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Quantizing a multi-pronged open string junction

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Covariant quantization of a multi-pronged open bosonic string junction is studied beyond static analysis. Its excited states are described by a set of ordinary bosons as well as some sets of twisted bosons on the worldsheet. The system is characterized by a certain large algebra of twisted type that includes a single Virasoro algebra as a subalgebra. By properly defining the physical states, one can show that there are no ghosts in the Hilbert space.

Subject Index B29, B39

1. Introduction

Since the 1990s, when D-branes and various string dualities were found, string junctions have been studied by many authors in the context of superstrings and M-theory.[1] These analyses mainly focused on the static properties such as BPS conditions or stability, with a few exceptions (see, e.g., Refs. [3,4].) String junctions are dynamical objects formed by dynamical strings, so that one can naturally ask their dynamical properties such as the spectra of their excited states and other quantum features beyond static properties.

Going back to the 1970s, some earlier works studying classical motions [5–7] and a simple-minded quantization [8] of string junctions appeared. In those days a string junction was considered as a model of the baryon, and they naively tried to quantize a three-string junction. In the analyses, they did not reach a physical spectrum, which is largely due to the non-closed property of the constraint algebra. In 1984, Ref. [9] analyzed the constraint structure of the same system more carefully, and got a deeper insight into classical solutions, but still the full quantum spectrum had been left undetermined.

In the present paper, the authors are going to revisit the problem. In particular, we propose a set of physical state conditions under which we can show there are no ghosts in the spectrum. In the following, we treat multi-pronged open bosonic string junctions in flat space-time. An f-pronged open string junction is an object consisting of f open string segments, whose one ends are tied together at a point and the other ends are free (see Fig. 1). We quantize such a system based on so-called “old covariant quantization” (OCQ).

[1]We only cite work [1,2] that explains the essential points since there are so many papers and we cannot list them all.

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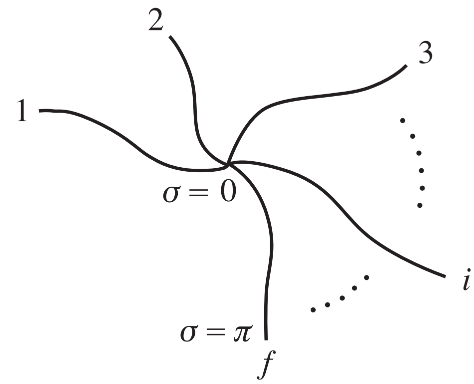

Fig. 1. f-pronged open string junction.

Our system is different from that of multi free strings in the sense that string segments are somehow interacting with each other through the connecting point. Therefore the constraint algebra governing this system is not just a multiple of Virasoro algebra, but rather an infinite-dimensional open algebra that includes a single Virasoro algebra as a subalgebra. Also, the state space is not just a multiple of single string Fock space. One needs to introduce some sets of twisted bosons in addition to a set of ordinary bosons on the worldsheet. Accordingly, the operator algebra of the constraints is of twisted type; namely it contains both periodic and anti-periodic parts, and we find many interesting and non-trivial features worth investigating in their own right.

The paper is organized as follows. In Sect. 2, we formulate an f-pronged open bosonic string junction starting from a Nambu–Goto-type action. We determine the mode expansion of the variables in the orthonormal gauge and study the structure of the primary constraints. In Sect. 3, we investigate the physical state conditions in detail. In Sect. 4, we concretely determine the physical spectrum and discuss its properties. There are many things to be clarified, some of which are discussed in the final section. Some useful algebraic relations are collected in Appendices A and B. Some details of the physical states are given in Appendix C. We make a remark on the light-cone gauge in Appendix D.

2. f-pronged open string junction

Let us consider the f-pronged open bosonic string junction shown in Fig. 1. We denote a coordinate variable of the ith string segment by $X^{(i)\mu}(\tau,\sigma)(i=1,2,\ldots,f)$, where $\mu(=0,1,\ldots,D-1)$ is a space-time index and $\tau,\sigma(\sigma\in[0,\pi])$ are the worldsheet parameters.[2] Each string is connected to the other strings at $\sigma=0$, i.e.,

$$X^{(i)\mu}(\tau,0)=X^{(j)\mu}(\tau,0)\qquad \text{for arbitrary } i \text{ and } j, \tag{1}$$

while $\sigma=\pi$ is a free end. We also use a notation $\xi^\alpha=(\tau,\sigma)(\alpha=0,1)$ for the worldsheet parameters. Then the Nambu–Goto-type action for the system is

$$S=-T\sum_{i=1}^{f}\int d\tau d\sigma\sqrt{-\det[\partial_\alpha X^{(i)}{}_\mu(\tau,\sigma)\,\partial_\beta X^{(i)\mu}(\tau,\sigma)]}, \tag{2}$$

where $T(=\frac{1}{2\pi\alpha'})$ is the string tension and the determinant is taken with respect to the indices α,β of the partial derivatives. In the following, we sometimes abbreviate a contraction of the

[2] We adopt $\eta^{\mu\nu}=\text{diag.}(-1,1,\ldots,1)$ for the target space metric.

space-time indices simply by a dot; e.g., $A \cdot B \equiv A_\mu B^\mu$. We also denote $\dot{X} \equiv \partial_\tau X$ and $X' \equiv \partial_\sigma X$ as a customary use. We can write the action in a more explicit way as

$$S = -T \sum_{i=1}^{f} \int d\tau d\sigma \sqrt{-(\dot{X}^{(i)} \cdot \dot{X}^{(i)})(X'^{(i)} \cdot X'^{(i)}) + (\dot{X}^{(i)} \cdot X'^{(i)})^2}. \tag{3}$$

The action has a reparametrization invariance under the (local) transformation

$$\delta X^{(i)\mu}(\tau,\sigma) = -\epsilon^{(i)\alpha}(\tau,\sigma)\, \partial_\alpha X^{(i)\mu}(\tau,\sigma), \tag{4}$$

as long as the infinitesimal transformation parameters $\epsilon^{(i)\alpha}(\tau,\sigma)$ satisfy

$$\epsilon^{(i)1}(\tau, 0) = 0 \qquad \text{and} \qquad \epsilon^{(i)1}(\tau, \pi) = 0. \tag{5}$$

Since the ith term in the action only depends on the ith variable $X^{(i)\mu}$, $\epsilon^{(i)\alpha}(\tau,\sigma)$ can be taken independently of each i except for the boundary condition

$$\epsilon^{(i)0}(\tau, 0) = \epsilon^{(j)0}(\tau, 0) \qquad \text{for arbitrary } i \text{ and } j, \tag{6}$$

which keeps the condition (1).

Taking a variation $\delta X^{(i)\mu}$ in the action, we obtain a set of equations of motion

$$\partial_\tau \left(\frac{\dot{X}^{(i)\mu}(X'^{(i)} \cdot X'^{(i)}) - X'^{(i)\mu}(\dot{X}^{(i)} \cdot X'^{(i)})}{\sqrt{-(\dot{X}^{(i)} \cdot \dot{X}^{(i)})(X'^{(i)} \cdot X'^{(i)}) + (\dot{X}^{(i)} \cdot X'^{(i)})^2}} \right) + \partial_\sigma \left(\frac{X'^{(i)\mu}(\dot{X}^{(i)} \cdot \dot{X}^{(i)}) - \dot{X}^{(i)\mu}(\dot{X}^{(i)} \cdot X'^{(i)})}{\sqrt{-(\dot{X}^{(i)} \cdot \dot{X}^{(i)})(X'^{(i)} \cdot X'^{(i)}) + (\dot{X}^{(i)} \cdot X'^{(i)})^2}} \right) = 0 \tag{7}$$

and boundary conditions

$$\left. \frac{X'^{(i)\mu}(\dot{X}^{(i)} \cdot \dot{X}^{(i)}) - \dot{X}^{(i)\mu}(\dot{X}^{(i)} \cdot X'^{(i)})}{\sqrt{-(\dot{X}^{(i)} \cdot \dot{X}^{(i)})(X'^{(i)} \cdot X'^{(i)}) + (\dot{X}^{(i)} \cdot X'^{(i)})^2}} \right|_{\sigma=\pi} = 0, \tag{8}$$

$$\sum_{i=1}^{f} \left. \frac{X'^{(i)\mu}(\dot{X}^{(i)} \cdot \dot{X}^{(i)}) - \dot{X}^{(i)\mu}(\dot{X}^{(i)} \cdot X'^{(i)})}{\sqrt{-(\dot{X}^{(i)} \cdot \dot{X}^{(i)})(X'^{(i)} \cdot X'^{(i)}) + (\dot{X}^{(i)} \cdot X'^{(i)})^2}} \right|_{\sigma=0} = 0 \tag{9}$$

as a stationary condition for the action.

The canonical conjugate momentum for each variable $X^{(i)\mu}$ is

$$P^{(i)}_\mu = T \frac{\dot{X}^{(i)}_\mu(X'^{(i)} \cdot X'^{(i)}) - X'^{(i)}_\mu(\dot{X}^{(i)} \cdot X'^{(i)})}{\sqrt{-(\dot{X}^{(i)} \cdot \dot{X}^{(i)})(X'^{(i)} \cdot X'^{(i)}) + (\dot{X}^{(i)} \cdot X'^{(i)})^2}}, \tag{10}$$

from which we obtain the primary constraints

$$P^{(i)}_\mu P^{(i)\mu} + T^2 X'^{(i)}_\mu X'^{(i)\mu} = 0, \tag{11}$$

$$P^{(i)}_\mu X'^{(i)\mu} = 0. \tag{12}$$

The Hamiltonian given by the Legendre transform of the Lagrangian in Eq. (3) vanishes. Therefore the total Hamiltonian consists only of primary constraints multiplied by arbitrary parameter functions $u^{(i)}_1(\tau,\sigma)$ and $u^{(i)}_2(\tau,\sigma)$:

$$H_T = \sum_{i=1}^{f} \int_0^\pi d\sigma \left[u^{(i)}_1 (P^{(i)2} + T^2 X'^{(i)2}) + u^{(i)}_2 P^{(i)} \cdot X'^{(i)} \right]. \tag{13}$$

The above constraints are all first-class and there are no more constraints coming from their time evolution.

Now let us take a gauge

$$u_1^{(i)} = \frac{1}{2T}, \quad u_2^{(i)} = 0, \tag{14}$$

which is equivalent to the so-called orthonormal gauge that imposes

$$\dot{X}^{(i)} \cdot \dot{X}^{(i)} + X'^{(i)} \cdot X'^{(i)} = 0, \tag{15}$$

$$\dot{X}^{(i)} \cdot X'^{(i)} = 0, \tag{16}$$

or alternatively

$$(\dot{X}^{(i)} \pm X'^{(i)})^2 = 0. \tag{17}$$

In this gauge, the equations of motion and the boundary conditions are largely simplified as follows:

$$\text{(Eq. of motion)} \qquad (\partial_\tau^2 - \partial_\sigma^2) X^{(i)\mu} = 0, \tag{18}$$

$$\text{(Boundary cond.)} \qquad \partial_\sigma X^{(i)\mu}|_{\sigma=\pi} = 0, \tag{19}$$

$$\sum_{i=1}^{f} \partial_\sigma X^{(i)\mu}|_{\sigma=0} = 0. \tag{20}$$

These equations combined with Eq. (1) are enough to determine the mode expansion of $X^{(i)\mu}$. Note that in this gauge the canonical momentum variable becomes simply $P^{(i)\mu} = T\dot{X}^{(i)\mu}$. In the following, after canonically quantizing the system, we will basically impose that physical states should satisfy the relation

$$\langle \text{phys}|(P^{(i)} \pm TX'^{(i)})^2|\text{phys}\rangle = 0. \tag{21}$$

Of course, in the quantized version, some central term can appear in the constraint algebra, so we have to be careful about treating the zero-mode part of the constraints at the operator level, which will be discussed later.

2.1 *Mode expansion*

A general solution $X^{(i)\mu}(\tau, \sigma)$ of the equations of motion (18) consists of left-moving and right-moving modes:

$$X^{(i)\mu}(\tau, \sigma) = X_L^{(i)\mu}(\tau + \sigma) + X_R^{(i)\mu}(\tau - \sigma). \tag{22}$$

The boundary conditions also restrict the form of the functions of each mode: Eq. (19) gives

$$\dot{X}_L^{(i)\mu}(\tau + \pi) - \dot{X}_R^{(i)\mu}(\tau - \pi) = 0, \tag{23}$$

while Eq. (20) gives

$$\sum_{i=1}^{f} \left(\dot{X}_L^{(i)\mu}(\tau) - \dot{X}_R^{(i)\mu}(\tau) \right) = 0. \tag{24}$$

Equation (1) leads to

$$X_L^{(1)\mu}(\tau) + X_R^{(1)\mu}(\tau) = \cdots = X_L^{(f)\mu}(\tau) + X_R^{(f)\mu}(\tau), \tag{25}$$

which tells us that each term $X_L^{(i)\mu}(\tau) + X_R^{(i)\mu}(\tau)$ is equal to a certain i-independent function $\phi^\mu(\tau)$. Thus,

$$X_R^{(i)\mu}(\tau) = -X_L^{(i)\mu}(\tau) + \phi^\mu(\tau). \tag{26}$$

Substituting this relation into Eq. (24), we have

$$2\sum_{i=1}^{f} \dot{X}_L^{(i)\mu}(\tau) = f\dot{\phi}^\mu(\tau), \tag{27}$$

which means

$$\phi^\mu(\tau) = \frac{2}{f}\sum_{i=1}^{f} X_L^{(i)\mu}(\tau) + c^\mu, \tag{28}$$

where c^μ is a constant. Then the right-mover is completely determined by the left-mover up to the constant:

$$X_R^{(i)\mu}(\tau) = \sum_{j=i}^{f} A_{ij} X_L^{(j)\mu}(\tau) + c^\mu \tag{29}$$

where $A_{ij} = \frac{2}{f} - \delta_{ij}$. Also the periodicity of the left-mover is obtained through Eq. (23) as

$$\dot{X}_L^{(i)\mu}(\tau + 2\pi) = \sum_{j=1}^{f} A_{ij}\dot{X}_L^{(j)\mu}(\tau) = \dot{X}_R^{(i)\mu}(\tau). \tag{30}$$

It is easy to obtain eigenvalues and eigenvectors of the $f \times f$ matrix $A = (A_{ij})$. Writing $A = \frac{2}{f}Z - I$ with the matrix Z whose every element is 1 ($Z_{ij} = 1$) and the unit matrix I, we can first solve the eigenvalue problem for Z. Apparently, a vector v_1 whose every element is $\frac{1}{\sqrt{f}}$ is a normalized eigenvector of Z with eigenvalue f, and $f - 1$ vectors v_a ($a = 2, \ldots, f$) orthogonal to v_1 and themselves (i.e., $v_1 \cdot v_a = 0$, $v_a \cdot v_b = \delta_{ab}$) are those with eigenvalue 0. Therefore, they are also the eigenvectors for A as

$$Av_1 = v_1, \qquad Av_a = -v_a \quad (a = 2, \ldots, f). \tag{31}$$

The eigenvectors v_i are determined by

$$v_1^T = \left(\frac{1}{\sqrt{f}} \cdots \frac{1}{\sqrt{f}}\right), \qquad v_1 \cdot v_a = 0, \quad v_a \cdot v_b = \delta_{ab}. \tag{32}$$

Note that one simple representation for v_a ($a = 2, \ldots, f$) can be chosen as follows:

$$(v_a)_i = \begin{cases} \frac{1}{\sqrt{a(a-1)}} & (i = 1, \ldots, a-1) \\ -\sqrt{\frac{a-1}{a}} & (i = a) \\ 0 & (i = a+1, \ldots, f). \end{cases} \tag{33}$$

By defining a real orthogonal matrix U whose ith row is just v_i ($i = 1, \ldots, f$), i.e., $U_{ij} = (v_i)_j$, diagonalization of A is expressed as

$$UAU^T = \Gamma \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}. \tag{34}$$

Now let us define

$$Y^{i\mu}(\tau) \equiv \sum_{j=1}^{f} U_{ij} X_L^{(j)\mu}(\tau). \tag{35}$$

Conversely,

$$X_L^{(i)\mu}(\tau) = \sum_{j=1}^{f} U_{ji} Y^{j\mu}(\tau). \tag{36}$$

Then from Eq. (30),

$$\dot{Y}^{i\mu}(\tau + 2\pi) = \begin{cases} \dot{Y}^{1\mu}(\tau) & (i = 1) \\ -\dot{Y}^{i\mu}(\tau) & (i \neq 1). \end{cases} \tag{37}$$

If we also define for the right-mover

$$\tilde{Y}^{i\mu}(\tau) \equiv \sum_{j=1}^{f} U_{ij} X_R^{(j)\mu}(\tau), \tag{38}$$

then

$$\tilde{Y}^{i\mu}(\tau) = \begin{cases} Y^{1\mu}(\tau) + \tilde{c}^{\mu} & (i = 1) \\ -Y^{i\mu}(\tau) & (i \neq 1) \end{cases} \tag{39}$$

where

$$\tilde{c}^{\mu} = c^{\mu} \sum_{i=1}^{f} U_{1i} = c^{\mu}\sqrt{f}. \tag{40}$$

Now we see from Eq. (37) that $Y^{1\mu}(\tau)$ is periodic (up to constant) with periodicity 2π whereas the $Y^{a\mu}(\tau)$ ($a = 2, \ldots, f$) are anti-periodic. Therefore their mode expansions become

$$Y^{1\mu}(\tau) = \frac{1}{\sqrt{2}}\left[q^{\mu} + p^{\mu}\tau + i\sum_{n\neq 0} \frac{\alpha_n^{\mu}}{n} e^{-in\tau}\right], \tag{41}$$

$$Y^{a\mu}(\tau) = \frac{i}{\sqrt{2}} \sum_{r\in \mathbf{Z}+\frac{1}{2}} \frac{\alpha_r^{a\mu}}{r} e^{-ir\tau} \qquad (a = 2, \ldots, f). \tag{42}$$

Bringing them all back to the original variables, we obtain (also making q^{μ} absorb a constant ambiguity c^{μ})

$$X^{(i)\mu}(\tau,\sigma) = \sqrt{\frac{2}{f}}\left[q^{\mu} + p^{\mu}\tau + i\sum_{n\neq 0} \frac{\alpha_n^{\mu}}{n} e^{-in\tau}\cos(n\sigma)\right] + \sqrt{2}\sum_{a=2}^{f}(v_a)_i \sum_{r\in \mathbf{Z}+\frac{1}{2}} \frac{\alpha_r^{a\mu}}{r} e^{-ir\tau}\sin(r\sigma). \tag{43}$$

One can easily check that this expression satisfies the equations of motion (18) and all the boundary conditions (1), (19), and (20) by noting the property $\sum_{i=1}^{f}(v_a)_i = 0$, which comes from $v_a \cdot v_1 = 0$.

To quantize the system, we will set a canonical equal-time commutation relation in the interval $\sigma, \sigma' \in [0, \pi]$ with the canonical conjugate momentum $P^{(i)\mu}(\tau,\sigma) = \frac{1}{2\pi}\dot{X}^{(i)\mu}(\tau,\sigma)$:

$$\left[X^{(i)\mu}(\tau,\sigma), P^{(j)\nu}(\tau,\sigma')\right] = i\eta^{\mu\nu}\delta_{ij}\delta(\sigma - \sigma'). \tag{44}$$

Here (and hereafter) we have taken $T = \frac{1}{2\pi}$ ($\alpha' = 1$) for brevity. This relation leads to

$$\left[\alpha_n^\mu, \alpha_m^\nu\right] = n\delta_{n+m,0}\eta^{\mu\nu}, \qquad \left[\alpha_r^{a\mu}, \alpha_s^{b\nu}\right] = r\delta_{r+s,0}\delta^{ab}\eta^{\mu\nu}, \qquad [q^\mu, p^\nu] = i\eta^{\mu\nu}, \tag{45}$$

where $n, m \in \mathbf{Z}$, $r, s \in \mathbf{Z} + \frac{1}{2}$ and $\alpha_0^\mu \equiv p^\mu$. Thus our total Fock space is described by D ordinary bosons and $(f-1)D$ twisted bosons.

2.2 *Constraints*

Let us turn to the constraints (11) and (12). They can be combined into

$$(2\pi P^{(i)} \pm X'^{(i)})^2 \approx 0, \tag{46}$$

where the wavy equal is used in the sense of Eq. (21). Each of these can be written in terms of the left- and right-movers:

$$\left(\dot{X}_L^{(i)}(\tau)\right)^2 \approx 0, \qquad \left(\dot{X}_R^{(i)}(\tau)\right)^2 \approx 0. \tag{47}$$

Furthermore, in terms of the mode-diagonal variables $Y^{i\mu}$, they become

$$\sum_{j,k=1}^{f} K^i_{jk}\, \dot{Y}^j(\tau)\cdot\dot{Y}^k(\tau) \approx 0, \qquad \sum_{j,k=1}^{f} \tilde{K}^i_{jk}\, \dot{Y}^j(\tau)\cdot\dot{Y}^k(\tau) \approx 0, \tag{48}$$

$$K^i_{jk} \equiv U_{ji}U_{ki}, \qquad \tilde{K}^i \equiv \Gamma K^i \Gamma. \tag{49}$$

Here we have used the matrix notation $(K^i)_{jk} \equiv K^i_{jk}$ and Γ was defined in Eq. (34). The constraints defined by K^i and those of $\tilde{K}^i$ are interchanged when τ goes to $\tau + 2\pi$ since $\dot{Y}^{i\mu}(\tau + 2\pi) = (\Gamma\dot{Y})^{i\mu}(\tau)$. If we recombine them into those of $K^i + \tilde{K}^i$ and $K^i - \tilde{K}^i$, then each set of constraints has definite periodicity. So one may think that there are f periodic and f anti-periodic independent constraints. The situation, however, is not so simple. To see this, let us look at the structure of K^i.

We first define the following three types of $f \times f$ symmetric matrices:

$$P = \left(\begin{array}{c|c} \frac{1}{f} & \\ \hline & \end{array}\right), \qquad Q^{ij} = \left(\begin{array}{c|c} & \\ \hline & \frac{1}{2}(v_a)_i(v_b)_j + \frac{1}{2}(v_a)_j(v_b)_i \end{array}\right), \tag{50}$$

$$R^i = \left(\begin{array}{c|c} & \frac{1}{\sqrt{f}}(v_b)_i \\ \hline \frac{1}{\sqrt{f}}(v_a)_i & \end{array}\right). \tag{51}$$

In other words, $(P)_{11} = \frac{1}{f}$, $(Q^{ij})_{ab} = \frac{1}{2}(v_a)_i(v_b)_j + \frac{1}{2}(v_a)_j(v_b)_i$, $(R^i)_{1a} = (R^i)_{a1} = \frac{1}{\sqrt{f}}(v_a)_i$, and all other elements are vanishing. Then K^i and $\tilde{K}^i$ can be decomposed as

$$K^i = P + Q^{ii} + R^i, \qquad \tilde{K}^i = P + Q^{ii} - R^i. \tag{52}$$

One can see that the matrices for the anti-periodic constraints $K^i - \tilde{K}^i = 2R^i$ are not totally independent because of the relation $\sum_{i=1}^f R^i = 0$ as a result of $\sum_{i=1}^f (v_a)_i = 0$. We may choose, e.g., R^a as independent ones. Thus, we have f periodic and $f-1$ anti-periodic constraints:

$$\sum_{j,k=1}^{f} (P + Q^{ii})_{jk}\, \dot{Y}^j(\tau)\cdot\dot{Y}^k(\tau) \approx 0, \qquad \sum_{j,k=1}^{f} R^a_{jk}\, \dot{Y}^j(\tau)\cdot\dot{Y}^k(\tau) \approx 0. \tag{53}$$

In order to investigate the constraints in more detail, we introduce the so-called Fubini–Veneziano fields $\varphi^{i\mu}(z)$ with a complex variable $z(=e^{i\tau})$:

$$\varphi^{1\mu}(z) \equiv \sqrt{2}\, Y^{1\mu}(-i\ln z) = q^\mu - ip^\mu \ln z + i\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}z^{-n}, \tag{54}$$

$$\varphi^{a\mu}(z) \equiv \sqrt{2}\, Y^{a\mu}(-i\ln z) = i\sum_{r\in\mathbf{Z}+\frac{1}{2}}\frac{\alpha_r^{a\mu}}{r}z^{-r}. \qquad (a=2,\dots,f). \tag{55}$$

They satisfy the periodicity

$$\varphi^{1\mu}(e^{2\pi i}z) = \varphi^{1\mu}(z) + 2\pi p^\mu, \qquad \varphi^{a\mu}(e^{2\pi i}z) = -\varphi^{a\mu}(z). \tag{56}$$

It is convenient to define $A^{i\mu}(z) \equiv i\partial_z\varphi^{i\mu}(z)$, i.e.,

$$A^{1\mu}(z) = \sum_{n\in\mathbf{Z}}\alpha_n^\mu\, z^{-n-1}, \qquad A^{a\mu}(z) = \sum_{r\in\mathbf{Z}+\frac{1}{2}}\alpha_r^{a\mu}\, z^{-r-1}. \tag{57}$$

$A^{1\mu}(z)$ is periodic whereas the $A^{a\mu}(z)$ are anti-periodic under the replacement $z \to e^{2\pi i}z$. If we define the operator $T_M(z)$ for a given matrix M as[3]

$$T_M(z) \equiv \frac{1}{2}\sum_{i,j=1}^{f} M_{ij} : A^i(z)\cdot A^j(z) : + \frac{D}{16z^2}\mathrm{tr}(MP_-), \tag{58}$$

where $:\mathcal{O}:$ is a normal order of $\mathcal{O}$ with respect to the oscillators α_n^μ, $\alpha_r^{a\mu}$ (and $P_- = \frac{1}{2}(1-\Gamma)$), then the operators corresponding to the primary constraints (53) are given by $T_{P+Q^{ii}}(z)$ and $T_{R^a}(z)$. These operators satisfy the relations

$$T_{P+Q^{ii}}(e^{2\pi i}z) = T_{P+Q^{ii}}(z), \qquad T_{R^a}(e^{2\pi i}z) = -T_{R^a}(z), \tag{59}$$

and they have a formal Laurent expansion with an integer power of z and a half odd integer power respectively:

$$T_{P+Q^{ii}}(z) = \sum_{n\in\mathbf{Z}} L_n^{P+Q^{ii}} z^{-n-2}, \tag{60}$$

$$T_{R^a}(z) = \sum_{r\in\mathbf{Z}+\frac{1}{2}} L_r^{R^a} z^{-r-2}. \tag{61}$$

In terms of the oscillators, each mode operator is written as follows:

$$L_n^{P+Q^{ii}} = \frac{1}{2f}\sum_{m\in\mathbf{Z}} :\alpha_{n-m}\cdot\alpha_m: + \frac{1}{2}\sum_{a,b=2}^{f}(v_a)_i(v_b)_i\sum_{s\in\mathbf{Z}+\frac{1}{2}} :\alpha_{n-s}^a\cdot\alpha_s^b: + \frac{D}{16}\frac{f-1}{f}\delta_{n,0}, \tag{62}$$

$$L_r^{R^a} = \frac{1}{\sqrt{f}}\sum_{b=2}^{f}(v_b)_a\sum_{m\in\mathbf{Z}} :\alpha_{r-m}^b\cdot\alpha_m: . \tag{63}$$

Among the above operators, one special combination is

$$V_n \equiv \sum_i L_n^{P+Q^{ii}} \tag{64}$$

$$= \frac{1}{2}\sum_{m\in\mathbf{Z}} :\alpha_{n-m}\cdot\alpha_m: + \frac{1}{2}\sum_{a=2}^{f}\sum_{s\in\mathbf{Z}+\frac{1}{2}} :\alpha_{n-s}^a\cdot\alpha_s^a: + \frac{D}{16}(f-1)\delta_{n,0}. \tag{65}$$

[3]Note that the term $\frac{D}{16z^2}\mathrm{tr}(MP_-)$ in the definition of $T_M(z)$ is added in order that the OPE relations can be written in a unified manner (see Appendix A).

They satisfy the Virasoro algebra with a central charge Df:

$$[V_n,\ V_m] = (n-m)V_{n+m} + \frac{Df}{12}n(n^2-1)\delta_{n+m,0}. \tag{66}$$

Thus we have a single Virasoro algebra as a closed subalgebra of our constraints.

3. Physical state condition for an open string junction

3.1 *Preliminary discussion*

For the physical state condition, it is possible to impose that the positive modes of the constraint operators should annihilate physical states just as in the case of the old covariant quantization for a string:

$$L_n^{P+Q^{ii}}|\mathrm{phys}\rangle = 0, \qquad (n>0, \quad i=1,\ldots,f) \tag{67}$$

$$L_r^{R^a}|\mathrm{phys}\rangle = 0. \qquad (r>0, \quad a=2,\ldots,f) \tag{68}$$

Note that the algebras of $L_n^{Q^{i-1\,i-1}-Q^{ii}}$ and $L_r^{R^a}$ are not closed among themselves, as will be seen in Appendix B. However, if $A|\mathrm{phys}\rangle = 0$ and $B|\mathrm{phys}\rangle = 0$, then $[A, B]|\mathrm{phys}\rangle = 0$, so that requiring the above conditions will be sufficient. On the other hand, those of the zero-mode operators $L_0^{Q^{ii}}$ must be chosen more carefully. It is not straightforward to choose appropriate conditions for general f since the set of $f-1$ operators $L_0^{Q^{i-1\,i-1}-Q^{ii}}$ is not closed. We first study the special case of $f=2$, and go into the general discussion afterward.

3.2 *f=2 case*

We consider the $f=2$ case where the physical object is not a junction but an open string. In fact, in this case, there are only one positive integer and half-integer mode oscillators α_n^μ and $\alpha_r^{a=2,\mu}$, and the physical state conditions (67) and (68) are reduced to the following two simple ones:

$$V_n|\mathrm{phys}\rangle = 0, \qquad (n>0) \tag{69}$$

$$L_r^{R^{a=2}}|\mathrm{phys}\rangle = \left(\frac{1}{2}\sum_{m\in\mathbf{Z}} :\alpha_{r-m}^{a=2}\cdot\alpha_m:\right)|\mathrm{phys}\rangle = 0 \qquad (r>0) \tag{70}$$

since $P+Q^{11} = P+Q^{22}$. Also, there is only one zero-mode operator V_0, which counts the level of the state (plus $\frac{1}{2}p^2$ and a constant) as usual for strings, and the zero-mode condition should be taken as $V_0|\mathrm{phys}\rangle = (a_0 + \frac{D}{16})|\mathrm{phys}\rangle$ with a normal-order constant or an intercept parameter a_0. If we replace $\alpha_n^\mu \to \frac{1}{\sqrt{2}}\tilde{\alpha}_{2n}^\mu$ and $\alpha_r^{a=2,\mu} \to \frac{1}{\sqrt{2}}\tilde{\alpha}_{2r}^\mu$, they satisfy $[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\delta_{n+m,0}\eta^{\mu\nu}$. Writing $\tilde{L}_{2n} = 2(V_n - \frac{D}{16}\delta_{n,0})$ and $\tilde{L}_{2n+1} = 2L_{n+\frac{1}{2}}^{R^{a=1}} = -2L_{n+\frac{1}{2}}^{R^{a=2}}$, we can easily check that the $\tilde{L}_n$ satisfy the Virasoro algebra with a central charge D, and the physical state condition can be collected as

$$\left(\tilde{L}_n - \tilde{a}\delta_{n,0}\right)|\mathrm{phys}\rangle = 0 \qquad (n\geq 0) \tag{71}$$

where $\tilde{a} = 2a_0$ and

$$\tilde{L}_n = \frac{1}{2}\sum_{m\in\mathbf{Z}} :\tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m: . \tag{72}$$

This is exactly the same condition as for the old covariant quantization of open string theory. As is well known, the constant $\tilde{a}$ should be taken as $\tilde{a}\leq 1$ and the dimension as $D\leq 26$ in order to ensure that there is no ghost (negative norm) state in the physical spectrum (see, e.g., Ref.

[10, 11]). From various discussions like the one-loop level unitarity or the modern BRS quantization, we know that the consistent choice should be $\tilde{a} = 1$ and $D = 26$.

3.3 *Zero-mode condition*

Now we deal with the general $f \geq 3$ case and discuss the physical spectrum and its properties.

We have already proposed a reasonable condition for positive modes of the constraint operators as Eqs. (67) and (68). Here we consider the remaining discussion on zero modes of the constraint operators given by

$$L_0^{P+Q^{ii}} = \frac{1}{2f}\sum_{m\in\mathbf{Z}} :\alpha_{-m}\cdot\alpha_m: + \frac{1}{2}\sum_{a,b=2}^{f}(v_a)_i(v_b)_i \sum_{s\in\mathbf{Z}+\frac{1}{2}} :\alpha^a_{-s}\cdot\alpha^b_s: + \frac{D}{16}\frac{f-1}{f}. \tag{73}$$

These f operators are all independent and the commutation relations are

$$\begin{aligned}\left[L_0^{P+Q^{ii}}, L_0^{P+Q^{jj}}\right] &= [L_0^{Q^{ii}}, L_0^{Q^{jj}}] \\ &= -\frac{1}{f}\sum_{a,b=2}^{f}(v_a)_i(v_b)_j \sum_{r\in\mathbf{Z}+\frac{1}{2}} r :\alpha^a_{-r}\cdot\alpha^b_r:,\end{aligned} \tag{74}$$

which shows that the algebra is not closed within the f operators $L_0^{P+Q^{ii}}$. Thus we cannot impose a condition such that a state $|\phi\rangle$ should be a vector in the space of some finite representation of $L_0^{P+Q^{ii}}$. In fact, in order to obtain a closed algebra including all f operators $L_0^{P+Q^{ii}}$, we have to involve an infinite number of additional operators. On the other hand, the operator $V_0 = \sum_{i=1}^{f} L_0^{P+Q^{ii}}$ in itself has a desirable property

$$V_0|\phi\rangle = \left(N_{\text{level}} + \frac{1}{2}p^2 + \frac{D(f-1)}{16}\right)|\phi\rangle \tag{75}$$

where $|\phi\rangle$ is a state whose level and the square of the momentum are N_{level} and p^2 respectively. (Note that we take $\alpha' = 1$.) We see that one reasonable physical state condition involving the zero-mode operators is

$$V_0|\text{phys}\rangle = \left(a_0 + \frac{D(f-1)}{16}\right)|\text{phys}\rangle \tag{76}$$

with some constant a_0. In addition, for other zero-mode operators, it is at least possible to impose the condition that $L_0^{P+Q^{ii}}|\phi\rangle$ is always physical if $|\phi\rangle$ is physical.

3.4 *Physical state condition*

From the above discussion, the most reasonable physical state condition for the present system is to impose the following set of conditions (I), (II), and (III) for arbitrary non-negative integer N, positive integer n, and positive half-integer r:

$$\text{(I)}\ L_n^{P+Q^{ii}}\left(\prod_{k=j_1,\ldots,j_N} L_0^{P+Q^{kk}}\right)|\text{phys}\rangle = 0, \qquad (n>0,\quad i,\ j_\alpha = 1,\ldots,f) \tag{77}$$

$$\text{(II)}\ L_r^{R^a}\left(\prod_{k=j_1,\ldots,j_N} L_0^{P+Q^{kk}}\right)|\text{phys}\rangle = 0, \quad (r>0,\quad a=2,\ldots,f,\quad j_\alpha=1,\ldots,f) \tag{78}$$

$$\text{(III)}\ V_0|\text{phys}\rangle = \left(a_0 + \frac{D(f-1)}{16}\right)|\text{phys}\rangle \quad \left(\Leftrightarrow \left(N_{\text{level}} + \frac{1}{2}p^2 - a_0\right)|\text{phys}\rangle = 0\right). \tag{79}$$

In the above equations, $\prod_{k=j_1,\cdots,j_N} L_0^{P+Q^{kk}}$ is meant to be the product of N $L_0^{P+Q^{kk}}$:

$$\prod_{k=j_1,\cdots,j_N} L_0^{P+Q^{kk}} \equiv L_0^{P+Q^{j_1j_1}} L_0^{P+Q^{j_2j_2}} \cdots L_0^{P+Q^{j_Nj_N}} \tag{80}$$

for positive N and arbitrary j_α ($\alpha = 1, \ldots, N$), and $\prod_{k=j_1,\cdots,j_N} L_0^{P+Q^{kk}} \equiv 1$ for $N = 0$. In what follows, assuming that the physical state condition is indeed given by the set of these conditions, we will identify the space of physical states.

First, we examine the properties of each of the above three conditions in detail. The simplest condition (III) determines the relation between the level and the mass of the states. This restriction does not affect the other two conditions since

$$\left[V_0, L_m^{P+Q^{ii}}\right] = -mL_m^{P+Q^{ii}}, \qquad \left[V_0, L_r^{R^a}\right] = -rL_r^{R^a} \tag{81}$$

hold for any integer m or half-integer r. Thus we can continue the discussion within the space of fixed level (and mass) states separately. Now we consider the conditions (I) and (II) for $N = 0$, i.e., the conditions $L_n^{P+Q^{ii}}|\text{phys}\rangle = 0$ and $L_r^{R^a}|\text{phys}\rangle = 0$. In this case, from the commutation relations (B7), (B8), and (B9), we can prove that if we assume that condition (I) of $n = 1, 2$ and condition (II) of $r = \frac{1}{2}, \frac{3}{2}$ for $N = 0$ are satisfied, all the other conditions (I) and (II) for $N = 0$ are satisfied. In the process of this discussion, we can also show that a state $|\phi_0\rangle$ satisfying conditions (I) and (II) for $N = 0$ always satisfies $L_n^P|\phi_0\rangle = L_n^{Q^{ij}}|\phi_0\rangle = 0$ for $n \geq 2$. Next, we impose conditions (I) and (II) for $N = 1$ on a state $|\phi_0\rangle$ satisfying the conditions for $N = 0$. From the relations (B11) and (B12), the additional conditions that we have to impose on $|\phi_0\rangle$ are

$$L_r^{R^a} L_0^P|\phi_0\rangle = 0, \qquad L_n^P|\phi_0\rangle = 0, \qquad L_n^{Q^{ii}} L_0^{Q^{jj}}|\phi_0\rangle = 0. \quad (r, n > 0) \tag{82}$$

In fact, it is sufficient to impose the conditions for $r = 1/2$ and $n = 1, 2$ since the other equations result from them. By continuing a similar discussion, we conclude that conditions (I) and (II) are rewritten by the following simpler set of conditions for any $i, j, j_\alpha \in \{1, 2, \ldots, f\}$ and $a \in \{2, \ldots, f\}$:

$$L_n^P|\phi\rangle = L_n^{Q^{ij}}|\phi\rangle = 0, \qquad (n > 0) \tag{83}$$

$$L_r^{R^a}\left(L_0^P\right)^k|\phi\rangle = 0, \qquad (r > 0, \quad k = 0, 1, 2, \ldots) \tag{84}$$

$$L_n^{Q^{ii}}\left(L_0^{Q^{j_1j_1}} L_0^{Q^{j_2j_2}} \cdots L_0^{Q^{j_Nj_N}}\right)|\phi\rangle = 0. \qquad (n > 0, \ N = 0, 1, 2, \ldots) \tag{85}$$

In particular, since $L_0^P = \frac{1}{2f}p^2 + \frac{1}{f}N_{\text{level}}^{(\text{e})}$ where $N_{\text{level}}^{(\text{e})}(\equiv \frac{1}{2}\sum_{m\in\mathbf{Z}} : \alpha_{-m}\cdot\alpha_m :)$ counts the level for integer mode oscillators, the condition (84) must be satisfied independently for terms with different $N_{\text{level}}^{(\text{e})}$ in $|\phi\rangle$. Thus, by noting that Eq. (84) is equivalent to

$$L_r^{G^{(a)}}\left(L_0^P\right)^k|\phi\rangle = 0$$

where $L_r^{G^{(a)}} = \frac{1}{2}\sum_r : \alpha_s^a\cdot\alpha_{r-s} :$ as given in Eq. (A17), we see that Eq. (84) is reduced to the set of conditions

$$L_r^{G^{(a)},s}|\phi\rangle = 0 \qquad \left(r > 0, \ s \in \mathbf{Z} + \frac{1}{2}, \ a = 2, \ldots, f\right). \tag{86}$$

From the relations (B13), (B18), (B19), and (B21), we can show that any state satisfying Eqs. (83) and (86) also satisfies the remaining condition (85).

Thus, the physical state conditions (I) and (II) are reduced to the simpler conditions (83) and (86), which we name (I′), (I″), and (II′) respectively:

$$\begin{array}{lll} (\mathrm{I}') & L_n^P|\phi\rangle = 0, & (n>0) \\ (\mathrm{I}'') & L_n^{Q^{ij}}|\phi\rangle = 0, & (n>0;\ i,j=1,\dots,f) \\ (\mathrm{II}') & L_r^{G^{(a)},s}|\phi\rangle = 0. & (r>0,\ s\in\mathbf{Z}+\tfrac{1}{2},\ a=2,\dots,f) \end{array}$$

In fact, it is sufficient to impose (I′) for $n = 1, 2$, (I″) for $n = 1$, (II′) for $r = \frac{1}{2}$ since the other conditions can be derived from them. Note that the conditions (I′) and (II′) can be respectively represented by $L_n^E|\phi\rangle = 0$ and $L_n^{F^{(ab)}}|\phi\rangle = 0\,(a, b = 2, \cdots, f)$. Here L_n^E and $L_n^{F^{(ab)}}$ are given in Eqs. (A15) and (A16) respectively.

4. The physical spectrum and its properties

We will explicitly solve the physical state condition given by (I′), (I″), (II′), and (III) and study the properties of the physical spectrum. We represent a level N_{level} state as $|\phi\rangle_{N_{\text{level}}}$ and the vacuum state that is annihilated by all the positive frequency oscillators with momentum p as $|0, p\rangle$.

4.1 *First three levels:* $N_{\text{level}} = 0,\ \frac{1}{2},\ 1$

We first identify the explicit form of the physical states for $N_{\text{level}} = 0, \frac{1}{2}, 1$ and find a restriction on a constant a_0.

For $N_{\text{level}} = 0$, there is only one state $|\phi\rangle_0 = |0, p\rangle$ and p^2 is determined by the condition (III) as $\frac{1}{2}p^2 = a_0$.

For $N_{\text{level}} = \frac{1}{2}$, a general state satisfying the mass-shell condition (III) is given by

$$|\phi\rangle_{\frac{1}{2}} = \sum_{a=2}^{f} f_\mu^a \alpha_{-\frac{1}{2}}^{a,\mu}|0, p\rangle \tag{87}$$

with $\frac{1}{2}p^2 = a_0 - \frac{1}{2}$. Only the non-trivial condition for this state is given by (II′) with $r = \frac{1}{2}$ and $s = \frac{1}{2}$. This restricts the $f - 1$ coefficient vectors f_μ^a to satisfy

$$p\cdot f^a = 0. \tag{88}$$

We see that the norm of the physical state $|f^a, p\rangle_{\frac{1}{2}} = f_\mu^a\alpha_{-\frac{1}{2}}^{a,\mu}|0, p\rangle$ is

$${}_{\frac{1}{2}}\langle f^a, p|f^a, p'\rangle_{\frac{1}{2}} = \frac{1}{2}f^a\cdot f^a\,\delta^D(p-p') \tag{89}$$

where $\langle 0, p| = (|0, p\rangle)^\dagger$ with $(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$, $(\alpha_r^{a,\mu})^\dagger = \alpha_{-r}^{a,\mu}$ and $\langle 0, p'|0, p\rangle = \delta^D(p - p')$. In this case, if we take $a_0 (= \frac{1}{2}p^2 + \frac{1}{2}) > \frac{1}{2}$, we can choose $p_0 = 0$ since $p^2 > 0$. Then, from the same discussion as in the case of the old covariant quantization of open string theory, f^a can be taken as a time-like vector and the corresponding state $|f^a, p\rangle_{\frac{1}{2}}$ has negative norm. On the other hand, if we choose $a_0 \le \frac{1}{2}$, the physical states always have non-negative norm since $p^2 \le 0$. In particular, for $a_0 = \frac{1}{2}$, there are $f - 1$ zero-norm physical states $p_\mu\alpha_{-\frac{1}{2}}^{a,\mu}|0, p\rangle$ and $(f - 1)(D - 2)$ transverse positive norm physical states. Thus, in order to ensure the no-ghost theorem for this system, we at least have to choose

$$a_0 \le \frac{1}{2}, \tag{90}$$

which leads to $-\frac{1}{2}p^2\,(= \frac{1}{2}m^2) \ge N_{\text{level}} - \frac{1}{2}$. If we assume this condition, all the states with a level higher than $\frac{1}{2}$ have a real mass ($m^2 > 0$). We later show that this assumption is plausible from the discussion of the ζ-function regularization calculation of the normal-ordering constant.

For $N_{\text{level}} = 1$, general states are given by

$$|\phi\rangle_1 = \left(g_\mu \alpha_{-1}^{\mu} + \sum_{a,b} h_{\mu\nu}^{ab} \alpha_{-\frac{1}{2}}^{a,\mu} \alpha_{-\frac{1}{2}}^{b,\nu} \right) |0, p\rangle \tag{91}$$

where $h_{\mu\nu}^{ab} = h_{\nu\mu}^{ba}$ and the mass-shell condition is $\frac{1}{2}p^2 = a_0 - 1$. After imposing the set of non-trivial conditions for this level, i.e., (II′) with $r = \frac{1}{2}$ and $s = \pm\frac{1}{2}$ and (I″) with $n = 1$, we obtain the general form of the physical states as

$$|\phi_{\text{phys}}, p\rangle_1 = \sum_{a,b} h_{\mu\nu}^{ab} \alpha_{-\frac{1}{2}}^{a,\mu} \alpha_{-\frac{1}{2}}^{b,\nu} |0, p\rangle \tag{92}$$

where

$$\eta^{\mu\nu} h_{\mu\nu}^{ab} = 0, \quad p^\mu h_{\mu\nu}^{ab} = 0, \quad \frac{1}{2}p^2 = a_0 - 1. \tag{93}$$

We see that the norm of this state is calculated as

$${}_1\langle \phi_{\text{phys}}, p | \phi_{\text{phys}}, p' \rangle_1 = \frac{1}{2} \sum_{a,b} h_{\mu\nu}^{ab} h^{ab,\mu\nu}\, \delta^D(p - p'). \tag{94}$$

If we assume Eq. (90), the mass-shell condition for this (and all the higher) level(s) ensures $p^2 < 0$, and we can choose the frame $p_\mu = \delta_\mu{}^0 p_0$ ($p_0 \neq 0$ and $p_{i(\neq 0)} = 0$). Then, from the conditions (93), $h_{\mu\nu}^{ab} = 0$ if μ or ν is equal to 0, and the norm (94) becomes positive for any non-trivial $h_{\mu\nu}^{ab}$. Note that there is no zero-norm physical state for any $a_0 \leq \frac{1}{2}$.

4.2 *Physical states for an arbitrary level N_{level}*

We now investigate the physical state condition and obtain the general form of physical states for an arbitrary level N_{level}. Fortunately, the physical state condition that we set is strong enough to completely identify the space of physical states for any level. The result is the following: Any state satisfying the physical state condition has the form

$$h^{a_1 a_2 \cdots a_K}_{[\mu_1^M \mu_2^M \cdots \mu_f^M \cdots\cdots \mu_1^2 \mu_2^2 \cdots \mu_f^2 \mu_1^1 \mu_2^1 \cdots \mu_f^1]\sigma_1\sigma_2\cdots\sigma_K} \left(\prod_{m=1}^{M} \alpha_{-m}^{\mu_1^m} \alpha_{-(m-1/2)}^{2,\mu_2^m} \cdots \alpha_{-(m-1/2)}^{f,\mu_f^m} \right) \prod_{i=1}^{K} \alpha_{-\frac{1}{2}}^{a_i,\sigma_i} |0, p\rangle \tag{95}$$

or

$$h^{a_1 a_2 \cdots a_K}_{[\mu_2^{M+1} \cdots \mu_f^{M+1} \mu_1^M \mu_2^M \cdots \mu_f^M \cdots\cdots \mu_1^2 \mu_2^2 \cdots \mu_f^2 \mu_1^1 \mu_2^1 \cdots \mu_f^1]\sigma_1\sigma_2\cdots\sigma_K}$$
$$\times\, \alpha_{-(M+1/2)}^{2,\mu_2^{M+1}} \cdots \alpha_{-(M+1/2)}^{f,\mu_f^{M+1}} \left(\prod_{m=1}^{M} \alpha_{-m}^{\mu_1^m} \alpha_{-(m-1/2)}^{2,\mu_2^m} \cdots \alpha_{-(m-1/2)}^{f,\mu_f^m} \right) \prod_{i=1}^{K} \alpha_{-\frac{1}{2}}^{a_i,\sigma_i} |0, p\rangle. \tag{96}$$

Here, M and K are non-negative integers. Also, $h_{[\cdots],\cdots}$ is a tensor field with all the space-time indices μ_a^m within the bracket $[\cdots]$ being anti-symmetric. The tensor field $h_{\cdots}$ given in Eq. (95) should satisfy the relations

$$p^{\mu_a^m} h^{a_1 a_2 \cdots a_K}_{[\mu_1^M \mu_2^M \cdots \mu_1^1 \mu_2^1 \cdots \mu_f^1]\sigma_1\sigma_2\cdots\sigma_K} = p^{\sigma_i} h^{a_1 a_2 \cdots a_K}_{[\mu_1^M \mu_2^M \cdots \mu_1^1 \mu_2^1 \cdots \mu_f^1]\sigma_1\sigma_2\cdots\sigma_K} = 0 \tag{97}$$

$$\eta^{\sigma_i \mu_k^1} h^{a_1 a_2 \cdots a_K}_{[\mu_1^M \mu_2^M \cdots \mu_1^1 \mu_2^1 \cdots \mu_f^1]\sigma_1\sigma_2\cdots\sigma_K} = \eta^{\sigma_i \sigma_j} h^{a_1 a_2 \cdots a_K}_{[\mu_1^M \mu_2^M \cdots \mu_1^1 \mu_2^1 \cdots \mu_f^1]\sigma_1\sigma_2\cdots\sigma_K} = 0, \tag{98}$$

$$h^{a_1 a_2 \cdots a_K}_{[\mu_1^M \mu_2^M \cdots \mu_1^1 \mu_2^1 \cdots \mu_f^1]\sigma_1\sigma_2\cdots\sigma_K} \quad \text{is symmetric under a permutation } (a_i, \sigma_i) \leftrightarrow (a_j, \sigma_j) \tag{99}$$

for any i and j ($i \neq j$). Also, $h_{\dots}$ given in Eq. (96) satisfies the corresponding relations. Note that the level of the state (95) is

$$\frac{K}{2} + \sum_{m=1}^{M} (m + (f-1)(m-1/2)) = \frac{1}{2}(K + fM(M+1) - Mf + M), \tag{100}$$

and that of Eq. (96) is $\frac{1}{2}(K + (M+1)(Mf + f - 1))$. The proof that all the physical states are given by states of the form (95) or (96) is given in Appendix C.

As a non-trivial example, we represent all the physical states for $f = 3$ and $N_{\text{level}} = 5$. There are three types of physical states:

$$h_{[\mu_2^2\mu_3^2\mu_1^1\mu_2^1\mu_3^1]}\alpha_{-\frac{3}{2}}^{2,\mu_2^2}\alpha_{-\frac{3}{2}}^{3,\mu_3^2}\alpha_{-1}^{\mu_1^1}\alpha_{-\frac{1}{2}}^{2,\mu_2^1}\alpha_{-\frac{1}{2}}^{3,\mu_3^1}|0, p\rangle, \tag{101}$$

$$h^{a_1a_2\cdots a_6}_{[\mu_1^1\mu_2^1\mu_3^1]\sigma_1\sigma_2\cdots\sigma_6}\alpha_{-1}^{\mu_1^1}\alpha_{-\frac{1}{2}}^{2,\mu_2^1}\alpha_{-\frac{1}{2}}^{3,\mu_3^1}\alpha_{-\frac{1}{2}}^{a_1,\sigma_1}\alpha_{-\frac{1}{2}}^{a_2,\sigma_2}\cdots\alpha_{-\frac{1}{2}}^{a_6,\sigma_6}|0, p\rangle, \tag{102}$$

$$h^{a_1a_2\cdots a_{10}}_{\sigma_1\sigma_2\cdots\sigma_{10}}\alpha_{-\frac{1}{2}}^{a_1,\sigma_1}\alpha_{-\frac{1}{2}}^{a_2,\sigma_2}\cdots\alpha_{-\frac{1}{2}}^{a_{10},\sigma_{10}}|0, p\rangle \tag{103}$$

where $a_i = 2$ or 3. Each field satisfies the following relations:

$$p^{\mu_2^2}h_{[\mu_2^2\mu_3^2\mu_1^1\mu_2^1\mu_3^1]} = 0, \tag{104}$$

$$\begin{aligned} p^{\mu_1^1}h^{a_1a_2\cdots a_6}_{[\mu_1^1\mu_2^1\mu_3^1]\sigma_1\sigma_2\cdots\sigma_6} &= p^{\sigma_i}h^{a_1a_2\cdots a_6}_{[\mu_1^1\mu_2^1\mu_3^1]\sigma_1\sigma_2\cdots\sigma_6} = 0, \\ \eta^{\mu_1^1\sigma_i}h^{a_1a_2\cdots a_6}_{[\mu_1^1\mu_2^1\mu_3^1]\sigma_1\sigma_2\cdots\sigma_6} &= \eta^{\sigma_i\sigma_j}h^{a_1a_2\cdots a_6}_{[\mu_1^1\mu_2^1\mu_3^1]\sigma_1\sigma_2\cdots\sigma_6} = 0, \end{aligned} \tag{105}$$

$$p^{\sigma_i}h^{a_1a_2\cdots a_{10}}_{\sigma_1\sigma_2\cdots\sigma_{10}} = 0, \quad \eta^{\sigma_i\sigma_j}h^{a_1a_2\cdots a_{10}}_{\sigma_1\sigma_2\cdots\sigma_{10}} = 0 \tag{106}$$

and the appropriate symmetric properties.

4.3 *Properties of general physical states*

Now that we have fully identified the possible form of the physical states, we investigate and summarize their general properties.

First, we consider the constant a_0 that appears in Eq. (76) and has not been determined yet. In the case of string theory, the corresponding constant can be identified by calculating the zero-point energy of the sum of all physical degrees of freedom and the result is confirmed by the discussion of the BRS quantization method. For our system, a similar calculation can be performed if we assume that the number of physical degrees of freedom is the same as that of the transverse degrees of freedom. The sum of the zero-point energy corresponding to the degrees of freedom for one space-time direction for our system is given by

$$\sum_{n=0}^{\infty} n + (f-1)\sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right) \tag{107}$$

multiplied by $-\frac{1}{2}$. This type of summation can be performed by using the ζ-function regularization method and the result is given by

$$\sum_{n=0}^{\infty} n \to -\frac{1}{12}, \qquad \sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right) \to \frac{1}{24}. \tag{108}$$

By using the result,

$$\sum_{n=0}^{\infty} n + (f-1)\sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right) \to \frac{f-3}{24}, \tag{109}$$

and if we assume that the number of physical degrees of freedom is same as that of the transverse degrees of freedom, which is the case for string theory, a_0 is given by Eq. (109) multiplied by $-\frac{D-2}{2}$ as

$$a_0 = -(D-2)\frac{f-3}{48}. \tag{110}$$

For any $f \geq 3$ (and for the string case $f = 2$ and $D = 26$), this result indeed leads to $a_0 \leq \frac{1}{2}$. This is the same condition that ensures that all the level $N_{\text{level}} = \frac{1}{2}$ physical states have positive norm. In particular, if we assume $D = 26$, we have $a_0 = -\frac{f-3}{2}$, and the on-shell condition (76) becomes

$$-\frac{1}{2}p^2\left(=\frac{1}{2}m^2\right) = N_{\text{level}} + \frac{f-3}{2}, \tag{111}$$

which means that all states satisfying the on-shell condition should have integer or half-integer $-\frac{1}{2}p^2$ for every f including the string $f = 2$ case. In particular, $m^2 = f - 3$ for $N_{\text{level}} = 0$ and $m^2 = f - 2$ for $N_{\text{level}} = \frac{1}{2}$. Thus, the physical spectrum for $f \geq 3$ is limited to the massive one except for $f = 3$ and $N_{\text{level}} = 0$ in which case the ground state $|0, p\rangle$ is massless and physical.

From Eq. (111), we can also study the relation between mass and spin for each level. In general, the highest spin state for level N is given by a state of the form

$$h^{a_1a_2\cdots a_{2N}}_{(\sigma_1\sigma_2\cdots\sigma_{2N})}\alpha^{a_1,\sigma_1}_{-\frac{1}{2}}\alpha^{a_2,\sigma_2}_{-\frac{1}{2}}\cdots\alpha^{a_{2N},\sigma_{2N}}_{-\frac{1}{2}}|0, p\rangle. \tag{112}$$

Any other physical state for the level has spin less than that of this state: $0 \leq J \leq 2N(= J_{\max})$. Thus, for arbitrary $f \geq 3$, the relation between mass m and spin J is represented by

$$J \leq m^2 - f + 3, \tag{113}$$

which is also applied to the $f = 2$ open string case. (Remember that we take $\alpha' = 1$.)

Next, we will show the no-ghost theorem that ensures that all the physical states have positive norm under the assumption $a_0 \leq \frac{1}{2}$ for general $f \geq 3$ and $D \geq 2$. We have already shown that the theorem is indeed met for $N_{\text{level}} \leq 1$ in Sect. 4.1. As for the discussion on general N_{level}, we first note that any on-shell physical state has $-\frac{1}{2}p^2 > 0$ for $f \geq 3$ and $N_{\text{level}} > 0$ under the assumption $a_0 \leq \frac{1}{2}$. Then, choosing the momentum frame as $p_\mu = \eta_{\mu 0}p^0$, we see that any component of the tensor field $h_{\nu_1\nu_2\cdots}$ corresponding to any physical state of the form (95) or (96) always vanishes if any space-time index $\nu_j = 0$. That is, any non-zero physical state (95) or (96) includes only space-like oscillators α^i_{-n} and α^{ai}_{-s}. From the properties of the commutation relations of the oscillators, we easily see that all such states have positive norm. Thus we have proven the no-ghost theorem.

To summarize, for any $f \geq 3$ and $D \geq 2$, the no-ghost theorem is satisfied only if we have the condition $a_0 \leq \frac{1}{2}$. In particular, zero-norm physical states only appear in $N_{\text{level}} = \frac{1}{2}$ if $a_0 = \frac{1}{2}$. For other cases, all the physical states have positive norm. Thus, if we believe the result (110), we see that there are no zero-norm physical states in the spectrum for any $f \geq 3$, which is unlike the case of the string theory case $f = 2$. Zero-norm physical states for string theory play an important role in the theory being equipped with gauge symmetry. For $f \geq 3$, there seem to be no gauge degrees of freedom in the physical spectrum.

5. Discussion

We have revisited the covariant quantization problem of the f-pronged open string junction and found that its excitation is described by a set of ordinary bosons as well as $f - 1$ sets of twisted bosons on the worldsheet. The constraints form an open algebra with operators obeying both periodic and anti-periodic boundary conditions. We have, for the first time, succeeded in giving the physical state condition and identified the physical states that indeed have positive norm. Several remarks and/or questions are in order.

One may wonder whether our result of physical states coincides with that of the light-cone gauge analysis, which is familiar in the ordinary single string case. This question is not so easy to answer because taking a light-cone gauge is a non-trivial issue for a string junction, as described in Appendix D. Indeed, naive truncation to the transverse oscillators gives a slightly different number of states in each level. For example, in the $N_{\rm level} = 1$ (with $D \geq 3, f \geq 2$) case the number of our physical states is

$$\frac{1}{2}(f-1)(D-2)[(f-1)(D-2)+1] + (f-1)^2(D-2), \tag{114}$$

while naive truncation to the transverse mode gives

$$\frac{1}{2}(f-1)(D-2)[(f-1)(D-2)+1] + (D-2). \tag{115}$$

The number of constraints seems enough to eliminate light-cone degrees of freedom, but the structure of the physical states is not the same as naive truncation to the transverse oscillators.

As explained in, e.g., Sect. 3.3, the constraint algebra is not closed. If we include all operators newly appearing in the commutator one by one, then we eventually obtain a very large algebra. It is interesting to understand this algebra and to interpret our physical states in terms of it. In fact, by construction, our physical states in each level will become some sort of representation of a zero-mode subalgebra of that. If we define

$$B^{(p)}_{ab} = \sum_{r\in\mathbf{Z}+\frac{1}{2}} r^{p-1} : \alpha^a_{-r}\cdot\alpha^b_r : \qquad p = 1, 2, \dots, \tag{116}$$

which appears in the zero-mode part of the above-mentioned large algebra, they satisfy

$$\left[B^{(p)}_{ab},\ B^{(q)}_{cd}\right] = \delta_{bc}B^{(p+q)}_{ad} + (-1)^{p-1}\delta_{ac}B^{(p+q)}_{bd} + (-1)^{q-1}\delta_{bd}B^{(p+q)}_{ac} + (-1)^{p+q-2}\delta_{ad}(-1)^{q-1}B^{(p+q)}_{bc}, \tag{117}$$

$$B^{(p)}_{ab} = (-1)^{p-1}B^{(p)}_{ba}. \tag{118}$$

Note that $B^{(1)}_{ab} = 2L_0^{F^{(ab)}} - \frac{D}{8}\delta^{ab}$ in terms of the operator defined in Eq. (A16). This algebra is isomorphic to a twisted $(\mathbf{Z}_{>0})$-graded version of gl$(f-1)$, i.e., $\{E^+_{ab}\otimes u^{2k-1},\ E^-_{ab}\otimes u^{2k} \mid k = 1, 2, \dots;\ E^\pm_{ab} = \pm E^\pm_{ba} \in \mathrm{gl}(f-1);\ u \in \mathbf{C}\}$. We hope that this point will be clarified in the future.

Our analysis in the present paper has been within the so-called old covariant quantization (OCQ). One can find other fine structures from new covariant or BRST quantization, which will be our next task. After getting it done, we are also able to construct a free field theory for string junctions. The authors have previously studied extended string field theory [12,13] where multiple string Fock spaces are utilized to describe massless higher-spin modes with massive towers, just as closed string field theory can be formulated by the doubled Fock space of an open string with a suitable matching condition. The Fock space structure there is very similar to the current one, so that the adaptation of the formalism to the string junction may be straightforward. Furthermore, if we analyze the system based on the BRST quantization, we may identify the critical space-time dimension like $D = 26$ for the string case, which cannot be obtained only from the spectrum analysis of the old covariant quantization.

Another future task is to consider interactions. In Ref. [7] a single string emission vertex from the free end of a string segment was considered. There may be more varieties of interactions, some of which may need the introduction of another type of junction. In any case, they need to be classified.

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Appendix A. OPE for $T_M(z)$

We study the OPE properties of the operators $T_M(z)$ defined by Eq. (58) for general M. The product of two operators $A^{i\mu}(z)$ and $A^{i\mu}(w)$ is rewritten in a normal-ordered form by using the commutation relations (45) as

$$A^{1\mu}(z)A^{1\nu}(w) = \frac{1}{(z-w)^2}\eta^{\mu\nu} + \ :A^{1\mu}(z)A^{1\nu}(w): \tag{A1}$$

and

$$A^{a\mu}(z)A^{b\nu}(w) = \left(\frac{1}{(z-w)^2} + \epsilon(z,w)\right)\delta^{ab}\eta^{\mu\nu} + \ :A^{a\mu}(z)A^{b\nu}(w): \tag{A2}$$

where

$$\epsilon(z,w) = \frac{1}{2\sqrt{zw}(\sqrt{z}+\sqrt{w})^2}\ . \tag{A3}$$

By using these relations, the product of $T_M(z)$ and $T_N(w)$ for general $f \times f$ symmetric matrices M and N can be calculated as

$$\begin{aligned} T_M(z)T_N(w) \sim {} & \frac{D}{2}\frac{1}{(z-w)^4}\mathrm{tr}(M\circ N) + \frac{2}{(z-w)^2}T_{M\circ N}(w) + \frac{1}{z-w}\partial_w T_{M\circ N}(w) \\ & + \frac{1}{4}\frac{1}{z-w}([M,N])_{ij}\left(:(\partial_w A^i(w))\cdot A^j(w): - :A^i(w)\cdot\partial_w A^j(w):\right) \end{aligned} \tag{A4}$$

where

$$M\circ N \equiv \frac{1}{2}(MN+NM). \tag{A5}$$

Note that, on the right-hand side of Eq. (A4), there remain terms that cannot be represented only by T unless $[M,N]=0$.

We investigate the properties of Eq. (A4) for general M and N. First, note that any symmetric $f\times f$ matrix M can be expanded by the base matrices $H^{AB} = H^{BA} = (H^{AB})^T$ $(A, B = 1, 2, \ldots, f)$:

$$(H^{(AB)})_{ij} = \frac{1}{2}\left(\delta_{iA}\delta_{jB} + \delta_{iB}\delta_{jA}\right). \tag{A6}$$

We also use the following expressions:

$$E = H^{11}, \qquad F^{(ab)} = H^{ab}, \qquad G^{(a)} = H^{(1a)} \tag{A7}$$

where $a = 2, \ldots, f$. We can divide these base matrices $H^{(AB)}$ into two classes and define

$$\mathcal{M}^+ = \mathrm{Span}\{H^{11}, H^{ab}\}, \qquad \mathcal{M}^- = \mathrm{Span}\{H^{1a}\}. \tag{A8}$$

For any matrix $M^{\pm} \in \mathcal{M}^{\pm}$, from the definition of $T_M(z)$ given by Eq. (58), the following relation holds:

$$T_{M^{\pm}}(e^{2\pi i}z) = \pm T_{M^{\pm}}(z). \tag{A9}$$

The mode expansion of $T_{M^{\pm}}(z)$ is given by

$$T_{M^+}(z) = \sum_{n\in\mathbf{Z}} L_n^{M^+} z^{-n-2}, \qquad T_{M^-}(z) = \sum_{r\in\mathbf{Z}+\frac{1}{2}} L_r^{M^-} z^{-r-2}. \tag{A10}$$

Now that we have prepared the appropriate base matrices H^{AB} and the mode expansion for $T_M(z)$, we can calculate the commutation relation $[L_\xi^M, L_\eta^N]$ by operating

$$\oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{\xi+1} w^{\eta+1} \tag{A11}$$

on Eq. (A4) for any two matrices M and N. Here C_0 and C_w represent contour integration around $w = 0$ and $z = w$ respectively. Note that ξ (or η) is an integer if the matrix M (or N) belongs to $\mathcal{M}^+$, and a half-integer if M (or N) belongs to $\mathcal{M}^-$. After performing the integration calculation, we obtain the commutation relation

$$\begin{aligned}\left[L_\xi^M, L_\eta^N\right] &= (\xi - \eta) L_{\xi+\eta}^{M\circ N} + \frac{D}{12}\mathrm{Tr}(M \circ N)(\xi^3 - \xi)\delta_{\xi+\eta,0} \\ &\quad + \frac{1}{4}\left([M, N]\right)_{ij} \sum_\zeta (2\zeta - \xi - \eta) : \alpha^i_{\xi+\eta-\zeta} \cdot \alpha^j_\zeta : .\end{aligned} \tag{A12}$$

Here ζ is an integer for $j = 1$ and and a half-integer for $j = a$.

In the following, we explicitly present the commutation relations for each pair of base matrices $H^{(AB)}$ (or E, $F^{(ab)}$ and $G^{(a)}$) after collecting the related useful relations.

The base matrices satisfy the relations

$$H^{(AB)} \circ H^{(CD)} = \frac{1}{4}\left(\delta_{AC}H^{BD} + \delta_{AD}H^{BC} + \delta_{BC}H^{AD} + \delta_{BD}H^{AC}\right), \tag{A13}$$

$$\begin{aligned}\left[H^{(AB)}, H^{(CD)}\right]_{ij} &= \frac{1}{4}\left(\delta_{AC}(\delta_{Bi}\delta_{Dj} - \delta_{Bj}\delta_{Di}) + \delta_{AD}(\delta_{Bi}\delta_{Cj} - \delta_{Bj}\delta_{Ci})\right. \\ &\quad \left. + \delta_{BC}(\delta_{Ai}\delta_{Dj} - \delta_{Aj}\delta_{Di}) + \delta_{BD}(\delta_{Ai}\delta_{Cj} - \delta_{Aj}\delta_{Ci})\right).\end{aligned} \tag{A14}$$

The mode operators for base matrices are given by

$$L_n^E = \frac{1}{2}\sum_{m\in\mathbf{Z}} : \alpha_{n-m} \cdot \alpha_m :, \tag{A15}$$

$$L_n^{F^{(ab)}} = \frac{1}{2}\sum_{r\in\mathbf{Z}+\frac{1}{2}} : \alpha^a_{n-r} \cdot \alpha^b_r : + \frac{D}{16}\delta^{ab}\delta^{n,0}, \tag{A16}$$

$$L_r^{G^{(a)}} = \frac{1}{2}\sum_{m\in\mathbf{Z}} : \alpha^a_{r-m} \cdot \alpha_m : \frac{1}{2}\sum_{s\in\mathbf{Z}+\frac{1}{2}} : \alpha^{(a)}_s \alpha_{r-s} : . \tag{A17}$$

The commutation relations are summarized as follows:

$$\left[L_m^E, L_n^E\right] = (m-n)L^E_{m+n} + \delta_{m+n,0}\frac{D}{12}m(m^2-1), \tag{A18}$$

$$\left[L_m^{F^{(a_1a_2)}}, L_n^{F^{(b_1b_2)}}\right] = \sum_{i=1}^{2}\sum_{j=1}^{2}\delta^{a_{i+1},b_{j+1}}\left\{\frac{1}{4}(m-n)L_{m+n}^{F^{a_ib_j}} + \frac{1}{8}r\left(:\alpha_{m+n-r}^{a_i}\cdot\alpha_r^{b_j}: - :\alpha_{m+n-r}^{b_j}\cdot\alpha_r^{a_i}:\right)\right\} + \delta_{m+n,0}\left(\delta^{a_1,b_1}\delta^{a_2,b_2}+\delta^{a_1,b_2}\delta^{a_2,b_1}\right)\frac{D}{24}m\left(m^2-1\right), \tag{A19}$$

$$\left[L_r^{G^{(a)}}, L_s^{G^{(b)}}\right] = \frac{1}{4}(r-s)\left(L_{r+s}^{F^{(ab)}}+\delta^{a,b}L_{r+s}^{E}\right) + \frac{1}{8}\sum_{t\in\mathbf{Z}+\frac{1}{2}} t\left(:\alpha_{r+s-t}^{a}\cdot\alpha_t^{b}: - :\alpha_{r+s-t}^{b}\cdot\alpha_t^{a}:\right) + \delta_{r+s,0}\delta^{ab}\frac{D}{24}r\left(r^2-1\right), \tag{A20}$$

$$\left[L_n^{E}, L_s^{G^{(a)}}\right] = \frac{1}{2}\sum_{m\in\mathbf{Z}}(n-m) :\alpha_{s+n-m}^{a}\cdot\alpha_m:, \tag{A21}$$

$$\left[L_n^{F^{(ab)}}, L_s^{G^{(c)}}\right] = \delta^{a,c}\frac{1}{4}\sum_{m\in\mathbf{Z}}(m-s) :\alpha_{s+n-m}^{b}\cdot\alpha_m: + \delta^{b,c}\frac{1}{4}\sum_{m\in\mathbf{Z}}(m-s) :\alpha_{s+n-m}^{a}\cdot\alpha_m: \ . \tag{A22}$$

Note that i and j are taken as mod 2 values in Eq. (A19). We see that the operators L_n^E form a Virasoro algebra with a central charge D. On the other hand, the set of all the operators L_n^E, $L_n^{F^{(ab)}}$, and $L_r^{G^{(a)}}$ does not form a closed algebra because of the last term of Eq. (A12).

Commutation relations for general matrices are obtained from the above relations by using

$$L_\xi^{M+N} = L_\xi^M + L_\xi^N, \tag{A23}$$

which follows from

$$T_M(z) + T_N(z) = T_{M+N}(z). \tag{A24}$$

Appendix B. Algebra of L_ξ^M and related useful relations

We further investigate the properties of the algebra given by the operators L_ξ^M obtained by the mode expansion of $T_M(z)$ especially for $M = P, Q^{ij}, R^i$. The matrices P, Q^{ij}, and R^i defined in Eqs. (50) and (51) are expanded by the base matrices E, $F^{(ab)}$, and $G^{(a)}$ as

$$P = \frac{1}{f}E, \tag{B1}$$

$$Q^{ij} = \sum_{a=2}^{f}\sum_{b=2}^{f}(v_a)_i(v_b)_jF^{(ab)}, \tag{B2}$$

$$R^i = \sum_{a=2}^{f}\frac{2}{\sqrt{f}}(v_a)_iG^{(a)}. \tag{B3}$$

We see that the relations

$$\sum_{i=1}^{f}Q^{ij} = 0, \quad \sum_{i=1}^{f}R^i = 0 \tag{B4}$$

hold since

$$\sum_{i=1}^{f}(v_a)_i = 0, \qquad \sum_{i=1}^{f}(v_a)_i(v_b)_i = \delta_{ab}. \tag{B5}$$

Note that the following relation is also useful:

$$\sum_{a=2}^{f}(v_a)_i(v_a)_j = \delta_{ij} - \frac{1}{f} \ \left(= \mathrm{Tr}\, Q^{ij} \right) \ . \tag{B6}$$

From the general formula of commutation relations (A12), we can derive the following set of relations that are useful for analyzing the physical state conditions:

$$\frac{1}{2}\left[L_m^{Q^{ii}}, L_n^{Q^{jj}}\right] + \frac{1}{2}\left[L_m^{Q^{jj}}, L_n^{Q^{ii}}\right] = (m-n)\left(\delta_{ij} - \frac{1}{f}\right)L_{m+n}^{Q^{ij}} + \delta_{m+n,0}\left(\delta^{ij} - \frac{1}{f}\right)^2 \frac{D}{12} m\left(m^2-1\right), \tag{B7}$$

$$\frac{1}{2}\left[L_r^{R^i}, L_s^{R^j}\right] + \frac{1}{2}\left[L_r^{R^j}, L_s^{R^i}\right] = (r-s)\left(\frac{1}{f}L_{r+s}^{Q^{ij}} + \left(\delta_{ij} - \frac{1}{f}\right)L_{r+s}^{P}\right) + \delta_{r+s,0}\frac{1}{f}\left(\delta_{ij} - \frac{1}{f}\right)\frac{D}{6} r\left(r^2-1\right), \tag{B8}$$

$$\frac{1}{f}\left[L_n^{Q^{ij}}, L_s^{R^k}\right] + \frac{1}{2}\left(\delta_{jk} - \frac{1}{f}\right)\left[L_n^{P}, L_s^{R^i}\right] + \frac{1}{2}\left(\delta_{ik} - \frac{1}{f}\right)\left[L_n^{P}, L_s^{R^j}\right] = \frac{1}{2f}(n-s)\left\{\left(\delta_{jk} - \frac{1}{f}\right)L_{n+s}^{R^i} + \left(\delta_{ik} - \frac{1}{f}\right)L_{n+s}^{R^j}\right\}. \tag{B9}$$

Also, note that

$$\left[L_m^P, \ L_n^P\right] = (m-n)\frac{1}{f}L_{m+n}^P + \delta_{m+n,0}\frac{D}{12f^2}m(m^2-1). \tag{B10}$$

Next, we give several useful commutation relations between zero-mode operators and non-zero-mode operators:

$$\left[L_r^{R^j}, L_0^{P+Q^{ii}}\right] = \left(\delta_{ij} - \frac{1}{f}\right) r L_r^{R^i} + \frac{1}{2}\left(2 - \delta_{ij} f\right)\left[L_r^{R^i} + L_r^{R^j}, L_0^{P}\right], \tag{B11}$$

$$\left[L_m^{P+Q^{ii}}, L_0^{P+Q^{jj}}\right] = \frac{1}{f} m L_m^P + \left[L_m^{Q^{ii}}, L_0^{Q^{jj}}\right], \tag{B12}$$

$$\left[L_{m\neq 0}^{Q^{ij}}, L_0^{Q^{kk}}\right] = \frac{1}{2}\frac{\delta_{jk}f - 1}{\delta_{ik}f - 1}\left[L_m^{Q^{ii}}, L_0^{Q^{kk}}\right] + \frac{1}{2}\frac{\delta_{ik}f - 1}{\delta_{jk}f - 1}\left[L_m^{Q^{jj}}, L_0^{Q^{kk}}\right]. \tag{B13}$$

In particular, since $[L_m^{Q^{ii}}, L_0^{Q^{ii}}] = (1 - \frac{1}{f}) m L_m^{Q^{ii}}$,

$$\left[L_m^{P+Q^{ii}}, L_0^{P+Q^{ii}}\right] = \frac{1}{f} m L_m^P + \left(1 - \frac{1}{f}\right) m L_m^{Q^{ii}}. \tag{B14}$$

Further, if we define the operators

$$L_n^{E,m} = :\alpha_{n-m}\cdot\alpha_m:, \qquad L_n^{F^{(ab)},r} = :\alpha_{n-r}^a \cdot \alpha_r^b:, \qquad L_r^{G^{(a)},s} = :\alpha_s^a \cdot \alpha_{r-s}:, \tag{B15}$$

we can form a closed algebra (with central extension) by using them as generators by noting that

$$L_n^{E,m} = L_n^{E,n-m}, \qquad L_n^{F^{(ab)},r} = L_n^{F^{(ba)},n-r} \tag{B16}$$

and

$$L_n^E = \frac{1}{2}\sum_{m\in\mathbf{Z}} L_n^{E,m}, \qquad L_n^{F^{(ab)}} = \frac{1}{2}\sum_{r\in\mathbf{Z}+\frac{1}{2}} L_n^{F^{(ab)},r}, \qquad L_r^{G^{(a)}} = \frac{1}{2}\sum_{s\in\mathbf{Z}+\frac{1}{2}} L_r^{G^{(a)},s}. \tag{B17}$$

For example, the right-hand sides of Eqs. (A19)–(A22) can be written by using the operators given in Eq. (B15). Also, the following relations are useful for analyzing the physical state conditions:

$$\left[L_r^{R^j}, L_0^{Q^{ii}}\right] = \frac{1}{\sqrt{f}}\left(\delta_{ij} - \frac{1}{f}\right)\sum_{a=2}^{f}(v_a)_i \sum_{s\in\mathbf{Z}+\frac{1}{2}} s L_r^{G^{(a)},s}, \tag{B18}$$

$$\left[L_r^{G^{(a)},s}, L_0^{Q^{ii}}\right] = \frac{1}{\sqrt{f}}(v_a)_i \sum_{b=2}^{f}(v_b)_i s L_r^{G^{(b)},s}, \tag{B19}$$

and

$$\left[L_{m(\neq 0)}^{Q^{ii}}, L_0^{Q^{jj}}\right] = \left(\delta_{ij} - \frac{1}{f}\right)\sum_{a,b}(v_a)_i(v_b)_j \sum_{s\in\mathbf{Z}+\frac{1}{2}} (m-s) L_m^{F^{(ab)},m-s} \tag{B20}$$

$$= \left(\delta_{ij} - \frac{1}{f}\right)\left(L_m^{Q^{ij}} + \sum_{s\in\mathbf{Z}+\frac{1}{2}}\left[L_{m-\frac{1}{2}}^{G^{(a)},s},\ L_{\frac{1}{2}}^{G^{(b)},m-s}\right]\right) \tag{B21}$$

where

$$\left[L_r^{G^{(a)},s},\ L_{m-r}^{G^{(b)},t}\right] = \delta_{m,s+t}(r-s)\, L_m^{F^{(ab)},m-s} + \delta^{a,b}\delta_{s+t,0}\, s\, L_m^{E,r-s}. \tag{B22}$$

Appendix C. General solutions of the physical state condition

We give a proof that the space of states satisfying the physical state condition given in Sect. 3.4 is spanned by the states of the form (95) and (96) with the conditions (97)–(99). The condition that we have to impose is the set of relations (I′) for $n = 1, 2$, (I″) for $n = 1$, (II′) (or Eq. (86)) for $r = \frac{1}{2}$, and (III) as we discussed in Sect. 3.4.

We first consider the condition (86) for $r = \frac{1}{2}$, i.e., $L_{\frac{1}{2}}^{G^{(a)},s}|\phi\rangle = 0$. Since

$$\left[L_r^{G^{(a)},s}, \alpha_{-t}^{b,\mu}\right] = s\delta^{a,b}\delta_{s,t}\alpha_{r-s}^{\mu}, \qquad \left[L_r^{G^{(a)},s}, \alpha_{-t+\frac{1}{2}}^{\mu}\right] = (r-s)\delta_{s,r-t+\frac{1}{2}}\alpha_s^{a,\mu}, \tag{C1}$$

the operator $L_{\frac{1}{2}}^{G^{(a)},s}$ gives a non-trivial effect only on a state including the oscillator $\alpha_{-s}^{a,\mu}$ for $s > 0$, and $\alpha_{s-1/2}^{\mu}$ for $s < 0$. In fact, $L_{\frac{1}{2}}^{G^{(a)},s}$ causes the replacement of the oscillator as $\alpha_{-s}^{a,\mu} \to s\alpha_{-s+\frac{1}{2}}^{\mu}$ for $s > 0$, and $\alpha_{s-\frac{1}{2}}^{\mu} \to (\frac{1}{2} - s)\alpha_s^{a,\mu}$ for $s < 0$. From this property, a possible combination of oscillators $\alpha_{-s}^{a,\mu}$ and $\alpha_{-s+\frac{1}{2}}^{\mu}$ (for $s > 0$) or $\alpha_{s-\frac{1}{2}}^{\mu}$ and $\alpha_s^{a,\mu}$ (for $s < 0$) within a state $|\phi\rangle$ satisfying the condition $L_{\frac{1}{2}}^{G^{(a)},s}|\phi\rangle = 0$ can be determined as follows. For $s \geq \frac{3}{2}$, if there is any α_{-s}^{a,μ_a} in the state $|\phi\rangle$, then, in order to satisfy $L_{\frac{1}{2}}^{G^{(a)},s}|\phi\rangle = 0$, there should also be $f-1$ different types of oscillators α_{-s}^{b,μ_b} ($b \neq a$) and $\alpha_{-s+\frac{1}{2}}^{\mu}$, all of which should form the following anti-symmetric combination:

$$\alpha_{-s}^{2,[\mu_2}\alpha_{-s}^{3,\mu_3}\cdots\alpha_{-s}^{f,\mu_f}\alpha_{-s+\frac{1}{2}}^{\mu]}. \tag{C2}$$

Also, there should exist no other mode $-s$ oscillator $\alpha_{-s}^{a,\mu'}$ in the state. Similarly, for $-s \leq -\frac{1}{2}$, if there is any $\alpha_{-s-\frac{1}{2}}^{\mu}$ in the state $|\phi\rangle$, then, in order to satisfy $L_{\frac{1}{2}}^{G^{(a)},-s}|\phi\rangle = 0$, there should be $f-1$

oscillators α_{-s}^{a,μ_a} ($a = 2, \ldots, f$) and no other $\alpha_{-s-\frac{1}{2}}^{\mu'}$ within the state, and the f oscillators should form the combination

$$\alpha_{-s-\frac{1}{2}}^{[\mu}\alpha_{-s}^{2,\mu_2}\alpha_{-s}^{3,\mu_3}\cdots\alpha_{-s}^{f,\mu_f]}. \tag{C3}$$

Note that in this case any number of extra $\alpha_{-s}^{b,\nu}$ can be included within the state. Finally, for $s = \frac{1}{2}$, to satisfy $L_{\frac{1}{2}}^{G^{(a)},-s}|\phi\rangle = 0$, $|\phi\rangle$ can include any number K of oscillators $\alpha_{-\frac{1}{2}}^{a_i,\mu_i}$ ($i = 1, \ldots, K$). However, if we write the corresponding combination as

$$h_{\mu_1\mu_2\cdots\mu_K}^{a_1a_2\cdots a_K}\alpha_{-\frac{1}{2}}^{a_1,\mu_1}\alpha_{-\frac{1}{2}}^{a_2,\mu_2}\cdots\alpha_{-\frac{1}{2}}^{a_K,\mu_K}, \tag{C4}$$

the coefficient should satisfy $p^{\mu_1}h_{\mu_1\mu_2\cdots\mu_K}^{a_1a_2\cdots a_K} = 0$. From the above discussion, we see that any state satisfying the condition (II′) for $r = \frac{1}{2}$, i.e., $L_{\frac{1}{2}}^{G^{(a)},-s}|\phi\rangle = 0$ for all $s \in \mathbf{Z} + \frac{1}{2}$, must have the form (95) and (96) with the condition $p^{\sigma_i}h_{[\mu_1^M\mu_2^M\cdots\mu_1^1\mu_2^1\cdots\mu_f^1]\sigma_1\sigma_2\cdots\sigma_K}^{a_1a_2\cdots a_K} = 0$.

Next, we impose the condition (I″) for $n = 1$ on a state $|\phi_{(\mathrm{II}')_{\frac{1}{2}}}\rangle$ satisfying (II′) for $r = \frac{1}{2}$. The condition (II′) can be rewritten as the following simpler form:

$$L_n^{F^{(ab)}}|\phi\rangle = 0 \qquad (a, b = 2, \ldots, f) \tag{C5}$$

where $L_n^{F^{ab}}$ is given by Eq. (A16), and the condition for $r = \frac{1}{2}$ is explicitly given by

$$L_{\frac{1}{2}}^{F^{(ab)}}|\phi_{(\mathrm{II}')_{\frac{1}{2}}}\rangle = \left(\frac{1}{2}\alpha_{\frac{1}{2}}^{a}\cdot\alpha_{\frac{1}{2}}^{b} + \alpha_{-\frac{1}{2}}^{(a}\cdot\alpha_{\frac{3}{2}}^{b)} + \alpha_{-\frac{3}{2}}^{(a}\cdot\alpha_{\frac{5}{2}}^{b)} + \cdots\right)|\phi_{(\mathrm{II}')_{\frac{1}{2}}}\rangle \tag{C6}$$

$$= \frac{1}{2}\alpha_{\frac{1}{2}}^{a}\cdot\alpha_{\frac{1}{2}}^{b}|\phi_{(\mathrm{II}')_{\frac{1}{2}}}\rangle = 0. \tag{C7}$$

Here the second equality comes from the properties of $|\phi_{(\mathrm{II}')_{\frac{1}{2}}}\rangle$. This gives the traceless condition on each pair of indices corresponding to the coefficients of $\alpha_{-\frac{1}{2}}^{a\mu}$ as Eq. (98).

Finally, we consider the condition (I′) for $n = 1, 2$ on $|\phi_{(\mathrm{II}')_{\frac{1}{2}}}\rangle$. From the condition (II′), the integer mode oscillators of a state $|\phi_{(\mathrm{II}')_{\frac{1}{2}}}\rangle$ have to have the form

$$|\phi_{(\mathrm{II}')_{\frac{1}{2}}}\rangle = \alpha_{-1}^{[\mu_1^1}\alpha_{-2}^{\mu_1^2}\cdots\alpha_{-M}^{\mu_1^M]}\ \text{(half-integer mode oscillators)}\,|0, p\rangle. \tag{C8}$$

Since $L_n^P = \frac{1}{f}L_n^E$ and

$$[L_n^E, \alpha_{-m}^{\nu}] = m\alpha_{n-m}^{\nu}, \tag{C9}$$

the relations

$$L_1|\phi_{(\mathrm{II}')_{\frac{1}{2}}}\rangle = \alpha_0^{[\mu_1^1}\alpha_{-2}^{\mu_1^2}\cdots\alpha_{-M}^{\mu_1^M]}\ \text{(half-integer mode oscillators)}\,|0, p\rangle, \tag{C10}$$

$$L_2|\phi_{(\mathrm{II}')_{\frac{1}{2}}}\rangle = 2\alpha_{-1}^{[\mu_1^1}\alpha_0^{\mu_1^2}\alpha_{-3}^{\mu_1^3}\cdots\alpha_{-M}^{\mu_1^M]}\ \text{(half-integer mode oscillators)}\,|0, p\rangle \tag{C11}$$

hold. This gives the condition on the coefficient as

$$p^{\mu_1^1}h_{[\mu_1^M\mu_2^M\cdots\mu_1^1\mu_2^1\cdots\mu_f^1]\sigma_1\sigma_2\cdots\sigma_K}^{a_1a_2\cdots a_K} = 0. \tag{C12}$$

(or the equivalent condition $p^{\mu_1^2}h_{[\mu_1^M\mu_2^M\cdots\mu_1^1\mu_2^1\cdots\mu_f^1]\sigma_1\sigma_2\cdots\sigma_K}^{a_1a_2\cdots a_K} = 0$). This concludes the proof.

Appendix D. Remark on the light-cone gauge

Here we give a remark on the light-cone gauge. Let us first remind ourselves how the light-cone gauge is taken for the ordinary string case. After the orthonormal gauge is taken, we utilize residual gauge degrees of freedom that preserve the gauge condition to make light-cone oscillators $\alpha_n^+ = 0$. Then the Virasoro condition $L_n = 0$, together with $\alpha_n^+ = 0$, becomes a second-class constraint that can be solved by representing the remaining light-cone oscillators α_n^- in terms of the transverse modes (or by eliminating $\alpha_n^\pm$ with a Dirac bracket). Thus only transverse variables are left to us.

Now let us turn to our string junction case. We can easily find the reparametrization transformation (4), which preserves the orthonormal gauge condition (17). This should satisfy

$$\partial_+ \epsilon^{(i)-} = 0, \quad \partial_- \epsilon^{(i)+} = 0, \tag{D13}$$

where $\pm$ stands for the worldsheet light-cone directions. The boundary conditions (5) and (6) restrict them further:

$$\epsilon^{(i)\pm}(\tau, \sigma) = \epsilon(\tau \pm \sigma), \tag{D14}$$

where $\epsilon(\tau)$ is an i-independent function that satisfies $\epsilon(\tau) = \epsilon(\tau + 2\pi)$. Then we can use this degree of freedom to make $\alpha_n^+ = 0$ as in the string case. So we are left with $2f - 1$ sets of oscillators $\alpha_n^-, \alpha_r^{a+}, \alpha_r^{a-}$ other than transverse modes. Our remaining constraints are V_n, R_r^a, $L_n^{Q^{a-1\,a-1} - Q^{aa}}$ whose number of degrees of freedom is also $2f - 1$ in total. These $2f - 1$ constraints together with $\alpha_n^+ = 0$ can be considered as a set of $2f$ second-class constraints of the system. Thus if we were able to solve them in terms of transverse oscillators, then in principle we could reach the light-cone gauge. But these relations are so complicated that we have not succeeded in explicitly solving them yet. Indeed, even at the classical level, the authors of Ref. [9] concluded that it is not possible, though their treatment of the remaining second-class constraints is not clear.

Thus, although the number of physical degrees of freedom is likely to coincide with that of the light-cone gauge, the structure of the physical states may not be the same as the naive truncation to the transverse modes.

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