures from the six-dimensional (2,0) $\mathfrak{g} = A_{N-1}$ theories compactified on a Riemann surface $\Sigma_{g,n}$ with genus g and n total punctures.

Let Y have some number of columns of height h_i, and some number of rows of length ℓ_j, corresponding to a partition of N

$$N = \sum_i h_i = \sum_j \ell_j. \tag{C.14}$$

Let n_i be the number of columns of height h_i that appear in the sum. Then, the theory has an unbroken flavor symmetry

$$G = S\left[\prod_i U(n_i)\right], \tag{C.15}$$

which corresponds to the commutant of the embedding ρ.

We also assign a pole structure to the puncture [82], which can be read off of the Young diagram. Denote the pole structure by a set of N integers p_i, $i = 1, \ldots, N$. Label each of the N boxes in the Young diagram sequentially with a number from 1 to N, starting with 1 in the upper left corner and increasing from left to right across a row. Then, $p_i = i-$(height of ith box). For instance, $p_1 = 1 - 1 = 0$ always.

For example, a "maximal" puncture is labeled by a Young diagram with 1 row of length $\ell_1 = N$, or alternatively, N columns each of height $h_{1\leq j\leq N} = 1$. This is commonly denoted $Y = [1, \ldots, 1]$. The unbroken flavor symmetry is $G = SU(N)$, and the pole structure is $p_i = i - 1 = (0, 1, 2, \ldots, N-1)$. As another example, a "minimal" or "simple" puncture is labeled by a Young diagram with 1 row of length 2 and $N-2$ rows of length 1, or alternatively, 1 column of height $N-1$ and 1 column of height 1, denoted $Y = [N-1, 1]$. The unbroken flavor symmetry is $G = S[U(1) \times U(1)] = U(1)$, and the pole structure is $p_i = (0, 1, 1, \ldots, 1)$.

The effective number of vector multiplets that a regular puncture labeled by a Young diagram Y contributes to the theory is [83, 95]

$$n_v(P_Y) = -\frac{1}{2}\left(r_G + \frac{4}{3}d_G h_G\right) + \sum_{k=1}^{N}(2k-1)p_k \tag{C.16}$$

and the effective number of hypermultiplets is

$$n_h(P_Y) = \frac{1}{2}\left[\sum_{i=1}^{r}\ell_i^2 - 1\right] - \frac{1}{2}\left(r_G + \frac{4}{3}d_G h_G\right) + \sum_{k=1}^{N}(2k-1)p_k \tag{C.17}$$

For example, the maximal puncture contributes

$$n_v(P_{\max}) = -\frac{1}{2}(N^2-1), \qquad n_h(P_{\max}) = 0, \tag{C.18}$$

and the minimal puncture contributes

$$n_v(P_{\min}) = -\frac{1}{6}(4N^3 - 6N^2 - N + 3), \quad n_h(P_{\min}) = -\frac{1}{6}(4N^3 - 6N^2 - 4N). \tag{C.19}$$

An $SU(n_i)$ flavor group factor corresponds in the Young diagram to a nonzero difference of $n_i = \ell_i - \ell_{i+1}$ between the lengths of two rows. Then, the associated flavor central charge can be written

$$k_{SU(\ell_i - \ell_{i+1})} = 2\sum_{n\leq i}\ell_n. \tag{C.20}$$

C.3 Integral Identities

It is useful for us to understand integral properties of F_0 that are implied by (7.80) and (7.84). Recall that we describe the transverse space to the branes by the metric (7.64), where μ is the interval $[0,1]$, and that we use coordinates $x^{1,2}$ on the Riemann surface. Near each puncture, we subtract a small ball B_a^3 of size ε^a centered at the a'th monopole. Explicitly, the coordinate transformation near each ball is given by

$$(\mu-\mu^a)^2 = R_a^2(1-\tau_a^2), \qquad (x^1)^2 = (R_a\tau_a)^2\cos^2\varphi^a, \qquad (x^2)^2 = (R_a\tau_a)^2\sin^2\varphi^a, \quad \text{(C.21)}$$

where φ^a is the circle coordinate for the unbroken $U(1)^a$ at each source. We define $\widetilde{M}_3$ as the space $[\mu]\times\Sigma_{g,n}$, with the balls B_a^3 subtracted.

With μ as given in (C.21), it follows that for any function $h(\mu)$ and $n>0$,

$$\text{as } \varepsilon^a \to 0: \int_{\widetilde{M}^3} h(\mu)d\mu\wedge\rho_a^n dA = 0, \qquad \int_{\widetilde{M}^3} h(\mu)d\mu\wedge\rho_a^n F_0 = 0, \quad \text{(C.22)}$$

$$\int_{\widetilde{M}^3} h(\mu)d\mu\wedge d(\tau_a\rho_a)\wedge d\varphi^a = 0. \quad \text{(C.23)}$$

When no power of ρ_a appears in the integrand, we have that

$$\text{as } \varepsilon^a \to 0: \qquad \int_{\widetilde{M}^3} h(\mu)d\mu\wedge F_0 = -2\pi\chi(\Sigma_{g,1})\int h(\mu)d\mu. \quad \text{(C.24)}$$

The boundary of the balls B_a^3 is a sphere S_a^2. Integrating over the boundary S_a^2, we also have that

$$\int_{S_a^2} h(\mu)dA = 0, \qquad \int_{S_a^2} h(\mu)d\tau_a\wedge d\varphi^a(\rho_a)^n(\rho_b)^m = 4\pi\delta_{ab}(k^a)^{n+m}h(\mu^a). \quad \text{(C.25)}$$

These integrals are useful when evaluating the anomaly.

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