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Aspects of Scale Invariant Quantum Field Theory

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Aspects of Scale Invariant Quantum Field Theory

A dissertation submitted in partial satisfaction of the
requirements for the degree
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in

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by

Sridip Pal

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The dissertation of Sridip Pal is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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University of California San Diego

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DEDICATION

To my parents and my friends Ahanjit, Brato, Shouvik for being there *Always* .

EPIGRAPH

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.

—Sydney Coleman

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PUBLICATIONS

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ABSTRACT OF THE DISSERTATION

Aspects of Scale Invariant Quantum Field Theory

by

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We study various aspects of scale invariant quantum field theories, in particular, the non-relativistic ones. We explore consequences on symmetry algebra in terms of unitarity bound, existence of monotonic renormalization group flow. We leverage the non relativistic conformal symmetry algebra to construct an operator basis for Heavy Quark Effective Field theory and there by provide a framework for operator counting in this effective field theory. We investigate the large charge sector of non-relativistic conformal theory and in that limit, we find the scaling of operator dimensions with respect to the charge it carries. In the relativistic counterpart, we focus on 2D conformal field theories (CFT) and explore the consequences of modular invariance

of thermal partition function of a 2D CFT. In particular, we investigate the finer details of the asymptotic density of states in 2D CFT using modular invariance and put bounds on asymptotic gap in the spectra.

Chapter 1

Introduction

One of the central theme of the thesis is *Symmetry*. Symmetry is a powerful tool to describe nature. Since every symmetry is associated with a transformation under which the system under consideration looks the same, some physical quantity associated with the system has to be conserved or invariant under such transformation. For example, we can think of a perfectly round ball painted with a color uniformly, it would look exactly the same even when someone rotates the ball. If we look carefully at the figure 1.1, we can spot the rotation only because there is a dot which tells us that we have performed a rotation. Without the dot, the two picture of the ball look exactly the same. Without the dot, we will say the ball is spherically symmetric. The orange dot is what breaks the symmetry. This pattern of being same even when one rotates the object is related to rotational symmetry and the conserved quantity is the angular momentum of the object. In fact, human as a species has always been awed by patterns. Since the dawn of the civilization, we have been striving to find patterns, which is often poetically called finding *order in chaos*. We observe the nature with minute details, we find out patterns in events occurring repeatedly. We have found patterns in the cycle of day and night, the apparent motion of sun across the sky at different time of the year. The take home message in all of these is that whenever there is a pattern, there is a symmetry associated with it. Another example

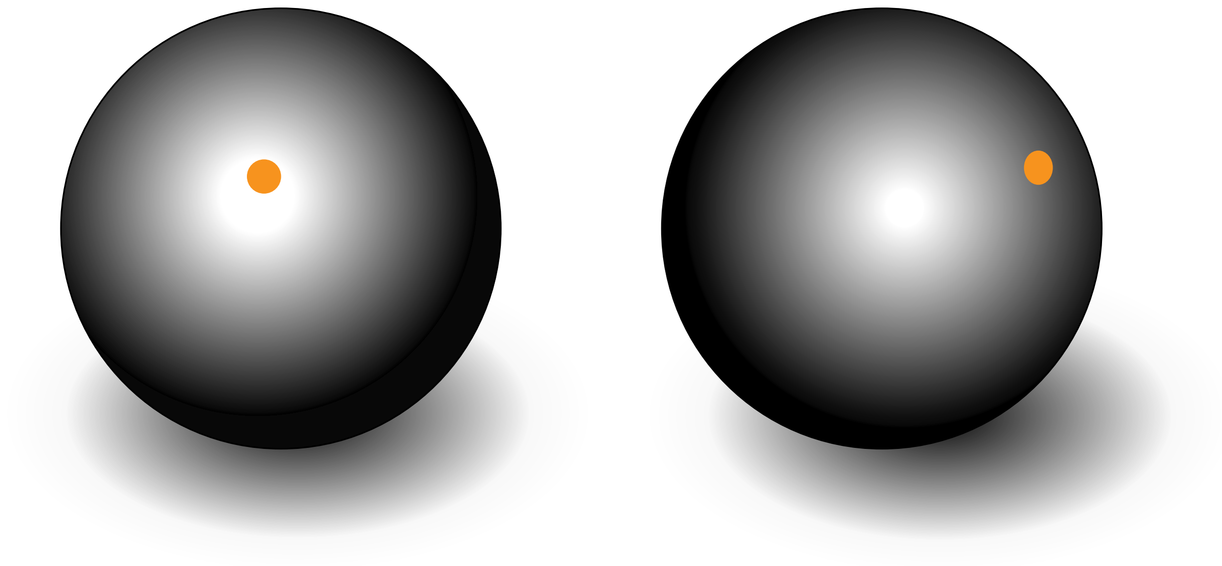

Figure 1.1: The rotation of a sphere with a dot, without the dot, the two scenarios are exactly the same.

closer to the particle physics is the observation that the lowest spin 0 mesons have similar masses and form an octet, this led to the discovery of $SU(3)$ flavor symmetry and hence quarks, the fundamental building block of subatomic elementary particles like proton, neutron etc. In short, symmetry turns out to be instrumental in explaining the nature. Even when one does not find a perfect pattern, one can always study the most symmetric scenario and systematically investigate the deviation from it. Going back to the painted ball example, the orange dot was what caused the deviation from perfect spherical symmetry.

To formalize the notion of symmetry, we define symmetry in the following way: we start with a system, we perform an operation on it, if it stays same or changes in a precise manner, we say that the system is *symmetric under the operation*. Another example, that one can think of and the one that will come along in this thesis over and over again is the scaling of space-time, which basically means zooming in or out while probing a system. Now if we have a self-similar pattern, for example fractals, zooming in or out would not effect the appearance of the system and we have *scale invariance*, see figure 1.2. To give a more intuitive picture, for a reader familiar with dimensional analysis, we can intimately relate the scale invariance to how we assign dimensions

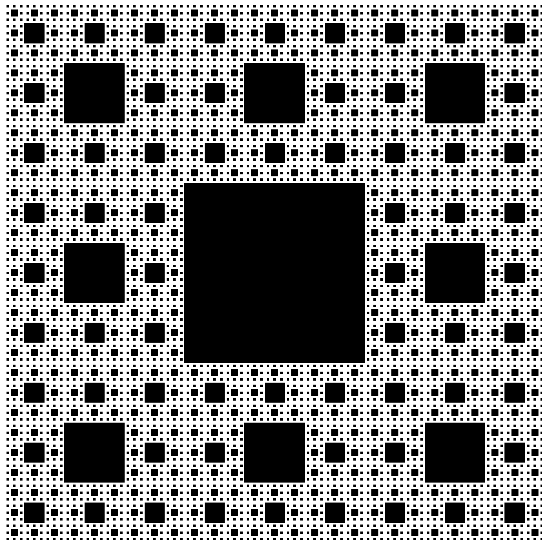

Figure 1.2: The image of a fractal (picture taken from Wikipedia). The fractal has a self similar pattern i.e it looks same whether we zoom in or zoom out.

to various physical quantities and perform dimension analysis. An instructive example is to consider the energy density of an electromagnetic wave, which can also be thought of as a gas of photon confined in a box of length L. We know that the energy should be extensive in volume and should go as L^3. Hence the energy density $\mathcal{E}$ should be independent of L. The only other physical quantity that $\mathcal{E}$ can depend on is temperature T and the fundamental constants: speed of light c and the reduced Planck's constant $\hbar$. From this, one can perform the dimension analysis and deduce that

$$\mathcal{E} = \alpha \frac{k_B T^4}{(\hbar c)^3} \tag{1.1}$$

where α is some undetermined dimensionless number. In the usual units, $k_B, \hbar$ and c are fdimensionful constants, nonetheless, we can choose a unit where they are 1 and we have

$$\mathcal{E} = \alpha T^4 \tag{1.2}$$

We can easily derive the above by using the fact that the theory describing photons in $3+1$D is scale invariant. In order to show that, we implement a scale transformation on space-time which takes $\boldsymbol{x} \to \lambda \boldsymbol{x}$ and $t \to \lambda t$. Now we assign dimensions to various physical quantities, keeping in mind that the $\hbar$ and c are constants and does not scale under such transformation. This leads to the following scaling assignment for $\mathcal{E}$ and temperature T.

$$\mathcal{E} \to \lambda^{-4} \mathcal{E} \tag{1.3}$$

$$T \to \lambda^{-1} T \tag{1.4}$$

Since $\mathcal{E}$ depends on T only, the only way they can be related and respect scale invariance[1] is to satisfy eq. (1.2). The input of scale invariance is coming from the fact there is no other dimensionful parameter which $\mathcal{E}$ can possibly depend on. This example can be thought of as a stepping stone for more sophisticated use of scale invariance that has been undertaken in this thesis. Naively, one can think if a theory does not have any intrinsic dimensionful parameter when we define it, the theory is scale invariant[2]. In principle one can consider more generic scale transformation:

$$x^i \to \lambda^{z_i} x^i \,, \quad t \to \lambda^z t \,. \tag{1.5}$$

If one wishes to preserve rotational invariance, one should also impose the restriction that all the z_i's are equal. In particular, to keep rotational invariance, without loss of generality, we can set $z_i = 1$, we will be interested in the following scale transformation:

$$\boldsymbol{x} \to \lambda \boldsymbol{x} \,, \quad t \to \lambda^z t \tag{1.6}$$

[1]Technically we call it covariance, which means both side of an equation transforms in a similar manner.

[2]A technical remark is in order: if a theory has dimensionful parameter, one can assign a scaling dimension to that parameter and transform it accordingly to restore scale invariance, but to decide whether a theory is invariant or not, we are not allowed to transform its parameter, for example, one can consider a massive scalar field, we say that the mass term breaks the scale invariance, this means that we are treating mass as a parameter and not allowed to assign a scaling dimension to mass and transform it accordingly.

Imposing Lorentz invariance on top of it would force $z = 1$. Unless otherwise mentioned, we will keep z to be arbitrary.

The second key theme of the thesis is *quantum field theory*. To understand quantum field theory, let us first review the concept of field. We start with the basic fact that any physical system can potentially be described by some quantities which evolves under some parameter. For example, we can consider a stream of water flowing through a pipe. We can describe the flow of the water by reading out the velocity of the water at each point along the stream. Thus for each point along the stream, we have a vector describing the motion of the water. This is what we call the velocity field. In general, a physical system can be described by some fields, which assigns numbers or ordered tuple of numbers to each point in space-time. Then the properties of the system are encoded within how the values of the field at different points in space-time are correlated with each other. It so turns out that the field can be quantum in nature i.e. each field can be thought of as a wave form of some quanta i.e. particle. Quantum field theory (QFT) is nothing but a quantized version of field theory which can also encode the particle nature of the wave. QFT is one of the most successful mathematical framework to investigate nature, from the theory of fundamental particles to collective phenomena in condensed matter. The concept of symmetry gets married to the concept of QFT via the assumption that the quantum fields and their correlation functions have to transform in a precise manner if the system that the QFT describes is symmetric under a transformation. As we will be interested in scale invariant field theories, we reiterate that a QFT is said to be scale invariant if we can assign scaling dimension to the fields such that all the correlation functions transforms with a scaling dimension under a scaling of space-time.

The research carried out in this thesis revolves around deeply understanding QFTs using symmetry arguments especially scale invariance. Even though we know a lot about weakly coupled QFTs, strongly coupled field theories are much less known. To provide a sense of strong and weak, let us introduce Alice, she is standing on the floor, the whole enormous sized

Earth is attracting Alice towards his center but Alice is not falling through (for him) because of the normal reaction coming from the floor. This normal reaction is nothing but an interaction, electromagnetic in nature. This, in turn, does show that the electromagnetic interaction is much stronger than the gravity. Similar phenomenon can be seen when in dry weather one can use static electricity to hold small pieces of paper against gravity with the comb.

In technical terms, the research undertaken here focuses primarily on the scale invariant systems, in particular those, described by the non-relativistic (NR) avatars of conformal field theories (a.k.a CFT, a genre of scale invariant theories) in different dimensions and 2D CFTs . From a practical standpoint, the scale invariance plays a pivotal role in understanding critical phenomena such as ferromagnetic phase transition, non relativistic fermions at unitarity, helium near superfluid transitions. The motivation to study scale invariance not only stems from its novel practical applications, but also from two fundamental cornerstones of physics: first of all, QFTs can be understood as renormalization group flows between scale invariant theories, so the latter organizes the space of QFTs, secondly the scale invariant QFTs offer the simplest and most tractable examples of the AdS-CFT correspondence, which offers one of our best hopes of understanding the ambitious holy grail, a nonperturbative theory of quantum gravity.

As mentioned earlier, renormalization group flows end at a point where the scale invariant field theories reside. In the space of relativistic QFTs, renormalization group flows are monotonic, which means that there is a function C, a function of couplings of the theory and the function decreases monotonically as the theory flows from the UV fixed point to the IR fixed point. One way to study/derive this monotonicity property is to couple the theory to gravity, and investigate the divergences in vacuum bubble diagrams following the work of Jack and Osborn for ϕ^4 theory, the work of Grinstein, Stergiou, Stone, Zhong for ϕ^3 theory. We have adapted this formalism and applied it to generic NR scale invariant systems to identify potential C-theorem candidates. The similar formalism got adapted by Auzzi et. al. to study NR scale invariant systems with Galilean boost invariance. This is also known as Schrödinger invariant field theory since the Schrödinger equation is invariant under this symmetry group. We went on showing that the free NR scalar field theory (which is an example of NR scale invariant system with Galilean boost invariance) does not have any Weyl anomaly using the heat kernel method. This work clarified lot of issues and subtleties associated with NR Schrödinger operator and corrected some of the previous results in literature. On this front, the long term goal is to understand the space of non-relativistic QFTs, prove/disprove the C-theorem like statement in non-relativistic scenarios. The chapter 2 and 3 expound on these.

In chapter 2, Weyl consistency conditions have been used in unitary relativistic quantum field theory to impose constraints on the renormalization group flow of certain quantities. We classify the Weyl anomalies and their renormalization scheme ambiguities for generic non-relativistic theories in $2+1$ dimensions with anisotropic scaling exponent $z=2$; the extension to other values of z are discussed as well. We give the consistency conditions among these anomalies. As an application we find several candidates for a C-theorem. We comment on possible candidates for a C-theorem in higher dimensions.

In chapter 3 we propose a method inspired from discrete light cone quantization (DLCQ) to determine the heat kernel for a Schrödinger field theory (Galilean boost invariant with $z=2$ anisotropic scaling symmetry) living in $d+1$ dimensions, coupled to a curved Newton-Cartan background, starting from a heat kernel of a relativistic conformal field theory ($z=1$) living in $d+2$ dimensions. We use this method to show the Schrödinger field theory of a complex scalar field cannot have any Weyl anomalies. To be precise, we show that the Weyl anomaly $\mathcal{A}^G_{d+1}$ for Schrödinger theory is related to the Weyl anomaly of a free relativistic scalar CFT $\mathcal{A}^R_{d+2}$ via $\mathcal{A}^G_{d+1} = 2\pi\delta(m)\mathcal{A}^R_{d+2}$ where m is the charge of the scalar field under particle number symmetry. We provide further evidence of vanishing anomaly by evaluating Feynman diagrams in all orders of perturbation theory. We present an explicit calculation of the anomaly using a regulated Schrödinger operator, without using the null cone reduction technique. We generalize our method to show that a similar result holds for one time derivative theories with even $z>2$.

In chapter 4 we prove a no-go theorem for the construction of a Galilean boost invariant and $z \neq 2$ anisotropic scale invariant field theory with a finite dimensional basis of fields. Two point correlators in such theories, we show, grow unboundedly with spatial separation. Correlators of theories with an infinite dimensional basis of fields, for example, labeled by a continuous parameter, do not necessarily exhibit this bad behavior. Hence, such theories behave effectively as if in one extra dimension. Embedding the symmetry algebra into the conformal algebra of one higher dimension also reveals the existence of an internal continuous parameter. Consideration of isometries shows that the non-relativistic holographic picture assumes a canonical form, where the bulk gravitational theory lives in a space-time with one extra dimension. This can be contrasted with the original proposal by Balasubramanian and McGreevy, and by Son, where the metric of a $d+2$ dimensional space-time is proposed to be dual of a d dimensional field theory. We provide explicit examples of theories living at fixed point with anisotropic scaling exponent $z = \frac{2\ell}{\ell+1}, \ell \in \mathbb{Z}$.

The chapters $5-9$ deal with a special non-relativistic (NR) avatars of conformal field theories, henceforth we call them NRCFT. NRCFT is a non relativistic scale invariant theory with Galilean boost invariance. It also admits another special symmetry known as special conformal invariance. The cutting edge experiments involving cold atoms provide us an excellent opportunity to realize systems like fermions at unitarity, which exhibits non-relativistic conformal invariance. One can measure the energy of ground state and excited states of these fermionic systems (e.g. systems consisting of $^{6}Li, {}^{39}K, {}^{133}Cs$) trapped in a harmonic potential. Since the energy is directly related to the scaling dimension of non-relativistic operators of underlying NRCFT, describing the system, we have a unique platform where one as a theorist can study how the symmetry constrains various physical data of NRCFT and subsequently experimentalists can verify them in a real life experiment. The research on this front aims at extracting these NRCFT data using symmetry principles on one hand while on the other hand it strives to connect these results to real life experiments.

In what follows, a novel connection between 1D CFT and non-relativistic CFT (NRCFT) to explore the neutral sector (containing physically important operators like current, charge density, Hamiltonian), which was otherwise unexplored due to subtleties in the representation theory of NRCFT. The connection arises because of the $SL(2,\mathbb{R})$ subgroup of the NR conformal group. The $SL(2,\mathbb{R})$ acts on the time co-ordinate and thus all the equal space correlators in NRCFT behave like the ones in 1D CFT. The work has shown the possibility of the use of 1D CFT to study NRCFT (bootstrapping NRCFT, deriving sum-rules, nonunitarity in fractional dimensions) and thus merits further exploration. This is specially relevant as 1D CFT has recently been analytically studied by Paulos et. al. and exact results have been derived. The connection with NRCFT allows us to investigate the implications of these fascinating exact results from 1D CFT in NRCFT.

In chapter 5 we relate the notion of unitarity of a $(0+1)$-D conformally ($SL(2,\mathbb{R})$) invariant field theory with that of a non-relativistic conformal (Schrödinger) field theory using the fact that $SL(2,\mathbb{R})$ is a subgroup of non-relativistic conformal (Schrödinger) group. Exploiting $SL(2,\mathbb{R})$ unitarity, we derive the unitarity bounds and null conditions for a Schrödinger field theory (for the neutral as well as the charged sector). In non integer dimensions the theory is shown to be non-unitary. The use of $SL(2,\mathbb{R})$ subgroup opens up the possibility of borrowing results from $(0+1)$-D $SL(2,\mathbb{R})$ invariant field theory to explore Schrödinger field theory, in particular, the neutral sector, which has otherwise been unexplored. This viewpoint of organizing the operator content of Schrödinger invariant field theory in terms of $SL(2,\mathbb{R})$ finds natural application in heavy quark effective field theory which we explore in chapter 6&7.

In chapter 6 we use a Hilbert series to construct an operator basis in the $1/m$ expansion of a theory with a nonrelativistic heavy fermion in an electromagnetic (NRQED) or color gauge field (NRQCD/HQET). We present a list of effective operators with mass dimension $d \leq 8$. Comparing to the current literature, our results for NRQED agree for $d \leq 8$, but there are some discrepancies in NRQCD/HQET at $d = 7$ and 8. In chapter 7 an operator basis of an effective theory with a heavy particle, subject to external gauge fields, is spanned by a particular kind of neutral scalar primary of the non-relativistic conformal group. We calculate the characters that can be used for generating the operators in a non-relativistic effective field theory, which accounts for redundancies from the equations of motion and integration by parts.

In chapter 8 we study Schrödinger invariant field theories (nonrelativistic conformal field theories) in the large charge (particle number) sector. We do so by constructing the effective field theory (EFT) for a Goldstone boson of the associated $U(1)$ symmetry in a harmonic potential. This EFT can be studied semi-classically in a large charge expansion. We calculate the dimensions of the lowest lying operators, as well as correlation functions of charged operators. We find universal behavior of three point function in large charge sector. We comment on potential applications to fermions at unitarity and critical anyon systems.

In chapter 9, we study operators in Schrödinger invariant field theories (non-relativistic conformal field theories or NRCFTs) with large charge (particle number) and spin. Via the state-operator correspondence for NRCFTs, such operators correspond to states of a superfluid in a harmonic trap with phonons or vortices. Using the effective field theory of the Goldstone mode, we compute the dimensions of operators to leading order in the angular momentum L and charge Q. We find a diverse set of scaling behaviors for NRCFTs in both $d = 2$ and $d = 3$ spatial dimensions. These results apply to theories with a superfluid phase, such as unitary fermions or critical anyon systems.

In chapter 10, we switch gears and focus on 2D conformal field theories. In 2 dimensions, the conformal algebra can be extended to a bigger infinite dimensional algebra known as *Virasoro* algebra. The infinite dimensionality of symmetry algebra provides us with immense control over the physical theories invariant under such algebra. On top of that, the thermal partition function of 2D CFTs on a spatial circle are modular invariant i.e. invariant under the exchange of thermal and spatial circle. This, in turn, relates the low temperature behavior to the high temperature behavior of partition function. Since the high temperature behavior is controlled by the density of states at high energy, the modular invariance reveals the density of states of a 2D CFT at high energy. This is known as *Cardy* formula. The mathematical methodology involves estimation and bounds of Laplace transform of distributions and comes under the umbrella of *Tauberian* theorems. The Cardy formula is an asymptotic expression for the density of states, that can be derived rigorously using Tauberian theorems. The basic features of Tauberian theorems can be explained in an elementary manner. The idea comes from an attempt to assign a sum to otherwise not summable series, where by not summable we mean where the partial sum S_n sum does not converge as $n \to \infty$. To cure this, what one does is to construct a hierarchy of notion of summability. Going up the hierarchy lets one *sum* series which are not summable in the lower hierarchy. The branch of mathematics dealing with these is Tauberian theory. To be more explicit, let us consider the sum $\sum_{k=0}(-1)^k(k+1)$, which is evidently not summable in the usual sense. Now one defines

$$f(\beta) \equiv \sum_{k=0} (-1)^k (k+1) e^{-n\beta} \tag{1.7}$$

For $\beta > 0$, however, this is summable in the usual sense and we find

$$f(\beta) = \frac{e^{2\beta}}{\left(e^{\beta}+1\right)^2} \tag{1.8}$$

It's easy to see, $e^{2\beta}/\left(e^{\beta}+1\right)^2$ is well defined at $\beta = 0$ and equals to $1/4$. Thus, one can say $f(\beta)$ goes to $1/4$ as β goes to 0. Now one defines,

$$\sum_{k=0} (-1)^k (k+1) \underbrace{=}_{\text{New notion of sum}} \frac{1}{4}. \tag{1.9}$$

This new notion of sum is called *Abel sum*. It's easy to see that summability in *Abel sense* follows directly from summability in the usual sense, but not the other way around. The Tauberian theorems specify the conditions under which the higher notion of summability (e.g. Abel summability) implies the lower notion of summability (e.g. the usual summability). In its generalized version, one can deduce the asymptotic behavior of the usual sum from the asymptotic behavior of *Abel sum*. We will be interested in partition function, which is Laplace transform of density of states and the Laplace transform is nothing but the continuous version of *Abel sum*. In particular, we will be interested in

$$Z(\beta) = \int d\Delta\, \rho(\Delta)\, e^{-(\Delta - c/12)\beta}, \qquad \rho(\Delta) = \sum_k \delta(\Delta - \Delta_k), \tag{1.10}$$

where β is the inverse temperature. Now, knowing the form of $Z(\beta)$ as β goes to 0 enables us to deduce the asymptotic form of $\int_0^\Delta d\Delta' \rho(\Delta')$ as Δ goes to ∞. The power of Tauberian theory comes from the fact that one does not have to impose any regularity condition on $\rho(\Delta)$.

The chapter 10 gives a brief introduction to the Cardy formula and motivates the use of sophisticated machinery of Tauberian theory. The chapter 11 expounds on finer details and subleading correction to this formula by methods inspired from Tauberian theorems. In particular, we improve the existing bounds on the $O(1)$ correction to the Cardy formula for the density of states in 2 dimensional conformal field theory at high energy. We prove a conjectured upper bound on the asymptotic gap between two consecutive Virasoro primaries for a central charge greater than 1, demonstrating it to be 1. Furthermore, a systematic method is provided to establish a limit on how tight the bound on the $O(1)$ correction to the Cardy formula can be made using bandlimited functions. The techniques and the functions used here are of generic importance whenever the Tauberian theorems are used to estimate some physical quantities.

Chapter 2

Weyl Consistency Conditions in Non-Relativistic Quantum Field Theory

Weyl consistency conditions have been used in unitary relativistic quantum field theory to impose constraints on the renormalization group flow of certain quantities. We classify the Weyl anomalies and their renormalization scheme ambiguities for generic non-relativistic theories in $2+1$ dimensions with anisotropic scaling exponent $z=2$; the extension to other values of z are discussed as well. We give the consistency conditions among these anomalies. As an application we find several candidates for a C-theorem. We comment on possible candidates for a C-theorem in higher dimensions.

2.1 Introduction

Aspects of the behavior of systems at criticality are accessible through renormalization group (RG) methods. Famously, most critical exponents are determined by a few anomalous dimensions of operators. However, additional information, such as dynamical (or anisotropic) exponents and amplitude relations can be accessed via renormalization group methods near but not strictly at criticality. Far away from critical points there are often other methods, *e.g.*, mean field approximation, that can give more detailed information. The renormalization group used away from critical points can valuably bridge the gap between these regions.

Systems of non-relativistic particles at unitarity, in which the S-wave scattering length diverges, $|a| \to \infty$, exhibit non-relativistic conformal symmetry. Ultracold atom gas experiments have renewed interest in study of such theories. In these experiments one can freely tune the S-wave scattering length along an RG flow [1, 2]: at $a^{-1} = -\infty$ the system is a BCS superfluid while at $a^{-1} = \infty$ it is a BEC superfluid. The BCS-BEC crossover, at $a^{-1} = 0$, is precisely the unitarity limit, exhibiting conformal symmetry. This is a regime where universality is expected, with features independent of any microscopic details of the atomic interactions. Other examples of non-relativistic systems with accidentally large scattering cross section include few nucleon systems like the deuteron [3, 4] and several atomic systems, including ^{85}Rb[5],^{138}Cs [6], ^{39}K [7].

In the context of critical dynamics the response function exhibits dynamical scaling. This is characterized by a dynamical scaling exponent which characterizes anisotropic scaling in the time domain. There has been recent interest in anisotropic scaling in systems that are non-covariant extensions of relativistic systems. The ultraviolet divergences in quantized Einstein gravity are softened if the theory is modified by inclusion of higher derivative terms in the Lagrangian. Since time derivatives higher than order 2 lead to the presence of ghosts,[1] Horava suggested extending Einstein gravity by terms with higher spatial derivatives but only order-2 time derivatives [13]. The mismatch in the number of spatial versus time derivatives is a version of anisotropic scaling, similar to that found in the non-relativistic context. This has motivated studies of extensions of relativistic quantum field theories that exhibit anisotropic scaling at short distances. Independently, motivated by the study of Lorentz violating theories of elementary

[1]Generically, the S-matrix in models with ghosts is not unitary. However, under certain conditions on the spectrum of ghosts and the nature of their interactions, a unitary S-matrix is possible [8, 9, 10, 11]. In theories of gravity Hawking and Hertog have proposed that ghosts lead to unitarity violation at short distances, and unitarity is a long-distance emergent phenomenon [12].

particle interactions [14], Anselmi found a critical point with exact anisotropic scaling, a so-called Lifshitz fixed point, in his studies of renormalization properties of interacting scalar field theories [15]; see Refs. [16, 17] for the case of gauge theories. Anomalous breaking of anisotropic scaling symmetry in the quantum Lifshitz model has been studied in Ref. [18, 19, 20, 21, 22]; see also Ref. [23] for an analysis using holographic methods.

Wess-Zumino consistency conditions for Weyl transformations have been used in unitary relativistic quantum field theory to impose constraints on the renormalization group flow of Weyl anomalies [24]. In 1+1 dimensions a combination of these anomalies gives Zamolodchikov's C-function [25], that famously decreases monotonically along flows towards long distances, is stationary at fixed points and equals the central charge of the 2D conformal field theory at the fixed point boundaries of the flow. Weyl consistency conditions can in fact be used to recover this result [24]. Along the same lines, in 3+1 dimensions Weyl consistency conditions can be used to show that a quantity $\tilde{a}$ satisfies

$$\mu\frac{d\tilde{a}}{d\mu} = \mathcal{H}_{\alpha\beta}\beta^{\alpha}\beta^{\beta} \tag{2.1}$$

where μ is the renormalization group scale, increasing towards short distances. The equation shows that at fixed points, characterized by $\mu dg^{\alpha}/d\mu \equiv \beta^{\alpha} = 0$, $\tilde{a}$ is stationary. It can be shown in perturbation theory that $\mathcal{H}_{\alpha\beta}$ is a positive definite symmetric matrix [26]. By construction the quantity $\tilde{a}$ is, at fixed points, the conformal anomaly a of Cardy, associated with the Euler density conformal anomaly when the theory is placed in a curved background [27]. This is then a 4-dimensional generalization of Zamolodchikov's C function, at least in perturbation theory. Going beyond 4 dimensions, Weyl consistency conditions can be used to show that in $d = 2n$ dimensions there is a natural quantity that satisfies (2.1), and that this quantity is at fixed points the anomaly associated with the d-dimensional Euler density [28]. Concerns about the viability of a C-theorem in 6-dimensions were raised by explicit computations of "metric" $\mathcal{H}_{\alpha\beta}$

in perturbation theory [29, 30, 31]. However it was discovered in Ref. [32] that there exists a one parameter family of extensions of the the quantity $\tilde{a}$ of Ref. [28] that obey a C-theorem perturbatively.

Weyl consistency conditions can also be used to constrain anomalies in non-relativistic field theories. The constraints imposed at fixed points have been studied in Ref. [18] for models with anisotropic scaling exponent $z = 2$ in 2-spatial dimensions; see Refs. [33, 34] for studies of the Weyl anomaly at $d = 4, z = 3$ and $d = 6$. Here we investigate constraints imposed along renormalization group flows. We recover the results of [18] by approaching the critical points along the flows. As mentioned above, there are questions that can only be accessed through the renormalization group methods applied to flows, away from fixed points. The additional information obtained from consideration of Weyl consistency conditions on flows can be used to ask a number of questions. For example, we may ask if there is a suitable candidate for a C-theorem.

A related issue is the possibility of recursive renormalization group flows. Recursive flows in the perturbative regime have been found in several examples in $4-\varepsilon$ and in 4 dimensional relativistic quantum field theory [35, 36, 37, 38, 39, 40]. Since Weyl consistency conditions imply $\tilde{a}$ does not increase along RG-flows it must be that $\tilde{a}$ remains constant along recursive flows. This can be shown directly, that is, without reference to the monotonicity of the flow; see [40]. In fact one can show that on recursive flows all physical quantities, not just $\tilde{a}$, remain constant: the recursive flow behaves exactly the same as a single fixed point. This is as it should be: the monotonicity of the flow of a implies that limit cycles do not exist in any physically meaningful sense [41, 42]; in fact, they may be removed by a field and coupling constant redefinition. However, it is well known that bona-fide renormalization group limit cycles exist in some non-relativistic theories [43, 44, 45]. The C-theorem runs afoul of limit-cycles, and an immediate question then is what invalidates it in models that exhibit recursive flows? Our analysis indicates some potential candidates for C-theorems but does not show whether generically the "metric" $\mathcal{H}_{\alpha\beta}$ has definite sign. The question of under what conditions the metric has definite sign, precluding recursive flows, is left open for further investigation.

The paper is organized as follows. In Sec.2.2 we set-up the computation, using a background metric and space and time dependent coupling constants that act as sources of marginal operators. In the section we also clarify the relation between the dynamical exponent and the classical anisotropic exponent. We then use this formalism in Sec. 2.3 where we analyze the consistency conditions for the case of 2-spatial dimensions and anisotropic exponent $z = 2$. The Weyl consistency conditions and scheme dependent ambiguities are lengthy, so they are collected in Apps. A.1 and A.2. In Sec. 2.4 we explore the case of arbitrary z, extending some of the results of the previous section and in Sec. 2.5 we propose a candidate C-theorem for any even spatial dimension. We offer some general conclusions and review our results in Sec. 2.6. There is no trace anomaly equation for the case of zero spatial derivatives, that is, particle quantum mechanics; we comment on this, and present a simple but useful theorem that does apply in this case, in the final appendix, App. A.3.

2.2 Generalities

We consider non-relativistic (NR) field theories with point-like interactions. Although not necessary for the computation of Weyl consistency conditions, it is convenient to keep in mind a Lagrangian description of the model. The Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, m, g)$ is a function of fields $\phi(t, \boldsymbol{x})$, mass parameters m and coupling constants g that parametrize interaction strengths. We restrict our attention to models for which the action integral,

$$S[\phi(\boldsymbol{x},t)] = \int dt\, d^d x\, \mathcal{L}$$

remains invariant under the rescaling

$$\boldsymbol{x} \mapsto \lambda \boldsymbol{x}, \qquad t \mapsto \lambda^z t, \tag{2.2}$$

that is,

$$S[\lambda^{\Delta}\phi(\lambda\boldsymbol{x},\lambda^z t)] = S[\phi(\boldsymbol{x},t)]\,.$$

Here Δ is the matrix of canonical dimensions of the fields ϕ. In a multi-field model the anisotropic scaling exponent z is common to all fields. Moreover, assuming that the kinetic term in $\mathcal{L}$ is local, so that it entails powers of derivative operators, z counts the mismatch in the number of time derivatives and spatial derivatives. In the most common cases there is a single time derivative and z spatial derivatives so that z is an integer.

For a simple example, useful to keep in mind for orientation, the action for a single complex scalar field with anisotropic scaling z in d dimensions is given by

$$S = \int dt\, d^d x\, \left[im\phi^* \overleftrightarrow{\partial_t} \phi - \boldsymbol{\nabla}_{i_1} \cdots \boldsymbol{\nabla}_{i_{z/2}} \phi^* \boldsymbol{\nabla}_{i_1} \cdots \boldsymbol{\nabla}_{i_{z/2}} \phi - g m^{z/d} |\phi|^{2N} \right], \tag{2.3}$$

where z is an even integer so that the Lagrangian density is local. If $N = 1 + z/d$ the scaling property (2.2) holds with $\Delta = d/2$ (alternatively, if $N \in \mathbb{Z}$, then $z = d(N-1) \in d\mathbb{Z}$). When (2.2) holds the coupling constant g is dimensionless. The mass parameters m have dimensions of T/L^z, where T and L are time and space dimensions, respectively. One may use the mass parameter to measure time in units of z-powers of length, and this can be implemented by absorbing m into a redefinition, $t = m\hat{t}$. In multi-field models one can arbitrarily choose one of the masses to give the conversion factor and then the independent mass ratios are dimensionless parameters of the model. In models that satisfy the scaling property (2.2), these mass ratios together with the coefficients of interaction terms comprise the set of dimensionless couplings that we denote by g^{α} below.

The above setup is appropriate for studies of, say, quantum criticality. However the calculations we present are applicable to studies of thermal systems in equilibrium since the imaginary time version of the action integral is equivalent to an energy functional in $d+1$ spatial dimensions. Taking $t = -iy$ in the example of Eq. (2.3) the corresponding energy integral is

$$H = \int dy\, d^d x \left[m\phi^* \overleftrightarrow{\partial_y} \phi + \boldsymbol{\nabla}_{i_1} \cdots \boldsymbol{\nabla}_{i_{z/2}} \phi^* \boldsymbol{\nabla}_{i_1} \cdots \boldsymbol{\nabla}_{i_{z/2}} \phi + g m^{z/d} |\phi|^{2N} \right].$$

The short distance divergences encountered in these models need to be regularized and renormalized. Although our results do not depend explicitly on the regulator used, it is useful to keep in mind a method like dimensional regularization that retains most symmetries explicitly. Thus we consider NR field theories in $1+n$ dimensions, where the spatial dimension $n = d - \varepsilon$, with d an integer. Dimensional regularization requires the introduction of a parameter μ with dimensions of inverse length, L^{-1}. Invariance under (2.2) is then broken, but can be formally recovered by also scaling μ appropriately, $\mu \mapsto \lambda^{-1}\mu$. For an example, consider the dimensionally regularized version of (2.3):

$$S[\phi_0(\boldsymbol{x},t);\mu] = \int dt\, d^n x \left[i m_0 \phi_0^* \overleftrightarrow{\partial_t} \phi_0 - \boldsymbol{\nabla}_{i_1} \cdots \boldsymbol{\nabla}_{i_{z/2}} \phi_0^* \boldsymbol{\nabla}_{i_1} \cdots \boldsymbol{\nabla}_{i_{z/2}} \phi_0 - g Z_g m_0^{z/d} \mu^{k\varepsilon} |\phi_0|^{2N} \right]. \tag{2.4}$$

We have written this in terms of bare field and mass, ϕ_0 and m_0, and have given the bare coupling constant explicitly in terms of the renormalized one, $g_0 = \mu^{k\varepsilon} Z_g g$. The coefficient $k = N - 1 = z/d$ is dictated by dimensional analysis. It follows that

$$S[\lambda^{n/2}\phi_0(\lambda\boldsymbol{x}, \lambda^z t); \lambda^{-1}\mu] = S[\phi_0(\boldsymbol{x},t);\mu] \tag{2.5}$$

In order to study the response of the system to sources that couple to the operators in the interaction terms of the Lagrangian, we generalize the coupling constants g^α to functions of space and time $g^\alpha(t,\boldsymbol{x})$. One can then obtain correlation functions of these operators by taking functional derivatives of the partition function with respect to the space-time dependent couplings, and then setting the coupling functions to constant values. Additional operators of interest are obtained by placing these systems on a curved background, with metric $\gamma_{\mu\nu}(t,\boldsymbol{x})$. One can then obtain correlations including components of the stress-energy tensor by taking functional derivatives with respect to the metric and evaluating these on a trivial, constant metric. For example, we then can define the components of the symmetric quantum stress energy tensor and finite composite operators in the following way:

$$T_{\mu\nu} = \frac{2}{\sqrt{\gamma}}\frac{\delta S_0}{\delta\gamma^{\mu\nu}} \qquad [O_\alpha] = \frac{1}{\sqrt{\gamma}}\frac{\delta S_0}{\delta g^\alpha} \tag{2.6}$$

The square bracket notation in the last term indicates that these are finite operators, possibly differing from $O_\alpha = \partial\mathcal{L}/\partial g^\alpha$ by a total derivative term.

Time plays a special role in theories with anisotropic scaling symmetry. Hence, it is useful to assume the background space-time, in addition to being a differential manifold $\mathcal{M}$, carries an extra structure — we can foliate the space-time with a foliation of co-dimension 1. This can be thought of a topological structure on $\mathcal{M}$ [13], before any notion of Riemannian metric is introduced on such manifold. Now the co-ordinate transformations that preserve the foliation are of the form:

$$t \mapsto \tau(t),\; x^i \mapsto \xi^i(\boldsymbol{x},t) \tag{2.7}$$

We will also assume the space-time foliation is topologically given by $\mathcal{M} = R\times\Sigma$. The foliation can be given Riemannian structure with three basic objects: h_{ij}, N_i and N. This is the ADM decomposition of the metric — one can generally think as writing the metric in terms of lapse and shift functions, $N(t,\boldsymbol{x})$ and $N_i(t,\boldsymbol{x})$, and a metric on spatial sections, $h_{ij}(t,\boldsymbol{x})$:

$$ds^2 = \gamma_{\mu\nu}dx^\mu dx^\nu = N^2dt^2 + 2N_i dt dx^i - h_{ij}dx^i dx^j \tag{2.8}$$

Here and below the latin indices run over spatial coordinates, $i, j = 1, \ldots, d$. We assume invariance of the theory under foliation preserving diffeomorphisms. In a non-relativistic set up, it is convenient to remove the shift N^i by a foliation preserving map $t \mapsto \tau(t)$ and $x^i \mapsto \xi^i(\boldsymbol{x}, t)$. The metric is then given by

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = N^2 dt^2 - h_{ij} dx^i dx^j \tag{2.9}$$

Once the shift functions are removed the restricted set of diffeomorphisms that do not mix space and time are allowed, $t \to \tau(t)$ and $x^i \to \xi^i(x)$, so that $N^i = 0$ is preserved.

In Euclidean space, the generating functional of connected Green's functions W is given by

$$e^W = \int [d\phi]\, e^{-S_0 - \Delta S}. \tag{2.10}$$

The action integral for these models is generically of the form

$$S_0 = \int dt\, d^n x N \sqrt{h} \mathcal{L}_0 , \tag{2.11}$$

where $h = \det(h_{ij})$. We have denoted by $\mathcal{L}_0$ the Lagrangian density with bare fields and couplings as arguments; these are to be expressed in terms of the renormalized fields and couplings, so as to render the functional integral finite. The term ΔS contains additional counter-terms that are solely functionals of g^α and $\gamma_{\mu\nu}$ that are also required in order to render W finite. In a curved background the scaling (2.2) can be rephrased in terms of a transformation of the metric,

$$N(\boldsymbol{x}, t) \mapsto \lambda^z N(\boldsymbol{x}, t) , \qquad h_{ij}(\boldsymbol{x}, t) \mapsto \lambda^2 h_{ij}(\boldsymbol{x}, t) . \tag{2.12}$$

Then the generalization of the formal invariance of Eq. (2.5) is

$$S_0[\lambda^z N(\boldsymbol{x}, t), \lambda^2 h_{ij}(\boldsymbol{x}, t), \lambda^{\Delta_0} \phi_0(\boldsymbol{x}, t); \lambda^{-1}\mu] = S_0[N(\boldsymbol{x}, t), h_{ij}(\boldsymbol{x}, t), \phi_0(\boldsymbol{x}, t); \mu] \tag{2.13}$$

for a suitable matrix of canonical dimensions Δ_0 of the bare fields (appropriate to $n = d - \varepsilon$ spatial dimensions).

We assume that when introducing a curved background the action integral is suitably modified so that the formal symmetry of Eq. (2.13) holds locally, that is, it holds when replacing $\lambda \to \exp(-\sigma(\boldsymbol{x},t))$. The modification to the action integral consists of additional terms that couple the fields ϕ to the background curvature.

For example, the model in Eq. (2.4) for $z = 2$ is modified to include, in addition to coupling to a background metric, additional terms

$$\int dt\, d^n x N\sqrt{h}\left[im_0\xi_K\phi_0^*\phi_0 K + \xi_{N\phi}\left(\phi_0^*\frac{\partial_i N}{N}\partial^i\phi_0 + \phi_0\frac{\partial_i N}{N}\partial^i\phi_0^*\right) + \xi_{NN}\frac{\partial_i N}{N}\frac{\partial^i N}{N}\phi_0^*\phi_0 + \xi_R R\phi_0^*\phi_0\right].$$

Here $K_{ij} = \frac{1}{2}\partial_t h_{ij}/N$ is the extrinsic curvature of the $t=$ constant hypersurfaces in the $N^i = 0$ gauge and $K = h^{ij}K_{ij}$ (with h^{ij} the inverse of the metric h_{ij}), and R is the d-dimensional Ricci scalar for the metric h_{ij}. Under the transformation (2.12) with $\lambda = \exp(-\sigma)$ one has $K \to e^{2\sigma}(K + n\partial_t\sigma/N)$, $R \to e^{2\sigma}(R + 2(n-1)\nabla^2\sigma - (n-1)(n-2)\nabla_i\sigma\nabla^i\sigma)$ and $N \to e^{-2\sigma}N$, so that choosing $\xi_K = 1/2$ and ensuring

$$2(n-1)\xi_R + 2\xi_{N\phi} + \frac{n}{2} = 0 \qquad (n+2)\xi_{N\phi} - 4\xi_{NN} + \frac{n}{2} = 0, \tag{2.14}$$

the action integral remains invariant. Thus, we have a one parameter family of parameters that preserves invariance of the action under anisotropic scaling. For arbitrary even z and arbitrary spatial dimension n, in the example (2.4) we first integrate by parts the spatial covariant derivatives:

$$\boldsymbol{\nabla}_{i_1}\cdots\boldsymbol{\nabla}_{i_{z/2}}\phi^*\boldsymbol{\nabla}_{i_1}\cdots\boldsymbol{\nabla}_{i_{z/2}}\phi \to (-1)^{z/2}\phi^*(\nabla^2)^{z/2}\phi\,.$$

Then we replace the operator $(\nabla^2)^{\frac{z}{2}}$ by $O^{(n+2z-4)}O^{(n+2z-8)}\cdots O^{(n+4)}O^{(n)}$ with $O^{(p)}$ defined as

$$O^{(p)} \equiv \left[\nabla^2 - \frac{p}{4(n-1)}R + \frac{2+p-n}{z}\frac{\partial_i N}{N}h^{ij}\partial_j + \frac{n}{4z^2}(2+p-n)\frac{\partial_i N}{N}h^{ij}\frac{\partial_j N}{N}\right] \tag{2.15}$$

Under $h_{ij} \to e^{-2\sigma}h_{ij}$, $N \to e^{-z\sigma}N$ and $\psi \to e^{\frac{p}{2}\sigma}\psi$, this operator transform covariantly, in the sense that

$$O^{(p)}\psi \to e^{(\frac{p}{2}+2)\sigma}O^{(p)}\psi. \tag{2.16}$$

Hence, under the Weyl rescaling $h_{ij} \to e^{-2\sigma}h_{ij}$, $N \to e^{-z\sigma}N$ and $\phi \to e^{\frac{n}{2}\sigma}\phi$ we have following, transforming covariantly

$$\phi_0^* O^{(n+2z-4)}O^{(n+2z-8)}\cdots O^{(n+4)}O^{(n)}\phi_0 \to e^{(n+z)\sigma}\phi_0^* O^{(n+2z-4)}O^{(n+2z-8)}\cdots O^{(n+4)}O^{(n)}\phi_0 \tag{2.17}$$

For $z=2$, this construction gives

$$N\sqrt{h}\phi_0^* O^{(n)}\phi_0 = N\sqrt{h}\phi_0^*\left[\nabla^2 - \frac{n}{4(n-1)}R + \frac{\partial_i N}{N}h^{ij}\partial_j + \frac{n}{8}\frac{\partial_i N}{N}h^{ij}\frac{\partial_j N}{N}\right]\phi_0 \tag{2.18}$$

$$= N\sqrt{h}\left[-\partial_i\phi_0^*\partial^i\phi_0 - \frac{n}{4(n-1)}R\phi_0^*\phi_0 + \frac{n}{8}\frac{\partial_i N}{N}h^{ij}\frac{\partial_j N}{N}\phi_0^*\phi_0\right] \tag{2.19}$$

This solves Eq. (2.14) with

$$\xi_R = -\frac{n}{4(n-1)}, \qquad \xi_{N\phi} = 0, \qquad \xi_{NN} = \frac{n}{8}. \tag{2.20}$$

The extra freedom for $z=2$ arises from the fact that $\phi_0^*\left[R + (n-1)\frac{\nabla^2 N}{N} - \frac{(n-1)(n+2)}{4}\frac{\partial_i N}{N}\frac{\partial^i N}{N}\right]\phi_0$ is Weyl invariant. This special invariant quantity is available only for $z=2$.

Having constructed a classically Weyl invariant curved space action, we have that

$\tilde{W} = W - W_{\text{c.t.}} = W + \Delta S$ is invariant under these local transformations:

$$\tilde{W}[e^{-z\sigma}N, e^{-2\sigma}h_{ij}, g^\alpha(e^{-\sigma}\mu)] = \tilde{W}[N, h_{ij}, g^\alpha(\mu)] \tag{2.21}$$

We have suppressed the explicit dependence on space and time and have assumed the only dependence on the renormalization scale μ is implicitly through the couplings: using μ-independence of bare couplings, $g_0 = \mu^{k\varepsilon} g(\mu) Z_g(g(\mu)) = (\lambda\mu)^{k\varepsilon} g(\lambda\mu) Z_g(g(\lambda\mu))$ so that $(\lambda^{-1}\mu)^{k\varepsilon} g(\mu) Z_g(g(\mu)) = \mu^{k\varepsilon} g(\lambda\mu) Z_g(g(\lambda\mu))$.

The generating functional W is not invariant in the sense of Eq. (2.21). The anomalous variation of W arises purely from the counter-terms: under an infinitesimal transformation,

$$\begin{aligned}\Delta_\sigma W &= W_{\text{c.t.}}[(1-z\sigma)N, (1-2\sigma)h_{ij}, g^\alpha - \sigma\mu\, dg^\alpha/d\mu] - W_{\text{c.t.}}[N, h_{ij}, g^\alpha] \\ &= \int dt\, d^dx\, N\sqrt{h}\big(\text{terms with derivatives on } N,\ h_{ij},\ g^\alpha \text{ and } \sigma\big)\end{aligned} \tag{2.22}$$

does not vanish. Using Eqs. (2.6) and choosing σ to be an infinitesimal local test function, this reads

$$z\langle T^0{}_0\rangle + \langle T^i{}_i\rangle - \beta^\alpha\langle[O_\alpha]\rangle = \big(\text{terms with derivatives on } N,\ h_{ij},\ g^\alpha \text{ and } \sigma\big)\,. \tag{2.23}$$

Evaluating at space and time independent coupling constants and on a flat metric, so that the right hand side vanishes, we recognize this as the trace anomaly for NRQFT.

Since the Weyl group is Abelian, consistency conditions follow from requiring that

$$[\Delta_\sigma, \Delta_{\sigma'}]\, W = 0\,. \tag{2.24}$$

These consistency conditions impose relations on the various anomaly terms on the right hand side of Eq. (2.22). In the following sections we classify all possible anomaly terms and derive the relations imposed by these conditions.

2.2.1 Dynamical exponent

In the theory of critical phenomena the dynamical exponent ζ characterizes how a correlation length scales with time in time dependent correlations. At the classical level (the gaussian fixed point) this just corresponds to the anisotropic exponent z introduced above. To understand the connection between these we must retain explicitly the dependence on the mass parameter(s) m in Eqs. (2.13) and (2.21). We consider for simplicity the case of a single mass parameter. In particular, we have

$$\tilde{W}[e^{-z\sigma}N, e^{-2\sigma}h_{ij}, g^{\alpha}(e^{-\sigma}\mu), m(e^{-\sigma}\mu)] = \tilde{W}[N, h_{ij}, g^{\alpha}(\mu), m(\mu)]\,. \tag{2.25}$$

By dimensional analysis and translational and rotational invariance, the correlator of fundamental fields is given by

$$\langle\phi(\boldsymbol{x},t)\phi(0,0)\rangle = \frac{1}{|\boldsymbol{x}|^{2\Delta}}F(\ln(m(\mu)|\boldsymbol{x}|^{z}/t), \ln(\mu|\boldsymbol{x}|))\,,$$

for some dimensionless function of two arguments, $F(x,y)$. This function is further constrained by the renormalization group equation. At a fixed point, $\beta^{\alpha} = 0$, it takes the form

$$\left(\mu\frac{\partial}{\partial\mu} + \gamma_m m\frac{\partial}{\partial m} + 2\gamma\right)\langle\phi(\boldsymbol{x},t)\phi(0,0)\rangle = 0\,,$$

where γ_m and γ are the mass anomalous dimension and the field anomalous dimension, respectively. These are generally dimensionless functions of the dimensionless coupling constants, g^{α}, here evaluated at their fixed point values, say, g^{α}_{*}. It follows that

$$\langle\phi(\boldsymbol{x},t)\phi(0,0)\rangle = \frac{1}{\mu_0^{2\gamma}|\boldsymbol{x}|^{2(\Delta+\gamma)}}f(m(\mu_0)\mu_0^{-\gamma_m}|\boldsymbol{x}|^{z-\gamma_m}/t)\,.$$

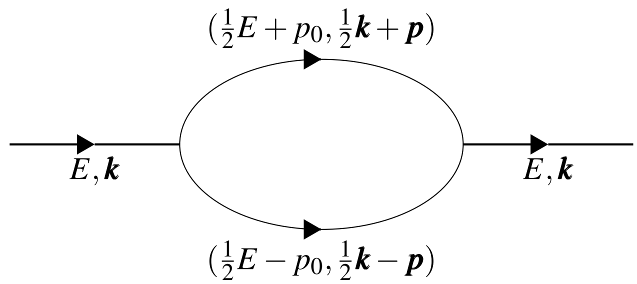

Figure 2.1: Self energy correction to propagator at one loop. The ingoing momenta is denoted by $(E,\boldsymbol{k})$ while the internal loop momenta is given by the variable p.

Here μ_0 is a reference renormalization point and f is a dimensionless function of one variable. This shows that at the fixed point the fields scale with dimension $\Delta+\gamma$ and the dynamical exponent is $\zeta = z-\gamma_m$. It is important to understand that while ζ can be thought of as running along flows, the exponent z is fixed to its classical (gaussian fixed point) value.

As an example consider the following Lagrangian for a $z=2$ theory in $4+1$ dimensions:

$$\mathcal{L} = \left[iZ_m m Z_\phi \phi^* \overleftrightarrow{\partial_t} \phi - Z_\phi \boldsymbol{\nabla}\phi^* \boldsymbol{\nabla}\phi - \tfrac{1}{2} Z_g g \mu^{\frac{\varepsilon}{2}} \sqrt{Z_m m} Z_\phi^{3/2} |\phi|^2 (\phi+\phi^*) \right], \tag{2.26}$$

The renormalization factors in dimensional regularization in $n+1$ dimensions, with $n = 4-\varepsilon$, have the following form:

$$Z_X = 1 + \sum_{n=1} \frac{a_n^X}{\varepsilon^n}, \tag{2.27}$$

where the residues a_n^X are functions of the renormalized coupling constant g. Independence of the bare parameters on the scale μ requires

$$0 = \mu \frac{d}{d\mu}\left(Z_g g \mu^{\frac{\varepsilon}{2}} \right) = \frac{\partial Z_g}{\partial g} \hat{\beta} g \mu^{\frac{\varepsilon}{2}} + Z_g \hat{\beta} \mu^{\frac{\varepsilon}{2}} + \frac{\varepsilon}{2} Z_g g \mu^{\frac{\varepsilon}{2}} \tag{2.28}$$

where $\hat{\beta} \equiv \mu dg/d\mu$ has $\hat{\beta}(g,\varepsilon) = -\frac{\varepsilon}{2} g + \beta(g)$, and

$$0 = \mu \frac{d}{d\mu}(Z_m m) = \frac{\partial Z_m}{\partial g} \hat{\beta} m + \mu \frac{dm}{d\mu} Z_m. \tag{2.29}$$

It follows that

$$\gamma_m = \mu \frac{d\ln(m)}{d\mu} = \tfrac{1}{2} g \, \frac{da_1^m}{dg} \,. \tag{2.30}$$

At one loop the self-energy correction to the propagator, represented by the Feynman diagram in Fig. 2.1, reads

$$i\Sigma(E,\boldsymbol{k}) = -\tfrac{1}{2} m g^2 \mu^{\varepsilon} \int \frac{dp_0}{2\pi} \frac{d^n p}{(2\pi)^n} \, D\left(\tfrac{1}{2}E - p_0, \tfrac{1}{2}\boldsymbol{k} - \boldsymbol{p}\right) \, D\left(\tfrac{1}{2}E + p_0, \tfrac{1}{2}\boldsymbol{k} + \boldsymbol{p}\right) \tag{2.31}$$

where the propagator is given by

$$D(E,\boldsymbol{p}\,) = \frac{i}{(2mE - \boldsymbol{p}^{\,2} + i0^+)} \,. \tag{2.32}$$

The integration over p_0 and then over $\boldsymbol{p}$ gives

$$\Sigma(E,\boldsymbol{k}\,) = \tfrac{1}{8} g^2 \mu^{\varepsilon} \int \frac{d^n p}{(2\pi)^n} \, \frac{1}{\left(mE - \frac{1}{4}\boldsymbol{k}^2 - \boldsymbol{p}^{\,2}\right)} = -\frac{1}{\varepsilon} \frac{g^2}{64\pi^2} (mE - \tfrac{1}{4}\boldsymbol{k}^{\,2}) + \cdots \,, \tag{2.33}$$

where the ellipses stand for finite terms. We read off

$$Z_\phi - 1 = \frac{g^2}{256\pi^2 \varepsilon} \qquad \text{and} \qquad Z_m - 1 = \frac{g^2}{256\pi^2 \varepsilon} \,.$$

Form which it follows that

$$\gamma_m = \frac{g^2}{256\pi^2} \,. \tag{2.34}$$

2.3 $d = 2, z = 2$ Non Relativistic theory

2.3.1 Listing out terms

We first consider $2+1$ NRCFT with $z = 2$. It is convenient to catalogue the possible terms on the right hand side of Eq. (2.22) by the number of space and time derivatives acting on the metric, the couplings and the transformation parameter σ. Rotational invariance implies that space derivatives always appear in contracted pairs. We must, in addition, insure the correct dimensions. Table 2.1 summarizes the dimensions of the basic rotationally invariant quantities; R stands for the curvature scalar constructed from the spatial metric h_{ij}. Since h_{ij} is the metric of a 2 dimensional space, rotational invariants constructed from the Riemann and Ricci tensors can be expressed in terms of R only.

Table 2.1: Basic rotationally invariant operators and their dimensions. They are made out of N, g^α and curvature R.

Operators	N	g^α	R
Length Dimension	0	0	2
Time Dimension	1	0	0

In order to match up the dimension of the Lagrangian, terms that only contain spatial derivatives must have exactly four derivatives. The derivatives can act on the metric or on on the dimensionless variation parameter σ. Hence we have following 2-spatial-derivatives components:

$$\frac{\partial_i N}{N}\frac{\partial^i N}{N}, \quad \frac{\partial_i N}{N}\partial^i g^\alpha, \quad \partial_i g^\alpha \partial^i g^\beta, \quad \frac{\nabla^2 N}{N}, \quad \nabla^2 g^\alpha, \quad R \tag{2.35}$$

$$\nabla^2 \sigma \tag{2.36}$$

$$\partial_i \sigma \frac{\partial^i N}{N}, \quad \partial_i \sigma \partial^i g^\alpha \tag{2.37}$$

where we note that in the term $\frac{\partial_i N}{N}$ the denominator serves to cancel off the time dimension of the numerator. To form a 4 derivative term out of above terms, we can (i) choose two terms among

Table 2.2: Summary of four spatial derivative terms that can enter the counterterm functional $W_{\text{c.t.}}$ or the anomaly on the right hand side of Eq. (2.22). The terms in $W_{\text{c.t.}}$ are the products of the first six entries of the first column and the first six of the first row, and their coefficients are the first of the entries listed in the table (uppercase letters). Those in the anomaly extend over the whole table; in the first 6×6 block they correspond to the second entry (lowercase characters) and for those a factor of σ is implicit. The red *NA* labels denote terms that are second order in infinitesimal parameter σ, hence dropped. Latin indices are contracted with the inverse metric h^{ij} when repeated, eg, $\partial_i N\partial^i N = h^{ij}\partial_i N\partial_j N$.

∇^4 Sector	$\partial_i N\partial^i N$	$\partial_i g^\alpha\partial^i g^\beta$	$\partial_i N\partial^i g^\alpha$	$\nabla^2 N$	$\nabla^2 g^\alpha$	R	$\nabla^2\sigma$	$\partial_i\sigma\partial^i N$	$\partial_i\sigma\partial^i g^\alpha$
$\partial_i N\partial^i N$	P_3, p_3	$X_{\alpha\beta}, x_{\alpha\beta}$	$P_{1\alpha}, \rho_{9\alpha}$	P_4, p_4	Y_α, y_α	Q, χ_4	χ_3	ρ_{11}	$\rho_{8\alpha}$
$\partial_i g^\alpha\partial^i g^\beta$	$X_{\alpha\beta}, x_{\alpha\beta}$	$X_{\alpha\beta\gamma\delta}, x_{\alpha\beta\gamma\delta}$	$X_{\alpha\beta\gamma}, x_{\alpha\beta\gamma}$	$X_{2\alpha\beta}, x_{2\alpha\beta}$	$T_{2\alpha\beta\gamma}, t_{2\alpha\beta\gamma}$	$Y_{5\alpha\beta}, y_{5\alpha\beta}$	$a_{3\alpha\beta}$	$\rho_{1\alpha\beta}$	$t_{\alpha\beta\gamma}$
$\partial_i N\partial^i g^\alpha$	$P_{1\alpha}, \rho_{9\alpha}$	$X_{\alpha\beta\gamma}, x_{\alpha\beta\gamma}$	$P_{5\alpha\beta}, p_{5\alpha\beta}$	$P_{25\alpha}, \rho_{25\alpha}$	$P_{26\alpha\beta}, \rho_{26\alpha\beta}$	χ_α	$\chi_{1\alpha}$	$\rho_{10\alpha}$	$x_{1\alpha\beta}$
$\nabla^2 N$	P_4, p_4	$X_{2\alpha\beta}, x_{2\alpha\beta}$	$P_{25\alpha}, \rho_{25\alpha}$	P_{23}, ρ_{23}	$P_{24\alpha}, \rho_{24\alpha}$	H, c^2	h_2	ρ_{12}	$\rho_{13\alpha}$
$\nabla^2 g^\alpha$	Y_α, y_α	$T_{2\alpha\beta\gamma}, t_{2\alpha\beta\gamma}$	$P_{26\alpha\beta}, \rho_{26\alpha\beta}$	$P_{24\alpha}, \rho_{24\alpha}$	$P_{22\alpha\beta}, \rho_{22\alpha\beta}$	$A_{5\alpha}, a_{5\alpha}$	$a_{4\alpha}$	$\rho_{7\alpha}$	$\rho_{21\alpha\beta}$
R	Q, χ_4	$Y_{5\alpha\beta}, y_{5\alpha\beta}$	$Q_{1\alpha}, \chi_\alpha$	$H, c^{\ 1}$	$A_{5\alpha}, a_{5\alpha}$	A, a	n	h_1	$a_{7\alpha}$
$\nabla^2\sigma$	χ_3	$a_{3\alpha\beta}$	$\chi_{1\alpha}$	h_2	$a_{4\alpha}$	n	NA	NA	NA
$\partial_i\sigma\partial^i N$	ρ_{11}	$\rho_{1\alpha\beta}$	$\rho_{10\alpha}$	ρ_{12}	$\rho_{7\alpha}$	h_1	NA	NA	NA
$\partial_i\sigma\partial^i g^\alpha$	$\rho_{8\alpha}$	$t_{\alpha\beta\gamma}$	$x_{1\alpha\beta}$	$\rho_{13\alpha}$	$\rho_{21\alpha\beta}$	$a_{7\alpha}$	NA	NA	NA

(2.35) with repetition allowed: there are $6^2 - {}^6C_2 = 21$ such terms; (ii) (2.36) can combine with any of (2.35) giving 6 additional terms; and (iii) we can choose one of (2.37) and choose another from (2.35), yielding an additional $2*6 = 12$ terms. Hence we will have $21+12+6 = 39$ terms with four space derivatives. Terms with derivatives of R, such as

$$\partial_i R\partial^i g^\alpha \qquad \text{and} \qquad \partial_i R\frac{\partial^i N}{N}\,,$$

are not independent. Integrating by parts, the term $\partial_i R\partial^i g^\alpha$ can written in terms of $R\nabla^2 g^\alpha$ and $R\partial_i\sigma\partial^i g^\alpha$, and the term $R\nabla^2 N$ can be expressed in terms of $\partial_i R\frac{\partial^i N}{N}$. The 39 four derivative terms, which we call the ∇^4 sector, appear on the right hand side of (2.22) with dimensionless coefficients that are functions of the couplings g^α, and with a factor of σ if the term does not already contain one. Table 2.2 gives our notation for the coefficients of these terms in Eq. (2.22).

Two time derivatives are required for the sector with pure time derivatives, which we label ∂_t^2. The terms must still have length dimension -4. The dimensions of the basic building

Table 2.3: Basic building blocks for operators in the ∂_t^2 sector and their dimensions. They are made out of K, g^α and $(K_{ij} - \frac{1}{2}Kh_{ij})$.

Operators	K	g^α	$(K_{ij} - \frac{1}{2}Kh_{ij})$
Length Dimension	0	0	0
Time Dimension	1	0	1

Table 2.4: Summary of two time derivative terms that can enter the counterterm functional $W_{\text{c.t.}}$ or the anomaly on the right hand side of Eq. (2.22). The terms in $W_{\text{c.t.}}$ are the products of the first, second, fourth entries of the first column and the first, second, fourth entry of the first row , and their coefficients are the first of the entries listed in the table (uppercase letters). Those in the anomaly extend over the whole table; in the first 2×2 block they correspond to the second entry (lowercase characters) and for those a factor of σ is implicit. The red NA labels denote terms that are either second order in infinitesimal parameter σ or terms that are not rotationally invariant.

∂_t^2 Sector	K	$\partial_t g^\alpha$	$\partial_t \sigma$	$K_{ij} - \frac{1}{2}Kh_{ij}$
K	D, d	W_α, w_α	f	NA
$\partial_t g^\alpha$	W_α, w_α	$X_{0\alpha\beta}$, $\chi_{0\alpha\beta}$	b_α	NA
$\partial_t \sigma$	f	b_α	NA	NA
$K_{ij} - \frac{1}{2}Kh_{ij}$	NA	NA	NA	E, e

blocks are given in Tab. 2.3, where $K_{ij} = \frac{1}{2}\partial_t h_{ij}/N$ is the extrinsic curvature of the t =constant hypersurfaces in the $N^i = 0$ gauge and $K = h^{ij}K_{ij}$ (with h^{ij} the inverse of the metric h_{ij}). The combination $(K_{ij} - \frac{1}{2}Kh_{ij})$ is convenient because it is Weyl invariant. Hence, for the ∂_t^2 sector we have the following basic one derivative terms:

$$K, \qquad \partial_t g^\alpha \tag{2.38}$$

$$\partial_t \sigma \tag{2.39}$$

$$K_{ij} - \frac{1}{2}Kh_{ij} \tag{2.40}$$

The term $\partial_t N$ is not included in the list because it is not covariant. The diffeomorphism invariant quantity is given by $\partial_t N - \Gamma^0_{00} N$ which vanishes identically 0.

Possible anomaly terms are constructed from the $2^2 - 1 = 3$ products of terms in (2.38); from 2 terms by combining (2.39) and one from (2.38); and we can have (2.40) contracted

Table 2.5: Summary of one-time, two-space derivative terms that can enter the counterterm functional $W_{\text{c.t.}}$ or the anomaly on the right hand side of Eq. (2.22). The terms in $W_{\text{c.t.}}$ are the products of the entries that have no explicit σ factor, and their coefficients are the first of the entries listed in the table (uppercase letters). Those in the anomaly extend over the whole table; terms without explicit σ have coefficients that correspond to the second entry (lowercase characters) and for those a factor of σ must be included. Latin indices are contracted with the spatial metric as necessary to make the product of the first column and first row entries rotationally invariant; for example, ρ_4 denotes the coefficient of $K\partial_i N\partial^i N$. For last entry in the first column, indices are contracted with those in the terms in first row. The red NA labels denote terms that are second order in infinitesimal parameter σ, hence dropped. The blue NA one denotes a term that is identically 0 since $K_{ij} - \frac{1}{2}Kh_{ij}$ vanishes upon contraction via h^{ij}.

$\partial_t\nabla^2$ Sector	$\partial^i N\partial^j N$	$\partial^i g^\alpha\partial^j g^\beta$	$\partial^i N\partial^j g^\alpha$	$\nabla^i\nabla^j N$	$\nabla^i\nabla^j g^\alpha$	R	$\nabla^i\nabla^j\sigma$	$\partial^i\sigma\partial^j N$	$\partial^i\sigma\partial^j g^\alpha$
K	P, ρ_4	$X_{5\alpha\beta}, x_{5\alpha\beta}$	P_α, ρ_α	L, j[3]	$P_{3\alpha}, b_{8\alpha}$	B, b	m	l_1	$b_{7\alpha}$
$\partial_t g^\alpha$	$X_\alpha, \rho_{6\alpha}$	$X_{3\alpha\beta\gamma}, x_{3\alpha\beta\gamma}$	$P_{4\alpha\beta}, p_{4\alpha\beta}$	$B_{6\alpha}, b_{6\alpha}$	$X_{4\alpha\beta}, x_{4\alpha\beta}$	$B_{5\alpha}, b_{5\alpha}$	$B_{9\alpha}, b_{9\alpha}$	$\rho_{5\alpha}$	$x_{6\alpha\beta}$
$\partial_t\sigma$	ρ_3	$b_{3\alpha\beta}$	$\rho_{1\alpha}$	l_2	$b_{4\alpha}$	k	NA	NA	NA
$K_{ij} - \frac{1}{2}Kh_{ij}$	F_1, f_1	$F_{2\alpha\beta}, f_{2\alpha\beta}$	$F_{3\alpha}, f_{3\alpha}$	F_4, f_4	$F_{5\alpha}, f_{5\alpha}$	NA	f_6	f_7	$f_{8\alpha}$

with itself. Thus in total there are $3+2+1=6$ terms listed in Tab. 2.4 that also gives the corresponding coefficients.

The sector with mixed derivatives has terms with one time and two spatial derivatives. For this $\partial_t\nabla^2$ sector we can form terms by combining one of (2.38) or (2.39) with one of (2.35), (2.36) or (2.37), excluding terms quadratic in σ. This gives $3*9-3=24$ terms, as displayed with their coefficients in Tab. 2.5. Finally, we have terms that are not constructed as products of rotationally invariant quantities. Coefficient of those terms are listed in the last row of Tab. 2.5.

2.3.2 Using counter-terms

One can similarly list all possible terms in $W_{\text{c.t.}}$. The requirements imposed by dimensional analysis and rotational invariance are as before, the only difference being that these terms are built from the metric and the couplings but not the parameter of the Weyl transformation σ. Therefore the list of possible counterterms is obtained from the one for anomalies by replacing $\sigma \to 1$. Tables. 2.2, 2.4 and 2.5 give, as uppercase letters, our notation for the coefficients of

[3] $K\nabla^2 N$ can be written as $\partial_i K\partial^i N$ by doing integration by parts, and it is for this operator that we use the coefficient j.

these operators in $W_{\text{c.t.}}$.

The counterterms in $W_{\text{c.t.}}$ are not completely fixed by requiring finiteness of the generating functional. The ambiguity consists of the freedom to include arbitrary finite contributions to each term. This freedom to add finite counter-terms does not affect the consistency conditions but does change the value of the individual terms related by them. We can use this freedom to set some anomalies to zero, simplifying the analysis of the consequences of the Weyl consistency conditions. In particular, in searching for an a-theorem we can use this freedom to simplify the consistency conditions. It may be possible to show then that there exist some class of subtraction schemes for which there exists a possible candidate for an a-theorem, but a general, counter-term and scheme independent statement may not be possible.

To illustrate this, consider the variation of the K^2 and $K\partial_t g^\alpha$ terms in $W_{\text{c.t.}}$:

$$\Delta_\sigma \int dt\, d^2x N\sqrt{h}\left(DK^2\right) = \int dt\, d^2x N\sqrt{h}\left(-4\frac{1}{N}\partial_t\sigma DK - \sigma\beta^\alpha\partial_\alpha DK^2\right),$$

$$\Delta_\sigma \int dt\, d^2x N\sqrt{h}\,(W_\alpha K\partial_t g^\alpha) = \int dt\, d^2x N\sqrt{h}\Big(-\sigma\left[\beta^\alpha\partial_\alpha W_\gamma + W_\alpha\partial_\gamma\beta^\alpha\right] K\partial_t g^\gamma$$
$$- \frac{1}{N}\partial_t\sigma\,\beta^\alpha\, W_\alpha K - 2\frac{1}{N}\partial_t\sigma\, W_\alpha\partial_t g^\alpha\Big)$$

Inspecting Tabs. 2.2, 2.4 and 2.5 we see that the f anomaly gets contributions only from these variations, so that the change in f induced by finite changes in the counterterms is given by

$$\delta f = -4D - \beta^\alpha W_\alpha\,. \tag{2.41}$$

With a slight abuse of notation we have denoted here the arbitrary, finite, additive change to the coefficients of counterterms by the same symbol we have used for the counterterm coefficients themselves. From Eq. (2.41) we see that one can always choose D so as to set f arbitrarily, and it is often convenient to set $f = 0$. For a second example consider the R^2 anomaly, a. A similar computation gives

$$\delta a = -\beta^\alpha\partial_\alpha A \tag{2.42}$$

In this case we may solve this equation so as to set $a = 0$ only if $a = 0$ at fixed points, where $\beta^\alpha = 0$. As we will see below, the Weyl consistency conditions constrain some anomalies to vanish at fixed points.

We give in App. A.2 the complete set of ambiguities for models with $z = 2$ in $d = 2$ spatial dimensions. Terms in the effective actions whose coefficients can be varied at will are not properly anomalies, since the coefficients can be set to zero. With a slight abuse of language they are commonly referred to as *trivial anomalies* and we adopt this terminology here. Table 2.6 summarizes the trivial anomalies found in each sector.

Table 2.6: Trivial anomalies for each sector. Finite ambiguities in counter-terms give sufficient freedom to set all these anomalies arbitrarily; setting them to zero is often convenient. For anomalies grouped within parenthesis, all but one of them can be set arbitrarily.

Sector	Trivial Anomalies
∂_t^2	f, b_α
$\nabla^2\partial_t$	(ρ_3, l_1), $x_{6\alpha\beta}$, $\rho_{5\alpha}$, $b_{3\alpha\beta}$,$b_{4\alpha}$, $b_{9\alpha}$, (k, m, l_2), $(b_{7\alpha}, \rho_{1\alpha})$, f_6, f_7, $f_{8\alpha}$
∇^4	χ_3, ρ_{11}, $(\rho_{10\alpha}, \rho_{13\alpha}, \rho_{8\alpha})$, $a_{3\alpha\beta}$, $\rho_{1\alpha\beta}$, $t_{\alpha\beta\gamma}$, $\chi_{1\alpha}$, $x_{1\alpha\beta}$, h_2, ρ_{12}, $a_{4\alpha}$, $\rho_{7\alpha}$, $\rho_{21\alpha\beta}$, n, h_1, $a_{7\alpha}$

Table 2.7: Vanishing anomalies for each sector. The Weyl consistency conditions imply these anomalies, or combination of anomalies, vanish at fixed points (where $\beta^\alpha = 0$). An anomaly is conditionally vanishing if it is vanishing only for a particular choice of counterterms.

Sector	Vanishing Anomalies	Conditionally Vanishing Anomalies
∂_t^2	d	w_α
$\nabla^2\partial_t$	f_4, f_1, ρ_4, $b_{7\alpha}$ b, j, $2\rho_3 - l_1 + 2l_2$, $k + m - l_2$	$b_{6\alpha} + \rho_{6\alpha}$, $b_{5\alpha} - b_{6\alpha}$, $b_{7\alpha} - \rho_{1\alpha}$ $x_{5\alpha\beta}$, $f_{3\alpha}$, $b_{8\alpha}$
∇^4	$\chi_4 - p_4$, $2p_3 + p_4$, $c - \chi_4$ $2a + c$, $p_4 + 2\rho_{23}$, $2\rho_{23} + c$	$h_1 + 2h_2 + 2\chi_3 - c - \rho_{12}$, $x_{\alpha\beta} + x_{2\alpha\beta}$, $\rho_{13\alpha}$ $y_{5\alpha\beta} - x_{2\alpha\beta}$, $y_\alpha + \rho_{24\alpha}$, $a_{5\alpha} - \rho_{24\alpha}$, $\rho_{25\alpha} + \rho_{9\alpha}$

2.3.3 Consistency conditions and vanishing anomalies

In computing the consistency condition (2.24) one finds a functional that is a combination of linearly independent "operators" (combinations of σ, $\gamma_{\mu\nu}$ and g^α), each with a coefficient that is a linear combination of the coefficients in Tabs. 2.2, 2.4 and 2.5 and their derivatives. Thus the consistency conditions can be expressed as a set of equations among these coefficients and their derivatives. The full set of consistency conditions for $d = 2$, $z = 2$ are listed in App. A.1. On the left of each condition we have listed the operator the condition arises from. We have verified that these conditions reduce to the ones computed in Ref. [18] at fixed points. In the $\partial_t\nabla^2$ sector the consistency conditions, Eqs. (A.1), are given for arbitrary z, while for the ∂_t^2 and ∇^4 sectors, Eqs. (A.2) and (A.3), respectively, the value $z = 2$ has been used.

At fixed points the consistency conditions imply some anomalies vanish. These are known as *vanishing anomalies*. For example, setting $\beta^\alpha = 0$ in Eq. (A.2a) gives $d = 0$. Table 2.7 summarizes the vanishing anomalies found in each sector. The table also shows *conditionally* *vanishing anomalies*. These are vanishing anomalies but only for a specific choice of counterterms. For example, setting $\beta^\alpha = 0$ in Eq. (A.2b) gives $-2w_\alpha + b_\gamma \partial_\alpha \beta^\gamma = 0$, and Eq. (A.4c) shows that we can choose the counterterm W_α to set $b_\alpha = 0$.

As explained above, some vanishing anomalies can be set to zero. For example, from Tab. 2.7 we see that d is a vanishing anomaly, and then Eq. (A.4e) informs us that one may choose D to enforce $d = 0$. We note, however, that by Eqs. (A.4a) and (A.4e) one may either choose f or d to vanish, but not both.

2.3.4 Applications

While there are many avenues for analysis in light of the relations imposed by Weyl consistency conditions on the anomalies, we concentrate on finding candidates for a C-theorem. We search for a combination of anomalies, C, a local function in the space of dimensionless coupling constants that flows monotonically, $\mu dC/d\mu \geq 0$. We try to establish this by judiciously setting some anomalies to zero by the freedom explained above and looking for a relation of the form

$$\beta^\alpha \partial_\alpha C = -\mathcal{H}_{\alpha\gamma} \beta^\alpha \beta^\gamma.$$

Our first three candidates arise from the ∇^4 sector. Consider Eq. (A.3l), here reproduced:

$$-a_{5\alpha}\beta^\alpha + 4a + 2c + \beta^\alpha \partial_\alpha n = 0$$

The combination $2a + c$ is a vanishing anomaly. One may then use (A.71) and (A.62) to set $2a + c = 0$. Equation (A.68) shows $a_{4\alpha}$ is a trivial anomaly and one may set $a_{4\alpha} = 0$. Combining with Eq. (A.3c) we have

$$\beta^\alpha \partial_\alpha n = \rho_{22\alpha\gamma} \beta^\alpha \beta^\gamma + \rho_{24\alpha} \beta^\alpha$$

Similarly, Eq. (A.3i) shows $2\rho_{23} + c$ is a vanishing anomaly and using (A.60) we may set

$2\rho_{23}+c=0$. We then have from Eq. (A.3i) again that

$$\beta^\gamma \partial_\gamma h_2 = \beta^\gamma \rho_{24\gamma}$$

The difference of these equations then gives us our first candidate for a C-theorem, with $C = n - h_2$:

$$\beta^\alpha \partial_\alpha (n - h_2) = \rho_{22\alpha\gamma}\beta^\alpha\beta^\gamma. \tag{2.43}$$

A second candidate can be found as follows. Eq. (A.3s) shows $\chi_4 - p_4$ is a vanishing anomaly. Then $Q - P_4$ can be chosen so that $\chi_4 - p_4 = 0$; see Eqs. (A.41) and (A.39). Using Eq. (A.3k) with $\rho_{7\alpha} = 0$ as it is a trivial anomaly, we obtain

$$-\beta^\alpha \partial_\alpha \chi_3 = \tfrac{1}{4}\rho_{26\alpha\gamma}\beta^\alpha\beta^\gamma + \rho_{24\alpha}\beta^\alpha$$

It follows that

$$\beta^\alpha \partial_\alpha (n + \chi_3) = (\rho_{22\alpha\gamma} - \tfrac{1}{4}\rho_{26\alpha\gamma})\beta^\alpha\beta^\gamma \tag{2.44}$$

Combining Eqs. (A.3n), (A.3j) and (A.3r) while setting $\chi_{1\alpha} = 0$, $p_4 + 2\rho_{23} = 0$ and $c - \chi_4 = 0$ gives what appears to be yet another candidtae in the ∇^4 sector:

$$\beta^\alpha \partial_\alpha (c + \rho_{12} - h_1) = -\tfrac{1}{2}\rho_{26\alpha\gamma}\beta^\alpha\beta^\gamma \tag{2.45}$$

However, setting the trivial anomalies $\rho_{1\alpha}$ and $\chi_{1\alpha}$ to zero, Eq. (A.3o) gives

$$h_2 + \chi_3 = \tfrac{1}{2}(c + \rho_{12} - h_1)$$

which shows that the candidates given by eq (2.43),(2.44),(2.45) are not linearly independent in the scheme with $2a+c = 2\rho_{23}+c = \chi_4 - c = \chi_4 - p_4 = p_4 + \rho_{23} = 0$ and $a_{4\alpha} = \rho_{1\alpha} = \rho_{7\alpha} =$

$\chi_{1\alpha} = 0$.

We find one candidate for a C-theorem in the ∂_t^2 sector. Equation (A.2a) shows d is a vanishing anomaly and use Eqs. (A.4e) and (A.4c) to set $d = b_\alpha = 0$. Combining (A.2a) and (A.2b) gives

$$\beta^\alpha \partial_\alpha f = -\chi_{0\alpha\gamma}\beta^\alpha\beta^\gamma. \tag{2.46}$$

In the $\partial_t \nabla^2$-sector we find the following candidates for a C-theorem:

$$\beta^\alpha \partial_\alpha m = -\tfrac{1}{2} x_{4\alpha\gamma}\beta^\alpha\beta^\gamma \tag{2.47}$$

$$\beta^\alpha \partial_\alpha l_1 = -\tfrac{1}{2} p_{4\gamma\alpha}\beta^\gamma\beta^\alpha \tag{2.48}$$

$$\beta^\alpha \partial_\alpha (\rho_3 + l_2) = -\tfrac{1}{2z} p_{4\alpha\gamma}\beta^\gamma\beta^\alpha \tag{2.49}$$

$$\beta^\alpha \partial_\alpha \left(f_6 + \frac{z}{2} f_7 - \beta^\gamma f_{5\gamma} \right) = \beta^\alpha\beta^\gamma \left(f_{2\alpha\gamma} - \partial_\alpha f_{5\gamma} \right) \tag{2.50}$$

We have kept the explicit dependence on z in these equations. As we will see below the $\partial_t \nabla^2$-sector is special in that the Weyl anomalies and the relations from consistency conditions hold for arbitrary z. Hence, the C-candidates in this sector are particularly interesting since they are candidates for any z. To derive (2.47) we have used that j and b are vanishing anomalies, as evident from Eqs. (A.1d) and (A.1f), and used B and L to set $b = j = 0$ in Eq. (A.1f) and $P_{3\alpha}$ to set $b_{4\alpha} = 0$ in Eq. (A.1b). For (2.48) we used $j = 0$ in Eq. (A.1a) and (A.1n), deduce that ρ_4 is a vanishing anomaly and use P to set $\rho_4 = 0$ in Eq. (A.1n) and P_α to set $\rho_{1\alpha} = 0$ in Eq. (A.1a). For (2.49), we set $j = \rho_4 = 0$ as before and in addition we set $\rho_{5\alpha} = 0$ using X_α in (A.1e), and use Eqs. (A.1d), (A.1e) and (A.1m). In the scheme, $j = \rho_{1\alpha} = 0$, Eq (A.1o) implies that the candidates given by (2.49) and (2.48) are linearly dependent. Last but not the least, (2.50) is derived from Eqs. (A.1p)–(A.1r) by using $F_{3\alpha}$ to set $f_{8\alpha} = 0$ and setting to zero the vanishing anomalies f_1 and f_4 using F_1 and F_4.

Two comments are in order. First, we have not established any C-theorem. To do so would require showing that the two index symmetric tensor appearing on at least one of the right hand side of Eqs. (2.43)–(2.46) is positive definite, so that it acts as a metric in the space of flows. In addition, the interpretation of C as counting degrees of freedom is better supported if it is a monotonic function of the number of degrees of freedom at a gaussian fixed point. And second, we do not expect a positive definite metric can be found in generality, since cyclic flows are known to appear in NR quantum systems. Cyclic flows appear in relativistic systems too, but they differ from NR ones in that there is scaling symmetry all along the cyclic flows and, in fact, the C quantity is constant along the cyclic flow [40]. Investigating the conditions under which a theory gives positive definite metric(s) in the space of flows is beyond the scope of this work; we hope to return to this problem in the future.

2.4 Generalisation to arbitrary z value

In this section, we will explore the possibility to generalize the work for arbitrary z value. It is clear that the formalism fails for non-integer values of z since in that case, we can not make up for dimensions with regular analytic functions of curvature and coupling constants. This is because the quantities constructed out of geometry and coupling constants always have integer length and time dimension. Furthermore, in a Lagrangian formulation a non-integer z requires non-analyticity of Lagrangian. So we begin by recalling under what conditions a Lagrangian with local interactions allows for integer z values.

Consider first the case of $d = 2$ at arbitrary z value. In constructing $\Delta W_{\text{c.t.}}$, rotational invariance implies even number of spatial derivatives, say $2n$. Along with m time derivatives, we must have

$$mz + 2n = z + 2\,.$$

We look for solutions with integer values for m and n. For $m = 1$ we must have $n = 1$ and this satisfies the equation for any z. Else, for $m \neq 1$ we have

$$z = \frac{2(1-n)}{(m-1)}\,.$$

For $z > 0$ we must have either $m = 0$ with $n > 1$ or $n = 0$ with $m > 1$. For $m = 0$ solutions exist only if $z = 2k$ is even, with $2n = 2(1+k)$ spatial derivatives. On the other hand, with $n = 0$, we have solutions for $z = 2/k$, with $m = k+2$ time derivatives. To summarize, for $z > 0$ we can classify the counterterms by sector as follows:

- There is a pure ∇^2 sector for $z = 2k$, $k \in \mathbf{Z}$. It has precisely $2(k+1)$ spatial derivatives. We have discussed in detail the case $k = 1$. Higher values of k can be similarly analyzed, but it it involves an ever increasing number of terms as z increases.

- There is a pure ∂_t sector for $z = 2/k$, $k \in \mathbf{Z}$, with $k+1$ time derivatives. We have analyzed the $k = 1$ case. Higher values of k can be similarly analyzed, but it involves an ever increasing number of terms as z decreases.

- There is a $\partial_t \nabla^2$ sector for arbitrary z. It has 1-time and 2-spatal derivatives regardless of z. Therefore, the classification of anomalies and counterterms is exactly as in the $z = 2$ case, and the consistency conditions and derived C-candidates are modified by factors of $z/2$ relative the $z = 2$ case.

2.5 A candidate for a C-theorem in $d+1$D

In relativistic $2n$-dimensional QFT the quantity that is believed to satisfy a C-theorem is associated with the Euler anomaly, that is, it is the coefficient of the Euler density E_{2n} in the conformal anomaly [28].[4] It would seem natural to seek for analogous candidates in non-

[4] There is no known local C-function candidate for odd-dimensional relativistic field theory. Jafferis has proposed a non-local F-function for 3D relativistic theories that shares the monotonicity properties of a C-function[46]

relativistic theories. The obvious analog involves the Euler density for the spatial sections $t =$ constant; by dimensional analysis and scaling it should be constructed out of $z+d = 2n$ spatial derivatives acting on the metric h_{ij}. However, for a d-dimensional metric the Euler density E_{2n} with $2n-d = z > 0$ vanishes. Hence, we are led to consider an anomaly of the form XE_d, that is the Euler density computed on the spatial sections $t =$ constant times some quantity X with the correct dimensions, $[X] = z$. This construction is only valid for even spatial dimension, $d = 2n$. The most natural candidate for X is K: it is the only choice if z is odd. If z is even it can be constructed out of spatial derivatives. For example, if $z = dk = 2nk$ for some integers k and n, one may take $X = (E_d)^k$.

The variation of the Euler density yields the Lovelock tensor [47], H_{ij}, a symmetric 2-index tensor that satisfies

$$\nabla_i H^{ij} = 0$$

In looking for a candidate C-theorem we consider a set of operators that close under Weyl-consistency conditions, starting from XE_d. Since $\delta_\sigma(\sqrt{h}E_d) = \sqrt{h}H^{ij}\nabla_i\partial_j\sigma$, and $[XH^{ij}] = z+d-2$, we are led to include terms with the Lovelock tensor and two spatial derivatives. In order to compute the consequences of the Weyl consistency conditions we assume

$$\delta X = z\sigma X + \cdots \tag{2.51}$$

where the ellipses denote terms that depend on derivatives of σ and are therefore independent of X. Consider therefore a subset of terms in the anomaly that appear in the consistency conditions that lead to a potential C-theorem:

$$\Delta_\sigma W = \int d^d x dt\, N\sqrt{h}\left[\sigma\left\{aXE_d + bXH^{ij}R_{ij} + \chi_4 XH^{ij}\frac{\partial_i N}{N}\frac{\partial_j N}{N} + \chi_\alpha XH^{ij}\frac{\partial_i N}{N}\partial_j g^\alpha \right.\right.$$
$$\left. + y_{5\alpha\beta}XH^{ij}\partial_i g^\alpha \partial_j g^\beta + cH^{ij}\partial_i X\frac{\partial_i N}{N} + a_{5\alpha}H^{ij}\partial_i X\partial_j g^\alpha\right\}$$

$$+\partial_i\sigma\left\{n\partial_j X H^{ij} + h_1 \frac{\partial_j N}{N} H^{ij} X + a_{7\alpha}\partial_j g^\alpha H^{ij} X\right\}\Bigg] \quad (2.52)$$

Correspondingly there are metric and coupling-constant dependent counter-terms with coefficients denoted by uppercase symbols:

$$W_{\text{c.t.}} = \int d^d x dt\, N\sqrt{h}\left[AXE_d + BXH^{ij}R_{ij} + X_4 X H^{ij}\frac{\partial_i N}{N}\frac{\partial_j N}{N} + X_\alpha X H^{ij}\frac{\partial_i N}{N}\partial_j g^\alpha \right.$$
$$\left. + Y_{5\alpha\beta} X H^{ij}\partial_i g^\alpha \partial_j g^\beta + C H^{ij}\partial_i X \frac{\partial_i N}{N} + A_{5\alpha} H^{ij}\partial_i X \partial_j g^\alpha\right] \quad (2.53)$$

Freedom to choose finite parts of counter-terms leads to ambiguities in the anomaly coefficients as follows:

$$\delta a = -\beta^\alpha \partial_\alpha A \quad (2.54a)$$

$$\delta\chi_4 = -\beta^\alpha\partial_\alpha X_4 \quad (2.54b)$$

$$\delta\chi_\alpha = -\beta^\gamma\partial_\gamma X_\alpha - X_\gamma\partial_\alpha\beta^\gamma \quad (2.54c)$$

$$\delta y_{5\alpha\beta} = -\beta^\gamma\partial_\gamma Y_{5\alpha\beta} - Y_{5\gamma\beta}\partial_\alpha\beta^\gamma - Y_{5\alpha\gamma}\partial_\beta\beta^\gamma \quad (2.54d)$$

$$\delta c = -\beta^\alpha\partial_\alpha C \quad (2.54e)$$

$$\delta b = -\beta^\alpha\partial_\alpha B \quad (2.54f)$$

$$\delta a_{5\alpha} = -\beta^\gamma\partial_\gamma A_{5\alpha} - A_{5\gamma}\partial_\alpha\beta^\gamma \quad (2.54g)$$

$$\delta n = -A - (d-2)B - Cz - \beta^\alpha A_{5\alpha} \quad (2.54h)$$

$$\delta h_1 = -2zX_4 - \beta^\alpha X_\alpha + Cz - A - (d-2)B \quad (2.54i)$$

$$\delta a_{7\alpha} = -\partial_\alpha(A + (d-2)B) - zX_\alpha - 2\beta^\gamma Y_{5\gamma\alpha} + zA_{5\alpha} \quad (2.54j)$$

In addition to the Euler density, E_d, there are several independent scalars one can construct out of d derivatives of the metric in d dimensions (except for $d = 2$, for which the only 2-derivative invariant is the Ricci scalar and hence $E_d \propto R$). E_d is special in that it is the only quantity that gives just the Lovelock tensor under an infinitesimal Weyl trasformation, $\delta_\sigma(\sqrt{h}E_d) = \sqrt{h}H^{ij}\nabla_i\partial_j\sigma$. In general some other d-derivative invariant[5] $\mathcal{E}$ constructed out of $d/2$ powers of the Riemann tensor will instead give $\delta_\sigma(\sqrt{h}\mathcal{E}) = \sqrt{h}\mathcal{H}^{ij}\nabla_i\partial_j\sigma$ where $\mathcal{H}^{ij} \neq 0$ is not divergence-less, $\nabla_i\mathcal{H}^{ij} \neq 0$. We have given an example of such a term above, $H^{ij}R_{ij}$, both in the anomaly and among the counter-terms. Given a basis of d-derivative operators $\mathcal{E}$ and $d-2$ derivative 2-index symmetric tensors $\mathcal{H}^{ij}$ one can derive Weyl consistency conditions by demanding that the coefficients of each linearly independent operator in $[\Delta_\sigma, \Delta_{\sigma'}]W$ vanish. Suppose $\Delta_\sigma W \supset \int \sigma[aE_d + b\mathcal{E}]$: a change of basis by $\mathcal{E} \to \mathcal{E} + \xi E_d$ results in shifting $a \to a + \xi b$ in the consistency conditions that arise from terms involving H^{ij}. Similarly, a change of basis of $d-2$ derivative 2-index symmetric tensors $\mathcal{H}^{ij} \to \mathcal{H}^{ij} + \xi H^{ij}$ shifts by a common amount all the consistency conditions that arise from terms involving H^{ij}. So while we have not retained all the anomalies that can contribute to the consistency conditions that lead to a potential C-theorem, they give a common contribution to all those consistency conditions and therefore effectively shift the contribution of a to the potential C-theorem —and the shift is immaterial since it is basis dependent. Consider, for example, the coefficient b of the anomaly term $H^{ij}R_{ij}$ which we have retained precisely to demonstrate these points. Since $\delta_\sigma R_{ij} = (d-2)\nabla_i\partial_j\sigma + h_{ij}\nabla^2\sigma$ it is natural to define $\mathcal{H}_{ij}$ by $\delta_\sigma(\sqrt{h}H^{ij}R_{ij}) = \sqrt{h}[(d-2)H^{ij} + \mathcal{H}^{ij}]\nabla_i\partial_j\sigma$. With this definition of a basis of operators the consistency conditions in Eqs. (2.55) below all contain the combination $a + (d-2)b$; had we defined instead a basis with the operator $H^{ij}R_{ij} - (d-2)E_d$ or defined the basis of 2-index tensors through $\delta_\sigma(\sqrt{h}H^{ij}R_{ij}) = \sqrt{h}\mathcal{H}^{ij}\nabla_i\partial_j\sigma$, the anomaly b would not have appeared in Eqs. (2.55) at all. Similarly the ambiguity due to finite counter-terms in anomalies associated with the Lovelock tensor all enter in the combination $A + (d-2)B$.

Imposing $[\Delta_{\sigma'}, \Delta_\sigma]W = 0$ we find three conditions,

$$(\sigma\partial_j\sigma' - \sigma'\partial_j\sigma)H^{ij}\partial_i X: \quad \beta^\alpha\partial_\alpha n = zc + a_{5\alpha}\beta^\alpha + a + (d-2)b \tag{2.55a}$$

[5] Weyl variations of d-derivative scalars constructed from less than $d/2$ powers of the Riemann tensor do not contribute to the consistency condition we are considering.

$$(\sigma\partial_j\sigma' - \sigma'\partial_j\sigma)H^{ij}\partial_i NX: \quad \beta^\alpha\partial_\alpha h_1 = a + (d-2)b + 2z\chi_4 + \beta^\alpha\chi_\alpha - cz \tag{2.55b}$$

$$(\sigma\partial_j\sigma' - \sigma'\partial_j\sigma)H^{ij}\partial_i g^\alpha X: \quad \partial_\alpha(a + (d-2)b) - \beta^\gamma\partial_\gamma a_{7\alpha} - a_{7\gamma}\partial_\alpha\beta^\gamma = za_{5\alpha} - z\chi_\alpha - 2y_{5\alpha\gamma}\beta^\gamma \tag{2.55c}$$

Here we have listed on the left the independent operators in $[\Delta_{\sigma'}, \Delta_\sigma]W$ whose coefficients must vanish yielding the condition correspondingly listed on the right. We have checked that the conditions in Eqs. (2.55) are invariant under the ambiguities listed in Eqs. (2.54). The freedom represented by these ambiguities allows us to set $a + (d-2)b + zc = 0$ in Eq. (2.55a). To see this note that $a + (d-2)b + zc$ is a vanishing anomaly per Eq. (2.55a), and Eqs. (2.54a), (2.54e) and (2.54f) give $\delta(a + (d-2)b + zc) = -\beta^\alpha\partial_\alpha(A + (d-2)B + zC)$ which can be integrated. A similar argument using Eq. (2.55b) shows that $a + (d-2)b + 2z\chi_4 - cz$ is a vanishing anomaly. Using this freedom we have a simpler version of the consistency conditions:

$$(\sigma\partial_j\sigma' - \sigma'\partial_j\sigma)H^{ij}\partial_i X: \quad \beta^\alpha\partial_\alpha n = a_{5\alpha}\beta^\alpha$$

$$(\sigma\partial_j\sigma' - \sigma'\partial_j\sigma)H^{ij}\partial_i NX: \quad \beta^\alpha\partial_\alpha h_1 = \beta^\alpha\chi_\alpha$$

$$(\sigma\partial_j\sigma' - \sigma'\partial_j\sigma)H^{ij}\partial_i g^\alpha X: \quad \partial_\alpha(a + (d-2)b) - \beta^\gamma\partial_\gamma a_{7\alpha} - a_{7\gamma}\partial_\alpha\beta^\gamma = za_{5\alpha} - z\chi_\alpha - 2y_{5\alpha\gamma}\beta^\gamma$$

Combining these we arrive at the candidate for a C-theorem:

$$\beta^\alpha\partial_\alpha\left[a + (d-2)b + zh_1 - zn - \beta^\gamma a_{7\gamma}\right] = -2y_{5\alpha\gamma}\beta^\gamma\beta^\alpha \tag{2.56}$$

Establishing a C-theorem requires in addition demonstrating positivity of the "metric" $-2y_{5\alpha\gamma}$ in Eq. (2.56). While we have not attempted this, it may be possible to demonstrate this in generality working on a background with positive definite Lovelock tensor and using the fact that $y_{5\alpha\gamma}$ gives the RG response of the contact counter-term to the obviously positive definite correlator $\langle O_\alpha O_\gamma\rangle$. In addition, one should check that, when computed at the gaussiaan fixed point, the quantity $a + (d-2)b + zh_1 - zn - \beta^\gamma a_{7\gamma}$ is a measure of the number of degrees of freedom. We hope to come back to this questions in the future, by performing explicit calculations (at and away from fixed points) of these quantities — but such extensive computations are beyond the scope of this work.

The limit $d = 2$ is special since $H^{ij} = 2h^{ij}$. In our analysis, the term $H^{ij}R_{ij} = 2R = 2E_2$ so a and b appear in the combination $a+2b$ throughout. The potential C theorem reads

$$\beta^\alpha \partial_\alpha \left[a + 2b + zh_1 - zn - \beta^\gamma a_{7\gamma}\right] = -2y_{5\alpha\gamma}\beta^\gamma\beta^\alpha \tag{2.57}$$

As we have seen in Sec. 2.4, potential C-theorems in $d = 2$ for any z can be found only in the $\nabla^2\partial_t$ sector. Consulting Tab. 2.5 we see the only candidate for X in our present discussion is $X = K$. None of the potential C-theorems listed in Eqs. (2.47)–(2.50) (nor linear combinations thereof) reproduce the potential C-theorem in Eq. (2.57). The reason for this is that in Sec. 2.3.4 we looked for C-theorems from consistency conditions that included, among others, tems with $\sigma\nabla_i\partial_j\sigma' - \sigma'\nabla_i\partial_j\sigma$, wheras in this section we integrated such terms by parts. The difference then corresponds to combining the consistency conditions given in the appendix with some of their derivatives.

In fact we have found a scheme for deducing aditional C-theorem candidates in $d = 2$ by taking derivatives of some of our consistency conditions. The method is as follows. Take $X \in \{R, \nabla^2 N, \partial_i N \partial^i N, K\}$; the first three instances apply to the case $z = 2$ while the last is applicable for arbitrary z. Then :

- Consider the consistency condition involving $\sigma\nabla^2\sigma' X$, and take a derivative to obtain an equation, say T_1.

- Take the consistency condition involving $\sigma\nabla_i\sigma'\partial^i N X$. From this one may deduce a linear combination of anomalies is vanishing. Set that to 0 using the ambiguity afforded by counter-terms. The remaining terms in the equation (all proportional to β^α) give an equation we denote by T_2.

- Take the consistency condition involving $\sigma\nabla_i\sigma'\partial^i g^\alpha X$, contract it with β^α, to get an equation, say, T_3.
- Combine T_1, T_2, T_3 in a manner such that there are no terms of the form $\beta^\alpha r_{\alpha\cdots}$ and $r_{\gamma\cdots}\beta^\alpha\partial_\alpha\beta^\gamma$.

Following this scheme we obtain four new C-theorem candidates. In the following the expressions for $T_{1,2,3}$ refer to the equation numbers of the consistency conditions in the appendix:

(i) $X = R$. $T_1 = \text{A.3l}, T_2 = A.3r, T_3 = \text{A.3m}$. Set $c - \chi_4 = 0$. Then

$$\beta^\alpha\partial_\alpha\left[8a + 2c + 2h_1 + 2\beta^\gamma\partial_\gamma n - \beta^\gamma a_{7\gamma}\right] = 2\beta^\alpha\beta^\gamma\left[\partial_\alpha a_{5\gamma} - y_{5\alpha\gamma}\right] \tag{2.58}$$

(ii) $X = \nabla^2 N$. $T_1 = \text{A.3i}, T_2 = \text{A.3j}, T_3 = \text{A.3d}$. Set $4p_4 + 8\rho_{23} = 0$. Then

$$\beta^\alpha\partial_\alpha\left[8\rho_{23} + 4c + 2\rho_{12} + 2\beta^\gamma\partial_\gamma h_2 - \beta^\gamma\rho_{13\gamma}\right] = 2\beta^\alpha\beta^\gamma\left[\partial_\alpha\rho_{24\gamma} - x_{2\alpha\gamma}\right] \tag{2.59}$$

(iii) $X = \nabla_i N\nabla^i N$. $T_1 = \text{A.3s}, T_2 = \text{A.3p}, T_3 = \text{A.3t}$. Set $8p_3 + 4p_4 = 0$. Then

$$\beta^\alpha\partial_\alpha\left[4\chi_4 - 4p_4 + 2\rho_{11} + 2\beta^\gamma\partial_\gamma\chi_3 - \beta^\gamma\rho_{8\gamma}\right] = 2\beta^\alpha\beta^\gamma\left[\partial_\alpha y_\gamma - x_{\alpha\gamma}\right] \tag{2.60}$$

(iv) $X = K$. $T_1 = \text{A.1f}, T_2 = \text{A.1n}, T_3 = \text{A.1h}$. Set $j - \rho_4 = 0$. Then

$$\beta^\alpha\partial_\alpha\left[4b + zj + zl_1 + 2\beta^\gamma\partial_\gamma m - \beta^\gamma\rho_{7\gamma}\right] = 2\beta^\alpha\beta^\gamma\left[\partial_\alpha b_{8\gamma} - x_{5\alpha\gamma}\right] \tag{2.61}$$

We have verified that after accounting for differences in basis and notation Eq. (2.61) is precisely the same as the general C-theorem candidate of this section given in Eq. (2.57).

2.6 Summary and Discussion

Wess-Zumino consistency conditions for Weyl transformations impose constraints on the renormalization group flow of Weyl anomalies. As a first step in studying these constraints in non-relativistic quantum field theories we have classified the anomalies that appear in $d = 2$ (spatial dimensions) at $z = 2$ (dynamical exponent at gaussian fixed point). There are many more anomalies than in the comparable relativistic case (3+1 dimensions): there are 39 anomalies associated with 4-spatial derivatives (Table 2.2), 6 with 2-time derivatives (Table 2.4) and 32 more that contain 1-time and 2-spatial derivaties (Table 2.5). Freedom to add finite amounts to counterterms gives in turn freedom to shift some anomalies arbitrarily. "Trivial Anomalies" are those that can thus be set to zero. We then classified all counterterms (Tables 2.2–2.5), gave the shift in Weyl anomalies produced by shifts in counterterms (in App. A.2), and then listed the trivial anomalies (Table 2.6).

The consistency conditions among these $39+6+32$ anomalies do not mix among the three sectors. They are listed by sector in App. A.1, and from these we can read-off "Vanishing Anomalies" — those that vanish at fixed points; see Table. 2.7. As an application of the use of these conditions we find 6 combinations that give C-function candidates. That is, we find (combinations of) anomalies $\tilde{a}$ and $\mathcal{H}_{\alpha\beta}$ that satisfy $\mu d\tilde{a}/d\mu = \mathcal{H}_{\alpha\beta}\beta^{\alpha}\beta^{\beta}$, where $\beta^{\alpha} = \mu dg^{\alpha}/d\mu$ give the flow of the dimensionless coupling constants; then $\tilde{a}$ flows monotonically provided $\mathcal{H}_{\alpha\beta}$ is positive definite. We have not endeavored to attempt to prove that any of our $\mathcal{H}_{\alpha\beta}$ functions are positive definite, and hence our candidates remain just that, candidates. Exploring positivity of these functions in specific examples would be of interest, and determining model-independently under which conditions positivity holds would be more so.

It is important to appreciate the generality, or lack of it thereof, of our results. While we have used some specific form of the Lagrangian in setting up and contextualizing the computation, there is in fact no need to assume this in order to classify the anomalies and compute the consistency conditions. On the other hand we have made a fairly strong assumption, that the classical action integral is invariant under the anisotropic scale transformation $\boldsymbol{x} \mapsto \lambda \boldsymbol{x}, t \mapsto \lambda^z t$. All our couplings correspond to marginal deformations. In the $3+1$-dimensional relativistic case relevant deformations do modify the consistency conditions, but the candiate C-theorem is not affected, at least by a class of relevant deformations [24]. Clearly, another interesting direction of future study is to investigate the effect of relevant deformations on our consistency conditions: perhaps some of the 6 C-candidates survive even in the presence of relevant deformations, much as in the relativistic case.

While we have performed a detailed analysis only for the $z = 2$ case in $2+1$ dimensions, our results can be readily used in other cases too. For theories in $2+1$ dimensions with $z > 0$ and neither $z = 2k$ nor $z = 2/k$ where k is an integer, only the sector of anomalies with 1-time and 2-spatial derivatives remains. Moreover, the classification of anomalies and the consistency conditions for that sector that were derived assuming $z = 2$ are valid for arbitrary z, with minor modifications in the form of a sprinkling of factors of $z/2$; we have retained explicit z dependence in the consistency conditions in this sector, Eqs. (A.1). This means, in particular, that the 4 C-candidates in this sector, in Eqs. (2.47)–(2.50), are C-candidates for arbitrary z. For $z = 2k \geq 4$ there are anomalies with $2(k+1)$ spatial derivatives; their classification depends on z, so a case-by-case analysis is required. For $z = 2/k \leq 2$ there are anomalies with $k+1$ time derivatives; again their classification depends on z and a case-by-case analysis is required.

For spatial dimensions $d > 2$, if d is even a C-theorem candidate, in Eq. (2.57), becomes available that mimics the one in relativistic theories. Again it relies on assuming only marginal operators are present, but it is possible that, just as in the $3+1$ realtivistic case, the conclusion is not modified by inclusion of relevant deformations. The candidate is based on the anomaly associated with the d-dimensional Euler density for the theory on a curved background. Here again it would be interesting to have an explicit example, to test whether the putative metric in coupling constant space, $\mathcal{H}_{\alpha\beta}$, is positive definite. The analysis of a potential C-theorem in the case of general dimensions d yields four additional potential C-theorems in $d = 2$, three for $z = 2$ given in Eqs. (2.58)–(2.60) and one more for arbitrary z, given in Eq. (2.61). It deserves mention that all of our proposed C theorem candidates are scheme dependent even at a fixed point. Hence, the value of them at a fixed point can be shifted using counter-terms F.

If any of these candidates yields a bona-fide C-theorem the presence of limit cycles in non-relativistic quantum field theories is called into question. Limit cycles in relativistic 3+1 dimensional theories physically correspond to critical points, and the recursive flow corresponds to what amounts to a simultaneous rotation among fundamental fields and marginal operators and their coefficients. Cyclic behavior in non-relativistic quantum systems, on the other hand, do not display continuous scale invariance, so there is no reason to expect that C would remain constant along the flow. The resolution may be that there are no C-theorems at all. Or that there are C-theorems only under conditions that do not apply to systems that exhibit cycles. We look forward to developments in this area.

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Chapter 3

On the Heat Kernel and Weyl Anomaly of Schrödinger invariant theory

We propose a method inspired from discrete light cone quantization (DLCQ) to determine the heat kernel for a Schrödinger field theory (Galilean boost invariant with $z = 2$ anisotropic scaling symmetry) living in $d+1$ dimensions, coupled to a curved Newton-Cartan background, starting from a heat kernel of a relativistic conformal field theory ($z = 1$) living in $d+2$ dimensions. We use this method to show the Schrödinger field theory of a complex scalar field cannot have any Weyl anomalies. To be precise, we show that the Weyl anomaly $\mathcal{A}^G_{d+1}$ for Schrödinger theory is related to the Weyl anomaly of a free relativistic scalar CFT $\mathcal{A}^R_{d+2}$ via $\mathcal{A}^G_{d+1} = 2\pi\delta(m)\mathcal{A}^R_{d+2}$ where m is the charge of the scalar field under particle number symmetry. We provide further evidence of vanishing anomaly by evaluating Feynman diagrams in all orders of perturbation theory. We present an explicit calculation of the anomaly using a regulated Schrödinger operator, without using the null cone reduction technique. We generalise our method to show that a similar result holds for one time derivative theories with even $z > 2$.

3.1 Introduction

The Weyl anomaly in relativistic Conformal Field Theory (CFT) has a rich history [48, 49, 50, 51, 52, 53, 54, 55]. In $1+1$ dimensions irreversibility of RG flows has been established by Zamoldchikov [25] who showed monotonicity of a quantity C that equals the Weyl anomaly c at fixed points. Remarkably, the anomaly c equals the central charge of the CFT. In $3+1$ dimension, there is a corresponding "a-theorem" [56, 57, 41, 58] where a again appears in the Weyl anomaly, and there is strong evidence for a similar a-theorem in higher, even dimensions [28, 29, 30, 32]. In contrast, much less is known in the case of non-relativistic field theories admitting anisotropic scale invariance under the following transformation

$$\boldsymbol{x} \to \lambda \boldsymbol{x}, \qquad t \to \lambda^z t\,. \tag{3.1}$$

Nonetheless, non-relativistic conformal symmetry does emerge in various scenarios. For example, fermions at unitarity, in which the S-wave scattering length diverges, $|a| \to \infty$, exhibit non-relativistic conformal symmetry. In ultracold atom gas experiments, the S-wave scattering length can be tuned freely along an RG flow and this has renewed interest in the study of the RG flow of such theories [1, 2]. In fact, at $a^{-1} = -\infty$ the system behaves as a BCS superfluid while at $a^{-1} = \infty$ it becomes a BEC superfluid. The BCS-BEC crossover, at $a^{-1} = 0$, is precisely the unitarity limit, exhibiting non-relativistic conformal symmetry [59, 60]. In this regime, we expect universality, with features independent of any microscopic details of the atomic interactions. Other examples of non-relativistic systems exhibiting scaling symmetry come with accidentally large scattering cross section. Examples include various atomic systems, like ^{85}Rb[5],^{138}Cs [6], and few nucleon systems like the deuteron [3, 4].

Galilean CFT, which enjoys $z = 2$ scaling symmetry is special among Non-Relativistic Conformal Field Theories (NRCFTs). On group theoretic grounds, there is a special conformal generator for $z = 2$ that is not present for $z \neq 2$ theories [61, 62]. The coupling of such theories to the Newton Cartan (NC) structure is well understood [62, 63, 64, 65]. The generic discussion of anomalies in such theories has been initiated by Jensen in [22]. Moreover, there have been recent works classifying and evaluating Weyl anomalies at fixed points [18, 66, 20, 67, 68] and even away from the fixed points; the latter have resulted in proposed C-theorem candidates [21, 69].

It has been proposed in [22], using the fact that Discrete Light Cone Quantization (DLCQ) of a relativistic CFT living in $d+2$ dimensions yields a non-relativistic Galilean CFT in $d+1$ dimensions with $z=2$, that the Weyl anomaly of the relativistic CFT survives in the non-relativistic theory. The conjecture states that the Weyl anomaly $\mathcal{A}^G$ for a Schrödinger field theory (Galilean boost invariant with $z=2$ scale symmetry and special conformal symmetry) is given by

$$\mathcal{A}^G_{d+1} = aE_{d+2} + \sum_n c_n W_n \tag{3.2}$$

where E_{d+2} is the $d+2$ dimensional Euler density of the parent space-time and W_n are Weyl covariant scalars with weight $(d+2)$. The right hand side is computed on a geometry given in terms of the $d+2$ dimensional metric; this will be explained below, see Eq. (3.19). A specific example of particular interest is

$$\mathcal{A}^G_{2+1} = aE_4 - cW^2 \tag{3.3}$$

where W^2 stands for the square of the Weyl tensor.

The purpose of this work is twofold. First, we show that these proposed relations must be corrected to include a factor of $\delta(m)$, when the Schrödinger invariant theory involves a single complex scalar field having charge m under the $U(1)$ symmetry. To be precise, we show that

$$\mathcal{A}^G_{d+1} = 2\pi\delta(m)\mathcal{A}^R_{d+2} \tag{3.4}$$

where $\mathcal{A}^R_{d+2}$ is the Weyl anomaly of the corresponding relativistic CFT in $d+2$ dimensions. This is derived explicitly for the case of a bosonic (commuting) scalar field, but the derivation applies equally to the case of a fermionic (anti-commuting) scalar field. The second purpose is to develop a framework inspired from DLCQ to evaluate the heat kernel of a theory with one time derivative kinetic term in a non-trivial curved background. This framework enables us to calculate not only the heat kernel but also the anomaly coefficients. In fact, using this method and its appropriately modified form enables us to generalise Eq. (3.4) to one time derivative theories with arbitrary even z, where the parent $d+2$ dimensional theory enjoys $SO(1,1)\times SO(d)$ symmetry with scaling symmetry acting as $t\to\lambda^{z/2}t, x^{d+2}\to\lambda^{z/2}x^{d+2}, x^i\to\lambda x^i$, $(i=1,\dots,d+1)$.

The paper is organised as follows. We will briefly review coupling of a Schrödinger field theory to the Newton-Cartan structure in Sec. 3.2. In Sec. 3.3, we sketch how DLCQ can be used to obtain Schrödinger field theories following the procedure of [22] and propose its modified cousin, that we call Lightcone Reduction (LCR), to obtain a Schrödinger field theory. In Sec. 3.4 we determine the heat kernel for free Galilean CFT coupled to a flat NC structure in two different ways, on the one hand using LCR and on the other without the use of DLCQ, providing a check on our proposed method for determining the heat kernel for Galilean field theory coupled to a curved NC geometry. We then proceed to evaluate the heat kernel on curved spacetime according to the proposal and subsequently derive the Weyl anomaly for Schrödinger field theory of a single complex scalar. In Sec. 3.5 we reconsider the computation using perturbation theory; we find that for a wide class of models on a curved background all vacuum diagrams vanish. In fact, we show that an anomaly is not induced in the more general case that $U(1)$ invariant dimensionless couplings are included, regardless of whether we are at a fixed point or away from it, in all orders of a perturbative expansion in the dimensionless coupling and metric. In Sec. 3.6, we give a formal proof of our prescription and generalise the framework to calculate the heat kernel and anomaly for theories with one time derivative and arbitrary even z. We conclude with a brief summary of the results obtained and discuss future directions of investigation. Technical aspects of defining heat kernel for one time derivative theory in flat space-time are explored in App. B.1, and on a curved background in App. B.2. Finally, in App. B.3 we present an explicit calculation of the anomaly using a regulated Schrödinger operator, without using the null cone reduction technique.

3.2 Newton-Cartan Structure & Weyl Anomaly

The study of the Weyl anomaly necessitates coupling of non-relativistic theory to a background geometry, which can potentially be curved. Generically, the prescription for coupling to a background can depend on the global symmetries of the theory on a flat background. Of interest to us are Galilean and Schrodinger field theories. The algebra of the Galilean generators is given by [61]

$$\begin{gathered}
[M_{ij},N]=0\,, \qquad [M_{ij},P_k]=\imath(\delta_{ik}P_j-\delta_{jk}P_i)\,, \qquad [M_{ij},K_k]=\imath(\delta_{ik}K_j-\delta_{jk}K_i)\,, \\
[M_{ij},M_{kl}]=\imath(\delta_{ik}M_{jl}-\delta_{jk}M_{il}+\delta_{il}M_{kj}-\delta_{jl}M_{ki})\,, \\
[P_i,P_j]=[K_i,K_j]=0\,, \qquad [K_i,P_j]=\imath\delta_{ij}N\,, \\
[H,N]=[H,P_i]=[H,M_{ij}]=0\,, \quad [H,K_i]=-\imath P_i\,,
\end{gathered} \tag{3.5}$$

and the commutators of dilatation generator with that of Galilean ones are given by

$$\begin{gathered}
[D,P_i]=\imath P_i\,, \qquad [D,K_i]=(1-z)\imath K_i\,, \quad [D,H]=z\imath H\,, \\
[D,N]=\imath(2-z)N\,, \quad [M_{ij},D]=0
\end{gathered} \tag{3.6}$$

where $i,j=1,2,\ldots,d$ label the spatial dimensions, z is the anisotropic exponent, P_i, H and M_{ij} are generators of spatial translations, time translation spatial rotations, respectively, K_i generates Galilean boosts along the x^i direction, N is the particle number (or rest mass) symmetry generator and D is the generator of dilatations. The generators of Schrödinger invariance include, in addition, a generator of special conformal transformations, C. The Schrödinger algebra consists of the $z=2$ version of (3.5),(3.6) plus the commutators of C,

$$[M_{ij},C]=0\,,\qquad [K_i,C]=0\,,\qquad [D,C]=-2\imath C\,,\qquad [H,C]=-\imath D. \tag{3.7}$$

In what follows, by Schrödinger invariant theory we will mean a $z=2$ Galilean, conformally invariant theory. For $z\neq 2$ we only discuss anisotropic scale invariant theories invariant under a group generated by P_i, M_{ij}, H, D and N such that the kinetic term involves one time derivative only. The most natural way to couple Galilean (boost) invariant field theories to geometry is to use the Newton-Cartan (NC) structure [62, 63, 64]. In what follows we briefly review NC geometry, following Ref. [22].

The NC structure defined on a $d+1$ dimensional manifold $\mathcal{M}_{d+1}$ consists of a one form n_μ, a symmetric positive semi-definite rank d tensor $h_{\mu\nu}$ and an $U(1)$ connection A_μ, such that the metric tensor

$$g_{\mu\nu}=n_\mu n_\nu+h_{\mu\nu} \tag{3.8}$$

is positive definite. The upper index data v^μ and $h^{\mu\nu}$ is defined by

$$v^\mu n_\mu=1,\qquad v^\nu h_{\mu\nu}=0,\qquad h^{\mu\nu}n_\nu=0,\qquad h^{\mu\rho}h_{\rho\nu}=\delta^\mu_\nu-v^\mu n_\nu \tag{3.9}$$

Physically v^μ defines a local time direction while $h_{\mu\nu}$ defines a metric on spatial slice of $\mathcal{M}_d$.

As prescribed in [62], while coupling a Galilean invariant field theory to a NC structure, we demand

1. Symmetry under reparametrization of co-ordinates. Technically, this requirement boils down to writing the theory in a diffeomorphism invariant way.

2. $U(1)$ gauge invariance. The fields belonging to some representation of Galilean algebra carry some charge under particle number symmetry, which is an $U(1)$ group. Promoting this to a local symmetry requires a gauge field A_μ that is sourced by the $U(1)$ current.

3. Invariance under Milne boost under which $(n_\mu, h^{\mu\nu})$ remains invariant, while

$$v^\mu \to v^\mu + \psi^\mu, \quad h_{\mu\nu} \to h_{\mu\nu} - \left(n_\mu \psi_\nu + n_\nu \psi_\mu\right) + n_\mu n_\nu \psi^2, \quad A_\mu \to A_\mu + \psi_\mu - \frac{1}{2} n_\mu \psi^2 \tag{3.10}$$

where $\psi^2 = h^{\mu\nu}\psi_\mu\psi_\nu$ and $v^\nu\psi_\nu = 0$.

The action of a free Galilean scalar ϕ_m with charge m, coupled to this NC structure satisfying all the symmetry conditions listed above is given by

$$\int d^{d+1}x\sqrt{g}\left[\imath m v^\mu \left(\phi_m^\dagger D_\mu \phi_m - \phi_m D_\mu \phi_m^\dagger\right) - h^{\mu\nu} D_\mu \phi_m^\dagger D_\nu \phi_m\right] \tag{3.11}$$

where $D_\mu = \partial_\mu - \imath m A_\mu$ is the appropriate gauge invariant derivative.

From a group theory perspective, a Galilean group can be a subgroup of a larger group that includes dilatations. That is, besides the symmetries mentioned earlier, a Galilean invariant field theory coupled to the flat NC structure can also be scale invariant, *i.e.,* invariant under the following transformations

$$\boldsymbol{x} \to \lambda \boldsymbol{x}, \qquad t \to \lambda^z t, \tag{3.12}$$

where z is the dynamical critical exponent of the theory. As mentioned earlier, for $z = 2$, the symmetry algebra may further be enlarged to contain a special conformal generator, resulting in the Schrödinger group. On coupling a Galilean CFT with arbitrary z to a nontrivial curved NC structure, the scale invariance can be thought of as invariance under following scaling of NC data (also known as anisotropic Weyl scaling; henceforth we omit the word *anisotropic*, and by Weyl transformation it should be understood that we mean the transformation with appropriate z):

$$n_\mu \to e^{z\sigma} n_\mu, \qquad h_{\mu\nu} \to e^{2\sigma} h_{\mu\nu}, \qquad A_\mu \to e^{(2-z)\sigma} A_\mu, \tag{3.13}$$

where σ is a function of space and time.

Even though classically a Galilean CFT may be scale invariant, it is not necessarily true that it remains invariant quantum mechanically. Renormalisation may lead to anomalous breaking of scale symmetry much like in the Weyl anomaly in relativistic CFTs (where $z = 1$). The anomaly $\mathcal{A}$ is defined from the infinitesimal Weyl variation (3.13) of the connected generating functional W:

$$\delta_\sigma W = \int d^{d+1}x \sqrt{g}\, \delta\sigma\, \mathcal{A}, . \tag{3.14}$$

We mention in passing that away from the fixed point the coupling is scale dependent, that is, the running of the coupling under the RG must be accounted for, hence the variation δ_σ on the couplings needs to be incorporated. The generic scenario has been elucidated in Ref. [69].

In this work, we are interested in anomalies at a fixed point. Even in the absence of running of the coupling, the background metric can act as an external operator insertion on vacuum bubble diagrams leading to new UV divergences that are absent in flat space-time. Removing these new divergences can potentially lead to anomalies. The anomalous ward identity for anisotropic Weyl transformation is given by[22]

$$z n_\mu \mathcal{E}^\mu - h^{\mu\nu} T_{\mu\nu} = \mathcal{A}, \tag{3.15}$$

where $n_\mu \mathcal{E}^\mu$ and $h^{\mu\nu} T_{\mu\nu}$ are respectively diffeomorphic invariant measure of energy density and trace of spatial stress-energy tensor.

In what follows, we will be interested in evaluating the quantity appearing on the right hand side of Eq. (3.15). A standard method is through the evaluation of the heat kernel in a curved background. Hence, our first task is to figure out a way to obtain the heat kernel for theories with kinetic term involving only one time derivative. In the next few sections we will introduce methods for computing heat kernels and arrive at the same result from different approaches.

3.3 Discrete Light Cone Quantization (DLCQ) & its cousin Lightcone Reduction (LCR)

One elegant way to obtain the heat kernel is to use Discrete Light Cone Quantization (DLCQ). This exploits the well known fact that a $d+1$ Galilean invariant field theory can be constructed by starting from a relativistic theory in $d+2$ dimensional Minkowski space in light cone coordinates

$$ds^2 = 2dx^+dx^- + dx^i dx^i \tag{3.16}$$

where $i = 2,3,\ldots,d+1$ and $x^\pm = \frac{x^1 \pm t}{\sqrt{2}}$ define light cone co-ordinates, followed by a compactification in the null co-ordinate x^- on a circle. From here on, by *reduced* theory we will mean the theory in $d+1$ dimensions while by *parent* theory we will mean the $d+2$ dimensional theory on which this DLCQ trick is applied. We first present a brief review of DLCQ.

The generators of $SO(d+1,1)$ which commute with P_-, the generator of translation in the x^- direction, generate the Galilean algebra. P_- is interpreted as the generator of particle number of the reduced theory. In light cone coordinates the mass-shell condition for a massive particle becomes[1]

$$p_+ = \frac{|\boldsymbol{p}|^2}{2(-p_-)} + \frac{M^2}{4(-p_-)} \tag{3.17}$$

Eq. (3.17) can be interpreted as the non-relativistic energy of a particle, p_+, with mass $m = -p_-$ in a constant potential. The reduced mass-shell condition (3.17) is Galilean invariant, that is, invariant under boosts ($\boldsymbol{v}$) and rotations ($\mathbf{R}$):

$$\boldsymbol{p} \to \mathbf{R}\boldsymbol{p} - \boldsymbol{v}p_-, \qquad p_+ \to p_+ + \boldsymbol{v}\cdot(\mathbf{R}\boldsymbol{p}) - \frac{1}{2}|\boldsymbol{v}|^2 p_-$$

Setting $M = 0$, the dispersion relation is of the form

$$\omega = \frac{k^2}{2m} \tag{3.18}$$

and enjoys $z = 2$ scaling symmetry. To rephrase, setting $M = 0$ will allow one to append a dilatation generator, which acts as follows:

$$p_+ \to \lambda^2 p_+, \quad p_- \to p_-, \quad \boldsymbol{p} \to \lambda\boldsymbol{p}$$

Had we not compactified in the x^- direction, p_- would be a continuous variable. The parameter p_- can be changed using a boost in the $+-$ direction, but compactification in the x^- direction spoils relativistic boost symmetry and the eigenvalues of p_- become discretized, $p_- = \frac{n}{R}$, where R is the compactification radius. We note that Lorentz invariance is recovered in the $R \to \infty$ limit. For convenience, by appropriately rescaling the generators of spatial translations and of special conformal transformations, as well as P_-, we can set $R = 1$.

One can technically perform DLCQ even in a curved space-time as long as the metric admits a null isometry. This guarantees that we can adopt a coordinate system with a null

[1]The unusual sign convention in our definition of x^- results in the peculiar sign in Eq. (3.17).

coordinate x^- such that all the metric components are independent of x^-. To be specific, we will consider the following metric:

$$ds^2 = G_{MN}dx^M dx^N, \qquad G_{\mu-} = n_\mu, \qquad G_{\mu\nu} = h_{\mu\nu} + n_\mu A_\nu + n_\nu A_\mu, \qquad G_{--} = 0 \tag{3.19}$$

where $M,N = +,-,1,2,\ldots,d$ run over all the indices in $d+2$ dimensions, the index $\mu = +,1,2,\ldots,d$ runs over $d+1$ dimensions and $h_{\mu\nu}$ is a rank d tensor. Ultimately, $h_{\mu\nu}, n_\mu, A_\mu$ are to be identified with the NC structure, and just as above we can construct $h^{\mu\nu}$ and v^μ such that Eq. (3.9) holds. Moreover, these quantities transform under Milne boost symmetry as per Eq. (3.10). Hence, the boost invariant inverse metric is given by

$$G^{-\mu} = v^\mu - h^{\mu\nu}A_\nu, \qquad G^{\mu\nu} = h^{\mu\nu}, \qquad G^{--} = -2v^\mu A_\mu + h^{\mu\nu}A_\mu A_\nu\,. \tag{3.20}$$

Reduction on x^- yields a Galilean invariant theory coupled to an NC structure given by $(n_\mu, h^{\mu\nu}, A_\mu)$, with metric given by (3.8). Moreover, all the symmetry requirements listed above Eq. (3.10) are satisfied by construction.

This prescription allows us to construct Galilean QFT coupled to a non trivial NC structure starting from a relativistic QFT placed in a curved background with one extra dimension. For example, we can consider DLCQ of a conformally coupled scalar field in $d+2$ dimensions,

$$S_R = \int d^{d+2}x\sqrt{-G}\left[-G^{MN}\partial_M\Phi^\dagger\partial_N\Phi - \xi\mathcal{R}\Phi^\dagger\Phi\right], \qquad \xi = \frac{d}{4(d-1)} \tag{3.21}$$

where $\mathcal{R}$ stands for the Ricci scalar corresponding to the G_{MN} metric. We compactify x^- with periodicity 2π and expand Φ in fourier modes as

$$\Phi = \frac{1}{\sqrt{2\pi}}\sum_m \phi_m(x^\mu)e^{\imath m x^-}, \qquad \phi_m = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} dx^-\ \Phi e^{-\imath m x^-}\,. \tag{3.22}$$

In terms of ϕ_m , we recast the action, Eq. (3.21) in following form using Eq. (3.20)

$$S_R = \sum_m \int d^{d+1}x\sqrt{g}\left[\imath m v^\mu\left(\phi_m^\dagger D_\mu\phi_m - \phi_m D_\mu\phi_m^\dagger\right) - h^{\mu\nu}D_\mu\phi_m^\dagger D_\nu\phi_m - \xi\mathcal{R}\phi_m^\dagger\phi_m\right] \tag{3.23}$$

where $D_\mu = \partial_\mu - \imath m A_\mu$ and where each of the ϕ_m carry charge m under the particle number symmetry and sit in distinct representations of the Schrödinger group. The theory described by Eq. (3.23) is not Lorentz invariant because we have a discrete sum over m, breaking the boost invariance along the null direction.

The point of DLCQ is to break Lorentz invariance to Galilean invariance. As explained above, one can work in the uncompactified limit, and still break the Lorentz invariance by dimensional reduction. In the uncompactified limit, the sum over eigenvalues of P_- becomes integration over the continuous variable p_-. Nonetheless, one can focus on any particular Fourier mode. Technically, we can implement this by performing a Fourier transformation with respect to x^- of quantities of interest. This procedure also yields a Galilean invariant field theory where the elementary field is the particular Fourier mode under consideration. Henceforth we will refer to this modified version of DLCQ as Lightcone Reduction (LCR).

Taking a cue from the relation between the actions given by Eqs. (3.21) and (3.23) we propose the following prescription to extract the heat kernel in the reduced theory:
The heat kernel operator K_G in $d+1$ dimensional Galilean theory is related to the heat kernel operator K_R of the parent $d+2$ dimensional relativistic theory via

$$\langle(\boldsymbol{x}_2,t_2)|K_G|(\boldsymbol{x}_1,t_1)\rangle = \int_{-\infty}^{\infty} dx^- \ \langle\boldsymbol{x}_2,x_2^-,x_2^+|K_R|\boldsymbol{x}_1,x_1^-,x_1^+\rangle\ e^{-\imath m x_{12}^-} \tag{3.24}$$

where $x_{12}^- = x_2^- - x_1^-$ and the time t in the reduced theory is to be equated with x^+ in the parent theory.

We will postpone the proof of our prescription to Sec. 3.6. In the next section, we will lend support to our prescription by verifying our claim using two different methods of calculating the heat kernel. We emphasize that the reduction prescription, described above, is applicable to the $z = 2$ case of Galilean and scale invariant theories. The generic reduction procedure for arbitrary z (though not Galilean boost invariant) is discussed later in sec. 3.6.2.

3.4 Heat Kernel for a Galilean CFT with $z = 2$

3.4.1 Preliminaries: Heat Kernel, Zeta Regularisation

We start by briefly reviewing the heat kernel and zeta function regularisation method [70, 71, 57, 30]. A pedagogical discussion can be found in [72, 73]. Let us consider a theory with partition function $\mathcal{Z}$, formally given by

$$\mathcal{Z} = \int [\mathcal{D}\phi][\mathcal{D}\phi^\dagger]\, e^{-\int d^d x \phi^\dagger \mathcal{M} \phi} \tag{3.25}$$

where the eigenvalues of the operator $\mathcal{M}$ have positive real part.[2]The path integral over the field variable ϕ suffers from ultraviolet (UV) divergences and requires proper regularization and renormalisation to be rendered as a meaningful finite quantity. Similarly, the quantum effective action $W = -\ln \mathcal{Z}$ corresponding to this theory, given by a formal expression

$$W = \ln(\det(\mathcal{M}))$$

requires regularization and renormalisation.[3]

The method of zeta-function regularization introduces several quantities; the heat kernel operator

$$\mathcal{G} = e^{-s\mathcal{M}}\,, \tag{3.26}$$

[2]Positivity is required for convergence of the gaussian integral.

[3]For anti-commuting fields $W = -\ln(\det(\mathcal{M}))$; the minus sign is the only difference between commuting and anti-commuting cases, so that in what follows we restrict our attention to the case of commuting fields.

its trace K over the space L^2 of square integrable functions

$$K(s,f,\mathcal{M}) = \mathrm{Tr}_{L^2}\left(f\mathcal{G}\right) = \mathrm{Tr}_{L^2}\left(fe^{-s\mathcal{M}}\right), \tag{3.27}$$

where $f \in L^2$, and the zeta-function, defined as

$$\zeta(\varepsilon, f, \mathcal{M}) = \mathrm{Tr}_{L^2}\left(f\mathcal{M}^{-\varepsilon}\right). \tag{3.28}$$

K and ζ are related via Mellin transform,

$$K(s,f,\mathcal{M}) = \frac{1}{2\pi\imath}\int_{c-\imath\infty}^{c+\imath\infty} d\varepsilon\, s^{-\varepsilon}\Gamma(\varepsilon)\zeta(\varepsilon,f,\mathcal{M}) \quad \text{and} \quad \zeta(\varepsilon,f,\mathcal{M}) = \frac{1}{\Gamma(\varepsilon)}\int_0^\infty ds\, s^{\varepsilon-1}K(s,f,\mathcal{M}). \tag{3.29}$$

As is customary, below we use $f = 1$. However this should be understood as taking the limit $f \to 1$ at the end of the computation to ensure all expressions in intermediate steps are well defined.

Formally W is given by the divergent expression

$$W = -\int_0^\infty ds \frac{1}{s} K(s,1,\mathcal{M})$$

The regularized version, W_ε, is defined by shifting the power of s

$$W_\varepsilon = -\tilde{\mu}^{2\varepsilon}\int_0^\infty ds \frac{1}{s^{1-\varepsilon}} K(s,1,\mathcal{M}) = -\tilde{\mu}^{2\varepsilon}\Gamma(\varepsilon)\zeta(\varepsilon,1,\mathcal{M}) \tag{3.30}$$

where the parameter $\tilde{\mu}$ with length dimension -1 is introduced so that W_ε remains adimensional. In this context, the parameter ε behaves like a regulator, the divergences re-appearing as $\varepsilon \to 0$.

In this limit

$$W_\varepsilon = -\left(\frac{1}{\varepsilon} - \gamma_E + \ln(\tilde{\mu}^2)\right)\zeta(0,1,\mathcal{M}) - \zeta'(0,1,\mathcal{M}) + O(\varepsilon)\,,$$

so that subtracting the $\frac{1}{\varepsilon}$ term gives the renormalized effective action

$$W^{\text{ren}} = -\zeta'(0,1,\mathcal{M}) - \ln(\mu^2)\zeta(0,1,\mathcal{M})\,. \tag{3.31}$$

where $\mu^2 = \tilde{\mu}^2 e^{-\gamma_E}$ and γ_E is the Euler constant. On a compact manifold $\zeta(\varepsilon,1,\mathcal{M})$ is finite as $\varepsilon \to 0$ and the renormalized effective action given by (3.31) is finite, as it should. For non-compact manifolds the standard procedure for computing a renormalized effective action is to subtract a reference action that does not modify the physics. One may, for example, define $W = \ln(\det(\mathcal{M})/\det(\mathcal{M}_0))$, where the operator $\mathcal{M}_0$ is defined on a trivial (flat) background. This amounts to replacing $K(s,1,\mathcal{M}) \to K(s,1,\mathcal{M}) - K(s,1,\mathcal{M}_0)$ in Eq. (3.30) and correspondingly $\zeta(\varepsilon,1,\mathcal{M}) \to \zeta(\varepsilon,1,\mathcal{M}) - \zeta(\varepsilon,1,\mathcal{M}_0)$. The expression for W^{ren} in (3.31) remains valid if it is understood that this subtraction is made before the $\varepsilon \to 0$ limit is taken.

Classical symmetry under Weyl variations (both in the relativistic case and the anisotropic one) guarantees $\mathcal{M}$ transforms homogeneously, *i.e.,* $\delta_\sigma\mathcal{M} = -\Delta\sigma\mathcal{M}$ under $\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}$ where Δ is the scaling dimension of $\mathcal{M}$. Hence, we have

$$\delta_\sigma\zeta(\varepsilon,1,\mathcal{M}) = -\varepsilon\mathrm{Tr}_{L^2}\left(\delta\mathcal{M}\,\mathcal{M}^{-\varepsilon-1}\right) = \Delta\varepsilon\zeta(\varepsilon,\sigma,\mathcal{M})\,. \tag{3.32}$$

Consequently, the anomalous variation of W is given by

$$\delta_\sigma W^{ren} = -\Delta\zeta(0,\sigma,\mathcal{M})\,. \tag{3.33}$$

In the relativistic case, using the fact that

$$\delta_\sigma W = \frac{1}{2}\int d^{d+1}x\sqrt{g}T_{\mu\nu}\delta g^{\mu\nu} = -\int d^{d+1}x\sqrt{g}T^{\mu}{}_{\mu}\delta\sigma\,, \tag{3.34}$$

one has the trace anomaly equation

$$\mathcal{A} = -T^{\mu}{}_{\mu} = -\frac{1}{\sqrt{g}}\Delta\left(\frac{\delta\zeta(0,\sigma,\mathcal{M})}{\delta\sigma}\right)_{\sigma=0}. \tag{3.35}$$

In the non-relativistic case, the Weyl anisotropic scaling is given by $h_{\mu\nu} \to e^{2\sigma}h_{\mu\nu}$ and $n_\mu \to e^{z\sigma}n_\mu$. We have

$$\delta_\sigma W = \int d^{d+1}x\sqrt{g}\left(\frac{1}{2}T_{\mu\nu}\delta h^{\mu\nu} - \mathcal{E}_\mu\delta n^\mu\right) = \int d^{d+1}x\sqrt{g}\left(h^{\mu\nu}T_{\mu\nu} - zn^\mu\mathcal{E}_\mu\right)\delta\sigma \tag{3.36}$$

leading to

$$\mathcal{A} = zn_\mu\mathcal{E}^\mu - h^{\mu\nu}T_{\mu\nu} = -\frac{1}{\sqrt{g}}\Delta\left(\frac{\delta\zeta(0,\sigma,\mathcal{M})}{\delta\sigma}\right)_{\sigma=0}. \tag{3.37}$$

One can evaluate $\delta\zeta(0,\sigma,\mathcal{M})/\delta\sigma\big|_{\sigma=0}$ using the asymptotic form ($s \to 0$) of the heat kernel, K. The asymptotic expansion depends on the operator $\mathcal{M}$ and its scaling dimension. Schematically, one has

$$K(s,1,\mathcal{M}) = \frac{1}{s^{d_\mathcal{M}}}\sum_{n=0}^{\infty} s^{\kappa(n)}\sqrt{g}a_n,$$

where $\kappa(n)$ is a linear function of n. The singular pre-factor, $\frac{1}{s^{d_\mathcal{M}}}$, is determined by the heat kernel in the background-free, flat space-time limit while the expansion accounts for corrections from background fields or geometry. The asymptotic expansion is guaranteed to exist if the heat kernel is well behaved for $s > 0$ in the flat space-time limit, that is, if $\sum_i e^{-s\lambda_i}$, with λ_i, the eigenvalues of the operator $\mathcal{M}$, is convergent. The convergence requires that λ_i have, at worst, a power law growth and positive real part [74].

We are interested in operators $\mathcal{M}$ of generic form

$$\mathcal{M} = 2\imath m\partial_{t'} - (-1)^{z/2}(\partial_i\partial_i)^{z/2},$$

for which the heat kernel has a small s expansion of the following form

$$K(s,1,\mathcal{M}) = \frac{1}{s^{1+d/z}}\sum_{n=0}^{\infty} s^{2n/z}\int d^{d+1}x\sqrt{g}\,a_n, \tag{3.38}$$

where d is number of spatial dimension and z is dynamical exponent.[4] Then the zeta function is given by

$$\zeta(0,f,\mathcal{M}) = \int d^{d+1}x\sqrt{g}\,f\,a_{(d+z)/2}, \tag{3.39}$$

so that we arrive at an expression for the Weyl anomaly

$$\mathcal{A} = -\Delta\, a_{(d+z)/2}. \tag{3.40}$$

Hence, in order to determine the Weyl anomaly, one has to calculate the coefficient $a_{(d+z)/2}$ of the heat kernel expansion (3.38).[5] In subsequent sections, we will find out a way to evaluate the heat kernel in flat space-time and then in curved space-time for a Schrödinger invariant field theory. We will be doing this first without using DLCQ/LCR, and then again with LCR (modified cousin of DLCQ) using the prescription introduced above.

[4]In next few sections, we explicitly find this asymptotic form for $z = 2$ while the arbitrary z case is handled separately in 3.6.2.

[5]Incidentally, this shows that the anomaly is absent when $d+z$ is odd.

3.4.2 Heat Kernel in Flat Space-time

Direct calculation (without use of DLCQ)

The action for a free Galilean CFT on a flat space-time (which is in fact invariant under the Schrödinger group) is given by

$$S = \int dt\ d^d x \phi^\dagger \left[2m\imath\partial_t + \nabla^2\right] \phi \tag{3.41}$$

In order to improve convergence of the functional integral defining the partition function we perform a continuation to imaginary time :

$$e^{\imath \int dt d^d x \phi^\dagger \left[2m\imath\partial_t + \nabla^2\right]\phi} \underset{t=-\imath\tau}{\mapsto} e^{-\int d\tau d^d x \phi^\dagger \left[2m\partial_\tau - \nabla^2\right]\phi} \tag{3.42}$$

Hence, the Euclidean version of $\mathcal{M} = 2m\imath\partial_t + \nabla^2$ is given by

$$\mathcal{M}_E = 2m\partial_\tau - \nabla^2\,, \tag{3.43}$$

and it is this operator for which we will compute the heat kernel. The prescription $t = -\imath\tau$ is equivalent to adding $+\imath\varepsilon$ to the propagator in Minkowskian flat space. In fact, the same $+\imath\varepsilon$ prescription is obtained by deriving the non-relativistic propagator as the non-relativistic limit of the relativistic propagator.

The Heat kernel for $\mathcal{M}_E$ is a solution to the equation[6]

$$(\partial_s + \mathcal{M}_E)\,\mathcal{G} = 0\,, \tag{3.44}$$

[6]Even though $\mathcal{M}_E$ is not a hermitian operator, the heat kernel is well defined for any operator as long as $\mathrm{Re}(\lambda_k) > 0$ where λ_k are its eigenvalues. We explore this technical aspect in appendix.

that is

$$(\partial_s + 2m\partial_{\tau_2} - \nabla^2_{x_2})\,\mathcal{G}(s;(\boldsymbol{x}_2,\tau_2),(\boldsymbol{x}_1,\tau_1)) = 0\,, \tag{3.45}$$

with boundary condition $\mathcal{G}(0;(\boldsymbol{x}_2,\tau_2),(\boldsymbol{x}_1,\tau_1)) = \delta(\tau_2-\tau_1)\delta^d(\boldsymbol{x}_2-\boldsymbol{x}_1)$. Equation (3.45) is solved by

$$\mathcal{G}(s;(\boldsymbol{x}_2,\tau_2),(\boldsymbol{x}_1,\tau_1)) = \delta\left(2ms-(\tau_2-\tau_1)\right)\frac{e^{-\frac{|\boldsymbol{x}_2-\boldsymbol{x}_1|^2}{4s}}}{(4\pi s)^{\frac{d}{2}}} \tag{3.46}$$

Consequently, the Eulcidean two point correlator is given by

$$G((\boldsymbol{x}_2,\tau_2),(\boldsymbol{x}_1,\tau_1)) = \int_0^\infty ds\,\mathcal{G}(s) = \frac{\theta(\tau)}{2m}\frac{e^{-\frac{m|\boldsymbol{x}|^2}{2\tau}}}{(2\pi\frac{\tau}{m})^{\frac{d}{2}}} \tag{3.47}$$

where $\tau = \tau_2 - \tau_1$ and $\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. The same two point correlator can be obtained by Fourier transform from the Minkowski momentum space propagator G_M, or its imaginary time version,

$$G_M(p,\omega) = \frac{\imath}{2m\omega - |\boldsymbol{p}|^2 + i0^+} \underset{\substack{t=-\imath\tau \\ \omega=\imath\omega_E}}{\mapsto} G = \frac{1}{2m\omega_E + \imath|\boldsymbol{p}|^2} \tag{3.48}$$

In the coincidence limit the heat kernel of (3.46) contains a Dirac-delta factor, $\delta(ms)$. Since this non-analytic behavior is unfamiliar, it is useful to re-derive this result by directly computing the trace K, Eq. (3.26). One can conveniently choose the test function $f = e^{-|\eta\omega|}$. Hence

$$K(s,f,\mathcal{M}_{E,g}) = \mathrm{Tr}\,\left(fe^{-s\mathcal{M}_{E,g}}\right) = \int\left(\frac{d^dk}{(2\pi)^d}e^{-sk^2}\right)\left(\int\frac{d\omega}{2\pi}e^{-2m\imath s\omega-|\eta\omega|}\right)$$

The integral over k gives the factor of $1/s^{d/2}$, while the integral over ω gives

$$\frac{1}{\pi}\frac{\eta}{4m^2s^2+\eta^2}$$

that tends to $\delta(2ms)$ as $\eta \to 0$. Before taking the limit, this factor gives a well behaved function for which the Mellin transform that defines ζ, Eq. (3.29), is well defined for $d/2 < \mathrm{Re}(\varepsilon) < d/2+2$ and can be analytically continued to $\varepsilon = 0$.

One may be concerned that the derivation above is only formal as it does not involve an elliptic operator. This is easily remedied by considering the elliptic operator[7] $\mathcal{M}' = \imath\eta\sqrt{-\partial_t^2} + \imath(2m)\partial_t + \nabla^2$. Its spectrum, $\left(2m\omega - k^2 + \imath\eta|\omega|\right)$, tends to that of the Minkowskian Schrödinger operator $\mathcal{M}$ as $\eta \to 0$. Consequently, the spectrum for the Euclidean avatar[8] ($\mathcal{M}'_{E,g}$) of $\mathcal{M}'$ becomes $\left(k^2 + 2m\imath\omega + |\eta\omega|\right)$ and the heat kernel for that operator is given by

$$K(s,1,\mathcal{M}'_{E,g}) = \mathrm{Tr}\,\left(e^{-s\mathcal{M}'_{E,g}}\right) = \int\left(\frac{d^dk}{(2\pi)^d}e^{-sk^2}\right)\left(\int\frac{d\omega}{2\pi}e^{-2m\imath s\omega - s|\eta\omega|}\right)$$

The integral over k gives the factor of $1/s^{d/2}$ as before, while the integral over ω gives

$$\frac{1}{\pi s}\left(\frac{\eta}{4m^2+\eta^2}\right)$$

that tends to $\frac{1}{s}\delta(2m)$ as $\eta \to 0$. As we will see later, Light Cone Reduction technique indeed reproduces this factor of $\delta(2m)$.

Derivation using LCR

In Euclidean, flat $d+2$ dimensional space-time, the heat kernel $\mathcal{G}_{R,E}$ of a relativistic scalar field at free fixed point is given by [75]

$$\mathcal{G}_{R,E}(s;x_2^M,x_1^M) = \frac{1}{(4\pi s)^{d/2+1}}e^{-\frac{(x_1-x_2)^2}{4s}} \tag{3.49}$$

[7] The choice of regulator is suggested naturally, as it can ultimately be linked to the Minkowski form of the propagator $G = \frac{\imath}{2m\omega - k^2 + \imath|\eta\omega|} \to \frac{\imath}{2m\omega - k^2 + \imath 0^+}$

[8] Alternatively, one can think of introducing the regulator, only after going over to the Euclidean version. The unregulated Euclidean operator, $\mathcal{M}_{E,g} = 2m\partial_\tau - \nabla^2$ is regulated to $\mathcal{M}'_{E,g} = 2m\partial_\tau - \nabla^2 + \eta\sqrt{-\partial_\tau^2}$.

where the superscript reminds us that this is the relativistic case and $(x_1 - x_2)^2 = (x_1^M - x_2^M)(x_1^N - x_2^N)\delta_{MN}$.

In preparation for using LCR, we rewrite the expression (3.49) by first reverting to Minkowski space, $t = -\imath x^0$, and then switching to light-cone coordinates.[9] Using $x^\pm = x_2^\pm - x_1^\pm$ we have:

$$\mathcal{G}_{R,M}(s;(x_2^+,x_2^-,\boldsymbol{x}_2),(x_1^+,x_1^-,\boldsymbol{x}_1)) = \frac{1}{(4\pi s)^{d/2+1}} e^{-\frac{x^+x^-}{2s} - \frac{|\boldsymbol{x}|^2}{4s}} \tag{3.50}$$

where $\mathcal{G}_{R,M}$ is the heat kernel in Minkowski space. Now, in the reduced theory, the co-ordinate x^+ becomes the time coordinate t. Going to imaginary time, $t \to \tau = \imath t$, and Fourier transforming we obtain the heat kernel $\mathcal{G}_{g,E}$ for the Galilean invariant theory in Euclidean space:

$$\begin{aligned}\mathcal{G}_{g,E}(s;(\boldsymbol{x}_2,\tau_2),(\boldsymbol{x}_1,\tau_1))) &= \int_{-\infty}^{\infty} \frac{1}{(4\pi s)^{d/2+1}} e^{\frac{\imath\tau x^-}{2s} - \frac{|\boldsymbol{x}|^2}{4s}} e^{-\imath m x^-} dx^- \\ &= 2\pi\delta\left(\frac{\tau}{2s} - m\right) \frac{1}{(4\pi s)^{d/2+1}} e^{-\frac{|\boldsymbol{x}|^2}{4s}}\end{aligned} \tag{3.51}$$

where $\tau = \tau_2 - \tau_1$, in detailed agreement with Eq. (3.46). For later use we note that in the coincidence limit we have

$$\mathcal{G}_{g,E}((\boldsymbol{x},\tau),(\boldsymbol{x},\tau))) = \frac{2\pi\delta(m)}{(4\pi s)^{d/2+1}}. \tag{3.52}$$

It is interesting to note that LCR directly gives $\sim \delta(m)/s^{d/2+1}$ while the direct computations gives $\sim \delta(ms)/s^{d/2}$. Our main result, below, follows from the coincidence limit of the heat kernel expansion in Eq. (3.57), which is useful only for $s \neq 0$, since it is used to extract the coefficients of powers of s in the expansion. The limiting behavior as $s \to 0$ of the function $\mathcal{G}_{g,E}$ is a delta function enforcing coincidence of the points, by construction (and this is why $a_0 = 1$ at coincidence), and therefore the behavior as $s \to 0$ is correct but of no significance.

[9] Recall, in the parent theory $x^\pm = \frac{1}{\sqrt{2}}(x^1 \pm t)$. Note that we are using a non-standard sign convention in the definition of x^-.

The spectral dimension of the operator $\mathcal{M}_E$ is given by

$$d_{\mathcal{M}} = -\frac{d\ln(K)}{d\ln(s)} = \frac{d}{2}+1 \tag{3.53}$$

which explains why there can not be any trace anomaly when the spatial dimension d is odd. This has to be contrasted with the relativistic case where the spectral dimension of the laplacian operator is given by $\frac{d+1}{2}$, so that in the relativistic case the anomaly is only present when the spatial dimension d is odd.

3.4.3 Heat Kernel in Curved spacetime

Now that we know that LCR works in flat space-time, we can go ahead and implement it in curved space-time exploiting the known fact that for relativistic field theories coupled to a curved geometry, the heat kernel can be obtained as an asymptotic series. The method is explained in, *e.g.,* Refs. [70, 75, 30].

The method, first worked out by DeWitt [76], starts with an Ansatz for the form of the heat kernel taking a cue from the form of the solution in flat space-time for the heat equation. For small enough s the Ansatz for the heat kernel, corresponding to a relativistic theory in $d+2$ dimensions, reads:

$$\mathcal{G}_{R,E}(x_2,x_1;s) = \frac{\Delta^{1/2}_{\rm VM}(x_2,x_1)}{(4\pi s)^{d/2+1}} e^{-\sigma(x_2,x_1)/2s} \sum_{n=0}^{\infty} a_n(x_2,x_1)\, s^n\,, \qquad a_0(x_1,x_2)=1 \tag{3.54}$$

with $a_n(x_2,x_1)$ the so-called Seeley–DeWitt coefficients and where $\sigma(x_2,x_1)$ is the biscalar distance-squared measure (also known as the geodetic interval, as named by DeWitt), defined by

$$\sigma(x_2,x_1) = \tfrac{1}{2}\left(\int_0^1 d\lambda \sqrt{G_{MN}\frac{dy^M}{d\lambda}\frac{dy^N}{d\lambda}}\right)^2, \qquad y(0)=x_1\,,\ y(1)=x_2\,, \tag{3.55}$$

with $y(\lambda)$ a geodesic. The bi-function $\Delta_{\rm VM}(x_2,x_1)$ is called the van Vleck-Morette determinant;

this biscalar describes the spreading of geodesics from a point and is defined by

$$\Delta_{\mathrm{VM}}(x_2,x_1) = G(x_2)^{-1/2} G(x_1)^{-1/2} \det\left(-\frac{\partial^2}{\partial x_2^M \partial x_1^{N'}}\sigma(x_2,x_1)\right). \tag{3.56}$$

where G is the negative of determinant of metric G_{MN}.

Now, to implement LCR, recall that a Schrödinger invariant theory coupled to a generic curved NC structure is obtained by reducing from the $d+2$ dimensional metric G_{MN} in Eq. (3.19). In taking the coincident limit we must keep x_1^- and x_2^- arbitrary in order to Fourier transform with respect to x^- per the prescription (3.24). Therefore, we work in the coincident limit where $x_1^\mu = x_2^\mu$, with $\mu = +,1,2,\cdots,d$. Now, since x^- is a null direction, in this limit we have $\sigma((x_1^-,x^\mu),(x_2^-,x^\mu)) = 0$ or $[\sigma] = 0$ for brevity. Furthermore, null isometry guarantees that metric components are independent of x^- and so are $[a_n]$ and $[\Delta_{VM}]$. Thus the coincident limit is equivalent to the coincident limit of the parent theory, hence $\left[\Delta_{VM}\right] = 1$. We refer to appendix B.2 for details.

Thus, in the coincidence limit, we have the following expression for the heat kernel corresponding to the reduced theory:

$$\mathcal{G}_{g,E}(s;(\tau,\boldsymbol{x}),(\tau,\boldsymbol{x})) = \frac{2\pi\delta(m)}{(4\pi s)^{d/2+1}} \sum_{n=0}^{\infty} a_n((\tau,\boldsymbol{x}),(\tau,\boldsymbol{x}))\, s^n\,, \qquad a_0((\tau_1,\boldsymbol{x}_1),(\tau_2,\boldsymbol{x}_2)) = 1 \tag{3.57}$$

where to define τ, we have proceeded just as in flat space: first revert to a Minkowski metric, then switch to light cone coordinates, and finally go over to imaginary x^+ time, τ. Subsequently, using Eq. (3.40) the anomaly is given by

$$\mathcal{A}_{d+1}^G = -4\pi\delta(m)\frac{a_{d/2+1}}{(4\pi)^{d/2+1}}. \tag{3.58}$$

From Eq. (3.57) it is clear that only the zero mode of P_- can contribute to the anomaly; the anomaly vanishes for fields with non-zero $U(1)$ charge. We already know that the anomaly for the relativistic complex scalar case is given by

$$\mathcal{A}^R_{d+2} = -\frac{2a_{d/2+1}}{(4\pi)^{d/2+1}} \,. \tag{3.59}$$

Thereby we establish the result advertised in the introduction, giving the Weyl anomaly of a $d+1$ dimensional Schrödinger invariant field theory of a single complex scalar field carrying charge m under $U(1)$ symmetry), $\mathcal{A}^G_{d+1}$, in terms of the anomaly in the relativistic theory in $d+2$ dimensions, $\mathcal{A}^R_{d+2}$:

$$\mathcal{A}^G_{d+1} = 2\pi\delta(m)\mathcal{A}^R_{d+2} \,, \tag{3.60}$$

computed on the class of metrics given in Eq. (3.19).

At this point, we pause to remark on the interpretation of the $\delta(m)$ factor. While it trivially shows that the anomaly is absent for $m \neq 0$, the interpretation becomes subtle when $m = 0$. The apparent divergence in the anomaly is just an artifact of the usual zero mode problem associated with null reduction. A similar issue has been pointed out in [62] in reference to [77, 78]. The reduced theory in the $m \to 0$ limit becomes infrared divergent; the fields become non-dynamical in that limit. The infrared divergence is also evident from Eq. (3.24). One may further understand the presence of $\delta(m)$ by letting m be a continuous parameter and considering a continuous set of fields ϕ_m, of charge m. The anomaly arising from the continuous set of fields is given by summing over their contributions:

$$\frac{1}{2\pi}\int dm\,\mathcal{A}^G_{d+1} = \mathcal{A}^R_{d+2}\int dm\,\delta(m) = \mathcal{A}^R_{d+2}$$

The right hand side is exactly what we expect since allowing the parameter m to continuously vary restores the Lorentz invariance: consulting Eq. (3.23) we see that this continuous sum corresponds to restoring the relativistic theory of Eq. (3.21).

That the constant of proportionality relating $\mathcal{A}^R_{d+2}$ to $\mathcal{A}^G_{d+1}$ vanishes for $m \neq 0$ can be verified by an all-orders computation of $\mathcal{A}^G_{d+1}$, to which we now turn our attention.

3.5 Perturbative proof of Vanishing anomaly

The fact that the anomaly vanishes for non-vanishing m can be shown perturbatively taking the background to be slightly curved. In flat space-time, wavefunction renormalization and coupling constant renormalization are sufficient to render a quantum field theory finite. Defining composite operators requires further renormalization. Therefore, when the model is placed on a curved background additional short distance divergences appear since the background metric can act as a source of operator insertions. To cure these divergences, new counter-terms are required that may break scaling symmetry even at a fixed point of the renormalization group flow. In this section, we will treat the background metric as a small perturbation of a flat metric so that we compute in a field theory in flat space-time with the effect of curvature appearing as operator insertions of the perturbation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. To be specific, we will look at the vacuum bubble diagrams with external metric insertions. It turns out that all of these Feynman diagrams vanish at all orders of perturbation theory, leading to a vanishing anomaly. In fact, we will show that these anomalies vanish even away from the fixed point as long as the theory satisfies some nice properties.

Suppose we have a rotationally invariant field theory such that:

1. The theory includes only rotationally invariant ("scalar") fields.

2. At free fixed point, the theory admits an $U(1)$ symmetry under which the scalar fields are charged.

3. The free propagator is of the form $\frac{\imath}{2m\omega - f(|\boldsymbol{k}|) + \imath\varepsilon}$, where, generically, $f(|\boldsymbol{k}|) = |\boldsymbol{k}|^z$.

4. The interactions are perturbations about the free fixed point by operators of the form $g(\phi,\phi^*)|\phi|^2$, where g is a polynomial of the scalar field ϕ.

An elementary argument presented below shows that, under these conditions, all the vacuum bubble diagrams vanish to all orders in perturbation theory.

Before showing this, a few comments are in order. First, the argument is valid in any number of spatial dimensions. Second, assumption 4 precludes terms like $\phi^4+(\phi^*)^4$ or $K\phi^2$ in the Lagrangian. To be precise, $F(\phi)+\text{h.c.}$ can evade this theorem for any holomorphic function F of ϕ. This is because assumption 4 implies that each vertex of the Feynman diagrams of the theory has at least one incoming scalar field into it and one outgoing scalar field line from it; having both incoming and outgoing lines at each vertex is at the heart of this result. Thirdly, it should be understood that all interactions that can be generated via renormalization, that is, not symmetry protected, are to be included. For example, were we to consider a single scalar field with only the interaction $\phi^3\phi^*+\text{h.c.}$, the interactions $\phi^4+(\phi^*)^4$ and $(\phi\phi^*)^2$ will be generated along the RG flow. Nonetheless, $U(1)$ symmetry will always prohibit a holomorphic interaction $F(\phi)+h.c.$ Lastly, assumption 3 can be relaxed to include a large class of functions $f(|\boldsymbol{k}|^2)$; this means one can recast this result in terms of perturbation theory along the RG-flow rather than about fixed points.

To prove this claim, notice first that a vacuum diagram is a connected graph without external legs (hanging edges). Moreover, since we are considering a complex scalar field, the vertices are connected by directed line segments. These directed segments form directed closed paths. To see this, recall that by assumption each vertex has at least one ingoing and one outgoing path. Starting from any vertex, we have at least one outgoing path. Any one of these paths must have a second vertex at its opposite end, since by assumption there are not hanging edges. Take any one outgoing path and follow it to the next vertex. Now, at this second vertex repeat this argument: follow the outward path to a third vertex. And so on. Since a finite graph has a finite number of vertices, at some point in the process we have to come back to a vertex we have already visited. For example, assume that we first revisit the i-th vertex. This means that starting from the vertex i we have a directed path which loops back to the i-th vertex itself. The simplest example is that of a path starting and ending on the first vertex, corresponding to a self contraction of the elementary field in the operator insertion.

Let us call this directed loop Γ. We use the freedom in the choice of loop energy and momentum in the evaluation of the Feynman diagram to assign a loop energy ω in a way such that ω loops around Γ. In performing the integral over ω it suffices to consider the Γ subdiagram only. The resulting integration is of the form:

$$\int d\omega P(\omega, \boldsymbol{k}, \{\omega_n, \boldsymbol{k}_n\}) \prod_{n\in\Gamma} \frac{1}{(\omega + \omega_n - f(|\boldsymbol{k} + \boldsymbol{k}_n|)/2m + i\varepsilon)} \tag{3.61}$$

where the product is over all vertices in Γ and correspondingly over all line segments in Γ out of these vertices. Energy ω_n and momentum $\boldsymbol{k}_n$ enter Γ at the vertex n. The factor $P(\omega, \boldsymbol{k}, \{\omega_n, \boldsymbol{k}_n\})$ is polynomial in momentum and energy and may arise if there are derivative interactions. Note that every propagator factor has the same sign $i\varepsilon$ prescription, that is, all poles in complex-ω lie in the lower half plane (have negative imaginary part). The integral over the real ω axis can be turned into an integral over a closed contour in the complex plane, by closing the contour on an infinite radius semicircle on the upper half plane, using the fact that for two or more propagators the integral over the semicircle at infinity vanishes. Then Cauchy's theorem gives that the integral over the closed contour vanishes as there are no poles inside the contour.

This proves the claim, except for the singular case of a self-contraction, that is, a propagator from one vertex to itself. Self contractions can be removed by normal ordering, again giving a vanishing result. For an alternative way of seeing this note that this integral is independent of external momentum and energy, and is formally divergent in the ultraviolet (as $|\omega| \to \infty$). The integral results in a constant (independent of external momentum and energy) that must be subtracted to render it finite, and can be chosen to be subtracted completely, to give a vanishing result.

The computation in the case of anti-commuting fields differs only in that a factor of -1 is introduced for each closed fermionic loop. Hence the claim applies equally to the case of anti-commuting scalar fields.

We now return to the derivation of our main result, Eq. (3.4). The conditions above are satisfied for the theories considered in Sec. 3.4.3, namely, free theories of complex scalars, with the free propagator given by $\frac{\imath}{2m\omega-|\boldsymbol{k}|^2+\imath 0^+}$. Recall that we are to put the theory on a curved background which is assumed to be a small perturbation from flat background. The perturbations act as insertions on vacuum bubble diagrams, but since they preserve the $U(1)$ symmetry the model still satisfies the assumptions above. Hence all the bubble diagrams vanish, and we conclude there are no divergences coming from metric insertion on bubble diagrams. Consequently, there is no scale anomaly. We emphasize that the absence of the Weyl anomaly is valid in all orders of perturbation in both the coupling and the metric. The result holds true even if we make the couplings to be space-time dependent so that every coupling insertion injects additional momentum and energy to the bubble diagram. Physically, the anomaly vanishes because the absence of antiparticles in non-relativistic field theories and the conservation of $U(1)$ charge forbid pair creation, necessary for vacuum fluctuations that may give rise to the anomaly.

This perturbative proof holds for theories which need not be Galilean invariant, and the question arises as to whether one may use LCR to make statements about anomalies for theories with kinetic term involving one time derivative and $z \neq 2$. We will take up this task in following section, starting by giving the promised proof of our prescription in Eq. (3.24).

We remark that perturbative proof works for $m \neq 0$. For $m = 0$, the integrand becomes independent of ω, and one can not perform the contour integral to argue the diagrams vanish. In fact, the integral over ω is divergent, as expected from our earlier expectation that at $m = 0$ one encounters IR divergences. One way to see the presence of $\delta(m)$, as explained earlier, is to take a continuous set of fields ϕ_m, labelled by continuous parameter m. If we exchange the sum over

(1-loop) bubble diagrams and the integral over m, then each of the propagator can be thought of as a relativistic propagator with m, playing the role of p_-. Thus the whole calculation formally becomes that of the relativistic anomaly.

One can verify our results by explicit calcualation in specific cases. In a slightly curved space-time, one can treat the deviation from flatness as background field sources. This also serves the purpose of checking that the η-regularization is appropriate, obtaining the anomaly as a function of η. Since, as $\eta \to 0$, for $m \neq 0$, the flat space heat kernel vanishes, one expects the anomaly to be vanishing. In fact, one can check that a $\delta(m)$ is recovered as $\eta \to 0$. We refer to the App. B.3 for an explicit calculation; it verifies our results in detail, and shows the vanishing anomaly regardless of the order of limits $\eta \to 0$ and $m \to 0$.

3.6 Modified LCR and Generalisation

3.6.1 Proving the heat kernel prescription

In this subsection we will explain why our proposed method to determine the heat kernel for Schrödinger field theory ($z = 2$) worked in a perfect manner, as evidenced by the agreement between Eqs. (3.46) and (3.51). We will see that one can use LCR to relate the heat kernel of a theory living in $d+1$ dimensions with that of a parent theory living in $d+2$ dimensions, as long as the parent theory has $SO(1,1)$ invariance.[10] Furthermore, if the parent theory has a dynamical scaling exponent given by z, then the theory living in $d+1$ dimension has $2z$ as its dynamical exponent. We will make these statements precise in what follows.

Suppose the operator D defined in $d+2$ dimensional space-time is diagonal in the eigenbasis of P_-, the conjugate momenta to x^-:

$$\langle x_2^+, x_2^i, m_2 | D | x_1^+, x_1^i, m_1 \rangle = \langle x_2^+, x_2^i | D_{m_2} | x_1^+, x_1^i \rangle \delta(m_2 - m_1)\,, \tag{3.62}$$

[10] One may as well assume that both parent and reduced theories have, in addition, $SO(d)$ rotational symmetry.

where $m_{1,2}$ label the eigenvalues of P_-. The example worked out in Sec. 3.4.2 had $D = \mathcal{M}$, and it does satisfy this requirement. It follows that

$$\begin{aligned}\langle x_2^+, x_2^i, x_2^- | e^{-sD} | x_1^+, x_1^i, x_1^- \rangle &= \frac{1}{2\pi} \int dm_1\, dm_2 e^{-\imath m_1 x_1^- + \imath m_2 x_2^-} \langle x_2^+, x_2^i, m_2 | e^{-sD} | x_1^+, x_1^i, m_1 \rangle \\ &= \frac{1}{2\pi} \int dm_1 e^{\imath m_1 x_{12}^-} \langle x_2^+, x_2^i | e^{-sD_{m_1}} | x_1^+, x_1^i \rangle\,, \end{aligned} \tag{3.63}$$

from which we obtain

$$\langle x_2^+, x_2^i | e^{-sD_m} | x_1^+, x_1^i \rangle = \int dx^- e^{-\imath m x_{12}^-} \langle x_2^+, x_2^i, x_2^- | e^{-sD} | x_1^+, x_1^i, x_1^- \rangle\,. \tag{3.64}$$

This is precisely the prescription we gave in Eq. (3.24).

3.6.2 Generalisation

Since the LCR (or DLCQ) trick requires null cone reduction, it may seem necessary that the parent theory have $SO(d+1,1)$ symmetry, and that this will result necessarily in a Galilean invariant reduced theory, that is, with $z = 2$. This is not quite right: one may relax the condition of $SO(d+1,1)$ symmetry and obtain reduced theories with $z \neq 2$. The key observation is that for null cone reduction only two null coordinates are needed, with the rest of the coordinates playing no role. Hence, we consider null cone reduction of a $d+2$ dimensional theory which enjoys $SO(1,1) \times SO(d)$ symmetry. The reduced theory will be a $d+1$ dimensional theory with $SO(d)$ rotational symmetry and a residual $U(1)$ symmetry that arises from the null reduction. The point is that the theory can enjoy anisotropic scaling symmetry. Consider, for example, the following class of operators

$$\mathcal{M}_{rc;d+2} = \left(-\partial_t^2 + \partial_x^2\right) - (-1)^{z/2} (\partial_i \partial^i)^{z/2}\,, \tag{3.65}$$

where $t = x^0$ and $x = x^{d+1}$ and for the reminder of this section there is an implicit sum over repeated latin indices, over the range $i = 1, \ldots, d$. These operators transform homogeneously under

$$x^i \to \lambda x^i, \qquad t \to \lambda^{z/2} t \qquad \text{and} \qquad x \to \lambda^{z/2} x\,. \tag{3.66}$$

Introducing null coordinates as before, $x^{\pm} = \frac{1}{\sqrt{2}}(x \pm t)$, null reduction of this operator yields

$$\mathcal{M}_{gc;d+1} = 2\imath m \partial_{t'} - (-1)^{z/2} (\partial_i \partial_i)^{z/2}\,, \tag{3.67}$$

where $t' = x^+$ is the time coordinate of the reduced theory. From the dispersion relation of the reduced theory, $2m\omega = |\boldsymbol{k}|^z$, we read off that the dynamical exponent is z. Here we are interested in even z to insure that the operator $\mathcal{M}_{gc;d+1}$ is local. For $z = 2$, we recover the case discussed in earlier sections with the parent theory being Lorentz invariant and the reduced theory being Schrödinger invariant.

Following the prescription (3.64), we can relate the matrix element of the heat kernel operator for $\mathcal{M}_{r;d+2}$ to that of $\mathcal{M}_{g;d+1}$, via[11]

$$\mathcal{G}_{\mathcal{M}_{gc;d+1}} = \int_{-\infty}^{\infty} dx^- \; e^{-\imath m x^-} \langle x_0^- + x^- | \mathcal{G}_{\mathcal{M}_{rc;d+2}} | x_0^- \rangle\,. \tag{3.68}$$

This should be viewed as an operator relation: thinking of the basis on which the operator $\mathcal{G}_{\mathcal{M}_{r;d+2}}$ acts as given by the tensor product of $|x^+\rangle$, $|x^-\rangle$ and $|x^i\rangle$ for $i = 1, 2, \ldots, d$, then $\langle x_0^- + x^- | \mathcal{G}_{\mathcal{M}_{r;d+2}} | x_0^- \rangle$ is an operator acting on the complement of the space spanned by $|x^-\rangle$. Taking the trace on both sides of Eq. (3.68), we obtain the heat kernel of the reduced theory:

$$K_{\mathcal{M}_{gc;d+1}} = \int_{-\infty}^{\infty} dx^- \; e^{-\imath m x^-} \; \mathrm{Tr}_{x^+, x^i} \langle x_0^- + x^- | \mathcal{G}_{\mathcal{M}_{rc;d+2}} | x_0^- \rangle \tag{3.69}$$

[11] Provided these heat kernels are well defined. We postpone this technical aspect to the appendix.

Equations (3.68) or (3.69) are useful in practice only when we know either left or right hand sides by some other means. Hence, the next meaningful question to be asked is whether we can calculate $\mathcal{G}_{\mathcal{M}_r}$ explicitly for a curved space-time for any z. The case for $z = 2$, that in which the parent theory is relativistic and the reduced theory is Schrödinger invariant, is well known and was presented in Sec. 3.4.2. For generic z, the answer is yes to some extent. We will find a closed form expression when the slice of constant (t, x) in space-time is described by a metric that does not depend on t or x:

$$ds^2 = -dt^2 + (dx)^2 + h_{ij}(x^i)dx^i dx^j \tag{3.70}$$

With this choice, the heat kernel equation for the curved background version of the operator $\mathcal{M}_{rc;d+2}$ of Eq. (3.65) admits a solution by separation of variables, into the product of the relativistic heat kernel in $1+1$ dimensions and the heat kernel for an operator acting only on the d-dimensional slice [18]. Specifically, we consider operators

$$\mathcal{M}_{rc;d+2} = \nabla^2_{t,x} - D^{z/2} \tag{3.71}$$

where $\nabla^2_{t,x} = (-\partial_t^2 + \partial_x^2)$ and D is a second order scalar differential operator on the slice of constant (t, x), *e.g.*, $D = -\nabla^2 = -1/\sqrt{h}\partial_i\sqrt{h}h^{ij}\partial_j$. With these choices,

$$\mathcal{G}_{\mathcal{M}_{rc;d+2}} = \mathcal{G}_{\nabla^2_{t,x}}\,\mathcal{G}_{D^{z/2}}\,. \tag{3.72}$$

Gilkey has shown that the heat kernel expansion for D^k can be computed from that for D [74] for $k > 0$. The argument is based on the observation that the ζ-functions for the two operators are related:

$$\zeta(\varepsilon, f, D^k) = \mathrm{Tr}_{L^2}\left(f(D^k)^{-\varepsilon}\right) = \mathrm{Tr}_{L^2}\left(fD^{-k\varepsilon}\right) = \zeta(k\varepsilon, f, D)\,.$$

Gilkey's result is as follows: If D has heat kernel expansion

$$K_D = \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{n\geq 0} s^{n-\frac{d}{2}} a_n^{(d)} \tag{3.73}$$

then the heat kernel expansion of D^k is

$$\begin{aligned} K_{D^k} = \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{n\geq 0} s^{\frac{2n-d}{2k}} \frac{\Gamma(\frac{d-2n}{2k})}{k\Gamma(\frac{d}{2}-n)} a_n^{(d)} &= \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{\substack{n\geq 0 \\ 2n\neq d (\mathrm{mod}\ 2k)}} s^{\frac{2n-d}{2k}} \frac{\Gamma(\frac{d-2n}{2k})}{k\Gamma(\frac{d}{2}-n)} a_n^{(d)} \\ &+ \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{\substack{n\geq 0 \\ 2n= d (\mathrm{mod}\ 2k)}} s^{\frac{2n-d}{2k}} (-1)^{\frac{(2n-d)(1-k)}{2k}} a_n^{(d)} \end{aligned} \tag{3.74}$$

Hence, $\mathcal{M}_{rc;d+2} = (-\partial_t^2 + \partial_x^2) - (-\nabla^2)^{z/2}$ has heat kernel expansion

$$\langle x_2^+, x_2^-, x^i | \mathcal{G}_{\mathcal{M}_{rc;d+2}} | x_1^+, x_1^-, x^i \rangle = \frac{e^{\frac{-x_{12}^+ x_{12}^-}{2s}}}{4\pi s} \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{n\geq 0} s^{\frac{2n-d}{z}} \frac{\Gamma(\frac{d-2n}{z})}{\frac{z}{2}\Gamma(\frac{d}{2}-n)} a_n^{(d)} \tag{3.75}$$

where $x_{12}^{\pm} = x_2^{\pm} - x_1^{\pm}$ and $a_n^{(d)}$ are the well known coefficients of the heat kernel expansion of $-\nabla^2$.

Now, the reduced theory lives on $d+1$ dimensional space-time with curved spatial slice, *i.e.*, the background metric is given by

$$ds^2 = -dt^2 + h_{ij} dx^i dx^j \,, \tag{3.76}$$

where i runs from 1 to d. In order to extract the heat kernel of $\mathcal{M}_{gc;d+1} = 2\imath m \partial_t + (-\nabla^2)^{z/2}$, we need partial tracing of heat kernel of $\mathcal{M}_{rc;d+2}$,

$$\langle x_0^- + x^- | \mathrm{Tr}_{x^+, x^i} \mathcal{G}_{\mathcal{M}_{rc;d+2}} | x_0^- \rangle = \left(\frac{1}{\sqrt{4\pi}}\right)^d \frac{1}{4\pi s} \sum_{n\geq 0} s^{\frac{2n-d}{z}} \frac{\Gamma(\frac{d-2n}{z})}{\frac{z}{2}\Gamma(\frac{d}{2}-n)} a_n^{(d)} \,, \tag{3.77}$$

leading to

$$K_{\mathcal{M}_{gc;d+1}} = 2\pi\delta(m)\frac{1}{4\pi s}\left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{n\geq 0} s^{\frac{2n-d}{z}} \frac{\Gamma(\frac{d-2n}{z})}{\frac{z}{2}\Gamma(\frac{d}{2}-n)} a_n^{(d)} . \tag{3.78}$$

Adding conformal coupling modifies $a_n^{(d)}$ but the pre-factor stays $2\pi\delta(m)\frac{1}{4\pi s}\left(\frac{1}{\sqrt{4\pi}}\right)^d$. Hence, we have the generalised result

$$\mathcal{A}^g_{d+1} = 2\pi\delta(m)\mathcal{A}^r_{d+2} \tag{3.79}$$

where $\mathcal{A}^g_{d+1}$ is the Weyl anomaly of a theory of a single complex scalar field of charge m under a $U(1)$ symmetry living in $d+1$ dimensions with dynamical exponent z and $\mathcal{A}^r_{d+2}$ is the Weyl anomaly of a field theory living in $d+2$ dimension such that it admits a symmetry under $t \to \lambda^{z/2}t, x^{d+2} \to \lambda^{z/2}x^{d+2}$and $x^i \to \lambda x^i$ for $i = 1,\ldots,d+1$. Thus we have shown that theories with one time derivative on a time independent curved background do not have any Weyl anomalies. This is consistent with the perturbative result obtained previously.

It deserves mention that the operator $\mathcal{M}_{rc;d+2}$ of Eq. (3.71) does not transform homogeneously under Weyl transformations. In order to construct a Weyl covariant operator consider generalizing the metric (3.70) to the following form

$$ds^2 = Ndx^+dx^- + h_{ij}dx^i dx^j . \tag{3.80}$$

If N is independent of x^- the metric for the reduced theory will include a general lapse function N. Then we replace $(\nabla^2)^{\frac{z}{2}}$ by $O^{(d+2z-4)}O^{(d+2z-8)}\cdots O^{(d+4)}O^{(d)}$ with $O^{(p)}$ defined as

$$O^{(p)} \equiv \nabla^2 - \frac{p}{4(d-1)}R + \frac{2+p-d}{z}\frac{\partial_i N}{N}h^{ij}\partial_j + \frac{d}{4z^2}(2+p-d)\frac{\partial_i N}{N}h^{ij}\frac{\partial_j N}{N} \tag{3.81}$$

Under $h_{ij} \to e^{2\sigma}h_{ij}$, $N \to e^{z\sigma}N$ and $\psi \to e^{-\frac{p}{2}\sigma}\psi$, this operator transforms covariantly, in the sense that

$$O^{(p)}\psi \to e^{-(\frac{p}{2}+2)\sigma} O^{(p)}\psi. \tag{3.82}$$

Therefore, under the Weyl rescaling $h_{ij} \to e^{2\sigma} h_{ij}$, $N \to e^{z\sigma} N$ and $\phi \to e^{-\frac{d}{2}\sigma}\phi$ we have that

$$N\sqrt{h}\phi^* O^{(d+2z-4)} O^{(d+2z-8)} \cdots O^{(d+4)} O^{(d)}\phi \tag{3.83}$$

is invariant under under Weyl transformations.

Adding the conformal coupling will modify the expressions for $a_n^{(d)}$, but scaling with respect to s will remain unmodified. Hence we can enquire about existence or absence of potential Weyl anomalies. To have a non-vanishing Weyl anomaly, we need to have an s independent term in the heat kernel expansion. This is possible only when $\frac{2n-d}{z} = 1$, *i.e.*, when $d+z$ is even; see Eqs. (3.75) and (3.78). Since for a local Lagrangian z must be even, this condition corresponds to even d [12]. This is expected because of the following reason: the scalars we can construct out of geometrical data (that can potentially appear as a trace anomaly) have even dimensions and the volume element scales like λ^{d+z}, so that in order to form a scale invariant quantity $d+z$ has to be even. Now when d is even, we have s independence for $n = (d+z)/2$ and the coefficient of s^0 is given by $\left(\frac{1}{\sqrt{4\pi}}\right)^d (-1)^{1-\frac{z}{2}} a^d_{\frac{d+z}{2}}$. Hence, the result relating anomalies in the parent and reduced theory, Eq. (3.79), still holds.

3.7 Summary, Discussion and Future directions

We have shown that for a $d+1$ dimensional Schrödinger invariant field theory of a single complex scalar field carrying charge m under $U(1)$ symmetry, the Weyl anomaly, $\mathcal{A}^G_{d+1}$, is given

[12] Giving up on the requirement of locality allows z to be any positive real number. In this case, the anomaly is expected to be present whenever $d+z$ is even. It might be of potential interest to look at these cases carefully and make sure that non-locality does not provide any obstruction in the anomaly calculation and that the renormalization process can be done in a consistent manner.

in terms of that of a relativistic free scalar field living in $d+2$ dimensions, $\mathcal{A}^R_{d+2}$, via

$$\mathcal{A}^G_{d+1} = 2\pi\delta(m)\mathcal{A}^R_{d+2}\,. \tag{3.84}$$

Here the parent $d+2$ theory lives in a space-time with null isometry generated by the Killing vector ∂_- so that the metric can be given in terms of a $d+1$ dimensional Newton-Cartan structure. The result is shown to be generalised to

$$\mathcal{A}^g_{d+1} = 2\pi\delta(m)\mathcal{A}^r_{d+2}\,, \tag{3.85}$$

where $\mathcal{A}^g_{d+1}$ is the Weyl anomaly of a theory of a single complex scalar field of charge m under an $U(1)$ symmetry living in $d+1$ dimensions with dynamical exponent z, while $\mathcal{A}^r_{d+2}$ is the Weyl anomaly of an $SO(1,1)\times SO(d)$ invariant theory living in $d+2$ dimension such that it admits symmetry under $t\to\lambda^{z/2}t$, $x^{d+2}\to\lambda^{z/2}x^{d+2}$ and $x^i\to\lambda x^i$ for $i=1,\ldots,d+1$.

To obtain information regarding the anomaly, we introduced a method to systematically handle the heat kernel for a theory with kinetic term involving one time derivative only. We provided crosschecks and consistency checks on our heat kernel prescription. One may worry that to properly define a heat kernel the square of the derivative operator must be considered. This would also be the case for, say, the Dirac operator. In fact, one can properly define it this way; see, for example, Ref. [79].

The result obtained regarding the anomaly of Schrödinger field theory is consistent with the one by Jensen [62]. Auzzi *et al*, [80] have studied the anomaly for a Euclidean operator given by

$$\mathcal{M}'_{E,g} = 2m\sqrt{-\partial_t^2} - \nabla^2\,, \tag{3.86}$$

with eigenspectra given by $|\boldsymbol{k}|^2 + 2m|\omega| \geq 0$. One can define the heat kernel for this operator as well, but the eigenspectra of this operator is not analytically related to that of $\mathcal{M}_{M,g} = 2\imath m\partial_t + \nabla^2$, which is $-k^2 + 2m\omega$. As a result the propagator in ω-$\boldsymbol{k}$ space has a cut on the complex ω plane with branch point at the origin, making the analytic continuation to Minkowski space problematic. It is known that the two point correlator of Schrödinger field theory is constrained and has a particular form as elucidated in Ref. [59, 81]. While our prescription and the resulting Euclidean correlator conforms to that form, it is not clear how the Euclidean Schrödinger operator defined in Ref. [80] does, if at all. Finally, we note that the operator $\sqrt{-\partial_t^2}$ is non-local (in the sense that the kernel, defined by $\sqrt{-\partial_t^2} f(t) = \int dt' K(t-t') f(t')$, has non-local support, $K(t) = 2\partial_t \mathrm{P}\frac{1}{t}$).

There are several avenues of investigation suggested by this work:

1. What happens in the case of several scalar fields with different charge interacting with each other while preserving Schrödinger invariance in flat space-time? How is the pre-factor $\delta(m)$ modified?

2. It is not obvious how null reduction of a theory of a Dirac spinor in $d+2$ dimensions can result in a Lagrangian in $d+1$ dimensions of the form $\mathcal{L} = 2\imath m\psi^\dagger\partial_t\psi + \psi^\dagger\nabla^2\psi$, let alone one with $\mathcal{L} = 2\imath m\psi^\dagger\partial_t\psi - \psi^\dagger(-\nabla^2)^{z/2}\psi$ for $z \neq 2$. On the other hand, as we have seen, the functional integral over non-relativistic anti-commuting fields yields the same determinant as that of commuting fields (only a positive power). Hence, the anomaly of the anti-commuting field is the negative of that of the commuting field.

3. Calculations using the same Euclidean operator as in Ref. [80] give a non-vanishing entanglement entropy in the ground state [82]. By contrast, for the operator $\mathcal{M}_{M,g} = 2\imath m\partial_t + \nabla^2$, the entanglement entropy in the ground state vanishes, since for this local non-relativistic field theory $\phi(x)|0\rangle = 0$ and hence the ground state is a product state. It would be of interest to verify this result by direct computation using a method based on our prescription.

4. The method described in Sec. 3.6.2 to compute Weyl anomalies in theories with $z \neq 2$

is not sufficiently general in that, by assuming the metric is time independent and has constant lapse, it neglects anomalies involving extrinsic curvature or gradients of the lapse function. A future challenge is to develop a more general computational method.

We hope to come back to these questions in the future.

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Chapter 4

Unitarity and Universality in non relativistic Conformal Field theory

We relate the notion of unitarity of a $(0+1)$-D conformally ($SL(2,\mathbb{R})$) invariant field theory with that of a non-relativistic conformal (Schrödinger) field theory using the fact that $SL(2,\mathbb{R})$ is a subgroup of non-relativistic conformal (Schrödinger) group. Exploiting $SL(2,\mathbb{R})$ unitarity, we derive the unitarity bounds and null conditions for a Schrödinger field theory (for the neutral as well as the charged sector). In non integer dimensions the theory is shown to be non-unitary. The use of $SL(2,\mathbb{R})$ subgroup opens up the possibility of borrowing results from $(0+1)$-D $SL(2,\mathbb{R})$ invariant field theory to explore Schrödinger field theory, in particular, the neutral sector, which has otherwise been unexplored.

4.1 Introduction

The conformal field theory [83] has a rich literature with wide application in describing physics at relativistic fixed points. Much of its armory stem from the early papers on the representation theory of $SL(2,\mathbb{R})$, a subgroup of the conformal group [84, 85, 86]. The unitarity bound along with the null condition is one of the many consequences of the representation theory of $SL(2,\mathbb{R})$ algebra [87]. The conformal bootstrap program also relies on knowing conformal ($SL(2,\mathbb{R})$) blocks [88, 89, 90, 91, 92]. The task we take up here is to use this arsenal of $SL(2,\mathbb{R})$ algebra to hammer a class of non-relativistic conformal theories (NRCFT), which are $SL(2,\mathbb{R})$ invariant.

The non-relativistic conformal invariance emerges at fixed points without Lorentz invariance, in particular, in a scenario, where the symmetry involves scaling time and space in a separate way. If the theory permits, one can have Galilean boost invariance and invariance under special conformal transformations as well. A prime example of such kind is the theory described by Schrödinger equation, where the maximal kinetic invariance group is the Schrödinger group[93]. Fermions at the unitarity limit (when the S-wave scattering length $a \to \infty$) are also described by Schrödinger field theory[94, 60, 59, 1, 2]. Examples of approximate non-relativistic conformal field theories include systems involving ^{85}Rb [5],^{133}Cs[6],^{39}K [7], deuterons [3, 4] and spin chain models[95].

Much like its relativistic cousin, progress has been made regarding the form of correlators and convergence of operator product expansion (OPE) in such theories for a sector with non-zero charge using the symmetry algebra only [96, 93, 94, 97, 98, 59, 61, 81, 99] (which is Schrödinger algebra). The state-operator correspondence invoking the harmonic potential is available for the charged sector. Nonetheless, the neutral sector has remained elusive since the representation theory along with the concept of primary and descendant breaks down for the neutral sector[99]. Thus, there is no state-operator correspondence available for the neutral sector, neither there is a proof of OPE convergence if the four point correlator involves neutral operator(s). On the other hand, physically relevant operators like Hamiltonian, number current, stress-energy tensor are neutral. This motivates us in first place to use $SL(2,\mathbb{R})$ to explore the neutral sector as one can organize the operator content according to $SL(2,\mathbb{R})$ representation, which is applicable to both the neutral as well as the charged sector. To our favor, it so turns out that $SL(2,\mathbb{R})$ provides strong constraints on properties of Schrödinger field theories even for the charged sector on top of solving all the puzzles mentioned before in context of the neutral sector.

The purpose of this work is multifold. The most important point that we make is that $SL(2,\mathbb{R})$ establishes a powerful and novel link between $(0+1)$-D conformal field theory (CFT) and NRCFTs. Thus results proven for $(0+1)$-D CFTs immediately apply to NRCFTs and vice versa. In fact, using $SL(2,\mathbb{R})$, we come up with state-operator map, subsequently, derive the unitarity bound, the null condition for the neutral sector for the first time. Secondly, we reformulate the notion of unitarity in the charged sector and re-derive the unitarity bound without invoking the standard map to harmonic oscillator. This, in turn helps us to identify the non-unitary sector in fractional dimensions, which has otherwise not been known previously. Moreover, we explore the universal features of Schrödinger field theories including the convergence of the operator product expansion (OPE) in the neutral sector. Convergence of OPE in all the sectors also opens up the possibility of bootstrapping these theories. We deduce the universal behavior of three point coefficient and establish for the first time that even in NRCFT, there exist infinite number of $SL(2,\mathbb{R})$ primaries. In short, we explicitly unveil a complete equivalence between correlators of NRCFTs on $(\tau,\mathbf{0})$ slice and $(0+1)$-D CFTs via the notion of $SL(2,\mathbb{R})$ primaries and descendants. Last but not the least, the use of $SL(2,\mathbb{R})$ primaries/descendants proves to be quintessential in operator counting of heavy particle effective field theory, where neutral scalar operators appear in the Lagrangian. Only with the aid of $SL(2,\mathbb{R})$, it is possible to organize the operator basis of heavy particle effective field theory in Schrödinger representation [100].

The paper is organized as follows. In sec. 4.2, we derive the unitary bounds and null conditions for both the charged sector as well as the neutral sector of Schrödinger algebra. Non-unitarity in non integer dimensions has been explored in sec. 4.3. The sec. 4.4 deals with the universality, in particular, the OPE convergence, the asymptotic behavior of three point coefficients in Schrödinger invariant field theory. We conclude with an elaborate discussion pointing out potential avenues of future research. To aid the main flow of the paper, the details of $SL(2,\mathbb{R})$ invariant theory and representation of Schrödinger algebra have been relegated to appendix C.1 and C.2 respectively. The role of time reversal and parity is elucidated in appendix C.3. The appendix C.4 expounds on defining the Euclidean Schrödinger field theory, which comes out as a byproduct of organizing the operator content of Schrödinger field theory according to $SL(2,\mathbb{R})$ algebra.

4.2 Unitarity bounds & Null conditions

The Schrödinger group acts on space-time as follows [96, 93, 94, 98]:

$$t \mapsto \frac{at+b}{ct+d}, \quad \boldsymbol{r} \mapsto \frac{\mathbb{R}\boldsymbol{r}+\boldsymbol{v}t+f}{ct+d}. \tag{4.1}$$

where $ad-bc=1$, $\mathbb{R}$ is a d dimensional rotation matrix, $\boldsymbol{v}$ denotes the Galilean boost and f is a spatial translation. For the sector with non-zero charge, the representation is built by translating all the operators to the origin and considering the little group generated by dilatation operator D, Galilean boost generator K_i, and special conformal transformation generator C. The highest weight states (ϕ_α) are annihilated by C and K_i i.e.

$$[C,\phi_\alpha(0,\mathbf{0})]=0, \quad [K_i,\phi_\alpha(0,\mathbf{0})]=0. \tag{4.2}$$

These are called primary operators. The commutators with D and particle number symmetry generator $\hat{N}$ dictate the charge and the dimension of these operators ϕ_α i.e. $[D,\phi_\alpha(0,\mathbf{0})] = \imath\Delta_\alpha\phi(0,\mathbf{0})$ and $[\hat{N},\phi_\alpha(0,\mathbf{0})]=N_\alpha\phi_\alpha(0,\mathbf{0})$. The time and space translation generators H and P_i create descendant operators by acting upon primary operators and raising the dimension by 2 and 1 respectively. The concept of primaries and descendants breaks down within the neutral sector. Since K_i and P_j commute in this sector, P_j acting on a primary spits out a primary in stead of a descendant.

The subgroup, $SL(2,\mathbb{R})$ is defined by $\mathbb{R} = \mathbb{I}$, $\boldsymbol{v} = 0$, $f = 0$ and generated by H, D and C. Evidently, the $(t,0)$ slice is an invariant domain of $SL(2,\mathbb{R})$. Using this $SL(2,\mathbb{R})$ algebra, one can reorganize the operator content. A $SL(2,\mathbb{R})$ primary O is defined by requiring $[C, O(0,\mathbf{0})] = 0$. Thus all the primaries defined by (4.2) are $SL(2,\mathbb{R})$ primaries but not the other way around. The situation is reminiscent of 2D conformal field theory where we have Virasoro primaries as well as $SL(2,\mathbb{R})$ primaries and the $SL(2,\mathbb{R})$ primaries are called quasi-primaries. We will borrow that nomenclature and call the Schrödinger primaries as *primaries* while we name $SL(2,\mathbb{R})$ primaries, *quasi-primaries*. Remarkably the notion of quasi-primaries goes through even for a zero charge sector. Henceforth, by $\phi(t)$ (or $O(t)$), we will mean the operator $\phi(t,\mathbf{0})$(or $O(t,\mathbf{0})$).

For a $SL(2,\mathbb{R})$ invariant field theory, there is a notion of unitarity/reflection positivity, which guarantees that the two point correlator of two operators inserted at imaginary time $-\tau$ and τ is positive definite. We will exploit the $SL(2,\mathbb{R})$ subgroup of Schrödinger group to borrow the notion of *reflection positivity* in Schrödinger field theory. We consider the following states for $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$:

$$
\begin{aligned}
|\psi_\alpha(\tau_1)\rangle &= \int d\tau \, \left[\delta(\tau-\tau_1) + 2\tau_1\alpha^{-1}\delta'(\tau-\tau_1)\right] O(\tau)|0\rangle\,, \\
&= \left| \left[O(\tau_1) - 2\tau_1\alpha^{-1}(\partial_\tau O)(\tau_1)\right] \right\rangle. \\
|\Psi_\beta(\tau_1)\rangle &= \int d\tau \left[\left(\frac{Nd}{2\Delta} - \frac{N}{\beta}\right) \delta'(\tau-\tau_1)\phi^\dagger(\tau)|0\rangle \right. \\
&\quad \left. + \delta(\tau-\tau_1)\mathcal{A}^\dagger(\tau)|0\rangle \right] = \left| \left[\frac{N}{\beta}\partial_\tau\phi^\dagger - \frac{1}{2}\nabla^2\phi^\dagger\right] \right\rangle_{\tau=\tau_1}
\end{aligned}
\tag{4.3}
$$

where O is a quasi-primary, $\phi^\dagger$ is a primary with charge $-N$ and $\mathcal{A}^\dagger \equiv \left(\frac{Nd}{2\Delta}\partial_\tau\phi^\dagger - \frac{1}{2}\nabla^2\phi^\dagger\right)$ is a quasi-primary[1].

[1]The details of $SL(2,\mathbb{R})$ invariant field theory and Schrödinger algebra can be found in the appendix C.1 and C.2.

To derive the unitarity bound for quasi-primary, we demand that the state $|\psi_\alpha\rangle = |\psi_\alpha(1/2)\rangle$[2] has a positive norm:

$$\langle\psi_\alpha|\psi_\alpha\rangle \geq 0 \Leftrightarrow \Delta^2 + (2\alpha+1)\Delta + \alpha^2 \geq 0\,, \tag{4.4}$$

where we have used $\langle O(\tau_1)O(\tau_2)\rangle = (\tau_2-\tau_1)^{-\Delta}$. For $\alpha < -\frac{1}{4}$ there is no constraint on Δ. For

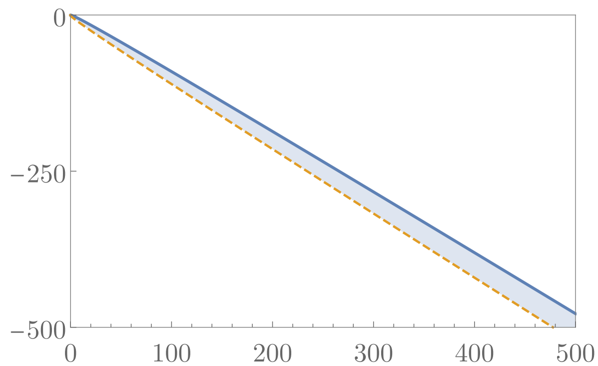

Figure 4.1: Unitarity bound on (α,Δ) plane: the projection of the region bounded by two curves onto the Y axis excludes $\Delta < 0$. The blue thick curve is $\Delta_+ = -\alpha - \frac{1}{2} + \sqrt{\alpha + \frac{1}{4}}$ while the orange dashed curve is $\Delta_- = -\alpha - \frac{1}{2} - \sqrt{\alpha + \frac{1}{4}}$.

$\alpha \geq -1/4$, the region (Δ_-, Δ_+) is excluded where

$$\Delta_\pm = (-\alpha - 1/2 \pm \sqrt{\alpha + 1/4}) \leq 0.\,. \tag{4.5}$$

As we vary α, the whole $\Delta < 0$ region gets excluded (fig. 4.1) since, $\Delta_+ - \Delta_- = 2\sqrt{\alpha + 1/4}$.

Now we will do the same for a primary and consider the norm of the state $|\Psi_\beta\rangle = |\Psi_\beta(1/2)\rangle$,

$$\langle\Psi_\beta|\Psi_\beta\rangle \geq 0 \Leftrightarrow \Delta^2 + \Delta(1-\beta d) + \frac{1}{4}\beta d(\beta(d+2)-4) \geq 0 \tag{4.6}$$

To find out the norm of Ψ_β, we have used the two point correlator of primaries, fixed by

[2]Technically, the ratio, α/τ_1 is dimensionless. So, in some suitable unit, one can choose $\tau_1 = 1/2$ and vary α in the same unit.

Schrödinger algebra. This is exactly where it becomes important that the actual symmetry group is bigger than $SL(2,\mathbb{R})$ and there are d spatial dimensions. The region, excluded (fig. 4.2) due to (4.6) is given by (Δ_-,Δ_+) where $\Delta_\pm = \frac{d\beta-1}{2} \pm \frac{1}{2}\sqrt{1+2d\beta-2d\beta^2}$.

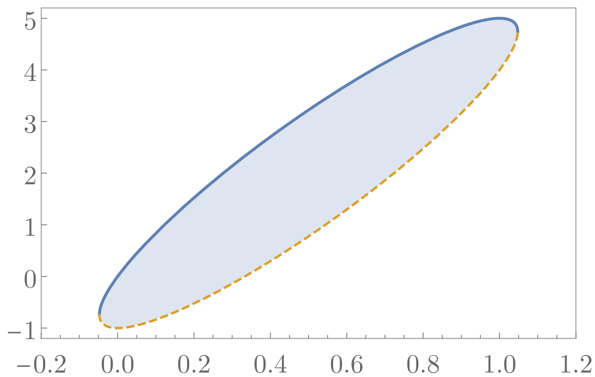

Figure 4.2: Unitarity bound on (β,Δ) plane: the projection of the region bounded by the ellipse onto the Y axis excludes $\Delta \in \left(-1,\frac{d}{2}\right)$. Here $d=10$. The blue thick curve is $\Delta_+ = \frac{d\beta-1}{2} + \frac{1}{2}\sqrt{1+2d\beta-2d\beta^2}$ while the orange dashed curve is $\Delta_- = \frac{d\beta-1}{2} - \frac{1}{2}\sqrt{1+2d\beta-2d\beta^2}$.

As we vary β, on the (β,Δ) plane, the excluded region is bounded by an ellipse. This, in turn, excludes $\Delta \in \left(-1,\frac{d}{2}\right)$. Thus we have (recalling a primary is a quasi-primary too and has to satisfy the bound for quasi-primary)

$$\Delta \notin \left(-1,\frac{d}{2}\right) \cup (-\infty,0) \Rightarrow \Delta \geq \frac{d}{2}. \tag{4.7}$$

The bound is saturated when $\Delta_+ = \frac{d}{2}$, which implies that $\beta = 1$ and we arrive at the null condition:

$$\langle\Psi_1|\Psi_1\rangle = 0 \Leftrightarrow N\partial_\tau\phi^\dagger - \frac{1}{2}\nabla^2\phi^\dagger = 0 \tag{4.8}$$

The unitarity bound and the null condition, thus obtained for the charged sector, is consistent with the results in [60]. A technical remark is in order: setting $\beta = 1$ to begin with, would not suffice to derive the unitarity bound. This is because merely demanding$\langle\Psi_1|\Psi_1\rangle \geq 0$ would exclude the region $\left(\frac{d}{2}-1,\frac{d}{2}\right)$ only.

For a sector with $N=0$, the unitarity bound becomes the one obtained by using $|\psi_\alpha\rangle$, thus the null condition is achieved when $\Delta_+(\alpha) = 0 \Rightarrow \alpha = 0$. Thus the null condition for neutral sector reads $\partial_\tau O = 0$. The bound in the neutral sector is lower compared to the bound in charged sectors, thus in free Schrödinger field theory there's no neutral operator satisfying the bound except the Identity operator. The identity operator by definition creates the neutral vacuum state, has 0 dimension and trivially time independent. It would be interesting to find an operator besides the identity operator, which saturates the bound or improve the bound for the non-identity operators. One might hope to come up with stronger bound for the neutral sector by considering the norm of the state $A|\psi_\alpha\rangle + B|\Psi_\beta\rangle$, but this is given by $A^2\langle\psi_\alpha|\psi_\alpha\rangle + B^2\langle\Psi_\beta|\Psi_\beta\rangle$ since $|\Psi_\beta\rangle$ is charged whereas $|\psi_\alpha\rangle$ is neutral, leading to $\langle\psi_\alpha|\Psi_\beta\rangle = 0$. Now, $A^2\langle\psi_\alpha|\psi_\alpha\rangle + B^2\langle\Psi_\beta|\Psi_\beta\rangle \geq 0$ by previous bounds.

Subtleties associated with Null condition & Non-Renormalization: The derivation of the null condition assumes that the only operator that can annihilate the vacuum is the null operator (denoted as $\hat{0}$ henceforth). This is not necessarily true in a non-relativistic set up. For example, the canonical way of quantizing free Schrödinger field theory starts with the existence of an operator ϕ such that ϕ annihilate the vacuum Thus for $\tau > 0$, we have $\langle 0|\phi^\dagger(0)\phi(\tau)|0\rangle = 0$. But this does not imply that ϕ is a null operator. In a theory with anti-particles, ϕ can not annihilate the vacuum since its Fourier decomposition consists of several particle annihilation operators and anti-particle creation operators. But a non-relativistic field theory admits a quantization process without having any anti-particle in its spectrum. Thus, non-trivial operators like ϕ can have the vacuum state as their kernel.

To state a generic null condition, we consider the set of operators $\mathcal{S}_N$, defined by $s_N \in \mathcal{S}_N$ iff $|0\rangle \in \ker(s_N)$, $[\hat{N}, s_N] = N s_N$. The null condition then reads:

$$N\partial_\tau\phi^\dagger - \frac{1}{2}\nabla^2\phi^\dagger \in \mathcal{S}_{-N} \cup \{\hat{0}\}\,. \tag{4.9}$$

We see that, unlike relativistic CFT, the Eq. (4.9) can be satisfied at an interacting fixed point. It has, therefore, consequences in terms of anomalous dimension of ϕ. For example, let us consider a free Schrödinger field theory and perturb by an operator of the form $s(x)\phi(x)$ where $s \in \mathcal{S}_N$. If the theory flows to another fixed point such that Eq. (4.9) holds, the field ϕ can not acquire an anomalous dimension. This happens because the null condition (4.9) implies that even at the non-trivial fixed point ϕ has dimension $\frac{d}{2}$, which equals the dimension at the free fixed point.

The non-renormalization theorem can be utilized in following way: consider a free Schrödinger field theory with free elementary fields ϕ_α (the ones that appear in Lagrangian at free fixed point) and $[\hat{N}, \phi_\alpha] = N_\alpha \phi_\alpha$ with $N_\alpha < 0$. We further assume without loss of generality that $\phi_1^\dagger$ has the minimum positive charge given by $-N_1 > 0$. The absence of anti-particles mean ϕ_α annihilates the vacuum. Now we perturb the theory by adding a classically marginal $s_{-N_1}\phi_1 + h.c$ term, where s_{-N} carries charge $-N > 0$ and annihilates the vacuum. Assuming that the theory flows to a another fixed point invariant under Schrödinger symmetry, we can show that the field ϕ_1 does not acquire any anomalous dimension at the non trivial fixed point.

We proceed by observing that all the terms that might get generated due to renormalization group flow preserve $U(1)$. Furthermore, we only look for the operators of the form $s'\phi_1$, as they contribute to the equation motion of $\phi_1^\dagger$. Now, the $U(1)$ charge conservation guarantees that s' has $-N_1$ charge. We need to show that s' annihilates the vacuum. This would not be the case if $s' = \phi_1^\dagger$, but this operator can not be generated from a classically marginal term. So we are left with the other option which requires having at least two elementary field operators such that their charges add up to $-N_1$. Since $-N_1$ is the least possible positive charge, there exists at least one operator with negative charge and this implies that $s' \in \mathcal{S}_{-N_1}$ i.e s' annihilates the vacuum. Thus the null condition $(N_1\partial_\tau + \nabla^2)\phi^\dagger|0\rangle = 0$ is always satisfied for the field with the least possible charge and the corresponding field operator does not acquire any anomalous dimension at the nontrivial fixed point. For example, fermions at unitarity is described by two equivalent theories living at a non-trivial Wilson-Fisher fixed point: one in $2+\varepsilon$ dimensions, another one in $4-\varepsilon$ dimensions. It is easy to verify from [59] that both of them conform to the above theorem. The one fermion operator ψ does not acquire anomalous dimension in both $2+\varepsilon$ and $4-\varepsilon$ dimensions whereas in $4-\varepsilon$ dimensions, the two fermion operator ϕ does acquire a anomalous dimension, which should be the case since even at tree level the equation motion of $\phi^\dagger$ does not belong to $\mathcal{S}_{-N}$ where

$$s_{-N} \in \mathcal{S}_{-N} \quad \text{iff} \quad |0\rangle \in \ker(s_{-N}) \ \& \ [\hat{N}, s_{-N}] = -N s_{-N} \,. \tag{4.10}$$

4.3 Non Unitarity in non integer dimensions

The unitarity of a $SL(2,\mathbb{R})$ invariant field theory can be defined in non-integer dimensions by analytically continuing the appropriate correlator. Relativistic CFTs in non integer dimensions can have *Evanescent operators*, corresponding to states with negative norm, thus has a non-unitary sector. These operators cease to exist whenever d becomes integer, nonetheless they are present and non-trivial whenever one extends the theory away from integer (spatial) dimensions [101, 102, 103, 104]. Here we consider a free Schrödinger field theory in $d+1$ dimensions and show the presence of such operators. In particular, we consider the following set of operators for $n \geq 2$,

$$R_n(t,\boldsymbol{x}) := \delta^{i_1[j_1}\delta^{|i_2|j_2}\cdots\delta^{|i_n|j_n]} : M_{i_1j_1}M_{i_2j_2}\cdots M_{i_nj_n} :,$$

where $M_{ij} = \partial_i\partial_j\phi(t,\boldsymbol{x})$, ϕ is a primary operator with dimension $d/2$ and all the j indices are anti-symmetrized. For example, $R_2(t,\boldsymbol{x}) := (\delta^{i_1j_1}\delta^{i_2j_2} - \delta^{i_1j_2}\delta^{i_2j_1}) : M_{i_1j_1}M_{i_2j_2} :$. For integer $d < n$ dimensions, at least one of the indices has to repeat itself, thus the operator becomes trivially zero. For non integer $d < n-1$ and for $d \geq n$, the operators are indeed nontrivial.

The operator R_2 produces a negative norm state (we are using the notion of *state* borrowed from $SL(2,\mathbb{R})$ invariant field theory, as explained in the appendix C.1) in a theory living on

$(1-\varepsilon)+1$ dimensions with $1>\varepsilon>0$. The norm of R_2 is given by

$$\left\langle R_2(-\tfrac{1}{2},0)R_2^\dagger(\tfrac{1}{2},0)\right\rangle = \#\,(d+2)\,(d+1)\,d\,(d-1) \tag{4.11}$$

where # is a positive number, determined by the two point correlator of ϕ and number of independent ways to contract. Here we have also set $\tau=\frac{1}{2}$ without any loss of generality. As expected, the norm becomes zero as $d=0,1$. The norm is negative when $0<d<1$. Similarly, we find that

$$\begin{aligned}
&\left\langle R_3(-\tfrac{1}{2},0)R_3^\dagger(\tfrac{1}{2},0)\right\rangle = \#\,(d+2)\,(d+1)\,d^2\,(d-1)\,(d-2)\\
&\left\langle R_4(-\tfrac{1}{2},0)R_4^\dagger(\tfrac{1}{2},0)\right\rangle\\
&\quad = \#\,(d+2)\,(d+1)\,d^2\,(d-1)^2\,(d-2)\,(d-3)\\
&\left\langle R_5(-\tfrac{1}{2},0)R_5^\dagger(\tfrac{1}{2},0)\right\rangle\\
&\quad = \#\,(d+2)\,(d+1)\,d^2\,(d-1)^2\,(d-2)^2\,(d-3)\,(d-4)
\end{aligned}$$

where # is a positive number, determined by the two point correlator of ϕ and number of independent ways to contract. In general, the operator R_n produces a negative norm state:

$$\left\langle R_n(-\tfrac{1}{2},0)R_n^\dagger(\tfrac{1}{2},0)\right\rangle = \#\,(d-n+3)_n \prod_{j=0}^{n-1}(d-j)\ ,$$

where $(d-n+3)_n = \frac{\Gamma(d+3)}{\Gamma(d-n+3)}$ is the Pochhammer symbol. The norm becomes negative when $(n-2)<d<(n-1)$.

These negative norm states are robust and do survive at the Wilson-Fisher fixed point as long as the fixed point can be reached perturbatively i.e. $\varepsilon << 1$[103].

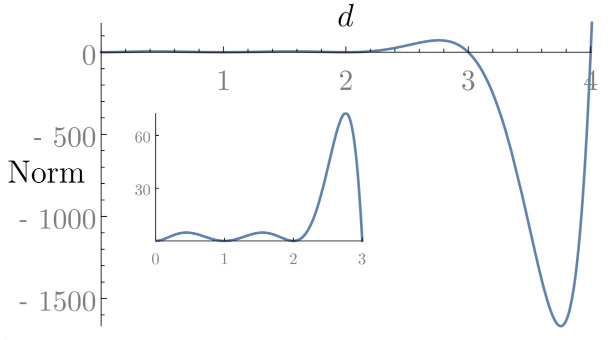

Figure 4.3: The norm of R_5 as a function of d. This becomes negative for $3 < d < 4$. The zoomed in version shows that the norm is non negative when $d \in (0,3)$ and becomes zero iff d is an integer below 5 i.e. $d = 0,1,2,3,4$.

4.4 Universality

$(0+1)$-D $SL(2,\mathbb{R})$ invariant field theory has universal features, irrespective of the details of the theory. They come out as a natural consequence of $SL(2,\mathbb{R})$ invariance and crossing symmetry. Schrödinger field theory, by virtue of being $SL(2,\mathbb{R})$ invariant theory as well, inherits these universal features, specially in the neutral sector. Following [105], we consider the 4 point correlator of four Hermitian operator $\langle O(0)O(\tau)O(1)O(\infty)\rangle = \tau^{-\Delta_O}\mathcal{G}(\tau)$, where, from the $SL(2,\mathbb{R})$ algebra, it follows that

$$\mathcal{G}(\tau) = \int_0^\infty d\Delta p(\Delta) G_\Delta(\tau)\,, \tag{4.12}$$

$$G_\Delta(\tau) = \tau^{\frac{\Delta}{2}}\ {}_2F_1\left(\frac{\Delta}{2},\frac{\Delta}{2},\Delta,\tau\right) \tag{4.13}$$

where $p(\Delta)$ is the weighted spectral density and given by $|c_{OO\Delta}|^2\rho(\Delta)$. Here $\rho(\Delta)$ is the density of quasi-primaries at Δ, $c_{OO\Delta}$ is the three point coefficient and G_Δ is $SL(2,\mathbb{R})$ block. We refer to the (C.12) in appendix C.1 for the generic form of three point correlator. The convergence of this integral for finite τ holds true for the same reason it holds true in $(0+1)$-D conformal field theory. Now, as $\tau \to 1$, the operator product expansion (OPE) of $O(\tau)O(1)$ is dominated by the contribution from the identity operator, thus we have

$$\mathcal{G}(1-\tau) \sim \tau^{-\Delta_O}\,, \quad \tau \to 0\,. \tag{4.14}$$

Using the fact,

$$G_\Delta(1-\tau) \simeq 2^\Delta \sqrt{\frac{\Delta}{2\pi}} K_0\left(\sqrt{\tau}\Delta\right)\,, \tag{4.15}$$

one can obtain [105]:

$$p(\Delta) \underset{\Delta\to\infty}{\sim} 2^{-\Delta}\sqrt{\frac{2\pi}{\Delta}}\frac{4^{1-\Delta_O}}{\Gamma(\Delta_O)^2}\Delta^{2\Delta_O-1} \tag{4.16}$$

The difference in factors of 2, as compared to Ref. [105], is coming from the definition of the dilatation operator and Δ in Schrödinger field theory. We remark that Schrödinger group has $U(1)$ subgroup, invariance under which implies that each of the operator O carries zero charge under $U(1)$. Thus, we are in fact probing the neutral sector, where one can not define the notion of Schrödinger primary. Furthermore, the non-zero asymptotics of $p(\Delta)$ in Eq. (4.16) directly implies that there has to be infinite number of quasi-primaries.

Infinite number of quasi-primaries: One can prove the existence of infinite number of quasi-primaries in the $OO \to OO$ OPE channel using the crossing symmetry as well. It might seem that the existence of infinite number of quasi-primaries are trivial as in the charged sector, operators that are some number of spatial derivatives acting on a primary do appear and they can be written down as a linear combination of $SL(2,\mathbb{R})$ descendants and quasi-primaries. But, here we consider $\langle O(0)O(\tau)O(1)O(\infty)\rangle$ and all the operators lie at $\boldsymbol{x}=0$. As a result, the operators that appear in the OPE are not of the form of some spatial derivative acting on a primary.

The proof goes by noting that the crossing symmetry implies

$$(1-\tau)^{\Delta_O}\mathcal{G}(\tau)=\tau^{\Delta_O}\mathcal{G}(1-\tau) \tag{4.17}$$

where Δ_O is the dimension of the operator O.

As $\tau\to 0$, the leading contribution to the left hand side of (4.17) comes from identity, i.e. we have $(1-\tau)^{\Delta_O}\mathcal{G}(\tau)=1+\cdots$. If we look at the right hand side in terms of blocks, we realize that each $G_\Delta(1-\tau)$ goes like $\log(\tau)$, thus the each term in the block decomposition of $\tau^{\Delta_O}\mathcal{G}(1-\tau)$ behaves like $\tau^{\Delta_O}\log(\tau)$, which goes to 0 as $\tau\to 0$. If we have finite number of quasi-primaries, since each of the summands goes to 0, we could never have (4.17) satisfied in $\tau\to 0$ limit. This proves the existence of infinite number of quasi-primaries which are not descendants of a primary. Similar argument works for $SL(2,\mathbb{R})$ primaries of conformal field theories as well. This line of argument has first appeared in [92] (See also[106]).

Analyticity of Three point function: The three point function of Schrödinger primaries are given by

$$\begin{aligned}&\langle\phi_1(\boldsymbol{x}_1,t_1)\phi_2(\boldsymbol{x}_2,t_2)\phi_3(\boldsymbol{x}_3,t_3)\rangle\\&=\exp\left[-N_1\frac{|\boldsymbol{x}_{13}|^2}{\tau_{13}}-N_2\frac{|\boldsymbol{x}_{23}|^2}{\tau_{23}}\right]\left(\prod_{i<j}\tau_{ij}^{\frac{\Delta_k-\Delta_i-\Delta_j}{2}}\right)F(v_{123})\end{aligned} \tag{4.18}$$

where $v_{123}=\frac{1}{2}\left(\frac{|\boldsymbol{x}_{23}|^2}{\tau_{23}}+\frac{|\boldsymbol{x}_{12}|^2}{\tau_{12}}-\frac{|\boldsymbol{x}_{13}|^2}{\tau_{13}}\right)$ and $\boldsymbol{x}_{ij}=\boldsymbol{x}_j-\boldsymbol{x}_i$, $\tau_{ij}=\tau_j-\tau_i$. F is a model dependent function and zero if $\sum_i N_i\neq 0$, where N_i is the charge carried by ϕ_i. By translation invariance, we can set $\tau_3=0$ and $\boldsymbol{x}_3=0$. As $\phi(\mathbf{0},\tau)$ is a quasi-primary, upon setting $\boldsymbol{x}_1=\boldsymbol{x}_2=\boldsymbol{x}_3=0$, we immediately obtain that $F(0)$ is a finite number and given by three point coefficient c_{123}.

For simplicity, let us work in $d=1$ dimension and set $x_1=x_3=0$ and investigate the behavior of F as a function of x_2. The $SL(2,\mathbb{R})$ algebra guarantees that F is infinitely differentiable at $x_2 = 0$. This follows from noting that

$$\begin{aligned}
&\langle \phi_1(0,t_1)\phi_2(x_2,t_2)\phi_3(0,0)\rangle \\
&= \sum_\beta \frac{1}{\beta!} x_2^\beta \langle \phi_1(0,t_1)\partial_{x_2}^\beta \phi_2(0,t_2)\phi_3(0,0)\rangle
\end{aligned} \tag{4.19}$$

and finiteness of $\langle \phi_1(0,t_1)\partial_{x_2}^\beta \phi_2(0,t_2)\phi_3(0,0)\rangle$. The finiteness follows from finiteness of norm and the fact that $\partial_{x_2}^\beta \phi_2(0,t_2)$ can be written down as a linear combination of quasi-primaries and descendants of quasi-primary. For example, we list out the first two quasi-primaries (in d spatial dimensions), given ϕ is a primary (which is trivially a quasi-primary too):

$$\mathcal{B}_i^{(1)} \equiv \partial_i \phi\,, \quad \mathcal{B}^{(2)} \equiv \frac{Nd}{2\Delta}\partial_\tau \phi + \frac{1}{2}\nabla^2 \phi\,.$$

4.5 Discussion & Outlook

We have shown that the features of $(0+1)$-D conformal field theory is inherited by the Schrödinger field theory. $SL(2,\mathbb{R})$ algebra can be leveraged to derive the unitary bounds and null conditions, to prove the convergence of operator product expansion in the kinematic limit, where all the operators are inserted at same $\boldsymbol{x}$, but at different times. Moreover, if we consider the four point correlator of Schrödinger primaries with all but one inserted on $(\tau, \mathbf{0})$ slice and one operator inserted at some different $\boldsymbol{x} \neq \mathbf{0}$, we can still prove the OPE convergence by using $SL(2,\mathbb{R})$ invariance. This happens because $\boldsymbol{x}$ dependence of the four point correlator is simply given by $\exp\left[\frac{N|\boldsymbol{x}|^2}{2\tau}\right]$ where $N < 0$ is the charge of the operator. The use of $SL(2,\mathbb{R})$ reveals the universal behavior of the weighted spectral density function and the existence of infinitely many quasi-primaries. We emphasize the salient role of $SL(2,\mathbb{R})$ in this context, as concept of Schrödinger primaries and descendants break down in the neutral sector. Moreover, one can easily deduce the analyticity of three point co-efficient function as a consequence of

$SL(2,\mathbb{R})$. It worths mentioning that the usual oscillator picture also relies on $SL(2,\mathbb{R})$ algebra in hindsight. In fact, the state-operator correspondence using the oscillator picture work beyond the primary operator: for every quasi-primary operator, one can define a state $|O\rangle = e^{-H} O|0\rangle$ such that $(H+C)\,|O\rangle = \Delta|O\rangle$, where $H+C$ can be interpreted as a Hamiltonian for the same system under harmonic trap.

The use of $SL(2,\mathbb{R})$ algebra provides us with a neat way to define the Euclidean Schrödinger theory. We refer to the appendix C.4 for more details. This justifies the Wick rotation done in [107] to evaluate the heat kernel and the Weyl anomaly. Moreover, the use of Euclidean Schrödinger operator in [80, 82] comes under question in this light as the correlator obtained from the heat kernel of such operators do not satisfy the constraint coming from $SL(2,\mathbb{R})$ algebra. In this connection, it deserves a remark that the notion of parity ($\tau \to -\tau$) and time reversal ($\tau \to -\tau$ with charge conjugation) is subtle in $(0+1)$-D conformal field theory and the same subtlety is also present in Schrödinger field theories (the details have been relegated to the appendix C.3). If one can consistently impose parity invariance beyond $(\tau, \mathbf{0})$ slice, such theories should have anti-particles and are suspected to have a non-zero entanglement entropy in the vacuum in contrast with its cousin where anti-particles are absent. One also wonders about presence of Weyl anomalies in such parity invariant theories on coupling to a non trivial curved background in the same spirit of [20, 21, 69, 22, 62, 64].

The most important take home message is that bootstrapping the Schrödinger field theory on $(\tau, 0)$ slice exactly amounts to bootstrapping $0+1$ D conformal field theory. Thus one can extend the analysis for 4 point correlator of operators with different dimensions, not necessarily the Hermitian ones with an aim to use $SL(2,\mathbb{R})$ bootstrap [108, 109, 110, 111] program to derive useful constraints for Schrödinger field theories. Furthermore, the four point correlator $\langle O(0)O(t)O(1)O(\infty)\rangle$ is analytic in complex t domain. One might hope to gain more mileage for $(0+1)$-D conformal field theory as well as the Schrödinger field theory using analyticity in the complex plane [108, 109]. In fact, if one is interested in knowing the spectra of the dilatation operator, then bootstrapping on $(\tau, 0)$ slice is sufficient as well. Should one consider a four point correlator of operators O_i inserted at different $\boldsymbol{x}$, the OPE would have operators $[P_{i_1}, [P_{i_2}, \cdots [P_{i_n}, O_k]]]$ while on the $(\tau, 0)$ slice, we would only have O_k operator. But the dimension of $[P_{i_1}, [P_{i_2}, \cdots [P_{i_n}, O_k]]]$ is completely fixed by O_k. This feature elucidates why it is sufficient to bootstrap on the $(\tau, \mathbf{0})$ slice to know the spectra of the dilatation operator. Similar argument applies for knowing the OPE coefficients.

On a different note, the operator basis for the heavy quark effective field theory (HQEFT), non relativistic QED/QCD [112] can be organized according to the representation of the Schrödinger algebra (or of $SL(2,\mathbb{R})$ algebra) like it is done for the Standard Model effective field theory [113, 114, 115]. As the operators appearing in the Lagrangian of HQEFT are necessarily neutral, the concept of quasi-primary is quintessential in that context as reported on a separate paper [100] with an application towards construction of an operator basis[116, 117, 118] for heavy particle effective field theory.

There are further questions which requires more attention. Fermions at unitarity [59] is described by a nontrivial fixed point in $4-\varepsilon$ dimensions, it is important to investigate whether there is any imprint of non-unitarity in the physics of that fixed point. A step towards this would be to find out whether heavy enough operators acquire complex anomalous dimension at WF fixed point. For a relativistic scenario, this has been done in [103]. It is also worthwhile to investigate whether $SL(2,\mathbb{R})$ constrains the properties of a thermal Schrödinger field theory [119]. At 0 temperature, one can calculate all the correlators using the OPE coefficients. For $T > 0$, the OPE is expected to hold true for time $|t| << \frac{\hbar}{k_B T}$[120]. Thus using the $SL(2,\mathbb{R})$ algebra, it seems possible to obtain sum rules involving conductivities as done in [120, 121, 122], particularly for CFTs. Furthermore, the idea presented here is extendable to the theories invariant under a symmetry group which contains $SL(2,\mathbb{R})$ as subgroup. The natural question is to ask whether the generalized z $(z \neq 2)$ group can have a bound. It is shown[61] that the algebra does not close with the special conformal generator C, if one has the particle number symmetry generator $\hat{N}$.

Thus $SL(2,\mathbb{R})$ subgroup is absent and they can not be realized with finite dimensional basis of operators [123]. Nonetheless, if one does not have the $U(1)$ associated with particle number symmetry, the algebra closes with C and it does have a $SL(2,\mathbb{R})$ piece, so similar analysis can be done for field theories invariant under such group. For sake of completeness, we write down the algebra so that $SL(2,\mathbb{R})$ becomes manifest:

$$
\begin{aligned}
&[D,C] = -2\imath C\,, [D,H] = 2\imath H\,, [H,C] = -\imath D \\
&[D,P_i] = \imath\alpha P_i\,, [D,K_i] = 2\imath(\alpha-\alpha^{-1})K_i\,, [H,K_i] = -\imath P_i \\
&[H,P_i] = [P_i,P_j] = [K_i,P_j] = 0
\end{aligned}
$$

where $\alpha = \frac{1}{z}$. The commutation relations of these with the generators of rotation group are the usual ones. Last but not the least, in $1+1$ dimensions, $SL(2,\mathbb{R})$ algebra gets extended to infinite Virasoro algebra. One can then introduce Virasoro conformal blocks and one has more analytical control over such theories. One wonders whether there exists any such extension for the Schrödinger algebra. If exists, it would imply the possibility of borrowing the arsenal of Virasoro algebra.

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Chapter 5

Existence and Construction of Galilean invariant $z \neq 2$ Theories.

We prove a no-go theorem for the construction of a Galilean boost invariant and $z \neq 2$ anisotropic scale invariant field theory with a finite dimensional basis of fields. Two point correlators in such theories, we show, grow unboundedly with spatial separation. Correlators of theories with an infinite dimensional basis of fields, for example, labeled by a continuous parameter, do not necessarily exhibit this bad behavior. Hence, such theories behave effectively as if in one extra dimension. Embedding the symmetry algebra into the conformal algebra of one higher dimension also reveals the existence of an internal continuous parameter. Consideration of isometries shows that the non-relativistic holographic picture assumes a canonical form, where the bulk gravitational theory lives in a space-time with one extra dimension. This can be contrasted with the original proposal by Balasubramanian and McGreevy, and by Son, where the metric of a $d+2$ dimensional space-time is proposed to be dual of a d dimensional field theory. We provide explicit examples of theories living at fixed point with anisotropic scaling exponent $z = \frac{2\ell}{\ell+1}, \ell \in \mathbb{Z}$.

5.1 Introduction

Gravity duals of non-relativistic field theories have been proposed in [61, 124]. It has been observed in Ref. [61], that one can consistently define an algebra with Galilean boost invariance and arbitrary anisotropic scaling exponent z. While the metric having isometry of this generalized Schrödinger group has been used with the holographic dictionary to construct correlators of a putative field theory[125, 126, 127, 128, 129, 130, 131], there is no explicit field theoretic realization of such a symmetry for $z \neq 2$.[1] One surprising feature, noted as a "strange aspect" in Ref. [61], is that, unlike in the canonical AdS/CFT correspondence, where the CFT in d dimensions is dual to the gravity in $d+1$-dimensions, in the non-relativistic case the metric is of a space-time with two additional dimensions. The $(d+2)$-dimensional metric, having isometries of the d-dimensional generalized Schrödinger group, is given by[61, 124]

$$ds^2 = L^2 \left[-\frac{dt^2}{r^{2z}} + \frac{2d\xi dt + dx^2}{r^2} + \frac{dr^2}{r^2} \right], \tag{5.1}$$

where ξ is the extra dimension having no analogous appearance in the relativistic AdS-CFT correspondence. The metric is invariant under the required anisotropic scaling symmetry

$$x_i \to \lambda x_i \,,\; t \to \lambda^z t \,,\; r \to \lambda r \,,\; \xi \to \lambda^{2-z} \xi, \tag{5.2}$$

and under Galilean boosts

$$x_i \to x_i + v_i t \,, \qquad \xi \to \xi - \frac{1}{2} \left(2 v_i x_i + v^2 t \right). \tag{5.3}$$

For $z = 2$, an explicit construction of Galilean boost invariant field theory in $(d-1)+1$ dimensions has been known. Thus a question arises naturally as to whether one can get rid of the

[1] We note that just matching the isometries is necessary but not sufficient for the existence of a holographic description. Here we just seek a group invariant field theory, which may or may not have a gravity dual.

extra ξ direction and reduce the correspondence down to a canonical correspondence between a d-dimensional quantum field theory on flat space and a $(d+1)$-dimensional gravitational theory. This was answered positively in Ref. [132]. But for $z \neq 2$ we do not know of any explicit d-dimensional field theoretic example having the generalized Schrödinger symmetry, nor do we know an example of a $(d+1)$-dimensional metric having the same set of isometries. Thus the "strange aspect" of d-$(d+2)$ correspondence appears to persist for $z \neq 2$.

In this paper, we initiate a field theoretic study of $z \neq 2$ theories.[2] We prove a no-go theorem for the construction of a space-time translation invariant, rotation invariant, Galilean boost invariant,[3] and $z \neq 2$ anisotropic scale invariant field theory with a finite number[4] of fields in d dimensions. Two point correlators in such theories, we show, grow unboundedly with spatial separation. By contrast, correlators of theories with an infinite number of fields, *e.g.*, labeled by a continuous parameter, do not necessarily exhibit this bad behavior. Hence, such theories behave effectively as a $(d+1)$-dimensional theory. In the context of holography, this explains the "strange aspect"; the $z \neq 2$ theories indeed provide us with the possibility of a canonical realization of holography, *i.e.*, a $(d+1)$-dimensional theory is dual to a $(d+2)$-dimensional geometry. The $z=2$ case is special in that respect since it is possible to obtain a d-dimensional theory with a finite number of fields such that the symmetries on field theory side match onto the isometries of a $(d+2)$-dimensional geometry. The special role of $z=2$ has been emphasized in the context of the holographic dictionary in Refs. [130, 131]. For $z=2$, the dual space-time can be made into a $(d+1)$-dimensional one via Kaluza-Klein reduction of the $(d+2)$-dimensional metric [132]. This is possible since, for $z=2$, the extra direction ξ does not scale by the transformations given in Eq. (5.2). The scaling of ξ given in Eq. (5.2) can be verified on the field theory side of the duality by embedding the d-dimensional generalized Schrödinger group

[2]Theories with $z=2$ have been studied from a field theoretic point of view in many works; see, *e.g.*, Refs. [59, 60, 81, 99]. The $z=\infty$ case without particle number symmetry has been explored in Ref. [133, 134].

[3]Here by Galilean boost invariance, we mean invariance under both the boost and a $U(1)$ particle number symmetries. The $U(1)$ naturally arises as a commutator of generators of boosts and translations.

[4]More precisely, a finite-dimensional basis of operators as defined below Eq. (5.12)

into the conformal group of one higher dimension *i.e.*, $SO(d,2)$. By contrast, since for $z \neq 2$ the ξ direction does scales, any attempt to compactify the extra direction ξ is at odds with the continuous scaling symmetry. The no-go theorem that we have proved is consistent with the argument in Ref. [135], based on consistency of thermodynamic equation of state, that a perfect fluid with $z \neq 2$ Schrödinger symmetry and discrete spectrum for the energy and particle number, H and N, can not exist. In Sec. 5.4, we present some fixed point theories with $z = \frac{2\ell}{\ell+1}$, with $\ell \in \mathbb{Z}$.

Before delving into a technical proof, we present a physical argument for our main result.[5] Consider a theory invariant under $z = 2$ Schrödinger symmetry, where, under a boost[61]

$$\phi(\boldsymbol{x},t) \mapsto \exp\left[-\imath n(\tfrac{1}{2}v^2 t + \boldsymbol{v}\cdot\boldsymbol{x})\right]\phi(\boldsymbol{x}-\boldsymbol{v}t,t)\,, \tag{5.4}$$

where $[N,\phi] = n\phi$. In turn, the state of a particle with momentum $\boldsymbol{k} = 0$ *i.e.*, $\phi^\dagger_{\boldsymbol{k}=0}|0\rangle$ transforms under the boost by $\boldsymbol{v}$ as follows:

$$\begin{aligned} |\boldsymbol{v}\rangle &\equiv e^{-\imath \boldsymbol{K}\cdot\boldsymbol{v}}\phi^\dagger_{\boldsymbol{k}=0}|0\rangle \\ &= \int d\boldsymbol{x}\,\exp\left[\imath n(\tfrac{1}{2}v^2 t + \boldsymbol{v}\cdot\boldsymbol{x})\right]\phi^\dagger(\boldsymbol{x}-\boldsymbol{v}t,t|0\rangle \\ &= \exp\left[\imath \tfrac{nv^2}{2}t\right]\phi^\dagger_{\boldsymbol{k}=n\boldsymbol{v}}|0\rangle\,. \end{aligned} \tag{5.5}$$

This has the interpretation of having a boosted particle moving with momentum $n\boldsymbol{v}$ and kinetic energy $-\frac{1}{2}n\boldsymbol{v}^2$. A positive value of n results in decreasing energy with increasing boost. Therefore, negative semi-definiteness of n is required for stability. In case of more than a single species of particle, the matrix $\mathbb{N}$ appearing in $[N,\Phi^\dagger] = -\mathbb{N}\Phi^\dagger$ has to be negative semi-definite. As we will see, from the symmetry algebra it follows that for a theory with finite number of fields with $z \neq 2$ the trace of $\mathbb{N}$ must vanish, spoiling the negative semi-definiteness and the stability in the

[5] We thank John McGreevy for discussions leading to this argument.

sense discussed above; by contrast, for $z=2$ there is no constraint on the trace of $\mathbb{N}$. The above is merely a heuristic argument, giving intuition behind the technical result presented below.

5.2 Generalized Schrödinger algebra and its Representation

The Galilean algebra consists of generators corresponding to spatial translations, P_i, time translation, H, Galilean boosts, K_i, rotations, M_{ij}, along with a particle number generator, N, such that they satisfy the following commutation relations [59, 60, 81, 99]:

$$\begin{gathered}
[M_{ij},N]=[P_i,N]=[K_i,N]=[H,N]=0 \\
[M_{ij},P_k]=\imath(\delta_{ik}P_j-\delta_{jk}P_i)\,, \\
[M_{ij},K_k]=\imath(\delta_{ik}K_j-\delta_{jk}K_i)\,, \\
[M_{ij},M_{kl}]=\imath(\delta_{ik}M_{jl}-\delta_{jk}M_{il}+\delta_{il}M_{kj}-\delta_{jl}M_{ki})\,, \\
[P_i,P_j]=[K_i,K_j]=0\,, \qquad [K_i,P_j]=\imath\delta_{ij}N\,, \\
[H,N]=[H,P_i]=[H,M_{ij}]=0\,, \quad [H,K_i]=-\imath P_i\,.
\end{gathered} \tag{5.6}$$

The algebra can be enhanced by appending a dilatation generator D,[6] which scales space and time separately, in the following way:

$$x_i\to\lambda x_i, \qquad t\to\lambda^z t\,. \tag{5.7}$$

The commutators of D with the rest of the generators are given by

$$\begin{gathered}
[D,P_i]=\imath P_i\,, \quad [D,K_i]=(1-z)\imath K_i\,, \quad [D,H]=z\imath H\,, \\
[D,N]=\imath(2-z)N\,, \quad [M_{ij},D]=0\,.
\end{gathered} \tag{5.8}$$

[6]This enhanced algebra corresponds to that of deformed ISIM(2) [136], with the following identification: $H\mapsto P_+$, $N\mapsto P_-$, $K_i\mapsto M_{+i}$ and $D\mapsto -\frac{1}{b}N$ where $b(z-1)=1$.

The physical interpretation of N is subtle. For $z = 2$ it is usually thought of as a particle number symmetry generator. The subtlety in the context of holography has been explored in [132]. For rest of this work, we take an agnostic viewpoint and treat N as a generator of symmetry without specifying its physical origin. This will enable us to explore all the possibilities, as allowed by symmetries. The case $z = 2$ is very special in that one can append an additional generator C of special conformal transformations. Thus one can have the full Schrödinger algebra for $z = 2$ [96, 93, 94, 98, 81, 99, 137]. When $z \neq 2$, the generator corresponding to special conformal transformation is not available.

In what follows, we will assume (unless otherwise specified) that the field theory lives in $d = (d-1)+1$ dimensions and that the vacuum is invariant under Galilean boosts, *i.e.*, $K_i|0\rangle = \langle 0|K_i = 0$

The field representation is built by defining local operators Φ such that H and P act canonically,

$$[H,\Phi] = -\imath\partial_t\Phi\,,\ [P_i,\Phi] = \imath\partial_i\Phi\,. \tag{5.9}$$

We consider representations of the little group, generated by D, K_i, N and M_{ij}, that keeps the origin, $(\mathbf{0},0)$, invariant. The fields Φ have definite transformation properties under D, K_i and N,[7]

$$[D,\Phi(\boldsymbol{x}=\mathbf{0},t=0)] = \imath\mathcal{D}\Phi(\boldsymbol{x}=\mathbf{0},t=0)\,, \tag{5.10}$$

$$[N,\Phi(\boldsymbol{x}=\mathbf{0},t=0)] = \mathcal{N}\Phi(\boldsymbol{x}=\mathbf{0},t=0)\,, \tag{5.11}$$

$$[K_i,\Phi(\boldsymbol{x}=\mathbf{0},t=0)] = \mathcal{K}_i\Phi(\boldsymbol{x}=\mathbf{0},t=0)\,. \tag{5.12}$$

where $\mathcal{D}$, $\mathcal{N}$, and $\mathcal{K}_i$ are linear operators. We refer to the smallest non-trivial irreducible representation in Eqs. (5.10)–(5.12) as "the basis of operators".[8] For Lagrangian theories the

[7]The fields Φ also have definite transformation properties under M_{ij}, but this will not play a role in the discussion below.

[8]For example, the free Schrödinger field theory is invariant under $z = 2$ Schrödinger algebra and the single field

basis of operators corresponds to the elementary fields from which the Lagrangian is constructed. Henceforth, we restrict our attention to the basis of operators, and continue to denote by $\mathcal{D}$, $\mathcal{N}$, and $\mathcal{K}_i$ their linear representation. In the finite dimensional case, we denote these by finite dimensional matrices $\mathbb{A}$, $\mathbb{N}$ and $\mathbb{K}_i$, $\mathbb{A}$ respectively.

Consider $G_{\alpha\beta} \equiv \langle 0|\Phi_\alpha(x,t)\Phi_\beta(0,0)|0\rangle$. Using Eqs. (5.9), the commutator in (5.12) translates, in the finite dimensional case, to

$$[K_i,\Phi] = (-\imath t\partial_i \mathbf{I} + x_i\mathbb{N} + \mathbb{K}_i)\,\Phi \tag{5.13}$$

where $x_i = x^i$. Galilean boost invariance of the vacuum, $K_i|0\rangle = \langle 0|K_i = 0$, then gives

$$\langle 0|\left[K_i,\Phi_\alpha(x,t)\Phi_\beta(0,0)\right]|0\rangle = 0$$
$$\Rightarrow (-\imath t\partial_i\delta_{\alpha\sigma} + x_i\mathbb{N}_{\alpha\sigma} + \mathbb{K}_{i\alpha\sigma})\,G_{\sigma\beta} + \mathbb{K}_{i\beta\sigma}G_{\alpha\sigma} = 0\,.$$

Using the fact that $[\mathbb{N},\mathbb{K}_i] = 0$, the solution to the above differential equation is given by

$$G_{\alpha\beta} = \left(e^{-\imath\frac{|\boldsymbol{x}|^2}{2t}\mathbb{N}}e^{-\imath\frac{\boldsymbol{x}\cdot\mathbb{K}}{t}}C(t)e^{-\imath\frac{\boldsymbol{x}\cdot\mathbb{K}^T}{t}}\right)_{\alpha\beta}$$

where $C(t)$ is an as yet undetermined matrix function of t. The norm is defines as $|\boldsymbol{x}|^2 \equiv \sum_i (x^i)^2$ while the dot product is defined as $\boldsymbol{x}\cdot\boldsymbol{K} = \sum_i x_i K_i$. Similarly, one can consider $G'_{\alpha\beta} \equiv \langle 0|\Phi_\alpha(x,t)\Phi^\dagger_\beta(0,0)|0\rangle$ which is given by

$$G'_{\alpha\beta} = \left(e^{-\imath\frac{|\boldsymbol{x}|^2}{2t}\mathbb{N}}e^{-\imath\frac{\boldsymbol{x}\cdot\mathbb{K}}{t}}C'(t)e^{\imath\frac{\boldsymbol{x}\cdot\mathbb{K}^\dagger}{t}}\right)_{\alpha\beta}$$

where $C'(t)$ is as yet undetermined.

ϕ forms a one dimensional irreducible representation of the little group i.e. $[D,\phi(\mathbf{0},0)] = \imath\frac{d}{2}\phi(\mathbf{0},0)$, $[N,\phi(\mathbf{0},0)] = N\phi(\mathbf{0},0)$ and $[K_i,\phi(\mathbf{0},0)] = 0$.

Since $[D,N] = \imath(2-z)N$, we have

$$[\mathbb{\Delta}, \mathbb{N}] = (2-z)\mathbb{N}\,.$$

and this leads to $\mathrm{Tr}(\mathbb{N}) = 0$; similarly, for $z \neq 1$ we have $\mathrm{Tr}(\mathbb{K}_i) = 0$. Now using Jordan-Chevalley decomposition, we can write

$$\mathbb{N} = \mathbb{N}_1 + \mathbb{N}_2\,,\ [\mathbb{N}_1, \mathbb{N}_2] = 0\,,$$
$$\mathbb{K} = \mathbb{K}_1 + \mathbb{K}_2\,,\ [\mathbb{K}_1, \mathbb{K}_2] = 0\,.$$

where $\mathbb{K}_1$ and $\mathbb{N}_1$ are diagonalizable matrices while $\mathbb{N}_2$ and $\mathbb{K}_2$ are nilpotent matrices (here and below we suppress the vector index in $\mathbb{K}$ and $\mathbb{K}_{1,2}$ to avoid clutter). Let us define diagonal matrices D_{N_1} and D_{K_1} such that

$$P_N D_{N_1} P_N^{-1} = \mathbb{N}_1, \quad P_K D_{K_1} P_K^{-1} = \mathbb{K}_1 \tag{5.14}$$

where P_N and P_K diagonalize $\mathbb{N}_1$ and $\mathbb{K}_1$ respectively. The zero trace condition leads to $\mathrm{Tr}(\mathbb{K}_1) = \mathrm{Tr}(D_{K_1}) = 0$ and $\mathrm{Tr}(\mathbb{N}_1) = \mathrm{Tr}(D_{N_1}) = 0$, which in turn implies that either all the diagonal entries of D_{N_1} (or D_{K_1}) are zero, in which case $\mathbb{N}$ (or $\mathbb{K}$) is a nilpotent matrix, or there has to be both positive and negative entries. We can then recast the correlators as follows:

$$G_{\alpha\beta} = \left(e^{\frac{|\boldsymbol{x}|^2}{2\imath t}\mathbb{N}_1} e^{\frac{|\boldsymbol{x}|^2}{2\imath t}\mathbb{N}_2} e^{\frac{\boldsymbol{x}\cdot\mathbb{K}_1}{\imath t}} e^{\frac{\boldsymbol{x}\cdot\mathbb{K}_2}{\imath t}} C(t) e^{\frac{\boldsymbol{x}\cdot\mathbb{K}_1^T}{\imath t}} e^{\frac{\boldsymbol{x}\cdot\mathbb{K}_1^T}{\imath t}}\right)_{\alpha\beta} \tag{5.15}$$

$$G'_{\alpha\beta} = \left(e^{\frac{|\boldsymbol{x}|^2}{2\imath t}\mathbb{N}_1} e^{\frac{|\boldsymbol{x}|^2}{2\imath t}\mathbb{N}_2} e^{\frac{\boldsymbol{x}\cdot\mathbb{K}_1}{\imath t}} e^{\frac{\boldsymbol{x}\cdot\mathbb{K}_2}{\imath t}} C'(t) e^{-\frac{\boldsymbol{x}\cdot\mathbb{K}_1^\dagger}{\imath t}} e^{-\frac{\boldsymbol{x}\cdot\mathbb{K}_2^\dagger}{\imath t}}\right)_{\alpha\beta} \tag{5.16}$$

It follows that when $\mathbb{N}_1 \neq 0$, $e^{\frac{|\boldsymbol{x}|^2}{2\imath t}\mathbb{N}_1} = P_N e^{\frac{|\boldsymbol{x}|^2}{2\imath t}D_{N_1}} P_N^{-1}$ has exponential growth for imaginary time irrespective of how we do the analytical continuation of the correlator to imaginary time. This growth can not be overcome by any of the other terms as nilpotency of $\mathbb{N}_2$ guarantees that

$$e^{-\imath\frac{|\boldsymbol{x}|^2}{2t}\mathbb{N}_2} = \sum_{\ell=0}^{\ell=M-1}\left(-\imath\frac{|\boldsymbol{x}|^2}{2t}\right)^{\ell}\mathbb{N}_2^{\ell} \tag{5.17}$$

where $\mathbb{N}_2^M = 0$ for some integer M. Also, terms like $e^{\imath\frac{\boldsymbol{x}\cdot\mathbb{K}}{t}}$ cannot suppress the exponential growth arising from $\mathbb{N}_1$.

If instead $\mathbb{N}_1 = 0$ then $e^{-\imath\frac{|\boldsymbol{x}|^2}{2t}\mathbb{N}_2}$ gives polynomial growth with x. We employ the same technique to establish the effect of $e^{\imath\frac{\boldsymbol{x}\cdot\mathbb{K}}{t}}$. If $\mathbb{K}_1 \neq 0$, there will be exponential growth for some entries, while terms involving $\mathbb{K}_2$ are polynomial in nature, giving exponential growth as a whole. Alternatively, if $\mathbb{K}_1 = 0$ then $\mathbb{K}$ is nilpotent and we have polynomial growth.

We note that only when $z = 2$ or the representation is infinite, we can not implement the $\mathrm{Tr}(\mathbb{N}) = 0$ condition and the above argument fails. This is expected for $z = 2$ since the two point correlator is well behaved in this case, that corresponds to Schrödinger field theory [96, 93, 94, 98]. We conclude that in the finite dimensional case for $z \neq 2$ a quantum field theory with the symmetry of the algebra in Eqs. (5.6) and (5.8) is ill-behaved. For example, since correlators grow with spatial separation cluster decomposition fails. The same conclusion can be drawn via an independent argument in the case that $\mathbb{\Delta}$ is diagonal; see App. D.1.

Therefore, for $z \neq 2$ we are left to consider infinite dimensional representations. In this case we can display explicitly an example that does not obviously lead to problematic quantum field theories. To achieve this, we introduce fields ψ labeled by a new non-compact variable ξ, such that

$$[N,\psi] = \imath\partial_\xi\psi\,, \tag{5.18}$$

$$[D,\psi] = \imath\left(zt\partial_t + x^i\partial_i + (2-z)\xi\partial_\xi + \Delta_\psi\right)\psi\,, \tag{5.19}$$

$$[K_i,\psi] = \left(-\imath t\partial_i + \imath x_i\partial_\xi\right)\psi. \tag{5.20}$$

Thus, $\mathcal{D} = (2-z)\xi\partial_\xi + \Delta_\psi$, $\mathcal{N} = \imath\partial_\xi$, and $\mathcal{K}_i = 0$. Note that ξ must be a non-compact variable, else scaling symmetry is broken. To be concrete,

$$\left[\xi, \partial_\xi\right] = -1\,, \quad \left[\xi\partial_\xi, \partial_\xi\right] = -\partial_\xi\,, \tag{5.21}$$

are well defined only when ξ is a non-compact variable. If we take a Fourier transform with respect to ξ, it becomes obvious that $\mathcal{N}$ is diagonal while $\mathcal{D}$ is not diagonal. This, however, is immaterial, since in terms of a new variable $\xi' = \ln|\xi|$, $\mathcal{N}$ is non-diagonal and $\mathcal{D}$ is diagonal.

We say ψ is a *primary* operator if $[K_i, \psi(\boldsymbol{x}=0, t=0; \xi)] = 0$, that is, $\mathcal{K}_i = 0$; this was assumed in the commutation relations (5.20). Once again, one can invoke the Galilean boost invariance of the vacuum to obtain the form of the two point correlator of primaries ψ and ϕ. This is most easily computed in terms of the the Fourier transformed operators, *e.g.*, $\psi(\boldsymbol{x}, t, m_1) = \int d\xi\, \psi(\boldsymbol{x}, t, \xi) e^{\imath m\xi}$; we obtain

$$\begin{aligned}&\langle 0|\psi(\boldsymbol{x}, t, m_1)\phi(0, 0, m_2)|0\rangle \\ &= \begin{cases} h(t)\delta(m_1+m_2)\, f(t^{2-z}m_1^z)\exp\left(\frac{\imath m_1|\boldsymbol{x}|^2}{2t}\right), & z \neq 0 \\ h(t)\delta(m_1+m_2)\, f(m_1)\exp\left(\frac{\imath m_1|\boldsymbol{x}|^2}{2t}\right), & z = 0 \end{cases}\end{aligned} \tag{5.22}$$

where $h(t)$ is an as yet undetermined function of t. Evidently, Eq. (5.22) is consistent with the correlator of the $z=2$ theory [81, 99]. For $z \neq 2$, rewriting in terms of ξ, we obtain:

$$\begin{aligned}&\langle 0|\psi(\boldsymbol{x}, t, \xi)\phi(0, 0, 0)|0\rangle \\ &\propto \begin{cases} h(t)t^{1-2/z}\tilde{g}\left(\frac{|\boldsymbol{x}|^2-2t\xi}{2t^{2/z}}\right), & z \neq 0 \\ \tilde{h}(t)\tilde{f}\left(\frac{|\boldsymbol{x}|^2}{2t} - \xi\right) = \tilde{h}(t)\left(\frac{|\boldsymbol{x}|^2}{2t} - \xi\right)^{-\Delta/2}, & z = 0 \end{cases}\end{aligned} \tag{5.23}$$

where $\tilde{g}(s) = \int dy\, e^{-\imath ys} g(y)$, $g(y) = f(y^z)$ and $y^z = m^z t^{2-z}$. When $z = 0$, we use the fact that $\tilde{f}$

has to scale covariantly under $z=0$ scaling, where $\tilde{f}$ is the Fourier transform of f; here $h(t)$ must be a power law of t with $t^{-\alpha}$ such that the scaling dimensions of ψ and ϕ add up to $\alpha z+(2-z)$ for $z\neq 0$ and Δ for $z=0$ with $\tilde{h}(t)$ being any function of t.

5.3 Null reduction and Embedding into Conformal group

A standard trick to obtain a d dimensional $z=2$ Schrödinger invariant theory is to start with a conformal field theory in $d+1$ dimensions and perform a null cone reduction [77, 22, 138, 139, 107, 140]. This is possible because the Schrödinger group, Sch(d), can be embedded into $SO(d,2)$. Next we show that the generalized Schrödinger group can also be embedded into $SO(d,2)$. A similar embedding has been considered in [136] in the context of the Lie algebra of the deformed ISIM(2) group.

If the generators of $SO(d,2)$ are given by $P_\mu^{(r)}, M_{\mu\nu}^{(r)}, D^{(r)}, C_\mu^{(r)}$ where $P_\mu^{(r)}$ are translation generators, $M_{\mu\nu}^{(r)}$ are Lorentz generators, $D^{(r)}$ is the relativistic scaling generator and $C_\mu^{(r)}$ are special conformal generators (here the superscript "(r)" denotes the relativistic generators), then following generators generate the generalized Schrödinger algebra:

$$K_i = M_{i-}^{(r)}, H = P_+^{(r)}, N = P_-^{(r)} \tag{5.24}$$

$$M_{ij} = M_{ij}^{(r)}, P_i = P_i^{(r)} \tag{5.25}$$

$$D = D^{(r)} + (1-z)M_{+-}^{(r)} \tag{5.26}$$

It is straightforward to verify that D scales $x^- \to \lambda^{2-z}x^-$. Only for $z=2$, does x^- not scale and one is able to do a null cone reduction via compactification in the x^- direction, yielding a discrete spectra for N. On the other hand, for $z\neq 2$, even via null cone reduction one can not truly get rid of the x^- direction since any compactification in the x^- direction would spoil the scaling symmetry. As a result, for $z\neq 2$ the null reduction always leaves a continuous spectra for the generator N.

5.4 Explicit $d+1$ dimensional examples

5.4.1 $z=0$

Here we provide with an explicit example of a generalized Schrödinger invariant theory in $(d-1)+1$ dimensions with $z=0$ and verify that the two point correlator indeed conforms to the general form given in Eq. (5.23).

We consider a Lagrangian model given by

$$\mathcal{L} = \phi^\dagger \left(2\partial_t\partial_\xi - \nabla^2 + 2\imath\partial_\xi\right)\phi \tag{5.27}$$

and the two point correlator is given by[9]

$$\langle\phi\phi^\dagger\rangle \propto \left(\frac{1}{t}\right)^{\frac{d-1}{2}} \exp\left[-\imath t\right] \left(\frac{|\boldsymbol{x}|^2}{2t} - \xi\right)^{-\frac{d-1}{2}} \tag{5.28}$$

In $d+1$ dimensions, $\frac{d-1}{2} = \frac{(d+1)-2}{2}$ is precisely the dimension of a free relativistic scalar. This is because the generalized Schrödinger algebra can be embedded into the conformal group of one higher dimension, as mentioned in Sec. 5.3.

For $z=0$, t does not scale. One may contemplate perturbing the gaussian fixed point by marginal operators constructed out of powers of ∂_t, for example, $\phi^\dagger \exp\left(\imath\partial_t\right)\partial_\xi\phi$. However, Galilean boost invariance requires that ∂_t appears in the combination with other derivatives shown in Eq. (5.27). By contrast, in the models presented in Refs. [133, 134], where $N=0$ and the Lagrangian is invariant under $\boldsymbol{x} \to \boldsymbol{x}$ and $t \to \lambda t$, arbitrary powers of spatial derivatives are allowed.

[9]The correlator in (5.28) follows from (5.27) only after restricting the field ϕ to positive ξ-Fourier modes; see the footnote below Eq. (5.30) for more details.

5.4.2 $z = \frac{2\ell}{\ell+1}, \ell \in \mathbf{Z}, \ell \geq 1$

These series of examples are given by following Lagrangian

$$\mathcal{L}_\ell = \phi^\dagger \left(2\partial_t\partial_\xi - \nabla^2 + 2g\left(\imath\partial_\xi\right)^{\ell+1}\right)\phi \tag{5.29}$$

The two point correlators, after partial Fourier transformation is given by

$$G(\boldsymbol{x},t,m) \propto t^{-\frac{d-1}{2}} m^{\frac{d-3}{2}} \exp\left[\imath\left(\frac{m|\boldsymbol{x}|^2}{2t} - gm^\ell t\right)\right] \tag{5.30}$$

where $z = \frac{2\ell}{\ell+1}$. One can Fourier transform[10] to obtain the correlator in position space-time only depending on the analytical ease to do so. For $d = 3$, $\ell = 2$ i.e $z = \frac{4}{3}$, we have

$$G(\boldsymbol{x},t,\xi) \propto t^{-1}\frac{1}{\sqrt{gt}} \exp\left[\frac{\imath(x^2 - 2\xi t)^2}{16gt^3}\right] \tag{5.32}$$

which is consistent with Eq. (5.23) for $z \neq 0$. After performing a Euclidean rotation, $t \to -i\tau$, $\xi \to i\xi$, one finds good behavior of this correlator at large spatial separation (as long as $g < 0$).

One can add classically marginal interactions to the model in (5.29). For example, one may add $(\phi\phi^\dagger)^{n-1}\phi(\imath\partial_\xi)^k\phi^\dagger$ with $k = (\ell+1)[(d-1)\beta + d - 2]$ and $n = 2\beta + 3$, where β is a non-negative integer. Furthermore, one can have supersymmetric generalizations of $z \neq 2$ theories,

[10]Care is needed regarding the allowed values of m. The correlator in (5.30) is most readily obtained by Fourier transform of

$$G(\boldsymbol{k},t,m) \propto \exp\left[-\imath t\left(\frac{|\boldsymbol{k}|^2}{2m} + gm^\ell\right)\right]. \tag{5.31}$$

For even, positive ℓ, the integral over $\boldsymbol{k}$ is well defined only for $Im(t/m) < 0$, and the result can be analytically continued to all values of t/m. The integral over m requires $Im(t) < 0$ (for $g > 0$), and again one analytically continues to all values of t.

For odd (and positive) ℓ, the Fourier transform with respect to m is ill behaved for any value of t, because there is no deformation of the contour of integration that can render the integral of $\exp\left[\imath\left(\frac{m|\boldsymbol{x}|^2}{2t} - gm^\ell t\right)\right]$ over m finite. Both for ℓ odd and for $\ell = 0$, a sensible way to make this integral well defined is to restrict it to $m > 0$. This is, in fact, how we obtained the correlator for the $z = \ell = 0$ in Eq. (5.28). Strictly speaking, these are not Lagrangian theories; these systems are close analogues of the chiral boson, where the Fourier modes are restricted [141].

much like the $z = 2$ case presented in [142] where supersymmetry is an internal symmetry exchanging Fermionic and Bosonic fields.

5.5 Conclusion

The most natural way to realize the Schrödinger algebra and its $z \neq 2$ avatar in a gravity dual of a d-dimensional non-relativistic field theory with Galilean boost and scale invariance with dynamical exponent z is via isometries of the bulk metric. As it turns out, the dual metric is of a $(d+2)$-dimensional space-time [61, 124]. By contrast, for the canonical notion of gauge-gravity duality the bulk gravitational theory lives in one extra dimensional space-time. Above we have expounded the presence of the two extra dimensions in the duality. We showed that on the field theory side of the duality, for $z \neq 2$, one needs to have an internal continuous parameter, effectively making the field theory $(d+1)$-dimensional. Any attempt to construct a $z \neq 2$ non-relativistic field theory with Galilean boost and scale invariance with finite number of fields is bound to run into trouble, since correlators will grow with separation and will fail to exhibit cluster decomposition. This result follows solely from constrains that the symmetry algebra places on two point correlators. It is important to have the particle number symmetry for the no-go theorem. Without particle number symmetry, there are indeed examples of Galilean boost invariant $z \neq 2$ theories [143]. Examples of theories with $z = \infty$ anisotropic scaling symmetry based on warped conformal field theories, are discussed in Ref. [133, 134].

Only for $z = 2$ is a consistent d-dimensional field theoretic realization of the symmetry, with finite number of fields, possible, and therefore a conventional $(d+1)$-dimensional gravity dual is available. On the gravity side, the metric dual to a $z = 2$ Schrödinger theory has a direction ξ which does not scale, and can therefore be compactified. The Kaluza-Klein reduction of the momentum conjugate to ξ generates a discrete spectrum for N that matches onto a d-dimensional field theory. The ξ direction for $z \neq 2$ duals scales, forbidding any such compactification. One can also see this by embedding the generalized Schrödinger group into $SO(d,2)$; see Sec. 5.3.

That there is no impediment to constructing a sensible $z \neq 2$ non-relativistic field theory with Galilean boost and scale invariance for an infinite number of fields is most easily established by giving explicit examples. Above we presented explicit examples of Galilean boost invariant theories, with $z = \frac{2\ell}{\ell+1}$.

Given that we have explicit examples and the generic form of the correlator, several new questions come to mind. One can ask how one may couple these theories to gravity. Non-relativistic theory coupled to gravity gives a natural framework to study Ward identity anomalies, and scale anomalies [144, 145, 62, 64, 63, 18, 22, 66, 20, 69, 21, 107]. Since these theories are intrinsically $(d+1)$-dimensional, the use of Newton-Cartan geometry is not a natural choice. It would also be interesting to understand the dispersion relation of Goldstone bosons, arising from spontaneous breaking of $z \neq 2$ scale symmetry; the $z = 2$ case has been studied in [146].

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Chapter 6

Hilbert Series and Operator Basis for NRQED and NRQCD/HQET

We use a Hilbert series to construct an operator basis in the $1/m$ expansion of a theory with a nonrelativistic heavy fermion in an electromagnetic (NRQED) or color gauge field (NRQCD/HQET). We present a list of effective operators with mass dimension $d \leq 8$. Comparing to the current literature, our results for NRQED agree for $d \leq 8$, but there are some discrepancies in NRQCD/HQET at $d = 7$ and 8.

6.1 Introduction

An operator basis for an effective field theory is a set containing all operators that give rise to different scattering matrix elements, invariant under relevant symmetries of the theory. The Hilbert series[1] can be used as a tool for enumerating the elements of an operator basis for effective field theories. With it, one can impose symmetry requirements [147, 148, 149] and account for redundancies between operators coming from the equations of motion and

[1]The Hilbert series is a generic concept, defined on any graded vector space. In our context, it is defined over the ring of operators of the effective theory under consideration, as in Refs. [113, 114]. Our working definition is given in Section 6.3.

integration by parts [113, 114]. So far, the focus has been on relativistic theories, e.g., especially the effective theory of the Standard Model [114, 150, 151], and nonrelativistic effective theories have been unexplored using Hilbert-series methods.

We examine the specific effective theories of a single nonrelativistic fermion in an electromagnetic field or color field. These effective theories are called non-relativistic QED (NRQED) and non-relativistic QCD (NRQCD), respectively. NRQCD is the same theory described by the heavy quark expansion (HQET) [112], and we refer to this theory as NRQCD/HQET. One can construct a list of effective operators, where operators are suppressed by the appropriate powers of $1/m$ (where m is the fermion's mass), in the rest frame of the heavy fermion without loss of generality. These two effective theories have been used extensively over the past few decades. For example, NRQED was originally formulated in Ref. [152], where higher-dimensional operators were listed by the authors of Refs. [153, 112, 116], and is used to explore the proton radius puzzle (see, for example, Ref. [154], and references therein). NRQCD/HQET is a tool that can be used to extract the value of $|V_{cb}|$ in inclusive semileptonic B decays, and the higher-order terms in $1/m$ have been discussed in Refs. [155, 112, 156, 118]. These high-order terms in $1/m$ may become important when analyzing the high-luminosity data from the upcoming Belle-II experiment [157, 158].

There is currently no disagreement in the literature regarding the number of NRQED operators up to and including order $1/m^4$, and the Hilbert series we construct for NRQED agrees with the results in Refs. [152, 153, 112, 116]. Also, our results for NRQCD/HQET agree with those in Refs. [155, 112] up to order $1/m^3$, but we find discrepancies with other analyses at $1/m^3$ and $1/m^4$. Specifically, we count 11 operators at $1/m^3$ (as does Ref. [112]), and 25 operators at $1/m^4$. However, at order $1/m^3$, Ref. [159] says there are 5, and Refs. [156, 118] claim there are 9. At order $1/m^4$, Refs. [156, 118] claim there are 18 operators. The differences between our results and those found in Refs. [156, 118] could be explained by there being two symmetric $SU(3)$ color singlets for operators with two gauge bosons. We discuss this further in Section 6.5.

6.2 Effective Theory for a Nonrelativistic Fermion

We consider a system where the relevant dynamics of a massive fermion in an external, dynamical, gauge field occurs at energy scales well below the rest mass, m, of a fermion.[2] The following effective Lagrangian can be used to describe such a system with a heavy fermion:

$$\mathcal{L} = \psi^\dagger iD_t\psi + \sum_{k=1}^{\infty} c_k \psi^\dagger O_k \psi. \tag{6.1}$$

Here, ψ is a two-component Pauli spinor, c_k is a coupling constant, and O_k are Hermitian operators, suppressed by the appropriate powers of $1/m$. All operators O_k must be rotationally and translationally invariant, contain either zero or one spin vector s^i, and are built from time and spatial components of covariant derivatives, i.e., iD_t and $iD_\perp$, respectively.

Listing all operators that satisfy only these conditions leads to over counting, since some operators can be related to others via integration by parts or relations associated with the equations of motion. In particular, operators with derivatives that act on $\psi^\dagger$ can be related to other operators with derivatives that act on ψ by integrating by parts:

$$\psi^\dagger i\overleftarrow{\partial_t} O\psi + \psi^\dagger i\partial_t O\psi = 0, \tag{6.2}$$

$$\psi^\dagger i\overleftarrow{\partial_\perp} O\psi + \psi^\dagger i\partial_\perp O\psi = 0, \tag{6.3}$$

where O is some operator. Also, the equation of motion for ψ is

$$iD_t\psi + \sum_{k=1}^{\infty} c_k O_k\psi = 0. \tag{6.4}$$

Therefore, if D_t acts on ψ, it can be replaced by a series of operators, all at higher powers in $1/m$:

$$\psi^\dagger O iD_t\psi = -\sum_{k=1}^{\infty} c_k\psi^\dagger O O_k\psi, \tag{6.5}$$

[2]Physical theories are diffeomorphic, so if an operator is zero in one reference frame, it is zero in all other frames. Therefore, we choose to work in the rest frame of the nonrelativistic fermion for the purposes of enumerating operators and constructing a basis.

where O is some Hermitian operator. A similar argument holds for the equations of motion associated with $\psi^\dagger$. There are also equations of motion associated with the external gauge field. We refer to antisymmetric combinations of covariant derivatives as $E^i = \frac{i}{g}[D_t, D_\perp^i]$ and $B^i = -\frac{i}{2g}\varepsilon^{ijk}[D_{\perp j}, D_{\perp k}]$. The equations of motion for E^i and B^i are Maxwell's equations, or its non abelian version:

$$\boldsymbol{D}_\perp \cdot \boldsymbol{E} = \rho, \tag{6.6}$$

$$\boldsymbol{D}_\perp \cdot \boldsymbol{B} = 0, \tag{6.7}$$

$$\boldsymbol{D}_\perp \times \boldsymbol{E} = -D_t \boldsymbol{B}, \tag{6.8}$$

$$\boldsymbol{D}_\perp \times \boldsymbol{B} = \boldsymbol{J} + D_t \boldsymbol{E}, \tag{6.9}$$

where ρ and $\boldsymbol{J}$ are the external charge and current densities, respectively. In summary, correct enumeration of operators, when accounting for redundancies associated with integration by parts and the equations of motion, amounts to removing: (1) total derivatives, (2) those of the form $\psi^\dagger i\overleftarrow{D}_t O\psi$ and $\psi^\dagger OiD_t\psi$, (3) those with $\boldsymbol{D}_\perp \cdot \boldsymbol{B}$, and (4) those with either $\boldsymbol{D}_\perp \times \boldsymbol{E}$ or $D_t\boldsymbol{B}$.

More symmetry is expected in a theory with a nonrelativistic fermion, such as reparameterization invariance [160, 112] or residual Lorentz symmetry [161]. Imposing this invariance would require establishing relationships between the coefficients of operators at different orders in $1/m$. For this work, however, we focus only on a rotationally- and translationally-invariant theory, since this can be readily encoded into a Hilbert series. In the particular examples of NRQED and NRQCD/HQET, invariance under parity and time reversal transformations are also expected, since the underlying theories are invariant under parity and time reversal, which we discuss in Sections 6.4 and 6.5.

6.3 Hilbert series for a nonrelativistic theory

The Hilbert series can be used to count the number of invariants under a group transformation, utilizing the plethystic exponential, defined as

$$PE_{\phi}^{\text{bosons}} \equiv \exp\left[\sum_{n=1}^{\infty} \frac{\phi^n}{n} \chi_R(z_1^n, z_2^n, ..., z_k^n)\right], \tag{6.10}$$

$$PE_{\psi}^{\text{fermions}} \equiv \exp\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\psi^n}{n} \chi_R(z_1^n, z_2^n, ..., z_k^n)\right]. \tag{6.11}$$

Here, χ_R is the character of the representation R of group G of rank k, ϕ and ψ are spurions (complex numbers taken to have modulus less than unity) corresponding to the field associated with the representation R, and the z_i's are complex numbers with unit modulus (called fugacities) that parameterize the maximal torus of G. The plethystic exponentials are defined so as to ensure, if Taylor expanded in ϕ or ψ, that the nth power of ϕ or ψ will have a coefficient equal to the character of symmetric (in the case of bosonic statistics) or antisymmetric (in the case of fermionic statistics) tensor products, constructed out of representation R, n times. The Hilbert series that counts the total number of group invariants is generated by performing the following integral (often called the Molien-Weyl formula):

$$HS = \oint [d\mu]_G \, PE_x, \tag{6.12}$$

where the contour integral is done over the maximal torus of the group G with respect to the Haar measure, $[d\mu]_G$, associated with the group G. The Hilbert series, as defined by Eq. (6.12), is a polynomial in the spurions such that the coefficient of different powers of the spurions counts the number of invariants under the group G.[3] For further details, we refer to [148, 149, 147, 113, 114].

Using the machinery of the Hilbert series, we can construct all possible operators O_k in

[3] The invariants are counted using the character orthogonality relation:

$$\oint [d\mu]_G \, \chi_R \, \chi_{R'} = \delta_{RR'}, \tag{6.13}$$

Eq. (6.1). The characters for E, B, ψ, $\psi^\dagger$, and s are (note that P_0, $P_\perp$, $\mathcal{D}_t$, and $\mathcal{D}_\perp$ are defined in Eqs. (6.21) and (6.22) and in the text thereafter):

$$\chi_E = P_0 P_\perp \chi_E^C \left(\chi_{\mathbf{3}}^{SO(3)} - \mathcal{D}_\perp \chi_{\mathbf{3}}^{SO(3)} + \mathcal{D}_\perp^2\right), \tag{6.16}$$

$$\chi_B = P_0 P_\perp \chi_B^C \left(\chi_{\mathbf{3}}^{SO(3)} - \mathcal{D}_\perp\right), \tag{6.17}$$

$$\chi_\psi = P_0 P_\perp \chi_\psi^C \chi_{\mathbf{2}}^{SU(2)} (1 - \mathcal{D}_t), \tag{6.18}$$

$$\chi_{\psi^\dagger} = P_0 P_\perp \chi_{\psi^\dagger}^C \chi_{\mathbf{2}}^{SU(2)} (1 - \mathcal{D}_t), \tag{6.19}$$

$$\chi_s = \chi_{\mathbf{3}}^{SO(3)} \chi_{\mathbf{3}}^{SU(2)}, \tag{6.20}$$

where $\chi_{\mathbf{3}}^{SO(3)}$, $\chi_{\mathbf{2}}^{SU(2)}$, and $\chi_{\mathbf{3}}^{SU(2)}$ are the characters for a **3** of $SO(3)$, a **2** of $SU(2)$, and a **3** of $SU(2)$, respectively. Explicit expressions for these characters can be found in Appendix E.1. The characters χ^C represent the way E, B, ψ, and $\psi^\dagger$ are charged under the external gauge field. For example, if the fermion has color, then χ_E^C and χ_B^C are both the characters for the adjoint representation of $SU(3)$, and χ_ψ^C ($\chi_{\psi^\dagger}^C$) is the character for the fundamental (antifundamental) representation of $SU(3)$. P_0 and $P_\perp$ generate all symmetric products of temporal and spatial derivatives, respectively:

$$P_0 \equiv \exp\left[\sum_{n=1}^{\infty} \frac{\mathcal{D}_t^n}{n}\right] = \frac{1}{1 - \mathcal{D}_t}, \tag{6.21}$$

$$P_\perp \equiv \exp\left[\sum_{n=1}^{\infty} \frac{\mathcal{D}_\perp^n}{n} \chi_{\mathbf{3}}^{SO(3)}(z^n)\right] = \frac{1}{(1 - z\mathcal{D}_\perp)(1 - \mathcal{D}_\perp)(1 - \mathcal{D}_\perp/z)}, \tag{6.22}$$

where $\mathcal{D}_t$ and $\mathcal{D}_\perp$ are the spurions that correspond to time and spatial derivatives in the operator,

---

where χ_R and $\chi_{R'}$ are characters of irreducible representations, R and R', of G. When R' is a trivial singlet representation, using $\chi_{\text{singlet}} = 1$, we have

$$\oint [d\mu]_G \, \chi_R = 1, \quad \text{iff } \chi_R = \chi_{\text{singlet}}. \tag{6.14}$$

Therefore, using the definition of the plethystic exponentials, the Hilbert series we use, which counts the invariants under group G is

$$HS = \oint [d\mu]_G \, PE. \tag{6.15}$$

respectively. The characters in Eqs. (6.16) - (6.19) take the form they do so as to remove terms that are zero according to the equations of motion for the external gauge field, where we choose to construct operators with $\partial \boldsymbol{B}/\partial t$, in lieu of $\nabla \times \boldsymbol{E}$, according to Eq. (6.8). We note that, we have to add back $\mathcal{D}_\perp^2$ in Eq. (6.16), to enforce the constraint that $\nabla \cdot (\nabla \times \boldsymbol{E}) = 0$. Without it, Hilbert series will erroneously subtract off $\nabla \cdot (\nabla \times \boldsymbol{E})$, which was not there to begin with before the subtraction.

The general Hilbert series for a theory with a heavy fermion is

$$HS = \oint [d\mu]_{SO(3)} \oint [d\mu]_{SU(2)} \oint [d\mu]_C \, \frac{1}{P_0 P_\perp} \, PE_E \, PE_B \, PE_\psi \, PE_{\psi^\dagger} \, PE_s. \tag{6.23}$$

The bosonic plethystic exponential, i.e., Eq. (6.10), is used for E, B, and s, while the fermionic one, i.e., Eq. (6.11), is used for ψ and $\psi^\dagger$. The expressions for the Haar measures can be found in Appendix E.1. The factor of $1/P_0 P_\perp$ removes operators that are total time derivatives and total spatial derivatives.[4] This method, however, will over-subtract operators that are total derivatives, but which have already been subtracted by the equations of motion. Thus, this Hilbert series will, in general, produce some terms with negative signs, all of which are redundant operators, and can be ignored. One can expand the plethystic exponentials for ψ and $\psi^\dagger$ to first order, and perform the $SU(2)$ integral by hand, which results in the Hilbert series for the operators O_k in Eq. (6.1):

$$HS = \oint [d\mu]_{SO(3)} \oint [d\mu]_C \, \frac{P_\perp}{P_0} (1 + s\chi_{\mathbf{3}}^{SO(3)}) \, \chi^C_{\psi^\dagger} \chi^C_\psi \, PE_E \, PE_B. \tag{6.24}$$

Explicit expressions for the Hilbert series NRQED and NRQCD/HQET will be given in Sections 6.4 and 6.5, respectively, including discussions on how to impose invariance under parity and time reversal.

[4]One can justify introducing the factor of $1/P_0 P_\perp$, by noting that one can always choose a basis of operators where no time or spatial derivatives act on $\psi^\dagger$, using integration by parts. This procedure should remove the $P_0 P_\perp$ in the definition of the character for $\psi^\dagger$. The factor $1/P_0 P_\perp$ can also be justified using differential forms, as discussed in Ref. [114].

6.4 NRQED

In NRQED, the relevant gauge symmetry group is $U(1)$. Here, $\chi_E^C = \chi_B^C = 1$, since photons do not have any $U(1)$ charge, and $\chi_\psi^C = \chi_\psi^{U(1)} = 1/\chi_{\psi^\dagger}^{U(1)}$. Because of this, we have

$$\oint [d\mu]_{U(1)}\, \chi_{\psi^\dagger}^{U(1)} \chi_\psi^{U(1)} = \oint [d\mu]_{U(1)} = 1 \tag{6.25}$$

The Hilbert series for O_k in Eq. (6.1) in NRQED is

$$HS \;=\; \oint [d\mu]_{SO(3)}\, \frac{P_\perp}{P_0}(1 + s\chi_{\mathbf{3}}^{SO(3)})\; PE_E\; PE_B. \tag{6.26}$$

Again, we ignore any negative terms generated by this Hilbert series, since they are both total derivatives and related to other operators by the equations of motion, as discussed in Section 6.3. Since parity is a symmetry of QED, one can demand that O_k respects parity by requiring that it is composed of any number of parity-even objects, i.e., $\mathcal{D}_t$, B, and s, and an even number of parity-odd objects, i.e., $\mathcal{D}_\perp$ and E. This can be automated without explicitly constructing the operators O_k by hand.

The output for this Hilbert series for dimensions 5, 6, 7, and 8, before imposing invariance under time reversal, is

$$HS_{d=5} \;=\; \mathcal{D}_\perp^2 + sB, \tag{6.27}$$

$$HS_{d=6} \;=\; 2E\mathcal{D}_\perp + sE\mathcal{D}_\perp, \tag{6.28}$$

$$HS_{d=7} \;=\; \mathcal{D}_\perp^4 + E^2 + B^2 + B\mathcal{D}_\perp^2 + 5sB\mathcal{D}_\perp^2, \tag{6.29}$$

$$HS_{d=8} \;=\; sB^2\mathcal{D}_t + sE^2\mathcal{D}_t + 2EB\mathcal{D}_\perp + 3sE\mathcal{D}_\perp^3 + 5E\mathcal{D}_\perp^3 + 7sEB\mathcal{D}_\perp. \tag{6.30}$$

While the Hilbert series can count the number of operators that are invariant under the given symmetries, it does not say how the indices within each operator are contracted. In general, this needs to be done by hand. To do this, we choose to organize operators according to what objects the derivatives are acting on. E and B have no electric charge, so derivatives acting on

E and B are only partial derivatives. As such, objects of the form $[\partial_t ... \partial_\perp E]$ and $[\partial_t ... \partial_\perp B]$ are Hermitian, where the square brackets indicate that the derivatives only act on E or B. So as not to introduce terms like $\nabla \cdot \mathbf{B}$ and $\nabla \times \mathbf{E}$, we require that the $SO(3)$ index of B cannot be be symmetric with any index of $\partial_\perp$ acting on it, and the index on E must be symmetric with the index of any $\partial_\perp$ acting on it. Because ψ does have electromagnetic charge, the derivatives acting on it are covariant derivatives. Only spatial derivatives can act on ψ, due to the equations of motion, and we use anticommutator brackets $\{A,B\} \equiv AB + BA$ to construct fully Hermitian operators O. One can impose invariance under time reversal by hand, as shown in Table 6.2 for d = 5, 6, 7, and 8. T-even operators are those with any number of T-even objects, i.e., E and $\partial_\perp$, and an even numbers of T-odd objects, i.e., ∂_t, $iD_\perp$, B, and s.

In the special case of NRQED, where the group is abelian, there is a method to impose T invariance that is easily automated. This is done by modifying the Hilbert series to distinguish those spatial derivatives $\partial_\perp$ acting only on E and B from the spatial derivatives $iD_\perp$ that act on ψ. Here, the former ones are always T-even, while the latter are always T-odd. This results in:

$$HS_{d=5} = \mathcal{D}_\perp^2 + sB, \tag{6.31}$$

$$HS_{d=6} = E\mathcal{D}_\perp + sE\mathcal{D}_\perp, \tag{6.32}$$

$$HS_{d=7} = \mathcal{D}_\perp^4 + E^2 + B^2 + B\mathcal{D}_\perp^2 + 3sB\mathcal{D}_\perp^2, \tag{6.33}$$

$$HS_{d=8} = sB^2\mathcal{D}_t + sE^2\mathcal{D}_t + EB\mathcal{D}_\perp + 2sE\mathcal{D}_\perp^3 + 3E\mathcal{D}_\perp^3 + 4sEB\mathcal{D}_\perp. \tag{6.34}$$

This method agrees with the result when explicitly constructing operators and selecting by hand only those that are T-even, and it agrees with the lists of operators up to and including $d = 8$ in Refs. [152, 153, 112, 116]. It is straight forward, using the Hilbert series as a guide, to explicitly list operators for $d > 8$. We show in Fig. 6.1 the total number of operators in NRQED up to $d = 18$, when T invariance is imposed, and list the total number of operators in Table 6.1.

Table 6.1: The total number of effective operators in NRQED with mass dimension d up to and including $d = 18$, which are invariant under parity and time reversal transformations.

Mass dimension (d)	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Number of operators	2	2	7	12	37	73	218	474	1303	3077	7896	19359	48023	117625

Table 6.2: The output of the NRQED Hilbert series for mass dimensions $d = 5, 6, 7$, and 8. We list the possible Hermitian combinations of these operators, distinguishing between those that are even and odd under time reversal, where $i, j, k, l, m, \ldots$ signify $SO(3)$ indices. Note that these Hermitian operators O are those in the bilinear $\psi^\dagger O \psi$, and the square brackets indicate that the derivative acts only on the object in the square bracket. Also, in the special case of NRQED, time-reversal symmetry can be imposed in an automated way, without constructing Hermitian operators by hand; see the text at the end of Section 6.4 for details.

Order	HS	T even	T odd
$\frac{1}{m}$	$\mathcal{D}_\perp^2$	$(iD_\perp)^2$	
	sB	$s^i B^j \delta_{ij}$	
$\frac{1}{m^2}$	$2E\mathcal{D}_\perp$	$[\partial^i E^j]\delta_{ij}$	$\{E^i, iD_\perp^j\}\delta_{ij}$
	$sE\mathcal{D}_\perp$	$s^i\{E^j, iD_\perp^k\}\varepsilon_{ijk}$	
$\frac{1}{m^3}$	$\mathcal{D}_\perp^4$	$(iD_\perp)^4$	
	E^2	E^2	
	B^2	B^2	
	$B\mathcal{D}_\perp^2$	$\{[\partial^i B^j], iD_\perp^k\}\varepsilon_{ijk}$	
	$5sB\mathcal{D}_\perp^2$	$\{s^i B^j, (iD_\perp^k)^2\}\delta_{ij}$ $\{s^i B^j, iD_\perp^k iD_\perp^l\}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $s^i[\partial^2 B^j]\delta_{ij}$	$\{s^i[\partial^j B^k], iD_\perp^l\}\delta_{ij}\delta_{kl}$ $\{s^i[\partial^j B^k], iD_\perp^l\}\delta_{ik}\delta_{jl}$
$\frac{1}{m^4}$	$sB^2\mathcal{D}_t$	$s^i B^j[\partial_t B^k]\varepsilon_{ijk}$	
	$sE^2\mathcal{D}_t$	$s^i E^j[\partial_t E^k]\varepsilon_{ijk}$	
	$2EB\mathcal{D}_\perp$	$\{E^i B^j, iD_\perp^k\}\varepsilon_{ijk}$	$E^i[\partial^j B^k]\varepsilon_{ijk}$
	$3sE\mathcal{D}_\perp^3$	$\{s^i E^j, iD_\perp^k (iD_\perp)^2\}\varepsilon_{ijk}$ $\{s^i[\partial^j\partial^k E^l], iD_\perp^m\}(\varepsilon_{ijm}\delta_{kl} + \varepsilon_{ikm}\delta_{jl} + \varepsilon_{ilm}\delta_{jk})$	$\{s^i[\partial^j E^k], iD_\perp^l iD_\perp^m\}(\varepsilon_{ijl}\delta_{km} + \varepsilon_{ijm}\delta_{kl} + \varepsilon_{ikl}\delta_{jm} + \varepsilon_{ikm}\delta_{jl})$
	$5E\mathcal{D}_\perp^3$	$\{[\partial^i E^j], (iD_\perp)^2\}\delta_{ij}$ $\{[\partial^i E^j], iD_\perp^k iD_\perp^l\}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $[\partial^i\partial^j\partial^k E^l](\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$	$\{E^i, iD_\perp^j iD_\perp^k iD_\perp^l\}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $\{[\partial^i\partial^j E^k], iD_\perp^l\}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
	$7sEB\mathcal{D}_\perp$	$s^i E^j[\partial^k B^l]\delta_{ik}\delta_{jl}$ $s^i E^j[\partial^k B^l]\delta_{il}\delta_{jk}$ $s^i B^j[\partial^k E^l](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $s^i B^j[\partial^k E^l]\delta_{ij}\delta_{kl}$	$\{s^i E^j B^k, iD_\perp^l\}\delta_{ij}\delta_{kl}$ $\{s^i E^j B^k, iD_\perp^l\}\delta_{ik}\delta_{jl}$ $\{s^i E^j B^k, iD_\perp^l\}\delta_{il}\delta_{jk}$

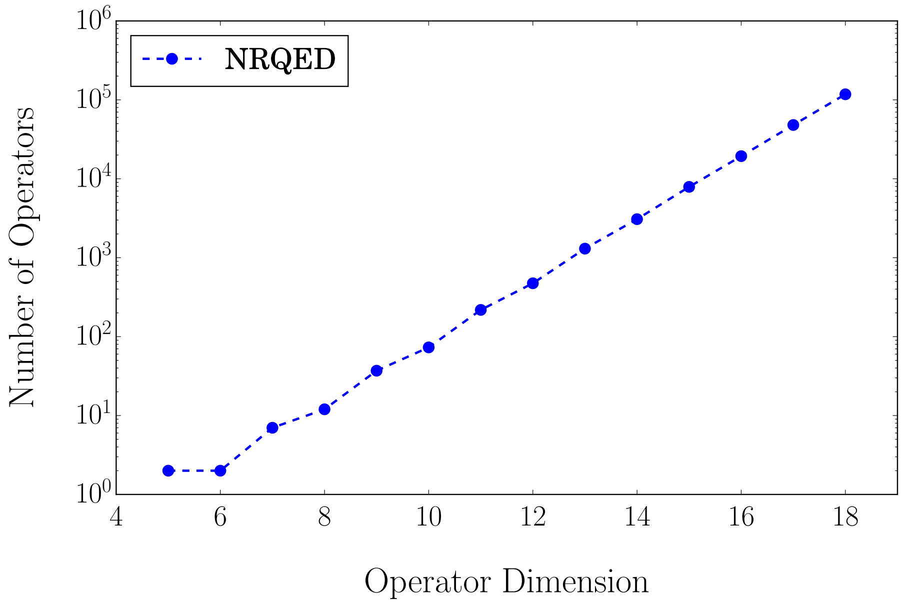

Figure 6.1: The total number of rotationally-invariant operators in NRQED, which are even under parity and time reversal, as a function of the operator dimension d. Explicit form of the operators for $d = 5, 6, 7, 8$ can be found in Table 6.2.

6.5 NRQCD/HQET

The construction of the Hilbert series for NRQCD/HQET is very similar to that of NRQED, where now $\chi_E^C = \chi_B^C = \chi_{\mathbf{8}}^{SU(3)}$, $\chi_\psi^C = \chi_{\mathbf{3}}^{SU(3)}$, and $\chi_{\psi^\dagger}^C = \chi_{\bar{\mathbf{3}}}^{SU(3)}$. The Hilbert series for the operators O_k in Eq. (6.1) is

$$HS = \oint [d\mu]_{SO(3)} \oint [d\mu]_{SU(3)} \frac{P_\perp}{P_0}(1 + s\chi_{\mathbf{3}}^{SO(3)})\, \chi_{\mathbf{3}}^{SU(3)}\, \chi_{\bar{\mathbf{3}}}^{SU(3)}\, PE_E\, PE_B. \qquad (6.35)$$

When invariance under parity is imposed, the output from this Hilbert series for operators of mass dimension d = 5, 6, 7, and 8 is:

$$HS_{d=5} = \mathcal{D}_\perp^2 + sB, \qquad (6.36)$$

$$HS_{d=6} = 2E\mathcal{D}_\perp + sE\mathcal{D}_\perp, \qquad (6.37)$$

$$HS_{d=7} = \mathcal{D}_\perp^4 + 2E^2 + 2B^2 + sE^2 + sB^2 + B\mathcal{D}_\perp^2 + 5sB\mathcal{D}_\perp^2, \qquad (6.38)$$

$$HS_{d=8} = B^2\mathcal{D}_t + E^2\mathcal{D}_t + 2sB^2\mathcal{D}_t + 2sE^2\mathcal{D}_t + 6EB\mathcal{D}_\perp + 3sE\mathcal{D}_\perp^3 + 5E\mathcal{D}_\perp^3 + 21sEB\mathcal{D}_\perp \qquad (6.39)$$

Unlike NRQED, we have not found an automated way to implement invariance under time reversal in NRQCD/HQET, because T acts as an anti-unitary operator, and counting T-invariant operators requires keeping track of factors i while constructing Hermitian operators. This is not an issue when constructing invariants in NRQED, since it has an abelian $U(1)$ symmetry, but when the group is non-abelian, like $SU(3)$, the algebra's structure constants, e.g., f_{abc}, bring with them a factor of i, and imposing T-symmetry is no longer straight-forward.

We take the output from this Hilbert series and explicitly contract indices by hand, separating those that are even and odd under time reversal. The prescription is very close to the one we used in NRQED. We choose to suppress color indices, and express $E = E_a T^a$ and $B = B_a T^a$, where T^a are the eight generators of $SU(3)$, which satisfy:

$$[T^a, T^b] = i f^{abc} T_c, \tag{6.40}$$

$$\{T^a, T^b\} = \frac{1}{3}\delta^{ab} + d^{abc} T_c. \tag{6.41}$$

We utilize the following notation, where the letters $i, j, k, l, m, \ldots$ are used for $SO(3)$ indices, and the letters $a, b, c, \ldots$ are used to signify the $SU(3)$ generators:

$$\psi^\dagger [D_\perp^i E^j]_a \delta_{ij} T^a \psi \equiv \psi^\dagger \left([\partial_\perp^i E_a^j] + g (A_\perp)^{ib} E^{jc} f_{abc} \right) \delta_{ij} T^a \psi, \tag{6.42}$$

where $A^\mu \equiv A_a^\mu T^a$ is the gauge field. When there are two $SU(3)$ generators in an operator, one can use the relation that results in adding Eqs. (6.40) and (6.41) together:

$$T^a T^b = \frac{1}{6}\delta^{ab} + \frac{1}{2}\left(d^{abc} T_c + i f^{abc} T_c \right). \tag{6.43}$$

From this, one can see that, for example, the operator E^2 in the Hilbert series can be contracted in two ways:

$$E_a^i E_b^j \delta_{ij} T^a T^b \;\rightarrow\; E_a^i E_b^j \delta_{ij} \delta^{ab} \;\text{ and }\; E_a^i E_b^j \delta_{ij} d^{abc} T_c. \tag{6.44}$$

A third contraction with f_{abc} is completely antisymmetric in a, b, c, which results in an operator equal to zero, in this case. Finally, it should be noted that f^{abc} should be thought of a odd under time reversal.[5] The two contractions in Eq. (6.44) give rise to different matrix elements, since the contraction of color indices would be different. A complete list of NRQCD/HQET operators can be found in Table 6.3 for $d \leq 8$. Extending the list to higher orders would be straight-forward.

Our results agree with those in Ref. [112] for NRQCD/HQET operators up to and including operators of order $1/m^3$. However, we find a different number compared to Refs. [156, 118] for operators at order $1/m^3$ and $1/m^4$. Specifically, Refs. [156, 118] claim there are 9 operators at $1/m^3$, and 18 operators at $1/m^4$, while we find 11 and 25, respectively. These discrepancies are consistent with the possibility that Refs. [156, 118] count only once the two symmetric terms, i.e., contractions with δ^{ab} and d^{abc}, in Eq. (6.43).

6.6 Discussion and Conclusions

We construct a Hilbert series for an effective theory with a single non-relativistic fermion in an external, and dynamical, gauge field, defining characters and using a method to subtract operators that are related to others via the equations of motion associated with the heavy fermion and the external gauge bosons, as well as integration by parts. We consider the specific examples of NRQED and NRQCD/HQET, where the heavy fermion has electric charge or color, respectively. Imposing invariance under parity can be easily automated. Invariance under time reversal also can be automated in the case of NRQED, since it is an abelian theory, but not for NRQCD/HQET, in which case we separate T-even and -odd operators by hand. For both effective theories, we construct explicit contractions for effective operators at dimensions $d \leq 8$ that are invariant under parity and time reversal transformations, as enumerated in Table 6.2 for NRQED and Table 6.3 for NRQCD/HQET. In a theory with a nonrelativistic fermion, additional symmetry, e.g., reparameterization invariance [160, 112] or residual Lorentz symmetry [161], is expected, in general. However, we do not impose such additional constraints, since it remains an

[5] This can be heuristically understood by noting that color is an internally symmetry, and must therefore be invariant under spacetime transformations. Therefore, the matrix multiplication between two $SU(3)$ generators must be even under time reversal, which requires $T^{-1} i f^{abc} T = i f^{abc}$, and f^{abc} can be therefore be thought of as T-odd.

Table 6.3: Same as Table 6.2, but for NRQCD/HQET, separating those operators that are even and odd under time reversal. See the text at the end of Section 6.5 for a discussion regarding notation.

Order	HS	T even	T odd
$\frac{1}{m}$	$\mathcal{D}_\perp^2$	$(iD_\perp)^2$	
	sB	$s^i B_a^j \delta_{ij} T^a$	
$\frac{1}{m^2}$	$2E\mathcal{D}_\perp$	$[\Delta^i E^j]_a \delta_{ij} T^a$	$\{E_a^i, iD_\perp^j\}\delta_{ij} T^a$
	$sE\mathcal{D}_\perp$	$s^i\{E_a^j, iD_\perp^k\}\varepsilon_{ijk} T^a$	
$\frac{1}{m^3}$	$\mathcal{D}_\perp^4$	$(iD_\perp)^4$	
	$2E^2$	$E_a^i E_b^j \delta_{ij} d^{abc} T_c$ $E_a^i E_b^j \delta_{ij} \delta^{ab}$	
	$2B^2$	$B_a^i B_b^j \delta_{ij} d^{abc} T_c$ $B_a^i B_b^j \delta_{ij} \delta^{ab}$	
	sE^2	$s^i E_a^j E_b^k \varepsilon_{ijk} f^{abc} T_c$	
	sB^2	$s^i B_a^j B_b^k \varepsilon_{ijk} f^{abc} T_c$	
	$B\mathcal{D}_\perp^2$	$\{[\Delta^i B^j]_a, iD_\perp^k\}\varepsilon_{ijk} T^a$	
	$5sB\mathcal{D}_\perp^2$	$\{s^i B_a^i, (iD_\perp^k)^2\}\delta_{ij} T^a$ $\{s^i B_a^j, iD_\perp^k iD_\perp^l\}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) T^a$ $s^i[\Delta^2 B^j]_a \delta_{ij} T^a$	$\{s^i[\Delta^j B^k]_a, iD_\perp^l\}\delta_{ij}\delta_{kl} T^a$ $\{s^i[\Delta^j B^k]_a, iD_\perp^l\}\delta_{ik}\delta_{jl} T^a$
$\frac{1}{m^4}$	$B^2\mathcal{D}_t$	$B_a^i [D_t B^j]_b \delta_{ij} f^{abc} T_c$	
	$E^2\mathcal{D}_t$	$E_a^i [D_t E^j]_b \delta_{ij} f^{abc} T_c$	
	$2sB^2\mathcal{D}_t$	$s^i B_a^j [D_t B^k]_b \varepsilon_{ijk} \delta^{ab}$ $s^i B_a^j [D_t B^k]_b \varepsilon_{ijk} d^{abc} T_c$	
	$2sE^2\mathcal{D}_t$	$s^i E_a^j [D_t E^k]_b \varepsilon_{ijk} \delta^{ab}$ $s^i E_a^j [D_t E^k]_b \varepsilon_{ijk} d^{abc} T_c$	
	$6EB\mathcal{D}_\perp$	$\{E_a^i B_b^j, iD_\perp^k\}\varepsilon_{ijk} \delta^{ab}$ $\{E_a^i B_b^j, iD_\perp^k\}\varepsilon_{ijk} d^{abc} T_c$ $E_a^i [\Delta^j B^k]_b \varepsilon_{ijk} f^{abc} T_c$	$\{E_a^i B_b^j, iD_\perp^k\}\varepsilon_{ijk} f^{abc} T_c$ $E_a^i [\Delta^j B_b^k]_b \varepsilon_{ijk} \delta^{ab}$ $E_a^i [\Delta^j B^k]_b \varepsilon_{ijk} d^{abc} T_c$
	$3sE\mathcal{D}_\perp^3$	$\{s^i E_a^j, iD_\perp^k (iD_\perp)^2\}\varepsilon_{ijk} T^a$ $\{s^i[\Delta^j \Delta^k E^l]_a, iD_\perp^m\} T^a(\varepsilon_{ijm}\delta_{kl} + \varepsilon_{ikm}\delta_{jl} + \varepsilon_{ilm}\delta_{jk})$	$\{s^i[\Delta^j E^k]_a, iD_\perp^l iD_\perp^m\} T^a(\varepsilon_{ijl}\delta_{km} + \varepsilon_{ijm}\delta_{kl} + \varepsilon_{ikl}\delta_{jm} + \varepsilon_{ikm}\delta_{jl})$
	$5E\mathcal{D}_\perp^3$	$\{[\Delta^i E^j]_a, (iD_\perp)^2\}\delta_{ij} T^a$ $\{[\Delta^i E^j]_a, iD_\perp^k iD_\perp^l\} T^a(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $[\Delta^i \Delta^j \Delta^k E^l]_a T^a(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$	$\{E_a^i, iD_\perp^j iD_\perp^k iD_\perp^l\} T^a(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $\{[\Delta^i \Delta^j E^k]_a, iD_\perp^l\} T^a(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
	$21sEB\mathcal{D}_\perp$	$\{s^i E_a^j B_b^k, iD_\perp^l\}\delta_{ij}\delta_{kl} f^{abc} T_c$ $\{s^i E_a^j B_b^k, iD_\perp^l\}\delta_{ik}\delta_{jl} f^{abc} T_c$ $\{s^i E_a^j B_b^k, iD_\perp^l\}\delta_{il}\delta_{jk} f^{abc} T_c$ $s^i E_a^j [\Delta^k B^l]_b \delta_{ik}\delta_{jl} d^{abc} T_c$ $s^i E_a^j [\Delta^k B^l]_b \delta_{il}\delta_{jk} d^{abc} T_c$ $s^i E_a^j [\Delta^k B^l]_b \delta_{ik}\delta_{jl} \delta^{ab}$ $s^i E_a^j [\Delta^k B^l]_b \delta_{il}\delta_{jk} \delta^{ab}$ $s^i B_a^j [\Delta^k E^l]_b d^{abc} T_c(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $s^i B_a^j [\Delta^k E^l]_b \delta_{ij}\delta_{kl} d^{abc} T_c$ $s^i B_a^j [\Delta^k E^l]_b \delta^{ab}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $s^i B_a^j [\Delta^k E^l]_b \delta^{ab}\delta_{ij}\delta_{kl}$	$\{s^i E_a^j B_b^k, iD_\perp^l\}\delta_{ij}\delta_{kl} d^{abc} T_c$ $\{s^i E_a^j B_b^k, iD_\perp^l\}\delta_{ik}\delta_{jl} d^{abc} T_c$ $\{s^i E_a^j B_b^k, iD_\perp^l\}\delta_{il}\delta_{jk} d^{abc} T_c$ $\{s^i E_a^j B_b^k, iD_\perp^l\}\delta_{ij}\delta_{kl} \delta^{ab}$ $\{s^i E_a^j B_b^k, iD_\perp^l\}\delta_{ik}\delta_{jl} \delta^{ab}$ $\{s^i E_a^j B_b^k, iD_\perp^l\}\delta_{il}\delta_{jk} \delta^{ab}$ $s^i E_a^j [\Delta^k B^l]_b \delta_{ik}\delta_{jl} f^{abc} T_c$ $s^i E_a^j [\Delta^k B^l]_b \delta_{il}\delta_{jk} f^{abc} T_c$ $s^i B_a^j [\Delta^k E^l]_b f^{abc} T_c(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $s^i B_a^j [\Delta^k E^l]_b \delta_{ij}\delta_{kl} f^{abc} T_c$

open question regarding how to encode such requirements with Hilbert-series methods.

Our results agree with those presented in Refs. [152, 153, 112, 116] for NRQED, which discuss operators up to and including $d = 8$. The total number of operators in NRQED grows exponentially, as shown Fig. 6.1 and listed in Table 6.1 for mass dimension $d \leq 18$. When using a Hilbert series for NRQCD/HQET, we count a total of 2 operators each at orders $1/m$ and $1/m^2$, which agrees with Ref. [155], 11 operators at $1/m^3$, which agrees with Ref. [112], and 25 operators at $1/m^4$. However, at order $1/m^3$, other analyses claim that there are either 5 [159], or 9 [156, 118] total operators, and Refs. [156, 118] claim there are a total of 18 operators at $1/m^4$. The differences between our results and those found in Refs. [156, 118] can be explained by the existence of two symmetric $SU(3)$ color singlets for operators with two gauge bosons, as discussed at the end of Section 6.5. It is possible that analyses using the results in Refs. [159, 156, 118], may need to be reevaluated, e.g., Refs. [157, 162, 158].

The authors of Refs. [113, 114] discuss a connection between enumerating operators in a relativistic effective theory and the representations of the relativistic conformal group. Here, selecting only primary operators constructed out of tensor products of the conformal group's short representations correctly accounts for redundancies between operators via integration by parts and the equations of motion. We strongly suspect that our results can be reformulated in terms of the non-relativistic conformal group [61, 81, 59, 22, 107], and we take this up as future work.

Note: While this article was in review for publication, the authors of Ref. [118] updated their work, and their results now agree with our enumeration of NRQCD/HQET effective operators for $d \leq 8$.

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and author of this paper.

Chapter 7

Conformal Structure of the Heavy Particle EFT Operator Basis

An operator basis of an effective theory with a heavy particle, subject to external gauge fields, is spanned by a particular kind of neutral scalar primary of the non-relativistic conformal group. We calculate the characters that can be used for generating the operators in a non-relativistic effective field theory, which accounts for redundancies from the equations of motion and integration by parts.

7.1 Introduction

If one can say that a particle, and not its antiparticle, exists in the laboratory, then the length scale of its spatial wave function Δx is parametrically larger than its Compton wavelength $1/M$. This hierarchy of scales leads to heavy particle effective field theories (heavy particle EFTs), where one can systematically include higher powers of $1/(\Delta x\, M)$. Such systems are, in fact, fairly common. For example, the b quark can be located anywhere within a B meson, which has a spatial size of $\sim 1/\Lambda_{\text{QCD}}$, and heavy quark effective field theory is an expansion in powers of $\Lambda_{\text{QCD}}/m_b \sim 0.3$. A more dramatic example is the electron in a hydrogen atom, whose wave function has a size $\Delta x \sim 10^{-10}$ m, and $1/(\Delta x\, m_e) \sim 10^{-3}$, which is why the Schrödinger equation for single-partial quantum mechanics works so well in describing this system, using only the first-order expansion in $1/M$. Sometimes these theories are called non-relativistic effective field theories, insofar as there is a inertial frame in which there are non-relativistic particles.

Even though heavy particle EFTs describe common physical scenarios, enumerating the independent operators that appear in the Lagrangian, i.e., defining the operator basis, takes considerable effort. The reason for this is that defining the operator basis is more than just requiring that the operators preserve certain symmetries - it also involves accounting for non-trivial redundancies between operators from the classical equations of motion and integration by parts [163, 164]. There are popular EFTs with Lagrangians containing heavy fields, e.g., NRQED (external abelian gauge fields), and HQET and NRQCD (external color gauge fields). The operator basis for NRQED was written to order $O(1/M^3)$ in Ref. [153] and to $O(1/M^4)$ in Ref. [116]. The HQET/NRQCD operator basis was enumerated up to $O(1/M^3)$ in Ref. [112], and to $O(1/M^4)$ by Ref. [117], which was later confirmed in Ref. [118].

A huge stride was taken recently by the authors of Refs. [113, 114, 115], where they noticed that the operator basis for a relativistic EFT can be organized according to the representations of the conformal group. In particular, accounting for the redundancies from the classical equations of motion can be mapped to the null conditions that saturate unitarity in the conformal group, and choosing the operator basis to be spanned only by primaries of the conformal algebra removes any redundancies associated with integration by parts. By embedding operators into representations of the conformal group, one can use characters as inputs into a Hilbert series, which then can generate the operator basis, counting the number of operators in the EFT with the given field content. Constructing the explicit operators with contracted internal and Lorentz indices, however, needs to be done by hand. Even so, the Hilbert series output provides an invaluable tool for constructing a bonafide operator basis. For example, a Hilbert series aided in constructing the first correct operator basis for dimension-7 operators in the standard model EFT [114] and dimension-8 operators in the HQET/NRQCD Lagrangian [117].

In practice, the characters of particular group representations are used as inputs for the

Hilbert series, which generates all possible tensor products of these representations. We refer the reader to Ref. [150] for an introductory and pedagogical discussion regarding Hilbert series and the underlying group theory. In Ref. [117], we constructed such a Hilbert series to help write down the operator basis for NRQED and HQET/NRQCD to order $O(1/M^4)$. However, we did not use any organizational principle associated with the representation of the conformal group, since the equations of motion for non-relativistic fields are not those for relativistic fields. Instead, the "characters" we used in the Hilbert series were constructed by hand. This begs the question: *Are they characters of a group representation?* Perhaps unsurprisingly, the answer is: "*Yes*." In this work, we show that the characters used in Ref. [117] are those associated with "shortened" representations of the non-relativistic conformal group, and the operator basis for heavy particle effective field theories are spanned by a special category of primary operators. While this does have some analogy to the relativistic scenario studied in Refs. [113, 114, 115], there are important subtleties with non-relativistic theories, which we discuss in some detail.

7.2 Operator Basis for Heavy Particle EFTs

We consider operators that comprise an effective field theory that are singlets under the symmetries of the theory, each constructed out of the relevant degrees of freedom and any number of derivatives acting on them. Of such operators, two or more may give rise to identical S-matrix elements, in which case they ought to not be counted as distinct. This occurs when two or more operators: (1) differ by a total derivative, or (2) can be related via the classical equations of motion (a kind of field redefinition) [163, 164]. Accounting for these redundancies amounts to the program of constructing operator bases in effective field theories.

How these redundancies apply to heavy-particle EFTs is described in Ref. [117], and we will briefly recapitulate it here. Consider every possible rotationally- and gauge-invariant operator in the rest frame[1] of a theory with only one heavy particle ψ:

$$\mathcal{L} = \psi^\dagger iD_t\psi + \frac{1}{M^{d-1}}\sum_k^\infty c_k\psi^\dagger O_k^{[d]}\psi\,, \tag{7.1}$$

where the c_k's are dimensionless constants, M is the mass of the heavy particle, and the operators $O_k^{[d]}$ are Hermitian operators of mass dimension $d \geq 2$, constructed out of field strength tensors of gauge fields, covariant time derivatives, D_t, covariant spatial derivatives, $\mathbf{D}_\perp$ (and these derivatives can act to the right as well at the left), and spin vectors. All covariant derivatives must be symmetric under exchange of spatial indices, otherwise they are proportional to field strength tensors, which have already been included.

Consider a set of operators that contain only one derivative. For a given operator O, the relationship between operators by integrating by parts is

$$\psi^\dagger O\partial\psi + [\partial\psi^\dagger]O\psi + \psi^\dagger[\partial O]\psi \;\;=\;\; \partial(\psi^\dagger O\psi) = \text{total derivative}\,, \tag{7.2}$$

where ∂ is a partial time or spatial derivative, and the square brackets indicate that the derivative only acts on the operator within the brackets. Combined with the identity

$$\psi^\dagger OD\psi + [D\psi^\dagger]O\psi + \psi^\dagger[DO]\psi = D(\psi^\dagger O\psi)\,, \tag{7.3}$$

where D is a covariant time or spatial derivative, and with the fact that $D(\psi^\dagger O\psi) = \partial(\psi^\dagger O\psi)$ since $\psi^\dagger O\psi$ was defined to be a gauge singlet, we have the constraint:

$$\psi^\dagger OD\psi + [D\psi^\dagger]O\psi + \psi^\dagger[DO]\psi = \text{total derivative}\,. \tag{7.4}$$

This equation relates three operators to a total derivative, so to account for this redundancy, we need to ignore one of them. One easy option is to only ignore operators O that contains derivative that act on $\psi^\dagger$. But this solution cannot be generalized to the case with more derivatives in the operator, and accounting for the redundancies associated from integration by parts between

[1]In general, a covariant derivative D_μ can be written in terms of the velocity 4-vector v^μ: $D_\mu = (v\cdot D)v^\mu + D_\perp^\mu$. In the rest frame, i.e., where $v^\mu = (1,0,0,0)$, then $D_\mu = (D_t, \mathbf{D}_\perp)$.

operators with more derivatives becomes more challenging.

The equations of motion for ψ also give relationship between operators:

$$iD_t\psi + \frac{1}{M^{d-1}}\sum_k^{\infty} c_k O_k^{[d]}\psi = 0\,, \tag{7.5}$$

and multiplying from the left by $\psi^\dagger O_j^{[d']}$:

$$\psi^\dagger O_j^{[d']} iD_t\psi = -\frac{1}{M^{d-1}}\sum_k^{\infty} c_k \psi^\dagger O_j^{[d']} O_k^{[d]}\psi\,. \tag{7.6}$$

Any operator O that contains a covariant time derivative that acts on ψ can be related to an infinite set of other operators at higher order. Therefore, this single equation that relates operators can be imposed if one ignores any operator that contains a covariant time derivative that acts on ψ. The same argument follows for $\psi^\dagger$.

Lastly, there are relationships between operators due to the equations of motion of the field strength tensors associated with *external* gauge fields, i.e., $D_\mu F^{\mu\nu} = j^\nu$ and $D_\mu \widetilde{F}^{\mu\nu} = 0$, where $j^\mu = (\rho, \mathbf{J})$. Because the effective theory defined in Eq. (7.1) is restricted to only the sector with one matter degree of freedom, there exists the possibility that whatever gauge fields appear in Lagrangian may have equations of motion that include external sources. Representing the covariant derivative as $D_\mu = (D_t, \mathbf{D}_\perp)$ in the rest frame of the heavy particle, we have the non-abelian generalizations of Maxwell's equations:

$$\mathbf{D}_\perp \cdot \mathbf{E} = \rho\,, \tag{7.7}$$

$$\mathbf{D}_\perp \cdot \mathbf{B} = 0\,, \tag{7.8}$$

$$\mathbf{D}_\perp \times \mathbf{E} = -D_t\mathbf{B}\,, \tag{7.9}$$

$$\mathbf{D}_\perp \times \mathbf{B} = \mathbf{J} + D_t\mathbf{E}\,. \tag{7.10}$$

So, if the operator O in Eq. (7.1) is constructed out of $\mathbf{E}$ and $\mathbf{B}$ (and covariant derivatives acting on them), then Maxwell's equations will make some operators vanish, as well as provide relationships between different operators. Accounting for these constraints, one must impose $\mathbf{D}_\perp \cdot \mathbf{B} = 0$, and choose to either express the operator O in terms of $\mathbf{D}_\perp \times \mathbf{E}$ or $D_t\mathbf{B}$, but not both.

In summary, imposing the constraints on operators from integration by parts and equations of motion of ψ, $\mathbf{E}$, and $\mathbf{B}$ on the rotationally- and gauge-invariant Lagrangian density in Eq. (7.1) will provide the operator basis for HQET. That is, every operator gives rise to different S-matrix elements. There are additional symmetry constraints on such an EFT from residual relativistic boost symmetry [160, 161]. This amounts to relating the coupling constants c_k in Eq. (7.1) to one another, but does not alter the operator basis.

7.3 Operator Basis and the Schrödinger Algebra

We show that the characters in Ref. [117], which were constructed by hand in order to generate the operator basis for a heavy-particle effective theory, are, in fact, characters of irreducible representations of the non-relativistic conformal group (this group is also referred to as the Schrödinger group, and we will use these terms interchangeably). Furthermore, from this one can determine that the operator basis for a heavy-particle effective field theory is spanned by particular kinds of primary operators of the non-relativistic conformal group. For those readers not familiar with symmetries of non-relativistic systems or the Schrödinger algebra, we invite them to read the Appendix, which is an introductory review to some of its well-known features that are relevant to the following discussion.

The Lie algebra of the Schrödinger group is:

$$
\begin{gathered}
= i\varepsilon_{ijk}J_k\,, \quad [J_i,K_j] = i\varepsilon_{ijk}K_k\,, \quad [J_i,P_j] = i\varepsilon_{ijk}P_k\,, \\
[H,K_i] = -iP_i\,, \quad [K_i,P_j] = iN\delta_{ij}\,, \\
[D,K_i] = -iK_i\,, \quad [D,P_i] = iP_i\,, \quad [D,H] = 2iH\,, \\
[C,P_i] = iK_i\,, \quad [C,H] = iD\,, \\
[K_i,K_j] = [H,P_i] = [H,J_i] = [P_i,P_j] = [N,\text{any}] = 0\,, \\
[D,J_i] = [C,J_i] = [C,K_i] = 0\,,
\end{gathered}
\tag{7.11}
$$

where these are the generators of rotations (J_i), non-relativistic boosts (K_i), time translations (H), spatial translations (P_i), scaling transformations (D), special conformal transformations (C), and number charge (N). States that transform as irreducible representations of the Schrödinger algebra can be labeled with the eigenvalues of the Cartan generators for the group, i.e., D, N, and J_3:

$$D\,|\Delta,n,m\rangle = i\Delta\,|\Delta,n,m\rangle\,, \tag{7.12}$$

$$N\,|\Delta,n,m\rangle = n\,|\Delta,n,m\rangle\,, \tag{7.13}$$

$$J_3\,|\Delta,n,m\rangle = m\,|\Delta,n,m\rangle\,. \tag{7.14}$$

The Schrödinger algebra has raising and lowering operators, analogous to those for angular momentum, which raise and lower the scaling dimension Δ:

$$DP_i\,|\Delta,n,m\rangle = i(\Delta+1)P_i\,|\Delta,n,m\rangle\,, \tag{7.15}$$

$$DK_i\,|\Delta,n,m\rangle = i(\Delta-1)K_i\,|\Delta,n,m\rangle\,, \tag{7.16}$$

$$DH\,|\Delta,n,m\rangle = i(\Delta+2)H\,|\Delta,n,m\rangle\,, \tag{7.17}$$

$$DC\,|\Delta,n,m\rangle = i(\Delta-2)C\,|\Delta,n,m\rangle\,. \tag{7.18}$$

P_i and H raise the scaling dimension, and K_i and C lower it. Using the linear combinations:

$$P_{\pm} \equiv P_1 \pm iP_2 \,, \qquad K_{\pm} \equiv K_1 \pm iK_2 \,, \tag{7.19}$$

the following eigenvalue equations following directly from the algebra:

$$\begin{aligned} J_3 P_{\pm} |\Delta, n, m\rangle &= (m \pm 1) P_{\pm} |\Delta, n, m\rangle \,, & (7.20) \\ J_3 P_3 |\Delta, n, m\rangle &= m P_3 |\Delta, n, m\rangle \,, & (7.21) \\ J_3 K_{\pm} |\Delta, n, m\rangle &= (m \pm 1) K_{\pm} |\Delta, n, m\rangle \,, & (7.22) \\ J_3 K_3 |\Delta, n, m\rangle &= m K_3 |\Delta, n, m\rangle \,, & (7.23) \\ J_3 H |\Delta, n, m\rangle &= m H |\Delta, n, m\rangle \,, & (7.24) \\ J_3 C |\Delta, n, m\rangle &= m C |\Delta, n, m\rangle \,. & (7.25) \end{aligned}$$

More such relations exist, but we only list the ones here that we will use. The Cartan generator N commutes with everything, and the action of other generators on the state does not change its number-charge. This allows us to consider states sector-wise depending on its number charge. In the sector of states with number charge $n \neq 0$, one can lower the scaling dimension by action of K_i and C, but the unitarity bound restricts the lowest possible dimension. For details, see Appendix F.3. There can be lowest-weight states of scaling dimension (in group theory literature, this is known as highest-weight state), such that:

$$\begin{aligned} K_i |\Delta_*, n, m\rangle &= 0 \,, & (7.26) \\ C |\Delta_*, n, m\rangle &= 0 \,, & (7.27) \end{aligned}$$

where $m = -j, -j+1, \cdots, j$ where j is the total spin of the highest-weight state. We note that even though the highest-weight states can be assigned a total spin j, this is no longer true once we act on these states by $P_{\pm}$ or P_3, since acting on a state with spin j, they produce a linear combination of spin $j+1, j, \cdots, |j-1|$. If the state's scaling dimension is $\Delta_* > d/2$, where

d is the number of spatial dimensions, then an irreducible representation of the Schrödinger algebra can generated by acting repeatedly on $|\Delta_*, n, m\rangle$ with P_i and H. Thus all the states in the representation are of the form:

$$\left|\Delta, n, m'\right\rangle = H^\ell P_+^r P_-^p P_3^q \left|\Delta_*, n, m\right\rangle \,, \tag{7.28}$$

where, specifically, $\Delta = 2\ell + r + p + q + \Delta_*$, $m' = r - p + m$ and $m = -j, -j+1, \cdots, j$. The character for this representation is a trace over all its states (following the procedure for relativistic conformal representation, as detailed in Ref. [165]):

$$\begin{aligned}
\chi_{[\Delta>d/2, n\neq 0, j]} &= \mathrm{Tr}\left[e^{i\theta_D D + i\theta_N N + i\theta_3 J_3}\right], && (7.29)\\
&= e^{in\theta_N} \sum_{\substack{|m|\leq j\\ \ell,r,p,q\geq 0}} \langle \mathrm{adjoint}|\, e^{i\theta_D D + i\theta_3 J_3} H^\ell P_+^r P_-^p P_3^q \left|\Delta_*, n, m\right\rangle, && (7.30)\\
&= \frac{e^{in\theta_N}\, \Lambda^{\Delta_*}\, \chi_{(j)}^{SU(2)}(z)}{(1-z^2\Lambda)(1-\Lambda)(1-\Lambda/z^2)(1-\Lambda^2)}, && (7.31)
\end{aligned}$$

where $\Lambda \equiv e^{-\theta_D}$, $z \equiv e^{i\theta_3/2}$, and $\langle \mathrm{adjoint}|$ means the complex conjugate of the state, created by the action of $H^\ell P_+^r P_-^p P_3^q$ on $|\Delta_*, n, m\rangle$, such that the norm of the state is unity. Here, $\chi_{(j)}^{SU(2)}(z)$ is the character for an $SU(2)$ $j-$plet, i.e.,

$$\chi_{(j)}^{SU(2)}(z) \equiv \sum_{|m|\leq j} \langle j, m|\, z^{2J_3} \left|j, m\right\rangle. \tag{7.32}$$

For example, the character for an $SU(2)$ doublet is $\chi_{\mathbf{2}}^{SU(2)} = z + 1/z$, and the character for an $SU(2)$ triplet is $\chi_{\mathbf{3}}^{SU(2)} = z^2 + 1 + 1/z^2$, and so on. Since P_i and H are the generators for spatial and time translations, respectively, we can identify the term $[(1-z^2\Lambda)(1-\Lambda)(1-\Lambda/z^2)]^{-1}$ as the generating functional for all possible *symmetric* products of spatial derivatives, and $(1-\Lambda^2)^{-1}$ as the generating functional for all possible products of time derivatives. To make this more clear, we can put in the numbers $\mathcal{D}_t$ and $\mathcal{D}_\perp$ (of modulus less than unity) to flag where and how many derivatives are generated, e.g.,

$$P_0(\Lambda) \equiv \frac{1}{(1-\Lambda^2\mathcal{D}_t)} = 1+\Lambda^2\mathcal{D}_t+\Lambda^4\mathcal{D}_t^2+\ldots$$
$$P_\perp(\Lambda) \equiv \frac{1}{(1-z^2\Lambda\mathcal{D}_\perp)(1-\Lambda\mathcal{D}_\perp)(1-\Lambda\mathcal{D}_\perp/z^2)} \tag{7.33}$$
$$= 1+\Lambda\mathcal{D}_\perp\chi_{\mathbf{3}}^{SU(2)}+\Lambda^2\mathcal{D}_\perp^2\left(1+\chi_{\mathbf{5}}^{SU(2)}\right)+\ldots$$

We can illustrate the behavior of these generating functionals with an example. Consider that the generating function for spatial derivatives acts on an object that is a singlet under rotation, call it ϕ, then the rotational indices can be reintroduced by hand, and generating derivatives can be represented as: $P_\perp\phi = \phi+\partial_i\phi+\partial_i\partial_i\phi+\partial_i\partial_j\phi+\cdots$. Note that there is no term like $\varepsilon_{ijk}\partial_i\partial_j\phi$ generated; it is trivially zero.

If the scaling dimension of the highest-weight state $|\Delta_*,n,m\rangle$ in the representation is $\Delta_*=d/2$, then the unitarity bound is saturated, leading to the fact that the following state has zero norm (see Appendix F.3 and Refs. [59, 81, 137]):

$$\left(H-\frac{P_i^2}{2n}\right)|\Delta_*=d/2,n,m\rangle = 0\,. \tag{7.34}$$

Therefore, the character for the representation when the highest-weight state has $\Delta_*=d/2$ should not contain the contribution coming from the state $\left(H-\frac{P_i^2}{2n}\right)|\Delta_*=d/2,n,m\rangle$ and any power of H or P_i acting on it. This can be achieved by removing the tower of states generated by H acting on $|\Delta_*=d/2,n,m\rangle$. As discussed in Section 7.2, this is precisely the requirement that when defining an operator basis with a heavy particle, taking into account the equations of motion, that one can choose a basis with no time derivatives act on heavy field ψ. The character for such a shortened representation is can be easily calculated:

$$\chi_{[\Delta_*=d/2,n\neq0,j]} = e^{in\theta_N}\,\Lambda^{d/2}\,P_\perp(\Lambda)\,\chi_{(j)}^{SU(2)}(z)\,. \tag{7.35}$$

This is the character used in Ref. [117] for the heavy particle degree of freedom, modulo the multiplicative factor of $e^{in\theta_N}\,\Lambda^{d/2}$. Therefore, for the sake of defining an operator basis, one can say that the heavy-particle state is a highest-weight state with scaling dimension $\Delta_* = d/2$ and $n \neq 0$. And, in particular, if it is a heavy fermion, then $\chi^{SU(2)}_{(j)} = \chi^{SU(2)}_{\mathbf{2}}$. This is a scenario when the equation of motion can be derived using the algebra and the constraint from unitarity, though this does not always have to the case.[2] The representation for $\psi^\dagger$ is the same as ψ, but with the sign of n flipped.

In the heavy particle EFT, the Lagrangian is expressed using *external* electric and magnetic fields (or their non-abelian generalizations). We are interested in embedding these fields within an representation of the Schrödinger group, where they would have well-defined charges under scaling transformations, number charge, and z-component of angular momentum. To begin, one must take care to reinstate the location of the speed of light constant c. Because space time scale differently under scaling transformations, i.e., $x \to \lambda x$ and $t \to \lambda^2 t$, the speed of light is not invariant, behaving as an intrinsic scale in the theory. As such, $\boldsymbol{\mathcal{E}} \equiv \mathbf{E}/c$ and $\mathbf{B}$ are the fundamental fields that appear in the field strength tensor $F^{\mu\nu}$. Since they are externally defined, they can be taken to scale in a similar way, both with scaling dimension $\Delta = 2$. Also, $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$ transform as vectors under rotation, so they are both spin-1. Lastly, since the electric and magnetic fields are Hermitian, they can not carry any number-charge, so they have $n = 0$. As noted in Appendix F.2, the representation of the Schrödinger group for operators with $n = 0$ differs from the $n \neq 0$ sector. For example, consider a highest-weight state $|\Delta_*, n = 0, m\rangle$ such that $K_i\,|\Delta_*, n = 0, m\rangle = C\,|\Delta_*, n = 0, m\rangle = 0$. The Schrödinger algebra then leads to the following:

$$K_j P_i\,|\Delta_*, n = 0, m\rangle \quad = \quad 0\,, \tag{7.36}$$

$$C P_i\,|\Delta_*, n = 0, m\rangle \quad = \quad 0\,. \tag{7.37}$$

Therefore, the state $P_i\,|\Delta_*, n = 0, m\rangle$ is also a highest-weight state in scaling dimension. In order to embed $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$ in the Schrödinger representation, we can choose to define the following kind

[2]To cite a specific example, at the interacting fixed point of a relativistic ϕ^4 theory, the equation of motion is not associated with a unitarity bound, since, in $4 - \varepsilon$ dimensions, the field ϕ acquires an anomalous dimension, which no longer saturates the unitarity bound [166].

of state $|\Delta_*, n=0, m\rangle$, where: (1) $C\,|\Delta_*, n=0, m\rangle = 0$, and (2) $|\Delta_*, n=0, m\rangle \neq P_i\,|\Delta'_*, n=0, m'\rangle$, where $|\Delta'_*, n=0, m'\rangle$ is some other state in the Hilbert space. This is the definition of the state in the Schrödinger group that we associate with the electric and magnetic fields. As before, one can build up a representation of the Schrödinger group (two towers of states) by acting by P_i and H on this $|\Delta_*, n=0, m\rangle$. In $d=3$ spatial dimensions there is no constraint from unitarity regarding how high these towers can go. However, these towers do not extend forever, since two of Maxwell's equations are:

$$\mathbf{D}_\perp \cdot \mathbf{B} = 0\,, \qquad \mathbf{D}_\perp \times \boldsymbol{\mathcal{E}} = -\frac{1}{c} D_t \mathbf{B}\,. \tag{7.38}$$

The other two Maxwell's equation with source term do not constrain or relate the tower of states, since both the current and charge density are externally defined. If we take the $c \to \infty$ limit, then Eqs. (7.38) gets contracted from the Poincaré representation to the $N=0$ representation of the Galilean group (for some details regarding this $c \to \infty$ contraction, see Appendix F.1), and end up being invariant under scaling and special conformal transformations:

$$\mathbf{D}_\perp \cdot \mathbf{B} = 0\,, \qquad \mathbf{D}_\perp \times \boldsymbol{\mathcal{E}} = 0\,. \tag{7.39}$$

Therefore, these are the shortening conditions for the states in the Schrödinger group associated with the electric and magnetic fields. So, the the characters for $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$ are:

$$\chi^{\mathcal{E}}_{[\Delta_*=2,n=0,j=1]} = \Lambda^2\, P_0(\Lambda)\, P_\perp(\Lambda) \left(\chi_{\mathbf{3}}^{SU(2)} - \Lambda \mathcal{D}_\perp \chi_{\mathbf{3}}^{SU(2)} + \Lambda^2 \mathcal{D}_\perp^2 \right), \tag{7.40}$$

$$\chi^{B}_{[\Delta_*=2,n=0,j=1]} = \Lambda^2\, P_0(\Lambda)\, P_\perp(\Lambda) \left(\chi_{\mathbf{3}}^{SU(2)} - \Lambda \mathcal{D}_\perp \right). \tag{7.41}$$

The additional $\Lambda^2 \mathcal{D}_\perp^2$ term in the character for $\boldsymbol{\mathcal{E}}$ is due to the fact that if one subtracts out $\Lambda \mathcal{D}_\perp \chi_{\mathbf{3}}^{SU(2)}$, then one will also subtract out $\Lambda^2 \mathcal{D}_\perp^2$, but this term was never there to begin with, since derivatives are symmetric under interchange of their spatial indices. These are precisely the characters used in Ref. [117] for the external gauge fields.

We have established how the requirements of defining an operator basis for a heavy particle EFT can be associated with the relevant degrees of freedom in the theory, i.e., the heavy particles, electric and magnetic fields, and the time and spatial derivatives that act on them, falling into irreducible representations of the Schrödinger group. In particular, we have shown that characters of certain representations of the Schrödinger group match those we used in Ref. [117], where we only had in mind the constraints from the equations of motion. At last, we can take tensor products between these representations of the Schrödinger group to generate operators that appear in the Lagrangian. Illustrating this with the following cartoon (including the details of the shortening conditions for $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$ is a bit cumbersome):

$$\begin{pmatrix} \psi^\dagger \\ \mathcal{D}_\perp \psi^\dagger \\ \mathcal{D}_\perp^2 \psi^\dagger \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{\mathcal{E}} \\ \mathcal{D}_\perp \boldsymbol{\mathcal{E}} \\ \mathcal{D}_\perp^2 \boldsymbol{\mathcal{E}}, \mathcal{D}_t \boldsymbol{\mathcal{E}} \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} \mathbf{B} \\ \mathcal{D}_\perp \mathbf{B} \\ \mathcal{D}_\perp^2 \mathbf{B}, \mathcal{D}_t \mathbf{B} \\ \vdots \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \psi \\ \mathcal{D}_\perp \psi \\ \mathcal{D}_\perp^2 \psi \\ \vdots \end{pmatrix} = \begin{pmatrix} O \\ \mathcal{D}_\perp O \\ \mathcal{D}_\perp^2 O, \mathcal{D}_t O \\ \vdots \end{pmatrix} + .. \tag{7.42}$$

Because the right-hand side of Eq. (7.42) are also representations of the Schrödinger group, only the highest-weight operators are not associated with total derivatives. These highest-weight operators have the properties that $[C, O] = 0$, and $O \neq [P_i, O']$, where O' is some other operator in the Hilbert space. Therefore, it is exactly these highest-weight operators that span the operator basis for the EFT with one heavy particle. This connection between representations of the Schrödinger group and a heavy particle EFT can be made for EFTs with multiple heavy fields. If there are no interactions that cause heavy fields to transform into other types, then each heavy field can be labeled with a different charge n, and operator basis is required be invariant under N.

7.4 Discussion and Conclusions

The operator basis for a heavy particle EFT, subject to external gauge fields, can be organized according to the representations of the non-relativistic conformal group (often called the Schrödinger group). Such an organization allows one to easily remove any redundancy between operators due to integration by parts and the equations of motion for the individual degrees of freedom. Specifically, we discuss that the heavy particle states are highest-weight states that saturate unitarity in $d = 3$ spatial dimensions, and this leads to a representation of the Schrödinger group that removes any time derivatives acting on the heavy field, which amounts to the same imposition on the operator basis due to the equations of motion of the heavy field. The external gauge fields are associated with the $n = 0$ sector of the Schrödinger group, and this necessitates an extended classification of highest-weight states [137]. Maxwell's equations in the $c \to \infty$ limit produce the shortening conditions for the external gauge fields. Taken together, the tensor products of these Schrödinger representations is itself a representation of the Schrödinger group, and the highest-weight (with $n = 0$ and $j = 0$) operators of this are exactly those which are not total derivatives, and therefore are the operators that span the operator basis for an EFT with a heavy particle.

We have shown the characters of representation for the neutral sector and charged sector of the Schrödinger group to be precisely those used in Ref. [117], which used a Hilbert series to help tabulate the operator basis for NRQED and HQET/NRQCD, up to and including operators at $O(1/M^4)$. An analogous situation occurs in relativistic theories, and the authors of Refs. [113, 114, 115] discuss why it may not be unreasonable to intuit a connection between an operator basis and the representations of a conformal group. It is also worth mentioning that since the Schrödinger group is non-compact, it has similar subtleties associated with character orthogonality as in the relativistic conformal group. Operationally, this does not hamper the operator counting program, nonetheless it would be interesting to explore further the subtleties associated with character orthogonality of Schödinger algebra in the same spirit as done in relativistic case.

The methodology introduced here could be leveraged to also describe the operator content beyond the realm of heavy particle effective field theory with a single heavy field, for example, fermions at unitarity [2, 1], two nucleon systems [4, 3], and it could have potential application in writing an operator basis for anisotropic Weyl anomaly, i.e., the anomaly associated with nonrelativistic scaling upon coupling the theory with curved space-time [20, 69, 21].

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Chapter 8

Nonrelativistic Conformal Field Theories in the Large Charge Sector

We study Schrödinger invariant field theories (nonrelativistic conformal field theories) in the large charge (particle number) sector. We do so by constructing the effective field theory (EFT) for a Goldstone boson of the associated $U(1)$ symmetry in a harmonic potential. This EFT can be studied semi-classically in a large charge expansion. We calculate the dimensions of the lowest lying operators, as well as correlation functions of charged operators. We find universal behavior of three point function in large charge sector. We comment on potential applications to fermions at unitarity and critical anyon systems.

8.1 Introduction and Summary

Symmetry has always been a guiding principle in characterizing physical systems. While weakly coupled field theories are known to be tractable in terms of perturbation theory in coupling, often the strongly coupled ones can only be constrained by symmetry arguments. For example, the physics of low-energy quantum chromo dynamics (QCD) is captured by an effective theory of pions, whose low-energy interactions are fixed by the broken chiral symmetry.

Conformal field theories (CFTs) are especially beautiful examples of how one can leverage the symmetry group. While generically strongly coupled, conformal symmetry almost completely fixes the behavior of correlation functions and gives non-trivial insights into the structure of their Hilbert spaces. In some cases, the conformal bootstrap [92] can provide us with rich physics of such theories entirely based on symmetry principles. However, we are still lacking many concrete calculational tools for these theories. In CFTs with an additional global $U(1)$, recent progress has been made by constructing effective field theories for their large charge (Q) sector. Generically, the large charge sector can be horribly complicated in terms of elementary fields and their interactions, but one can set up a systematic $1/Q$ expansion to probe this strongly coupled regime. This has been useful in finding the scaling of operator dimensions, and many other meaningful physical quantities [167, 168, 169, 170, 171, 172].

In this work, we will be dealing with systems with non relativistic scale and conformal invariance i.e. systems invariant under Schrödinger symmetry. While in CFT, one needs to have a external global symmetry to talk about large charge expansion, the nonrelativistic conformal field theories (NRCFTs) come with a "natural" $U(1)$, the particle number symmetry. The Schödinger symmetry group and its physical consequences have been studied in [94, 60, 59, 173, 99, 137]. The physical importance of Schrödinger symmetry lies in varied realisation of the symmetry group, starting from fermions at unitarity[1, 2] to examples including spin chain models [95], systems consisting of deuterons [3, 4], ^{133}Cs[6], ^{85}Rb [5],^{39}K [7].

Such theories, similar to CFTs, admit a state-operator correspondence[174, 173] in which the dimensions of operators correspond to energy of a state in a harmonic potential[1]. Specifically, the scaling generator D, which scales $\boldsymbol{x} \mapsto \lambda x$ and $t \mapsto \lambda^2 t$ for $\lambda \in \mathbf{R}$ gets mapped to the Hamiltonian (H_ω) in the harmonic trap i.e. $H_\omega \equiv H + \omega^2 C$ where $C = \frac{1}{2}\int d^d x\, x^2 n(x)$ is the special conformal generator and $n(x)$ is the number density and H is the time translation generator of the Schrödinger group. The parameter ω determines the strength of the potential and plays an analogous role to the radius of the sphere in the relativistic state-operator correspondence.[2].

Given this set up, we consider an operator Φ with large number charge Q. For example,

[1]This state-operator map is different from the one discussed in [137] to explore the neutral sector. In [137], the map is more akin to the $(0+1)$ dimensional CFT.

[2]Here and also subsequently, we will be working in non-relativistic "natural" units of $m = \hbar = 1$

one can think of $\phi^{\frac{N}{2}}$ for $\phi(x) = :\psi_{\uparrow}^{\dagger}(x)\psi_{\downarrow}^{\dagger}(x)$: in the case of fermions at unitarity in $d = 3$ dimensions. By the state-operator correspondence, the operator is related to a state $|\Phi\rangle$ with finite density of charge (n) in the harmonic trap. There's an energy scale set by the density $\Lambda_{UV} \sim \mu \sim n^{\frac{2}{d}}$, μ being the chemical potential which fixes the total charge to Q. There is also a scale set by the trap $\Lambda_{IR} \sim \omega$ which controls the level spacing of H_{ω}. The limit of large charge $Q \gg 1$ then implies a parametric separation of these scales. This allows us to set up a perturbatively controlled expansion in $1/Q$ and probe the large charge sector of a theory invariant under Schrödinger symmetry.

In this limit it becomes appropriate to ask, what state of *matter* describes the large charge sector? Such a state with finite density of charge necessarily breaks some of the space-time symmetries e.g. scale transformations, (Galilean) boosts, special-conformal transformations. That these symmetries are spontaneously broken also implies that they must be realized non-linearly in the effective field theory (EFT) describing the large charge sector. We expect the low-energy degrees of freedom to be Goldstones.

One possibility is that the $U(1)$ symmetry remains unbroken. This is the case for a system with a Fermi surface. There the low-energy degrees of freedom would also include fermionic matter in addition to any Goldstones. The simplest candidate EFT, Landau Fermi-Liquid theory, is incompatible with the non-linearly realized Schrödinger symmetry[175] and therefore this is a fairly exotic possibility.

Another possibility is that the $U(1)$ symmetry is also spontaneously broken, leading to superfluid behavior. This has been the case most studied in the literature and seems like the most obvious possibility for a bosonic NRCFT. Additionally, both unitary fermions and the scale invariant anyon gas at large density are suspected to be superfluids. Therefore we focus exclusively on this symmetry breaking pattern.

Summary of Results

We compute the properties of the ground state $|\Phi\rangle$ with finite density of charge, under the assumption it describes a rotationally invariant superfluid, via an explicit path integral representation:

$$\langle\Phi|e^{-H_\omega T}|\Phi\rangle = \int \mathcal{D}\chi\, e^{-S_{eff}[\chi]+\mu\int d^dx\, n(x)} \tag{8.1}$$

where χ is a Goldstone boson describing excitations above the ground state, μ is the chemical potential and $n(x)$ is the number density which is canonically conjugate to χ. This integral can then be computed by saddle point in the large μ limit. The chemical potential μ can then be fixed semi-classically in terms of the charge Q. Thus self-consistently, we are obtaining a large Q expansion. We employ the coset construction to write down the most general effective action for the Goldstone which is consistent with the non-linearly realized Schrödinger symmetry.

- For the case with magnetic vector potential $\boldsymbol{A}=0$ (the one that is relevant for the NRCFT in harmonic trap), we find the effective Lagrangian given by

$$\begin{aligned}\mathcal{L}_{eff} = c_0 X^{\frac{d}{2}+1} + c_1\frac{X^{\frac{d}{2}+1}}{X^3}\partial_i X\partial^i X + c_2\frac{X^{\frac{d}{2}+1}}{X^3}(\partial_i A_0)^2 \\ + c_3\frac{X^{\frac{d}{2}+1}}{X^2}\partial_i\partial^i A_0 + c_4\frac{X^{\frac{d}{2}+1}}{X^2}(\partial_i\partial^i\chi)^2\end{aligned} \tag{8.2}$$

  where $X=\partial_t\chi - A_0 - \frac{1}{2}\partial_i\chi\partial^i\chi$. However this is not the full set of constraints. It can be shown that imposing ‘general coordinate invariance’ will reduce the number of independent Wilson coefficients even further[176]. In particular there are the additional constraints: $c_2=0$ and $c_3=-d^2c_4$. Additionally, in $d=2$, one can have parity violating operator at this order:

$$c_5\frac{1}{X}\varepsilon^{ij}(\partial_i A_0)(\partial_j X) \tag{8.3}$$

  The details can be found in Section 8.4.

- The dispersion relation of low energy excitation above the ground state is found out to be:

$$\varepsilon(n,\ell) = \pm\omega\left(\frac{4}{d}n^2 + 4n + \frac{4}{d}\ell n - \frac{4}{d}n + \ell\right)^{\frac{1}{2}} \tag{8.4}$$

where ℓ is the angular momentum and n is a non-negative integer and $\varepsilon(n,\ell)$ is the excitation energy. The dispersion determines the low-lying operator dimensions explicitly. Since, $\varepsilon(n=0,\ell=1) = \pm\omega$ and $\varepsilon(n=1,\ell=0) = \pm 2\omega$, they can be identified with two different kinds of descendant operators appearing in the Schrödinger algebra. The details can be found in Section 8.6.2.

- In the leading order in Q, we find the ground state energy i.e. dimension Δ_Q of the corresponding operator Φ:

$$\Delta_Q = \left(\frac{d}{d+1}\right)\xi Q^{1+\frac{1}{d}}\,, \quad \text{where} \quad \frac{1}{c_0} = \frac{\Gamma(\frac{d}{2}+2)}{\Gamma(d+1)}(2\pi\xi^2)^{\frac{d}{2}}\,. \tag{8.5}$$

where c_0 is UV parameter of the theory, appearing in the Lagrangian (8.2).

Specifically, we have

$$\Delta_Q = \frac{2}{3}\left(\xi Q^{3/2}\right) + c_1\frac{4\pi}{3}\xi\left(Q^{\frac{1}{2}}\log Q\right) + O\left(Q^{\frac{1}{2}}\right) \quad \text{for } d=2\,. \tag{8.6}$$

$$\Delta_Q = \left(\frac{3}{4}\right)\xi Q^{4/3} - \left(c_1 + \frac{c_3}{2}\right)(3\sqrt{2}\pi^2)\xi^2 Q^{2/3} + O\left(Q^{5/9}\right) \quad \text{for } d=3\,. \tag{8.7}$$

The details can be found in Section 8.6.1.

- We find the structure function F appearing in three point function of two operators with large charge Q and $Q+q$ and one operator ϕ_q with small charge q goes as follows:

$$F(v = \mathbf{i}\omega y^2) \propto Q^{\frac{\Delta_\phi}{2d}}\left(1 - \frac{\omega y^2}{2\xi}Q^{-1/d}\right)^{\frac{\Delta_\phi}{2}} e^{-\frac{1}{2}q\omega y^2} \tag{8.8}$$

where y is the insertion point of ϕ_q in the oscillator co-ordinate and Δ_ϕ is the dimension ϕ_q. The details can be found in Section 8.7.2.

Note: While this work was being completed a paper appeared with some overlap[177]. They identify many of the same operators we do, through different means and without couplings to the background gauge field. The primary tool we utilize is the state-operator correspondence for NRCFTs, therefore directly compute properties of the NRCFTs in harmonic trap in large charge limit.

8.2 Lightning Review of Schrödinger Algebra

The Schrödinger algebra has been extensively explored in [94, 60, 59, 173, 99, 137]. Here we take the readers through a quick tour of the essential features of Schrödinger algebra, that we are going to use through out this paper. The most important subgroup of Schrödinger group is the Galilean group, generated by time translation generator H, spatial translation generators P_i, rotation generators J_{ij} and boost generators K_i. One can centrally extend this group by appending another $U(1)$ generator N, which generates the particle number symmetry. As a whole, these generators constitute what we call Galilean algebra and they satisfy:

$$\begin{aligned}
[J_{ij},N] = [P_i,N] = [K_i,N] = [H,N] = 0 \\
[J_{ij},P_k] = \mathbf{i}(\delta_{ik}P_j - \delta_{jk}P_i)\,, \\
[J_{ij},K_k] = \mathbf{i}(\delta_{ik}K_j - \delta_{jk}K_i)\,, \\
[J_{ij},J_{kl}] = \mathbf{i}(\delta_{ik}J_{jl} - \delta_{jk}J_{il} + \delta_{il}J_{kj} - \delta_{jl}J_{ki})\,, \\
[P_i,P_j] = [K_i,K_j] = 0\,, \qquad [K_i,P_j] = \mathbf{i}\delta_{ij}N\,, \\
[H,N] = [H,P_i] = [H,M_{ij}] = 0\,, \quad [H,K_i] = -\mathbf{i}P_i\,.
\end{aligned} \tag{8.9}$$

The Galilean group is enhanced to Schrödinger group by appending a scaling generator D and a special conformal generator C such that they satisfy the following commutator relations:

$$[D,P_i] = \mathbf{i}P_i\,, \quad [D,K_i] = -\mathbf{i}K_i\,, \tag{8.10}$$

$$[D,H] = 2\mathbf{i}H\,, \quad [D,C] = -2\mathbf{i}C\,, [H,C] = -\mathbf{i}D\,, \tag{8.11}$$

$$[J_{ij},D] = 0\,, \quad [J_{ij},C] = 0\,, \quad [N,D] = [N,C] = 0\,. \tag{8.12}$$

The state-operator correspondence for an NRCFT is based on the following definition [173]:

$$|O\rangle \equiv e^{-\frac{H}{\omega}}O^\dagger(0)\,|0\rangle = O^\dagger\left(-\frac{\mathbf{i}}{\omega},0\right)|0\rangle \tag{8.13}$$

where $O^\dagger$ is a primary operator of number charge $Q_{O^\dagger} = -Q_O \geq 0$. By the Schrödinger algebra, this state satisfies:

$$N\,|O\rangle = Q_{O^\dagger}\,|O\rangle \qquad H_\omega\,|O\rangle = \omega\Delta_O\,|O\rangle \tag{8.14}$$

where $H_\omega = H + \omega^2 C$ is the Hamiltonian with the trapping potential.

It is natural to define a transformation from Galilean coordinates $x = (t,\boldsymbol{x})$ to the "oscillator frame" $y = (\tau,\boldsymbol{y})$ where the time translation $\tau \to \tau + a$ is generated by H_ω. Explicitly this is given by

$$\omega\tau = \arctan\omega t\,, \qquad \boldsymbol{y} = \frac{\boldsymbol{x}}{\sqrt{1+\omega^2 t^2}} \tag{8.15}$$

and allows us to map primary operators and their correlation functions in the oscillator frame to the Galilean frame via the map[173]:

$$\tilde{O}(y) = \left(1+\omega^2 t^2\right)^{\frac{\Delta_O}{2}} \exp\left[\frac{\mathbf{i}}{2}Q_O\frac{\omega^2|\boldsymbol{x}|^2 t}{1+\omega^2 t^2}\right] O(x) \tag{8.16}$$

$$O(x) = [\cos(\omega t)]^{\Delta_O} \exp\left[-\frac{\mathbf{i}}{2}Q_O\omega|\boldsymbol{y}|^2\tan(\omega\tau)\right]\tilde{O}(y) \tag{8.17}$$

In this paper, we will be interested in matrix elements of the form:

$$\langle\Phi|\phi_1(y_1)\cdots\phi_n(y_n)|\Phi\rangle \tag{8.18}$$

where $\Phi^\dagger$ is a primary of charge $Q \gg 1$ and ϕ_i are also charged [3] primaries with $q_i \ll Q$.[4]

In the Galilean frame, the general form of a two point function is fixed to be

$$\langle O_1(x_1) O_2(x_2) \rangle = c \delta_{\Delta_1,\Delta_2} \delta_{Q_1,-Q_2} \frac{\exp\left[\mathbf{i} Q_2 \frac{|\boldsymbol{x}|^2}{2t}\right]}{(t_1 - t_2)^{\Delta_1}} \tag{8.19}$$

where c is a numerical constant, Δ_i is the dimension of the operator O_i, Q_i is the charge of O_i. The symmetry algebra constrains the general form of a three-point function upto a arbitrary function of a cross-ratio v_{ijk} defined below:

$$\begin{aligned} \langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle &\equiv G(x_1; x_2; x_3) \\ &= F(v_{123}) \exp\left[-\mathbf{i} \frac{Q_1}{2} \frac{\boldsymbol{x}_{13}^2}{t_{13}} - \mathbf{i} \frac{Q_2}{2} \frac{\boldsymbol{x}_{23}^2}{t_{23}}\right] \prod_{i<j} t_{ij}^{\frac{\Delta}{2} - \Delta_i - \Delta_j} \end{aligned} \tag{8.20}$$

where $\Delta \equiv \sum_i \Delta_i$, $x_{ij} \equiv x_i - x_j$, and $F(v_{ijk})$ is a function of the cross-ratio v_{ijk} defined:

$$v_{ijk} = \frac{1}{2} \left(\frac{\boldsymbol{x}_{jk}^2}{t_{jk}} - \frac{\boldsymbol{x}_{ik}^2}{t_{ik}} + \frac{\boldsymbol{x}_{ij}^2}{t_{ij}} \right) \tag{8.21}$$

We note that the three point function becomes zero unless $\sum Q_i = 0$.

8.3 Lightning Review of Coset Construction

A symmetry is said to be spontaneously broken if the lowest energy state, the ground state, is not an eigenstate of the associated charge. The low-energy effective action, describing the physics above the ground state, is still invariant under the full global symmetry group but the broken subgroup is realized *non-linearly*. Typically this means the effective action describes some number of Goldstones.

The coset construction gives a general method for constructing effective actions with

[3] The state-operator correspondence breaks down for neutral operators as they actually trivially on the vacuum and their representation theory is not well understood. [137] explores how to circumvent this issue.

[4] Here we point out that if an operator is explicitly written as a function of oscillator co-ordinate, it is to be understood that we have already employed the mapping (8.16). Thus $\phi_i(y_1)$ in (8.18) should technically be written as $\tilde{\phi}_i(y_1)$, albeit we omit "tilde" sign for notational simplicity.

appropriate non-linearly realized symmetry actions. It was developed for internal symmetries by CCZW [178, 179] and later generalized to space-time symmetries[180]. Here we give a nimble review of the method and its application to the superfluid. We refer to the original literature and the recent review [181] for more details. The primary objective of the coset construction is to write down the most general action, invariant under a global symmetry group G but where only the subgroup G_0 is linearly realized. Let us consider a symmetry group which contains the group of translations, generated by P_a. Let us denote the broken generators as X_b corresponding to associated Goldstones $\pi_b(x)$. We denote unbroken generators as T_c.

We can define the exponential map from space-time to the coset space G/G_0

$$U \equiv e^{\mathbf{i}\bar{P}_a x^a} e^{\mathbf{i}X_b \pi^b(x)} \tag{8.22}$$

With this map we can define the 1-form, known as the Maurer-Cartan (henceforth we call it MC) form, on the coset space. Under a G-transformation (8.22) transforms as

$$g : U(x) \to e^{\mathbf{i}\bar{P}_a (x')^a} e^{\mathbf{i}X_b \pi'^b(x')} h(\pi(x), g) \tag{8.23}$$

where $h(\pi(x), g)$ is some element in G_0, determined by the Goldstones and $g \in G$, that "compensates" to bring $U(x)$ back to the form in (8.22). This determines how the Goldstone fields transform[5].

Expanded in a basis of generators the MC form looks like:

$$\Omega \equiv -\mathbf{i}U^{-1}\partial_\mu U \equiv E^a_\mu(\bar{P}_a + (\nabla_a \pi^b)X_b + A^c_a T_c) \tag{8.24}$$

where each of the tensors $\{E^a_\mu,\ \nabla_a\pi^b,\ T_c\}$ is a function of the Goldstone fields π_a. Here E^a_μ is a vierbein, $\nabla_a\pi^b$ are the covariant Goldstone derivatives and A^c_a transforms like a connection.

Several remarks are in order. Once space-time symmetries are broken the quantity d^dx is no longer necessarily a scalar under those transformations. However the quantity $d^dx\det E$

[5]For space-time symmetries there's a translation piece even though $\bar{P}_a$ are unbroken. This is because, on coordinates, translations are always non-linearly realized as $x \to (x+a)$

can be used to define an invariant measure for the action. On the other hand, contractions of the objects $\nabla_a \pi^b$, in a way which manifestly preserves the G_0 symmetry, also provides us with G invariants and form the Goldstone part of the effective action. The connection, A_a^c and the vierbein, can be used to define the following "higher" covariant derivative

$$\nabla_a^H \equiv (E^{-1})_a^\mu \partial_\mu + \mathbf{i} A_a^c T_c \tag{8.25}$$

An object like $\nabla_a^H \nabla_b \pi^c$ also transforms covariantly and G_0-invariant contractions with other tensors should be included. The other primary use of (8.25) is for defining covariant derivatives of "matter fields". For example, suppose ψ is a matter field transforming in a k-dimensional linear representation r of G_0 as $\psi \to \psi' = r(h)\psi$. The coset construction provides multiple ways to uplift G_0 representations to full G representations. The one of importance to us is when r appears in the decomposition of a K-dimensional representation R of G. Defining the field $\tilde{\psi} \equiv (\psi,\, 0)$ in the K-dimensional representation, one can show that the field $\Psi = R(\Omega)\tilde{\psi}$ transforms linearly under the full group G. If a subset of the symmetry is gauged then we just covariantly replace $\partial_\mu \to D_\mu = \partial_\mu + \mathbf{i}\bar{A}_\mu^d \bar{T}_d$ in the above. The tensors will then depend on the gauge fields $\bar{A}$ but otherwise everything goes through.

One last important aspect of space-time symmetry breaking is that not all the Goldstone bosons are necessarily independent [182]. This occurs when the associated currents differ only by functions of spacetime. A localized Goldstone particle is made by a current times a function of spacetime, so we can not sharply distinguish the resulting particles. This redundancy also appears in the coset construction. Suppose X and X' are two different broken generators in different G_0-multiplets and we denote their associated Goldstone bosons π and π'. Let $\bar{P}_\nu$ be an unbroken translation generator. Let us also assume that there's a non-trivial commutator of the form $[P_\nu, X] \supseteq X'$. One can see, from calculating the Maurer-Cartan form via the BCH identity, that this implies an undifferentiated π in the covariant Goldstone derivative $\nabla_\nu \pi'$. The quadratic term is then $(\nabla_\nu \pi')^2 \sim c^2 \pi^2$; this is an effective mass term for the π Goldstone. Thus we are justified in integrating it out by imposing its equation of motion. A simpler, but equivalent up to redefinitions, constraint is setting $\nabla_\nu \pi' = 0$. This is a covariant constraint, completely consistent with the symmetries. In the literature it is known as an “inverse Higgs constraint” .

8.4 Schrödinger Superfluid from Coset Construction

In this section, we will use the coset construction to construct the most general Goldstone action consistent with the broken symmetries of a rotationally invariant Schrödinger superfluid. For the purpose of determining local properties of the superfluid state in the trap we can first work in the thermodynamic limit defined by $\Lambda_{IR} \sim \omega \to 0$. The symmetry generators are then just those of the usual Schrödinger group.

The superfluid ground state $|\Phi\rangle$ spontaneously breaks the number charge N. As mentioned in the introduction, this state also breaks the conformal generators and boosts. It is simplest to describe such states in the grand canonical ensemble. We remark that in the thermodynamic limit, one can leverage the equivalence between canonical ensemble with fixed chrage and grand canonical ensemble[6]. Thus, in what follows, we define the operator $\bar{H} = H - \mu N$ such that $\bar{H}\,|\Phi\rangle = 0$. The parameter μ plays the role of a chemical potential; it is a Lagrange multiplier to be determined by the charge density. By assumption, $|\Phi\rangle$ is not an eigenstate of N. It therefore cannot be an eigenstate of H while satisfying $\bar{H}\,|\Phi\rangle = 0$. The unbroken ‘time’ translations are therefore generated by $\bar{H}$[183]. The symmetry breaking pattern is then given by:

$$\text{Unbroken: } \{\bar{H} \equiv H - \mu N\,, P_i\,, J_{ij}\} \qquad \text{Broken: } \{N\,, K_i\,, C\,, D\}\,, \tag{8.26}$$

for which we can parameterize the coset space as:

$$U = e^{\mathrm{i}\bar{H}t} e^{-\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{x}} e^{\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{K}} e^{-\mathrm{i}\lambda C} e^{-\mathrm{i}\sigma D} e^{\mathrm{i}\pi N} = e^{\mathrm{i}Ht} e^{-\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{x}} e^{\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{K}} e^{-\mathrm{i}\lambda C} e^{-\mathrm{i}\sigma D} e^{\mathrm{i}\chi N}\,. \tag{8.27}$$

Here we use 4 distinct Goldstone fields:

[6]As a result, one can always view the large charge expansion as a large chemical potential expansion

- π is the 'phonon', the Goldstone for the charge. It defines the shifted field $\chi \equiv \pi + \mu t$
- $\boldsymbol{\eta}$ is the 'framon', the Goldstone for (Galilean) boosts. It transforms as a vector.
- λ is the 'trapon', the Goldstone for special conformal transformations.
- σ is the 'dilaton', the Goldstone for dilations.

To allow for a background field A_μ, we define the covariant derivative $D_\mu = \partial_\mu + \mathbf{i} A_\mu N$. From this group element we can calculate the MC form:

$$-\mathbf{i} U^{-1} D_\mu U \equiv E_\mu^\nu [\bar{P}_\nu + (\nabla_\nu \eta^i) K_i - (\nabla_\nu \lambda) C - (\nabla_\nu \sigma) D + (\nabla_\nu \pi) Q] \tag{8.28}$$

where $\bar{P}_\mu \equiv (-\bar{H}, \boldsymbol{P})$, and we've anticipated the absence of a gauge field for J_{ij}. We remark that the relativistic notation is just for ease of writing; because space and time are treated differently we have to treat those components of the MC form separately. Explicitly we have the following:

$$E_0^0 = e^{-2\sigma}, \quad E_0^i = -\eta^i e^{-\sigma}, \quad E_i^0 = 0, \quad E_i^j = \delta_i^j e^{-\sigma}, \tag{8.29}$$

$$\nabla_0 \eta^j = e^{3\sigma} (\dot{\eta}^j + \boldsymbol{\eta} \cdot \boldsymbol{\partial} \eta^j), \quad \nabla_i \eta^j = e^{2\sigma} (\partial_i \eta^j - \lambda \delta_i^j), \tag{8.30}$$

$$\nabla_0 \lambda = e^{4\sigma} (\dot{\lambda} + \boldsymbol{\eta} \cdot \boldsymbol{\partial} \lambda + \lambda^2), \quad \nabla_i \lambda = e^{3\sigma} \partial_i \lambda, \tag{8.31}$$

$$\nabla_0 \sigma = e^{2\sigma} (\dot{\sigma} + \boldsymbol{\eta} \cdot \boldsymbol{\partial} \sigma - \lambda), \quad \nabla_i \sigma = e^{\sigma} \partial_i \sigma, \tag{8.32}$$

$$\nabla_0 \pi = e^{2\sigma} (\dot{\chi} - A_0 - \mu e^{-2\sigma} + \boldsymbol{\eta} \cdot \boldsymbol{\partial} \chi + \frac{1}{2} \eta^2), \quad \nabla_i \pi = e^{\sigma} (\partial_i \chi - A_i + \eta_i), \tag{8.33}$$

which can be used to construct the effective action.

There are 4 commutators that each imply a different constraint

$$[P_i, K_j] = -\mathbf{i} \delta_{ij} N \implies \nabla_i \pi = 0, \quad [\bar{H}, D] = -2\mathbf{i} (\bar{H} + \mu N) \implies \nabla_0 \pi = 0, \tag{8.34}$$

$$[\bar{H}, C] = -\mathbf{i}D \implies \nabla_0 \sigma = 0\,, \qquad [P_i, C] = -\mathbf{i}K_j \delta_{ij} \implies \nabla_i \eta^j = 0\,. \tag{8.35}$$

Imposing them allows everything to be written in terms of a single Goldstone field χ. Upon defining the gauge invariant derivatives:

$$D_t \chi \equiv \partial_t \chi - A_0\,, \quad D_i \chi \equiv \partial_i \chi - A_i\,, \tag{8.36}$$

the simplest pair can be solved as:

$$\nabla_i \pi = 0 \quad \implies \quad \eta_i = -D_i \chi\,, \tag{8.37}$$

$$\nabla_0 \pi = 0 \quad \implies \quad \mu e^{-2\sigma} = D_t \chi - \frac{1}{2} D_i \chi D^i \chi\,. \tag{8.38}$$

The other two involve the trapon λ:

$$\nabla_i \eta^j = 0 \quad \implies \quad \lambda \delta_i^j = \partial_i \eta^j = -\partial_i D^j \chi\,, \tag{8.39}$$

$$\nabla_0 \sigma = 0 \quad \implies \quad \lambda = \dot{\sigma} + \boldsymbol{\eta} \cdot \boldsymbol{\partial} \sigma\,, \tag{8.40}$$

which can be written together as:

$$\dot{\sigma} + \boldsymbol{\eta} \cdot \boldsymbol{\partial} \sigma - \frac{1}{d} \boldsymbol{\partial} \cdot \boldsymbol{\eta} = -\frac{1}{2} \frac{\partial_0 X}{X} + \frac{1}{2} \frac{D_i \chi \partial^i X}{X} + \frac{1}{d} \partial_i D^i \chi = 0\,. \tag{8.41}$$

This is simply the leading order equation of motion for χ as we will show below.

The leading order action comes from the vierbein (8.29) which can be expressed with χ as

$$\det E = e^{-(d+2)\sigma} \propto \left(D_t \chi - \frac{1}{2} D_i \chi D^i \chi \right)^{\frac{d}{2}+1}\,. \tag{8.42}$$

Defining the variable X as

$$X = D_t \chi - \frac{1}{2} D_i \chi D^i \chi\,, \tag{8.43}$$

we can write the leading order effective action as

$$S_0 = \int dtd^dx\, c_0\, \mathcal{O}_0 = \int dtd^dx\, c_0 X^{\frac{d}{2}+1}\,, \tag{8.44}$$

where c_0 is a dimensionless constant. The leading order theory (8.44) is time reversal invariant as it acts as:

$$T: \quad t \to -t\,, \quad \pi \to -\pi\,, \quad A_0 \to -A_0\,. \tag{8.45}$$

Higher derivative terms are constructable from contractions of the following objects:

$$\nabla_0 \eta^i\,, \quad \nabla_0 \lambda\,, \quad \nabla_i \lambda\,, \quad \nabla_i \sigma\,. \tag{8.46}$$

as well as contractions of the 'higher covariants'

$$\nabla_0^H = -e^{2\sigma}\partial_0 + e^{\sigma}\eta^i\partial_i\,, \qquad \nabla_i^H = e^{\sigma}\partial_i\,, \tag{8.47}$$

acting on the tensors (8.46). All of these objects can be expressed in terms of χ by the constraints (8.34) and (8.35). Even though we are interested in large Q expansion eventually, to touch the base with the EFT written in [176], we emphasize that the power counting is done with X, being taken to be $O(p^0)$, which implies that objects like $[(\partial_i\chi)(\partial_i\chi)]^k$, $\partial_t\chi$ and A_0 are also order one. Additional derivatives then increase the dimension. In what follows, the field strengthsE_i and F_{ij} are defined as

$$E_i \equiv \partial_0 A_i - \partial_i A_0 \quad F_{ij} \equiv \partial_i A_j - \partial_j A_i\,. \tag{8.48}$$

At $O(p^2)$ we have following operators:

$$\mathcal{O}_1 \equiv \det E\ \nabla_i\sigma\nabla^i\sigma\ \propto\ \frac{X^{\frac{d}{2}+1}}{X^3}\partial_i X\partial^i X\,, \tag{8.49}$$

$$\mathcal{O}_2 \equiv \det E\ (\nabla_0\eta_i - 2\nabla_i\sigma)^2\ \propto\ \frac{X^{\frac{d}{2}+1}}{X^3}[E^2 + 2E_iF_{ij}(D_j\chi) + F_{ij}F_{ik}(D_j\chi)(D_k\chi)]\,, \tag{8.50}$$

$$\mathcal{O}_3 \equiv \det E\ \nabla_i\sigma(\nabla_0\eta^i - 2\nabla^i\sigma)\ \propto\ \frac{X^{\frac{d}{2}+1}}{X^2}[\partial_i E^i + [\partial_i F_{ij}](D_j\chi) - \frac{1}{2}F_{ij}F^{ij}]\,, \tag{8.51}$$

$$O_4 \equiv \det E\ \nabla_0 \lambda \ \propto \ \frac{X^{\frac{d}{2}+1}}{X^2} (\partial_i D^i \chi)^2 \,, \tag{8.52}$$

where the second expression of (8.51) is obtained via integration-by-parts and the (8.52) is obtained by a straight forward application of the identity (8.41) and integration-by-parts. These operators were found in reference[176] for $d = 3$ by very different means. Additionally, in $d = 2$, one can construct following parity violating operators at this order:

$$O_5 \equiv \det E\ \varepsilon^{ij} (\nabla_0 \eta_i)(\nabla_j \sigma) \propto \frac{X^{\frac{d}{2}+1}}{X^3} \varepsilon^{ij} \left[E_i - F_{jk}(D_k \chi) \right] (\partial_j X) \,, \tag{8.53}$$

$$O_6 \equiv \det E\ \varepsilon^{ij} \nabla_i^H (\nabla_0 \eta_j - 2 \nabla_j \sigma) \propto \frac{X^{\frac{d}{2}+1}}{X^2} \varepsilon^{ij} \partial_i (E_j - F_{jk}(D_k \chi)) \,. \tag{8.54}$$

Similarly in $d = 3$ we have ε^{ijk} but that means the parity violating operators will be higher order in the derivative expansion.

8.5 Superfluid Hydrodynamics

In this section, we study the superfluid hydrodynamics. As a warm up, we first consider the fluid without the trap, thus there is no intrinsic length scale associated with such a system. The leading order superfluid Lagrangian is known to take the form [176]:

$$\mathcal{L} = P(X) \tag{8.55}$$

where P stands for 'pressure' as function of the chemical potential μ at zero temperature and X is the same as defined in the previous section. Due to the absence of any internal scale, dimensional analysis dictates that:

$$P = c_0 \mu^{\frac{d}{2}+1} \,, \tag{8.56}$$

which we get from (8.44) by evaluating on the groundstate solution $\chi_{cl} = \mu t$. The number density is conjugate to the Goldstone field χ and at leading order is:

$$n \equiv \frac{\partial \mathcal{L}}{\partial \dot{\chi}} = P'(X) = c_0 \left(\frac{d}{2} + 1 \right) X^{\frac{d}{2}} . \tag{8.57}$$

One can then define the superfluid velocity in terms of the Goldstone as:

$$v_i \equiv -D_i \pi = -D_i \chi = \eta_i \tag{8.58}$$

where we have used the inverse Higgs constraint (8.37). This gives a simple interpretation of the equation of motion:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} = \partial_t n + \partial_i (n v^i) = 0 , \tag{8.59}$$

which is the continuity equation of superfluid hydrodynamics. Using equations (8.37), we can write:

$$\partial_\mu n = c_0 \frac{d}{2} \left(\frac{d}{2} + 1 \right) X^{\frac{d}{2}-1} (\partial_\mu X) = -dn(\partial_\mu \sigma) \qquad \partial_i v^i = -\partial_i D^i \chi = \boldsymbol{\partial} \cdot \boldsymbol{\eta} \tag{8.60}$$

The equation of motion (8.59) thus comes out to be as follows:

$$\partial_t n + \partial_i (n v^i) = -dn\dot{\sigma} - dn(\boldsymbol{\eta} \cdot \boldsymbol{\partial} \sigma) + n \boldsymbol{\partial} \cdot \boldsymbol{\eta} = 0 \tag{8.61}$$

and becomes equivalent to the constraint (8.41). Thus the superfluid EFT is consistent with the symmetry breaking pattern we discussed in the previous section.

8.5.1 Superfluid in a Harmonic Trap

Now we turn on the harmonic trap and study this superfluid EFT in the trapping potential by taking:

$$A_0 = \frac{1}{2} \omega^2 r^2 , \qquad \boldsymbol{A} = 0 . \tag{8.62}$$

In the presence of a harmonic potential, the ground state density is no longer uniform. The number density is given by the conjugacy relation (8.57) and to leading order is:

$$n(x) = c_0 \left(\frac{d}{2}+1\right)(\mu - \frac{1}{2}\omega^2 r^2)^{\frac{d}{2}}, \tag{8.63}$$

which is vanishing at the "cloud radius" $R = \sqrt{\frac{2\mu}{\omega^2}}$. This defines an IR cutoff for the validity of our EFT in the trap. Semi-classically, we can fix μ in terms of the number charge Q by imposing[7]:

$$Q = \langle Q|\hat{N}|Q\rangle = \int d^d x \langle Q|n(x)|Q\rangle = \frac{c_0 (2\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)\left(\frac{\mu}{\omega}\right)^d}{\Gamma(d+1)} \implies \frac{\mu}{\omega} \equiv \xi Q^{\frac{1}{d}} \tag{8.64}$$

The naive effective Lagrangian up to next-leading order is then:

$$\mathcal{L}_{eff} = c_0 X^{\frac{d}{2}+1} + c_1 \frac{X^{\frac{d}{2}+1}}{X^3}\partial_i X \partial^i X + c_2 \frac{X^{\frac{d}{2}+1}}{X^3}(\partial_i A_0)^2 + c_3 \frac{X^{\frac{d}{2}+1}}{X^2}\partial_i \partial^i A_0 + c_4 \frac{X^{\frac{d}{2}+1}}{X^2}(\partial_i \partial^i \chi)^2 \tag{8.65}$$

For $d = 2$ we have an additional parity violating operator at this order:

$$\mathcal{L}_{eff} \ni c_5 \varepsilon^{ij} \frac{(\partial_i A_0)(\partial_j X)}{X} \tag{8.66}$$

However, this is not the full set of constraints. It can be shown that imposing 'general coordinate invariance' will reduce the number of independent Wilson coefficients even further[176]. In particular there are the additional constraints:

$$c_2 = 0 \quad c_3 = -d^2 c_4 \tag{8.67}$$

Obtaining these from the coset construction would require additionally gauging the space-time symmetries [184]. The requirement of gauging the space-time symmetries is expected as a consequence of the number operator being part of the spacetime symmetry algebra and the fact that the number symmetry has been gauged. We leave this refinement for future work. For reasons that will become clear in the next section it is not necessary to work beyond this order in

[7]This is equivalent to fixing Q by differentiating the free energy given by the action

the derivative expansion.

8.6 Operator Dimensions

8.6.1 Ground State Energy & Scaling of Operator Dimension

The ground state energy is readily computed by a Euclidean path integral, in the infinite Euclidean time separation, the path integral projects out the ground state, from which one can read off the ground state energy. A nice pedagogical example of this technique can be found in [169] in context of fast spinning rigid rotor. On the other hand, from the state operator correspondence, we know that the ground state energy translated to dimension of the corresponding operator. Thus, equipped with the effective Lagrangian (8.65) obtained, the operator dimensions can be calculated via the path integral (8.1):

$$\lim_{T\to\infty} \langle Q|e^{-H_\omega T}|Q\rangle \sim e^{-S_{eff}[\chi_{cl}]-\mu\int d^D x\, n(x)} \sim e^{-\Delta_Q \omega T}\,, \tag{8.68}$$

where to leading order we have

$$-S_{eff}[\chi_{cl}] = c_0\Omega_d T\int_0^R dr\, r^{d-1}\left(\mu-\frac{1}{2}\omega^2 r^2\right)^{\frac{d}{2}+1} = c_0\frac{(2\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}{\Gamma(d+2)}\left(\frac{\mu}{\omega}\right)^{d+1}\omega T\,. \tag{8.69}$$

Here, Ω_d is the volume factor. Combining the results of (8.69) and (8.64) then gives the leading order operator dimension:

$$\Delta_Q = \frac{\mu}{\omega}Q - \left(-\frac{S_{eff}}{\omega T}\right) = \frac{d}{d+1}\xi Q^{1+\frac{1}{d}}\,. \tag{8.70}$$

This predicts $\Delta_Q \sim Q^{\frac{3}{2}}$ in $d=2$ and $\Delta_Q \sim Q^{\frac{4}{3}}$ in $d=3$, as in the relativistic case. That these leading order results are finite implies we can trust the EFT prediction. In general, however, the ground state energy in the trap is an infrared (IR) sensitive quantity. This becomes apparent at higher orders in the derivative expansion.

For example, we consider the case of $d=2$. The simplest operator at next leading order is (8.49). To analyze its contribution, define the distance from the cloud s as $r=R-s$. Its contribution to the energy, and hence the operator dimension via (8.69), would go like:

$$\int \mathrm{d}^3x\, \frac{\partial_i X \partial^i X}{X} \sim \int_0^R \mathrm{d}r\, r \frac{\omega^4 r^2}{\mu - \frac{1}{2}\omega^2 r^2} \sim \mu \int \mathrm{d}s\, \frac{1}{s}\,, \tag{8.71}$$

which is log divergent for small s, close to the edge. For $d = 3$, noticed in reference [176], a divergence first appears at next-next leading order associated with the operator:

$$\det E (\nabla_i \sigma \nabla^i \sigma)^2 \propto \frac{(\partial_i X \partial^i X)^2}{X^{\frac{7}{2}}}\,. \tag{8.72}$$

This leads to a power-law divergence, implying an even greater sensitivity to IR physics compared to $d = 2$. Ultimately these divergences originate from the breakdown of our EFT as the superfluid gets less dense. This occurs in a small region before the edge of the cloud at radius $R^* \equiv R - \delta$ where δ is roughly the width of this region. Following [176], we can estimate the size of this region as follows. One interpretation of (8.63) is that the chemical potential is now effectively space dependent. At the cutoff radius R^*, there is then an "effective chemical potential"

$$\mu(r) \equiv \mu - \frac{1}{2}\omega^2 r^2\,, \quad \mu_{eff} \equiv \mu(r = R^*) = \frac{1}{2}\delta(2R - \delta)\omega^2 \approx R\omega^2\delta\,. \tag{8.73}$$

There is a length scale set by μ_{eff} which controls the EFT expansion parameter in this region. Once that length is comparable to the distance δ itself we cannot claim to control the calculation semi-classically. Using (8.73) this gives the estimate scaling:

$$\delta \sim \sqrt{\frac{1}{\mu_{eff}}} \implies \delta \sim \frac{1}{(\omega^2\mu)^{\frac{1}{6}}} \tag{8.74}$$

We can estimate the contribution of this region to the energy by cutting off the divergent integrals at R^*. For $d = 2$ the effective action contains a term:

$$-S_{eff} \ni c_1 (2\pi) T \int_0^{R^*} dr\, r \frac{\omega^4 r^2}{\mu - \frac{1}{2}\omega^2 r^2} = 4\pi T \mu c_1 \left(\frac{13}{8} - \log\left[\frac{2\mu}{\mu_{eff}}\right]\right) + \cdots \tag{8.75}$$

where the $\cdots$ terms vanish as $\delta \to 0$

Substituting the relations (8.64) and (8.74) gives:

$$\Delta_Q \ni -4\pi\xi Q^{\frac{1}{2}} c_1 \left(\frac{13}{8} - \frac{1}{2}\log 2 - \frac{1}{3}\log Q - \frac{2}{3}\log \xi \right) \tag{8.76}$$

Changing the cutoff relation (8.74) by a factor can then change the $O(Q^{\frac{1}{2}})$ contribution, but not the logarithmic divergence which is universal. This translates to an uncertainty of order $O(Q^{\frac{1}{2}})$ in the operator dimension in $d = 2$. A similar analysis[176] for $d = 3$ and (8.72) translates to uncertainty of order $O(Q^{\frac{5}{9}})$.

Unlike $d = 2$, the operator (8.49) gives a finite correction to leading order scaling of dimension of operator in $d = 3$. This can be found by figuring out the contibution to S_{eff} [see Eq. (8.65)]

$$-S_{eff} \ni c_1 \int \mathrm{d}\tau^E \int_0^R \mathrm{d}r\, 4\pi r^2 \left(\frac{\omega^4 r^2}{\sqrt{\mu - \frac{1}{2}\omega^2 r^2}} \right) = c_1 (3\sqrt{2}\pi^2) \left(\frac{\mu}{\omega}\right)^2 \omega T \tag{8.77}$$

Similar contribution[8] comes from (8.51):

$$-S_{eff} \ni c_3 \int \mathrm{d}\tau^E \int_0^R \mathrm{d}r\, 4\pi r^2 (\omega^2) \left(\mu - \tfrac{1}{2}\omega^2 r^2\right)^{\frac{1}{2}} = c_3 \left(\frac{3\pi^2}{\sqrt{2}}\right) \left(\frac{\mu}{\omega}\right)^2 \omega T \tag{8.78}$$

To summarize, using (8.70), we have

$$\Delta_Q = \frac{3}{4}\left(\xi Q^{4/3}\right) - \left(c_1 + \frac{c_3}{2}\right)(3\sqrt{2}\pi^2)\xi^2 Q^{2/3} + O(Q^{\frac{5}{9}}) \quad \text{for } d = 3\,, \tag{8.79}$$

$$\Delta_Q = \frac{2}{3}\left(\xi Q^{3/2}\right) + c_1 \frac{4\pi}{3}\xi \left(Q^{\frac{1}{2}} \log Q\right) + O\left(Q^{\frac{1}{2}}\right) \quad \text{for } d = 2\,. \tag{8.80}$$

The Eq. (8.70), (8.79) and (8.80) constitute the main findings of this subsection.

8.6.2 Excited State Spectrum

We can also analyze the low energy excitations above the ground state. These correspond to low lying operators in the spectrum at large charge. To compute their dimension, we expand the leading action (8.44) to quadratic order in fluctuations π about the semi-classical saddle,

[8] Contribution should have come from (8.50) as well, but as we mentioned earlier, $c_2 = 0$ [176].

$\chi = \mu t + \pi$. The spectrum of π can then be found by linearizing the equation of motion (8.59):

$$\ddot{\pi} - \frac{2}{d}\left(\mu - \frac{1}{2}\omega^2 r^2\right)\partial^2\pi + \omega^2 \boldsymbol{r}\cdot\boldsymbol{\partial}\pi = 0 \tag{8.81}$$

Expanding $\pi(t,x) = e^{\mathbf{i}\varepsilon t} f(r) Y_\ell$ where Y_ℓ is a spherical harmonic, one can show (8.81) reduces to a hypergeometric equation. Details can be found in Appendix A. The dispersion relation is given by:

$$\varepsilon(n,\ell) = \pm\omega\left(\frac{4}{d}n^2 + 4n + \frac{4}{d}\ell n - \frac{4}{d}n + \ell\right)^{\frac{1}{2}} \tag{8.82}$$

where ℓ is the angular momentum and n is a non-negative integer. In the NRCFT state-operator correspondence, there are two different operators which generate descendants. In the Galilean frame, these are the operators $\boldsymbol{P}$ and H. While $\boldsymbol{P}$ raises the dimension by 1 and carries angular momentum, acting by H raises the dimension by 2 and carries no angular momentum. In the oscillator frame, this corresponds to:

$$\boldsymbol{P}_\pm = \frac{1}{\sqrt{2\omega}}\boldsymbol{P} \pm \mathbf{i}\sqrt{\frac{\omega}{2}}\boldsymbol{K} \quad L_\pm = \frac{1}{2}\left(\frac{1}{\omega}H - \omega C \pm \mathbf{i}D\right) \tag{8.83}$$

which then satisfy

$$[H_\omega, \boldsymbol{P}_\pm] = \pm\omega\boldsymbol{P}_\pm \quad [H_\omega, L_\pm] = \pm 2\omega L_\pm \tag{8.84}$$

One can check by equation (8.82) that $\varepsilon(n=0,\ell=1) = \pm\omega$ and $\varepsilon(n=1,\ell=0) = \pm 2\omega$. This allows us to identify these Goldstone modes with the descendant operators in (8.83) as $\pi_{(n=0,\ell=1)} \sim P_\pm$ and $\pi_{(n=1,\ell=0)} \sim L_\pm$. The other modes generate distinct primaries and descendants, including higher spin. We remark that in a strict sense, the above is the leading order result for the difference in dimensions between low-lying operators in this sector and the dimension of the ground state found in the previous section. It is also subject to corrections suppressed in $1/Q$ from subleading operators and loop effects.

8.7 Correlation Functions

In a relativistic CFT, the form of two and three point correlators is entirely fixed by symmetry. However, the four-point function depends on two conformally invariant cross ratios of the coordinates. The Schrödinger symmetry is less constraining, as there exists an invariant cross ratio even for a three-point function. This implies only the two-point functions of (number) charged operators is completely determined by symmetry.

8.7.1 Two Point Function

Following [169], we start with analyzing two point function. In path integral approach, when the in and out states are well separated in time, we have

$$\langle \Phi_Q, \tau_2 | e^{-H_\omega(\tau_2^{(E)}-\tau_1^{(E)})} | \Phi_Q, \tau_1 \rangle = e^{-\Delta_O(\tau_2^{(E)}-\tau_1^{(E)})} \tag{8.85}$$

where $\tau^{(E)}$ is the Euclideanized oscillator time. This is obtained from τ by doing Wick rotation i.e. $\tau^{(E)} = \mathbf{i}\tau$. This is evidently consistent with (G.13) upon doing the Wick rotation and taking $(\tau_2^{(E)} - \tau_1^{(E)}) \to \infty$. One subtle remark is in order: the Hamiltonian H_ω generates the time (τ) translation in oscillator frame. Thus the states prepared by path integration corresponds to operators in oscillator frame.

8.7.2 Three Point Function

We consider the matrix element that defines the simplest charged[9] three-point function

$$\langle \Phi_{Q+q} | \phi_q(y) | \Phi_Q \rangle \tag{8.86}$$

where ϕ_q is a light charged scalar primary with charge q and both of Φ_Q and Φ_{Q+q} has $O(1)$ dimension, given by Δ_Q and Δ_{Q+q}. By assumption, ϕ_q transforms in a linear representation R of the unbroken rotation group. To enable calculation in our EFT, we can extend this to a linear

[9]The additional charge of $\langle \Phi|$ is required for the correlator to be overall neutral and therefore non-vanishing.

representation of the full Schroedinger group using the Goldstone fields. In what follows, we take ϕ_q as the "dressed" operator[169]:

$$\phi_q(y) = R\left[e^{\mathbf{i}\boldsymbol{K}\cdot\boldsymbol{\eta}}e^{-\mathbf{i}\lambda C}e^{-\mathbf{i}\sigma D}e^{\mathbf{i}\chi N}\right]\hat{\phi}_q \tag{8.87}$$

where, by the assumption of ϕ_q being a scalar primary, is trivially acted on by $\boldsymbol{K}$ and C. This, combined with (8.37) gives

$$\phi_q = c_q X^{\frac{\Delta_\phi}{2}} e^{\mathbf{i}\chi q} \tag{8.88}$$

where c_q is a constant, which depends on UV physics. Upon evaluating (8.86) semi-classically about the saddle we found before, the leading order result for the correlator comes out to be:

$$\begin{aligned}\langle\Phi_{Q+q}(\tau_2)|\phi_q(\tau,\boldsymbol{y})|\Phi_Q(\tau_1)\rangle &= c_q\left(\mu-\frac{1}{2}m\omega^2y^2\right)^{\frac{\Delta_\phi}{2}} e^{\mathbf{i}\mu q(\tau-\tau_2)}e^{-\mathbf{i}\Delta_Q(\tau_2-\tau_1)} \\ &= c_q\mu^{\frac{\Delta_\phi}{2}}\left(1-\frac{y^2}{R^2}\right)^{\frac{\Delta_\phi}{2}} e^{\mu q\tau^{(E)}}e^{\omega\left(-\Delta_{Q+q}\tau_2^{(E)}+\Delta_Q\tau_1^{(E)}\right)}\end{aligned} \tag{8.89}$$

where we have used the following identity, which can be derived using the leading order operator dimension (8.70) and (8.64):

$$\frac{\Delta_{Q+q}-\Delta_Q}{q} = \alpha_0\left(1+\frac{1}{d}\right)Q^{\frac{1}{d}} + O\left(\frac{1}{Q}\right) \approx \frac{\partial\Delta_Q}{\partial Q} = \frac{\mu}{\omega} \tag{8.90}$$

as expected since μ is a chemical potential and $\omega\Delta_Q$ is the energy. We note that the operator insertion should be away from the edge of the cloud $|y-R| \gg \delta$, where δ is the cut-off imposed to keep the divergences coming from the $y \to R$ limit at bay.

Now we use (the details can be found in appendix [G.2.1])

$$\begin{aligned}\lim_{\tau_2^{(E)}\to\infty}\frac{1}{(1+\omega^2t_2^2)^{\Delta_{Q+q}/2}}\exp\left(-\omega\Delta_{Q+q}\tau_2^{(E)}\right) &= 2^{-\Delta_{Q+q}}\omega^{\Delta_{Q+q}/2}, \\ \lim_{\tau_1^{(E)}\to-\infty}\frac{1}{(1+\omega^2t_1^2)^{\Delta_Q/2}}\exp\left(\omega\Delta_Q\tau_1^{(E)}\right) &= 2^{-\Delta_Q}\omega^{\Delta_Q/2},\end{aligned}$$

to write down the correlator in terms of operators in Galilean frame (we repeat that the path intergral in oscillator frame prepares a state corresponding to operator in oscillator frame):

$$\langle \Phi_{Q+q}(\mathbf{i}/\omega)|\phi_q(\tau,\mathbf{y})|\Phi_Q(-\mathbf{i}/\omega)\rangle = c_q \mu^{\frac{\Delta_\phi}{2}} \left(1-\frac{y^2}{R^2}\right)^{\frac{\Delta_\phi}{2}} e^{\mu q \tau^{(E)}} 2^{-\Delta_Q-\Delta_{Q+q}} \omega^{(\Delta_Q+\Delta_{Q+q})/2}. \tag{8.91}$$

This can be matched onto the three point function, which is constrained by Schrödinger algebra:

$$\langle \Phi_{Q+q}|\phi_q(\tau,\mathbf{y})|\Phi_Q\rangle = F(v)\exp\left(\frac{q}{2}\omega y^2\right)(2)^{\Delta_\phi}\left(\frac{\mathbf{i}\omega}{2}\right)^{\frac{\Delta}{2}} e^{-i\omega(\Delta_Q-\Delta_{Q+q})\tau}. \tag{8.92}$$

The appendix [G.2.2] has the necessary details. Now, upon comparing (8.92) and (8.91), we deduce the universal behavior of $F(v)$ in the large charge sector:

$$F(v=\mathbf{i}\omega y^2) \propto Q^{\frac{\Delta_\phi}{2d}}\left(1-\frac{\omega y^2}{2\xi}Q^{-1/d}\right)^{\frac{\Delta_\phi}{2}} e^{-\frac{1}{2}q\omega y^2} \tag{8.93}$$

which can be rewritten as following, using (8.44):

$$F(v=\mathbf{i}\omega y^2) \propto \Delta_Q^{\frac{\Delta_\phi}{2(d+1)}}\left(1-\frac{\omega y^2}{2\xi}(\tfrac{d+1}{d\xi}\Delta_Q)^{-\frac{1}{d+1}}\right)^{\frac{\Delta_\phi}{2}} e^{-\frac{1}{2}q\omega y^2} \tag{8.94}$$

The (8.93) and (8.94) are the main results of this subsection. This shows the universal scaling behavior of the structure function F in the large charge sector.

8.8 Conclusions and Future Directions

We have studied the large charge (Q) sector of theories invariant under Schrödinger group. We have employed coset construction to write down an effecive field theory (EFT) describing the large Q sector in any arbitrary dimension $d \geq 2$ assuming superfluidity and rotational invariance.

The effective Lagrangian is given by

$$\mathcal{L}_{eff} = c_0 X^{\frac{d}{2}+1} + c_1 \frac{X^{\frac{d}{2}+1}}{X^3}\partial_i X \partial^i X + c_2 \frac{X^{\frac{d}{2}+1}}{X^3}(\partial_i A_0)^2 + c_3 \frac{X^{\frac{d}{2}+1}}{X^2}\partial_i \partial^i A_0 + c_4 \frac{X^{\frac{d}{2}+1}}{X^2}(\partial_i \partial^i \chi)^2$$

where $X = \partial_t \chi - A_0 - \frac{1}{2}\partial_i \chi \partial^i \chi$ and χ is the Goldstone excitation of the superfluid ground state. We emphasize that the general co-ordinate invariance, as discussed in [176] will put more constraints on the Wilson coefficients, we leave that as a future project. The EFT is then studied perturbatively as an expansion in $1/Q$. This is to be contrasted with the EFT written down in [176]. While EFT in [176] is controlled by small momentum parameter, ours is controlled by $1/Q$ expansion, which enables us to probe and derive universal results and scaling behaviors in large Q sector. In particular, when Q is very large, we find the scaling behavior of operator dimension with charge, consistent with that found very recently in [177]. We also find that in the large charge sector, structure function of three point correlator has a universal behavior. Last but not the least we derived the dispersion relation for the low energy excitation over this state with large Q and identify the two different kind of descendents as two different modes of excitations. A summary of the results can be found in the introduction.

The theory of conformal, and even superconformal, anyons has been studied before in great detail [185, 186, 59, 187]. In these systems there exists a simple n-particle operator $O = (\Phi^\dagger)^n$ whose dimension is given as

$$\Delta_O = n + n(n-1)\theta \tag{8.95}$$

where θ is the statistics parameter that arises from the Chern-Simons term of level k as $\theta = \frac{1}{2k}$ for bosonic theories. For large k relative to n, close to the bosonic limit, this is known to be the ground state in the trap. It is known as the "linear solution" in the literature due to the linear dependence on θ. For the superconformal theories it is a BPS operator and the dimension (8.95) is exact. A state corresponding to such an operator is not a superfluid and our theory cannot capture the physics of the system in that regime. However, it is known there is a level crossing for smaller k where the ground state corresponds to an operator whose dimension is not protected by the BPS bound. For those operators the classical dimension scales as $n^{\frac{3}{2}}$, in agreement with our results. We are then led to believe the effective field theory we've constructed may apply to anyon NRCFTs in that regime.

Another family of NRCFTs can be defined by the holographic constructions of McGreevy, Balasubramanian[61] and Son[188]. It would be interesting to study these on the gravitational side in the large charge limit, as there might exist a regime where both the EFT and gravity descriptions are valid. The analog of this for the relativistic case was carried out recently[189].

One can envision to extend our results in several ways. One possible extension of these results would be to study operators with large spin as well as charge. If the superfluid EFT remains valid, for sufficiently large spin, one naively expects such operators correspond to vortex configurations in the trap. This was studied in CFT_3, where multiple distinct scaling regimes were shown to exist [190]. Moreover, one can generalize these results to NRCFTs with a larger internal global symmetry group or study systems where the symmetry breaking pattern is different. Potentially interesting examples include "chiral" superfluids [191], where the rotational symmetry is additionally broken by the superfluid order parameter, or the vortex lattice [192] where the translation symmetry is spontaneously broken.

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Chapter 9

The Spinful Large Charge Sector of Non-Relativistic CFTs: From Phonons to Vortex Crystals

We study operators in Schrödinger invariant field theories (non-relativistic conformal field theories or NRCFTs) with large charge (particle number) and spin. Via the state-operator correspondence for NRCFTs, such operators correspond to states of a superfluid in a harmonic trap with phonons or vortices. Using the effective field theory of the Goldstone mode, we compute the dimensions of operators to leading order in the angular momentum L and charge Q. We find a diverse set of scaling behaviors for NRCFTs in both $d = 2$ and $d = 3$ spatial dimensions. These results apply to theories with a superfluid phase, such as unitary fermions or critical anyon systems.

9.1 Introduction and Summary

Superfluid states of matter are one of most fundamental examples of spontaneous symmetry breaking and appear in countless systems from Helium-4 [193, 194, 195, 196] to neutron stars [197]. Superfluidity is also a possibility for finite density states of scale invariant critical systems [198]. Recently this observation has been used to perform explicit calculations of relativistic conformal field theory (CFT) data, despite strong coupling[167, 168, 169, 170, 171, 199]. The key idea behind this is the fact that the large charge operators of the CFT correspond to finite density states on the sphere, which spontaneously break the conformal invariance and $U(1)$ corresponding to the charge. Superfluid phenomenology then becomes relevant for describing the large charge sectors of these CFTs. For example, another hallmark of superfluidity is the formation of vortices upon insertion of angular momentum. Therefore states with vortices correspond to large charge operators with spin, and calculating the energy of these vortices reveal the spinning operator spectrum in CFT [190].

However, many interesting critical systems do not possess Lorentz symmetry. This includes ultracold fermi gases at "unitarity", where observation of vortex lattices is perhaps the most dramatic evidence for a superfluid ground-state in a system which exhibits an emergent scale invariance[200]. At this critical point the system has a non-relativistic conformal symmetry, or Schrödinger symmetry. This symmetry algebra plays a pivotal role in understanding numerous physical systems[1]. Examples include the aforementioned "fermions at unitarity"[1, 2], as well as systems comprised of deuterons [3, 4], ^{133}Cs[6], ^{85}Rb [5],^{39}K [7], and various spin chain models [95]. There has been significant progress in understanding the consequences of Schrödinger symmetry and its realization in field theory.[94, 60, 59, 173, 99, 137] These non-relativistic conformal field theories (NRCFTs) admit a state-operator correspondence akin to their relativistic cousins. Operators with "particle number" charge are related to states in a harmonic potential.[174] This has been exploited to calculate the energies of few-body quantum mechanics systems in a harmonic trap. This correspondence also implies a way that the spectrum of NRCFTs can be determined. The operators with large charge correspond to finite density states in the trap. These states of matter sometimes admit a simple effective field theory description, enabling semi-classical calculations controlled in the large charge limit [201, 177].

The simplest and most physically relevant possibility is that of a superfluid ground-state,

[1] It is important to mention that Schrödinger symmetry is not simply the non-relativistic limit of the conformal symmetry but rather an entirely distinct algebra [98].

which is the situation we will explore here.[2] Extending upon the results of [190, 201], we study NRCFT operators which have both large charge and spin. Such operators correspond to either phonon or vortex excitations of the superfluid. We then compute the leading order scaling of their dimensions $\Delta_{Q,L}$ as functions of their angular momentum L and number charge Q and find a diverse range of behaviors in $d = 2$ and $d = 3$.

Trailer of the Results:

We compute the leading scaling dimension $\Delta_{Q,L}$ of spinning operators of a non-relativistic conformal field theory as a function of $U(1)$ charge Q and angular momentum L in the large charge limit. The answers are determined up to a single Wilson coefficient c_0 in the EFT description. We leverage the state operator correspondence to arrive at the result that depending on the range of angular momentum, the spining operators correspond to different excitation modes of the superfluid. For a smaller range of angular momentum, we find that they correspond to phonon with angular momentum L. As we increase the angular momentum, we pass through a regime where a single vortex becomes energetically favorable. If we further increase the angular momentum, multiple vortices develop and the superfluid exhibits an effective "rigid body motion" where we can neglect the discrete nature of the vortices.

[2] It should be emphasized that this is not the only possibility. Ultimately the question "Given this NRCFT, what state of matter describes its large charge sector?" depends on the NRCFT, which we treat as UV physics. However we expect our results to be valid for a wide set of NRCFTs, including some of physical relevance such as unitary fermions [200].

In $d=2$, the leading behavior has 3 regimes and is given as follows:

$$d=2 \quad \Delta_{Q,L} = \begin{cases} \sqrt{L}+\Delta_Q & 0<L\leq Q^{1/3} \\ \\ \sqrt{\frac{c_0\pi}{2}}\sqrt{L}\log L+\Delta_Q & Q^{1/3}<L\leq Q \\ \\ \sqrt{\frac{9c_0\pi}{2}}\left(\frac{L^2}{Q^{3/2}}\right)+\Delta_Q & Q<L<Q^{3/2} \end{cases} \tag{9.1}$$

where $\Delta_Q = \frac{2}{3}\left(\frac{1}{\sqrt{2\pi c_0}}\right)Q^{3/2}$ is the contribution from ground state energy in $d=2$.

In $d=3$ dimensions, we have 4 regimes, given by:

$$d=3 \quad \Delta_{Q,L} = \begin{cases} \sqrt{L}+\Delta_Q & 0<L\leq Q^{2/9} \\ \\ \alpha\left(\frac{L}{Q^{1/9}}\right)+\Delta_Q & Q^{2/9}<L\leq Q^{1/3} \\ \\ \left(\frac{5\pi^4 c_0}{8\sqrt{2}}\right)^{1/3} L^{2/3}\log L+\Delta_Q & Q^{1/3}<L\leq Q \\ \\ \frac{1024}{25}\left(\frac{32c_0^2}{25\pi^4}\right)^{1/6}\left(\frac{L^2}{Q^{4/3}}\right)+\Delta_Q & Q<L<Q^{4/3} \end{cases} \tag{9.2}$$

where $\Delta_Q = \frac{3}{2}\frac{1}{\sqrt{2\pi}}\left(\frac{6}{15\sqrt{\pi}c_0}\right)^{1/3} Q^{4/3}$ is the contribution from ground state energy in $d=3$ and α is an undetermined $O(1)$ coefficient. We make two remarks at this point. The first one is that while for $d=2$, the transition happens from a single phonon regime to vortex regime at $L\sim Q^{1/3}$, for $d=3$, there is a regime $Q^{2/9}\leq L\leq Q^{3/9}$, where neither vortex nor the single phonon solution gives the lowest energy. It is a cross-over describing the physics of a vortex string forming near the boundary of the trap where our EFT is strongly coupled. The only well defined configuration in this angular momentum regime contains multiple phonons, and we determine the scaling from that. The second remark is that the EFT description breaks down whenever $\Delta_{Q,L} - \Delta_Q \sim \Delta_Q$, so we can not probe operators with larger angular momentum with this method.

The rest of the paper is organized as follows. We briefly review the superfluid hydrodynamics and large charge NRCFT in section 9.2. The section 9.3 details out the contribution coming from phonons and derives the regime where it is energetically favorable to have them. Subsequently, we discuss the single vortex in $d = 2$ and $d = 3$ in section 9.4. The multi-vortex and rigid body motion is elucidated in section 9.5 followed by a brief conclusion and future avenues to explore in section 9.6. Some of our results and validity regimes are more apparent in dual frame using particle-vortex duality which we elaborate on in appendix H.1. The appendix H.2 contains a contour integral useful for calculating interaction energy of multiple vortices in $d = 3$.

9.2 Superfluid Hydrodynamics and Large Charge NRCFT

In this section we briefly review the superfluid hydrodynamics in the Hamiltonian formalism, specialized to the case of a Schrödinger invariant system in a harmonic potential $A_0 = \frac{1}{2}\omega^2 r^2$. All of our results will be to leading order in the derivative expansion. For a more in-depth review of the formalism, we refer to [176, 201, 177].

The low-energy physics of a superfluid is determined by a single Goldstone field χ. The leading order Lagrangian determines the pressure of the system:

$$\mathcal{L} = c_0 X^{\frac{d+2}{2}} \equiv P(X) \qquad X \equiv \partial_0\chi - A_0 - \frac{1}{2}(\partial_i\chi)^2 \tag{9.3}$$

The number density and superfluid velocity are defined respectively as:

$$n = \frac{\partial L}{\partial \dot{\chi}} = c_0\left(\frac{d}{2}+1\right)X^{\frac{d}{2}} \qquad v_i = -\partial_i\chi \tag{9.4}$$

The action (9.3) has a $U(1)$ symmetry of $\chi \to \chi + c$ whose current can be written as:

$$j^{\mu} = \left(n, nv^{i}\right) \tag{9.5}$$

The Hamiltonian density comes out to be:

$$\mathcal{H} = n\dot{\chi} - \mathcal{L} = n\left(X + A_0 + \frac{1}{2}v^2\right) - P(X) \tag{9.6}$$

Now, using the thermodynamic relation $nX - P(X) \equiv \varepsilon(n)$: we can simplify (9.6) and express the Hamiltonian as:

$$H = \int \mathrm{d}^d x\, \mathcal{H} \quad \mathcal{H} = \frac{1}{2}nv^2 + \varepsilon(n) + nA_0 \tag{9.7}$$

.

Note that the presence of the harmonic trap implies the density is non-uniform and vanishes at radius $R_{TF} = \sqrt{\frac{2\mu}{\omega^2}}$. For most values of r the density is large and varies slowly compared to the UV length scale $\frac{1}{\sqrt{\mu}}$. However, the large charge expansion begins to break down at $R^* = R_{TF} - \delta$ where $\delta \sim \frac{1}{(\omega^2\mu)^{\frac{1}{6}}}$[201, 176]. There is a boundary layer of thickness δ where the superfluid effective field theory (EFT) cannot be trusted as it is no longer weakly coupled. At leading order in the derivative expansion this does not effect the observables but leads to divergences at higher orders.[3]

Given this set up, the ground-state at finite density corresponds to the classical solution of $\chi_{cl} = \mu t$. The number charge of this configuration is determined from μ by:

$$Q \equiv \int \mathrm{d}^d x\, n_{cl}(x) = c_0\left(\frac{d}{2}+1\right)\int \mathrm{d}^d x\, (\mu - A_0)^{\frac{d}{2}} = \frac{1}{\xi}\left(\frac{\mu}{\omega}\right)^d \tag{9.8}$$

where $\frac{1}{c_0} = \frac{\Gamma(\frac{d}{2}+2)}{\Gamma(d+1)}(2\pi\xi^2)^{\frac{d}{2}}$. We can then compute the ground-state energy as function of Q

[3]These are UV divergences which can be canceled by counter-terms localized at this edge, as suggested by Simeon Hellerman in a private communication.

using (9.7):

$$E_Q = \int \mathrm{d}^d x \, [\varepsilon(n_{cl}) + n_{cl}A_0] = \omega\xi \left(\frac{d}{d+1}\right) Q^{\frac{d+1}{d}} \tag{9.9}$$

Via the state-operator correspondence of NRCFTs, this semi-classical calculation determines the dimension of a charged scalar operator to leading order in Q as $\Delta_Q = \frac{E_Q}{\omega}$. In particular, we have obtained [201]:

$$\Delta_Q = \begin{cases} \frac{2}{3}\xi Q^{3/2} & \text{for} \quad d=2 \\ \frac{3}{4}\xi Q^{4/3} & \text{for} \quad d=3 \end{cases} \tag{9.10}$$

In this work, we'll be interested in excited state configurations which carry some angular momentum. These will correspond to spinful operators in the large charge sector of the NRCFTs which the superfluid EFT describes. The simplest of these excitations are phonons; smooth solutions of the equation of motion with $\chi_{cl} = \mu t + \pi$. Expanding π in modes $\pi_{n,\ell}$, the Hamiltonian can be written to leading order in the derivative expansion as:

$$H = H_0 + \sum_{n,\ell} \omega(n,\ell)\pi^\dagger_{n,\ell}\pi_{n,\ell} + \cdots \tag{9.11}$$

where $\omega(n,\ell)$ is the dispersion relation for phonons:

$$\omega(n,\ell) = \omega\left(\frac{4}{d}n^2 + \left(4 - \frac{4}{d}\right)n + \frac{4}{d}n\ell + \ell\right)^{\frac{1}{2}} \tag{9.12}$$

for n is a positive integer and ℓ is the total angular momentum. The phonon wavefunctions are given as $f_{n,\ell} \sim (\frac{r}{R_{TF}})^{\frac{\ell}{2}} G_{n,\ell}(r) Y_\ell$ where $G_{n,\ell}$ is a hypergeometric function and Y_ℓ is a spherical harmonic. A state with M phonon modes of $\{n=0, \ell=1\}$ can be identified as the descendant operator $\boldsymbol{\partial}^M O_Q$ with dimension $\Delta_Q + M$. Additionally, NRCFTs have another generator of descendants ∂_t which corresponds to the phonon with $\{n=1, \ell=0\}$. States that can be created by adding phonons with other values of n and ℓ correspond to distinct primaries [201].[4]

[4]They are primary as they are by construction annihilated by the lower operators K and C which correspond to $\pi_{n=0,\ell=1}$ and $\pi_{n=1,\ell=0}$ respectively.

The other configuration of a superfluid that can support angular momentum is a vortex, which gives rise to a singular velocity field of the condensate. This is a distinct semi-classical saddle point which is not simply related to the ground state. It must therefore correspond to a unique set of spinful charged operators present in all NRCFTs whose scalar large charge sector is described by the superfluid EFT.

These two excitations, phonons and vortices, are the configurations of the superfluid we know support angular momentum. In the rest of the paper we answer the question, what is the lowest energy configuration of the superfluid for a given angular momentum? By answering this and using the superfluid EFT defined above we compute the scaling behavior of operators carrying charge and angular momentum.

9.3 Phonons

The simplest excited state(s) with angular momentum are phonons. From the dispersion (9.12), we can see that the lowest energy configuration with angular momentum L is a single phonon with $n = 0$ and $\ell = L$. This is known as a "surface mode" as the wavefunction is nodeless and supported mostly at the end of the trap. The energy cost of this single phonon is given by[5]:

$$\Delta E = \omega L^{\frac{1}{2}} . \tag{9.13}$$

However the validity of (9.12) rests on the assumption that the phonon modes do not carry large amounts of momentum. In particular, the surface mode wavefunction has $f_\ell \sim (\frac{r}{R})^{\frac{\ell}{2}} Y_\ell$ which for large ℓ is increasingly concentrated at the edge of the trap. Once the support of the phonon wavefunction is mostly within the boundary region of thickness δ, we can no longer trust the solution or the dispersion (9.12). This occurs when $\frac{R_{TF}}{\ell}$ becomes comparable to δ [202]. This yields a maximum angular momentum for phonons: $\ell_{max} \sim Q^{\frac{2}{3d}}$.

[5]Note this is parametrically lower in energy than in the relativistic case studied in [190], as the phonon spectrum on the sphere is $\varepsilon(\ell) = \sqrt{\frac{1}{2}\ell(\ell+1)}$.

Thus we have the following scalings for operator dimensions:

$$d = 2 \quad \Delta_{Q,L} = L^{\frac{1}{2}} + \Delta_Q \qquad 0 < L \leq Q^{\frac{1}{3}} \tag{9.14}$$

$$d = 3 \quad \Delta_{Q,L} = L^{\frac{1}{2}} + \Delta_Q \qquad 0 < L \leq Q^{\frac{2}{9}} \tag{9.15}$$

where Δ_Q is the operator dimension determined from (9.9).

We can also consider multi-phonon configurations and ask ourselves whether it is energetically favorable to have a single phonon rather than multi phonon configuration, given total angular momentum. In order to answer this, we assume that phonon interactions are negligible, suppressed to leading order in the Q-expansion, so the energy and angular momentum of multiple phonons add linearly. In particular, suppose we have N_γ phonons, each carrying angular momentum ℓ. The energy and angular momentum to leading order is:

$$\Delta E = \omega N_\gamma \ell^{\frac{1}{2}} \qquad L = N_\gamma \ell \tag{9.16}$$

This tells us that for a given angular momentum L, it is energetically favourable to have a single phonon carrying the entire angular momentum rather than multiple phonons carrying it altogether.[6]

As we'll see below, naively a single phonon of $\ell = L$ would always be the most energeti-

[6]One can also arrive at the same conclusion by considering N_γ phonons, each carrying angular momentum $\boldsymbol{\ell_i}$. The energy and angular momentum to leading order is then given by:

$$\Delta E = \omega \sum_i |\boldsymbol{\ell}_i|^{\frac{1}{2}} \qquad L = \left|\sum_i \boldsymbol{\ell}_i\right| \tag{9.17}$$

We have

$$\Delta E = \omega \sum_i |\boldsymbol{\ell}_i|^{\frac{1}{2}} = \omega \sqrt{\sum |\boldsymbol{\ell}_i| + \sum \sqrt{|\boldsymbol{\ell}_i||\boldsymbol{\ell}_j|}} \geq \omega \sqrt{\left|\sum \boldsymbol{\ell}_i\right| + \sum \sqrt{|\boldsymbol{\ell}_i||\boldsymbol{\ell}_j|}} \geq \omega\sqrt{L} \tag{9.18}$$

Hence, the minimum value is obtained when all the $l_i = 0$ except one i.e. we land up with single phonon case. On the other hand, if all the $\boldsymbol{\ell}_i$'s are along same direction, then using Cauchy-Schwartz inequality, one can obtain $\Delta E \leq \omega (N_\gamma)^{1/2} \sqrt{L}$, which implies that the energy would be maximized if each phonon carries angular momentum of L/N_γ.

cally favorable configuration per angular momentum. However the cutoff of $\ell_{max} \sim Q^{\frac{2}{3d}}$ means we cannot trust this conclusion beyond $L = \ell_{max}$. Multi-phonon configurations are in principle valid for larger values of L.[7] The most energetically favorable of which has N_γ phonons with $\ell = \ell_{max}$, which gives the scaling:

$$\Delta E = \omega L \ell_{max}^{-1/2} \qquad \ell_{max} \sim Q^{\frac{2}{3d}} \tag{9.19}$$

where we cannot determine the dimensionless coefficient from ℓ_{max} as it depends on how we regulate the cutoff region of size δ. Nevertheless, the linear scaling in L means we can compare to other configurations such as vortices. In particular, we will arrive at the conclusion that whenever $L \geq Q^{1/3}$, the minimum energy configuration with a given angular momentum starts to be attained by vortex solutions.

For $d = 2$, the transition happens from a single phonon regime to vortex regime at $L \sim Q^{1/3}$, while for $d = 3$, there is a regime $Q^{2/9} \leq L \leq Q^{1/3}$, which is inaccessible by both the vortex string and the single-phonon configurations. The most energetically favorable configuration, consistent within the leading order EFT analysis, is therefore the multi-phonon configuration above with a macroscopic number of phonons $N_\gamma \sim Q^{\frac{1}{9}}$ at the upper bound $L \sim Q^{\frac{1}{3}}$.

This would imply the following scaling for the operator dimension:

$$d = 3 \quad \Delta_{Q,L} = \alpha L Q^{-1/9} + \Delta_Q \qquad Q^{2/9} < L \leq Q^{1/3} \tag{9.20}$$

where α is an unknown order one coefficient.

However the exact nature of this state appears to be related to UV physics of how a vortex string configuration forms from surface mode phonons in the boundary region of the condensate, which is inaccessible within our formalism. We therefore cannot give a full accounting of this regime of angular momentum. Beyond $L > Q^{\frac{1}{3}}$ we can be confident the lowest energy configuration is a vortex, as we'll now discuss.

[7] N_γ cannot be made arbitrarily large as the assumption that phonon interactions are suppressed breaks down.

9.4 Single Vortex in the Trap

A vortex is a configuration of the superfluid with a singular velocity field carrying angular momentum. The singular nature arises because of the relation (9.4) implying that v_i is necessarily irrotational except due to defects in the field χ; configurations where $\int_C d\chi = 2\pi s$ for some integer s. In $d = 2$ these are particle like excitations while in $d = 3$ they correspond to strings, these will be the dimensions we focus on in this work. In fact this language can be made precise via particle-vortex duality, where vortices are "charged" objects under some dual gauge field. Adapting this duality to the Schrödinger invariant superfluid has been done in Appendix H.1 but it is inessential for describing the leading order results.

The simplest configuration in the trap is a single static vortex for which the condensate order parameter changes by only 2π.[8] The approximate velocity profile v_i of such a configuration is:

$$v_i = \frac{\varepsilon^{ij}(r_j - R_j)}{(\boldsymbol{r} - \boldsymbol{R})^2} \tag{9.21}$$

where r is the radial coordinate in $d = 2$ or the axial coordinate in $d = 3$, and $\boldsymbol{R}$ is the location of the vortex and we assume that the vortex is streched along the z axis.

The presence of the vortex changes the semi-classical number density, making it singular at $r = R$. Before that point the density vanishes, implying a short distance cutoff for the superfluid EFT. This is the 'vortex core size' a whose scaling dimension we can determine as follows.

One interpretation of the non-uniform density (9.8) is that the effective chemical potential is distance dependent. In the presence of a vortex at $\boldsymbol{r} = \boldsymbol{R}$ it is given as:

$$\mu_{eff}(\boldsymbol{r}) \equiv \mu - \frac{1}{2}\omega^2 r^2 - \frac{1}{2}\frac{1}{|\boldsymbol{r} - \boldsymbol{R}|^2} \tag{9.22}$$

This determines a locally varying UV length scale $\frac{1}{\sqrt{\mu_{eff}}}$. The EFT, which is controlled in the

[8]This is in contrast to the CFT case where a minimum of two vortices are needed on the sphere to ensure compatibility with the Gauss law.

limit of large density, becomes strongly coupled at the length a when $a \sim \frac{1}{\sqrt{\mu_{eff}}}$. Solving this equation for a gives the scaling relations[9]

$$d=2 \quad a \sim \frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{1-\frac{R^2}{R_{TF}^2}}} \qquad d=3 \quad a \sim \frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{1-\frac{R^2}{R_{TF}^2}-\frac{z^2}{R_{TF}^2}}} \tag{9.23}$$

Near the center of the trap, a is on order the UV length scale $\frac{1}{\sqrt{\mu}}$. However as the vortex approaches the boundary of the trap, either in its placement $\boldsymbol{R}$ or along the length of the vortex string in $d=3$, the fact the density is depleted due to the trap implies the cutoff near the vortex string must happen sooner [203]. As mentioned previously, the EFT is already strongly coupled in the boundary region of size δ. Therefore the largest placements of the vortex we can confidently study have $R = R_{TF} - \delta$ where the core size scales as $a \sim \frac{1}{(\mu\omega^2)^{\frac{1}{3}}}$ which is still parametrically suppressed in μ.

Regulating this divergence as described above, the correction to the semi-classical number density due to the vortex is subleading in μ and therefore negligible for leading order results. This implies the dominant contribution to the energy of a vortex configuration comes from the kinetic energy of the velocity field.

The velocity field (9.21) does not define a stationary flow in the sense that $\partial_i(nv^i) \neq 0$ because of the inhomogeneity of the density. This inhomogenity will cause the vortex to precess in a circle [204]. However since the density varies slowly, as previously discussed, the correction to the velocity field due to this is suppressed in the large-charge expansion. Using particle-vortex duality, this is equivalent to the assumption that particle sourcing the gauge field in dual description has suppressed velocity, hence we are effectively dealing with an electrostatic scenario. The details are relegated to the appendix H.1, in particular, the discussion after (H.15).

We remark that in dual frame, the cloud boundary is like a conductor, hence the tangential electric field should be vanishing. This means the velocity field of the vortex should be such that

[9] This is an equivalent condition to cutting off the theory when the velocity field sourced by the vortex becomes comparable to the local speed of sound in the superfluid $c_s^2 \sim \frac{\partial P}{\partial n} \sim X$.

there is no radial outflow of particles out of the trap. Given this condition, one might worry that the velocity field above does not vanish at the boundary R_{TF}. However, since we require the normal component of the flow to vanish at the boundary i.e. $\hat{N}\cdot(n\boldsymbol{v}) = 0$ where $\boldsymbol{N}$ is a vector normal to the trap at boundary, the inhomogeneity of the superfluid comes to rescue and the condition is trivially met by the vanishing of the density $n(x)$ at R_{TF} [205]. [10]

In what follows, we will be evaluating the energy and angular momentum of vortex configurations in $d = 2$ and $d = 3$ spatial dimensions.

9.4.1 Single vortex in $d = 2$

Let's first work in $d = 2$ with the velocity field given by (9.21). The difference in energy between the vortex state and the ground state can then be computed from the kinetic energy of Hamiltonian (9.7) as:

$$\Delta E = \int \mathrm{d}^2x\, \frac{1}{2}nv^2 = c_0\mu \int \mathrm{d}^2x\, \left(1 - \frac{r^2}{R_{TF}^2}\right)\frac{1}{(\boldsymbol{r}-\boldsymbol{R})^2} \tag{9.24}$$

As mentioned, there is a divergence at $r = R$ which we will regulate by assuming a vortex core size of $a(R) \sim \frac{1}{\sqrt{\mu_{eff}}}$ where $\mu_{eff} = \mu\left(1 - \frac{R^2}{R_{TF}^2}\right)$. Evaluating the integral (9.24) gives:

$$\Delta E = 2c_0\pi\mu\left(1 - \frac{R^2}{R_{TF}^2}\right)\left[\log\left(\frac{R_{TF}}{2a(R)}\right) + \frac{1}{2}\log\left(1 - \frac{R^2}{R_{TF}^2}\right) - 1\right] + c_0\pi\mu + O(a) \tag{9.25}$$

We can also compute the angular momentum via the integral:

$$\boldsymbol{L} = \int \mathrm{d}^2x\, n\boldsymbol{v}\times\boldsymbol{r} \tag{9.26}$$

[10] This is generically known as a "soft boundary". Had we been dealing with homogenous fluid with non vanishing density at boundary, we ought to consider a mirror vortex configuration to ensure the imposition of $\hat{N}\cdot(n\boldsymbol{v}) = 0$, this is just like considering the mirror charge while solving for electric in the presence of a conductor. Regardless, such modifications to the velocity field in the boundary region of the inhomogenous condensate give suppressed corrections to our leading order results below.

For our configuration the angular momentum is entirely in the $\hat{z}$ direction with magnitude:

$$L = 4\pi c_0 \mu \int_R^{R_{TF}} \mathrm{d}r\, r \left(1 - \frac{r^2}{R_{TF}^2}\right) = 2\pi c_0 \frac{\mu^2}{\omega^2}\left(1 - \frac{R^2}{R_{TF}^2}\right)^2 \tag{9.27}$$

where we've used $\oint_r \boldsymbol{v} \cdot d\boldsymbol{\ell} = 2\pi$ for a circle centered at the origin of radius $r > R$, and otherwise vanishes.

As one can see, it is energetically favorable for the vortex to appear at the edge of the cloud $R \approx R_{TF}$. However we cannot trust the solution in the regime of low density near there for reasons previously discussed. Therefore the largest distance the vortex can be where we have confidence in the validity of the semi-classical approximation is $R^* = R_{TF} - \delta$. This gives a minimum angular momentum, of the vortex configuration $L_{min} \sim Q^{\frac{1}{3}}$. The largest value of the angular momentum occurs when the vortex is in the center at $R = 0$ with $L_{max} \sim Q$.

Combining these results gives the leading order expressions for the operator dimensions in terms of L and Q as:

$$d = 2 \quad \Delta_{Q,L} = \sqrt{\frac{c_0\pi}{2}}\sqrt{L}\log L + \Delta_Q \qquad Q^{\frac{1}{3}} < L \leq Q \tag{9.28}$$

9.4.2 Single vortex in $d = 3$

Let's consider the case of $d = 3$ now. The minimal energy excitation is a single vortex string. The string must necessarily break the spherical symmetry of the trap. We will consider the string being stretched along the z-axis, ensuring that all the angular momentum is $L = L_z$.[11]

The energy of the vortex string again comes from the kinetic energy and can be evaluated as:

$$\Delta E = \int \mathrm{d}^3x\, \frac{1}{2} n v^2 = \int_{-Z(R)}^{Z(R)} \mathrm{d}z\, T(z,R) \tag{9.29}$$

[11] A curved string will generically have to be longer in order to carry the same angular momentum, as parts of the velocity field it sources will cancel against each other. The longer strings will be energetically more expensive, making the straight line configuration energetically favorable to leading order.

where $T(z,R)$ is the tension of the string and $Z(R) = R_{TF}\sqrt{\left(1-\frac{R^2}{R_{TF}^2}\right)}$ defines the integration bound along the length of the string.

The tension can be computed via a similar integral in $d=2$ as:

$$\begin{aligned} T(z,R) &= \frac{1}{2}\int_0^{r(z)} \mathrm{d}r r \int_0^{2\pi} \mathrm{d}\phi \, n(r,z) \frac{1}{(\boldsymbol{r}-\boldsymbol{R})^2} \\ &= \pi n(R,z) \left[\log\left(\frac{r(z,R)}{a(z,R)}\right) - \log\left(1+\frac{r(z)}{r(z,R)}\right)\right] + \cdots \end{aligned} \tag{9.30}$$

where $\cdots$ refer to the non logarithimic pieces. Here $n(r,z) = \frac{5}{2}c_0\mu^{\frac{3}{2}}\left(1-\frac{1}{R_{TF}^2}(r^2+z^2)\right)^{\frac{3}{2}}$ is the number density, $r(z) = R_{TF}\sqrt{1-\frac{z^2}{R_{TF}^2}}$ is the radial (radius in cylindrical co-ordinate) size of the trap at a height z and $r(z,R) = R_{TF}\sqrt{1-\frac{z^2+R^2}{R_{TF}^2}}$. Integrating the leading logarithimic piece along the string length gives the energy:

$$\begin{aligned} \Delta E &= \int_{-Z(R)}^{Z(R)} \mathrm{d}z \, \pi n(R,z) \log\left(\frac{r(z,R)}{a(z,R)}\right) \\ &= \frac{15}{16}\pi^2 c_0 \mu^{3/2} R_{TF}\left(1-\frac{R^2}{R_{TF}^2}\right)^2 \left[\log\left(1-\frac{R^2}{R_{TF}^2}\right) + \log\left(R_{TF}\sqrt{\mu}\right)\right] \end{aligned} \tag{9.31}$$

Evaluating the angular momentum of this configuration is similar to $d=2$ and yields:

$$L = \int_{-Z(R)}^{Z(R)} \mathrm{d}z \int_R^{r(z)} \mathrm{d}r \, r \left[\frac{5}{2}c_0\mu^{\frac{3}{2}}\left(1-\frac{r^2+z^2}{R_{TF}^2}\right)^{\frac{3}{2}}\right] = \frac{5\pi c_0}{8\sqrt{2}}\left(\frac{\mu}{\omega}\right)^3\left(1-\frac{R^2}{R_{TF}^2}\right)^3 \tag{9.32}$$

Again the lowest allowed value of the angular momentum occurs for a vortex at $R^* = R_{TF} - \delta$ and scales as $L_{min} \sim Q^{\frac{1}{3}}$ while the maximum occurs at $R=0$ with $L_{max} \sim Q$.

Together these results imply the scaling:

$$d = 3 \quad \Delta_{Q,L} = \left(\frac{5\pi^4 c_0}{8\sqrt{2}}\right)^{1/3} L^{2/3}\log L + \Delta_Q \qquad Q^{\frac{1}{3}} < L \leq Q \tag{9.33}$$

This determines the leading order dimension for the operator which creates the vortex string but we can also study the spectrum of operators above it. For example, the presence of a vortex string along the $\hat{z}$-direction should split the phonon m degeneracy in (9.12). Treating this perturbatively, such a splitting is suppressed in the charge[12] Q [203].

Besides phonons, there are unique excitations of the vortex string related to displacements of position. These are known as "Kelvin modes" and they define another set of low-lying operators above the one which created the vortex string. These modes are basically the radial displacement of the vortex core from the original axis. For long wavelength modes $ka(z,R) \ll 1$ and in the regime where $z \ll R_{TF}$, we can effectively assume that density is uniform[13]. Under this assumption, followed by considering a situation where the amplitude of the displacement is small, we have the standard result quoted in superfluid literature i.e. $\omega(k) \approx \frac{1}{2}k^2 \log \frac{1}{|k|a(R)}$, where $a(R) = \frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{1-\frac{R^2}{R_{TF}^2}}}$ is the vortex core size via (9.23). We remark that the boundary conditions on the string should quantize $k \sim \frac{n}{R_{TF}}$, so there is an approximate continuum of such operators above the gap to create a single vortex string. The spacing of these modes and exact dimensions are only visible at higher orders in the Q expansion.

9.5 Multi-Vortex Profile

Consider a collection of N_v vortices at locations $\boldsymbol{R}_i$ with winding numbers s_i. The velocity field of such a contribution is additive and described by:

$$\boldsymbol{v} = \sum_i \boldsymbol{v}_i = \sum_i s_i \frac{\hat{z} \times (\boldsymbol{r} - \boldsymbol{R}_i)}{|\boldsymbol{r} - \boldsymbol{R}_i|^2} \implies \nabla \times \boldsymbol{v} = \sum_i s_i \delta(\boldsymbol{r} - \boldsymbol{R}_i) \tag{9.34}$$

Because the angular momentum is linear in the velocity field, this implies the total angular momentum of the system is given by the sum of the individual ones:

$$L = \sum_i L_i = \begin{cases} 2\pi c_0 \left(\frac{\mu}{\omega}\right)^2 \sum_i s_i \left(1 - \frac{R_i^2}{R_{TF}^2}\right)^2 & d = 2 \\ \frac{5\pi c_0}{8\sqrt{2}} \left(\frac{\mu}{\omega}\right)^3 \sum_i s_i \left(1 - \frac{R_i^2}{R_{TF}^2}\right)^3 & d = 3 \end{cases} \tag{9.35}$$

[12]One could also consider the energy of a vortex-phonon configuration. The "interaction energy" between the two is given as $\int \mathrm{d}^d x\, \boldsymbol{v}_{vortex} \cdot \boldsymbol{\partial}\pi$ which is also suppressed in the Q expansion.

[13]For work going beyond this approximation in non-uniform condensates, see [206].

For vortices far from the boundary, where $\frac{R_{TF}-R_i}{R_{TF}} \sim O(1)$ (as opposed to Q suppressed number), we have that $L_i \sim s_i Q$.

We can compute the energy of a generic multi-vortex configuration explicitly from this velocity field (9.34). The energy breaks up into single-vortex contributions and pair-wise interaction energies:

$$\Delta E = \frac{1}{2}\int \mathrm{d}^d x \, nv^2 = \sum_i E_i + \sum_{i\neq j}\sum_j E_{ij} \tag{9.36}$$

where the single vortex energy is already computed as

$$E_i = \frac{1}{2}\int \mathrm{d}^d x \, nv_i^2 = \begin{cases} \omega\sqrt{\frac{c_0\pi}{2}} s_i^2 \sqrt{L_i}\log L_i & d=2 \\ \omega\left(\frac{5\pi^4 c_0}{8\sqrt{2}}\right)^{1/3} s_i^2 L_i^{2/3}\log L_i & d=3 \end{cases} \tag{9.37}$$

and E_{ij} is the interaction energy given by:

$$E_{ij} = \int \mathrm{d}^d x \, nv_i \cdot v_j \tag{9.38}$$

In $d=2$ this integral evaluates to:

$$\begin{aligned} \frac{E_{ij}}{s_i s_j} &= \pi c_0 \mu \left(1 - \frac{\boldsymbol{R}_i\cdot\boldsymbol{R}_j}{R_{TF}^2}\right)\log\left(\frac{R_{TF}^4 + R_i^2 R_j^2 - 2R_{TF}^2 \boldsymbol{R}_i\cdot\boldsymbol{R}_j}{(\boldsymbol{R}_i - \boldsymbol{R}_j)^4}\right) \\ &- 2\pi c_0 \mu \frac{1}{R_{TF}^2}\left(R_i^2 + R_j^2 - R_{TF}^2\right) \\ &+ 2\pi c_0 \mu \frac{1}{R_{TF}^2}|\boldsymbol{R}_i \times \boldsymbol{R}_j| \arctan\left(\frac{|\boldsymbol{R}_i\times\boldsymbol{R}_j|}{R_{TF}^2 - \boldsymbol{R}_i\cdot\boldsymbol{R}_j}\right) \end{aligned} \tag{9.39}$$

where $\boldsymbol{R}_i$ and $\boldsymbol{R}_j$ are the positions of the vortex pair with $R_j > R_i$ assumed without loss of generality. To leading order in the charge and small vortex separation this simplifies to:

$$E_{ij} \sim s_i s_j \mu \log\frac{R_{TF}}{|\boldsymbol{R}_i - \boldsymbol{R}_j|} + \ldots \tag{9.40}$$

This piece is the result of the singular nature of the vortices and describes their interaction. The analogous result of (9.39) for $d=3$ is not analytically tractable, but the leading interaction piece in the charge and small vortex separation is given by:

$$E_{ij} \sim s_i s_j \mu^{\frac{3}{2}} R_{TF} \log \frac{R_{TF}}{|\boldsymbol{R}_i - \boldsymbol{R}_j|} + \cdots \tag{9.41}$$

One can extract several physical features of the multivortex profile using the expressions above for the energy. First of all, the minimum energy configurations per angular momentum will have $s_i = 1$ for every vortex as the energy scales quadratically in the charge but the angular momentum only scales linearly. The angular momentum for the entire configuration then scales as $L \sim N_v Q$ assuming $\frac{R_{TF}-R_i}{R_{TF}} \sim O(1)$. Secondly, we remark that the logarithmic terms (9.40) and (9.41) imply that the minimal energy configuration will generically be a triangular array of vortices[207, 208]. Empirically this structure persists as the number of vortices is made large, even in the presence of a harmonic trap[200].

In principle the energy, and therefore the operator dimension, should be found by fixing the angular momentum and varying over the positions R_i to find the minimum energy configuration. However, for $N_v \sim O(1)$ the interaction is negligible and the energy will scale as $E \sim N_v E_v$ where E_v is the energy of a single vortex placed in the center of the trap. To consider L parametrically larger than Q we must consider $N_v \gg 1$. While we cannot exactly analyze (9.36) in this limit, we are justified in approximating the vortex density as a continuous quantity, corroborated by the fact that in this limit the interaction energy dominates and has terms which go as $N_v^2 \mu^{\frac{d}{2}} R_{TF}^{d-2} \sim L^2/I$, where I is the moment of inertia, given later by Eq. (9.47).

Continuum Approximation: We can take advantage of the fact the vortices are dense to coarse grain (9.34) and replace it with a continuous velocity field which satisfies:

$$\oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{\ell} = 2\pi N_v(C) \tag{9.42}$$

where $N_v(C)$ is the number of vortices in the area enclosed by the curve C. Let L be the angular momentum (to be precise the z component of the angular momentum) of the configuration. We take a variational approach, minimizing the energy over smooth v with fixed L. To this end, define:

$$E_\Omega = \frac{1}{2}\int \mathrm{d}^d x\, nv^2 - \Omega\left(\int \mathrm{d}^d x\, n\,(\boldsymbol{r}\times\boldsymbol{v})\cdot\hat{z} - L\right) \tag{9.43}$$

$$= \frac{1}{2}\int \mathrm{d}^d x\, n\,(v - \Omega\hat{z}\times\boldsymbol{r})^2 - \frac{\Omega^2}{2}\int \mathrm{d}^d x\, nr^2 + \Omega L \tag{9.44}$$

where Ω is a Lagrange multiplier to fix the angular momentum. From (9.43), we can see that the minimum energy velocity field is that of a rotating rigid body with uniform vortex density:

$$\boldsymbol{v} = \Omega\hat{z}\times\boldsymbol{r} \implies \Delta E = \frac{\Omega^2}{2}\int \mathrm{d}^d x\, nr^2 = \frac{\Omega^2}{2} I \tag{9.45}$$

where I is the moment of inertia of the condensate, computed from the density as:

$$I = \int \mathrm{d}^d x\, n(r) r^2 \,. \tag{9.46}$$

and Ω can be determined via its relation to L as $\Omega = \frac{L}{I}$. Now the moment of inertia I evaluates to

$$I = \begin{cases} \frac{4}{3}\pi c_0 \frac{\mu^3}{\omega^4} = \frac{1}{\omega}\left(\frac{2}{3}\frac{1}{\sqrt{2\pi c_0}} Q^{\frac{3}{2}}\right) & d=2 \\ \\ \frac{5\pi^2}{8\sqrt{2}} c_0 \frac{\mu^4}{\omega^5} = \frac{1}{\omega}\left(\frac{25}{1024}\left(\frac{25\pi^4}{32c_0^2}\right)^{1/6} Q^{\frac{4}{3}}\right) & d=3 \end{cases} \tag{9.47}$$

Using (9.42) and (9.45) we can also determine that the angular momentum of the configuration scales as $L \sim N_v Q$ as expected from (9.35). Consequently, the energy is that of a rigid body with angular momentum L and is given by:

$$\Delta E = \frac{L^2}{2I}\,, \tag{9.48}$$

Notice that this leading order result is independent of the trap and the inhomogeneity of the density. Corrections will arise from the inhomogeneity of the trap and the discreteness of the vortices, but they are subleading in N_v and suppressed in R_{TF} [204]. Indeed, that there are terms in the energy which scale as N_v being neglected is visible in (9.36).

We remark that there are constraints of the vortex density of the system. The vortex spacing λ should be larger than the vortex core size i.e. $\lambda \gg a \sim \frac{1}{\sqrt{\mu_{eff}}}$. Beyond this limit we expect interactions to be strong and the EFT description to break down [209]. Now, in a scenario where we have multiple vortices, a rough estimation yields that

$$N_v \sim \frac{R_{TF}^2}{\lambda^2} \sim \begin{cases} \sqrt{Q\ell} & d=2 \\ (Q\ell)^{1/3} & d=3 \end{cases} \tag{9.49}$$

where ℓ is the typical angular momentum of a vortex in the multivortex configuration. Thus in $d=2$, the maximum angular momentum configuration that one can reach within the validity of the EFT corresponds to a maximum density of $N_v \sim Q$. Physically this means most of the vortices are near the center and $\ell \sim Q$ and the total angular momentum $L \sim Q^2$. For $d=3$ this corresponds to $N_v \sim Q^{2/3}$ which is less than Q because the vortices are extended objects and the total angular momentum amounts to $L \sim Q^{5/3}$. But our EFT breaks down before this. Using particle vortex duality as in H.1, one can see that the EFT breaks down when the electric field becomes comparable to magnetic field. This means that the EFT breaks down when the contribution coming from rigid body rotation becomes comparable to Δ_Q. Hence, the maximum angular momentum that can be attained within the validity of our EFT is $L \sim Q^{3/2}$ in $d=2$ and $L \sim Q^{4/3}$ in $d=3$.

These determine the absolute limits on the angular momentum accessible within our EFT and together with (9.48) and (9.47) imply the following operator dimension scaling:

$$d=2 \quad \Delta_{Q,L} = \sqrt{\frac{9c_0\pi}{2}}\left(\frac{L^2}{Q^{3/2}}\right) + \Delta_Q \qquad Q < L < Q^{3/2} \tag{9.50}$$

$$d = 3 \quad \Delta_{Q,L} = \frac{1024}{25}\left(\frac{32c_0^2}{25\pi^4}\right)^{1/6}\left(\frac{L^2}{Q^{4/3}}\right) + \Delta_Q \qquad Q < L < Q^{4/3} \tag{9.51}$$

The above constitute the main results of this section.

9.6 Conclusions and Future Directions

To summarize, we have calculated how the dimensions of operators in NRCFTs scale with number charge Q and spin L in the limit of $Q \gg 1$ via the state-operator correspondence. The NRCFTs under consideration exist in $d = 2$ and $d = 3$ and by assumption are described by the superfluid EFT. This allows for explicit calculations by studying phonon and vortex configurations of the superfluid. We expect applicability of our result to "fermions at unitarity" and certain conformal anyon theories, as well any other NRCFT with this symmetry breaking behavior in its large charge sector[174, 186, 185, 187]. In fact the superfluid state of unitary fermions in a harmonic trap has been experimentally observed, including the formation of vortices [200].

The most direct extension of these results would be to go to beyond the leading order scaling. To do so would require reasoning about the divergences associated with the vortex core, the size and structure of which is entirely determined by UV physics. It should be possible to regulate such divergences by considering operators localized on the vortex. Such a procedure in the relativistic effective string theory was worked out in [210, 211] and the effective string theory of vortex lines in superfluids was explored in [212]. A similar analysis has also been applied to divergences of the superfluid EFT near R_{TF}, associated with the dilute regime of size δ [213] .

It would be especially interesting to study other possible symmetry breaking patterns, such as those relevant for chiral superfluids [191]. As mentioned in this large angular momentum regime the vortices are arranged as a triangular lattice. Deformations of this vortex lattice are a novel excitation in this limit, known as 'Tkachenko modes' [214]. Presumably these excited states would correspond to a tower of low-lying operators above the operator which creates the vortex-lattice. However, treatment these modes and corrections to the results (9.50) and (9.51) would require us to think about a new EFT which captures the spontaneous breaking of spatial symmetry by the vortex lattice. This EFT has been worked out by [192] and may be adaptable to the Schrödinger invariant case in a trap. Especially interesting would be systems with a Fermi surface, however such a critical state must necessarily be a non-fermi liquid following the results of [175].

While our EFT is not valid at larger angular momentum[14], it is interesting to ask if there is an analog of the large spin expansion when $L \sim \Delta$ for NRCFTs. The techniques for NRCFT bootstrap are not well developed, but see ref [173]. It is interesting to note that unlike in CFT, there is no unitary bound restricting $L \leq \Delta$ as spin can be treated as an internal degree of freedom.

Another interesting direction would be to consider correlation functions of charged spinning operators in these NRCFTs. The universal scaling of the 3-point function and higher are all explicitly calculable within this EFT, as was done for scalar charged operators in [201]. In relativistic CFTs this was worked out in [190, 169] for certain operators. We leave this and other questions for future work.

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Chapter 9, in full, is a reprint of the material as it appears in Shauna M Kravec, Sridip Pal, JHEP **1905** (2019) 194. The dissertation/thesis author was one of the primary investigators and authors of this paper.

[14] A similar issue occurs in this approach to CFTs [190]

Chapter 10

Modular Invariance and Cardy Formula

The symmetry algebra of conformal field theory (CFT) in different dimension is highly constraining in determining the behavior of various physical quantities. In fact, knowing the operator spectra, the three point coefficients and making sure that crossing symmetry is satisfied is powerful enough to completely specify a CFT. In 2 dimensions, the conformal algebra gets extended to its infinite dimensional avatar, named as Virasoro algebra This provides us with immense analytical control over 2D CFT. It so turns out that one can consistently define 2D CFT on any Riemann surface of arbitrary genus, the simplest non trivial of which is a genus 1 surface, also known as torus.

In this chapter and the next one, we consider a 2 dimensional CFT on a torus. Physically this means that we are considering a thermal CFT living on a spatial circle of length L, at an inverse temperature β: it can be thought of as obtained by doing a path integral over a square torus. The two cycles of the torus represent the thermal cycle of length β and the spatial circle of length L. Since the CFT is scale invariant the thermal partition function satisfies following functional form:

$$Z(\beta, L) = Z\left(\frac{\beta}{L}\right) \tag{10.1}$$

On the other hand such a square torus can be assigned a modular parameter τ,which is defined as

$$\tau = \frac{\imath\beta}{L} \tag{10.2}$$

The eq. (10.1) implies that the partition function of a CFT on a torus is a function of modular parameter τ only.

One can generalize the square torus to a non-square one by extending the domain of definition τ from the positive imaginary axis to whole upper half plane. We require $\mathrm{Im}(\tau)$ to be positive since the cycle length is a positive number. The statement of the scale invariance of CFT remains the same: the partition function Z is still a function of τ. In fact, this fact is manifest in the definition of the torus partition function of 2 dimensional CFT. In particular, the torus partition function of 2D CFT is defined as

$$Z(\tau,\bar{\tau}) = \sum_{h,\bar{h}} q^{h-\frac{c}{24}}\bar{q}^{\bar{h}-\frac{c}{24}} \tag{10.3}$$

where we have used the fact that 2D CFT has two copies of Virasoro symmetry, the holomorphic and the antiholomorphic one. Here we have $q = \exp(2\pi\imath\tau)$ and $\bar{q} = \exp(-2\pi\imath\bar{\tau})$ and the sum is over the spectrum of the operators with conformal weight $(h,\bar{h})$. From here on, without loss of generality, we will assume $L = 2\pi$ and consider a square torus:

$$Z(\beta) = \sum_{\Delta_i} \exp\left[-\beta\left(\Delta_i - \frac{c}{12}\right)\right] \tag{10.4}$$

$$= \int_0^\infty d\Delta\, \rho(\Delta) \exp\left[-\beta\left(\Delta - \frac{c}{12}\right)\right], \quad \rho(\Delta) = \sum_i \delta(\Delta - \Delta_i) \tag{10.5}$$

where $\Delta_i = h_i + \bar{h}_i$ is the scaling dimension of the operators in the CFT spectra.

Now let us understand the torus in a more geometric way. A torus can be thought of as a plane with points identified if they are separated by a vector of the form $n_1\boldsymbol{\alpha}_1 + n_2\boldsymbol{\alpha}_2$ for two linearly independent vectors $\boldsymbol{\alpha}_i$ and two integers n_i. Now one can do a following transformation on the basis vectors $\boldsymbol{\alpha}_i$:

$$\begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{bmatrix}, \text{ where } ad - bc = 1\,,\ a,b,c,d \in \mathbb{Z} \tag{10.6}$$

This transformation forms a group $SL(2,\mathbb{Z})$ and maps the torus to itself. The action of $SL(2,\mathbb{Z})$ on the modular parameter τ is given by

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad a,b,c,d \in \mathbb{Z}\ \&\ ad - bc = 1 \tag{10.7}$$

and known as *modular transformation*. Since under modular transformation the torus gets mapped to itself, the partition function stays invariant. Hence we arrive at the statement of modular invariance of torus partition function:

$$Z(\tau,\bar{\tau}) = Z\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) \tag{10.8}$$

The Cardy formula [217] for asymptotic density of states is then derived by leveraging the above modular invariance of the torus partition function.

There are two basic kind of modular transformations, which can be used to compose any arbitrary modular transformation:

$$S : \tau \mapsto -\frac{1}{\tau} \tag{10.9}$$

$$T : \tau \mapsto \tau + 1 \tag{10.10}$$

For a square torus, which we will be interested in, the S transformation exchanges the thermal and spatial circle and does the following mapping (shown in fig. 10.1)

$$\beta \mapsto \frac{4\pi^2}{\beta} \tag{10.11}$$

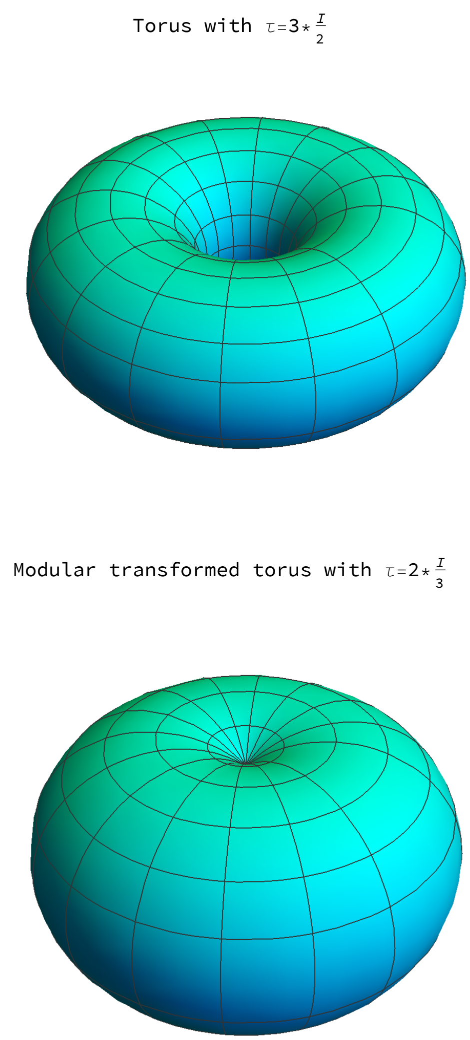

Figure 10.1: S modular transformation on square torus. The torus on the top has $\tau = \frac{3\imath}{2}$ while the bottom one has $\tau = \frac{2\imath}{3}$.

In terms of inverse temperature β, the statement of the modular invariance boils down to:

$$Z(\beta) = Z\left(\frac{4\pi^2}{\beta}\right) . \tag{10.12}$$

We emphasize that the above equation is at the heart of the Cardy's analysis. It relates the low temperature behavior of Z with its high temperature behavior. At low temperature $Z(\beta)$ is dominated by the ground state i.e. $\Delta = 0$ (and controlled by the universal parameter c, the central charge of the 2D CFT) and we have

$$Z(\beta) \underset{\beta\to\infty}{=} \exp\left[\beta \frac{c}{12}\right] \tag{10.13}$$

Now in order to obtain the high temperature expression, we note that $\beta \to 0$ implies that $\beta' = 4\pi^2/\beta \to \infty$, thus we have the high temperature behavior of the partition function:

$$Z(\beta) \underset{\beta\to 0}{=} Z\left(\beta'\right) \underset{\beta'\to\infty}{=} \exp\left[\beta' \frac{c}{12}\right] = \exp\left[\frac{\pi^2 c}{3\beta}\right] \tag{10.14}$$

On the other hand, the eq. (10.5) gives us a formal expression for density of states as an inverse Laplace transform of partition function:

$$\rho(\Delta) = \int d\beta \, Z(\beta) \exp\left[\beta\left(\Delta - \frac{c}{12}\right)\right] \tag{10.15}$$

For $\beta \to 0$ limit, the partition function $Z(\beta)$ is dominated by the heavy states with large scaling dimension $\Delta \to \infty$. Thus in the eq. (10.15), if we substitute $Z(\beta)$ with its high temperature approximation, given by the eq. (10.14) and are able to perform the integral, we should obtain the expression for asymptotic density of states. The result of these procedure gives us

$$\rho(\Delta) \simeq \pi\sqrt{\frac{c}{3}} \frac{I_1\left(\sqrt{\frac{c}{3}\left(\Delta - \frac{c}{12}\right)}\right)}{\sqrt{\frac{c}{3}\left(\Delta - \frac{c}{12}\right)}} \tag{10.16}$$

The true density of states is a sum over Dirac delta function and it can never be equal to the smooth function appearing in the eq. (10.16). To emphasize on this point, let us rename (following the convention in [232]) the density of states appearing in the right hand side of the eq. (10.16) as $\rho_0(\Delta)$ to distinguish it from the actual density of states i.e. we define

$$\rho_0(\Delta) = \pi\sqrt{\frac{c}{3}} \frac{I_1\left(\sqrt{\frac{c}{3}\left(\Delta - \frac{c}{12}\right)}\right)}{\sqrt{\frac{c}{3}\left(\Delta - \frac{c}{12}\right)}} \,. \tag{10.17}$$

As the eq. (10.16) is only an approximate formula, one legitimate question is to ask how accurate the formula. One way to circumnavigate this problem is to integrate the density of states upto some cut-off dimension Δ and then compare with the actual number of states below that cut off.

It is clear that even after integrating the $\rho_0(\Delta')$ upto some cut-off Δ, we should trust the expression only for large Δ. The question that arises naturally at this point is upto which order we should trust the Bessel function appearing in the expression for $\rho_0(\Delta)$. In short, the above method is completely blind to the error term. Nonetheless in the leading order, it does an amazing job as can explicitly be seen from the fig. 10.2 and fig. 10.3. On the vertical axis of the plots we have $F(\Delta)$ defined as

$$F(\Delta) = \int_0^{\Delta} d\Delta' \, \rho(\Delta') \tag{10.18}$$

On the other hand, if we try to go on including the subleading approximation of the Bessel function appearing in the eq. (10.16), we can see that the error goes up as witnessed in fig. 10.4. We remark that the difference between the red curve and the black curve is not a constant shift, rather a function Δ, which goes to 0 from the negative side as $\Delta \to \infty$, this can be seen from fig. 10.5. This motivates us to perform a more careful refinement of Cardy's analysis. This is where the techniques inspired from Tauberian theorems come handy. The usefulness of Tauberian theorems in the context of CFT is pointed out in [228]; subsequently, its importance was emphasized in Appendix C of [219], where the authors used Ingham's theorem [229]. The fact that going out to the complex plane while using Tauberian theorems would provide extra mileage in controlling the correction terms in various asymptotic quantities of CFT, has been pointed out in [230]. In particular, the use of [231] turned out to be extremely useful in this context.

In what follows, we consider an energy window of width 2δ, centered at some large Δ.

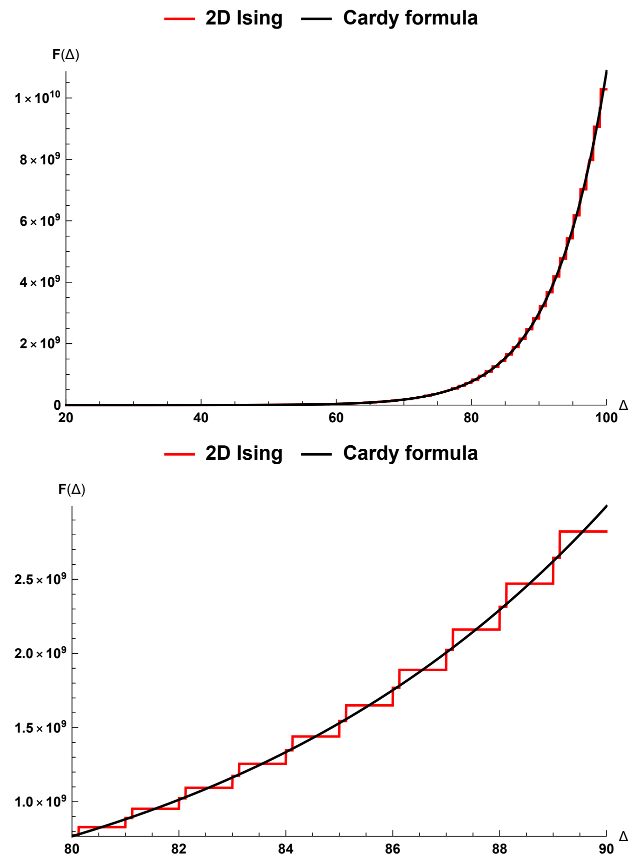

Figure 10.2: Integrated density of states upto Δ as a function of Δ and its approximation due to Cardy formula for 2D Ising model, the figure on the bottom is the zoomed version of the one on the top.

We estimate the number of states lying in that window i.e we estimate the following quantity:

$$\int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \, \rho(\Delta') = \int_{0}^{\infty} d\Delta' \, \rho(\Delta')\Theta\left(\Delta' \in (\Delta-\delta, \Delta+\delta]\right) \tag{10.19}$$

and the microcanonical entropy associated with the interval

$$S_\delta \equiv \log\left(\int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \, \rho(\Delta')\right) \tag{10.20}$$

The rest of this chapter is a brief review of [232], which acts a precursor of the following chapter.

The result of [232] which is going to be relevant for our purpose is as follows:

$$S_\delta = \log\left(\int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \, \rho(\Delta')\right) \underset{\Delta\to\infty}{\simeq} 2\pi\sqrt{\frac{c\Delta}{3}} + \frac{1}{4}\log\left(\frac{c\delta^4}{3\Delta^3}\right) + s(\delta,\Delta)\,, \tag{10.21}$$

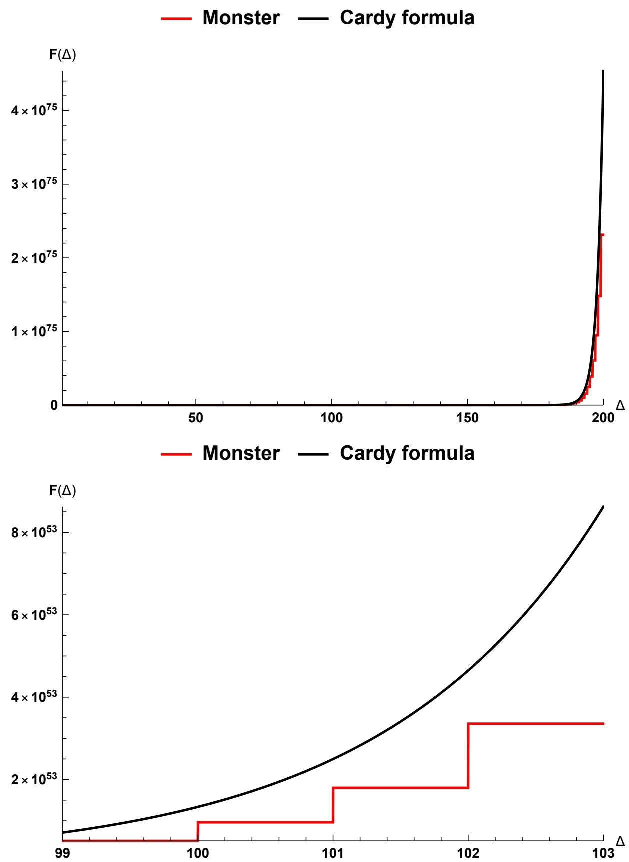

Figure 10.3: Integrated density of states upto Δ as a function of Δ and its approximation due to Cardy formula for Monster CFT with $c = 24$, the figure on the bottom is the zoomed version of the one on the top.

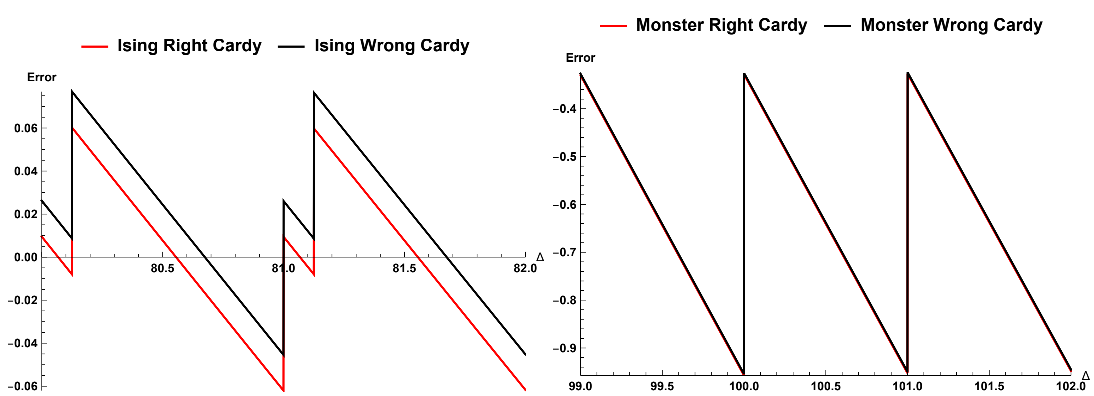

Figure 10.4: Difference between Integrated density of states upto Δ and its approximation due to Cardy formula as a function of Δ for 2D Ising model and Monster CFT, the figure shows that including the subleading term from the naive analysis actually increases the error.

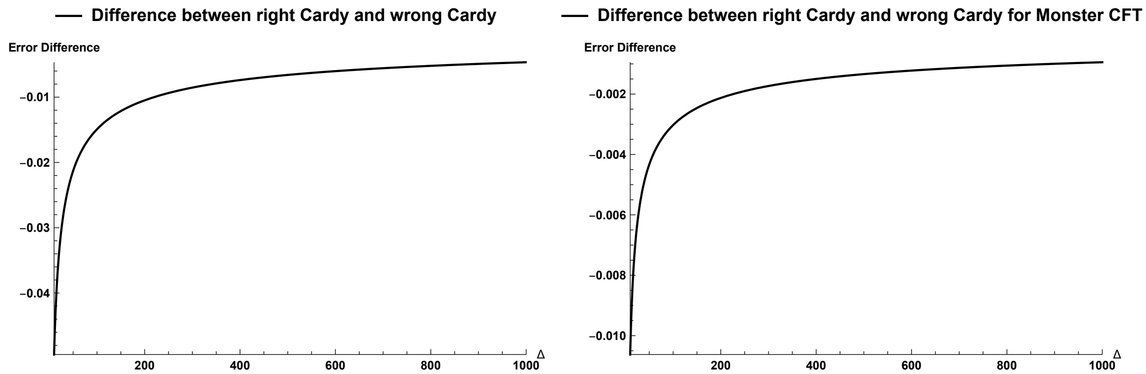

Figure 10.5: The difference between the error using the right Cardy formula and the wrong one as a function of Δ for 2D Ising model and Monster CFT, the figure shows that the difference goes to 0 from the negative side as $\Delta \to \infty$.

For $O(1)$ energy width, the $O(1)$ correction $s(\delta,\Delta)$ is bounded from above and below:

$$\delta = O(1): \ s_-(\delta,\Delta) \leq s(\delta,\Delta) \leq s_+(\delta,\Delta) \tag{10.22}$$

and these functions $s_\pm$ are given by

$$s_+ = MZ(\delta) \tag{10.23}$$

$$s_- = mz(\delta) \tag{10.24}$$

where $MZ(\delta)$ is defined as

$$MZ(\delta) = \begin{cases} \frac{\pi}{3}\left(\frac{\pi\delta}{2}\right)^3 \left(\sin\left(\frac{\pi\delta}{2}\right)\right)^{-4}, & \delta < \frac{a_*}{2\pi} \sim 0.54 \\ 2.02\,, & \delta > \frac{a_*}{2\pi} \sim 0.54\,. \end{cases} \tag{10.25}$$

Here, $a_* \sim 3.38$ satisfies $a_* = 3\tan(a_*/4)$. The function $mz(\delta)$ is defined as

$$mz(\delta) = \begin{cases} \frac{2\left(\delta^2 - \frac{3}{\pi^2}\right)}{3\delta^3}, & \frac{\sqrt{3}}{\pi} \leq \delta < \frac{3}{\pi} \sim 0.95, \\ \frac{4\pi}{27} \sim 0.46\,, & \delta \geq \frac{3}{\pi}. \end{cases} \tag{10.26}$$

Furthermore, it is shown in [232] for $\Delta \to \infty$:

$$\int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \, \rho(\Delta') > 0 \text{ if } \delta > \frac{\sqrt{3}}{\pi} \tag{10.27}$$

i.e. the asymptotic gap between operators is bounded above by $2\sqrt{\frac{3}{\pi^2}}$. Even though this is trivial due to the presence of descendant operators since within a window of width just greater than 1 there would always be one descendant, the nontriviality comes from the fact that the same bound applies to the asymptotic gap of Virasoro primaries. This led them to conjecture that the optimal asymptotic gap of Virasoro primaries is exactly 1. And this has to be the optimal one, since Monster CFT saturates the bound. The following chapter proves this conjecture made in [232] involving the asymptotic gap between primaries and improves the bound $s_\pm$ on $O(1)$ correction to the Cardy formula.

We start with two bandlimited functions $\phi_\pm$ such that they are just above and below the indicator function Θ centered at Δ with width of 2δ i.e. we have

$$\phi_-(\Delta') \leq \Theta \leq \phi_+(\Delta') \tag{10.28}$$

One can obtain from the above the following inequality

$$e^{\beta(\Delta-\delta)-\beta\Delta'}\phi_-(\Delta') \leq \Theta \leq e^{\beta(\Delta+\delta)-\beta\Delta'}\phi_-(\Delta'), \tag{10.29}$$

which, upon integrating against the density of states $\rho(\Delta)$ gives

$$e^{\beta(\Delta-\delta)}\int_0^\infty d\Delta' \, \phi_-(\Delta')\rho(\Delta')e^{-\beta\Delta'} \leq \exp\left[S_\delta\right] \leq e^{\beta(\Delta+\delta)}\int_0^\infty d\Delta' \, \phi_+(\Delta')\rho(\Delta')e^{-\beta\Delta'} \tag{10.30}$$

Let us go to the Fourier domain to analyze the terms that are bounding from the above and the below. This is exactly where the bandlimited nature of the functions $\phi_\pm$ plays a huge role. In terms of $\phi_\pm(\Delta') = \int_{-\infty}^{\infty} e^{-\imath\Delta t}\hat{\phi}_\pm(t)$, we have

$$e^{\beta(\Delta-\delta)}\int_{-\infty}^{\infty} dt\ \hat{\phi}_{-}(t)Y(\beta+\imath t)\leq \exp[S_{\delta}]\leq e^{\beta(\Delta+\delta)}\int_{-\infty}^{\infty} dt\ \hat{\phi}_{+}(t)Y(\beta+\imath t) \tag{10.31}$$

where we have

$$Y(\beta+\imath t)=\exp\left[(\beta+\imath t)\frac{c}{12}\right]Z(\beta+\imath t) \tag{10.32}$$

At this point β is arbitrary, but we know that at the end of the day we are interested in asymptotic density of states and hence we expect to set β to be very very small. In the naive analysis, we have seen that the expression for partition function for small β is obtained by doing modular transformation. Here also we implement a similar transformation and obtain:

$$e^{\beta(\Delta-\delta+c/12)}\int_{-\infty}^{\infty} dt\ \hat{\phi}_{-}(t)e^{\imath tc/12}Z\left(\frac{4\pi^2}{\beta+\imath t}\right)\leq e^{S_{\delta}}\leq e^{\beta(\Delta+\delta+c/12)}\int_{-\infty}^{\infty} dt\ \hat{\phi}_{-}(t)e^{\imath tc/12}Z\left(\frac{4\pi^2}{\beta+\imath t}\right) \tag{10.33}$$

Now we write the partition function as

$$Z(\beta)=Z_L(\beta)+Z_H(\beta) \tag{10.34}$$

$$Z_L(\beta)=\sum_{\Delta'<\Delta_H>\frac{c}{12}}\exp\left[-\beta\left(\Delta'-\frac{c}{12}\right)\right] \tag{10.35}$$

$$Z_H(\beta)=\sum_{\Delta'>\Delta_H>\frac{c}{12}}\exp\left[-\beta\left(\Delta'-\frac{c}{12}\right)\right] \tag{10.36}$$

Now from the naive analysis, we know that the smooth function $\rho_0(\Delta')$ actually reproduces the leading term of the partition function at small β. To make sure that this is indeed the leading expression i.e. to say that the contribution to the partition function coming from the heavy states i.e Z_H is suppressed, we need the bandlimited functions $\phi_{\pm}$. In fact, one can show that if the

$\hat{\phi}_{\pm}(t)$ has support within $[-2\pi, 2\pi]$, this is indeed the case and we have for $\beta = \pi\sqrt{\frac{c}{3\Delta}} \to 0$:

$$\exp\left[\beta(\Delta-\delta)\right]\int d\Delta'\ \rho_0(\Delta')e^{-\beta\Delta'}\phi_-(\Delta') \le e^{S_\delta} \le \exp\left[\beta(\Delta+\delta)\right]\int d\Delta'\ \rho_0(\Delta')e^{-\beta\Delta'}\phi_+(\Delta') \tag{10.37}$$

By evaluating the above integrals by saddle point approximation:

$$c_-\rho_0(\Delta) \le \int_{\Delta-\delta}^{\Delta+\delta} d\Delta'\ \rho(\Delta') \le c_+\rho_0(\Delta), \tag{10.38}$$

This equation would be one of the key equations that is used in the next chapter. Here $c_\pm$ is defined as

$$c_\pm = \frac{1}{2}\int_{-\infty}^{\infty} dx\ \phi_\pm(\Delta+\delta x). \tag{10.39}$$

The details of the the subleading nature of Z_H and the derivation of the eq. (10.39) can be found in [232]. The functions leading to the eq. (10.23) is given by

$$\phi_+(\Delta') = \left[\frac{\sin\left(\frac{\Lambda_+\delta}{4}\right)}{\frac{\Lambda_+\delta}{4}}\right]^{-4}\left[\frac{\sin\left(\frac{\Lambda_+(\Delta'-\Delta)}{4}\right)}{\frac{\Lambda_+(\Delta'-\Delta)}{4}}\right]^{4}, \tag{10.40}$$

$$\phi_-(\Delta') = \left[\frac{\sin\left(\frac{\Lambda_-(\Delta'-\Delta)}{4}\right)}{\frac{\Lambda_-(\Delta'-\Delta)}{4}}\right]^{4}\left(1-\frac{(\Delta'-\Delta)^2}{\delta^2}\right). \tag{10.41}$$

It turns out that the bound on the $O(1)$ correction to the Cardy formula, obtained in [232] is not the optimal one. One can improve the bounds by choosing different $\phi_\pm$ subject to the constraint (10.28) and the bandlimited nature of the functions $\phi_\pm$. This will be the main topic of our next chapter.

Chapter 11

Bounds on density of states and spectral gap in CFT_2

We improve the recently discovered bounds on the $O(1)$ correction to the Cardy formula for the density of states in 2 dimensional conformal field theory at high energy. We prove a conjectured upper bound on the asymptotic gap between two consecutive Virasoro primaries for a central charge greater than 1, demonstrating it to be 1. Furthermore, a systematic method is provided to establish a limit on how tight the bound on the $O(1)$ correction to the Cardy formula can be made using bandlimited functions. The techniques and the functions used here are of generic importance whenever the Tauberian theorems are used to estimate some physical quantities.

11.1 The premise and the results

Modular invariance is a powerful constraint on the data of 2D conformal field theory (CFT). It relates the low temperature data to the high temperature data. For example, using the fact that the low temperature behavior of the $2D$ CFT partition function is universal and controlled by a single parameter c, the central charge of the CFT, we can deduce the universal behavior of the partition function at high temperature and thereby deduce the asymptotic behavior of the density of states, which controls the high temperature behavior[1] of a 2D CFT [217]. Similar ideas can be extended to one point functions as well, where the low temperature behavior is controlled by the low lying spectra and three point coefficients [218, 219]. Yet another remarkable implication of the modular invariance of the partition function is the existence of infinite Virasoro primaries for CFT with $c > 1$. Significant progress has been made in recent years towards exploiting the modular invariance to deduce results in 2D CFT under the umbrella of modular bootstrap [220, 218, 219, 221, 222, 223, 224, 225, 226, 227].

Recently, with the use of complex Tauberian theorem, Mukhametzhanov and Zhiboedov [232] have explored the regime of validity, as well as corrections, to the Cardy formula with great nuance. In particular, they have investigated the entropy S_δ associated with a particular energy window of width δ around a peak value Δ, which is allowed to go to infinity, and found

$$S_\delta = \log\left(\int_{\Delta-\delta}^{\Delta+\delta} d\Delta'\, \rho(\Delta')\right) \underset{\Delta\to\infty}{\simeq} 2\pi\sqrt{\frac{c\Delta}{3}} + \frac{1}{4}\log\left(\frac{c\delta^4}{3\Delta^3}\right) + s(\delta,\Delta)\,, \tag{11.1}$$

where $\rho(\Delta)$ is the density of states, given by a sum of Dirac delta functions peaked at the positions of the operator dimensions. It is shown in [232] that for $O(1)$ energy width, the $O(1)$ correction $s(\delta,\Delta)$ is bounded from above and below:

$$\delta = O(1):\;\; s_-(\delta,\Delta) \leq s(\delta,\Delta) \leq s_+(\delta,\Delta) \tag{11.2}$$

The purpose of the current note is to improve the bound and provide a systematic way to estimate how tight the bounds can be made using bandlimited functions. We also prove the conjectured upper bound on the asymptotic gap between Virasoro primaries, which turns out to be 1. This gap is optimal since for the Monster CFT, the gap is precisely 1.

[1] The fact that the modular invariance of CFT can predict the asymptotic density of states is explicitly stated in [215]. One usually takes the inverse Laplace transform of the partition function to deduce such behavior; similar techniques also appeared in [216]. We thank Shouvik Datta for pointing this out.

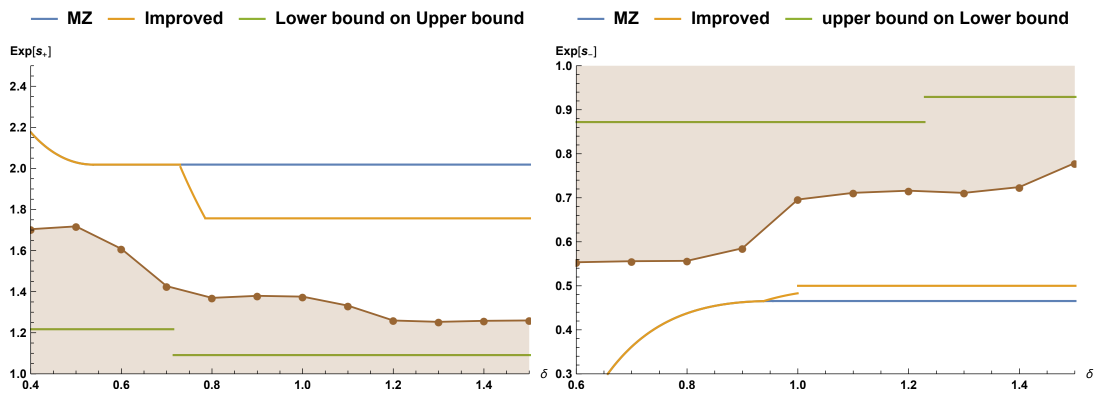

Figure 11.1: Exp$[s_\pm]$ as a function of δ, the half-width of the energy window. The blue line is the existing bound. The orange line denotes the improved bound that we report here. The green line is the analytical lower (upper) bound on the upper (lower) bound, while the brown dots stand for the lower (upper) bound on the upper (lower) bound obtained from enforcing the positive definiteness condition on the Fourier transform of $\pm(\phi_\pm - \Theta)$ via Matlab. The bound on bounds represented by the green line is thus weaker than that represented by the brown dots. The brown shaded region is **not** achievable by any bandlimited function.

Our results can be summarized by figure [11.1], where the green line and dots denote the lower (upper) bound on the upper (lower) bound. The orange lines denote the improved achievable bounds. The brown dots stand for the lower (upper) bound on the upper (lower) bound obtained from implementing the positive definiteness condition on the Fourier transform of $\pm(\phi_\pm - \Theta)$ via Matlab. The bound on bounds represented by the green line is thus weaker than that represented by the brown dots. In short, the brown shaded region is not achievable by any bandlimited function.

In particular, we show that the upper bound on $s(\delta,\Delta)$ is given by

$$\exp\left[s_+(\delta,\Delta)\right] = \begin{cases} MZ(\delta)\,, \quad \delta < 0.73 \\ \frac{3}{40\delta^3}\left(11\delta^2 + \frac{45}{\pi^2}\right)\,, \quad 0.73 < \delta \leq 0.785 \\ 1.7578\,, \quad \delta > 0.785 \end{cases} \tag{11.3}$$

where $MZ(\delta)$ is a function introduced in [232] and defined as

$$MZ(\delta) = \begin{cases} \frac{\pi}{3}\left(\frac{\pi\delta}{2}\right)^3\left(\sin\left(\frac{\pi\delta}{2}\right)\right)^{-4}, & \delta < \frac{a_*}{2\pi} \sim 0.54 \\ 2.02\,, & \delta > \frac{a_*}{2\pi} \sim 0.54\,. \end{cases} \tag{11.4}$$

Here, $a_* \sim 3.38$ satisfies $a_* = 3\tan(a_*/4)$. Eq.(11.3) is an improvement of the upper bound for $\delta > 0.73$, as evident from figure [11.4].

The lower bound $s_-(\delta,\Delta)$ is given by

$$\exp\left[s_-(\delta,\Delta)\right] = \begin{cases} mz(\delta)\,, & \frac{\sqrt{3}}{\pi} \leq \delta < \frac{\sqrt{\frac{165}{19}}}{\pi} \sim 0.94, \\ \frac{3\left(11\delta^2 - \frac{45}{\pi^2}\right)}{40\delta^3}\,, & \frac{\sqrt{\frac{165}{19}}}{\pi} < \delta \leq 1, \\ 0.5\,, & \delta > 1. \end{cases} \tag{11.5}$$

where $mz(\delta)$ is a function, introduced in [232]

$$mz(\delta) = \begin{cases} \frac{2\left(\delta^2 - \frac{3}{\pi^2}\right)}{3\delta^3}\,, & \frac{\sqrt{3}}{\pi} \leq \delta < \frac{3}{\pi} \sim 0.95, \\ \frac{4\pi}{27} \sim 0.46\,, & \delta \geq \frac{3}{\pi}. \end{cases} \tag{11.6}$$

The eq. (11.5) is an improvement of the lower bound for $\delta > 0.94$, as evident from figure [11.4].

One can verify the above bounds against 2D Ising model, Monster CFT and $k = 2$ extremal CFT, as witnessed in the series of figures 11.2 and 11.3 for different values of δ. The rest of the paper details the derivation of the above. In section 11.2, we derive the improvement on the bound on the $O(1)$ correction to the Cardy formula. Section 11.3 describes a systematic way to estimate how tight the bound can be made. We derive the optimal gap on the asymptotic spectra in section 11.4 and conclude with a brief discussion in section 11.5.

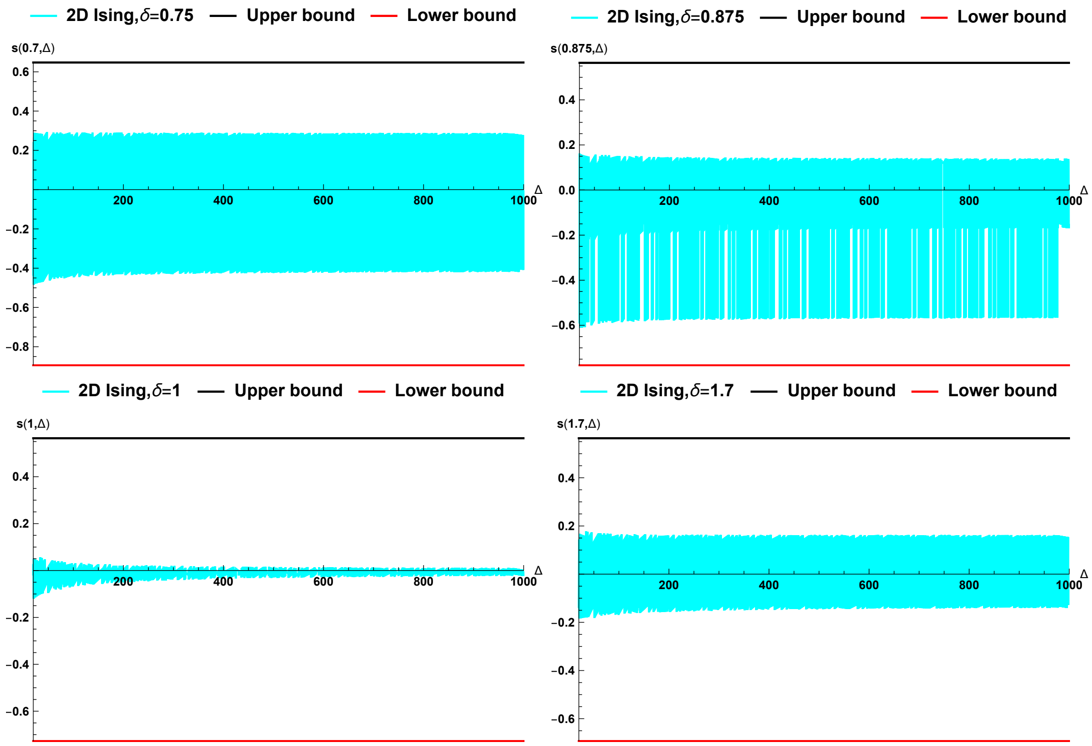

Figure 11.2: $s_{\pm}$ as a function of δ, the half-width of the energy window. We verify the bounds against the value obtained from 2D ising model, where the partition function and its q expansion is known explicitly. This is done for various arbitrary values of δ. The black line denotes the upper bound while the red line denotes the lower bound.

11.2 Derivation of the improvement

The basic ingredients for estimating the asymptotic growth of the density of states are two functions $\phi_{\pm}$ such that the following holds:

$$\phi_{-}(\Delta') < \Theta\left(\Delta' \in [\Delta-\delta, \Delta+\delta]\right) < \phi_{+}(\Delta')\,. \tag{11.7}$$

We refer the readers to section 4 of [232] for details of the procedure leading to a bound when Δ goes to infinity. The basic result can be summarized as:

$$c_{-}\rho_0(\Delta) \leq \int_{\Delta-\delta}^{\Delta+\delta} d\Delta'\, \rho(\Delta') \leq c_{+}\rho_0(\Delta), \tag{11.8}$$

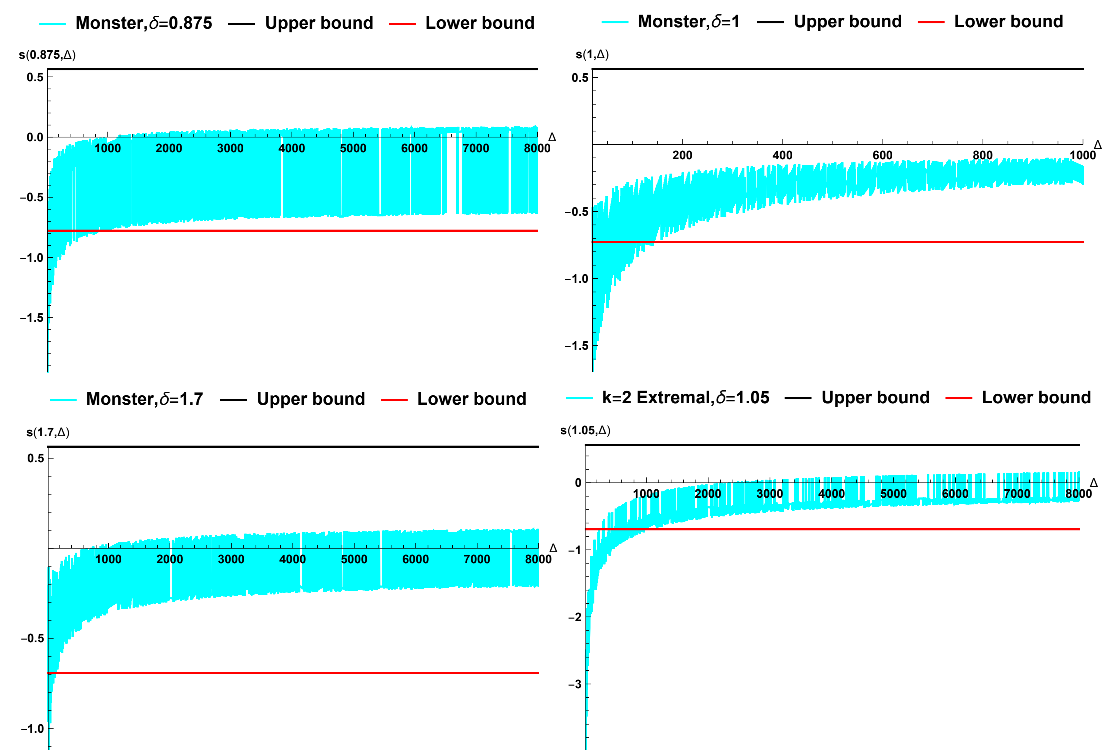

Figure 11.3: $s_{\pm}$ as a function of δ, the half-width of the energy window. We verify the bounds against the value obtained fromMonster CFT and $k = 2$ extremal CFT where the partition function and its q expansion is known explicitly. This is done for various arbitrary values of δ. The black line denotes the upper bound while the red line denotes the lower bound.

where $\rho_0(\Delta)$ reproduces the contribution from the vacuum at high temperature and is given by

$$\rho_0(\Delta) = \pi\sqrt{\frac{c}{3}}\frac{I_1\left(2\pi\sqrt{\frac{c}{3}\left(\Delta-\frac{c}{12}\right)}\right)}{\sqrt{\Delta-\frac{c}{12}}}\Theta\left(\Delta-\frac{c}{12}\right)+\delta\left(\Delta-\frac{c}{12}\right). \tag{11.9}$$

The above is in fact the leading result for the density of states at high energy. Furthermore, $c_\pm$ is defined as

$$c_\pm = \frac{1}{2}\int_{-\infty}^{\infty} dx\, \phi_\pm(\Delta+\delta x). \tag{11.10}$$

The eq. (11.8) holds if the Fourier transform of $\phi_\pm$ has a support on an interval which lies entirely within $[-2\pi, 2\pi]$. With this constraint in mind, we consider the following functions:

$$\phi_+(\Delta') = \left[\frac{\sin\left(\frac{\Lambda_+(\Delta'-\Delta)}{6}\right)}{\frac{\Lambda_+(\Delta'-\Delta)}{6}}\right]^6\left(1+\frac{(\Delta'-\Delta)^2}{\delta^2}\right), \tag{11.11}$$

$$\phi_-(\Delta') = \left[\frac{\sin\left(\frac{\Lambda_-(\Delta'-\Delta)}{6}\right)}{\frac{\Lambda_-(\Delta'-\Delta)}{6}}\right]^6\left(1-\frac{(\Delta'-\Delta)^2}{\delta^2}\right). \tag{11.12}$$

In order to ensure that the indicator function on the interval $[\Delta-\delta, \Delta+\delta]$ is bounded above by ϕ_+, we need to have

$$\delta\Lambda_+ \leq 4.9323\,. \tag{11.13}$$

The number in the eq. (11.13) is obtained by requiring that $\phi_+(\Delta\pm\delta) > 1$. The functions $\phi_\pm$ have Fourier transforms with bounded supports $[-\Lambda_\pm, \Lambda_\pm]$, respectively. Thus, in order for this support to lie within $[-2\pi, 2\pi]$, we also require that $\Lambda_\pm < 2\pi$. The bound is then obtained by minimizing (or maximizing)

$$c_\pm = \frac{1}{2\delta}\int dx\, \phi_\pm(\Delta+x) = \frac{3\pi\left(11\delta^2\Lambda_\pm^2 \pm 180\right)}{20\delta^3\Lambda_\pm^3} \tag{11.14}$$

for a given δ by varying $\Lambda_\pm$ subject to the constraint given by the eq. (11.13), as well as $\Lambda_\pm < 2\pi$. From the eq. (11.8), one can conclude [232] that

$$c_- \leq \exp[s(\delta,\Delta)] \leq c_+ . \tag{11.15}$$

Since for a fixed δ, c_+ is a monotonically decreasing function of Λ_+, we deduce that c_+ should be minimized by

$$\Lambda_+ = \min\left\{2\pi, \frac{4.9323}{\delta}\right\} = \begin{cases} 2\pi, & \delta < 0.785, \\ \frac{4.9323}{\delta}, & \delta > 0.785. \end{cases} \tag{11.16}$$

This explains the number 0.785 appearing in the bounds in the eq. (11.3). The final bound can be obtained by combining these results with the result of [232]. A similar analysis can be performed on c_-. These procedures yield the eq. (11.3) for the upper bound, while the lower bound is given by

$$\exp[s_-(\delta,\Delta)] = \begin{cases} mz(\delta)\,, & \frac{\sqrt{3}}{\pi} \leq \delta < \frac{\sqrt{\frac{165}{19}}}{\pi} \sim 0.94, \\ \frac{3\left(11\delta^2 - \frac{45}{\pi^2}\right)}{40\delta^3}, & \frac{\sqrt{\frac{165}{19}}}{\pi} < \delta < \frac{3\sqrt{\frac{15}{11}}}{\pi} \sim 1.12, \\ \frac{11}{60}\sqrt{\frac{11}{15}}\pi \sim 0.49\,, & \delta > \frac{3\sqrt{\frac{15}{11}}}{\pi} \sim 1.12. \end{cases} \tag{11.17}$$

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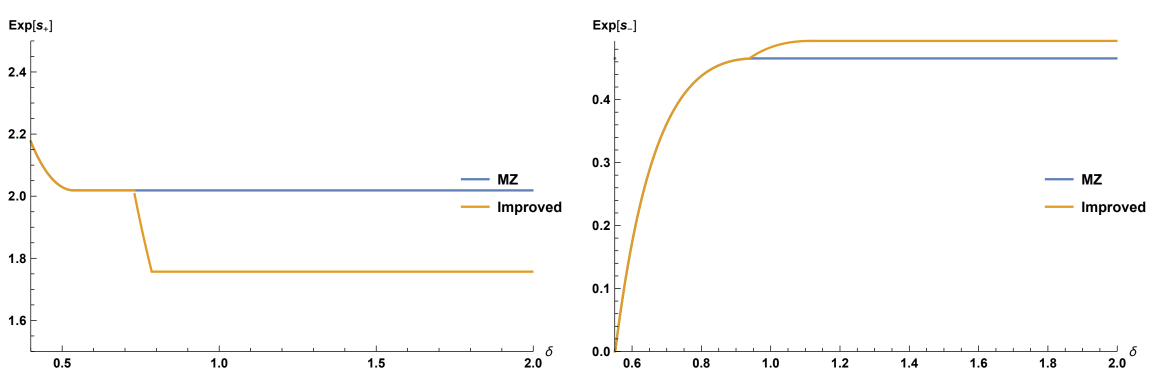

Figure 11.4: Exp$[s_\pm]$: The orange line denotes the improved lower (upper) bound while the blue line is the one from the literature.

The lower bound can be further improved for $\delta > 1$ by considering the following function whose Fourier transform has a support over $[-\frac{2\pi}{\delta}, \frac{2\pi}{\delta}]$.

$$\phi_-^{\text{Sphere}}(\Delta') := \frac{1}{1-\left(\frac{\Delta'-\Delta}{\delta}\right)^2}\left(\frac{\sin\left(\frac{\pi(\Delta'-\Delta)}{\delta}\right)}{\frac{\pi(\Delta'-\Delta)}{\delta}}\right)^2. \tag{11.18}$$

This yields $c_- = 0.5$, which is an improvement over the above; see figure 11.5.

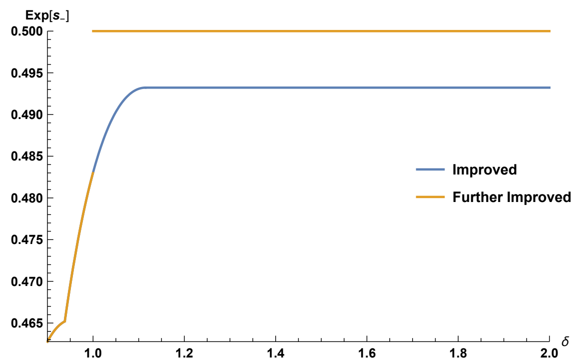

Figure 11.5: The orange line represents the improvement on the lower bound by using the function ϕ_-^{Sphere} appearing in the sphere packing problem.

Serendipity – connection to the sphere packing problem: The function in the eq. (11.18) also appears in the context of one dimensional sphere packing problem [233]. In fact, there is an uncanny similarity between the functions required in the two problems, especially if we look at the requirements on the function producing the lower bound[2]. In the sphere packing problem, one has a Fourier transform pair $f, \hat{f}$ satisfying

$$f(x) \leq 0 \text{ for } |x| > 1, \tag{11.19}$$

$$\hat{f}(k) \geq 0. \tag{11.20}$$

In our case, we have $x \leftrightarrow \Delta'$ and $k \leftrightarrow t$ and we require that $\hat{f}(k)$ has bounded support. In both scenarios, the goal is to maximize $\hat{f}(0)$. In the case of sphere packing, we also normalize $f(0)$ to one. For more details on the relevance of sphere packing to CFT, we refer the reader to the recent article [234].

[2]SP thanks John McGreevy for pointing to [234], where sphere packing plays a pivotal role.

It turns out that only in one dimension [233], where the sphere packing problem is trivial, the relevant function as given in the eq. (11.18) has bounded support in the Fourier domain and is positive[3]. This seems to suggest that if we want to further improve our bound, we need a bandlimited function whose Fourier transform becomes negative within the band.

Before moving on to the discussion of the bound on bounds, we pause to remark that the following class of functions parameterized by α can not be used to improve the bound from above:

$$\phi_+^{(\alpha)}(\Delta') = \left[\frac{\sin\left(\frac{\Lambda_+\delta}{\alpha}\right)}{\frac{\Lambda_+\delta}{\alpha}}\right]^{-\alpha}\left[\frac{\sin\left(\frac{\Lambda_+(\Delta'-\Delta)}{\alpha}\right)}{\frac{\Lambda_+(\Delta'-\Delta)}{\alpha}}\right]^{\alpha}, \ \alpha \geq 2. \tag{11.21}$$

Within this class of functions, $\alpha = 4$ gives the tightest bound as found in [232].

11.3 Bound on bounds

In this section, we provide a systematic algorithm to estimate how tight the bounds can be made using bandlimited functions $\phi_\pm$. This provides us with a quantitative estimate of the limitation of the procedure which produces these bounds on the $O(1)$ correction to the Cardy formula. If one drops the requirement that the function be bandlimited, one might hope to do better. For the rest of this section, we will restrict ourselves to bandlimited functions only.

We recall that the functions $\phi_\pm$ are chosen in such a way that they satisfy

$$\phi_-(\Delta') < \Theta\left(\Delta' \in [\Delta-\delta, \Delta+\delta]\right) < \phi_+(\Delta'). \tag{11.22}$$

[3]For higher dimensions too, bandlimited functions are used (see, for example, Proposition 6.1 in [233]); nonetheless, they do not provide the tightest bound for the higher dimensional sphere packing problem. For $n = 1$, the function appearing in the said proposition is related to the one that we have used. For other values of n, we obtain bounds strictly less than $1/2$. We thank Tom Hartman for pointing this out.

This inequality gives a trivial bound on $c_\pm$:

$$c_- \leq 1 \leq c_+ \,. \tag{11.23}$$

In what follows, we make this inequality tighter. In this context, the following characterization of the Fourier transform of a positive function in terms of a *positive definite* function turns out to be extremely useful. Before delving into the proof, let us define the notion of positive definiteness of a function. Unless otherwise specified, here we will be dealing with functions from the real line to the complex plane. A function $f(t)$ is said to be positive definite if for every positive integer n and for every set of distinct points $t_1, \ldots, t_n$ chosen from the real line, the $n \times n$ matrix A defined by

$$A_{ij} = f(t_i - t_j) \tag{11.24}$$

is positive definite. A function $g(\Delta)$ is said to be positive if $g(\Delta) > 0$ for every Δ. One can show that the Fourier transform of a positive function is positive definite[4]. Now, let us explore how this characterization can improve the eq. (11.23). Without loss of generality, we set $\Delta = 0$ henceforth, and define

$$g_\pm(\Delta') = \pm\left[\phi_\pm(\Delta') - \Theta\left(\Delta' \in [-\delta, \delta]\right)\right]. \tag{11.25}$$

At this point we use the fact that $\phi_\pm$ is a bandlimited function, i.e., it has a bounded support $[-\Lambda_\pm, \Lambda_\pm]$, and that $\Lambda_\pm < 2\pi$. This requirement stems from the procedure followed in [232]. Thus we arrive at the following:

$$\widetilde{g}_\pm(0) = \pm 2\delta\,(c_\pm - 1)\,, \tag{11.26}$$

$$\widetilde{g}_\pm(t) = \mp 2\delta \left(\frac{\sin(t\delta)}{t\delta}\right) \text{ for } |t| \geq 2\pi. \tag{11.27}$$

The eq. (11.23) states that $\tilde{g}(0)/2\delta > 0$. In order to improve this, we construct 2×2 matrices

[4]The proof is given in a box separately at the end of this subsection for those who are interested.

with $t_2 > 2\pi$:

$$G_{\pm}^{(2)} = \begin{bmatrix} \widetilde{g}_{\pm}(0) & \widetilde{g}_{\pm}(t_2) \\ \widetilde{g}_{\pm}(t_2) & \widetilde{g}_{\pm}(0) \end{bmatrix} . \tag{11.28}$$

For a fixed δ, we consider the first positive peak of $\tilde{g}_{\pm}$ outside $t > 2\pi$. If this occurs at $t = t(\delta)$, we choose $t_2 = t(\delta)$. Subsequently, the positive definiteness of the matrix $G_{\pm}^{(2)}$ boils down to the inequality

$$\widetilde{g}_{\pm}(0) > \widetilde{g}_{\pm}(t(\delta)) , \tag{11.29}$$

where $t(\delta)$ is the first positive peak of $\widetilde{g}_{\pm}$ outside $t > 2\pi$. For example, we can show that (see the green lines in Fig. 11.6):

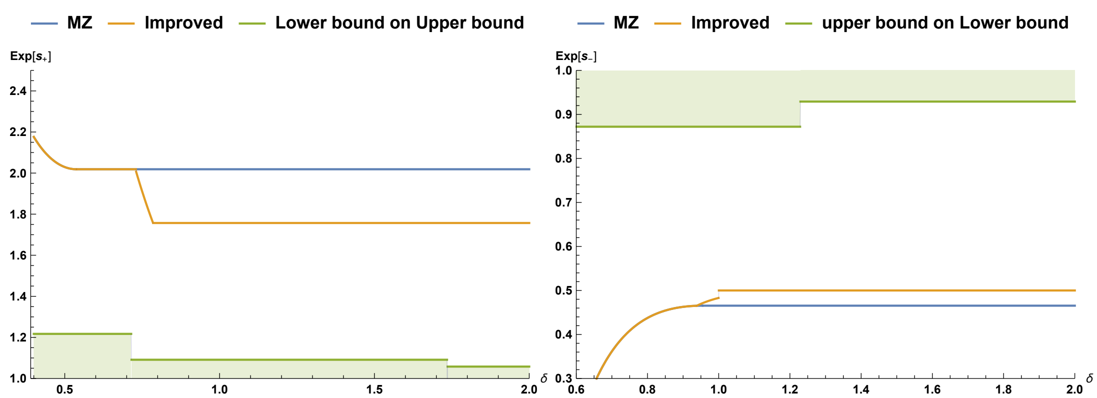

Figure 11.6: Exp$[s_{\pm}]$: The green line is the analytical lower and upper bound on upper and lower bound i.e. $c_{\pm}$ respectively. The green shaded region is not achievable by any bandlimited function.

$$c_{+} > \begin{cases} 1.2172\,, & \delta < 0.715, \\ 1.0913\,, & 1.735 > \delta > 0.715, \\ 1.0579\,, & 2.74 > \delta > 1.736, \end{cases} \tag{11.30}$$

$$c_- < \begin{cases} 0.872\,, & \delta < 1.229, \\ 0.9291\,, & 2.238 > \delta > 1.229. \end{cases} \tag{11.31}$$

We will take a detour now and show that the Fourier transform of an even and positive function is a positive definite function. Consider a function $g(\Delta)$ and let us define the Fourier transform as

$$\widetilde{g}(t) = \int_{-\infty}^{\infty} dt\; g(\Delta) e^{-\imath \Delta t} = 2\int_0^{\infty} dt\; \cos(\Delta t) g(\Delta). \tag{11.32}$$

Now, we construct the matrix

$$G_{ij} = g(t_i - t_j) = 2\int_0^{\infty} dt\; \cos\left[\Delta(t_i - t_j)\right] g(\Delta). \tag{11.33}$$

In order to show that G is a positive definite matrix, i.e., $\sum_{ij} v_i v_j G_{ij} > 0$ for $v_i \in \mathbb{R}$ such that $\sum_i v_i^2 \neq 0$, we think of an auxiliary 2 dimensional space with n vectors $\boldsymbol{v}_{(i)}$, (for clarity, we remark that i labels the vector itself, not its component) such that we have

$$\boldsymbol{v}_{(i)} \equiv \left(|v_i| \cos(\Delta t_i), |v_i| \sin(\Delta t_i)\right). \tag{11.34}$$

Thus, we have

$$\sum_{ij} v_i v_j G_{ij} = 2\int_0^{\infty} dt\; \left(\sum_{ij} v_i v_j \cos\left[\Delta(t_i - t_j)\right]\right) g(\Delta) \tag{11.35}$$

$$= 2\int_0^{\infty} dt\; \left(\boldsymbol{V}\cdot\boldsymbol{V}\right)\; g(\Delta) > 0 \tag{11.36}$$

if $t_1, \ldots, t_n$ are distinct. Here, $\boldsymbol{V}$ is given by

$$\boldsymbol{V} = \sum_i \text{sign}(v_i)\; \boldsymbol{v}_{(i)}\,. \tag{11.37}$$

This completes the proof that the Fourier transform of an even positive function is a positive definite function. First of all, it is easy to see that $c_\pm$, and hence the inequality, is insensitive to the midpoint of the interval, i.e., Δ, so we set it to 0 and this makes the functions $\phi_\pm$ and

Θ even. In particular, we will be applying this theorem to $\phi_+(\Delta') - \Theta(\Delta' \in [\Delta-\delta, \Delta+\delta])$ and $\Theta(\Delta' \in [\Delta-\delta, \Delta+\delta]) - \phi_-(\Delta')$. We make one more remark before exploring the consequences of this. The above result is true for any function, not necessarily even. The converse is also true due to *Bochner's Theorem*, but in what follows, we do not require the converse statement.

Matlab implementation

We implement the above argument using more than two points and making sure that $|t_i - t_j| \geq 2\pi$. For a fixed δ, we use a random number generator to sample the points t_i with the mentioned constraint. We do this multiple times and each time, we test the positive definiteness of the matrix G by providing as an input the value of $\pm(c_\pm - 1)$. The range of $\pm(c_\pm - 1)$ is chosen to be from the first peak $t(\delta)$ till some value larger than the achievable bound given in (11.3) and (11.17). This in turn yields a lower bound (or upper bound) for $c_\pm$ for each trial[5]. Subsequently, we pick out the best possible bound among all the trials. For example, we provide a table [11.1] showing the outputs from a typical run for improving the bound on the upper bound. The tables [11.1] and [11.2] improve the lower (upper) bound for $c_\pm$ and this is shown in the figure [11.1], where the brown dots are the stronger bounds over the green lines and disallow a larger region.

11.4 Bound on spectral gap: towards optimality

In this section, we switch gear and explore the asymptotic spectral gap. In [232], it has recently been shown that the asymptotic gap between Virasoro primaries are bounded above by $2\sqrt{\frac{3}{\pi^2}} \simeq 1.1$ and it has been conjectured that the optimal gap should be 1. The example of Monster CFT tells us that the gap can not be below than 1, hence 1 should be the optimal number. In this section, we show that the previous bound $2\sqrt{\frac{3}{\pi^2}}$ can be improved and made arbitrarily closer to the optimal value 1. Ideally, to prove this one should find out a function f (which

[5]We assume that the mesh size for $c_\pm - 1$ is small enough that one can safely find out a lower bound.

Table 11.1: Typical output from a run yielding lower bounds for the upper bound c_+. The $\mathrm{Max}(c_+)$ column contains a number that is greater than or equal to what can already be achieved.

δ	Number of iterations	# points	$\mathrm{Max}(c_+)$	Lower Bound
0.4	10000	300	2.2	1.7042
0.5	1000	300	2.02	1.6905
0.5	10000	200	2.02	1.7002
0.5	10000	300	2.02	1.7179
0.6	1000	200	2.02	1.6086
0.6	10000	200	2.02	1.5917
0.7	10000	200	2.02	1.4246
0.7	10000	250	2.02	1.4270
0.8	10000	200	1.757	1.3692
0.8	10000	200	2.757	1.3698
0.9	10000	200	2.757	1.3798
1	20000	200	1.757	1.3759
1.1	10000	200	2.757	1.3331
1.20	10000	150	2.757	1.2597
1.25	10000	150	2.757	1.2581
1.3	10000	170	2.757	1.2531
1.4	10000	150	2.757	1.2581
1.5	10000	150	1.757	1.2599
1.5	10000	150	2.757	1.2597
1.5	10000	150	2.757	1.2597
1.6	10000	150	1.757	1.2313
1.7	10000	150	1.757	1.1933

will eventually play the role of ϕ_- in this game, to be precise $f(\Delta') = \phi_-(\Delta + \Delta'))$ such that following holds:

$$f(\Delta') \leq \Theta\left(\Delta' \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]\right) \tag{11.38}$$

and

$$\tilde{f}(t) = 0 \text{ for } |t| \geq \frac{2\pi}{\varepsilon},\ \varepsilon > 1 \tag{11.39}$$

$$\tilde{f}(0) > 0 \tag{11.40}$$

Table 11.2: Typical output from a run providing upper bound for the lower bound c_-. The Min(c_-) column contains a number that is smaller than or equal to what can already be achieved.

δ	Iteration Number	# points	Min(c_-)	Upper Bound
0.6	1000	200	0.173	0.5738
0.6	10000	200	0.173	0.5535
0.7	10000	200	0.362	0.5604
0.7	10000	250	0.362	0.5559
0.8	10000	200	0.44	0.5567
0.9	10000	200	0.46	0.5853
1	10000	200	0.48	0.6960
1.1	10000	200	0.49	0.7112
1.2	10000	150	0.49	0.7161
1.2	10000	180	0.49	0.7161
1.3	10000	170	0.49	0.7111
1.4	10000	150	0.49	0.7243
1.5	10000	150	0.49	0.7788
1.6	20000	150	0.49	0.7895
1.7	20000	150	0.49	0.7861

This would have implied

$$\int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \rho(\Delta') > 0 \tag{11.41}$$

Now what would happen if $\tilde{f}(0) = 0$? One need to go back to the original derivation and reconsider it carefully. Hence instead of the eq. (11.8), we consider a more basic inequality[232]:

$$\begin{aligned} &\exp\left[\beta(\Delta-\delta)\right] \int d\Delta' \rho_0(\Delta') e^{-\beta\Delta'} \phi_-(\Delta') - Z_H \left(\frac{4\pi^2\beta}{\beta^2+\Lambda_-^2}\right) e^{-\beta\frac{c}{12}} \int_{-\Lambda_-}^{\Lambda_-} dt\ |\hat{\phi}(t)| \\ &\leq \int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \rho(\Delta') \end{aligned} \tag{11.42}$$

where $\Lambda_- = \frac{2\pi}{\varepsilon}$ and $Z_H(\beta)$ is the contribution from the heavy states and defined as

$$Z_H(\beta) = \sum_{\Delta > \Delta_H > \frac{c}{12}} e^{-\beta\left(\Delta - \frac{c}{12}\right)}. \tag{11.43}$$

Now we make the following choice for ϕ_-:

$$\phi_-(\Delta') = \frac{\cos^2\left(\frac{\pi(\Delta'-\Delta)}{\varepsilon}\right)}{1-4\left(\frac{\Delta'-\Delta}{\varepsilon}\right)^2}, \quad f(\Delta') = \frac{\cos^2\left(\frac{\pi\Delta'}{\varepsilon}\right)}{1-4\left(\frac{\Delta'}{\varepsilon}\right)^2} \tag{11.44}$$

This function f has following properties:

$$f(\Delta') \leq \Theta\left(\Delta' \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]\right) \tag{11.45}$$

$$\tilde{f}(t) = 0 \quad \text{for} \quad |t| \geq \frac{2\pi}{\varepsilon} \tag{11.46}$$

$$\tilde{f}(0) = 0 \quad \Rightarrow \quad c_- = 0 \tag{11.47}$$

Since $c_- = 0$, one can not readily evaluate the integral appearing in (11.42) by saddle point method and deduce $\exp[\beta(\Delta-\delta)] \int d\Delta' \rho_0(\Delta') e^{-\beta\Delta'} \phi_-(\Delta') = c_- \rho_0(\Delta)$, so we look for subleading corrections to the saddle point approximation. We find that the leading behavior is given by, after setting $\beta = \pi\sqrt{\frac{c}{3\Delta}}$,

$$\exp[\beta(\Delta-\delta)] \int d\Delta' \rho_0(\Delta') e^{-\beta\Delta'} \phi_-(\Delta') = C\rho_0(\Delta), \tag{11.48}$$

where C turns out to be

$$C = \int_0^\infty dx \, \left(\frac{\cos^2\left(\pi\frac{x}{\varepsilon}\right)}{1-4\frac{x^2}{\varepsilon^2}}\right) \exp\left[\frac{-x^2}{2\pi\sqrt{\frac{c}{3}}\Delta^{\frac{3}{2}}}\right]. \tag{11.49}$$

We remark that $C > 0$ for any finite Δ and it becomes 0 only at infinitely large Δ. The second piece in the eq. (11.42) for large Δ goes as $\rho_0(\Delta)^{1-\frac{1}{2}\left(1-\frac{1}{\varepsilon^2}\right)}$. The analysis for this second term is exactly same as done in [232]. For sufficiently large Δ, it can be numerically verified that $\rho_0(\Delta)^{1-\frac{1}{2}\left(1-\frac{1}{\varepsilon^2}\right)}$ is subleading compared to $C\rho_0(\Delta)$ as long as $\varepsilon > 1$ (we also provide an analytical proof later on). Here we have

$$\rho_0(\Delta) \underset{\Delta\to\infty}{=} \left(\frac{c}{48\Delta^3}\right)^{\frac{1}{4}} \exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right] \tag{11.50}$$

In fact one can analytically show that $\rho_0(\Delta)^{1-\frac{1}{2}\left(1-\frac{1}{\varepsilon^2}\right)}$ is subleading to $C\rho_0(\Delta)$ for large

Δ. One way to show this is to have an estimate for C. We start with the observation that the integrand is positive in $\left(0, \frac{\varepsilon}{2}\right)$ and negative in $\left(\frac{\varepsilon}{2}, \infty\right)$. Furthermore, we have

$$\int_0^\infty d\Delta' f(\Delta') = 0 \tag{11.51}$$

Using the above facts, one can always choose $0 < \varepsilon_1 < \frac{\varepsilon}{2}$ and $\frac{\varepsilon}{2} < \varepsilon_2 < \infty$ such that

$$\int_0^{\varepsilon_1} d\Delta' \, f(\Delta') = -\int_{\varepsilon_2}^\infty d\Delta' \, f(\Delta') \tag{11.52}$$

$$\int_{\varepsilon_1}^{\varepsilon_2} d\Delta' \, f(\Delta') = 0 \tag{11.53}$$

This is basically guaranteed by the continuity. We choose ε_1 such that $0 < \varepsilon_1 < \frac{\varepsilon}{2}$ and consider the function $F(y) = \int_{\varepsilon_1}^y dx \, f(x)$. Now $F(y)$ is a continuous function. It is positive when $y = \frac{\varepsilon}{2}$ and negative when $y \to \infty$. Thus by continuity, there exists $\frac{\varepsilon}{2} < \varepsilon_2 < \infty$ such that the eq. (11.52) holds. The shaded region in the figure. 11.7 is the area under the function f restricted to the interval $[\varepsilon_1, \varepsilon_2]$ so that the eq. (11.52) is satisfied.

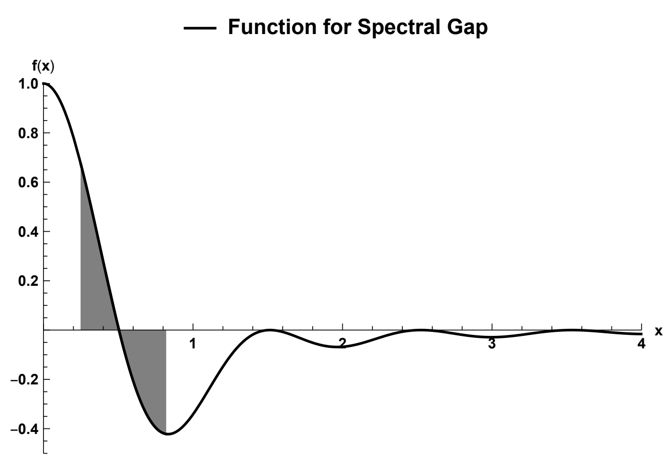

Figure 11.7: The function $\left(\frac{\cos^2\left(\pi \frac{x}{\varepsilon}\right)}{1-4\frac{x^2}{\varepsilon^2}}\right)$, the shaded region is the area under the function restricted to the interval $[\varepsilon_1, \varepsilon_2]$. Here $\varepsilon_1 = 0.25, \varepsilon = 1.01, \varepsilon_2 = 0.819$. These are chosen to ensure the shaded area is 0.

Now we note that

$$\int_{\varepsilon_1}^{\varepsilon_2} dx\, f(x) \exp\left[\frac{-x^2}{2\pi\sqrt{\frac{c}{3}}\Delta^{\frac{3}{2}}}\right] \geq 0 \tag{11.54}$$

and

$$\int_{0}^{\varepsilon_1} dx\, f(x) \exp\left[\frac{-x^2}{2\pi\sqrt{\frac{c}{3}}\Delta^{\frac{3}{2}}}\right] \geq \exp\left[\frac{-\varepsilon_1^2}{2\pi\sqrt{\frac{c}{3}}\Delta^{\frac{3}{2}}}\right] \int_{0}^{\varepsilon_1} dx\, f(x) \tag{11.55}$$

$$\int_{\varepsilon_2}^{\infty} dx\, f(x) \exp\left[\frac{-x^2}{2\pi\sqrt{\frac{c}{3}}\Delta^{\frac{3}{2}}}\right] \geq \exp\left[\frac{-\varepsilon_2^2}{2\pi\sqrt{\frac{c}{3}}\Delta^{\frac{3}{2}}}\right] \int_{\varepsilon_2}^{\infty} dx\, f(x) \tag{11.56}$$

where in the second inequality, we have used negativity of $f(x)$ for $x > \frac{\varepsilon}{2}$. Combining the last four equations i.e (11.52),(11.54),(11.55),(11.56) we can write

$$C \geq \Omega\left(\exp\left[\frac{-\varepsilon_1^2}{2\pi\sqrt{\frac{c}{3}}\Delta^{\frac{3}{2}}}\right] - \exp\left[\frac{-\varepsilon_2^2}{2\pi\sqrt{\frac{c}{3}}\Delta^{\frac{3}{2}}}\right]\right) \underset{\Delta\to\infty}{\simeq} \frac{\left(\varepsilon_2^2 - \varepsilon_1^2\right)\Omega}{2\pi\sqrt{\frac{c}{3}}\Delta^{\frac{3}{2}}} > 0 \tag{11.57}$$

where $\Omega = \int_0^{\varepsilon_1} dx\, f(x) > 0$ is an order one positive number. This clearly proves that as long as $\varepsilon > 1$, we can neglect the second piece i.e. contributions from the heavy states due to its subleading nature. In fact, one can do much better and show that[6] C falls like $\Delta^{-3/4}$ by noting the following:

$$\begin{aligned} C =\ & \frac{\varepsilon\pi}{8} \exp\left[-\frac{1}{8\pi\sqrt{\frac{c}{3}}\Delta^{3/2}}\right] \mathrm{Erfi}\left(\frac{1}{2\sqrt{2\pi\sqrt{\frac{c}{3}}\Delta^{3/4}}}\right) \\ & - \frac{\varepsilon\pi}{8} e^{-\frac{\sqrt{3}}{8\pi\sqrt{c}\Delta^{3/2}}} \mathrm{Im}\left[\mathrm{Erf}\left(\frac{\sqrt{\frac{\pi}{2}}\left(2\pi + \frac{i\sqrt{3}}{2\pi\sqrt{c}\Delta^{3/2}}\right)}{\sqrt[4]{3}\sqrt{\frac{1}{\sqrt{c}\Delta^{3/2}}}}\right)\right] \quad \underset{\Delta\to\infty}{\simeq} \frac{\varepsilon}{8}\left(\frac{3}{64c}\right)^{1/4} \Delta^{-3/4}. \end{aligned} \tag{11.58}$$

To summarize, we have proved that for sufficiently large Δ,

$$\int_{\Delta-\frac{\varepsilon}{2}}^{\Delta+\frac{\varepsilon}{2}} d\Delta'\, \rho(\Delta') \geq C\rho_0(\Delta) > 0 \tag{11.59}$$

Therefore we have been able to show that the asymptotic gap between two consecutive operators is bounded above by ε, where $\varepsilon > 1$. Now one can choose ε to be arbitrarily close to 1,

[6] We thank Alexander Zhiboedov for pointing this out in an email exchange.

which proves that the optimal bound is exactly 1. The analysis can be carried over to the case for Virasoro primaries, as pointed out in [232]. This implies that the asymptotic gap between two consecutive Virasoro primaries is bounded above by 1, thereby proves the conjecture made in [232].

11.5 Brief discussion

In this work, we have improved the existing bound on the $O(1)$ correction to the density of states in 2D CFT at high energy and proven the conjectured upper bound on the gap between Virasoro primaries. In particular, we have shown that there always exists a Virasoro primary in the energy window of width greater than 1 at large Δ.

We have provided a systematic way to estimate how tight the bound can be made using bandlimited functions. Since there is still a gap between the achievable bound and the bound on the bound, there is scope for further improvement. Ideally, one would like to close this gap, which might be possible either by sampling more points and leveraging the positive definiteness condition on a bigger matrix, or by choosing some suitable function which would make the achievable bound closer to the bound on the bound. Another possible way to obtain the bound on bound is to use a known 2D CFT partition functions, for example 2D Ising model and explicitly evaluate $s(\delta,\Delta)$. It would be interesting to see how the bound on bound obtained in this paper compares to the one which can be obtained from the 2D Ising model. For example, one can verify that the bound on bound obtained here is stronger than that could be obtained from 2D Ising model[7] for $\delta = 1$. It would be interesting to further explore this.

The utility of the technique developed here lies beyond the $O(1)$ correction to the

[7] We thank Alexander Zhiboedov for raising this question of how our bound compares to $s(1.7,\Delta)$ for the 2D Ising model, as found in [232].

Cardy formula. We expect the technique to be useful whenever one wants to leverage the complex Tauberian theorems. As emphasized in [232], the importance of Tauberian theorems lies beyond the discussion of 2D CFT partition functions, especially in investigating Eigenstate Thermalization Hypothesis [235, 236, 237, 238] in 2D CFTs[239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249]. We end with a cautious remark that if we relax the condition of using bandlimited functions, the bound on bounds would not be applicable and it might be possible to obtain nicer achievable bounds on the $O(1)$ correction to the Cardy formula.

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Chapter 11 in full, is a reprint of the material as it appears in Shouvik Ganguly, Sridip Pal, arXiv:1905.12636. The dissertation/thesis author was the primary investigator and author of this paper.

Appendix A

Weyl Consistency Conditions in Non Relativistic Quantum Field Theory

A.1 Consistency Conditions for $2+1$d NRCFT

We give below the consistency conditions for the $d=2$ theory. In the $\partial_t\nabla^2$ sector they are given for arbitrary z; else $z=2$ is assumed. The conditions in the $\partial_t\nabla^2$ sector are as follows:

$$\sigma\partial_t\sigma'\partial_i N\partial^i g^\alpha : \quad -\beta^\gamma\partial_\gamma\rho_{1\alpha}-\rho_{1\gamma}\partial_\alpha\beta^\gamma+2\rho_\alpha-2\partial_\alpha j+p_{4\gamma\alpha}\beta^\gamma=0 \tag{A.1a}$$

$$\sigma\partial_t\sigma'\nabla^2 g^\alpha : \quad -b_{4\sigma}\partial_\alpha\beta^\sigma-\beta^\sigma\partial_\sigma b_{4\alpha}+x_{4\gamma\alpha}\beta^\gamma+2b_{8\alpha}=0 \tag{A.1b}$$

$$\nabla^2\sigma\partial_t\sigma' : \quad 2k+2m-zl_2-b_{4\alpha}\beta^\alpha+\beta^\alpha b_{9\alpha}=0 \tag{A.1c}$$

$$-\sigma'\partial_t\sigma\nabla^2 N : \quad 2j+\beta^\alpha\partial_\alpha l_2=b_{6\alpha}\beta^\alpha \tag{A.1d}$$

$$\sigma'\partial_i\sigma\partial^i N\partial_t g^\alpha : \quad -2zb_{6\alpha}-2z\rho_{6\alpha}+\rho_{5\gamma}\partial_\alpha\beta^\gamma+\beta^\gamma\partial_\gamma\rho_{5\alpha}-\beta^\gamma p_{4\alpha\gamma}=0 \tag{A.1e}$$

$$\sigma'\nabla^2\sigma K : \quad 2b+\beta^\alpha\partial_\alpha m+zj=\beta^\alpha b_{8\alpha} \tag{A.1f}$$

$$\sigma'\partial_t\sigma R : \quad 2b-\beta^\alpha\partial_\alpha k+b_{5\alpha}\beta^\alpha=0 \tag{A.1g}$$

$$\begin{aligned}\sigma'\partial_i\sigma K\partial^i g^\alpha : \quad & -2x_{5\alpha\gamma}\beta^\gamma+\beta^\gamma\partial_\gamma b_{7\alpha}-z\rho_\alpha \\ & +b_{7\gamma}\partial_\alpha\beta^\gamma-2b_{8\gamma}\partial_\alpha\beta^\gamma+z\partial_\alpha j=0\end{aligned} \tag{A.1h}$$

$$\sigma'\nabla^2\sigma\partial_t g^\alpha : \quad -x_{4\alpha\gamma}\beta^\gamma+2b_{5\alpha}-zb_{6\alpha}+b_{9\gamma}\partial_\alpha\beta^\gamma+\beta^\gamma\partial_\gamma b_{9\alpha}=0 \tag{A.1i}$$

$$\partial_i\sigma\partial_t\sigma'\partial^i g^\alpha : \quad -2b_{4\gamma}\partial_\alpha\beta^\gamma+2b_{7\alpha}+x_{6\gamma\alpha}\beta^\gamma-2b_{3\alpha\gamma}\beta^\gamma-z\rho_{1\alpha}=0 \tag{A.1j}$$

$$\sigma\partial_t\sigma'\partial_i g^\alpha\partial^i g^\beta: \quad -\beta^\gamma\partial_\gamma b_{3\alpha\beta} - b_{3\gamma\beta}\partial_\alpha\beta^\gamma - b_{3\gamma\alpha}\partial_\beta\beta^\gamma - b_{4\gamma}\partial_\alpha\partial_\beta\beta^\gamma + x_{3\gamma\alpha\beta}\beta^\gamma + 2x_{5\alpha\beta} = 0 \tag{A.1k}$$

$$-\sigma\partial_i\sigma'\partial_t g^\alpha\partial^i g^\beta: \quad -2x_{3\alpha\gamma\beta}\beta^\gamma + x_{6\sigma\beta}\partial_\alpha\beta^\sigma + x_{6\alpha\gamma}\partial_\beta\beta^\gamma - 2x_{4\alpha\sigma}\partial_\beta\beta^\sigma + \beta^\gamma\partial_\gamma x_{6\alpha\beta} - zp_{4\alpha\beta} = 0 \tag{A.1l}$$

$$\sigma\partial_t\sigma'\frac{\partial^i N}{N}\frac{\partial_i N}{N}: \quad \beta^\gamma\partial_\gamma\rho_3 - 2\rho_4 - \beta^\alpha\rho_{6\alpha} = 0 \tag{A.1m}$$

$$\sigma\partial_i\sigma' K\partial^i N: \quad 2zj - \beta^\alpha\rho_\alpha + \beta^\alpha\partial_\alpha l_1 - 2z\rho_4 = 0 \tag{A.1n}$$

$$\partial_i\sigma\partial_t\sigma'\partial^i N: \quad 2z\rho_3 - 2l_1 + 2zl_2 + 2j + \rho_{1\alpha}\beta^\alpha = 0 \tag{A.1o}$$

$$\sigma'\partial^j\sigma\partial^i N\left(K_{ij} - \tfrac{1}{2}Kh_{ij}\right): \quad 2zf_1 - \beta^\alpha f_{3\alpha} + zf_4 - \beta^\alpha\partial_\alpha f_7 = 0 \tag{A.1p}$$

$$\sigma'\partial^j\sigma\partial^i g^\alpha\left(K_{ij} - \tfrac{1}{2}Kh_{ij}\right): \quad 2f_{2\alpha\gamma}\beta^\gamma + zf_{3\alpha} - \beta^\gamma\partial_\gamma f_{8\alpha} - f_{8\gamma}\partial_\alpha\beta^\gamma + 2f_{5\gamma}\partial_\alpha\beta^\gamma = 0 \tag{A.1q}$$

$$\sigma'\nabla^i\partial^j\sigma\left(K_{ij} - \tfrac{1}{2}Kh_{ij}\right): \quad zf_4 + \beta^\alpha f_{5\alpha} - \beta^\alpha\partial_\alpha f_6 = 0 \tag{A.1r}$$

The conditions coming from ∂_t^2 sector are as follows:

$$\sigma'\partial_t\sigma K: \quad 4d - \beta^\alpha\partial_\alpha f + \beta^\alpha w_\alpha = 0 \tag{A.2a}$$

$$\sigma'\partial_t\sigma\partial_t g^\alpha: \quad -2w_\alpha + \beta^\gamma\partial_\gamma b_\alpha + b_\gamma\partial_\alpha\beta^\gamma - 2\chi_{0\alpha\gamma}\beta^\gamma = 0 \tag{A.2b}$$

The conditions coming from the ∇^4 sector are given by:

$$\partial_i\sigma'\nabla^2\sigma\partial^i g^\alpha: \quad -2\rho_{13\alpha} - \beta^\gamma\rho_{21\alpha\gamma} + 2a_{7\alpha} + 2\chi_{1\alpha} + 2a_{3\alpha\gamma}\beta^\gamma + 2a_{4\gamma}\partial_\alpha\beta^\gamma = 0 \tag{A.3a}$$

$$\sigma'\nabla^2\sigma\partial_i g^\alpha\partial^i g^\beta: \quad -\beta^\gamma t_{2\gamma\alpha\beta} + \beta^\gamma\partial_\gamma a_{3\alpha\beta} + a_{3\alpha\gamma}\partial_\beta\beta^\gamma + a_{3\beta\gamma}\partial_\alpha\beta^\gamma + a_{4\gamma}\partial_\alpha\partial_\beta\beta^\gamma + 2y_{5\alpha\beta} - 2x_{2\alpha\beta} = 0 \tag{A.3b}$$

$$\sigma'\nabla^2\sigma\nabla^2 g^\alpha: \quad 2a_{5\alpha} + \beta^\gamma\partial_\gamma a_{4\alpha} + a_{4\gamma}\partial_\alpha\beta^\gamma - 2\rho_{22\alpha\gamma}\beta^\gamma - 2\rho_{24\alpha} = 0 \tag{A.3c}$$

$$\sigma'\partial_i\sigma\partial^i g^\alpha\nabla^2 N: \quad -2x_{2\alpha\gamma}\beta^\gamma + \beta^\gamma\partial_\gamma\rho_{13\alpha} + \rho_{13\gamma}\partial_\alpha\beta^\gamma - 2\rho_{24\gamma}\partial_\alpha\beta^\gamma - 2\rho_{25\alpha} = 0 \tag{A.3d}$$

$$\sigma'\partial_i\sigma\partial^i N\partial_j N\partial^j g^\alpha: \quad -2p_{5\beta\alpha}\beta^\beta+\beta^\gamma\partial_\gamma\rho_{10\alpha}+\rho_{10\gamma}\partial_\alpha\beta^\gamma-4\rho_{25\alpha}-4\rho_{9\alpha}=0 \tag{A.3e}$$

$$\begin{aligned}\sigma'\partial_i\sigma\partial^i g^\alpha\partial_j g^\beta\partial^j g^\gamma: \quad & -4x_{\alpha\sigma\beta\gamma}\beta^\sigma+\beta^\sigma\partial_\sigma t_{\alpha\beta\gamma}+t_{\sigma\beta\gamma}\partial_\alpha\beta^\sigma+t_{\alpha\sigma\gamma}\partial_\beta\beta^\sigma+t_{\alpha\sigma\beta}\partial_\gamma\beta^\sigma \\ & -2t_{2\sigma\beta\gamma}\partial_\alpha\beta^\sigma+\rho_{21\alpha\sigma}\partial_\beta\partial_\gamma\beta^\sigma-2x_{\alpha\beta\gamma}=0\end{aligned} \tag{A.3f}$$

$$\begin{aligned}\sigma'\partial_i\sigma\partial^i g^\alpha\partial_j N\partial^j g^\beta: \quad & -4p_{5\alpha\beta}-2\rho_{26\beta\gamma}\partial_\alpha\beta^\gamma \\ & +x_{1\alpha\gamma}\partial_\beta\beta^\gamma+x_{1\gamma\beta}\partial_\alpha\beta^\gamma+\beta^\gamma\partial_\gamma x_{1\alpha\beta}-2x_{\alpha\gamma\beta}\beta^\gamma=0\end{aligned} \tag{A.3g}$$

$$\begin{aligned}\sigma\partial_i\sigma'\partial^i g^\alpha\nabla^2 g^\beta: \quad & -\beta^\gamma\partial_\gamma\rho_{21\alpha\beta}-\rho_{21\gamma\beta}\partial_\alpha\beta^\gamma-\rho_{21\alpha\gamma}\partial_\beta\beta^\gamma \\ & +4\rho_{22\gamma\beta}\partial_\alpha\beta^\gamma+2\rho_{26\alpha\beta}+2t_{2\beta\gamma\alpha}\beta^\gamma=0\end{aligned} \tag{A.3h}$$

$$\sigma'\nabla^2\sigma\nabla^2 N: \quad -4\rho_{23}-\beta^\gamma\rho_{24\gamma}+\beta^\gamma\partial_\gamma h_2-2c=0 \tag{A.3i}$$

$$\sigma'\partial_i\sigma\partial^i N\nabla^2 N: \quad 4p_4-\beta^\alpha\partial_\alpha\rho_{12}+8\rho_{23}+\beta^\gamma\rho_{25\gamma}=0 \tag{A.3j}$$

$$\sigma'\partial_i\sigma\partial^i N\nabla^2 g^\alpha: \quad -4y_\alpha+\beta^\gamma\partial_\gamma\rho_{7\alpha}+\rho_{7\gamma}\partial_\alpha\beta^\gamma-4\rho_{24\alpha}-\beta^\gamma\rho_{26\gamma\alpha}=0 \tag{A.3k}$$

$$-\sigma'\nabla^2\sigma R: \quad -a_{5\alpha}\beta^\alpha+4a+2c+\beta^\alpha\partial_\alpha n=0 \tag{A.3l}$$

$$\begin{aligned}\sigma'\partial_i\sigma R\partial^i g^\alpha: \quad & -2y_{5\alpha\gamma}\beta^\gamma-2\chi_\alpha+2\partial_\alpha c \\ & +\beta^\gamma\partial_\gamma a_{7\alpha}+a_{7\gamma}\partial_\alpha\beta^\gamma-2a_{5\gamma}\partial_\alpha\beta^\gamma=0\end{aligned} \tag{A.3m}$$

$$\sigma\nabla^2\sigma'\partial^i g^\alpha\partial_i N: \quad -\beta^\gamma\partial_\gamma\chi_{1\alpha}-\chi_{1\gamma}\partial_\alpha\beta^\gamma+2\partial_\alpha c-2\chi_\alpha+\rho_{26\alpha\gamma}\beta^\gamma+2\rho_{25\alpha}=0 \tag{A.3n}$$

$$\partial_i\sigma\nabla^2\sigma'\partial^i N: \quad 2h_1+4h_2-2c+\beta^\alpha\chi_{1\alpha}+4\chi_3-\beta^\alpha\rho_{7\alpha}-2\rho_{12}=0 \tag{A.3o}$$

$$\sigma'\partial_i\sigma\partial^i N\partial_j N\partial^j N: \quad 8p_3-\beta^\alpha\partial_\alpha\rho_{11}+\rho_{9\alpha}\beta^\alpha+4p_4=0 \tag{A.3p}$$

$$\begin{aligned}\sigma\partial_i\sigma'\partial_j g^\alpha\partial^j g^\beta\partial^i N: \quad & 4x_{\alpha\beta}-\rho_{7\gamma}\partial_\beta\partial_\alpha\beta^\gamma-\rho_{1\gamma\beta}\partial_\alpha\beta^\gamma \\ & -\rho_{1\gamma\alpha}\partial_\beta\beta^\gamma-\beta^\gamma\partial_\gamma\rho_{1\alpha\beta}+\beta^\gamma x_{\gamma\alpha\beta}+4x_{2\alpha\beta}=0\end{aligned} \tag{A.3q}$$

$$\sigma\partial_i\sigma' R\partial^i N: \quad 4c+\beta^\alpha\partial_\alpha h_1-\beta^\alpha\chi_\alpha-4\chi_4=0 \tag{A.3r}$$

$$\sigma'\nabla^2\sigma\partial^i N\partial_i N: \quad 2\chi_4+\beta^\alpha\partial_\alpha\chi_3-\beta^\alpha y_\alpha-2p_4=0 \tag{A.3s}$$

$$\sigma'\partial_i\sigma\partial_j N\partial^j N\partial^i g^\alpha: \quad -2x_{\alpha\gamma}\beta^\gamma+\beta^\gamma\partial_\gamma\rho_{8\alpha}+\rho_{8\gamma}\partial_\alpha\beta^\gamma-2\rho_{9\alpha}-2y_\gamma\partial_\alpha\beta^\gamma=0 \tag{A.3t}$$

A.2 Anomaly ambiguities

As explained in Sec. 2.3.2 the freedom to shift counter-terms by finite amount makes anomaly coefficients ambiguous. We list here the precise form of these ambiguities, in the $\partial_t \nabla^2$ sector they are given for arbitrary z; else $z = 2$ is assumed :

A.2.1 ∂_t^2 Sector

$$\delta f = -4D - \beta^{\alpha} W_{\alpha} \tag{A.4a}$$

$$\delta w_{\alpha} = -\left[\beta^{\gamma}\partial_{\gamma} W_{\alpha} + W_{\gamma}\partial_{\alpha}\beta^{\gamma}\right] \tag{A.4b}$$

$$\delta b_{\alpha} = -2W_{\alpha} - 2X_{0\alpha\gamma}\beta^{\gamma} \tag{A.4c}$$

$$\delta \chi_{0\alpha\beta} = -\beta^{\gamma}\partial_{\gamma} X_{0\alpha\beta} - X_{0\alpha\gamma}\partial_{\beta}\beta^{\gamma} - X_{0\alpha\gamma}\partial_{\beta}\beta^{\gamma} \tag{A.4d}$$

$$\delta d = -\beta^{\alpha}\partial_{\alpha} D \tag{A.4e}$$

$$\delta e = -\beta^{\alpha}\partial_{\alpha} E \tag{A.4f}$$

A.2.2 $\partial_t \nabla^2$ Sector

$$\delta \rho_4 = -\beta^{\alpha}\partial_{\alpha} P \tag{A.5}$$

$$\delta x_{5\alpha\beta} = -P_{3\gamma}\partial_{\alpha}\partial_{\beta}\beta^{\gamma} - \beta^{\gamma}\partial_{\gamma} X_{5\alpha\beta} - X_{5\gamma\beta}\partial_{\alpha}\beta^{\gamma} - X_{5\gamma\alpha}\partial_{\beta}\beta^{\gamma} \tag{A.6}$$

$$\delta \rho_{\alpha} = -\beta^{\gamma}\partial_{\gamma} P_{\alpha} - P_{\gamma}\partial_{\alpha}\beta^{\gamma} \tag{A.7}$$

$$\delta j = -\beta^{\alpha}\partial_{\alpha} L \delta b_{8\alpha} = -\beta^{\gamma}\partial_{\gamma} P_{3\alpha} - P_{3\gamma}\partial_{\alpha}\beta^{\gamma} \tag{A.8}$$

$$\delta b = -\beta^{\alpha}\partial_{\alpha} B \tag{A.9}$$

$$\delta m = 2B + zL - P_{3\alpha}\beta^{\alpha} \tag{A.10}$$

$$\delta l_1 = -2zP + 2zL - \beta^{\alpha} P_{\alpha} \tag{A.11}$$

$$\delta b_{7\alpha} = -2P_{3\gamma}\partial_\alpha\beta^\gamma + z\partial_\alpha L - zP_\alpha - 2X_{5\alpha\gamma}\beta^\gamma \tag{A.12}$$

$$\delta\rho_{6\alpha} = -X_\gamma\partial_\alpha\beta^\gamma - \beta^\gamma\partial_\gamma X_\alpha \tag{A.13}$$

$$\delta x_{3\alpha\beta\gamma} = -\beta^\sigma\partial_\sigma X_{3\alpha\beta\gamma} - X_{3\sigma\beta\gamma}\partial_\alpha\beta^\sigma - X_{3\alpha\sigma\gamma}\partial_\beta\beta^\sigma - X_{3\alpha\sigma\beta}\partial_\gamma\beta^\sigma - X_{4\alpha\sigma}\partial_\gamma\partial_\beta\beta^\sigma \tag{A.14}$$

$$\delta p_{4\alpha\beta} = -\beta^\gamma\partial_\gamma P_{4\alpha\beta} - P_{4\gamma\beta}\partial_\alpha\beta^\gamma - P_{4\alpha\gamma}\partial_\beta\beta^\gamma \tag{A.15}$$

$$\delta b_{6\alpha} = -\beta^\gamma\partial_\gamma B_{6\alpha} - B_{6\gamma}\partial_\alpha\beta^\gamma \tag{A.16}$$

$$\delta x_{4\alpha\beta} = -X_{4\gamma\beta}\partial_\alpha\beta^\gamma - \beta^\gamma\partial_\gamma X_{4\alpha\beta} - X_{4\alpha\gamma}\partial_\beta\beta^\gamma \tag{A.17}$$

$$\delta b_{5\alpha} = -\beta^\gamma\partial_\gamma B_{5\alpha} - B_{5\gamma}\partial_\alpha\beta^\gamma \tag{A.18}$$

$$\delta b_{9\alpha} = 2B_{5\alpha} - zB_{6\alpha} - \beta^\gamma X_{4\alpha\gamma} \tag{A.19}$$

$$\delta\rho_{5\alpha} = -2zX_\alpha - 2zB_{6\alpha} - P_{4\alpha\gamma}\beta^\gamma \tag{A.20}$$

$$\delta x_{6\alpha\beta} = -X_{4\alpha\gamma}\partial_\beta\beta^\gamma - 2X_{3\alpha\gamma\beta}\beta^\gamma - zP_{4\alpha\beta} \tag{A.21}$$

$$\delta\rho_3 = -2P - \beta^\alpha X_\alpha \tag{A.22}$$

$$\delta b_{3\alpha\beta} = -2X_{5\alpha\beta} - X_{3\gamma\beta\alpha}\beta^\gamma \tag{A.23}$$

$$\delta\rho_{1\alpha} = -2P_\alpha + \partial_\alpha 2L - P_{4\gamma\alpha}\beta^\gamma \tag{A.24}$$

$$\delta l_2 = 2L - B_{6\alpha}\beta^\alpha \tag{A.25}$$

$$\delta b_{4\alpha} = -2P_{3\alpha} - X_{4\gamma\alpha}\beta^\gamma \tag{A.26}$$

$$\delta k = -2B - B_{5\alpha}\beta^\alpha \tag{A.27}$$

$$\delta f_1 = -\beta^\alpha\partial_\alpha F_1 \tag{A.28}$$

$$\delta f_{2\alpha\beta} = -\beta^\gamma\partial_\gamma F_{2\alpha\beta} - F_{2\gamma\beta}\partial_\alpha\beta^\gamma - F_{2\alpha\gamma}\partial_\beta\beta^\gamma - F_{5\gamma}\partial_\alpha\partial_\beta\beta^\gamma \tag{A.29}$$

$$\delta f_{3\alpha} = -\beta^\gamma\partial_\gamma F_{3\alpha} - F_{3\gamma}\partial_\alpha\beta^\gamma \tag{A.30}$$

$$\delta f_4 = -\beta^\gamma\partial_\gamma F_4 \tag{A.31}$$

$$\delta f_{5\alpha} = -\beta^\gamma\partial_\gamma F_{5\alpha} - F_{5\gamma}\partial_\alpha\beta^\gamma \tag{A.32}$$

$$\delta f_6 = -zF_4 - F_{5\alpha}\beta^\alpha \tag{A.33}$$

$$\delta f_7 = -2zF_1 - F_{3\alpha}\beta^\alpha - zF_4 \tag{A.34}$$

$$\delta f_{8\alpha} = -zF_{3\alpha} - 2F_{2\gamma\alpha}\beta^{\gamma} - 2F_{5\gamma}\partial_{\alpha}\beta^{\gamma} \tag{A.35}$$

A.2.3 ∇^4 Sector

$$\delta p_3 = -\beta^{\alpha}\partial_{\alpha}P_3 \tag{A.36}$$

$$\delta x_{\alpha\beta} = -Y_{\gamma}\partial_{\beta}\partial_{\alpha}\beta^{\gamma} - X_{\gamma\beta}\partial_{\alpha}\beta^{\gamma} - X_{\gamma\alpha}\partial_{\beta}\beta^{\gamma} - \beta^{\gamma}\partial_{\gamma}X_{\alpha\beta} \tag{A.37}$$

$$\delta\rho_{9\alpha} = -\beta^{\gamma}\partial_{\gamma}P_{1\alpha} - P_{1\gamma}\partial_{\alpha}\beta^{\gamma} \tag{A.38}$$

$$\delta p_4 = -\beta^{\alpha}\partial_{\alpha}P_4 \tag{A.39}$$

$$\delta y_{\alpha} = -\beta^{\gamma}\partial_{\gamma}Y_{\alpha} - Y_{\gamma}\partial_{\alpha}\beta^{\gamma} \tag{A.40}$$

$$\delta\chi_4 = -\beta^{\alpha}\partial_{\alpha}Q \tag{A.41}$$

$$\delta\chi_3 = 2Q - \beta^{\alpha}Y_{\alpha} - 2P_4 \tag{A.42}$$

$$\delta\rho_{11} = -8P_3 - P_{1\alpha}\beta^{\alpha} - 4P_4 \tag{A.43}$$

$$\delta\rho_{8\alpha} = -2P_{1\alpha} - 2X_{\alpha\gamma}\beta^{\gamma} - 2Y_{\gamma}\partial_{\alpha}\beta^{\gamma} \tag{A.44}$$

$$\begin{aligned}\delta x_{\alpha\beta\gamma\delta} = &-\beta^{\sigma}\partial_{\sigma}X_{\alpha\beta\gamma\delta} - X_{\sigma\beta\gamma\delta}\partial_{\alpha}\beta^{\sigma} - X_{\alpha\sigma\gamma\delta}\partial_{\beta}\beta^{\sigma} - X_{\alpha\beta\sigma\delta}\partial_{\gamma}\beta^{\sigma} \\ &- X_{\alpha\beta\gamma\sigma}\partial_{\gamma}\beta^{\sigma} - T_{2\sigma\alpha\beta}\partial_{\delta}\partial_{\gamma}\beta^{\sigma}\end{aligned} \tag{A.45}$$

$$\delta x_{\alpha\beta\gamma} = -\beta^{\sigma}\partial_{\sigma}X_{\alpha\beta\gamma} - X_{\sigma\beta\gamma}\partial_{\alpha}\beta^{\sigma} - X_{\alpha\sigma\gamma}\partial_{\beta}\beta^{\sigma} - X_{\alpha\beta\sigma}\partial_{\gamma}\beta^{\sigma} - P_{26\alpha\sigma}\partial_{\gamma}\partial_{\beta}\beta^{\sigma} \tag{A.46}$$

$$\delta x_{2\alpha\beta} = -X_{2\gamma\beta}\partial_{\alpha}\beta^{\gamma} - X_{2\gamma\alpha}\partial_{\beta}\beta^{\gamma} - \beta^{\gamma}\partial_{\gamma}X_{2\alpha\beta} - P_{24\gamma}\partial_{\alpha}\partial_{\beta}\beta^{\gamma} \tag{A.47}$$

$$\delta t_{2\alpha\beta\gamma} = -\beta^{\sigma}\partial_{\sigma}T_{2\alpha\beta\gamma} - T_{2\sigma\beta\gamma}\partial_{\alpha}\beta^{\sigma} - T_{2\alpha\sigma\gamma}\partial_{\beta}\beta^{\sigma} - T_{2\alpha\beta\sigma}\partial_{\gamma}\beta^{\sigma} - 2P_{22\sigma\alpha}\partial_{\gamma}\partial_{\beta}\beta^{\sigma} \tag{A.48}$$

$$\delta y_{5\alpha\beta} = -A_{5\gamma}\partial_{\alpha}\partial_{\beta}\beta^{\gamma} - Y_{5\alpha\gamma}\partial_{\beta}\beta^{\gamma} - Y_{5\beta\gamma}\partial_{\alpha}\beta^{\gamma} - \beta^{\gamma}\partial_{\gamma}Y_{5\alpha\beta} \tag{A.49}$$

$$\delta a_{3\alpha\beta} = -2X_{2\alpha\beta} - \beta^{\gamma}T_{2\gamma\alpha\beta} + 2Y_{5\alpha\beta} \tag{A.50}$$

$$\delta\rho_{1\alpha\beta} = -4X_{\alpha\beta} - X_{\gamma\alpha\beta}\beta^{\gamma} - 4X_{2\alpha\beta} \tag{A.51}$$

$$\delta t_{\alpha\beta\gamma} = -4X_{\alpha\sigma\beta\gamma}\beta^{\sigma} - 2T_{2\sigma\gamma\beta}\partial_{\alpha}\beta^{\sigma} - 2X_{\alpha\beta\gamma} \tag{A.52}$$

$$\delta p_{5\alpha\beta} = -\beta^{\gamma}\partial_{\gamma}P_{5\alpha\beta} - P_{5\gamma\beta}\partial_{\alpha}\beta^{\gamma} - P_{5\gamma\alpha}\partial_{\beta}\beta^{\gamma} \tag{A.53}$$

$$\delta\rho_{25\alpha} = -\beta^\gamma \partial_\gamma P_{25\alpha} - P_{25\gamma}\partial_\alpha \beta^\gamma \tag{A.54}$$

$$\delta\rho_{26\alpha\beta} = -\beta^\gamma \partial_\gamma P_{26\alpha\beta} - P_{26\gamma\beta}\partial_\alpha \beta^\gamma - P_{26\alpha\gamma}\partial_\beta \beta^\gamma \tag{A.55}$$

$$\delta\chi_\alpha = -\beta^\gamma \partial_\gamma Q_\alpha - Q_\gamma \partial_\alpha \beta^\gamma \tag{A.56}$$

$$\delta\chi_{1\alpha} = 2Q_\alpha - 2\partial_\alpha H - 2P_{25\alpha} - \beta^\gamma P_{26\alpha\gamma} \tag{A.57}$$

$$\delta\rho_{10\alpha} = -4P_{1\alpha} - 4P_{25\alpha} - 2P_{5\gamma\alpha}\beta^\gamma \tag{A.58}$$

$$\delta x_{1\alpha\beta} = -4P_{5\alpha\beta} - 2X_{\alpha\gamma\beta}\beta^\gamma - 2P_{26\beta\gamma}\partial_\alpha \beta^\gamma \tag{A.59}$$

$$\delta\rho_{23} = -\beta^\gamma \partial_\gamma P_{23} \tag{A.60}$$

$$\delta\rho_{24\alpha} = -\beta^\gamma \partial_\gamma P_{24\alpha} - P_{24\gamma}\partial_\alpha \beta^\gamma \tag{A.61}$$

$$\delta c = -\beta^\alpha \partial_\alpha H \tag{A.62}$$

$$\delta h_2 = -2H - 4P_{23} - \beta^\gamma P_{24\gamma} \tag{A.63}$$

$$\delta\rho_{12} = -4P_4 - 8P_{23} - P_{25\alpha}\beta^\alpha \tag{A.64}$$

$$\delta\rho_{13\alpha} = -2P_{24\gamma}\partial_\alpha \beta^\gamma - 2P_{25\alpha} - 2\beta^\gamma X_{2\alpha\gamma} \tag{A.65}$$

$$\delta\rho_{22\alpha\beta} = -\beta^\gamma \partial_\gamma P_{22\alpha\beta} - P_{22\alpha\gamma}\partial_\beta \beta^\gamma - P_{22\gamma\beta}\partial_\alpha \beta^\gamma \tag{A.66}$$

$$\delta a_{5\alpha} = -\beta^\gamma \partial_\gamma A_{5\alpha} - A_{5\gamma}\partial_\alpha \beta^\gamma \tag{A.67}$$

$$\delta a_{4\alpha} = 2A_{5\alpha} - 2\beta^\gamma P_{22\gamma\alpha} - 2P_{24\alpha} \tag{A.68}$$

$$\delta\rho_{7\alpha} = -4Y_\alpha - 4P_{24\alpha} - P_{26\gamma\alpha}\beta^\gamma \tag{A.69}$$

$$\delta\rho_{21\alpha\beta} = -2T_{2\alpha\gamma\beta}\beta^\gamma - 4P_{22\gamma\beta}\partial_\alpha \beta^\gamma - 2P_{26\alpha\beta} \tag{A.70}$$

$$\delta a = -\beta^\alpha \partial_\alpha A \tag{A.71}$$

$$\delta n = 4A + 2H - A_{5\alpha}\beta^\alpha \tag{A.72}$$

$$\delta h_1 = 4H - 4Q - \beta^\alpha Q_{1\alpha} \tag{A.73}$$

$$\delta a_{7\alpha} = -2A_{5\gamma}\partial_\alpha \beta^\gamma - 2Y_{5\alpha\gamma}\beta^\gamma - 2Q_\alpha + 2\partial_\alpha H \tag{A.74}$$

A.3 S-theorem: $0+1$D conformal quantum mechanics

One may wonder whether the formalism that leads to the Weyl anomaly and consistency conditions can be used for the case of $d=0$. One encounters an immediate obstacle when attempting this. There is no immediate generalization of the trace anomaly equation (2.23). The problem is that there is no extension of the action integral that gives invariance under the local version of rescaling transformations, because there is no extrinsic curvature tensor at our disposal. The naive generalisation of the Callan-Symanzik equation specialized to $d=0$, $H=\beta^{\alpha}O_{\alpha}$, cannot hold. In fact, for example, the free particle is a scale invariant system with $H\neq 0$.

The inverse square potential serves as a test ground for a simple realisation of the quantum anomaly, where the classical scale symmetry is broken by quantum mechanical effects [250] leading to dimensional transmutation *i.e,* after renormalization the quantum system acquires an intrinsic length scale [251, 252]. Studies have been made of non-self-adjointness of the Hamiltonian in the strongly attractive regime and how to obtain its self-adjoint extension, a procedure that effectively amounts to renormalisation [253, 254]. The system is also shown to exhibit limit cycle behaviour in renormalization group flows [255, 256]. This potential appears in different branches of physics, from nuclear physics [256, 257] and molecular physics [258] to quantum cosmology [259, 260, 261] and the study of black holes [262] Given this, it is of interest to understand how quantum effects break scale symmetry in non-relativist quantum mechanics. We will prove a general theorem concerning the breaking of scale symmetry.

In the quantum mechanical description of a scale invariant system, the Hamiltonian H and the generator of scale transformations D obey the following commutation relation:

$$[D,H]=izH \tag{A.75}$$

where z is the dynamical exponent of the theory. We will show an elementary *S-Theorem*, that

(A.75) is incompatible with H being Hermitian on a domain containing the state[1] $D|E\rangle$, where $|E\rangle$ is any non-zero energy eigenstate. The S-Theorem can be used to deduce that classically scale invariant systems, *e.g.,* the inverse square potential, cannot be quantized without loosing either unitarity or scale invariance if we insist on having bound states with finite non-zero binding energy.

To prove the theorem, we consider the eigenstates $|E\rangle$ of the Hamiltonian H and take expectation value of the $[D,H]$ in these eigenstates. We have

$$\langle E|[D,H]|E\rangle = \langle E|DH|E\rangle - \langle E|HD|E\rangle \tag{A.76}$$

Assuming H is hermitian and D is well defined we have

$$\langle E|[D,H]|E\rangle = 0 \tag{A.77}$$

On the other hand, scale invariance, Eq. (A.75), implies

$$\langle E|[D,H]|E\rangle = iz\langle E|H|E\rangle \neq 0 \tag{A.78}$$

Comparing (A.77) and (A.78), proves the theorem. It deserves mentioning that the mismatch is not due to the real part of the quantity $\langle E|\,[D,H]\,|E\rangle$ since,

$$\mathrm{Re}(\langle E|\,[D,H]\,|E\rangle) = 0 \tag{A.79}$$

is consistent with

$$\mathrm{Re}(\langle E|izH|E\rangle) = 0 \tag{A.80}$$

That the mismatch between (A.77) and (A.78) lies in the imaginary part hints at the fact that H can not be hermitian if we have scale invariance. We recall that hermiticity of H crucially depends on vanishing of a boundary term, which is imaginary when we consider quantities like $\langle E|H|E\rangle$.

[1] That is, the action of D on non-zero energy eigenstates is well defined.

For a simple illustration of S-theorem consider the free particle with one degree of freedom, $H = \frac{1}{2}p^2$ and $D = \frac{1}{2}(xp + px) - tH$. Consider first the particle in a finite periodic box with length L. The operator algebra of the free particle holds regardless of the presence of the periodic boundaries, so the S-theorem holds and it tells us that either H is not hermitian or $D|p\rangle$ is not a state. It is instructive to look carefully at the derivation of (A.77) and (A.78) in this context. An elementary computation gives

$$\langle p|\,(HD|p\rangle) - \langle p|\,(DH|p\rangle) = -ip^2 \qquad \text{(A.81)}$$

which is consistent with the scaling algebra

$$[D,H] = 2iH\,, \qquad \text{(A.82)}$$

but consistency comes at the expense of rendering H non-hermitian on a domain which contains the state $D|p\rangle$. Indeed, for the periodic box $D|p\rangle$ does not belong in the Hilbert space since $\langle x|D|p\rangle$ is not periodic. Hence, the apparent loss of hermiticity is irrelevant as it involves only functions that are not states. In the boundary free case ($L \to \infty$) the normalization of the continuum of energy eigenfunctions is by a Dirac-delta distribution, and the norm of the functions $\langle x|D|p\rangle$ involves up to two derivatives of the distribution. If we include these functions in the Hilbert space the Hamiltonian is not hermitian. On the other hand, if we choose the Hilbert space to be that of square integrable functions, then H is hermitian but neither $\langle x|p\rangle$ nor $\langle x|D|p\rangle$ are in the Hilbert space.

In contrast, consider the inverse square potential problem. For sufficiently strong attractive potential there are normalizable bound states $|E\rangle$, and the state $D|E\rangle$ is properly normalized. The Hamiltonian is hermitian, but this case requires reguralisation and renormalization and scale symmetry is broken.

This is in fact a statement of a more general result. A corollary of the S-theorem is that we cannot have (properly normalized) bound states with non-zero energy in a scale invariant system if we insist on the Hamiltonian being hermitian on the Hilbert space. As in the previous example, this follows from observing that if there exists a properly normalized state $|E\rangle$, then $D|E\rangle$ is also a properly normalized state since the wave-function vanishes sufficiently fast at infinity. This result is consistent with representation theory: a discrete spectrum $\{E_n\}$ cannot form a representation of a transformation which acts by $E \to \lambda^z E$ for continuous λ, (except if the only allowed finite energy value is $E = 0$). For example, it is well known that for the inverse square problem in the strongly attractive regime, continuous spectrum is an illusion since in that regime, H is no more Hermitian. To make H hermitian, we need to renormalize the problem, breaking the scale symmetry.

The S-theorem can be generalised to to any Hermitian operator A with non zero scaling dimension α, that is, $[D,A] = i\alpha A$. The operator A can not be Hermitian on a domain containing $D|A\rangle$ where $|A\rangle$ is the eigenstate of operator A. In particular, if we want A to be hermitian on a Hilbert space, $\mathcal{L}^2$, then the state $D|A\rangle$ can not belong to $\mathcal{L}^2$. For example, A can be the momentum operator p, which is hermitian on a rigged Hilbert space and has a non-zero scaling dimension. This generalized S-theorem implies that $D|p\rangle$ can not belong to the rigged Hilbert space, which is indeed the case.

Appendix B

On the Heat Kernel and Weyl Anomaly of Schrödinger invariant theory

B.1 Technical Aspects of Heat Kernel for one time derivative theory

Here's one more perspective of why $\delta(m)$ appears in heat kernel for one-time derivative theory using the eigenspectra of the operator $\mathcal{M}_g$ with one time derivative. The Minkowski $\mathcal{M}_{M,g}$ operator is given by

$$\mathcal{M}_{M,g} = 2\imath m\partial_t - (-\nabla^2)^{z/2} \tag{B.1}$$

and eigenspectra is given by $2m\omega - k^z$. Now, we can not directly define the heat kernel since the eigenvalues range from $-\infty$ to ∞, and therefore it blows up. A similar situation also arises in relativistic theory where the eigenspectra is given by $-\omega^2 + k^2$. There we define the heat kernel by Euclideanizing the time co-ordinate so that the eigenvalues become $\omega^2 + k^2 \geq 0$ and this positive definiteness allows for convergence. Technically, we can always define heat kernel for an operator M as long as the eigenvalues of M have positive real part. Building up on our experience to deal with the relativistic case, we use analytic continuation here as well. We define the Euclidean operator as

$$\mathcal{M}_{E,g} = 2m\partial_\tau + (-\nabla^2)^{z/2} \tag{B.2}$$

with eigenspectra given by $\lambda_{k,\omega} = -2\imath m\omega + k^z$. Evidently, Re $\left(\lambda_{k,\omega}\right) \geq 0$, hence we have a well defined heat kernel, given by

$$K_{\mathcal{M}_{E,g}} = \mathrm{Tr}e^{-s\mathcal{M}_{E,g}} = \int \frac{d^d k}{(2\pi)^d} e^{-sk^z} \int \frac{d\omega}{2\pi} e^{-2m\imath s\omega} = \frac{\delta(m)}{2s} \frac{2}{\Gamma(\frac{d}{2})} \frac{\Gamma\left(\frac{d}{z}+1\right)}{d\left(\sqrt{4\pi s^{\frac{2}{z}}}\right)^d} \tag{B.3}$$

Similarly, the Euclidean heat kernel is well defined for the operator $\mathcal{M}_{rc;d+2} = \nabla^2_{t,x} - (-\nabla^2_{x^i})^{z/2}$, where $i = 1, 2, \ldots d$ and $x \equiv x^{d+2}$. If we Wick rotate to Euclidean time τ, the eigenvalues of the operator $\mathcal{M}_{rc;d+2}$ are given by $\omega^2 + (k^{d+2})^2 + (|\boldsymbol{k}|^2)^{z/2} \geq 0$. The presence of $\delta(m)$ can more formally be treated with an extra regulizer η, as discussed in the last few paragraphs of 3.4.2 for $z = 2$; a similar argument, using the regulator η, applies to any z.

B.2 Riemann normal co-ordinate and coincident limit

In this appendix we show x^- independence of quantities relevant to the computation of the coincidence limit of the Heat Kernel when the light cone reduction technique is used. We assume that the daughter theory is coupled to a Newton Cartan structure, satisfying the Frobenius condition, *i.e.*, $\boldsymbol{n} \wedge d\boldsymbol{n} = 0$ is satisfied. This condition allows a foliation of the manifold globally. Thus, without loss of generality, the metric is given by

$$\begin{aligned} g_{\mu\nu} &= n_\mu n_\nu + h_{\mu\nu} \\ n_\mu &= (n, 0, 0, \cdots, 0)\ , \quad h_{\tau\nu} = 0. \end{aligned} \tag{B.4}$$

Using (3.9) and the fact h_{ij} is a positive definite matrix, we thus have

$$h^{\tau\nu} = 0\ , \quad v^\mu = \left(\tfrac{1}{n}, 0, 0, \cdots, 0\right). \tag{B.5}$$

The form of the metric, to which the reduced theory is coupled, corresponds to a parent spacetime metric G_{MN}, with non-vanishing components given by

$$G_{-+} = n\,, \quad G_{ij} = h_{ij}\,. \tag{B.6}$$

In addition, we assume that the parent space-time admits a null isometry so that h_{ij} and n are independent of x^-.

In what follows, we will work with this particular choice of metric G_{MN} (B.6). Without loss of generality, we choose $x_1 = (0,0,\cdots,0)$ (we call it point P) and construct the Riemann normal co-ordinate with the origin as the base point. The Riemann normal co-ordinate y^M, is given in terms of the original co-ordinate x^M as follows [263]:

$$y^M = x^M + f^M_{AB}x^A x^B + f^M_{ABC}x^A x^B x^C + \cdots, \tag{B.7}$$

where the index M runs over $+,-,1,2,3,\cdots,d$. In the coincident limit of the reduced theory, *i.e.*, $x_2^\mu \to 0$, for $\mu = +,1,2,\cdots,d$ (with x_2^- possibly different from 0), we claim that

$$\left[y_2^\mu\right] = 0, \qquad \left[y_2^-\right] = x_2^-, \tag{B.8}$$

where henceforth the square bracket is used to denote the coincident limit in the reduced theory.

We note that $[f^M_{ABC\cdots}x^A x^B x^C\cdots] = 0$ whenever any of the indices is not $-$. Recall that $f^M_{ABC\cdots}$ are constructed out of derivatives acting on metric. Thus, $f^M_{\underbrace{--\ldots-}_{N\text{ indices}}}$ can be non-zero only if it contains N factors of the metric tensor G_{-K_i}, where K_i is a running index with $i = 1,2,\cdots,N$. This is because $G_{--} = 0$ and derivatives can not carry the "$-$" index as the metric components are x^--independent. Moreover, by dimensional analysis $f^M_{\underbrace{--\ldots-}_{N}}$ has $N-1$ derivatives $f^M_{\underbrace{--\ldots-}_{N}}$. Schematically, this assumes one of the following forms

$$\partial_{A_1}\cdots\partial_{A_{N-1}}G_{-K_1}\cdots G_{-K_N}G^{MA_i}G^{A_{i_1}K_{j_1}}G^{A_{i_2}A_{j_2}}\cdots G^{K_{i_3}K_{j_3}}\cdots, \tag{B.9}$$

$$\partial_{A_1}\cdots\partial_{A_{N-1}}G_{-K_1}\cdots G_{-K_N}G^{MK_i}G^{A_{i_1}K_{j_1}}G^{A_{i_2}A_{j_2}}\cdots G^{K_{i_3}K_{j_3}}\cdots. \tag{B.10}$$

Here the derivatives are assumed to act on all possible combinations, resulting in different possible terms. For example, for $N=2$, one can have the following terms:

$$\begin{aligned}
&G^{MA_1}G^{K_1K_2}G_{-K_2}\partial_{A_1}G_{-K_1}\,,\\
&G^{MK_2}G^{A_1K_1}G_{-K_1}\partial_{A_1}G_{-K_2}\,,\\
&G^{MK_2}G^{A_1K_1}G_{-K_2}\partial_{A_1}G_{-K_1}\,.
\end{aligned} \tag{B.11}$$

There can not be any x^- derivative for a term to be non-vanishing. This implies the indices A_i are contracted among themselves, except possibly for one contracted with G^{MA_i}, and the indices K_i are contracted among themselves. But since $G_{-K}=0$ except for G_{-+}, and $G^{++}=0$, any term for which two factors of the metric tensor, $G_{-K_{i_1}}$ and $G_{-K_{i_2}}$, are contracted via $G^{K_{i_1}K_{i_2}}$ vanish.

Next, we show that $[\Delta_{VM}]=1$. The expression for Δ_{VM}, Eq, (3.56), involves bi-derivatives of the geodetic interval, Eq. (3.55), and the determinant of the metric. To begin with, we turn our attention to the determinant of the metric and note that

$$\left[G'(y_2)\right]=J^2(0,x_2^-,0,\cdots,0)G(0,x_2^-,0,\cdots,0)\,, \tag{B.12}$$

where a prime indicates quantities in Riemann normal co-ordinate and J is the Jacobian associated with the co-ordinate transformation (B.7). The x^- independence in the original co-ordinate guarantees that $G(0,x_2^-,0,\cdots,0)=G(0,0,0,\cdots,0)$, hence we have

$$\left[G'(y_2)\right]=\left(\frac{J(0,x_2^-,0,\cdots,0)}{J(0,0,0,\cdots,0)}\right)^2 G'(0). \tag{B.13}$$

Next consider the geodetic interval from point P to point Q. In Riemann normal co-ordinates [263]

$$y_2^M=y^M(Q)=y_1^M+s_Q\frac{dx^M}{ds}\bigg|_{s=0}, \tag{B.14}$$

where s_Q is the value of the affine parameter at Q and $s=0$ at P, with $y_1^M=y^M(P)$. Using Eq. (3.55), hence we have

$$2\sigma(y_2,y_1)=G_{MN}(0)(y_2^M-y_1^M)(y_2^N-y_1^N)=G'_{MN}(0)(y_2^M-y_1^M)(y_2^N-y_1^N) \tag{B.15}$$

where we have used $G'_{MN}(0) = G_{MN}(0)$. It follows that

$$\Delta_{VM} = \left(\frac{G'(y_2)}{G'(0)}\right)^{-1/2} . \tag{B.16}$$

We have continued back to Minkowskian signature (the definition in Eq. (3.56) is for metric with Euclidean signature). Since Δ_{VM} is a bi-scalar, use of Eqs. (B.13) and (B.16) and of $J(0,0,0,\cdots,0) = 1$ gives

$$[\Delta_{VM}] = \left(\frac{J(0,x_2^-,0,\cdots,0)}{J(0,0,0,\cdots,0)}\right)^{-1} = J^{-1}(0,x_2^-,0,\cdots,0) \tag{B.17}$$

in the original co-ordinate system, x^M. Equation (B.17) is consistent with the result that $\Delta_{VM} = 1$ when all the co-ordinates, including x^-, coincide, *i.e.*, when $x_2^- = 0$.

We aim to show that

$$\left[\det\left(\frac{\partial y^M}{\partial x^N}\right)\right] = \det\left(\left[\frac{\partial y^M}{\partial x^N}\right]\right) = 1 \tag{B.18}$$

From Eq. (B.7) we have

$$\left[\frac{\partial y^M}{\partial x^N}\right] = \delta^M_N + (f^M_{N-} + f^M_{-N})x^- + (f^M_{N--} + f^M_{-N-} + f^M_{--N})x^- x^- + \cdots \tag{B.19}$$

Consider first the lowest two terms in the expansion. Explicitly, we have [263]

$$2f^M_{N-} = 2f^M_{-N} = \Gamma^M_{N-} = -\frac{1}{2}G^{Mi}\partial_i G_{N-} - \frac{1}{2}G^{M+}\partial_+ G_{N-} + \frac{1}{2}G^{M+}\partial_N G_{+-} . \tag{B.20}$$

It follows that $f^M_{N-} \neq 0$ only for $M = -$ or $N = +$. Similarly, $f^M_{(N--)} \neq 0$ provided $M = -$ or $N = +$, since [263]

$$6f^M_{NIJ} = \Gamma^M_{NE}\Gamma^E_{IJ} + \partial_N \Gamma^M_{IJ} \tag{B.21}$$

By an argument analogous to that below Eqs. (B.11) one can show that $[f^M_{N--\dots-}] = 0$ (at least three − subscripts). Schematically

$$\left[\left(\frac{\partial y^M}{\partial x^N}\right)\right] = \begin{pmatrix} 1 & * & * & \dots & \dots & * \\ 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & * & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & * & 0 & 0 & 1 & 0 \\ 0 & * & 0 & 0 & 0 & 1 \end{pmatrix}$$

where a "$*$" means a non-zero entry. Thus, the matrix has unit determinant and we have, using Eq. (B.17),

$$[\Delta_{VM}] = 1\,. \tag{B.22}$$

Lastly, we turn to the heat kernel expansion coefficients, a_n. They are determined by the recursive relation [75],

$$na_n + \partial_M \sigma \partial^M a_n = -\Delta_{VM}^{-1/2} \mathcal{M}\left(\Delta_{VM}^{1/2} a_{n-1}\right)\,, \tag{B.23}$$

and $a_0 = 1$, where $\mathcal{M}$ is the relativistic operator in the parent theory. The condition of x^- independence of $[a_n]$, $[\partial_i a_n]$ and $[\partial_i \partial_j a_n]$ can be imposed on the recursion self-consistently. To show this one uses x^- independence of $[\Delta_{VM}]$, $[\partial_i \Delta_{VM}]$ and $\left[\partial_i \partial_j \Delta_{VM}\right]$, which follows from an argument similar to the one used to establish Eq. (B.22)

B.3 Explicit Perturbative Calculation of η-regularized Heat Kernel

In this appendix we give an explicit perturbative computation that shows the vanishing of the anomaly for a class of curved backgrounds. This serves to verify the general arguments presented in the body of the manuscript in a specific, simple example, and allows us to study explicitly the η regulated Heat Kernel asking in particular whether the $\eta \to 0$ limit is a well defined limit as $m \neq 0$. To be specific, we compute the heat kernel on a curved background, characterized by

$$n_\mu = \left(\frac{1}{1-n(x)}, 0, 0\right), \qquad v^\mu = (1-n(x), 0, 0) \tag{B.24}$$

$$h_{ij} = \delta_{ij}, \quad \sqrt{g} = \sqrt{\det(n_\nu n_\nu + h_\mu h_\nu)} = \frac{1}{1-n(x)}. \tag{B.25}$$

where $n(x)$ is a function of space only and $h_{i0} = 0$. The special choice is inspired by [80]and additionally serves the purpose of affording a direct comparison with that work. We will perform a perturbative calculation as an expansion in $n(x)$. We will specialize to a $2+1$ dimensional Schrödinger field theory coupled to this background. The action is given by

$$S = \int dt d^2x\, N \left(2m\phi^\dagger \imath \tfrac{1}{N}\partial_t \phi - h^{ij}\partial_i \phi^\dagger \partial_j \phi - \xi R \phi^\dagger \phi\right) \tag{B.26}$$

where $N(x) = \frac{1}{1-n(x)}$ and R is the Ricci scalar of the $3+1$ dimensional geometry, on which the parent theory lives.

As we will see, the result of this calculation is that the Weyl anomaly, corresponding to the theory described by Eq. (B.26) is given by

$$\mathcal{A}_G = 2\pi\delta(m)\left(-aE_4 + cW^2 + bR^2 + dD_M D^M R\right) \tag{B.27}$$

where the coefficients a, b, c, d are given by:

$$\begin{aligned} a &= \frac{1}{8\pi^2}\frac{1}{360}, \quad b = \frac{1}{8\pi^2}\frac{1}{2}\left(\xi - \frac{1}{6}\right)^2, \\ c &= \frac{1}{8\pi^2}\frac{1}{120}, \quad d = \frac{1}{8\pi^2}\left(\frac{1-5\xi}{30}\right). \end{aligned} \tag{B.28}$$

These are exactly the same as in the expression for the Weyl Anomaly of a relativistic complex scalar field theory[1] living in one higher dimension [49, 50, 51, 52, 53, 54, 55]:

$$\mathcal{A}_R = \left(-aE_4 + cW^2 + bR^2 + dD_M D^M R\right) . \tag{B.29}$$

To arrive at this result, we proceed by considering the heat kernel of the following Euclidean operator, corresponding to the action in Eq. (B.26), namely

$$\mathcal{M}_{E,c} = 2m\tfrac{1}{N}\partial_\tau - \mathcal{D}^2 + \xi R , \tag{B.30}$$

where we have

$$\mathcal{D}^2 = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}h^{ij}\partial_j\right) = \partial^2 + (1+n)\left(\partial_i n\right)\partial_i , \tag{B.31}$$

$$R = -2\partial^2 n - 2n\partial^2 n - \frac{7}{2}\partial_i n\partial_i n + \cdots , \tag{B.32}$$

$$-g^{1/4}\mathcal{D}^2\left(g^{-1/4}\delta(x)\right) = -\partial^2\delta(x) + \delta(x)\left(\frac{1}{2}\partial^2 n + \frac{1}{2}n\partial^2 n + \frac{3}{4}\partial_i n\partial_i n\right) . \tag{B.33}$$

The Euclidean operator can be expressed as the one in flat space-time, perturbed by the background field $n(x)$:

$$\begin{aligned}\langle \boldsymbol{x},\tau|\mathcal{M}_{E,c}|\boldsymbol{x}',\tau'\rangle &= \langle \boldsymbol{x},\tau|\mathcal{M}_{E,f}|\boldsymbol{x}',\tau'\rangle + mP_1(x)\partial_\tau\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\tau-\tau') \\ &\quad + P_2(x)\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\tau-\tau') ,\end{aligned} \tag{B.34}$$

where the subscript c and f denote the curved and flat space-time respectively while E denote the Euclidean nature of the operator. Here we have introduced

$$P_1(x) = 2n(x), \quad P_2(x) = \left(\frac{1}{2}\partial^2 n + \frac{1}{2}n\partial^2 n + \frac{3}{4}\partial_i n\partial_i n\right) - \xi\left(2\partial^2 n + 2n\partial^2 n + \frac{7}{2}\partial_i n\partial_i n\right) . \tag{B.35}$$

The heat kernel can be obtained as a perturbative expansion of the background fields as

[1] The Weyl anomaly of a complex scalar field is twice of that of a real scalar field.

follows:

$$K(s) = \exp\left[-s\left(\mathcal{M}_{E,f} + P\right)\right] = \sum_{N=0}^{\infty} (-1)^N K_N(s)\,. \tag{B.36}$$

The $K_N(s)$ is defined as follows:

$$K_N(s) = \int_0^s ds_N \int_0^{s_N} ds_{N-1} \cdots \int_0^{s_2} ds_1\; G(s-s_N)PG(s_N - s_{N-1})P \cdots G(s_2 - s_1)PG(s_1)\,. \tag{B.37}$$

where $G(s) = e^{-s\mathcal{M}_{E,f}}$ and P is the perturbation (B.34), explicitly given by

$$\langle \boldsymbol{x},\tau|P|\boldsymbol{x}',\tau'\rangle = mP_1(x)\partial_\tau \delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\tau-\tau') + P_2(x)\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\tau-\tau'). \tag{B.38}$$

One can now complete the calculation by using the matrix element of $G(s)$ as given by

$$\begin{aligned} \mathcal{G}_{g,E}\left(s;(\boldsymbol{x}_2,\tau_2),(\boldsymbol{x}_1,\tau_1)\right) &\equiv \langle \boldsymbol{x}_2,\tau_2|G(s)|\boldsymbol{x}_1,\tau_1\rangle \\ &= \frac{1}{\pi}\left(\frac{1}{4\pi s}\right)^{d/2}\left[\frac{s\eta}{(2ms-\tau_2+\tau_1)^2 + s^2\eta^2}\right] e^{-\frac{(\boldsymbol{x}_2-\boldsymbol{x}_1)^2}{4s}}, \end{aligned} \tag{B.39}$$

which corresponds to the heat kernel expression for the η-regulated Euclidean operator: $\mathcal{M}'_{E,g} = 2m\partial_\tau - \nabla^2 + \eta\sqrt{-\partial_\tau^2}$, as discussed in the last few paragraphs of 3.4.2.[2] This reproduces Eq. (3.46) as $\eta \to 0$.

The evaluation of Eq. (B.37) follows the procedure sketched out in the appendix of [80]. We separate the contributions from P_1 and P_2 to K_1 as follows:

$$K_{1P_1}(s) = \left(\frac{\frac{\eta}{2}}{m^2+\frac{\eta^2}{4}}\right)\left(\frac{-1}{4m^2+\eta^2}\right)\frac{8m^2}{(4\pi s)^2}\left(P_1 + \tfrac{s}{6}\partial^2 P_1 + \tfrac{s^2}{60}\partial^2\partial^2 P_1 + \cdots\right), \tag{B.40}$$

$$K_{1P_2}(s) = \left(\frac{\frac{\eta}{2}}{m^2+\frac{\eta^2}{4}}\right)\frac{2}{(4\pi s)^2}\left(sP_2 + \tfrac{s^2}{6}\partial^2 P_2 + \cdots\right), \tag{B.41}$$

[2] In curved space-time, $\mathcal{M}'_{E,g}$ includes a perturbation $n(x)\eta\sqrt{-\partial_\tau^2}$, that, however, does not contribute to the anomaly in the $\eta \to 0$ limit. This term's contribution to K_1 is proportional to $\frac{\eta(\eta^2-4m^2)}{(\eta^2+4m^2)^2}$ that vanishes as $\eta \to 0$, without giving a $\delta(m)$ (or any derivative of $\delta(m)$). This term's contributions to K_2 also vanish as $\eta \to 0$. We omit these terms for simplicity for rest of the appendix.

and for K_2, which gets contributions quadratic in P_1 and P_2, as follows:

$$K_{2P_1P_1}(s) = \frac{\left(24m^2 - 2\eta^2\right)}{\left(\eta^2 + 4m^2\right)^2}\left(\frac{2m^2\eta}{4m^2+\eta^2}\right)\frac{1}{(4\pi s)^2}\Bigg(P_1^2 + \tfrac{s}{3}P_1\partial^2 P_1 + \tfrac{s}{6}\partial_i P_1 \partial_i P_1$$
$$+ \frac{s^2}{180}\left(6P_1\partial^2\partial^2 P_1 + 5\partial^2 P_1\partial^2 P_1 + 12\partial_i\partial^2 P_1\partial_i P_1 + 4\left(\partial_i\partial_j P_1\right)\left(\partial_i\partial_j P_1\right)\right)\Bigg) \quad \text{(B.42)}$$

$$K_{2P_1P_2}(s) = \left(\frac{\frac{\eta}{2}}{m^2+\frac{\eta^2}{4}}\right)\left(\frac{-1}{4m^2+\eta^2}\right)\frac{8m^2}{(4\pi s)^2}\Big(\tfrac{s}{2}P_1P_2$$
$$+ \tfrac{s^2}{12}(P_2\partial^2 P_1 + P_1\partial^2 P_2 + \partial_i P_1\partial_i P_2) + \cdots\Big) \quad \text{(B.43)}$$

$$K_{2P_2P_1}(s) = K_{2P_1P_2}(s) \quad \text{(B.44)}$$

$$K_{2P_2P_2}(s) = \left(\frac{\frac{\eta}{2}}{m^2+\frac{\eta^2}{4}}\right)\frac{2}{(4\pi s)^2}\left(\tfrac{s^2}{2}P_2^2 + \cdots\right) \quad \text{(B.45)}$$

The anomaly is determined by the s-independent terms in K_N. In $\eta \to 0$ limit, factors of $\delta(m)$ arise, after use of the following easily verifiable limits

$$\lim_{\eta\to 0}\left(\frac{\frac{\eta}{2}}{m^2+\frac{\eta^2}{4}}\right)\left(\frac{8m^2}{4m^2+\eta^2}\right) = \pi\delta(m)\ ,$$
$$\lim_{\eta\to 0}\left(\frac{\frac{\eta}{2}}{m^2+\frac{\eta^2}{4}}\right) = \pi\delta(m)\ ,$$
$$\lim_{\eta\to 0}\frac{24m^2-2\eta^2}{(\eta^2+4m^2)^2}\left(\frac{2\eta m^2}{m^2+\frac{\eta^2}{4}}\right) = 2\pi\delta(m).$$

In $\eta \to 0$ limit, the s independent terms are given by

$$K_{1P_1}(s) \ni \frac{\delta(m)}{16\pi}\left[-\frac{1}{30}\partial^2\partial^2 n\right]\ ,$$
$$K_{1P_2}(s) \ni \frac{\delta(m)}{16\pi}\left[\frac{1}{3}\partial^2 P_2\right]$$
$$= \frac{\delta(m)}{16\pi}\Bigg[\frac{1}{3}\Bigg(\left(\frac{1}{2}-2\xi\right)\partial^2\partial^2 n + \left(\frac{1}{2}-2\xi\right)\partial^2 n\partial^2 n + \left(\frac{1}{2}-2\xi\right)n\partial^2\partial^2 n$$
$$+ \left(\frac{5}{2}-11\xi\right)\partial_i n\partial_i\partial^2 n + \left(\frac{3}{2}-7\xi\right)\left(\partial_i\partial_j n\right)\left(\partial_i\partial_j n\right)\Bigg)\Bigg]\ ,$$
$$K_{2P_1P_1} \ni \frac{\delta(m)}{16\pi}\left[\frac{1}{90}\left(6n\partial^2\partial^2 n + 5\partial^2 n\partial^2 n + 12\partial_i\partial^2 n\partial_i n + 4\left(\partial_i\partial_j n\right)\left(\partial_i\partial_j n\right)\right)\right]\ ,$$

$$K_{2P_1P_2} + K_{2P_1P_2} \ni \frac{\delta(m)}{16\pi} \left[\frac{-1}{3}(P_2\partial^2 n + n\partial^2 P_2 + \partial_i n \partial_i P_2)\right]$$
$$= \frac{\delta(m)}{16\pi} \left[\frac{-1}{3}\left(\frac{1}{2} - 2\xi\right)(\partial^2 n \partial^2 n + n\partial^2\partial^2 n + \partial_i n \partial_i \partial^2 n)\right] ,$$
$$K_{2P_2P_2} \ni \frac{\delta(m)}{16\pi} \left[P_2^2 + \cdots\right] = \frac{\delta(m)}{16\pi} \left[\left(\frac{1}{2} - 2\xi\right)^2 \partial^2 n \partial^2 n + \cdots\right] .$$

Using

$$R = -2\partial^2 n - 2n\partial^2 n - \frac{7}{2}\partial_i n \partial_i n + \cdots , \tag{B.46}$$

$$R^2 = 4(\partial^2 n)^2 + \cdots , \quad W^2 = \frac{1}{3}(\partial^2 n)^2 + \cdots , \tag{B.47}$$

$$E_4 = 2(\partial^2 n)^2 - 2(\partial_i \partial_j n)(\partial_i \partial_j n) + \cdots , \tag{B.48}$$

$$D_M D^M R = -2\partial^4 n - 2(\partial^2 n)^2 - 2n\partial^4 n - 13(\partial_j n)(\partial_j \partial^2 n) - 7(\partial_i \partial_j n)(\partial_i \partial_j n) + \cdots . \tag{B.49}$$

one verifies the anomaly expression in Eqs. (B.27) and (B.28). Since our calculation only fixes the value of $12b + c$, in oder to break the degeneracy we use the fact that for $\xi = \frac{1}{6}$ the Wess-Zumino consistency condition precludes an R^2 anomaly [80] and assume c is ξ-independent.

We emphasize that the calculation carried out here does not rely on any null cone reduction technique, hence, this lends further credence to the LCR prescription, which has correctly produced the $\delta(m)$ factor, as elucidated before.

Appendix C

Unitarity and Universality in non relativistic Conformal Field theory

C.1 $SL(2,\mathbb{R})$ invariant field theory

The $SL(2,\mathbb{R})$ group is generated by the three generators H,D,C satisfying the following algebra:

$$[D,H] = 2H\,,\ [D,C] = -2C\,,\ [H,C] = D. \tag{C.1}$$

The spectrum of D is real and physically represents the dimension. H raises the dimension and C lowers the same. The highest weight representation is the one, annihilated by C. A nice and brief exposition of $SL(2,\mathbb{R})$ invariant field theory can be found in the appendix of [105]. Here we discuss them for the sake of completeness and make the paper self-contained.

An $SL(2,\mathbb{R})$ invariant field theory is defined on a one dimensional manifold, parameterized by τ (say *time*), where $SL(2,\mathbb{R})$ group acts on the co-ordinate τ in following way:

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}\,, \quad \text{where } (ad-bc)=1 \tag{C.2}$$

In terms of the generators above, the H generates a time translation, C generates a special conformal transformation and D generates a scale transformation. The theory has a privileged class of operators (the highest weight representation) O_α satisfying

$$[C, O_\alpha(0)] = 0 \tag{C.3}$$

and carrying dimension Δ_α i.e $[D, O_\alpha(0)] = \Delta_\alpha O_\alpha(0)$. These are called $SL(2,\mathbb{R})$ primaries. They have following commutators with $SL(2,\mathbb{R})$ generators:

$$[D, O_\alpha(\tau)] = (2\tau\partial_\tau + \Delta_\alpha)\, O_\alpha(\tau)\,, \tag{C.4}$$

$$[C, O_\alpha(\tau)] = \left(-\tau^2\partial_\tau - \tau\Delta_\alpha\right) O_\alpha(\tau)\,, \tag{C.5}$$

$$[H, O_\alpha(\tau)] = \partial_\tau O_\alpha(\tau)\,. \tag{C.6}$$

It follows that under a finite $SL(2,\mathbb{R})$ transformation, $SL(2,\mathbb{R})$ primaries transform as

$$O(\tau) \mapsto \left[(c\tau+d)^{-2}\right]^{\frac{\Delta}{2}} O(\tau') \tag{C.7}$$

[1]

The correlators of the form $\langle O(\tau_1)O(\tau_2)O(\tau_3)\cdots O(\tau_n)\rangle$ will be of our "primary" interest. As $SL(2,\mathbb{R})$ preserves the ordering of time, the two distinct ordering of τ_i need not be related to each other. In what follows, we will be assuming $0 < \tau_1 < \tau_2 < \tau_3 < \cdots < \tau_n$. The unitarity for a $SL(2,\mathbb{R})$ invariant theory is defined via existence of an anti-linear conjugation map taking $O \mapsto O^\dagger$ such that following relations hold:

1. **Time reversal**:

$$\begin{aligned}&\langle O^\dagger(-\tau_n)\cdots O^\dagger(-\tau_3)O^\dagger(-\tau_2)O^\dagger(-\tau_1)\rangle \\ &= \langle O(\tau_1)O(\tau_2)O(\tau_3)\cdots O(\tau_n)\rangle^*\end{aligned} \tag{C.8}$$

[1]The presence of factor of $\frac{1}{2}$ in the weight, as compared to the weight, noted in [105] is due to the presence of extra 2 with $\tau\partial_\tau$ in the expression for $[D, O_\alpha(\tau)]$. This is done, in hindsight, to make the notation consistent with the dilatation operator in Schrödinger field theory i.e the dilatation operator acting on a Schrödinger primary at $\boldsymbol{x} = 0$ becomes the dilatation operator acting on a $SL(2,\mathbb{R})$ primary.

2. **Reflection positivity**: One can define a state $|\Psi\rangle$

$$|\psi\rangle = \int \left(\prod d\tau_i\right)\; g(\tau_1,\tau_2,\cdots,\tau_n)\;|O(\tau_1)O(\tau_2)\cdots O(\tau_n)|0\rangle \tag{C.9}$$

with

$$\langle\Psi|\Psi\rangle \geq 0 \tag{C.10}$$

In terms of correlator this reads:

$$\int \left(\prod d\tau_i'\right) \int \left(\prod d\tau_i\right) \tilde{g}(\tau_1',\tau_2',\cdots,\tau_n')g(\tau_1,\tau_2,\cdots,\tau_n)$$
$$\langle O^\dagger(\tau_n')\cdots O^\dagger(\tau_2')O^\dagger(\tau_1')O(\tau_1)O(\tau_2)\cdots O(\tau_n)\rangle \geq 0$$

where $\tilde{g}(\tau) = g(-\tau)^*$. Here g is an arbitrary function or distribution having support away from coincident points to avoid singularity.

The $SL(2,\mathbb{R})$ algebra fixes the functional form of the two point and the three point correlator. One can choose a Hermitian basis of operators $O = O^\dagger$ such that

$$\langle O_\alpha(\tau_1)O_\beta(\tau_2)\rangle = \frac{\delta_{\alpha\beta}}{(\tau_2-\tau_1)^\Delta} \tag{C.11}$$

$$\langle O_1(\tau_1)O_2(\tau_2)O_3(\tau_3)\rangle$$
$$= \frac{c_{123}}{(\tau_2-\tau_1)^{\frac{\Delta_2+\Delta_1-\Delta_3}{2}}(\tau_3-\tau_2)^{\frac{\Delta_3+\Delta_2-\Delta_1}{2}}(\tau_3-\tau_1)^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}}} \tag{C.12}$$

Time reversal symmetry guarantees that $c_{\gamma\beta\alpha} = c^*_{\alpha\beta\gamma}$. In general, $c_{\alpha\beta\gamma}$ can be complex numbers. All the higher point correlator can be obtained using the operator product expansion, which reads:

$$O_1(\tau_1)O_2(\tau_2) = \sum_\alpha c_{12\alpha}\frac{1}{(\tau_2-\tau_1)^{\frac{\Delta_1+\Delta_2-\Delta_\alpha}{2}}}\left[O_\alpha+\cdots\right] \tag{C.13}$$

where $\cdots$ are contributions coming from $SL(2,\mathbb{R})$ descendants and fixed by $SL(2,\mathbb{R})$ invariance. Thus the knowledge of the spectrum of D i.e set of $SL(2,\mathbb{R})$ primaries and three point coefficient

$c_{\alpha\beta\gamma}$ amounts to a complete knowledge of all the correlators.

C.2 Schrödinger algebra, Primaries and Quasi-primaries

Here we provide a detailed account of Schrödinger algebra. We expound the concepts of primaries and quasi-primaries. The Schrödinger group acts on space-time as follows[96, 93, 94, 98]:

$$t \mapsto \frac{at+b}{ct+d}, \quad \boldsymbol{r} \mapsto \frac{\mathbb{R}\boldsymbol{r}+\boldsymbol{v}t+f}{ct+d} \tag{C.14}$$

where $ad-bc=1$, $\mathbb{R}$ is a d dimensional rotation matrix, $\boldsymbol{v}$ denotes the Galilean boost and f is a spatial translation. For the sector with non-zero charge, the representation is built by translating all the operators to the origin and considering the little group generated by dilatation operator D, Galilean boost generator K_i, and special conformal transformation generator C. The highest weight states (ϕ_α) are annihilated by C and K_i i.e.

$$[C,\phi_\alpha(0,\mathbf{0})]=0, \quad [K_i,\phi_\alpha(0,\mathbf{0})]=0. \tag{C.15}$$

These are called primary operators. The commutators with D and particle number symmetry generator $\hat{N}$ dictate the charge and the dimension of these operators ϕ_α:

$$[D,\phi_\alpha(0,\mathbf{0})]=\imath\Delta_\alpha\phi(0,\mathbf{0}),\ [\hat{N},\phi_\alpha(0,\mathbf{0})]=N_\alpha\phi(0,\mathbf{0}) \tag{C.16}$$

The time and space translation generator H and P create descendant operators by acting upon primary operators, consequently raising the dimension by 2 and 1 respectively. It deserves mention that the concept of primaries and descendants breaks down within the neutral sector. Since K_i and P_j commute in this sector, P_j acting on a primary spits out a primary in stead of a descendant.

The subgroup defined by $\mathbb{R}=\mathbb{I}$, $\boldsymbol{v}=0$, $f=0$ is generated by H, D and C and is in fact

$SL(2,\mathbb{R})$. This acts as follows:

$$t \mapsto \frac{at+b}{ct+d}\,, \quad \boldsymbol{r} \mapsto \frac{\boldsymbol{r}}{ct+d}\,, \quad (ad-bc)=1\,. \tag{C.17}$$

It becomes evident that the $(t,0)$ slice is a invariant domain of $SL(2,\mathbb{R})$. Now one can reorganize the operator content using $SL(2,\mathbb{R})$ algebra. A $SL(2,\mathbb{R})$ primary O is defined by requiring $[C,O(0,\mathbf{0})]=0$. Thus all the primaries defined by (C.15) are $SL(2,\mathbb{R})$ primaries but not the other way around. As mentioned in the main text, the situation is reminiscent of 2D conformal field theory where we have Virasoro primaries as well as $SL(2,\mathbb{R})$ primaries and the $SL(2,\mathbb{R})$ primaries are called quasi-primaries. We have borrowed that nomenclature and called the Schrödinger primaries as *primaries* and $SL(2,\mathbb{R})$ primaries as *quasi-primaries*. We emphasize that the notion of quasi-primaries goes through even for the neutral sector. The commutator of quasi-primary $O(t)$ [for notational simplicity, $\phi(t)$ (or $O(t)$) implies the operator $\phi(t,\mathbf{0})$(or $O(t,\mathbf{0})$)] with the generators H, D, C are given by[81]:

$$[H,O(t)] = -\imath\partial_t O(t)\,, \tag{C.18}$$

$$[D,O(t)] = \imath(2t\partial_t+\Delta)O(t)\,, \tag{C.19}$$

$$[C,O(t)] = (-\imath t^2\partial_t - \imath t\Delta)O(t)\,. \tag{C.20}$$

This follows from $[C,O(0,\mathbf{0})]=0$ and

$$[D,H]=2\imath H\,,\ [D,C]=-2\imath C\,,\ [H,C]=-\imath D. \tag{C.21}$$

In terms of Euclidean time $\tau=\imath t$, and $D'=-\imath D$, we have

$$[D',H]=2H\,,\ [D',C]=-2C\,,\ [H,C]=D'\,. \tag{C.22}$$

and

$$[D',O(\tau)] = (2\tau\partial_\tau+\Delta)\,O(\tau)\,, \tag{C.23}$$

$$[C, O(\tau)] = \left(-\tau^2\partial_\tau - \tau\Delta\right) O(\tau)\,, \tag{C.24}$$

$$[H, O(\tau)] = \partial_\tau O(\tau)\,. \tag{C.25}$$

Thus we have a $SL(2,\mathbb{R})$ invariant theory defined on $(\tau, \mathbf{0})$ slice. $SL(2,\mathbb{R})$ acts on τ in the usual manner:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \quad (ad - bc) = 1\,. \tag{C.26}$$

We note that if ϕ is a primary operator, then

$$\mathcal{A} \equiv -\left(\frac{Nd}{2\Delta}\partial_\tau\phi + \frac{1}{2}\nabla^2\phi\right)\,,\ \mathcal{A}^\dagger \equiv \frac{Nd}{2\Delta}(\partial_\tau\phi^\dagger) - \frac{1}{2}\nabla^2\phi^\dagger \tag{C.27}$$

are quasi-primaries but not primaries unless $\Delta = \frac{d}{2}$. This follows from the commutation relations [81]:

$$[C, \phi(\tau, \boldsymbol{x})] = \left(-\tau^2\partial_\tau - \tau\Delta - \tau\boldsymbol{x}\cdot\boldsymbol{\nabla} + \frac{N|\boldsymbol{x}|^2}{2}\right)\phi$$

$$[K_j, \phi(\tau, \boldsymbol{x})] = \left(-\tau\partial_j + Nx_j\right)\phi$$

The operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ played a crucial role in proving the unitarity bound. In fact, $\mathcal{A}^\dagger$ annihilates the vacuum when $\Delta = \frac{d}{2}$ and at free fixed point, this is precisely the null operator.

C.3 Time reversal and Parity

The notion of time reversal and parity is subtle in $0+1$-D conformal field theories. The same subtlety is inherited by the Schr̈dinger field theory. Both of the symmetries come with a transformation of the form $\tau \to -\tau$, but the time reversal is realized as an anti-unitary operator acting on the fields whereas the parity does not involve any complex conjugation. To be precise, time reversal symmetry guarantees that

$$\begin{aligned}&\langle O^\dagger(-\tau_n)\cdots O^\dagger(-\tau_3)O^\dagger(-\tau_2)O^\dagger(-\tau_1)\rangle\\&=\langle O(\tau_1)O(\tau_2)O(\tau_3)\cdots O(\tau_n)\rangle^*\end{aligned} \tag{C.28}$$

while the parity invariance implies that

$$\begin{aligned}&\langle O(-\tau_n)\cdots O(-\tau_3)O(-\tau_2)O(-\tau_1)\rangle\\&=(-1)^{np}\langle O(\tau_1)O(\tau_2)O(\tau_3)\cdots O(\tau_n)\rangle\end{aligned} \tag{C.29}$$

where $p\in\{0,1\}$ is the parity of the $SL(2,\mathbb{R})$ primary operator O.

The three point correlators, as pointed out earlier, are given by

$$\begin{aligned}&\langle O_1(\tau_1)O_2(\tau_2)O_3^\dagger(\tau_3)\rangle\\&=\frac{C_{O_1O_2O_3}}{(\tau_2-\tau_1)^{\frac{\Delta_2+\Delta_1-\Delta_3}{2}}(\tau_3-\tau_2)^{\frac{\Delta_3+\Delta_2-\Delta_1}{2}}(\tau_3-\tau_1)^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}}}\end{aligned} \tag{C.30}$$

Now the time reversal implies that $C_{O_1O_2O_3}=C^*_{O_2^\dagger O_1^\dagger O_3^\dagger}$. Since cyclic ordering is preserved by $SL(2,\mathbb{R})$, we have $C_{O_2O_1O_3}=C_{O_3O_2O_1}$. Thus we have

$$C_{O_1O_2O_3}=C_{O_3^\dagger O_2^\dagger O_1^\dagger} \tag{C.31}$$

On the other hand, parity invariance implies that

$$C_{O_1O_2O_3}=(-1)^{p_1+p_2+p_3}C_{O_2O_1O_3} \tag{C.32}$$

where p_i is the parity of the field O_i.

We can easily show that the free Schrödinger field theory without any anti-particle (discussed later in Sec. **??**) does not satisfy the parity invariance. We recall that the two point correlator on the $(\tau,\mathbf{0})$ slice is given by

$$\langle\phi(0)\phi^\dagger(\tau)\rangle=\Theta(\tau)\tau^{-d/2} \tag{C.33}$$

If one assumes parity invariance with ϕ and $\phi^\dagger$ carrying opposite parity, the two point correlator satisfies:

$$\langle\phi(0)\phi^\dagger(\tau)\rangle = \langle\phi^\dagger(-\tau)\phi(0)\rangle = \langle\phi(0)\phi^\dagger(-\tau)\rangle \,. \tag{C.34}$$

Here the last equality follows because the cyclic order of the operator insertion is unaffected by $SL(2,\mathbb{R})$ invariance. Thus $\langle\phi(0)\phi^\dagger(\tau)\rangle$ has to be an even function of τ, which is not the case in (C.33). The presence of $\Theta(\tau)$ implies the absence of anti-particles.

On the other hand, when the parity symmetry is present one can write down a bootstrap equation (4.17) even for a charged sector. The section 4 of [264] elucidates on the scenario where a notion of parity is available. This motivates us to ask whether one can impose parity invariance on top of Schrödinger invariance. It indeed can be done by defining a free field theory such that the two point correlator takes the following form on $(\tau,\mathbf{0})$ slice:

$$\langle\psi(0,\mathbf{0})\psi^\dagger(\tau,\mathbf{0})\rangle = \frac{1}{|\tau|^{d/2}} \tag{C.35}$$

where $N<0$ is the $U(1)$ charge carried by the field ψ. All the higher point correlators are determined by Wick contraction. On $(\tau,0)$ slice, this theory is expected to behave like generalized Bose/Fermi theory in $0+1$ dimension [105]. We remark that even if it is possible to impose parity invariance on $(\tau,\mathbf{0})$ slice, it is not clear whether one can impose such invariance even away from the above mentioned temporal slice. This is because the boost invariance forces the $\boldsymbol{x}$ dependence of two point correlator to be $\exp\left(\imath N\frac{|\boldsymbol{x}|^2}{2t}\right)$ which is clearly not symmetric under $t\to -t$ unless one also imposes $N\to -N$ constraint. It would be interesting to consider another scenario, where one can have the following two point correlator:

$$\langle\psi(0,\mathbf{0})\psi^\dagger(\tau,\boldsymbol{x})\rangle = \frac{1}{|\tau|^{d/2}}\exp\left[\frac{N|\boldsymbol{x}|^2}{2|\tau|}\right] , \tag{C.36}$$

and it is obtained by different analytical continuation to imaginary time τ, depending on signature of real time t. It is not clear at present whether it carries any physical significance and leads to a well defined theory. Nonetheless, one can indeed have theories which enjoy parity invariance on the $(\tau, \mathbf{0})$ slice.

C.4 Free Schrödinger Field theory & Its Euclidean avatar

The free Schrödinger field theory in $d+1$ dimensions is described by

$$\mathcal{L} = 2\imath\phi^\dagger\partial_t\phi + \phi^\dagger\nabla^2\phi \,, \tag{C.37}$$

where t is the real time and we have taken the mass to be unity. The propagator in momentum space representation has a pole at

$$\omega = \frac{|\boldsymbol{k}|^2}{2}. \tag{C.38}$$

where ω is the energy and $\boldsymbol{k}$ is the momentum. This is in fact the dispersion relation of an on-shell particle, described the field theory (C.37). The presence of a pole brings in ambiguity in defining the position space propagator and thus necessitates a pole prescription. Similar ambiguity is present in the relativistic theory as well at free fixed point, where there are two poles at $\omega = \pm|\boldsymbol{k}|$. To circumnavigate the pole(s), one usually works in the imaginary time $\tau = \imath t, \omega_E = -\imath\omega$ (so called *Wick rotation*), where the propagator is uniquely defined. Upon analytic continuation back to the Minkowski space-time, this provides us with a pole prescription. In what follows, we will follow the same procedure for the free Schrödinger field theory and come up with an expression for the propagator consistent with the $SL(2,\mathbb{R})$ algebra.

The Wick rotated Schrödinger theory has a propagator of the following form

$$\frac{1}{|\boldsymbol{k}|^2 - 2\imath\omega_E} \tag{C.39}$$

This does not have a pole on the real axis of ω_E. Hence, the Fourier transform is well defined and unique and one obtains the Euclidean propagator G_E:

$$G_E(\tau_1, \boldsymbol{x}_1; \tau_2, \boldsymbol{x}_2) = \int \frac{d\omega_E}{2\pi} \int \frac{d^d k}{(2\pi)^d} \frac{e^{-\imath(\omega_E(\tau_1-\tau_2)-\boldsymbol{k}\cdot\boldsymbol{x})}}{|\boldsymbol{k}|^2 - 2\imath\omega_E}$$
$$= \frac{\Theta(\tau)}{2} \left(\frac{1}{2\pi\tau}\right)^{\frac{d}{2}} \exp\left(-\frac{\boldsymbol{x}^2}{2\tau}\right) \tag{C.40}$$

where $\tau = \tau_2 - \tau_1$ and $\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. Upon Wick rotation back to the real time, we have the following $\imath\varepsilon$ prescription:

$$\frac{\imath}{2\omega - |\boldsymbol{k}|^2 + \imath\varepsilon} \tag{C.41}$$

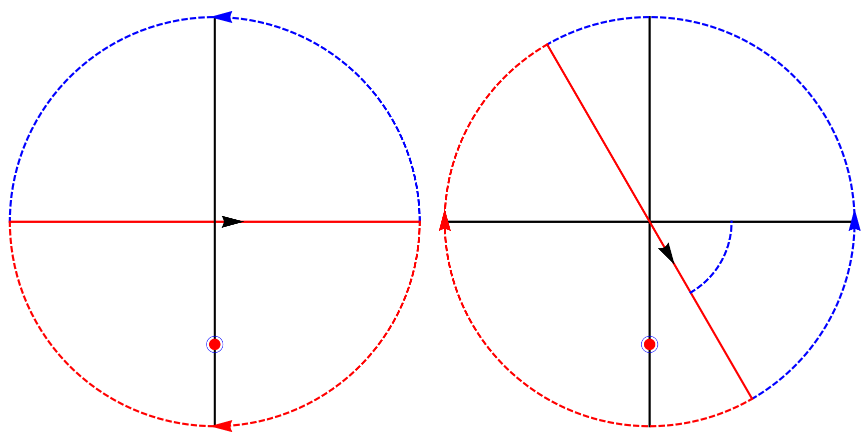

Figure C.1: Contour in the Wick rotated plane (ω_E plane). The wick rotation helps us to evaluate the integral.

The pole prescription, in momentum space, can be visualized through a series of diagrams, e.g. fig. (C.1,C.2). We recall that the Wick rotation involves defining $\tau = \imath t$ and $\omega_E = -\imath\omega$ such that $e^{\imath\omega t} = e^{\imath\omega_E\tau}$ holds true. One can rotate the contour clockwise by an angle of $\pi/2 - \varepsilon$ on the ω_E plane, where ε is very small but a positive number, without affecting the integral (See fig. C.1). At this point, one effects the wick rotation, which recasts the integral and the contour as shown in the fig. (C.2) and leads to the pole prescription, as in eq. (C.41).

Several remarks are in order. First of all, the physical significance of $\Theta(\tau)$ and hence, the $\imath\varepsilon$ prescription lies in the fact that there are no antiparticles in the theory. This has consequences

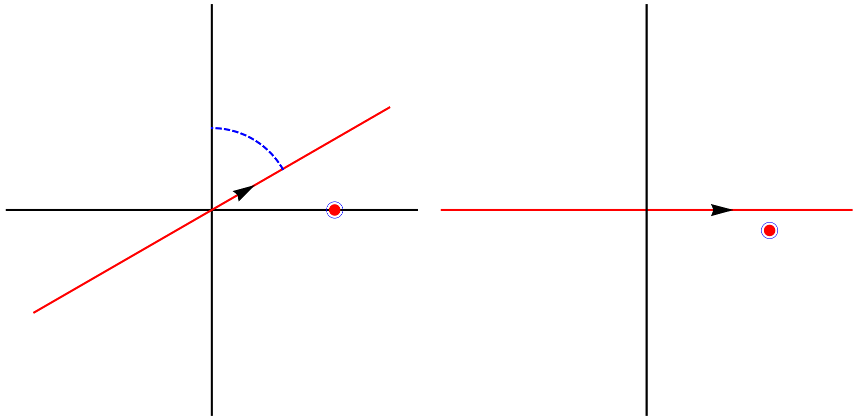

Figure C.2: Contour in the real ω plane. This is related to the contour in Euclidean plane by a Wick rotation.

e.g. the vacuum does not have any spatial entanglement entropy, the weyl anomaly is absent upon coupling the theory with a non trivial Newton-Cartan structure[107]. Furthermore, one can analytically continue the theory to live on a non integer dimensional space. The propagator as in eq. (C.40) can be defined by an analytic continuation in d. The analytical continuation is always understood to be in the number of spatial dimensions, without affecting the time co-ordinate.

The Schrödinger algebra constrains the real time two point correlator (G) of two primary operators of dimension Δ_O. It is given by

$$G(t_1,x_1;t_2,x_2) = ct^{-\Delta_O} e^{\imath \frac{N_O}{2}\frac{|\boldsymbol{x}|^2}{t}} \tag{C.42}$$

where $t = t_2 - t_1$ and $\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. We wick rotate the expression, introducing $\tau = \imath t$ and we choose $c = \frac{\Theta(\tau)}{2}\left(\frac{1}{2\pi\imath}\right)^{d/2}$ where $\tau = \tau_2 - \tau_1$. Upon comparing this expression with the Euclidean propagator, as in eq. (C.40), the dimension of free Schrödinger field is found out to be

$$\Delta_\phi = \frac{d}{2} \tag{C.43}$$

which is precisely the engineering dimension as evident from the Lagrangian (C.37).

We emphasize that the real time propagator, given in eq. (C.42), is generic in the sense it is suitable to describe any Schrödinger invariant fixed point including the interacting ones. Thus the take home message is that the usual trick of Wick rotation is consistent, as it should be, with the notion of the Euclidean Schrödinger group and the Euclidean Schrödinger operator is indeed given by $(\partial_\tau - \nabla^2)$.

Time ordering: The time-ordered (or anti-time ordered) real time correlator is obtained from the Euclidean correlator by proper analytic continuation. The Eq. (C.40) implies that

$$\langle \phi(0)\phi^\dagger(\tau)\rangle_E \propto \Theta(\tau) \tag{C.44}$$

where we have put in the subscript E to clearly specify that it is a Euclidean correlator. Now we will show that the time ordered correlator can be obtained by taking $\tau = -\imath t$ while the anti-time ordered one can be obtained by taking $\tau = \imath t$. For $t > 0$, we obtain the time ordered correlator by analytic continuation $(\tau = -\imath t)$

$$\langle 0|\phi^\dagger(t)\phi(0)|0\rangle \propto \Theta(-\imath t) = \imath\Theta(-t) = 0\,, \tag{C.45}$$

while for $t < 0$, we obtain

$$\langle 0|\phi(0)\phi^\dagger(t)|0\rangle \propto \Theta(-\imath t) = \imath\Theta(-t) \neq 0\,. \tag{C.46}$$

Similarly, for $t < 0$, the anti-time ordered correlator can be obtained by $(\tau = \imath t)$

$$\langle 0|\phi^\dagger(t)\phi(0)|0\rangle \propto \Theta(\imath t) = \imath\Theta(t) = 0 \tag{C.47}$$

while for $t > 0$, we have

$$\langle 0|\phi(0)\phi^\dagger(t)|0\rangle \propto \Theta(\imath t) = \imath\Theta(t) \neq 0 \tag{C.48}$$

It is easy to verify that all the equations (C.45),(C.46),(C.47),(C.48) conforms to the fact that the field ϕ annihilates the vacuum, which is a manifestation of absence of anti-particles. We also remark that (C.46) has interpretation of the amplitude associated with a particle being created at time $t < 0$ and subsequently propagating to $t = 0$. In the main text, we have used the analytic continuation $\tau = \imath t$, one could have equivalently choose the other analytic continuation $\tau = -\imath t$ and obtain similar results.

A different convention: The Euclidean time correlator can alternatively defined in following way where $\Theta(-\tau)$ appears in stead of (C.40):

$$
\begin{aligned}
G_E^{\mathrm{alt}}(\tau_1, \boldsymbol{x}_1; \tau_2, \boldsymbol{x}_2) &= \int \frac{d\omega_E}{2\pi} \int \frac{d^d k}{(2\pi)^d} \frac{e^{-\imath(\omega_E(\tau_2-\tau_1)-\boldsymbol{k}\cdot\boldsymbol{x})}}{|\boldsymbol{k}|^2 - 2\imath\omega_E} \\
&= \frac{\Theta(-\tau)}{2} \left(\frac{1}{2\pi\tau}\right)^{\frac{d}{2}} \exp\left(-\frac{\boldsymbol{x}^2}{2\tau}\right)
\end{aligned} \tag{C.49}
$$

where $\tau = \tau_2 - \tau_1$ and $\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. With this convention, $\tau = \imath t$ will yield time-ordered correlator while $\tau = -\imath t$ will give the anti–time ordered one. As mentioned earlier, in the main text, we have adopted the convention as in (C.40).

Appendix D

Existence and Construction of Galilean invariant $z \neq 2$ Theories

D.1 Diagonalizable and finite dimensional dilatation generator

In this appendix, we re-derive some of the results in Sec. 5.2 under the stronger assumption that the matrix $\mathbb{\Delta}$ is both diagonal and finite dimensional. This discussion is intended for clarity, since it is less abstract than the one presented in the main text.

We recall that

$$[D, \tilde{\Phi}(\boldsymbol{x} = \mathbf{0}, t = 0)] = \imath \mathcal{D} \tilde{\Phi}(\boldsymbol{x} = \mathbf{0}, t = 0) \tag{D.1}$$

and $\mathcal{D}$ is renamed as $\mathbb{\Delta}$ in the finite dimensional case.

To warm up, we show that both $\mathcal{D}$ and $\mathcal{N}$ are hermitian only if $z = 2$ or $\mathcal{N} = 0$. From $[D, N] = \imath(2 - z)N$, it follows that

$$[\mathcal{D}, \mathcal{N}] = (2 - z)\mathcal{N}. \tag{D.2}$$

Since $\mathcal{D}$ and $\mathcal{N}$ are assumed hermitian, $[\mathcal{D}, \mathcal{N}]^\dagger = -[\mathcal{D}, \mathcal{N}] = -(2 - z)\mathcal{N}$. Hence

$$-(2 - z)\mathcal{N} = [\mathcal{D}, \mathcal{N}]^\dagger = (2 - z)\mathcal{N}^\dagger = (2 - z)\mathcal{N},$$

which can only hold for $z = 2$ or $\mathcal{N} = 0$. If one assumes $\mathcal{N} \neq 0$ for some field, then $z = 2$. One can have $z \neq 2$ and hermiticity if $\mathcal{N} = 0$ for all fields. In this case the generator N is superfluous, and one can extend the algebra by including the generator of special conformal transformations.[1] Below we assume N does not identically vanish. Similarly, both $\mathcal{D}$ and $\mathcal{K}$ are hermitian only if $z = 1$ or $\mathcal{K} = 0$.

Now we consider the finite dimensional case where $\mathbb{\Delta}$ is diagonal. Alternatively, one can consider the case that $\mathbb{\Delta}$ is hermitian (and therefore, as just proved, $\mathbb{N}$ is not hermitian). In the finite dimensional, hermitian case, one can always choose to diagonalize $\mathbb{\Delta}$. Since $\mathbb{\Delta}$ is diagonal, $[\mathbb{\Delta}, \mathbb{N}] = (2-z)\mathbb{N}$ implies that $\left(\mathbb{\Delta}_{\alpha\alpha} - \mathbb{\Delta}_{\beta\beta} + z - 2\right)\mathbb{N}_{\alpha\beta} = 0$ (no summation over indices α, β is implicit), which, in turn, for $z \neq 2$ implies that $\mathbb{N}_{\alpha\alpha} = 0$ and at least one of $\mathbb{N}_{\alpha\beta}$ and $\mathbb{N}_{\beta\alpha}$ vanish. This implies that $\mathbb{N}$ is nilpotent,

$$\mathbb{N}^M = 0\,, \tag{D.3}$$

for some integer M no larger than the dimension of the representation. One can show this, without loss of generality, by arranging the components of the fields $\tilde{\Phi}_\alpha$ so that $\mathbb{N}$ is an upper triangular matrix. This result will play a pivotal role below.

Similarly, we assume that the field $\tilde{\Phi}(\mathbf{0}, 0)$ has the following commutation relation:

$$[K_i, \tilde{\Phi}(\boldsymbol{x} = \mathbf{0}, t = 0)] = \mathcal{K}_i \tilde{\Phi}(\boldsymbol{x} = \mathbf{0}, t = 0)\,. \tag{D.4}$$

For a finite dimensional case, we denote $\mathcal{K}_i$ by $\mathbb{K}_i$. By the same argument as above, one can show that either $z = 1$ or $\mathbb{K}_i = 0$ or $\mathbb{K}_i$ is nilpotent. Thus we have

$$\mathbb{K}_i^{L_i} = 0\,, \tag{D.5}$$

for some integer L_i no larger than the dimension of the representation. One can consider the

[1] There are indeed examples of $z \neq 2$ theories without particle number symmetry; see, for example, Refs. [133, 134, 143].

operator

$$\Phi \equiv \prod_{i=1}^{i=d-1} K_i^{L_i-1}(\tilde{\Phi}) , \tag{D.6}$$

where for any operator A and B, the action of the operator on the field is defined via

$$A(\tilde{\Phi}) \equiv [A, \tilde{\Phi}] , \tag{D.7}$$

$$BA(\tilde{\Phi}) \equiv B\big(A(\tilde{\Phi})\big) . \tag{D.8}$$

It can be easily verified that

$$[K_i, \Phi(\boldsymbol{x} = \mathbf{0}, t = 0)] = 0 , \tag{D.9}$$

$$[D, \Phi(\boldsymbol{x} = \mathbf{0}, t = 0)] = \imath \mathbb{\Delta}' \Phi(\boldsymbol{x} = \mathbf{0}, t = 0) , \tag{D.10}$$

$$[N, \Phi(\boldsymbol{x}, t)] = \mathbb{N} \Phi(\boldsymbol{x}, t) , \tag{D.11}$$

where $\mathbb{\Delta}' = (\mathbb{\Delta} - (z-1)\sum_i (L_i - 1))$. We call 'primary operators' those that satisfy (D.9). One could have considered operators obtained from these by analogous operations as above i.e. operators of the form $[N^{M-1}, \Phi]$, but that would not suffice to reveal the problems associated with finite dimensional representations.

Consider the two point correlator of primary operators in such a realization of the algebra, $G_{\alpha\beta} \equiv \langle 0| \Phi_\alpha(x,t) \Phi_\beta(0,0) |0\rangle$. Using Eqs. (5.9), the commutator in (D.9) translates to

$$[K_i, \Phi] = (-\imath t \partial_i \mathbf{I} + x_i \mathbb{N})\, \Phi . \tag{D.12}$$

Galilean boost invariance of the vacuum, $K_i|0\rangle = \langle 0| K_i = 0$, then gives

$$\begin{aligned} &\langle 0| \left[K, \Phi_\alpha(x,t) \Phi_\beta(0,0)\right] |0\rangle = 0 \\ &\Rightarrow (-\imath t \partial_i \delta_{\alpha\sigma} + x_i \mathbb{N}_{\alpha\sigma})\, G_{\sigma\beta} = 0 . \end{aligned} \tag{D.13}$$

The solution to the above differential equation is given by

$$G_{\alpha\beta} = \left[e^{-\imath\frac{|\boldsymbol{x}|^2}{2t}\mathbb{N}}\right]_{\alpha\gamma} C_{\gamma\beta}(t) = \sum_{\ell=0}^{M-1} \frac{1}{\ell!}\left(-\imath\frac{|\boldsymbol{x}|^2}{2t}\right)^{\ell} (\mathbb{N}^{\ell}C(t))_{\alpha\beta} \tag{D.14}$$

where $|\boldsymbol{x}|^2 = \sum_i (x^i)^2$, C is an as yet undetermined matrix function of t alone; the scaling symmetry implies that $C_{\alpha\beta}$ has a power law dependence on t. The above correlator (D.14) is consistent with the one given in (5.15) with $\mathbb{N}_1 = \mathbb{K}_1 = \mathbb{K}_2 = 0$. The exponential becomes a finite degree polynomial because $\mathbb{N}$ is nilpotent, and this is very specific to $z \neq 2$ theories. As explained above, the correlators are badly behaved: polynomial rather than exponential dependence on $|\boldsymbol{x}|$ leads to growth with spatial separation (and hence, cluster decomposition fails). In contrast, for $z = 2$ the matrix $\mathbb{N}$ is diagonal and there is no truncation of the expansion of the exponential. An additional constraint on the correlator follows from requiring that $\langle 0| \left[N, \Phi_\alpha(x,t)\Phi_\beta(0,0)\right] |0\rangle = 0$, which implies $NG + GN^T = 0$.

Consider next $G'_{\alpha\beta} = \langle 0|\Phi_\alpha(x,t)\Phi^\dagger_\beta(0,0)|0\rangle$. This is similarly given by

$$G'_{\alpha\beta} = \left[\exp\left(-\imath\frac{|\boldsymbol{x}|^2}{2t}\mathbb{N}\right) C'(t)\right]_{\alpha\beta} \tag{D.15}$$

for some undetermined matrix C' such that $C'_{\alpha\beta}$ is a function of t alone. Invariance under N implies that $NG' - G'N^\dagger = 0$. Notice that the condition on G' is different from that on G; one may have non-trivial solutions to one but not the other. For example, one can consider the two component field, $\Phi_{\alpha=1,2}$ characterized by:

$$\mathbb{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C' = g(t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; g(t) = t^{-(\Delta_{11}+\Delta_{22})/z}$$

$$\mathbb{\Delta} = \begin{bmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{bmatrix}, \quad \Delta_{22} - \Delta_{11} = (2-z),$$

$$G' = t^{-(\Delta_{11}+\Delta_{22})/z} \begin{bmatrix} 1 & -\imath\frac{|\boldsymbol{x}|^2}{2t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= t^{-(\Delta_{11}+\Delta_{22})/z} \begin{bmatrix} -\imath\frac{|\boldsymbol{x}|^2}{2t} & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that for this example $G_{\alpha\beta} = 0$ so consideration of the long distance behavior of this correlator alone does not, by itself, suggest the theory is ill-behaved.

Appendix E

Hilbert Series and Operator Basis for NRQED and NRQCD/HQET

E.1 Characters and Haar Measures

The characters for irreducible representations needed in the NRQED and NRQCD/HQET Hilbert series for $U(1)$, $SO(3)$, $SU(2)$, and $SU(3)$ are

$$\begin{aligned}
\chi^{U(1)}(x) &= x && \text{(E.1)}\\
\chi^{SO(3)}_{\mathbf{3}}(z) &= z^2+1+\frac{1}{z^2} && \text{(E.2)}\\
\chi^{SU(2)}_{\mathbf{2}}(y) &= y+\frac{1}{y} && \text{(E.3)}\\
\chi^{SU(2)}_{\mathbf{3}}(y) &= y^2+1+\frac{1}{y^2} && \text{(E.4)}\\
\chi^{SU(3)}_{\mathbf{3}}(x_1,x_2) &= x_2+\frac{x_1}{x_2}+\frac{1}{x_1} && \text{(E.5)}\\
\chi^{SU(3)}_{\bar{\mathbf{3}}}(x_1,x_2) &= x_1+\frac{x_2}{x_1}+\frac{1}{x_2} && \text{(E.6)}\\
\chi^{SU(3)}_{\mathbf{8}}(x_1,x_2) &= x_1x_2+\frac{x_2^2}{x_1}+\frac{x_1^2}{x_2}+2+\frac{x_1}{x_2^2}+\frac{x_2}{x_1^2}+\frac{1}{x_1x_2} && \text{(E.7)}
\end{aligned}$$

The contours integrals with respect to the Haar measures used in this analysis are

$$\oint [d\mu]_{U(1)} \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{1}{x} \qquad \text{(E.8)}$$

$$\oint [d\mu]_{SO(3)} \equiv \frac{1}{2\pi i}\oint_{|z|=1} \frac{1}{2z}(1-z^2)\left(1-\frac{1}{z^2}\right) \tag{E.9}$$

$$\oint [d\mu]_{SU(2)} \equiv \frac{1}{2\pi i}\oint_{|y|=1} \frac{1}{2y}(1-y^2)\left(1-\frac{1}{y^2}\right) \tag{E.10}$$

$$\oint [d\mu]_{SU(3)} \equiv \frac{1}{(2\pi i)^2}\oint_{|x_1|=1}\oint_{|x_2|=1} I \tag{E.11}$$

where I is given by

$$I = \frac{1}{6x_1x_2}(1-x_1x_2)\left(1-\frac{x_1^2}{x_2}\right)\left(1-\frac{x_2^2}{x_1}\right)\left(1-\frac{1}{x_1x_2}\right)\left(1-\frac{x_1}{x_2^2}\right)\left(1-\frac{x_2}{x_1^2}\right)$$

Appendix F

Conformal Structure of the Heavy Particle EFT Operator Basis

F.1 Normalsize Symmetries of Non-Relativistic Systems

In this appendix, we review the symmetries associated with non-relativistic systems. Much of the details that we present below can be found in, for example, Refs. [265, 93, 97, 94, 98, 59, 81, 99, 137]. Newton's equation of motion for a particle with mass m subject to an external force $\mathbf{F}$ at time t and position $\mathbf{x}$ is:

$$\mathbf{F}(t,\mathbf{x}) = m\frac{d^2\mathbf{x}}{dt^2}\,. \tag{F.1}$$

Consider a change in time and space coordinates $(t,\mathbf{x})$ to $(t',\mathbf{x}')$ defined by the following transformations:

$$t \mapsto t' = t + b\,, \qquad x_i \mapsto x_i' = R_{ij}x_j + v_i t + a_i\,. \tag{F.2}$$

Here, R_{ij} is a rotation matrix, v_i is the velocity, b is a translation in time, and a_i is a translation in space. Roman letters indicate space indices, and v_i, b, and a_i are all real constants, independent of time. Eq. (F.1) becomes the following under such a transformation:

$$F_i(t,\mathbf{x}) = m\frac{d^2x_i}{dt^2} \quad \mapsto \quad \widetilde{F}_i(t',\mathbf{x}') \equiv R_{ij}F_j(t,\mathbf{x}) = mR_{ij}\frac{d^2x_j}{dt^2} = m\frac{d^2x_i'}{dt'^2}\,, \tag{F.3}$$

which implies

$$\widetilde{F}_i(t',\mathbf{x}') = m\frac{d^2x_i'}{dt'^2}\,. \tag{F.4}$$

The transformations defined in Eq. (F.2) leave the *form* of Eq. (F.1) unchanged and are, therefore, symmetries associated with that equation of motion. The transformations in Eq. (F.2) are the most general *linear* transformations that leave time and space intervals (defined at the same moment in time) separately unchanged, i.e.,

$$t_1 - t_2 = \text{const}, \qquad |\mathbf{x}_1 - \mathbf{x}_2| = \text{const, if } t_1 = t_2. \tag{F.5}$$

These transformations furnish what is known as the *inhomogeneous Galilean group*. We refer to the inhomogeneous Galilean group as the Galilean group when no confusion is likely. The group multiplication laws and inverses of group elements are straightforward to work out.

The elements of the Galilean group can be represented as $\exp[i\theta_a X_a]$, where θ_a's are the 10 parameters needed to define a Galilean transformation, and the X_a's are the generators of the group. The following commutation relations define the Lie algebra:

$$\begin{gathered} = i\varepsilon_{ijk}J_k\,, \quad [J_i,K_j] = i\varepsilon_{ijk}K_k\,, \quad [J_i,P_j] = i\varepsilon_{ijk}P_k\,, \\ [H,K_i] = -iP_i\,, \quad [K_i,P_j] = 0\,, \\ [K_i,K_j] = [H,P_i] = [H,J_i] = [P_i,P_j] = 0\,, \end{gathered} \tag{F.6}$$

where P_i generates spatial translations, H generates time translations, J_i generates rotations, and K_i are generators of Galilean boosts (named as such to distinguish them from Lorentz boosts). Here, we use the convention that the antisymmetric tensor $\varepsilon_{123} = 1$. Note that the specific commutation relations $[K_i,K_j]$ and $[P_i,K_j]$ amount to the only differences compared to the Lie algebra of the relativistic Poincaré group.

The Lie algebra of the Galilean algebra can also be derived from the Poincaré algebra by reintroducing the speed of light, i.e., $H \to H/c$, $K_i \to cK_i$, and letting $c \to \infty$. For example, the generators of Lorentz boosts have the commutation relation $[K_i,K_j] = -i\varepsilon_{ijk}J_k$. Putting the factors of c back in: $[K_i, K_j] = -i\varepsilon_{ijk}J_k/c^2$, and in the limit $c \to \infty$, one induces the commutation relation for Galilean boosts: $[K_i, K_j] = 0$. Likewise, for the commutator $[K_i, P_j] = iH\delta_{ij} \to [K_i, P_j] = iH\delta_{ij}/c^2$ in the Poincaré group, this relation becomes $[K_i, P_j] = 0$ in the Galilean group, i.e., when $c \to \infty$.

There exists the possibility that the Galilean group can be augmented with an additional generator, called N, such that N commutes with all other generators:

$$[N, \text{any}] = 0\,, \tag{F.7}$$

and the commutation relation $[K_i, P_j]$ becomes:

$$[K_i, P_j] = 0 \qquad \longrightarrow \qquad [K_i, P_j] = iN\delta_{ij}\,. \tag{F.8}$$

The augmentation of this kind is known as *central extension*[1] [265]. When taking the $c \to \infty$ limit in the Poincaré algebra, one obtains the $N = 0$, i.e., the chargeless (neutral) sector of the Galilean group, which is its own algebra.

It is interesting to consider the special case when $\mathbf{F}(t, \mathbf{x}) = 0$. Here, the classical equation of motion is:

$$\frac{d^2\mathbf{x}}{dt^2} = 0\,. \tag{F.9}$$

This is invariant under the Galilean transformations. In addition to the transformations contained in the Galilean group, one could consider a kind of scaling transformation that acts like:

$$x_i \mapsto x_i' = \lambda x_i\,, \qquad t \mapsto t' = \lambda^z t\,. \tag{F.10}$$

For any value of z, the classical equation of motion is invariant under such transformations.

[1]Originally, the algebra without the central extension was called by Galilean algebra. Lately, the trend in the literature is to call the centrally-extended algebra the "Galilean algebra," and specify $N = 0$ as a special case.

However, if the classical action, which appears as $\exp(iS)$ in path integral,

$$S = \int dt \left[\sum_i \frac{1}{2} m \left(\frac{dx_i}{dt} \right)^2 \right] , \tag{F.11}$$

is to be invariant under such transformations, then it is necessary that $z = 2$. This difference in the requirements of the value of z illustrates the important point that the equation of motion, being associated with the extremum of the action, is less constraining than the action, since the latter contains information regarding all possible configurations of the system in time. There is one additional kind of transformation which leaves both the classical equation of motion and the classical action invariant:

$$x_i \mapsto x_i' = \frac{x_i}{(1+kt)} , \qquad t \mapsto t' = \frac{t}{(1+kt)} , \tag{F.12}$$

where k is a real number. These transformations are known as the *special conformal transformations*. Together, the Galilean transformations, the scaling transformations in Eq. (F.10) (when $z = 2$) and the special conformal transformations form what is called the *Schrödinger group*,[2] where temporal and spatial coordinates transform in the following way:

$$t \mapsto t' = \lambda^2 \left(\frac{t+b}{1+k(t+b)} \right) , \qquad x_i \mapsto x_i' = \lambda \left(\frac{R_{ij} x_j + v_i t + a_i}{1+k(t+b)} \right) . \tag{F.13}$$

The group multiplication laws and inverses are straightforward (albeit algebraically tedious) to calculate.[3]

The elements of the Schrödinger group can be represented as $\exp[i\theta_a X_a]$, where in the index a runs over the number of generators. The Lie algebra of the Schrödinger group is:[4]

[2]The group was popularized in the context of the free Schrödinger equation [93], hence the name. It does not necessarily have to do with quantum mechanics.

[3]We note that the transformation of the time coordinate [81, 137] can be mapped on to a SL(2,$\mathbb{R}$) group, i.e., transformations of the form

$$t \mapsto t' = \frac{at+b}{ct+d} , \qquad ad - bc = 1 . \tag{F.14}$$

In terms of these parameters, the inverse and the group multiplication is relatively less tedious to compute.

[4]These commutation relations are the same as those in Refs. [59, 81], but differ from those in Ref. [93] by some sign conventions.

$$
\begin{gathered}
= i\varepsilon_{ijk}J_k\,, \quad [J_i,K_j] = i\varepsilon_{ijk}K_k\,, \quad [J_i,P_j] = i\varepsilon_{ijk}P_k\,, \\
[H,K_i] = -iP_i\,, \quad [K_i,P_j] = iN\delta_{ij}\,, \\
[D,K_i] = -iK_i\,, \quad [D,P_i] = iP_i\,, \quad [D,H] = 2iH\,, \quad [D,C] = -2iC\,, \\
[C,P_i] = iK_i\,, \quad [C,H] = iD\,, \\
[K_i,K_j] = [H,P_i] = [H,J_i] = [P_i,P_j] = [N,\text{any}] = 0\,, \\
[D,J_i] = [C,J_i] = [C,K_i] = 0\,,
\end{gathered} \tag{F.15}
$$

where D is the generator of scaling transformations, and C is the generator of the special conformal transformations. The Cartan generators for the Schrödinger group are $E_1 \equiv -iD$, $E_2 \equiv J_3$, and $E_3 \equiv N$, i.e., the maximally commuting set of generators. The generators with definite weight under these Cartan generators are given by:

$$J_\pm \equiv J_1 \pm iJ_2\,, \tag{F.16}$$

$$P_\pm \equiv P_1 \pm iP_2\,, \tag{F.17}$$

$$K_\pm \equiv K_1 \pm iK_2\,, \tag{F.18}$$

$$P_3\,, H,\; K_3\,,\; C\,. \tag{F.19}$$

A generator X carries a weight w under a Cartan generator E if $[E,X] = wX$. The factor of $-i$ with generator D makes the weights real. The weights follow directly from the algebra, and are tabulated in Table F.1.

F.2 Symmetries, Operators, and States

If U is a group element, then assume there exists a Hilbert space with a vacuum state such that all U's leave the vacuum invariant:

$$U\,|0\rangle = |0\rangle\;. \tag{F.20}$$

Table F.1: Table for weights of generators $P_\pm,\ P_3,\ H,\ J_\pm,\ K_\pm,\ K_3,\ C$ under the Cartan generators $E_1 \equiv -iD$, $E_2 \equiv J_3$, and $E_3 \equiv N$. The ij-th entry is the weight of the i'th generator (i running over the possibilities $P_\pm,\ P_3,\ H,\ J_\pm,\ K_\pm,\ K_3,\ C$ under the Cartan generator E_j.

	$E_1 \equiv -iD$	$E_2 \equiv J_3$	$E_3 \equiv N$
$P_\pm$	1	± 1	0
P_3	1	0	0
H	2	0	0
$J_\pm$	0	± 1	0
$K_\pm$	-1	± 1	0
K_3	-1	0	0
C	-2	0	0

If U can be represented as $\exp[i\theta_a X_a]$, where X_a are the generators of the group's Lie algebra, then all X's annihilate the vacuum:

$$X_a \left|0\right\rangle = 0\,. \tag{F.21}$$

Consider that the Hilbert space is spanned by more states $\left|O_A\right\rangle$ (where A is just an arbitrary label) than just the vacuum, which are defined as local operators $O_A(\tau,\mathbf{x})$ acting on the vacuum where $\tau = it$.[5] For now, let us only consider (gauge invariant) states, created by operators acting at the origin:

$$O_A(0,\mathbf{0}) \left|0\right\rangle = \left|O_A\right\rangle\,. \tag{F.22}$$

Because acting on a state with a group transformation produces, in general, a linear combination of states still within that Hilbert space, one can say acting with a generator produces a linear combination of states:

$$X \left|O_A\right\rangle = L_{AB} \left|O_B\right\rangle\,, \tag{F.23}$$

where L is some matrix, which depends on what X was chosen. All the generators annihilate the vacuum, therefore

$$X \left|O_A\right\rangle = [X, O_A(0,\mathbf{0})] \left|0\right\rangle = L_{AB} O_B(0,\mathbf{0}) \left|0\right\rangle\,. \tag{F.24}$$

Consider that the Hilbert space is spanned by operators that transform non-trivially under

[5]The states are prepared in Euclidean time τ to ensure finite norm as we will see later.

the Schrödinger group transformations. The largest set of Cartan generators is $-iD$, N, and J_3, and we can always choose to label our states according the eigenvalues of these operators:

$$D\,|\Delta,n,m\rangle = i\Delta\,|\Delta,n,m\rangle\,, \tag{F.25}$$

$$N\,|\Delta,n,m\rangle = n\,|\Delta,n,m\rangle\,, \tag{F.26}$$

$$J_3\,|\Delta,n,m\rangle = m\,|\Delta,n,m\rangle\,. \tag{F.27}$$

The eigenvalue of D is complex, but we can see the significance of this in terms of the transformation properties of operators. Say that there exists an operator $O_{[\Delta,n,m]}$ that creates the state $|\Delta,n,m\rangle$ at the origin:

$$|\Delta,n,m\rangle \equiv O_{[\Delta,n,m]}(0,\mathbf{0})\,|0\rangle\,. \tag{F.28}$$

In terms of the $O_{[\Delta,n,m]}(0,\mathbf{0})$, we have

$$[D,O_{[\Delta,n,m]}(0,\mathbf{0})] = i\Delta\; O_{[\Delta,n,m]}(0,\mathbf{0}) \quad \to \quad e^{i\lambda D}O_{[\Delta,n,m]}(0,\mathbf{0})e^{-i\lambda D} = e^{-\lambda\Delta}O_{[\Delta,n,m]}(0,\mathbf{0})\,. \tag{F.29}$$

The factor of i ensures the operator $O_{[\Delta,n,m]}(0,\mathbf{0})$ gets scaled by a real number $\exp(-\lambda\Delta)$ under the transformation generated by D.

After some algebra, one can find the following relations, which follow directly from the Schrödinger algebra (see Table F.1):

$$DP_i\,|\Delta,n,m\rangle = i(\Delta+1)P_i\,|\Delta,n,m\rangle\,, \tag{F.30}$$

$$DK_i\,|\Delta,n,m\rangle = i(\Delta-1)K_i\,|\Delta,n,m\rangle\,, \tag{F.31}$$

$$DH\,|\Delta,n,m\rangle = i(\Delta+2)H\,|\Delta,n,m\rangle\,, \tag{F.32}$$

$$DC\,|\Delta,n,m\rangle = i(\Delta-2)C\,|\Delta,n,m\rangle\,, \tag{F.33}$$

or, written another way:

$$\big[D,[P_i,O_{[\Delta,n,m]}(0,\mathbf{0})]\big]\,|0\rangle = i(\Delta+1)[P_i,O_{[\Delta,n,m]}(0,\mathbf{0})]\,|0\rangle\,, \tag{F.34}$$

$$\left[D,[K_i,O_{[\Delta,n,m]}(0,\mathbf{0})]\right]|0\rangle = i(\Delta-1)[K_i,O_{[\Delta,n,m]}(0,\mathbf{0})]\,|0\rangle\,, \tag{F.35}$$

$$\left[D,[H,O_{[\Delta,n,m]}(0,\mathbf{0})]\right]|0\rangle = i(\Delta+2)[H,O_{[\Delta,n,m]}(0,\mathbf{0})]\,|0\rangle\,, \tag{F.36}$$

$$\left[D,[C,O_{[\Delta,n,m]}(0,\mathbf{0})]\right]|0\rangle = i(\Delta-2)[C,O_{[\Delta,n,m]}(0,\mathbf{0})]\,|0\rangle\,. \tag{F.37}$$

Therefore, P_i and H act as lowering operators, and K_i and C act as raising operators, in analogy to the representation of $SU(2)$ or $SO(3)$. If one assumes that the scaling dimension Δ of operators in this Hilbert space is bounded from below, then there will be a set operators O^P such that:

$$[K_i,O^P_{[\Delta,n,m]}(0,\mathbf{0})]\,|0\rangle = 0\,, \qquad \text{and} \qquad [C,O^P_{[\Delta,n,m]}(0,\mathbf{0})]\,|0\rangle = 0\,. \tag{F.38}$$

Such operators are called *primary operators*, which are associated with the highest-weight states.[6] Acting with P_i or H repeatedly on these primary operators produces a tower of operators, where P_i and H raise the scaling dimension by either one or two units. Acting with P_i and H are associated with space and time derivatives:

$$[P_i,O(t,\mathbf{x})]\,|0\rangle = i\partial_i O(t,\mathbf{x})\,|0\rangle\,, \qquad [H,O(t,\mathbf{x})]\,|0\rangle = -i\partial_t O(t,\mathbf{x})\,|0\rangle\,, \tag{F.39}$$

which is true for any operator O.

The Schrödinger algebra permits further categorization of different kinds of primary operators. Beginning with the Jacobi identity for any operators A, B, and C:

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]] = 0\,, \tag{F.40}$$

one can arrive at the following identities:

$$[K_i,[P_j,O^P_{[\Delta,n,m]}(0,\mathbf{0})]] = -in\delta_{ij}O^P_{[\Delta,n,m]}(0,\mathbf{0})\,, \tag{F.41}$$

$$[C,[P_i,O^P_{[\Delta,n,m]}(0,\mathbf{0})]] = 0\,. \tag{F.42}$$

For $n\neq 0$, $[P_i,O^P_{[\Delta,0,m]}]$ are not primary operators and are called *descendants*. Thus, when $n\neq 0$,

[6] These are actually states with lowest scaling dimension. Following the group theory literature, we call them the highest-weight state.

all descendants are total spatial or time derivatives of some primary operator. The distinction between primary and descendants become obfuscated in the $n = 0$ sector. This is because if $n = 0$, then $[P_i, O^P_{[\Delta,0,m]}]$ is also a primary operator, as evident from Eq. (F.41). In this scenario, one can reorganize the operators by being agnostic to the action of K_i, following Ref. [137]. In particular, one categorize the operators into *quasi-primaries* and its descendants, where the quasi-primaries satisfy $[C, O(0,\mathbf{0})] = 0$ without making any requirements for the value of $[K_i, O(0,\mathbf{0})]$, and the descendants are obtained by action of H on quasi-primaries. This categorization exploits the SL(2,$\mathbb{R}$) subgroup generated by H, D, C which applies for both the neutral and the charged sectors.

An operator can be written as a total time derivative of another operator if and only if it is a descendant of a quasi-primary, which leaves out the possibility that a quasi-primary can be a total spatial derivative of some operator. This necessitates further categorization: we define $O^{PA}_{[\Delta,n,m]}(0,\mathbf{0})]$ to be *quasi-primaries of type-A* if and only if satisfies

$$[C, O^{PA}_{[\Delta,n,m]}(0,\mathbf{0})] = 0\,, \tag{F.43}$$

$$O^{PA}_{[\Delta,n,m]}(0,\mathbf{0}) \neq [P_i, O]\,, \tag{F.44}$$

for any operator O. *Quasi-primaries of type-B* are those where Eq. (F.44) does not hold.

F.3 Constraints from Algebra and Unitarity bound

The Schrödinger algebra completely restricts the spacetime dependence of the two-point correlation function between two primary operators. Consider two primary operators, O_1 with scaling dimension Δ_1 and number charge n_1 and a second, O_2, with scaling dimension Δ_2 and number charge n_2. Begin with the following expressions, which are explicitly zero, since all generators annihilate the vacuum:

$$\langle 0|\,[K_i, O_2(t,\mathbf{x})]\,O_1(0,\mathbf{0})\,|0\rangle + \langle 0|\,O_2(t,\mathbf{x})[K_i, O_1(0,\mathbf{0})]\,|0\rangle \;=\; 0\,, \tag{F.45}$$

$$\langle 0| [C, O_2(t,\mathbf{x})] O_1(0,\mathbf{0}) |0\rangle + \langle 0| O_2(t,\mathbf{x})[C, O_1(0,\mathbf{0})] |0\rangle = 0\,, \tag{F.46}$$

$$\langle 0| [D, O_2(t,\mathbf{x})] O_1(0,\mathbf{0}) |0\rangle + \langle 0| O_2(t,\mathbf{x})[D, O_1(0,\mathbf{0})] |0\rangle = 0\,, \tag{F.47}$$

$$\langle 0| [N, O_2(t,\mathbf{x})] O_1(0,\mathbf{0}) |0\rangle + \langle 0| O_2(t,\mathbf{x})[N, O_1(0,\mathbf{0})] |0\rangle = 0\,. \tag{F.48}$$

One can use $O(t,\mathbf{x}) = e^{-i(Ht-P_jx_j)}O(0,\mathbf{0})e^{i(Ht-P_jx_j)}$ and Eq. (F.39) to generalize the relations in Eq. (F.38) to an arbitrary point in spacetime:

$$[K_i, O^P_{[\Delta,n,m]}(t,\mathbf{x})] |0\rangle = (nx_i - it\partial_i)\, O^P_{[\Delta,n,m]}(t,\mathbf{x}) |0\rangle \,, \tag{F.49}$$

$$[C, O^P_{[\Delta,n,m]}(t,\mathbf{x})] |0\rangle = \left(-it\Delta + \frac{1}{2}x^2 n - itx_j\partial_j - it^2\partial_t\right) O^P_{[\Delta,n,m]}(t,\mathbf{x}) |0\rangle \,, \tag{F.50}$$

$$[D, O^P_{[\Delta,n,m]}(t,\mathbf{x})] |0\rangle = i\left(\Delta + x_j\partial_j + 2t\partial_t\right) O^P_{[\Delta,n,m]}(t,\mathbf{x}) |0\rangle \,. \tag{F.51}$$

Inserting Eqs. (F.49) - (F.51) in to Eqs. (F.45) - (F.48):

$$\langle 0| (n_2 x_i - it\partial_i) O_2(t,\mathbf{x})\, O_1(0,\mathbf{0}) |0\rangle = 0\,, \tag{F.52}$$

$$\langle 0| \left(-it\Delta_2 + \frac{1}{2}x^2 n_2 - itx_j\partial_j - it^2\partial_t\right) O_2(t,\mathbf{x}) O_1(0,\mathbf{0}) |0\rangle = 0\,, \tag{F.53}$$

$$\langle 0| \left(x_j\partial_j + 2t\partial_t + \Delta_1 + \Delta_2\right) O_2(t,\mathbf{x}) O_1(0,\mathbf{0}) |0\rangle = 0\,, \tag{F.54}$$

$$(n_1 + n_2) \langle 0| O_2(t,\mathbf{x}) O_1(0,\mathbf{0}) |0\rangle = 0\,. \tag{F.55}$$

This system of differential equations can be simultaneously solved to give a non-trivial result only if $\Delta_1 = \Delta_2 \equiv \Delta$ and $n_2 = -n_1 \equiv n$, which can be satisfied if $O_1 = O_2^\dagger \equiv O^\dagger$, and if so [59, 81]:

$$\langle 0| O(t,\mathbf{x}) O^\dagger(0,\mathbf{0}) |0\rangle = \text{const} \cdot t^{-\Delta} \exp\left[\frac{-inx^2}{2t}\right]\,. \tag{F.56}$$

This is true whether t is positive or negative, therefore we have the following result that the time-ordered product $\langle 0| T\, O(t,\mathbf{x}) O^\dagger(0,\mathbf{0}) |0\rangle$ has the same form.

There are additional constraints on operators that come from the requirement of unitarity. There are various ways to arrive at the unitarity bound [59, 81, 137]. Here we will briefly sketch the method described in Ref. [137]. Let $\tau = it$, and assume the state $|\psi(\tau,\mathbf{x})\rangle$ can be associated with a local primary operator, with scaling dimension Δ and number charge n, acting on the vacuum: $|\psi(\tau,\mathbf{x})\rangle \equiv O^\dagger(\tau,\mathbf{x})\,|0\rangle$. Then, the requirement of unitarity can be written as:

$$\langle\psi(\tau,\mathbf{x})|\psi(\tau,\mathbf{x})\rangle \geq 0 \Leftrightarrow \lim_{\substack{\tau'\to-\tau \\ \mathbf{x}'\to\mathbf{x}}} \langle 0|\, O(\tau',\mathbf{x}')O^\dagger(\tau,\mathbf{x})\,|0\rangle \geq 0\,. \tag{F.57}$$

Because these are primary operators, the form of this two-point correlation function from Eq. (F.56) is

$$\langle 0|\, O(\tau',\mathbf{x}')O^\dagger(\tau,\mathbf{x})\,|0\rangle = \text{const}\cdot(\tau'-\tau)^{-\Delta}\exp\left[\frac{n(\mathbf{x}'-\mathbf{x})^2}{2(\tau'-\tau)}\right]. \tag{F.58}$$

The algebra does not constrain the overall constant in the above expression, so one is free to choose it such that

$$\text{const}\cdot(-\tau)^{-\Delta} \geq 0\,, \tag{F.59}$$

is always true. Because Eq. (F.57) is required for any state $|\psi\rangle$ in the Hilbert space, it must also hold for a state $|\widetilde{\psi}\rangle$ defined as any combination of partial (Euclidean) time and space derivatives acting on the original primary operator: $|\widetilde{\psi}\rangle \equiv (A\partial_\tau + B\partial_i + C\partial_i\partial_j + ...)\,O^\dagger(\tau,\mathbf{x})\,|0\rangle$, where A, B, C, ..., are constants. Consider a particular state:

$$|\widetilde{\psi}\rangle \equiv (\alpha\partial_\tau + \beta\partial_i\partial_i)\,O^\dagger(\tau,\mathbf{x})\,|0\rangle\,, \tag{F.60}$$

where α and β are real constants. The requirement that $\langle\widetilde{\psi}(\tau,\mathbf{x})|\widetilde{\psi}(\tau,\mathbf{x})\rangle \geq 0$ then leads to the following inequality:

$$\lim_{\substack{\tau'\to-\tau \\ \mathbf{x}'\to\mathbf{x}}} (-\alpha\partial_{\tau'} + \beta\partial_{i'}\partial_{i'})(\alpha\partial_\tau + \beta\partial_i\partial_i)\,\langle 0|\, O(\tau',\mathbf{x}')O^\dagger(\tau,\mathbf{x})\,|0\rangle \geq 0\,, \tag{F.61}$$

where the primes indicate that they only act on the primed spacetime variables. Using Eqs. (F.58) and (F.59), this leads to the inequality:

$$\beta^2 n^2(d^2+2d) + 2\alpha\beta n d(\Delta+1) + \alpha^2\Delta(\Delta+1) \geq 0\,, \tag{F.62}$$

where d is the number of spatial dimensions. For a fixed value of d, the Eq. (F.62) implies that $\Delta \notin (-1, d/2)$. Similarly, considering a state of the form $(\alpha' O^\dagger(\tau, \mathbf{x}) + \partial_\tau O^\dagger(\tau, \mathbf{x}))\,|0\rangle$, one can rule out $\Delta < 0$ [137]. Combining these two bounds, we have a bound on Δ, i.e., $\Delta \geq d/2$, which occurs when $\alpha/\beta = -2n$. If $\Delta = d/2$, then the state $|\widetilde{\psi}\rangle$ is a null state with zero norm, where, plugging these values for α and β back into Eq. (F.60), and setting $\Delta = d/2$:

$$\left(\partial_\tau - \frac{\partial_i^2}{2n}\right)|\psi(\tau, \mathbf{x})\rangle = 0\,. \tag{F.63}$$

Identifying $\tau = it$ and $-n$ (the charge of $O^\dagger$ is $-n > 0$) as the mass of the particle, we have

$$\left(-i\partial_t + \frac{\partial_i^2}{2m}\right) O^\dagger\,|0\rangle = 0\,, \quad \left(i\partial_t + \frac{\partial_i^2}{2m}\right) O\,|0\rangle = 0\,. \tag{F.64}$$

This is the Schrödinger equation. Because this is an example of a classical equation of motion for a well-defined quantum field theory, the bound $\Delta \geq d/2$ is therefore the strongest bound.[7]

[7] To prove this, assume there is a better (or as good) lower bound Δ^* such that $\Delta \geq \Delta^* \geq d/2$. There exists an example of a well-defined quantum field theory with $\Delta = d/2$, therefore $\Delta = \Delta^*$.

Appendix G

Nonrelativistic Conformal Field Theories in the Large Charge Sector

G.1 Appendix A: Phonons in the Trap

We are solving equation (8.81) in the range of $r \in [0,R]$ where $R^2 = \frac{2\mu}{\omega^2}$ is the cloud radius.

Inserting $\pi \propto e^{\mathbf{i}\varepsilon t} f(r) Y_\ell$ and expanding in spherical coordinates:

$$-\frac{\omega^2}{d}(R^2 - x^2)[\partial_r^2 f + \frac{(d-1)}{r}\partial_r f - \frac{1}{r^2}\ell(\ell+d-2)f] + \omega^2 r \partial_r f = \varepsilon^2 f \tag{G.1}$$

Defining the dimensionless variables $x \equiv \frac{r}{R}$ and $\lambda \equiv \frac{\varepsilon}{\omega}$ and changing variables to $z = x^2$

$$-\frac{1}{d}(1-z)[4z\partial_z^2 f + 2\partial_z f + 2(d-1)\partial_z f - \frac{1}{z}\ell(\ell+d-2)f] + 2z\partial_z f = \lambda^2 f \tag{G.2}$$

Equation (G.2) is a hypergeometric equation with two independent solutions

$$f(z) \sim c_1 z^{\frac{\ell}{2}}\, {}_2F_1(\alpha_-, \alpha_+, \gamma, z) + c_2 z^{\frac{1}{2}(2-d-\ell)}\, {}_2F_1(\alpha', \beta', \gamma', z) \tag{G.3}$$

Our solution should be valid on the interval $z \in [0,1]$ where it should be regular and finite at both $z = 0$ and $z = 1$. Regularity at the origin kills the second solution immediately.

Therefore we have:

$$f(z) \sim c_1 z^{\frac{\ell}{2}}\, {}_2F_1(\alpha_-, \alpha_+, \gamma, z) \tag{G.4}$$

where $\gamma = \ell + \frac{d}{2}$, $\alpha_{\pm} = \frac{1}{2}(\ell + d - 1) \pm \kappa$, and $\kappa = \frac{1}{2}(1 - 2d + d^2 - 2\ell + \ell d + \ell^2 + d\lambda^2)^{\frac{1}{2}}$

The function ${}_2F_1(\alpha_-, \alpha_+, \gamma, z)$ is finite at $z = 1$ under one of the following possibilities:

1. The values $\alpha_+ + \alpha_- < \gamma$ for any value of the arguments

2. If either $\alpha_{\pm}$ is equal to a non-positive integer

To see this, we use the following identity and regularity of ${}_2F_1$ around $(1 - z) = 0$:

$$
\begin{aligned}
{}_2F_1(\alpha_-, \alpha_+, \gamma, z) &= \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha_+ - \alpha_-)}{\Gamma(\gamma - \alpha_-)\Gamma(\gamma - \alpha_+)} {}_2F_1(\alpha_-, \alpha_+, \alpha_- + \alpha_+ + 1 - \gamma, 1 - z) \\
&+ \frac{\Gamma(\gamma)\Gamma(\alpha_+ + \alpha_- - \gamma)}{\Gamma(\alpha_-)\Gamma(\alpha_+)} (1 - z)^{\gamma - \alpha_- - \alpha_+} {}_2F_1(\gamma - \alpha_-, \gamma - \alpha_+, 1 + \gamma - \alpha_- - \alpha_+, 1 - z) \qquad \text{(G.5)}
\end{aligned}
$$

$$
{}_2F_1(\alpha_-, \alpha_+, \gamma, z \sim 1) \sim \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha_+ - \alpha_-)}{\Gamma(\gamma - \alpha_-)\Gamma(\gamma - \alpha_+)} + \frac{\Gamma(\gamma)\Gamma(\alpha_+ + \alpha_- - \gamma)}{\Gamma(\alpha_-)\Gamma(\alpha_+)} (1 - z)^{\gamma - \alpha_- - \alpha_+} \qquad \text{(G.6)}
$$

We can check explicitly that $\alpha_+ + \alpha_- = \ell + d - 1 \geq \gamma$ for $d \geq 2$, where the superfluid groundstate is possible. Therefore option (1) is ruled out.

Define $\alpha_- = -n$ where n is a non-negative integer.

The relation above implies $\alpha_+ = (\ell + d - 1) - \alpha_- = \ell + d + n - 1$

Consider the explicit product:

$$
\alpha_+\alpha_- = \frac{1}{4}(\ell + d - 1 + 2\kappa)(\ell + d - 1 - 2\kappa) = \frac{d}{4}(\ell - \lambda^2) \qquad \text{(G.7)}
$$

Substituting the integer relations for $\alpha_{\pm}$ turns equation (G.7) into a quadratic equation which can be solved for λ as:

$$
\lambda^2 = \frac{1}{d}(4n^2 + 4dn + 4\ell n - 4n + d\ell) \qquad \text{(G.8)}
$$

which yields the dispersion (8.82)

G.2 Appendix B: Correlation Functions in Oscillator Frame

G.2.1 Two point function

In Galilean frame the two point function is given by

$$\left\langle O(t=-\mathbf{i}/\omega)O^{\dagger}(t=\mathbf{i}/\omega)\right\rangle = c\left(-\frac{2\mathbf{i}}{\omega}\right)^{-\Delta_O} \tag{G.9}$$

Now we know

$$\begin{aligned}\left\langle O(t=-\mathbf{i}/\omega)O^{\dagger}(t=\mathbf{i}/\omega)\right\rangle &= \lim_{\substack{t_0\to\mathbf{i}\omega\\ \tau_0\to-\mathbf{i}\infty}} \frac{1}{(1+\omega^2 t_0^2)^{\Delta_O}}\left\langle O(\tau=\tau_0)O^{\dagger}(\tau=-\tau_0)\right\rangle \\ &= c\lim_{\tau_0\to\mathbf{i}\infty}\frac{1}{(1+\omega^2 t_0^2)^{\Delta_O}}\left(\frac{1}{\sin^2(2\omega\tau_0)}\right)^{\Delta_O/2} \\ &= c(2\mathbf{i})^{\Delta_O}\lim_{\tau_0^{(E)}\to\infty}\frac{1}{(1+\omega^2 t_0^2)^{\Delta_O}}\exp\left(-2\omega\Delta_O\tau_0^{(E)}\right) \end{aligned} \tag{G.10}$$

where $\omega t_0 = \tan(\omega\tau_0)$. Comparing (G.9) and (G.10), we obtain an identity:

$$\lim_{\substack{t_0\to\mathbf{i}\omega\\ \tau_0^{(E)}\to\infty}} \frac{1}{(1+\omega^2 t_0^2)^{\Delta_O/2}}\exp\left(-\omega\Delta_O\tau_0^{(E)}\right) = 2^{-\Delta_O}\omega^{\Delta_O/2} \tag{G.11}$$

where we have $\omega t = \tan(\omega\tau)$ and $\tau^{(E)} = \mathbf{i}\tau$. We note that $t=\pm\frac{\mathbf{i}}{\omega}$ corresponds to Oscillator frame Euclidean time $\tau_E = \mp\infty$, this follows from

$$\omega t = \tan\left(-\mathbf{i}\omega\tau_E\right) \tag{G.12}$$

Thus the operators are inserted at infinitely past and future Euclidean time.

In the oscillator frame, we have

$$\langle O(\tau_1)O^{\dagger}(\tau_2)\rangle = c\left[1+\tan^2(\omega\tau_1)\right]^{\frac{\Delta_O}{2}}\left[1+\tan^2(\omega\tau_2)\right]^{\frac{\Delta_O}{2}}\left(\tan(\omega\tau_1)-\tan(\omega\tau_2)\right)^{-\Delta_O},$$

which can be simplied into

$$\langle O(\tau_1)O^{\dagger}(\tau_2)\rangle = c\left[\sin(\omega(\tau_1-\tau_2))\right]^{-\Delta_O}, \tag{G.13}$$

using the identity

$$\frac{[1+\tan^2(\omega\tau_1)][1+\tan^2(\omega\tau_2)]}{[\tan(\omega\tau_1)-\tan(\omega\tau_2)]^2}=\frac{1}{\sin^2(\omega(\tau_1-\tau_2))}\,. \tag{G.14}$$

G.2.2 Three point function

In the Galilean frame, the general form of a three-point function is fixed to be:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle \equiv G(x_1;x_2;x_3)=F(v_{123})\exp\left[-\mathbf{i}\frac{Q_1}{2}\frac{\boldsymbol{x}_{13}^2}{t_{13}}-\mathbf{i}\frac{Q_2}{2}\frac{\boldsymbol{x}_{23}^2}{t_{23}}\right]\prod_{i<j}t_{ij}^{\frac{\Delta}{2}-\Delta_i-\Delta_j} \tag{G.15}$$

where $\Delta\equiv\sum_i\Delta_i$, $x_{ij}\equiv x_i-x_j$, and $F(v_{ijk})$ is a function of the cross-ratio v_{ijk} defined:

$$v_{ijk}=\frac{1}{2}\left(\frac{\boldsymbol{x}_{jk}^2}{t_{jk}}-\frac{\boldsymbol{x}_{ik}^2}{t_{ik}}+\frac{\boldsymbol{x}_{ij}^2}{t_{ij}}\right) \tag{G.16}$$

The matrix element (8.86) defines a 3-point function in this frame via (8.16) and (8.13)

$$\begin{aligned}
\langle\Phi_{Q+q}|\phi_q(\tau,\boldsymbol{y})|\Phi_Q\rangle &= (1+\omega^2t^2)^{\frac{\Delta_\phi}{2}}\exp\left[\frac{\mathbf{i}}{2}q\frac{x^2\omega^2t}{1+\omega^2t^2}\right]G\left(-\frac{\mathbf{i}}{\omega},0\,;t,\boldsymbol{x};\ \frac{\mathbf{i}}{\omega},0\right)\\
&= F(v)(1+\omega^2t^2)^{\frac{\Delta_\phi}{2}}\exp\left[\frac{\mathbf{i}}{2}q\frac{x^2\omega^2t}{1+\omega^2t^2}\right]\exp\left[-\frac{iqx^2}{2\left(t-\frac{i}{\omega}\right)}\right]\prod_{i<j}t_{ij}^{\frac{\Delta}{2}-\Delta_i-\Delta_j}\\
&= F(v)\exp\left[\frac{q}{2}\frac{\omega x^2}{1+\omega^2t^2}\right](1+\omega^2t^2)^{\frac{\Delta_\phi}{2}}\prod_{i<j}t_{ij}^{\frac{\Delta}{2}-\Delta_i-\Delta_j}\\
&= F(v)\exp\left[\frac{q}{2}\frac{\omega x^2}{1+\omega^2t^2}\right](2)^{\frac{1}{2}(-\Delta_{Q+q}+\Delta_\phi-\Delta_Q)}(\mathbf{i}\omega)^{\frac{\Delta}{2}}\left(\frac{1-\mathbf{i}\omega t}{1+\mathbf{i}\omega t}\right)^{\frac{\Delta_Q-\Delta_{Q+q}}{2}}\\
&= F(v)\exp\left(\frac{q}{2}\omega y^2\right)(2)^{\Delta_\phi}\left(\frac{\mathbf{i}\omega}{2}\right)^{\frac{\Delta}{2}}e^{-i\omega(\Delta_Q-\Delta_{Q+q})\tau}
\end{aligned}$$

where

$$v=\frac{1}{2}\left(\frac{x^2}{t-\frac{\mathbf{i}}{\omega}}+\frac{x^2}{-\frac{\mathbf{i}}{\omega}-t}\right)=\frac{\mathbf{i}\omega x^2}{1+\omega^2t^2} \tag{G.17}$$

Appendix H

The Spinful Large Charge Sector of Non-Relativistic CFTs: From Phonons to Vortex Crystals

H.1 Particle-Vortex Duality

Here we briefly review the particle vortex duality in nonrelativistic set up. The aim of the appendix is to cast the vortex dynamics in terms of an electrostatic (in $d = 3$ the gauge field is 2 form field, hence we coin the term "gaugostatic") problem, leveraging the duality. The idea is to solve the gaugostatic problem to figure out the field strength, which in turn gives us the velocity profile of the vortex, again using the dictionary of duality.

We consider the leading order superfluid Lagrangian in the presence of a potential $A_0 = \frac{1}{2}\omega^2 r^2, A_i = 0$:

$$\mathcal{L} = c_0 X^{\frac{d+2}{2}} \equiv P(X) \qquad X \equiv \partial_0\chi - A_0 - \frac{1}{2}(\partial_i\chi)^2 \tag{H.1}$$

The number density and superfluid velocity are defined respectively as:

$$n = \frac{\partial L}{\partial\dot{\chi}} = c_0\left(\frac{d}{2}+1\right)X^{\frac{d}{2}} \qquad v_i = -\partial_i\chi \tag{H.2}$$

The action (H.1) has a $U(1)$ symmetry of $\chi \to \chi + c$ whose current can be written as:

$$j^\mu = \left(n, nv^i\right) \tag{H.3}$$

For simplicity and physical relevance, we'll focus on the cases of $d = 2$ and $d = 3$. In $d = 2$, we can define:

$$j^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu a_\rho = \frac{1}{2}\varepsilon^{\mu\nu\rho} f_{\nu\rho} \tag{H.4}$$

for a one-form gauge field a_μ and field strength $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. This relates the superfluid variables (H.2) to the dual electric and magnetic fields as:

$$n = \varepsilon^{ij}\partial_i a_j \equiv b \qquad v^i = \frac{\varepsilon^{ij} f_{0j}}{b} \equiv \frac{\varepsilon^{ij} e_j}{b} \tag{H.5}$$

Similarly, in $d = 3$ we'll define the current in terms of a dual two-form gauge field $B_{\mu\nu}$

$$j^\mu = \varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} = \frac{1}{3}\varepsilon^{\mu\nu\rho\sigma} G_{\nu\rho\sigma} \tag{H.6}$$

where $G_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the three-form field strength. The superfluid variables are then expressible as:

$$n = \varepsilon^{ijk}\partial_i B_{jk} \equiv Y \qquad v^i = \frac{1}{3}\frac{\varepsilon^{ijk} G_{0jk}}{Y} \tag{H.7}$$

Vortices act as sources for the gauge fields and couple minimally as:

$$d = 2: \quad J^V_\mu a^\mu \qquad\qquad d = 3: \quad \frac{1}{4} J^V_{\mu\nu} B^{\mu\nu} \tag{H.8}$$

To implement the duality transformation, we note that internal energy $\varepsilon(n)$ is given by $nX - P(X)$ and so we can rewrite the (H.1) as

$$\mathcal{L} = nX - \varepsilon(n) = n\left(\dot{\chi} - A_0 - \frac{1}{2}(\partial_i\chi)(\partial^i\chi)\right) - \varepsilon(n) \tag{H.9}$$

$$= \frac{1}{2}nv^2 - \varepsilon(n) + n\left(\dot{\chi} + v^i\partial_i\chi\right) \tag{H.10}$$

where we have used $v_i = -\partial_i\chi$ and n is understood as a function of χ and its derivatives.

The internal energy is given by:

$$d = 2: \quad \varepsilon(n) = \frac{1}{4c_0}n^2 \qquad d = 3: \quad \varepsilon(n) = \frac{3}{5}\left(\frac{2}{5c_0}\right)^{\frac{2}{3}} n^{\frac{5}{3}} \tag{H.11}$$

Using the relation (H.5) we can express the Lagrangian in $d = 2$ as:

$$\mathcal{L} = \frac{1}{2}\frac{e^2}{b} - \frac{1}{4c_0}b^2 - bA_0 \tag{H.12}$$

This equation describes a kind of non-linear electrodynamics with a modified Gauss law:

$$\partial_i\left(\frac{e^i}{b}\right) = J_0^V \tag{H.13}$$

Similarly the Lagrangian in $d = 3$ is given via (H.7) as:

$$\mathcal{L} = \frac{1}{9}\frac{G_{0ij}G_0{}^{ij}}{Y} - \frac{3}{5}\left(\frac{2}{5c_0}\right)^{\frac{2}{3}} Y^{\frac{5}{3}} - YA_0 \tag{H.14}$$

with a "Gauss law" of:

$$\partial^i\left(\frac{G_{0ij}}{Y}\right) = J_{j0}^V \tag{H.15}$$

Now consider a motion of charged particle under the gauge field, sourced by J^V . In what follows, we will show that to leading order we can treat this as a "gaugostatic" problem and the velocity V_i of the charged particle is negligible. If V_i is negligible, one can potentially drop the kinetic term in the Lagrangian. As a result, the equation of motion for the particle turns out to be the one where there is no Lorentz force acting on the particle. This implies that V_i is of the same order as $|e|/b$ (in $d = 3$, this is $\frac{\sqrt{G_{0ij}G_0{}^{ij}}}{Y}$). For self consistency, we need to ensure V_i is very small i.e. the ratio $|e|/b$ is very small. This helps us to render the problem of vortex dynamics into a problem of "gaugostatics". In order to do that, we linearize (H.12) and (H.14) around parametrically large magnetic field b and Y and we see that the coupling goes as b in $d = 2$ and in $d = 3$, this goes like Y. Hence the electric field strength $|e|$ in $d = 2$ and $\sqrt{G_{0ij}G_0{}^{ij}}$

in $d=3$ goes like $\sqrt{b}$ and $\sqrt{Y}$ respectively and we have

$$\sqrt{V_i V^i} \sim \frac{|e|}{b} \sim \frac{1}{\sqrt{Q}}, \text{in}\, d=2 \tag{H.16}$$

$$\sqrt{V_i V^i} \sim \frac{\sqrt{G_{0ij}G_0{}^{ij}}}{Y} \sim \frac{1}{\sqrt{Q}}, \text{in}\, d=3 \tag{H.17}$$

Thus it is self consistent to assume that the charged particle is just drifting without any Lorentz force acting on it.

H.2 A Contour integral

This appendix contains the evauation of contour integrals, needed to figure out the vortex interaction energy in the multivortex scenario. In $d=2$, the vortex interaction energy goes like

$$\int \mathrm{d}r\, rn(r) \int \mathrm{d}\theta\, \boldsymbol{v}_i \cdot \boldsymbol{v}_j \tag{H.18}$$

wherre as for $d=3$, we have an extra integral along the z axis and r becomes the radius in cylindrical coordinate. In both cases, the θ integral can be done using contour integral and expressing $\boldsymbol{v}_i$ in terms of complex variables given by

$$v_i = \frac{i}{\bar{z}-\bar{z}_i},\; v_i^* = \frac{-i}{z-z_i} \tag{H.19}$$

Hence the integral evaluates to

$$I = \int \mathrm{d}\theta\, \boldsymbol{v}_i \cdot \boldsymbol{v}_j = Re\left(\int \mathrm{d}z\, \frac{-i}{z} v_i v_j^*\right) \tag{H.20}$$

$$= Re\left(\int \mathrm{d}z\, \frac{-i}{z} \frac{1}{(\bar{z}-\bar{z}_i)(z-z_j)}\right) \tag{H.21}$$

Now we note that $z\bar{z}=r^2$ and $z_i\bar{z}_i = R_i^2$ to rewrite the integral in following manner:

$$I = Re\left(\int \mathrm{d}z\; -i\frac{z_i}{(r^2 z_i - R_i^2 z)(z-z_j)}\right) = Re\left(\int \mathrm{d}z\, \frac{-i}{-R_i^2} \frac{z_i}{\left(z-\frac{r^2}{R_i^2}z_i\right)(z-z_j)}\right) \tag{H.22}$$

The poles are located at $z = z_j, z = \frac{r^2}{R_i^2} z_i$ i.e. they lie on the circle of radius $|z_j| = R_j$ and $|\frac{r^2}{R_i^2} z_i| = \frac{r^2}{R_i}$.

Without loss of generality, we consider $R_i < R_j$. Now there can be three scenarios:

1. $r < R_i < R_j$ implies $\frac{r^2}{R_i} < r < R_j$, hence the pole at $z = \frac{r^2}{R_i^2} z_i$ is picked, answer is

$$I = Re\left(-2\pi \frac{1}{r^2 - z_j z_i^*}\right) = -\frac{2\pi}{r^4 + R_j^2 R_i^2 - 2r^2 R_i R_j \cos(\phi)} \left(r^2 - R_i R_j \cos(\phi)\right) \quad \text{(H.23)}$$

2. $R_i < R_j < r$ implies $R_j < r < \frac{r^2}{R_i}$, hence the pole at $z = z_j$ is picked. and the answer is

$$I = \frac{2\pi}{r^4 + R_j^2 R_i^2 - 2r^2 R_i R_j \cos(\phi)} \left(r^2 - R_i R_j \cos(\phi)\right) \quad \text{(H.24)}$$

3. $R_i < r < R_j$ implies $r < R_j$ and $r < \frac{r^2}{R_i}$, so none of the poles is picked, the answer is 0.

Summing up we can write

$$I = \frac{\pi \left(r^2 - R_i R_j \cos(\phi)\right)}{r^4 + R_j^2 R_i^2 - 2r^2 R_i R_j \cos(\phi)} \left[sgn(r - R_i) + sgn(r - R_j)\right] \quad \text{(H.25)}$$

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