Correlation functions in supersymmetric gauge theories from supergravity fluctuations

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Abstract

Finding string duals of gauge theories is an important outstanding problem in theoretical physics. In recent years, some progress has been achieved: the AdS/CFT correspondence is a proposal of an exact duality between strings moving on an $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ Super Yang-Mills theory. The search is now on for finding string duals of more realistic gauge theories. A fruitful avenue of research in this context has been the study of theories involving stacks of D3-branes placed at conifold singularities. These theories have reduced supersymmetry and can also break conformal symmetry. Explicit supergravity solutions corresponding to these theories have been constructed. This dissertation is concerned with the study of one such solution, the warped deformed conifold, and its dual gauge theory. We develop a procedure for calculating correlation functions in the gauge theory by solving equations of motion for supergravity fluctuations in the warped conifold background. Using this procedure, we compute the high energy behavior of two point correlation functions of the gauge theory R-current and energy-momentum tensor, and show that these correlators are consistent with anomalous breaking of R-symmetry and dilatation symmetry. We also investigate a possible baryonic symmetry in the theory. We conclude by computing masses of low-lying glueball states in the gauge theory, explicitly demonstrating the presence of a mass gap.

Acknowledgments

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This work is dedicated to the memory of Jason Mizell, also known as Jam Master Jay.

Contents

Chapter 1

Introduction

1.1 Overview

The remarkable proposal [1, 2, 3] of an exact duality between $\mathcal{N} = 4$ superconformal Super Yang-Mills (SYM) theory in four dimensions and type IIB superstring theory on $AdS_5 \times S^5$ – known as the AdS/CFT correspondence – and the substantial evidence amassed to support that proposal, have revived interest in the idea that gauge theories have dual descriptions as strings. In order to find such dual descriptions for gauge theories of physical interest, two problems have to be addressed: first, finding the supergravity (SUGRA) backgrounds corresponding to these field theories, and second, going beyond the supergravity approximation to construct sigma models of strings moving on these backgrounds. The latter problem is very difficult and we will not comment on it in the present work. A particularly useful approach towards addressing the former problem has been the investigation of field theories, and corresponding supergravity solutions, obtained by placing stacks of D3-branes at conifold singularities. In a series of papers [4, 5, 6, 7, 8, 9][1], it was shown that the field theories (and their dual SUGRA solutions) obtained in this way break the extended supersymmetry (SUSY) of $\mathcal{N} = 4$ SYM down to $\mathcal{N} = 1$ SUSY [4]. Adding fractional D3-branes – D5-branes wrapped around the 2-cycle of

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[1]See also [10, 11].

the conifold – results in a supergravity solution, the Klebanov-Strassler (KS) solution [9], that corresponds to a field theory in which the conformal symmetry is also broken, and one of the coupling constants runs. This field theory is confining at low energies, and confinement can be seen directly from the SUGRA background. The investigation of this SUGRA solution and its correspondence with the dual gauge theory is therefore of obvious interest both in itself, and as a stepping stone on the way to understanding the string duals of realistic field theories.

The present dissertation is concerned with aspects of such an investigation. The original AdS/CFT correspondence provided a concrete prescription for computing correlation functions in the (conformal) quantum field theory by solving linearized equations of motion for fluctuations around the (AdS) SUGRA background [2, 3]. Mathematically, this prescription relied on properties of AdS space, and there is a challenge in extending it to non-asymptotically AdS spaces. In chapter 3, we show that such an extension is possible for the Klebanov-Strassler solution, whose asymptotics are described not by AdS but by the logarithmically warped space found by Klebanov and Tseytlin – the Klebanov-Tseytlin (KT) solution [8]. We develop a procedure for extracting the high energy behavior of field theory two point functions by solving equations of motion for fluctuations around the KT background. As an example, we compute the two point function of a minimal massless scalar; this is the first such computation for a non-asymptotically AdS background. In chapter 4, we apply this procedure to study the anomalous breaking of R and dilatation symmetries in the field theory. We derive and approximately solve the equations of motion for the SUGRA fields dual to the field theory R-current and energy-momentum tensor, and find the high energy behavior of two-point correlation functions for these operators. This behavior is consistent with anomalously broken symmetry. In chapter 5, we use the SUGRA background to ask if the field theory has a nontrivial conserved baryonic current. The answer is somewhat inconclusive: there is

no globally defined conserved current, but there may be an effective conserved current at high energies. The SUGRA vector dual to such a current disappears in a novel mechanism similar to but distinct from Higgsing. In chapter 6 we explicitly demonstrate the existence of a mass gap in the gauge theory by computing masses of low-lying glueball states. The remainder of the present chapter is devoted to a concise review of relevant preliminary material, which closely follows the perspective of [12]. Readers interested in a more extensive review may consult [13, 14].

Before we proceed, let us establish notation. Whenever we are in Minkowski space, the signature is $(-, +, \ldots, +)$; we will not be too careful about distinguishing Minkowski from Euclidean space. We will use Greek indices $\mu, \nu \ldots$ for flat 4-dimensional (Minkowski or Euclidean) space, lowercase Latin indices $i, j \ldots$ for the 5-dimensional space of noncompact dimensions (x^μ, r) and uppercase Latin indices $M, N \ldots$ for the full 10-dimensional space. The operator $\Box$ will denote the 4-dimensional flat space d'Alembertian, $\Box_{10}$ the full 10-dimensional Laplacian. h will always be used to denote the metric warp function $h(r)$ for 10-dimensional spacetimes; we will often suppress the r-dependence. For spacetimes with a reduced number of dimensions $D < 10$ we will use $H(r)$ to denote the metric warp function.

1.2 Black p-branes, D-branes and the AdS/CFT correspondence

String theory has its origins in an attempt to understand the strong interactions [15]. The mass $m(J)$ of the lowest-lying meson state of angular momentum J was found in experiments to roughly obey the Regge relation

$$m^2(J) = m_0^2 + \alpha' J.$$

This relation can be simply explained by supposing the mesons to be excitation modes of a rotating string. It inspired the earliest string models for the strong

interactions.

String theory has since become of interest to physicists as a candidate for a unified theory of all interactions, as it is a covariant, finite theory that includes gravity [16]. The strong interactions, as we now know, are correctly described by a non-abelian gauge theory with gauge group $SU(3)$, known as Quantum Chromodynamics or QCD. At low energies, however, this theory becomes strongly coupled, and a perturbative description in terms of gauge theory quanta is not useful in understanding such effects as color confinement and the presence of a mass gap. For this reason, physicists have continued to search for an effective description of strongly coupled gauge theory. In fact, the most natural gauge-invariant objects of the theory are not pointlike, but closed curves known as Wilson loops. Further, it can be shown that confinement of color charges follows from the fact that the expectation value of a Wilson loop obeys the area law: it is inversely proportional to the exponential of the area of the minimal area surface enclosed by the loop [17]. This behavior is naturally explained by assuming that the theory is effectively described by strings propagating subject to the boundary condition that the Wilson loop is the boundary of the string worldsheet. Thus, a string description of strongly coupled gauge theory naturally leads to confinement.

A very important step in the search for dual string descriptions of gauge theories was taken by 't Hooft [18]. He considered theories with N colors, where we now take N to be large (generalizing from the QCD case $N = 3$). He showed that if we take N to be large while keeping the 't Hooft coupling $\lambda = g_{YM}^2 N$ fixed, then each Feynman diagram in the perturbative expansion of the field theory carries a topological factor N^χ, where χ is the Euler characteristic of the graph representing the Feynman diagram. This suggests that we may think of the sum over diagrams of a given topology as a sum over string worldsheets of that topology. Now, since spheres, corresponding to string tree diagrams, are weighed by N^2, tori – the string 1-loop diagrams – by N^0 etc. the closed string coupling constant is of order $1/N$.

Thus, taking N large results in a weakly coupled string theory, with $1/N$ corrections corresponding to perturbative string corrections. Since $1/3$ is a relatively small number, we might even hope that the $1/N$ expansion would give a good approximation to QCD.

The above considerations suggested that gauge theories should have dual descriptions in terms of strings. However, it was still not clear what such string theories looked like. One clue came from work by Polyakov [19, 20] on strings moving in a non-critical number of dimensions: $D \neq 26$ for bosonic strings, or $D \neq 10$ for superstrings. Classically, string theory has worldsheet Weyl symmetry: it is invariant under the rescaling of the string worldsheet. However, in a noncritical number of dimensions, the Weyl symmetry develops a quantum anomaly. As a result, the field φ corresponding to the scale of the worldsheet, which classically can be gauged away, becomes a physical field, known as the Liouville field. This field effectively behaves as an extra dimension for the strings to move in. Moreover, the $(D+1)$-dimensional space consisting of the fields (X^μ, φ), where X^μ is the original (flat) target space of the non-critical string and φ is the Liouville field, typically becomes warped, with a metric that can be written as $ds^2 = d\varphi^2 + a^2(\varphi)(dX)^2$. Polyakov's insight was that the strings dual to four-dimensional gauge field theories should effectively move in such warped *five*-dimensional target spaces. However, it was still not clear what the warp factor $a^2(\varphi)$ looked like, or how to construct such string theories.

The answer emerged from seemingly unrelated research in superstring theory. The low energy effective supergravity theory of superstrings is known to contain form-field degrees of freedom known as Ramond-Ramond (RR) fields. The low energy effective action for type IIB string theory, which is the theory we will work with in the present dissertation, can be written as [21]

$$S = -\frac{1}{2\kappa^2} \int d^{10}x (\sqrt{-g}[R - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}e^{2\Phi}(\partial C)^2 - \frac{1}{12}g_s e^{-\Phi}H_3^2 - \frac{1}{12}g_s e^{\Phi}\tilde{F}_3^2 -$$
$$-\frac{1}{4\cdot 5!}g_s^2\tilde{F}_5^2] - g_s^2 C_4 \wedge F_3 \wedge H_3)(1.1)$$

where

$$F_3 = dC_2, \qquad H_3 = dB_2, \qquad F_5 = dC_4,$$

$$\tilde{F}_3 = F_3 - CH_3, \quad \tilde{F}_5 = F_5 + B_2 \wedge F_3.$$

Here C, C_2 and C_4 are the RR 0, 2 and 4-forms respectively, B_2 is the NS-NS 2-form and Φ is the dilaton. The gravitational coupling constant κ is related to the string tension α' and string coupling g_s by

$$\kappa = 8\pi^{7/2} g_s (\alpha')^2 \qquad (1.2)$$

The resulting equations of motion are:

$$R_{MN} = \frac{1}{2}\partial_M \Phi \partial_N \Phi + \frac{1}{2}e^{2\Phi}\partial_M C \partial_N C + \frac{1}{96}g_s^2 \tilde{F}_{MPQRS}\tilde{F}_N{}^{PQRS} +$$
$$+ \frac{g_s}{4}(e^{-\Phi}H_{MPQ}H_N{}^{PQ} + e^{\Phi}\tilde{F}_{MPQ}\tilde{F}_N{}^{PQ}) -$$
$$- \frac{1}{48}g_{MN}(e^{-\Phi}H_{PQR}H^{PQR} + e^{\Phi}\tilde{F}_{PQR}\tilde{F}^{PQR}), \qquad (1.3)$$

$$d \star (e^{\Phi}\tilde{F}_3) = g_s F_5 \wedge H_3,$$

$$d \star (e^{-\Phi}H_3 - Ce^{\Phi}\tilde{F}_3) = -g_s F_5 \wedge F_3, \qquad (1.4)$$

$$d \star d\Phi = e^{2\Phi}dC \wedge \star dC - \frac{g_s}{2}e^{-\Phi}H_3 \wedge \star H_3 + \frac{g_s}{2}e^{\Phi}\tilde{F}_3 \wedge \star\tilde{F}_3,$$

$$d(e^{2\Phi} \star dC) = -g_s e^{\Phi}H_3 \wedge \star\tilde{F}_3. \qquad (1.5)$$

These equations are supplemented by the self-duality condition

$$\star\tilde{F}_5 = \tilde{F}_5. \qquad (1.6)$$

It has long been known that the above action and equations of motion have solutions that are charged under the RR form fields [22, 23]. The typical form of a solution magnetically charged under an RR form field is that the field strength

$F_{q+1} = dC_q$ of the RR q-form field C_q has flux over a $q+1$-dimensional sphere S^{q+1}. The metric has the form

$$ds^2 = h^{-1/2}(r)(-dt^2 + \sum_{\mu=1}^{p}(dx^\mu)^2) + h^{1/2}(r)(dr^2 + r^2 d\Omega_{q+1}^2), \qquad (1.7)$$

so that $q = 7 - p$. $h(r)$ is a harmonic function of r in $10 - p$ dimensions. These solutions are known as (magnetically charged) black p-branes[2]. They can be dimensionally reduced to black holes in $10 - p$ spacetime dimensions by compactifying the p flat spacelike directions. They have a horizon at $r = 0$, and preserve 16 of the original 32 supersymmetries of the string theory. Although these solutions were well known, their precise role in string theory was not yet understood.

All this changed with the realization by Polchinski [24] that in addition to p-brane solutions, string theory has fundamental objects charged under Ramond-Ramond fields. The existence of such objects was required by string dualities [25], but Polchinski found a simple way to embed them in perturbative string theory. Type II string theories have only closed strings. Ordinarily, open strings, when they are present in a string theory, obey Neumann boundary conditions, meaning that the ends of the string have to move at the speed of light. Instead, one can consider adding to type II string theory a different kind of open strings, allowing a set of $9 - p$ directions $(X^{p+1}, \ldots, X^{10})$ to end on a $(p+1)$-dimensional hypersurface of 10-dimensional spacetime located at some point $((X^{p+1})', \ldots, (X^{10})')$. In other words, while $p + 1$ directions $(X^0, \ldots, X^p)$ of the open string continue to obey the usual (Neumann) boundary conditions, the remaining directions obey Dirichlet boundary conditions. The resulting object is known as a Dirichlet p-brane, or Dp-brane.

Polchinski showed that a Dp-brane carries a flux of the RR C_{p+1} form-field on its worldvolume. Thus D-branes are the sought-for fundamental objects that carry RR charges. They are BPS states, preserving 16 of the 32 supersymmetries of string theory. For a single D-brane, it can be shown that the open string degrees of freedom

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[2]For electrically charged branes, C_q has flux over the worldvolume of the brane, so $q = p + 1$.

ending on the brane are described by a $U(1)$ gauge field, together with $9 - p$ scalars and fermionic superpartners. We can also consider the situation of N parallel D-branes located at different points in the transverse space. In this case there are N^2 degrees of freedom corresponding to open strings beginning and ending on each brane (open strings are directed so the beginning and end are distinguishable). One can show that the mass of a vector corresponding to a string stretched between two branes is propotional to the separation between the branes. In the case of N branes at different points, we have N massless vectors generating a $U(1)^N$ gauge theory, as well as $N^2 - N$ massive vectors. If we bring all the branes together, then all vectors are massless. The minimal gauge symmetry required to describe N^2 massless vectors is $U(N)$. Thus we have arrived at the remarkable conclusion [26] that the low energy degrees of freedom of a stack of N coincident Dp-branes are described by the maximally supersymmetric $U(N)$ gauge theory in $p + 1$ spacetime dimensions.

Since our interest is mainly in 3+1 dimensional gauge theory, we will now focus on D3-branes, which are present in type IIB superstring theory. From the above, it follows that at low energies, the degrees of freedom a stack of N coincident D3-branes are described by $\mathcal{N} = 4$ SYM with gauge group $U(N)$. In addition to the massless gauge bosons, this theory contains 6 massless scalar fields in the adjoint representation of $U(N)$. Their geometrical interpretation is also clear: they describe the location of the stack of branes in transverse 6-dimensional space. Because of translation invariance, the 3-branes can be located anywhere, so there can be no potential for these scalar fields: their values are moduli of the theory.

Now, D-branes are tensile objects; the tension of a D3-brane is known to be

$$T = \frac{\sqrt{\pi}}{\kappa}, \tag{1.8}$$

where κ is the 10-dimensional gravitational constant. Because there is no force between parallel D-branes, the tension of a stack of N D3-branes is just N times

that, or $N\frac{\sqrt{\pi}}{\kappa}$. For large N, such a stack is a heavy object that curves the surrounding space. Thus, it must have a description as a SUGRA solution. Such a solution must have the same quantum numbers as the stack of 3-branes; it must preserve 16 supersymmetries and be charged under the RR 4-form field. All these properties are possessed by the black 3-brane solution described by the metric (1.7). The warp function for the metric in this solution takes the form

$$h(r) = 1 + \frac{R^4}{r^4}, \tag{1.9}$$

where R is a radius related to the RR charge of the solution. At small r, the metric (1.7) has the limiting form

$$ds^2 = \frac{r^2}{R^2}dx_\mu dx^\mu + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2. \tag{1.10}$$

This is a direct product of 5-dimensional Anti de-Sitter space (AdS_5) with a 5-sphere S^5, where both have radius of curvature R. This geometry is non-singular at small r, and, if R is large, all components of the curvature tensor are small when measured in terms of the string tension α'. Thus for $R \gg \sqrt{\alpha'}$, the SUGRA approximation accurately describes type IIB superstrings on this background. The full geometry (1.7) looks like a semi-infinite throat of radius R which for $r \gg R$ opens up into flat (9+1)-dimensional space.

To match the description in terms of D3-branes with the p-brane SUGRA solution, we equate the tension of a stack of N D3-branes with the ADM tension of the solution [27]. We obtain

$$\frac{2}{\kappa^2}R^4\Omega_5 = N\frac{\sqrt{\pi}}{\kappa}, \tag{1.11}$$

where $\Omega_5 = \pi^3$ is the volume of a unit 5-sphere. From the relation (1.2), and from the expression $g_{YM}^2 = 4\pi g_s$ for the Yang-Mills coupling on the D3-branes in terms of the string coupling g_s, we get

$$R^4 = g_{YM}^2 N (\alpha')^2, \tag{1.12}$$

so the size of the throat in units of the string tension is measured by the 't Hooft coupling $\lambda = g_{YM}^2 N$ of the gauge theory on the D3-branes. Also, the condition $R \gg \sqrt{\alpha'}$ for the validity of the SUGRA approximation translates into the condition of large 't Hooft coupling in the gauge theory. This is precisely when the theory is strongly coupled and ordinary perturbation theory is not applicable.

We have arrived, so far, at the following picture: the low energy degrees of freedom of a stack of coincident D3-branes are described by $\mathcal{N} = 4$ SYM with gauge group $U(N)$; at higher energies, these degrees of freedom also interact with the closed string modes of the ambient 10-dimensional space. On the other hand, the stack curves space and can be described by a SUGRA solution with a geometry that looks like a semi-infinite throat opening out into flat 10-dimensional space. The size of the throat in string units is given by the 't Hooft coupling of the gauge theory; at large 't Hooft coupling, SUGRA is a good description of type IIB superstring theory on this background.

What tests can we perform to see if this picture is actually correct? A natural set of tests is suggested by the fact the the gauge fields living on the branes interact with the closed string modes in the bulk. Such an interaction has the generic form

$$S_{int} = \int d^4x \phi(x) \mathcal{O}(x). \tag{1.13}$$

Here $\phi(x)$ denotes some closed string mode such as the dilaton or the graviton, evaluated at the position of the branes. $\mathcal{O}(x)$ is the gauge-invariant field theory operator that couples to the mode ϕ.

We would now like to consider the process consisting of a quantum of the mode ϕ being absorbed by the brane [28, 29, 30]. Such a process is allowed by the presence of the interaction term (1.13). If ϕ is a massless closed string mode such as the dilaton Φ, the operator $\mathcal{O}$ has naive dimension 4 on the worldvolume, so the bosonic part of the operator is typically quadratic in the field strength of the gauge field living on the branes (Φ in fact couples to trF^2). Thus one can think of the field ϕ being

swallowed by the brane, emitting two gluons that travel along the brane. The absorption cross-section for this process can be read off from the two-point function of the operator $\mathcal{O}$. Such a two-point function has an absorptive part, resulting in a discontinuity across the negative real axis in momentum space. It can be shown that the cross-section for the absorption of a mode of energy ω behaves as [30]

$$\sigma_{abs} \sim \frac{1}{\omega} Disc\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle, \tag{1.14}$$

where $k^2 = \omega^2$. For a dimension 4 operator, $\langle \mathcal{O}(x)\mathcal{O}(x')\rangle \sim \frac{\kappa^2}{|x-x'|^8}$, so in momentum space

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle \sim \kappa^2\omega^4 \log\omega. \tag{1.15}$$

The logarithm produces a discontinuity, so we have $\sigma_{abs} \sim \kappa^2\omega^3$. For a stack of N branes, there is an additional factor of order N^2 to account for the number of gluon degrees of freedom. The exact result for dilaton absorption is [28]

$$\sigma = \frac{\kappa^2\omega^3 N^2}{32\pi}. \tag{1.16}$$

To check the validity of the correspondence between the D-brane picture and the SUGRA picture, we would like to compare this result to a SUGRA computation. It is clear that in the SUGRA picture, it is the throat region that in some sense corresponds to the branes. Thus we can imagine a dilaton wave incident on the throat from $r = \infty$. Part of it gets reflected back, and a part is absorbed by the throat. Because the dilaton is a minimal massless scalar, it obeys the simple equation of motion

$$\Box_{10}\,\phi = 0, \tag{1.17}$$

where the 10-dimensional Laplacian $\Box_{10}$ is taken with respect to the black 3-brane background metric (1.7), with $h(r) = 1 + \frac{R^4}{r^4}$. To calculate the absorption cross-section, we need to compare the flux in the near horizon region $r \ll R$ to the flux

in the asymptotic region $r \gg R$. We will describe such calculations in detail in the next chapter. For now, we will simply quote the result [28]: a calculation of the cross-section for the absorption of such a massless scalar by the throat yields precisely eq. (1.16).

This is a spectacular confirmation of the correspondence between the two pictures. Similar calculations have been performed for absorption of other fields, such as the traceless graviton polarized along the brane and the RR scalar. In all cases, an exact agreement was found between the D-brane picture and supergravity [28, 29, 30, 31].

Let us pause to reflect on what exactly has happened here. The key is to note that eq. (1.14) not only allows us to read off the absorption cross-section of a closed string mode from the discontinuity of the two point function of the appropriate gauge theory operator, but conversely, we can derive the leading behavior of the two point function from the absorption cross-section [30, 31]. This is because the leading behavior is given by the non-analytic part of the two point function in momentum space, which is exactly the part that is picked up by the absorption cross-section. The leading behavior in position space can then be obtained by taking a Fourier transform. In this way, by doing a SUGRA calculation of the absorption of a mode such as the dilaton, we are effectively computing a correlation function in the field theory on the stack of D-branes!

The notion of an exact correspondence between the two pictures was decisively sharpened in a paper by Maldacena [1]. He realized that the universal region of the SUGRA solution is the throat $r \ll R$, and that this region should be directly identified with the low energy $\mathcal{N} = 4$ SYM on the stack of branes. The reason is that the low energy limit $\alpha' \to 0$ can be directly taken in the geometry, and is equivalent to the $r \to 0$ limit. In terms of the absorption considerations above, a particle incident from asymptotic infinity in the D-brane picture is converted into an excitation of the gauge theory degrees of freedom; as we saw, this is described in

SUGRA as a particle tunneling from large r into the throat $r \ll R$. This provides more support for the direct identification of the throat limit of the geometry with the low-energy limit of the theory on the D3-branes.

Maldacena was thus led to formulate the following famous proposal, known as the AdS/CFT correspondence: type IIB string theory on the space $AdS_5 \times S^5$, described by the metric (1.10), is exactly dual to the superconformal $\mathcal{N} = 4$ SYM in 4 spacetime dimensions.

In addition to the initial dynamical evidence for this proposal outlined above, it is possible to perform a detailed matching of the symmetries (for a review, see [13]). The isometry group of AdS_5 is $SO(2,4)$, and this is also the conformal group in $3+1$ dimensions. We also have the $SO(6)$ isometry group of the product S^5; this precisely corresponds to the R-symmetry group of $\mathcal{N} = 4$ SYM. If one includes all fermionic generators required by SUSY, the full isometry supergroup of $AdS_5 \times S^5$ is $SU(2,2|4)$ which is just the $\mathcal{N} = 4$ superconformal symmetry group. A final note on symmetry is that in both pictures, taking the low-energy, or throat, limit, produces enhanced supersymmetry: the full theory on the stack of branes, as well as the black 3-brane solution, have 16 supersymmetries, but $\mathcal{N} = 4$ SYM, as well as the $AdS_5 \times S^5$ geometry, have the maximal 32 supersymmetries of type IIB string theory.

The next step was to provide an exact prescription for matching the SUGRA excitations of the $AdS_5 \times S^5$ background with operators in the corresponding gauge theory. This was done in important papers by Gubser, Klebanov and Polyakov [2], and Witten [3]. Again, the motivation comes from the absorption calculations. A particle incident on the throat in the full 3-brane geometry has to go through the region where the throat opens out to asymptotic infinity. In the AdS limit, this region itself is asymptotic and forms the boundary of the AdS space. Thus the interface between the closed string modes corresponding to the SUGRA fluctuations, and the dual gauge theory correlation functions, must be on the boundary. As we

saw earlier, the interaction between the SUGRA modes and gauge theory operators is mediated by interaction terms generically described by eq. (1.13). Adding these terms to the gauge theory action yields the generating functional $W[\phi]$ of connected correlation functions in the gauge theory. This functional is a functional of 4-dimensional fields $\phi(x)$, which are in one-to-one correspondence with the SUGRA excitations. Since the SUGRA excitations depend on $4+1$ noncompact coordinates, it is natural to think of the 4-dimensional fields $\phi(x)$ as boundary conditions. The AdS/CFT correspondence then leads to the following proposal [2, 3]: The generating functional $W[\phi(x)]$ of connected correlation functions in $\mathcal{N} = 4$ SYM should be identified with the extremum of the classical SUGRA action $I[\phi(x, r)]$, where the fields $\phi(x, r)$ solve the SUGRA equations of motion subject to the boundary condition $\phi(x, r \to \infty) \sim \phi(x)$. The terms coupling to the gauge theory operators in the generating functional reproduce the boundary behavior of SUGRA fields. The exact scaling is determined by the dimension of the operator.

This prescription allows one to compute gauge theory correlation functions in $\mathcal{N} = 4$ SYM at large 't Hooft coupling by solving classsical SUGRA equations of motion on the $AdS_5 \times S^5$ background[3]. Many such calculations have now been performed on both sides, with exact agreement in every case. Thus the AdS/CFT correspondence provides the first realization, at least in the supergravity limit, of the long-held hope for finding dual descriptions of gauge theories in terms of strings. The main ideas outlined earlier in this section – 't Hooft's large N limit, and the propagation of strings in a higher-dimensional curved background – are all realized in this duality. At large λ, the $AdS_5 \times S^5$ solution has small curvature and is an accurate description of string theory. This solution is a compact 5-sphere times a warped space of non-compact $4 + 1$ dimensions. The fifth (radial) dimension of AdS_5 is analogous to the fifth Liouville dimension proposed by Polyakov. For $\mathcal{N} = 4$ SYM, this geometry answers the question: what is the target space that strings dual

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[3]This aspect of the correspondence is reviewed in more detail in section 3.2.

to gauge fields propagate in?

Although of some interest, $\mathcal{N} = 4$ SYM is not a very realistic gauge theory. For one thing, the maximal SUSY severely constrains the dynamics of the theory. Also, this theory is conformal, meaning that there is no interesting renormalization group flow. The coupling constant is the same at all scales; there is no confinement and no asymptotic freedom. The goal now is to use the powerful apparatus of matching stacks of D-branes and SUGRA solutions to try to obtain string duals of more realistic field theories.

In chapter 3 we will begin discussing the progress made towards this goal by considering theories living on branes placed at conifold singularities, rather than in flat space. Before doing that, we note that one way to break conformal invariance is actually to consider the full theory living on the stack on D3-branes in flat space, governed by the full DBI action which describes both the open string modes associated with the branes, and the bulk closed string modes that they interact with. The low energy limit of this theory is conformal; this is the limit taken in the AdS/CFT correspondence. However, at energies comparable to the string scale, the theory contains irrelevant operators that explicitly break conformal symmetry. In the next chapter we will study these operators and the corresponding SUGRA modes, the so-called fixed scalars.

Chapter 2

Fixed scalars and the breaking of conformal symmetry

As we indicated in the Introduction, the black p-brane solutions of superstring theory can be dimensionally reduced to black hole solutions by wrapping the p spacelike dimensions of the branes on a torus T^p. Research on such black hole backgrounds has shown that the spectrum of their fluctuations contains scalar fields known as *fixed scalars* that couple nonminimally to the background (their equations of motion contains potentials in addition to the usual Laplacian term of eq. (1.17)). Due to the nonminimal couplings, the low-energy absorption cross-sections for such fields are suppressed compared to those of the minimally coupled scalars [32, 33, 34, 35]. This suppression has a natural explanation in terms of the coupling (1.13) between the closed string modes corresponding to such scalars and operators living on the worldvolumes of branes: while the minimal scalars couple to marginal operators, the non-minimal ones couple to irrelevant operators [33, 34, 35]. Such operators are ignored in the conformal limit, but are well-known to be present in the non-polynomial actions of the DBI type that describe the full (not just low-energy) theory on the branes.

The theories of main interest to us are related to $(D = 7)$-dimensional black hole solutions corresponding to type IIB D3-branes, reviewed in the Introduction, as well

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as the so-called M2-brane and M5-brane backgrounds. We have not mentioned these solutions until now. The reason is that they are solutions, not of type IIB SUGRA in $D = 10$, but of the non-chiral maximal supergravity in $D = 11$ dimensions. This SUGRA has a 3-form field. In analogy to type IIB SUGRA, this implies the existence of electrically charged 2-brane solutions, corresponding to black holes in $D = 9$ dimensions, and magnetically charged 5-brane solutions, corresponding to black holes in $D = 6$ dimensions. 11-dimensional SUGRA is not the low-energy theory of any string theory, although it is suspected to be the low-energy theory of an as yet mysterious model known as M-theory. Thus, we have no explicit description of the theory on the worldvolume of M-branes in terms of string worldsheets with certain boundary conditions on the strings. What makes these theories particularly fascinating, and in some ways similar to the much better understood theory on the worldvolume of the D3-branes, is that in all cases, the low-energy limit of the theory is conformal. This can be seen from the fact that the near-horizon geometries of the solutions corresponding to these branes are all products of AdS spaces with spheres: $AdS_5 \times S^5$, $AdS_4 \times S^7$ and $AdS_7 \times S^4$ for the D3-brane, M2-brane and M5-brane solutions respectively. Another sign of conformality is that none of these solutions possess a nontrivial dilaton background (hence the term "nondilatonic branes"). In what follows, we will be able to treat all these solutions in terms of a single framework involving the dimensionally reduced action[1].

We obtain a charged black hole in $D = 10 - p$ by wrapping some number of Dirichlet p-branes over T^p. The part of the D-dimensional effective action that will be relevant for our calculations is [22]

$$S \sim \int d^D x \sqrt{-g} \left(R - \frac{1}{2} \partial_m \lambda \partial^m \lambda - e^{\beta \lambda} F_{mn} F^{mn} \right), \tag{2.1}$$

where

$$\beta = \sqrt{\frac{2(D-1)}{D-2}}, \tag{2.2}$$

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[1]This chapter is based on parts of the paper [36].

and F_{mn} is the field strength of the 1-form-field that results from the dimensional reduction of the RR $(p+1)$-form field C_{p+1} under which the branes are charged. The fixed scalar λ is a certain linear combination of $\log V$ and the 10-dimensional dilaton Φ (V is the internal volume of the brane measured in the 10-dimensional Einstein metric):

$$\lambda = \frac{D-7}{2\beta}\Phi - \frac{1}{2\beta}\log V. \tag{2.3}$$

The static charged black hole solution is

$$ds^2 = H^{\frac{1}{D-2}}(-H^{-1}dt^2 + dr^2 + r^2 d\Omega^2_{D-2}) \tag{2.4}$$

$$F_{rt} = \frac{1}{\sqrt{2}}\partial_r H^{-1}, \quad \lambda = \frac{\beta}{2}\log H \tag{2.5}$$

$$H(r) = 1 + \frac{R^{D-3}}{r^{D-3}} = 1 + \frac{2Q}{(D-3)r^{D-3}}. \tag{2.6}$$

This is the dimensionally reduced form of the black p-brane solutions discussed in chapter 1.

In considering fluctuations around this background, one may be concerned that the $D=7$ case should be treated separately because the solution also includes the 5-form background $G = \star F$. Thus, a priori the action is

$$\int d^7x \sqrt{-g}\left(R - \frac{1}{2}\partial_m\lambda\partial^m\lambda - e^{\beta\lambda}F_{mn}F^{mn} - \frac{2}{5!}e^{-\beta\lambda}G_{m_1...m_5}G^{m_1...m_5}\right).$$

However, we may dualize the G^2 term into the F^2 term, so that the action is equivalent to

$$\int d^7x \sqrt{-g}\left(R - \frac{1}{2}\partial_m\lambda\partial^m\lambda - 2e^{\beta\lambda}F_{mn}F^{mn}\right).$$

This makes it look essentially the same as the problem in $D \neq 7$. The extra factor of 2 in front of the F^2 term is compensated by the fact that the classical electric field has an extra $1/\sqrt{2}$: in this case

$$F_{rt} = \frac{1}{2}\partial_r H^{-1}.$$

However, it should be noted that this dualization is sufficient only for analyzing spherically symmetric fluctuations. A more careful treatment would be required to study the higher partial waves.

In studying the propagation of fixed scalars, special care needs to be taken to account for mixing with the gravitational field. This mixing can be traced to the fact that λ couples to background electric field of the black hole. Fortunately, the methods for disentangling this mixing have been developed in [35]. We now follow the steps outlined there.

We write the general spherically symmetric metric as

$$ds^2 = -e^{2A}dt^2 + e^{2B}dr^2 + r^2 e^{2U}d\Omega_{D-2}^2. \tag{2.7}$$

where we keep U constant and vary A and B. The electric field is then given by

$$F^{rt} = \frac{Qe^{-A-B-(D-2)U-\beta\lambda}}{r^{D-2}}. \tag{2.8}$$

The gravitational equations that follow from (2.1) are

$$R_{\mu\nu} - \frac{1}{2}\partial_\mu\lambda\partial_\nu\lambda - e^{\beta\lambda}(2F_{\mu\rho}F_{\nu\delta}g^{\rho\delta} - \frac{1}{D-2}g_{\mu\nu}F^2) = 0. \tag{2.9}$$

In particular, the 'rt' equation reads

$$-(D-2)r^{-1}(1+rU')\dot{B} + \frac{1}{2}\dot{\lambda}\lambda' = 0. \tag{2.10}$$

Varying, we find

$$\delta B = \frac{r\lambda_0'}{2(D-2)(1+rU')}\delta\lambda. \tag{2.11}$$

Varying the angular equation we get

$$\delta A' - \delta B' = -\frac{2\delta B}{r(1+rU')}[-r^2U'' - (D-2)rU' - (D-3)] +$$
$$+\frac{2\beta Q^2}{(D-2)(1+rU')r^{2D-5}}e^{-2(D-3)U-\beta\lambda_0}. \tag{2.12}$$

The fixed scalar equation is

$$\frac{1}{\sqrt{-g}}\partial_m(g^{mn}\sqrt{-g}\partial_n\lambda) + \beta e^{\beta\lambda}F^2 = 0.$$

(2.13)

Varying this and inserting the above expressions for the metric fluctuations, we obtain the general formula for the fixed scalar potential:

$$V(r) = -\frac{2(D-1)^2(D-3)^2 Q^2}{r^2[(D-1)Q+(D-2)(D-3)r^{D-3}]^2}.$$

(2.14)

We define the radius R in terms of the charge Q through $R^{D-3} = \frac{2Q}{D-3}$. Then the fixed scalar fluctuations in D dimensions obey the equation

$$[r^{-(D-2)}\partial_r r^{D-2}\partial_r + \omega^2(1+\frac{R^{D-3}}{r^{D-3}}) -$$

$$-\frac{2(D-1)^2(D-3)^2 R^{2(D-3)}}{r^2[(D-1)R^{D-3}+2(D-2)r^{D-3}]^2}]\delta\lambda = 0.$$

(2.15)

It remains to find an approximate solution of this equation for low energies and derive the absorption cross-section. As in previous work [32, 33, 34, 35], we divide space into three regions and match. These regions are the near region, where $r \ll R$, the far region, where $r \gg R$ and the intermediate region where the ω term can be neglected. In the near region the equation is

$$\left[\rho^{-(D-2)}\partial_\rho\rho^{D-2}\partial_\rho + \frac{(\omega R)^{D-3}}{\rho^{D-3}} - \frac{2(D-3)^2}{\rho^2}\right]\lambda_I = 0 \ .$$

(2.16)

where $\rho = \omega r$. Letting

$$\rho = Az^{2/(5-D)} \ , \qquad A^{D-5} = \frac{4(\omega R)^{D-3}}{(D-5)^2} \ ,$$

(2.17)

we find

$$\lambda_I = z^{(D-3)/(D-5)}H_\nu(z),$$

(2.18)

where

$$\nu = \frac{3(D-3)}{D-5} \ .$$

(2.19)

In the intermediate region the ω term is irrelevant and we get the solution

$$\lambda_{II}(r) = \frac{Br^{D-3}}{(D-1)R^{D-3} + 2(D-2)r^{D-3}}.$$ (2.20)

Matching with the near region, we obtain

$$B = [\frac{4}{(D-5)^2}]^{(3-D)/(D-5)} \frac{(D-1)\Gamma(\nu)2^\nu}{\pi}(\omega R)^{-2(D-3)/(D-5)}.$$ (2.21)

In the far region we have the equation

$$\left[r^{-(D-2)}\partial_r r^{D-2}\partial_r + \omega^2\right]\lambda_{III} = 0.$$ (2.22)

Its solution is

$$\lambda_{III} = C\rho^{-\mu}J_\mu(\rho),$$ (2.23)

where $\mu = (D-3)/2$. Matching to the intermediate region, we get

$$C = \frac{2^\mu\Gamma(\mu+1)}{2(D-2)}B.$$ (2.24)

The invariant flux is given by

$$\mathcal{F} = (1/2i)(\lambda^*\partial_r r^{D-2}\lambda - c.c.).$$ (2.25)

Taking the ratio of the flux at the horizon to the the incoming part of the flux at infinity, we get the absorption probability

$$P = \frac{4}{|C|^2}\frac{(D-5)}{2}A^{D-3},$$

which translates into

$$P = (D-5)\left[\frac{4}{(D-5)^2}\right]^{3(D-3)/(D-5)} \times$$
$$\times \frac{(D-2)^2}{(D-1)^2}\frac{8\pi^2}{2^{2(\mu+\nu)}(\Gamma(\nu))^2(\Gamma(\mu+1))^2}(\omega R)^{\frac{(D-3)(D+1)}{D-5}}.$$ (2.26)

The s-wave absorption cross-section is given by

$$\sigma = \frac{(2\sqrt{\pi})^{D-3}\Gamma\left(\frac{D-1}{2}\right)}{\omega^{D-2}}P.$$

Thus, we find

$$\sigma = (D - 5)[\frac{4}{(D-5)^2}]^{3(D-3)/(D-5)} \times$$

$$\times \frac{(D-2)^2}{(D-1)^2} \frac{8\pi^2(2\sqrt{\pi})^{D-3}}{2^{2(\mu+\nu)}(\Gamma(\nu))^2\Gamma(\mu+1)} R^{\frac{(D-3)(D+1)}{D-5}} \omega^{\frac{5D-13}{D-5}}. \tag{2.27}$$

Let us exhibit the scaling of the cross-section with the number of branes and the energy:

$$\sigma \sim N^{\frac{D+1}{D-5}} \omega^{\frac{5D-13}{D-5}}. \tag{2.28}$$

For N coincident D3 branes, which correspond to the $D = 7$ black hole, we find

$$\sigma_{D3} \sim \kappa_{10}^4 N^4 \omega^{11}. \tag{2.29}$$

For N coincident M5 branes, which correspond to the $D = 6$ black hole, we find

$$\sigma_{M5} \sim \kappa_{11}^{14/3} N^7 \omega^{17}. \tag{2.30}$$

For N coincident M2 branes, which correspond to the $D = 9$ black hole, we find

$$\sigma_{M2} \sim \kappa_{11}^{10/3} N^{5/2} \omega^8. \tag{2.31}$$

Let us now try to interpret these results in terms of the worldvolume theory on the stack of branes. The first question to ask is: what is the gauge theory operator $\mathcal{O}$ that couples to the fixed scalar fluctuations? The answer is essentially contained in eq. (2.3) which shows the origin of the fixed scalar λ. For $D = 7$, corresponding to the stack of 3-branes, we see that λ does not involve the dilaton, but only the volume V of the 3-branes. Now, in the case of M-branes, there is no dilaton in any case, so λ can only involve this volume. Thus in all cases of interest, λ corresponds to dilatation of the branes. The size of the branes is obviously governed by the trace of the graviton polarized along the branes. The worldvolume coupling of the graviton is given by

$$\int d^4x \frac{1}{2} h_{\mu\nu}(x) T^{\mu\nu}(x), \tag{2.32}$$

where $T^{\mu\nu}$ is the stress-energy tensor of the field theory on the branes.

Now, consider the trace h^μ_μ. This is the mode corresponding to the fixed scalar λ. Its dual operator T^μ_μ decouples in the conformal limit. It follows that, for D3 branes, h^μ_μ does not couple to a marginal (dimension 4) operator. We may deduce the leading operator to which it couples from the well-known structure of the DBI action [37, 38, 39]. To order F^4, we have

$$S_{\text{DBI}} = \frac{1}{4g^2_{YM}} \int d^4x \left[\text{Tr} F^2_{\mu\nu} - (2\pi\alpha')^2 \mathcal{O}_8 + \ldots \right].$$ (2.33)

The operator[2]

$$\mathcal{O}_8 = \frac{1}{3} \text{Tr}(F_{\mu\nu}F_{\rho\nu}F_{\mu\lambda}F_{\rho\lambda} + \frac{1}{2}F_{\mu\nu}F_{\rho\nu}F_{\rho\lambda}F_{\mu\lambda} -$$
$$-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}F_{\rho\lambda}F_{\rho\lambda} - \frac{1}{8}F_{\mu\nu}F_{\rho\lambda}F_{\mu\nu}F_{\rho\lambda})$$ (2.34)

has bare dimension 8 and obviously breaks conformal invariance. Thus, the trace of the stress-energy tensor calculated from this term is also of dimension 8, i.e. the lowest dimension coupling of the fixed scalar to the worldvolume is of the form

$$\int d^4x \, h^\mu_\mu \mathcal{O}_8 (2\pi\alpha')^2 \, .$$ (2.35)

The leading contribution to the 2-point function $\langle \mathcal{O}_8(x)\mathcal{O}_8(0)\rangle$ is a 3-loop diagram, which scales as

$$\langle \mathcal{O}_8(x)\mathcal{O}_8(0)\rangle \sim \frac{N^2 \left(Ng^2_{YM}\right)^2}{x^{16}}.$$ (2.36)

Let us compare this scaling to the absorption cross-section (2.29). Using once again the relation (1.14), performing the Fourier transform and isolating the discontinuity, we find that the absorption cross-section should behave as

$$\sigma_{abs} \sim N^4\kappa^4\omega^{11}.$$ (2.37)

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[2]We have not exhibited the dependence of this operator on the scalars and the fermions. We believe that these extra terms are determined by supersymmetry.

This has precisely the same scaling as the absorption cross-section found in (2.29). We have shown that the exact dimension of the operator $\mathcal{O}_8$, which the fixed scalar couples to, is in fact equal to 8. Thus, its anomalous dimension vanishes. We have, therefore, found a situation where gravity gives us a "proof" of a non-renormalization theorem for an operator in the worldvolume theory. We believe that in the gauge theory this theorem follows from the existence of the supersymmetric DBI action, and from the fact that insertions of $\mathcal{O}_8$ can be obtained by differentiating the path integral with respect to α'.

While our route towards the operator $\mathcal{O}_8$ involved using the DBI action, which breaks conformal invariance, the operator itself is expected to be one of the chiral operators of the $\mathcal{N} = 4$ SYM theory. This is required by the statement that the chiral operators are in one to one correspondence with the massless modes of type IIB supergravity [2, 40, 3]. Chiral operators involving trF^4 have indeed been found in [41]. From the form of the fixed scalar equation in the throat region (2.16), we find that the AdS mass-squared of the corresponding state is

$$m^2 = 32/R^2.$$

Thus, we believe that $\mathcal{O}_8$ should be identified with the $k = 0$ (the $SO(6)$ singlet) state in the tower

$$m^2 = (k + 4)(k + 8)/R^2,$$

which appears in type IIB supergravity on $AdS_5 \times S^5$ [42].

A parallel analysis may be performed also for the coincident M5 and M2 branes. In all cases we find that h_μ^μ is the fixed scalar field. From the results (2.30,2.31) we can then read off the dimension of the trace T_μ^μ on the worldvolume of the branes. Taking the Fourier transform, we find a dimension 12 operator on the 6-dimensional worldvolume of the M5-branes, and a dimension 6 operator on the 3-dimensional worldvolume of the M2-branes. Thus we see that in all three cases, the dilatation symmetry is explicitly broken by an operator having twice the marginal dimension.

For the M-branes, we do not currently have an explicit worldvolume action. so the fixed scalar absorption calculations can be taken as a prediction about the form of symmetry breaking operators in these theories.

Although the theories living on stacks of branes do break conformal symmetry. they are not of much help in our search for string duals of realistic field theories. This is because conformal symmetry is broken at energies where the gauge fields living on the branes no longer decouple from the closed string modes in the bulk. At such energies these theories are not really local 4-dimensional field theories at all. What we would like is to find situations where we can take the decoupling limit, so that we are dealing with bona-fide 4-dimensional local gauge theories. and still manage to break conformal symmetry and some supersymmetry. We will now describe one way of achieving this goal.

Chapter 3

Branes at conifold singularities

3.1 Introduction

Until now we have been looking at stacks of D3-branes placed in flat 10-dimensional space. A natural generalization of this is to consider string backgrounds of the form $R^{3,1} \times Y_6$, that is, products of flat Minkowski space with Calaby-Yau manifolds (i.e. manifolds that are Ricci-flat and preserve some supersymmetry) [43, 44, 4, 45][1]. A simple class of such manifolds are *cones* over 5-dimensional Einstein manifolds. A manifold X^5 is known as an Einstein manifold (of positive curvature), if its Ricci tensor and metric obey the equation $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$ (for $\Lambda > 0$). A cone over such a manifold has the metric

$$ds^2 = dr^2 + r^2 ds_5^2, \tag{3.1}$$

where ds_5^2 is the metric on X^5. The cone is Ricci-flat only if X^5 is an Einstein manifold. X^5 is known as the base of the cone. Note that flat 6-dimensional space R^6 can also be thought of as a cone; it has the metric (3.1) with base S^5. Of course, this is the maximally supersymmetric example. Another example is provided by the so-called conifold [47, 4]. This is a manifold that can be described by the following

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[1]See [46] for a review.

equation in four complex variables:

$$\sum_{a=1}^{4} z_a^2 = 0. \tag{3.2}$$

That this space is a cone can be seen from the fact that eq. (3.2) is invariant under the rescaling of all complex variables. It is a 3-dimensional complex space, so it has 6 real dimensions. It turns out that the base of the cone is the well-known space $T^{1,1} = (SU(2) \times SU(2))/U(1)$ [48, 4]. In what follows we will need the structure of the manifold $T^{1,1}$. This is a compact five-dimensional Einstein manifold with topology $S^2 \times S^3$; it has a nontrivial two-cycle and a nontrivial three-cycle. The coordinates on $T^{1,1}$ are the angles ψ, θ_1, θ_2, ϕ_1, ϕ_2 with θ_1, $\theta_2 \in [0, \pi]$, ϕ_1, $\phi_2 \in [0, 2\pi]$, $\psi \in [0, 4\pi]$. We define the 1-forms

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}}, \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}},$$
$$g^3 = \frac{e^1 + e^3}{\sqrt{2}}, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}},$$
$$g^5 = e^5, \tag{3.3}$$

where

$$e^1 = -\sin\theta_1 d\phi_1, \quad e^2 = d\theta_1,$$

$$e^3 = \cos\psi\sin\theta_2 d\phi_2 - \sin\psi d\theta_2,$$

$$e^4 = \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2,$$

$$e^5 = d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2. \tag{3.4}$$

The metric on $T^{1,1}$ is

$$ds_5^2 = \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2. \tag{3.5}$$

The closed nonexact forms corresponding to the nontrivial cycles are

$$\omega_2 = \frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4), \quad \omega_3 = g^5 \wedge \omega_2. \tag{3.6}$$

The metric of the conifold (3.2) is given by (3.1) with ds_5^2 given by (3.5).

We will need yet another description of the conifold. We define the variables

$$z_{ij} = \frac{1}{\sqrt{2}} \sum_a \sigma_{ij}^a z_a,$$

(3.7)

where the σ^a are Pauli matrices for $a = 1, 2, 3$ and σ^4 is i times the unit matrix. The eq. (3.2) becomes

$$det(z_{ij}) = 0.$$

(3.8)

This can be solved by setting

$$z_{ij} = A_i B_j,$$

(3.9)

where A_i, B_j are unconstrained. In terms of the description of the conifold as a cone over $T^{1,1}$, the overall scale of the A's and B's corresponds to the radial variable of the cone. Once this overall scale is set to 1, what remains is a product of two $SU(2)$ orbits. This needs to be divided by a $U(1)$ that rotates the two doublets by opposite phases and clearly leaves (3.9) unchanged. Thus we see again that the conifold is a cone over $(SU(2) \times SU(2))/U(1) = T^{1,1}$. The isometry group of this space is $SU(2) \times SU(2) \times U(1)$.

Let us now imagine placing a stack of N D3-branes at the singularity of the conifold, that is, at $r = 0$ [4]. What is the resulting low-energy gauge theory? Let us begin with a single brane. The moduli (A_i, B_j) are charged under the gauge fields corresponding to open strings ending on such a brane; the charges are separate for each doublet. Thus we must have a $U(1) \times U(1)$ gauge theory. A careful analysis of the situation with N branes shows that they are described by an $SU(N) \times SU(N)$ gauge theory. This theory is conformal, and has $\mathcal{N} = 1$ SUSY with a $U(1)$ R-symmetry group. It has an exactly marginal superpotential given by

$$W = \varepsilon^{ij} \varepsilon^{kl} tr A_i B_k A_j B_l,$$

(3.10)

determined up to an overall normalization. The chiral fields (A_1, A_2) transform in the $(N, \bar{N})$ representation of the gauge group, and the fields (B_1, B_2) in the $(\bar{N}, N)$ representation.

In analogy to flat space, to obtain the supergravity dual of the theory on the stack of 3-branes, we make the usual warped ansatz for the metric:

$$ds^2 = h^{-1/2}(r)dx_\mu dx^\mu + h^{1/2}(r)(dr^2 + r^2 ds_5^2),\tag{3.11}$$

which is again solved by the harmonic function $h(r) = 1 + \frac{R^4}{r^4}$. Once again, we can take the throat limit on both sides. On the stack of branes, this results in the usual decoupling of the gauge degrees of freedom from the closed string modes, that is, the throat limit is described by the $\mathcal{N} = 1$ $SU(N) \times SU(N)$ SCFT. On the SUGRA side, taking the throat limit results in the geometry

$$ds^2 = \frac{r^2}{R^2}dx_\mu dx^\mu + \frac{R^2}{r^2}dr^2 + R^2 ds_5^2.\tag{3.12}$$

This is the product space $AdS_5 \times T^{1,1}$. We have arrived at the conclusion, obtained by Klebanov and Witten [4], that the superconformal $SU(N) \times SU(N)$ $\mathcal{N} = 1$ gauge theory is dual to type IIB superstring theory on $AdS_5 \times T^{1,1}$. The conformal symmetry is evident from the fact that the SUGRA background is still AdS. A detailed matching of SUGRA excitations with gauge theory operators confirms this conclusion [49].

We have now been able to find a string dual of a less supersymmetric field theory. However, this is still a conformal theory, so there is no RG flow and no confinement. But, as realized by Klebanov and a series of collaborators [5, 7, 8, 9], it is in fact possible to break conformal symmetry in this setup. As we saw, the space $T^{1,1}$ has nontrivial 2- and 3-cycles. Consider wrapping a D5-brane around the 2-cycle of $T^{1,1}$ [5]; the remaining $3 + 1$ spacetime dimensions are along the flat $R^{3,1}$. From the point of view of the $3 + 1$ dimensions this D5-brane looks like a D3-brane, and will change the rank of the gauge group. But, because it is wrapped around one

of the two S^2s of $T^{1,1}$, open strings emanating from this brane are charged only under one of the factors of the gauge group. Such a wrapped D5-brane is known as a fractional D3-brane. A careful analysis shows that it changes the gauge group to $SU(N+1) \times SU(N)$ [5]. Wrapping M such branes around the same S^2 makes the gauge group $SU(N+M) \times SU(N)$.

To find the SUGRA dual of this gauge theory [7, 8, 9], we again take the warped ansatz (3.1). Also, to get the correct number of wrapped D5-branes, we require

$$F_3 = \frac{1}{2} M \alpha' \omega_3.$$

It turns out that to find a solution of the equations of motion, we must turn on the NS-NS 2-form B_2. As a result, the D3-brane charge cannot be kept constant and begins to flow. The full solution is given by [8]

$$h(r) = \frac{R^4 + 2L^4(\log(r/r_0) + 1/4)}{r^4}, \quad B_2 = \frac{2L^2}{3} \log(r/r_0) \omega_2, \quad F_3 = \frac{2L^2}{9 g_s} \omega_3,$$

$$F_5 = (\partial_r h^{-1}) d^4 x \wedge dr + \frac{R^4}{27} g^1 \wedge \ldots \wedge g^5, \quad \Phi = C = 0, \quad (3.13)$$

where $h(r)$ is the warp factor in the metric (3.1) and the radii R, L are given by

$$L^2 = \frac{9}{4} g_s M \alpha', \quad R^4 = \frac{27}{4} g_s N \pi (\alpha')^2. \tag{3.14}$$

This is the Klebanov-Tseytlin solution [8]. Note that in addition to the string scale α', the solution involves an arbitrary scale r_0. This scale is related to the confinement scale; we shall see the precise relation in the next section. As we flow toward the IR, the solution becomes singular at a radius $r = r_s$ where $h(r_s) = 0$, and is only reliable away from the singularity, or for $r \gg r_s$. In this region the curvature R satisfies $\alpha' R \sim \frac{1}{g_s M} \log(r/r_0)^{-3/2} \ll 1$, so supergravity is a good approximation to the dual gauge theory. Importantly, the KT solution is *not* asymptotically AdS in the UV, ot at large r: the warp function h differs from the AdS warp function by a logarithmic factor. Because the difference is only logarithmic, though, there is hope that some of the methods developed for AdS/CFT can be applied to the KT

background; we will see that this is in fact the case. For now let us note that the fact that the SUGRA background is non-AdS indicates that the dual field theory is no longer conformal.

A striking feature of the solution is that the D3-brane charge is scale dependent, namely

$$N_{eff}(r) = \frac{1}{(4\pi^2\alpha')^2} \int_{T^{1,1}} \tilde{F}_5 = N + \frac{3}{2\pi}g_s M^2 \log(r/r_0). \tag{3.15}$$

As shown in [9], this logarithmic running of the effective number of colors corresponds on the gauge theory side to a cascade of Seiberg duality transformations [50]. The basic point is that the field theory has two coupling constants, $1/g_1^2$ and $1/g_2^2$, one for each gauge group factor. The matching between the gauge coupling constants and corresponding SUGRA moduli is given by [4]

$$\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{\pi}{g_s e^\Phi}, \tag{3.16}$$

$$(\frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2})g_s e^\Phi = \frac{1}{2\pi\alpha'}(\int_{S^2} B_2) - \pi. \tag{3.17}$$

Since the dilaton is constant, we see that the sum of the two couplings does not run. However, because B_2 is turned on, the difference of the couplings does run, so they flow in opposite directions. If we follow the RG flow, there is a scale at which the $SU(N+M)$ coupling g_1 diverges. To continue past the infinite coupling, we perform an $\mathcal{N} = 1$ Seiberg duality transformation on this group factor. The $SU(N+M)$ gauge factor has $2N$ flavors in the fundamental representation (because of the two pairs of chiral fields (A_i, B_i) in an N-dimensional representation of the other gauge factor). Under a Seiberg duality transformation [50], this becomes an $SU(2N - (N+M)) = SU(N-M)$ gauge group. In this way, we get an $SU(N) \times SU(N-M)$ gauge theory, with the same properties as the one that we started with, but with the ranks of both gauge groups shifted by M. This theory undergoes the same kind of RG flow, the coupling of the larger gauge group diverges, and we have to perform Seiberg duality again, and so on. Thus this gauge theory undergoes a cascade of

Seiberg dualities, each of which changes the rank of each of the gauge factors by M. As a result, the effective number of colors in the theory flows with scale, and this is reflected in the behavior of $\tilde{F}_5$ in the SUGRA solution.

The process described above has to stop eventually, as we "run out of colors". And indeed, as the RR 5-form field strength flux approaches 0, the KT solution runs into a naked singularity. Normally, this might signal that SUGRA is no longer a good approximation to string theory at this point. Remarkably, the singularity of the KT solution can be resolved *in supergravity*, and a SUGRA solution that is nonsingular at all scales can be obtained [9]. It turns out that the right thing to do is to replace the conifold, (3.2), by the deformed conifold [48, 9], given by the equation

$$\sum_{a=1}^{4} z_a^2 = \epsilon^2. \tag{3.18}$$

The motivation for this comes from the fact that chiral symmetry is expected to be broken in the dual field theory. Eq. (3.18) is a natural way of breaking this $U(1)$ symmetry at low energies, while maintaining an effective symmetry at high energies. The deformed conifold described by eq. (3.18) has the metric [48, 51, 52]

$$ds_6^2 = \frac{1}{2}\epsilon^{4/3} K(\tau)\left[\frac{1}{3K^3(\tau)}(d\tau^2 + (g^5)^2) + \cosh^2(\tau/2)((g^3)^2 + (g^4)^2) + \right.$$
$$\left. + \sinh^2(\tau/2)((g^1)^2 + (g^2)^2)\right], \tag{3.19}$$

where

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh\tau}. \tag{3.20}$$

τ is a dimensionless radial variable. For large τ we may introduce a dimensionful radial coordinate r via

$$r^2 = \frac{3}{2^{5/3}}\epsilon^{4/3}e^{2\tau/3}. \tag{3.21}$$

In terms of this coordinate, $ds_6^2 \to dr^2 + r^2 ds_{T^{1,1}}^2$.

To find a SUGRA solution corresponding to branes moving on a deformed coni-
fold, we once again take the warped ansatz [9]

$$ds^2 = h^{-1/2}(\tau)dx_\mu dx^\mu + h^{1/2}(\tau)ds_6^2. \qquad (3.22)$$

Solving the SUGRA equations of motion, the metric warp factor $h(\tau)$ is found to be

$$h(\tau) = (g_s M \alpha')^2 2^{2/3} \epsilon^{-8/3} I(\tau), \qquad (3.23)$$

where

$$I(\tau) = \int_\tau^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3}. \qquad (3.24)$$

This is the Klebanov-Strassler, or KS solution [9]. In the UV, that is, for large τ,
it approaches the KT solution. At small τ we have $I(\tau) \to a_0$, where $a_0 \approx 0.7$. In
other words, the SUGRA solution described by (3.22) is nonsingular and has small
curvature everywhere; so it correctly describes the dual field theory at all scales.
The solution has no horizon in the IR. Consider now a Wilson contour positioned at
fixed τ, and calculate the expectation value of the Wilson loop. The minimal area
surface bounded by the loop will tend toward smaller τ. Because the coefficient of
$dx_\mu dx^\mu$ in the metric is finite at $\tau = 0$, a fundamental string with a surface extending
down to $\tau = 0$ will have finite tension, so the Wilson loop will obey the area law.

The Klebanov-Strassler solution is thus an example of a SUGRA background
dual to a confining $\mathcal{N} = 1$ SUSY gauge theory. So far, it is perhaps the closest
thing we have to a SUGRA dual of a realistic field theory. We now turn to an
investigation of various aspects of the correspondence between this gauge theory
and its supergravity dual.

3.2 Field theory correlators from the KT solution

Our purpose in this section, based largely on the paper [53], is to extend the stan-
dard AdS/CFT procedure for extracting gauge theory correlation functions from

supergravity to the KT background and its field theory dual, using as an example the massless scalar and its dual dimension 4 operator. We will outline that calculation and indicate how to use the same method to derive correlation functions of other field theory operators from this supergravity background.

Let us first recall how one extracts gauge theory correlation functions from the dual supergravity background in standard AdS/CFT. We follow the method of [2]. As explained in the Introsuction, for every SUGRA field ϕ there is a corresponding gauge theory operator $\mathcal{O}$ such that a term $W[\phi] = \int d^4x \phi(x)\mathcal{O}(x)$ can be added to the gauge theory action. The gauge theory/SUGRA correspondence then states

$$\left\langle e^{-W[\phi(x)]} \right\rangle = e^{-S[\phi(x)]}, \tag{3.25}$$

where $S[\phi(x)]$ is the classical SUGRA action evaluated on the field $\phi(x, r)$ that solves the supergravity equations of motion subject to the following boundary conditions: in the UV, i.e. for $r \to \infty$, $\phi(x, r) = r^\Delta \phi(x)$ where Δ is related to the dimension of the operator $\mathcal{O}$. We also require $\phi(x, r)$ be regular at the IR, i.e. for small r. In other words the classical SUGRA action evaluated on the classical solution $\phi(x, r)$ subject to these boundary conditions generates the connected gauge theory correlation functions of the operator $\mathcal{O}$.

In particular, suppose we want to calculate the two point function $\langle \mathcal{O}_4(x_1)\mathcal{O}_4(x_2) \rangle$ for an operator $\mathcal{O}_4$ corresponding to a minimal massless scalar ϕ propagating in the geometry (3.11), where ds_5^2 is the metric on some Einstein manifold X^5. The action for such a scalar is

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{g}[\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi] = \frac{V}{4\kappa^2} \int d^4x \int^\rho dr r^5[(\partial_r\phi)^2 + h(r)\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi], \tag{3.26}$$

where V is the volume of the X^5 and ρ is a UV cutoff to be taken to ∞ in the end. We have tacitly switched to Euclidean signature. The indices M, N run over the entire 10-dimensional space, the indices μ, ν over 4-dimensional Euclidean space.

The equation of motion resulting from this action is

$$(r^{-5}\partial_r r^5 \partial_r + h(r)\Box)\phi = 0. \tag{3.27}$$

Integrating by parts in the action (3.26), we get

$$S = \frac{V}{4\kappa^2} \int d^4x \int^{\rho} dr r^5 [-\phi(r^{-5}\partial_r r^5 \partial_r + h(r)\eta^{\mu\nu}\partial_\mu\partial_\nu)\phi + r^{-5}\partial_r(\phi r^5 \partial_r \phi)] =$$
$$= -\frac{V}{4\kappa^2}[\mathcal{F}(r)_{r=\rho} - \mathcal{F}(r)_{r=0}],$$

where $\mathcal{F}(r) = \phi(r)r^5\partial_r\phi(r)$ is the flux factor. We have used the equation of motion and the fact that there are no boundary terms from integrating by parts in the x^μ directions since the fields are assumed to vanish at 4-dimensional infinity. Going to momentum space, we find

$$S = \frac{V}{4\kappa^2} \int d^4k d^4q \phi_k \phi_q (2\pi)^4 \delta^{(4)}(k+q)\mathcal{F}_k, \tag{3.28}$$

where $\phi(x) = \int d^4k \phi_k e^{ikx}$ and

$$\mathcal{F}_k = [\tilde{\phi}_k r^5 \partial_r \tilde{\phi}_k]_0^{\rho}. \tag{3.29}$$

$\tilde{\phi}_k$ are momentum modes normalized to $\tilde{\phi}_k(\rho) = 1$. From (3.25), the corresponding 2 point function in momentum space is then

$$\langle \mathcal{O}_4(k)\mathcal{O}_4(q) \rangle = \frac{\partial^2 S}{\partial \phi_k \partial \phi_q} = (2\pi)^4 \delta^{(4)}(k+q)\frac{V}{4\kappa^2}\mathcal{F}_k. \tag{3.30}$$

Thus, to extract the 2 point function we need to solve the equations of motion for the momentum k Fourier mode of the field ϕ with the boundary conditions $\phi(\rho) = 1$, $\phi(r \to 0)$ regular, and find the flux factor $\mathcal{F}_k$. Note that ultimately, we are interested in terms nonanalytic in k, since the analytic terms correspond to contact terms in position space. However, there is a subtlety due to the fact that some of these contact terms diverge as $\rho \to \infty$, and need to be canceled by covariant counterterms [54, 55]. These covariant counterterms will generally change the prefactors in front of the 2-point functions [55]. In the particular case of the

minimal massless scalar, though, we will show that the prefactors are unchanged for both AdS and KT backgrounds[2].

In the standard AdS/CFT correspondence $h(r) = R^4/r^4$ and the equation of motion (3.27) in momentum space becomes

$$(r^{-5}\partial_r r^5 \partial_r - k^2 \frac{R^4}{r^4})\phi = 0.$$

Changing variables to $y = kR^2/r$ this is:

$$(y^3 \partial_y y^{-3} \partial_y - 1)\phi(y) = 0. \tag{3.31}$$

This is equivalent to a Bessel equation whose solution with the desired boundary conditions is

$$\phi(y) = \frac{y^2 K_2(y)}{k^2 \varepsilon^2 K_2(k\varepsilon)}$$

where $\varepsilon = R^2/\rho$ is a UV cutoff. This function has the small y expansion

$$\phi(y) = 1 - \frac{1}{4}y^2 - \frac{1}{16}y^4 \log y + \ldots \tag{3.32}$$

The logarithmic term gives the leading nonanalytic contribution, so that

$$\langle \mathcal{O}_4(k)\mathcal{O}_4(-k)\rangle \sim (k\varepsilon)^4 \log(k\varepsilon), \tag{3.33}$$

or

$$\langle \mathcal{O}(x_1)\mathcal{O}_4(x_2)\rangle \sim \frac{1}{|x_1 - x_2|^8}. \tag{3.34}$$

Let us note at this point that this is exactly the same correlator as eq. (1.15) derived in section 1.2 from the absorption cross-section of a minimal massless scalar. This is not surprising, since we are solving exactly the same equation (1.17) in both cases; the difference is that for the absorption calculation, we needed to solve the equation in the full (asymptotically flat) D3-brane background, whereas now it is enough to solve it in the near-horizon limit, where the background is an AdS space. The point

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[2]For a discussion of the AdS case see [6, 56].

is that all the information from the asymptotically flat region is encapsulated in the boundary conditions at the boundary of AdS.

Turning now to the Klebanov-Tseytlin background, our strategy will be simply to repeat the above steps. Consider again the minimal massless scalar. Starting from the action (3.26) with $X^5 = T^{1,1}$, we arrive in the same way as before at the result (3.30). With the warp factor $h(r)$ defined as in (3.13), the mode $\phi_k(r)$ now satisfies the equation

$$[r^{-5}\partial_r r^5 \partial_r - A^2 k^2 \frac{r_s^4}{r^4} \log(r/r_s)]\phi(r) = 0, \tag{3.35}$$

where we have defined

$$r_s = r_0 e^{-1/4 - R^4/2L^4}, \quad A^2 = \frac{2L^4}{r_s^4}. \tag{3.36}$$

Changing variables to

$$y = \frac{Akr_s^2}{r}, \quad Y = Akr_s, \tag{3.37}$$

eq. (3.35) becomes

$$[y^3 \partial_y y^{-3} \partial_y - \log \frac{Y}{y}]\phi(y) = 0. \tag{3.38}$$

To find the 2-point function (3.30), we need to solve eq. (3.38) with appropriate boundary conditions. This equation is valid for $y \ll Y$. As $y \to Y$, we run into a singularity. Recall that we would like to impose the boundary condition that $\phi(y)$ is regular in the IR, i.e. for large y. The rigorous way of doing this would be to look at the full Klebanov-Strassler spacetime [9] which resolves the KT singularity. But the KS solution is rather complicated and one would have no hope of solving the equations analytically. Instead, note that for large enough k, i.e. at high energies, Y is a large number. Thus there is a region where $0 \ll y \ll Y$. If we can solve eq. (3.38) in this region, we can impose the boundary condition that ϕ be regular at large y. If $Y \gg 1$ then this boundary condition will mimic the correct one, whatever the details of singularity resolution are. Next, note that if we take $1/Y \ll y \ll Y$,

then $|\log y| \ll |\log Y|$ and eq. (3.38) reduces to

$$(y^3 \partial_y y^{-3} \partial_y - \log Y)\phi = 0. \tag{3.39}$$

This is Bessel's equation, just like (3.31). Now we take the solution that is regular at large y. This is the same solution as we needed in (3.32):

$$\phi_{IR} = B(1 - \frac{1}{4}y^2 \log Y - \frac{1}{16}y^4 \log^2 Y \log(\sqrt{\log Y}y) + \ldots), \tag{3.40}$$

where B is an arbitraty constant. In the UV, i.e. for sufficiently small y, we solve (3.38) by expanding in y, and treating the $\log(Y/y)$ term as a perturbation. Namely, we make the ansatz

$$\phi = \phi_0 + \phi_1 + \phi_2 + \ldots. \tag{3.41}$$

where

$$[y^3 \partial_y y^{-3} \partial_y]\phi_{n+1} = [\log(Y/y)]\phi_n, \quad \phi_{-1} = 0. \tag{3.42}$$

As before, we impose the boundary condition $\phi(0) = 1$, where we have already taken the UV cutoff to infinity. We find

$$\phi_{UV} = (1 - \frac{1}{4}y^2 \log \frac{Y}{y} + y^4[\frac{1}{48} \log^3 \frac{Y}{y} + \frac{1}{64} \log^2 \frac{Y}{y} + \frac{1}{128} \log \frac{Y}{y} + C_k] + \ldots) \tag{3.43}$$

where C_k is an undetermined constant. The information about the 2 point function is hidden in the constant C_k since all other parts of the above expression are analytic in k (note that Y/y doesn't depend on k). We will now match ϕ_{UV} to ϕ_{IR}. Let us first identify the overlap region. We said before that the solution (3.43) is valid for small y. By looking at this solution we see that it has the form of an expansion in $y^2 \log(Y/y)$, so we are allowed to use this solution when $y^2 \log(Y/y) \ll 1$. On the other hand, the condition for the validity of eq. (3.39) is $1/Y \ll y \ll Y$. We see that when Y is large, these conditions are compatible and there is an overlap region $1/Y \ll y \ll 1/\sqrt{\log Y}$. In this region we can drop the $\log y$ terms in (3.43) since $|\log y| \ll |\log Y|$. Matching (3.43) to (3.40) order by order, the first two terms match if we set $B = 1$. However, if we look at the terms multiplying y^4, we see that

ϕ_{UV} has a $\log^3 Y$ term, whereas the leading term in ϕ_{IR} is a $\log^2 Y \log \log Y$ term. We must now use the undetermined constant C_k to cancel this leading $\log^3 Y$ term. Thus, we find

$$C_k = -\frac{1}{48} \log^3 Y + \dots = -\frac{1}{48} \log^3 Akr_s + \dots \tag{3.44}$$

where we have kept only the leading nonanalytic term. Using equations (3.43,3.44) and (3.29,3.30), we are now ready to compute the 2 point function. It is of the form

$$\langle \mathcal{O}_4(k) \mathcal{O}_4(-k) \rangle \sim \frac{A^4 k^4 r_s^8}{\kappa^2} \log^3 Akr_s. \tag{3.45}$$

After Fourier transforming, this produces a position space 2-point function that behaves as follows:

$$\langle \mathcal{O}_4(x_1) \mathcal{O}_4(x_2) \rangle \sim g_s^2 M^4 \frac{\log^2(r_s^2 |x_1 - x_2|^2/(g_s M \alpha')^2)}{|x_1 - x_2|^8}. \tag{3.46}$$

The range of validity of this result is $Y \gg 1$, which using eqs.(3.36,3.14) translates into $k \gg r_s/(g_s M \alpha')$. The new scale $\Lambda \sim r_s/(g_s M \alpha')$ is the only scale that appears in the field theory correlation functions; this is the confinement scale. Our result for the 2-point function is valid at energies higher than this scale, i.e. in the deconfined phase.

Equations (3.45,3.46) are our first encounter with powers of the $\log(x)$ appearing in the numerator of the 2-point correlation function. The above derivation shows how the $\log(r)$ factor in the KT metric warp function translates into position (or momentum) space logarithms in the 4-dimensional field theory. These logarithms, with varying powers, will appear in all the correlation functions that we compute. We will discuss their interpretation in the concluding chapter.

The above calculation has both important similarities with, and interesting differences from the RG flow backgrounds investigated in [57]-[61]. On the one hand, the calculation follows, step by step, the method of holographic renormalization [54, 55, 56]: we first solve the equations of motion in the UV. The solution involves an undetermined coefficient which encodes all information about the 2-point function. To obtain this coefficient, we need input from the IR. The difference is mainly

that the KT metric is not asymptotically AdS, meaning that the dual gauge theory is non-conformal at arbitrarily high energies. As a result, even the extreme UV behavior of correlation functions such as (3.45,3.46) is nontrivial, and that is indeed what we will be interested in. In contrast, the geometries studied in [57]–[61] are asymptotically AdS in the UV; the breaking of conformal and R-symmetries are IR phenomena, achieved by either adding relevant operators to the AdS Lagrangian, or turning on VEVs of scalar fields. This fact gives our work a somewhat different flavor from that of [57]–[61]. The authors of those papers were concerned with the IR behavior of correlation functions, since in the models they considered the UV behavior is conformal. They found that to obtain a sensible IR behavior with the correct pole structure in momentum space, they needed to go beyond the naive AdS cutoff regularization [58, 59] and develop a holographic renormalization scheme [60, 61] (for a review, see [56]). This scheme involves adding covariant counterterms to the regularized SUGRA action to cancel divergent contact terms. In the UV, these counterterms do not change the qualitative behavior of "naive" 2-point functions obtained by simply throwing out the contact terms, though they may renormalize the numerical coefficients [55]. Since in our paper we are concerned with the UV behavior of correlators, and since holograhic renormalization does not qualitatively change that behavior, we will not be careful about including these covariant counterterms. Moreover, in one case of interest (the minimal massless scalar) we will argue that the numerical prefactor is not renormalized. In general, though, one must include covariant counterterms to obtain the exact correlators[3]. Another feature of the present work is that unlike in [58, 59, 60, 61], the fluctuation equations we derive are not exactly solvable, but we have developed an iteration and matching procedure that allows us to extract the leading high-energy behavior of the correlators.

Having said this, let us briefly address the issue of renormalization. In terms of the variable $z = Ar_s^2/r$, the on-shell action (3.28) needed to compute the 2-point

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[3] I am indebted to Kostas Skenderis for pointing this out to me.

function is proportional to

$$S \sim \phi(z)z^{-3}\partial_z\phi(z), \tag{3.47}$$

where the action is to be evaluated on the surface $z = \varepsilon$. Since $\phi(z) = \phi_0(1 - \frac{1}{4}k^2z^2\log z + C_k k^4 z^4 + \ldots)$, where ϕ_0 is determined by the boundary condition at $z = 0$, all terms in this action, starting from the leading divergent contact term $\phi_0^2 k^2/\varepsilon^2$, will be proportional to k^2. To cancel them, we need to introduce covariant counterterms that are local on the surface $z = \varepsilon$ and use the induced metric on that surface. The most divergent such counterterm involving two derivatives (i.e. a power of k^2) is $\sim k^2\phi(z)^2/(z^2\log z)$. We see that the leading contribution to this counterterm from the term $C_k k^4 z^4$ in the expansion of $\phi(z)$ is at order $z^2 \to 0$, so C_k does not get renormalized. Thus, in the case of the minimal massless scalar, holographic renormalization does not change the leading order behavior of the 2-point function, including the overall coefficient (which we do not explicitly derive here).

It is straightforward in principle to extend our method to modes other than the minimal massless scalar. Suppose first that we have succeded in isolating a single mode ϕ whose flucutation equation decouples from other modes. Then it is still true that after we perform the change of variables (3.37), in the region $1/Y \ll y \ll Y$ we can replace all logarithms by constants, obtaining exactly solvable equations. We will then choose the solutions of these equations that are regular at large y. At small y we can solve the equations by the same sort of iterative expansion as in (3.41,3.42) with appropriate boundary conditions. Again, we will find that all the terms in the expansion are analytic in k with the exception of an undetermined constant C_k. Matching the UV and IR solutions, we will find as a rule that to match the behavior of the IR solution, we will have to choose C_k so as to cancel the leading log of the UV solution. From this we can then extract the 2-point functions.

In some cases we will encounter the following situation: we are interested in obtaining the correlator for an operator $\mathcal{O}$ whose dual field ϕ couples to other SUGRA fluctuations collectively denoted by φ_i. Our task is then to solve the fluctuation equations for ϕ, φ_i subject to the boundary condition

$$\phi(x, r \to \infty) \sim \hat{\phi}(x), \quad \varphi_i(x, r \to \infty) \to 0, \tag{3.48}$$

where $\hat{\phi}$ denotes the boundary condition for the field ϕ. We will find that we can still perform the iterative expansion (3.41,3.42) and solve in the UV for the fields ϕ, φ_i with boundary conditions (3.48), but that the solution of the equations in the IR limit becomes too cumbersome. Nevertheless, emboldened by our experience with the diagonal modes, we will assume that as before, the arbitrary constant that will appear on our UV expansion must be chosen so as to cancel the leading log coefficient in the critical term when the IR limit is taken. This is an extremely plausible assumption that yields sensible results for the correlation functions; unfortunately, in such cases we are only able to compute the leading order correlator up to a numerical factor[4].

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[4]Note that this is related *not* to renormalization, but to the difficulty of solving the IR equations.

Chapter 4

Correlation functions of operators in the R-symmetry multiplet

4.1 The R-current and its dual vector

In this chapter, based largely on the paper [65], we will use the method developed in the previous chapter to calculate new correlation functions from the KT background. In particular, we consider the SUGRA modes dual to the gauge-theory R-current J_μ^R and the gauge theory energy-momentum tensor $T_{\mu\nu}$. By solving the appropriate fluctuation equations, we are able to extract the leading high-energy behavior of the 2-point functions $\langle J_\mu^R(k)J_\nu^R(-k)\rangle$ and $\langle T_{\mu\nu}(k)T_{\rho\sigma}(-k)\rangle$. Because R-symmetry and conformal symmetry are broken, these correlators are expected to have longitudinal and trace parts, respectively, in addition to the transverse parts present in CFT. Indeed, these parts are found. Because the operators J_μ^R and $T_{\mu\nu}$ belong to the same supermultiplet (the supercurrent), their 2-point functions should be related to each other by supersymmetry Ward identities [61]. While we do not check these identities in detail, the form of the correlators we find suggests that they are in fact satisfied. We will have more to say on this in the concluding chapter.

The cascading $SU(N + M) \times SU(N)$ gauge theory has a classical $U(1)$ R-symmetry that gets broken down to Z_{2M} at the quantum level. As pointed out

in [67], this quantum phenomenon of the gauge theory can be described classically in the supergravity dual. In our SUGRA solution, the R-symmetry corresponds to translation of the angular coordinate ψ. Naively, the solution (3.13) is invariant under this gauge symmetry. However, this is not exactly true, because of a subtlety involving the RR 3-form field strength F_3. The 3-form given in (3.13) comes from a 2-form potential

$$C_2 = \frac{1}{2} M \alpha' \psi \omega_2. \tag{4.1}$$

ψ is periodic with period 4π, so this C_2 is not single-valued as a function of ψ; but it is single-valued up to a gauge transformation. Under a translation $\psi \to \psi + \varepsilon$,

$$C_2 \to C_2 + \frac{1}{2} M \alpha' \varepsilon \omega_2. \tag{4.2}$$

As discussed in [67], a gauge transformation can only shift C_2 by an integer multiple of $\pi \alpha' \omega_2$, so $\psi \to \psi + \varepsilon$ is a symmetry if ε is an integer multiple of $2\pi/M$. Since ε is defined *mod* 4π, a Z_{2M} subgroup of $U(1)$ remains a symmetry of the solution. As usual, the global R-symmetry of the gauge theory becomes gauged in supergravity. As described in [67] this gauge symmetry is spontaneously broken via a Higgs mechanism, and the vector field dual to the gauge theory R-current acquires a mass[1]. In this section we will derive the equation of motion for this vector and compute its mass. We will then use the method of the previous section to compute the 2-point correlation function of the R-currents. The most general form of this correlation function allowed by the symmetries is

$$\langle J_\mu^R(k) J_\nu^R(-k) \rangle = A(k^2) \pi_{\mu\nu}(k) + B(k^2) \frac{k_\mu k_\nu}{k^2}, \tag{4.3}$$

where

$$\pi_{\mu\nu}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \tag{4.4}$$

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[1] For a supergravity description of spontaneous R-symmetry breaking and R-current correlators in the Coulomb branch of N=4 SYM see [66].

is the transverse projector in 4 dimensions. A and B are the form factors we would like to compute. Note that if R symmetry is conserved, $\partial \cdot J^R = 0$, so $B = 0$ in this case. Thus a nonzero B indicates R-symmetry breaking.

The natural metric ansatz for fluctuations of the gauge field is

$$ds^2 = h^{-1/2}(r)dx_\mu dx^\mu + h^{1/2}(r)(dr^2 + r^2[\frac{1}{9}\chi^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2]), \tag{4.5}$$

where, following [67], we have defined the 1-form

$$\chi = g^5 - 2A_i dx^i, \tag{4.6}$$

which is invariant under the gauge transformations

$$\psi \to \psi + 2\lambda, \quad A \to A + d\lambda. \tag{4.7}$$

The RR 3-form field strength varies as:

$$F_3 = \frac{2L^2}{9g_s}(g^5 + 2\partial_i\theta dx^i) \wedge \omega_2 = \frac{2L^2}{9}(\chi + 2W_i dx^i) \wedge \omega_2, \tag{4.8}$$

where

$$W_i = A_i + \partial_i\theta \tag{4.9}$$

is a gauge-invariant vector field. In the above formulae the index i ranges over the 5 dimensions (x^μ, r). To obtain self-consistent equations of motion, we must also vary the RR scalar C, and the RR 4-form C_4. The most general variation of the RR 4-form C_4 consistent with the symmetries of the problem is

$$\delta C_4 = K^0 g^1 \wedge \dots \wedge g^4 + K^1 \wedge g^5 \wedge dg^5 + K^2 \wedge dg^5 + K^3 \wedge g^5, \tag{4.10}$$

where the K^rs are r-forms. In what follows we will be considering the linearized equations of motion for the fluctuations W_i, θ, $K^r_{i_1\dots i_r}$ around the KT background. The relevant equations of motion are the self-duality condition for $\tilde{F}_5$, the Einstein equations, and the equations of motion for the RR scalar C and the RR 2-form C_2:

$$\delta\tilde{F}_5 = \delta \star \tilde{F}_5, \tag{4.11}$$

$$\delta R_{ix} = \delta(\frac{g_s}{4}F_{iPQ}F_\chi^{PQ} - \frac{g_s}{4}CH_{iPQ}F_\chi^{PQ} + \frac{g_s^2}{96}\tilde{F}_{iPQRS}\tilde{F}_\chi^{PQRS}). \tag{4.12}$$

$$\delta(d \star dC) = -g_s H_3 \wedge \delta \star (F_3 - CH_3). \tag{4.13}$$

$$\delta d \star (F_3 - CH_3) = g_s \delta F_5 \wedge H_3. \tag{4.14}$$

Let us turn to the self-duality equation first. We define the following forms:

$$d^0 x^{\mu\nu\rho\sigma} = \eta^{\mu\mu'}\eta^{\nu\nu'}\eta^{\rho\rho'}\eta^{\sigma\sigma'}\epsilon_{\mu'\nu'\rho'\sigma'} = -1, \quad dx^{\mu\nu\rho} = \eta^{\mu\mu'}\eta^{\nu\nu'}\eta^{\rho\rho'}\epsilon_{\mu'\nu'\rho'\sigma}dx^\sigma,$$

$$d^2 x^{\mu\nu} = \frac{1}{2}\eta^{\mu\mu'}\eta^{\nu\nu'}\epsilon_{\mu'\nu'\rho\sigma}dx^\rho \wedge dx^\sigma, \quad d^3 x^\mu = \frac{1}{6}\eta^{\mu\mu'}\epsilon_{\mu'\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma,$$

$$d^4 x = \frac{1}{24}\epsilon_{\mu\nu\rho\sigma}dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \tag{4.15}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor in 4 dimnsions, and $\eta_{\mu\nu}$ is the flat Minkowski metric. The following identities are helpful:

$$dg^5 \wedge dg^5 = -2g^1 \wedge \ldots \wedge g^4,$$

$$\star(dr \wedge g^1 \wedge \ldots \wedge g^4) = -\frac{12}{r^3 h^2}d^4 x \wedge g^5,$$

$$\star(dx^\mu \wedge g^1 \wedge \ldots \wedge g^4) = -\frac{12}{r^3 h}d^3 x^\mu \wedge dr \wedge g^5,$$

$$\star(dx^\mu \wedge dr \wedge g^5 \wedge dg^5) = \frac{3}{rh}d^3 x^\mu \wedge dg^5,$$

$$\star(dx^\mu \wedge dx^\nu \wedge g^5 \wedge dg^5) = -\frac{3}{r}d^2 x^{\mu\nu} \wedge dr \wedge dg^5. \tag{4.16}$$

With δC_4 given by (4.10), and using the identities (4.16), the variation $\delta\tilde{F}_5$ of the RR 5-form field strength is

$$\delta\tilde{F}_5 = d\delta C_4 + B_2 \wedge \delta F_3 = (dK^0 + 2K^1) \wedge g^1 \wedge \ldots \wedge g^4 + dK^1 \wedge g^5 \wedge dg^5 + (dK^2 - K^3) \wedge dg^5 + dK^3 \wedge g^5. \tag{4.17}$$

The variation of its dual is

$$\delta(\star\tilde{F}_5) = \star(d\delta C_4 + B_2 \wedge \delta F_3) + (\delta\star)\tilde{F}_5 =$$

$$= (-\frac{12}{hr^3}(dK^0 + 2K^1)_\mu - \frac{8}{9}\frac{R^4 + 2L^4 \log(r/r_0)}{h(r)r^3}W_\mu)d^3 x^\mu \wedge dr \wedge g^5 +$$

$$+(-\frac{12}{h^2 r^3}(dK^0 + 2K^1)_r - \frac{8}{9}\frac{R^4 + 2L^4 \log(r/r_0)}{h^2 r^3}W_r)d^4x \wedge g^5 +$$

$$+\frac{3}{hr}(dK^1)_{\mu r}d^3x^\mu \wedge dg^5 - \frac{3}{r}(dK^1)_{\mu\nu}d^2x^{\mu\nu} \wedge dr \wedge dg^5 +$$

$$+\frac{r}{3}(dK^2 - K^3)_{\mu\nu r}d^2x^{\mu\nu} \wedge g^5 \wedge dg^5 + \frac{rh}{3}(dK^2 - K^3)_{\mu\nu\rho}dx^{\mu\nu\rho} \wedge dr \wedge g^5 \wedge dg^5 -$$

$$-(\frac{hr^3}{12}(dK^3)_{\mu\nu\rho r}dx^{\mu\nu\rho} + \frac{2}{27}(R^4 + 2L^4 \log(r/r_0))W_\mu dx^\mu) \wedge g^1 \wedge \ldots \wedge g^4 +$$

$$+(\frac{h^2 r^3}{12}(dK^3)_{\mu\nu\rho\sigma}d^0x^{\mu\nu\rho\sigma} + \frac{2h}{27}(R^4 + 2L^4 \log(r/r_0))W_r)dr \wedge g^1 \wedge \ldots \wedge g^4, \quad (4.18)$$

where we have used the identities (4.16), and set the gauge $\theta = 0$. The terms involving W_i come form the variation $\delta \star$ of the Hodge dual, which depends on the metric. At this point, it is convenient to introduce a slightly unusual version of the 5-dimensional Hodge dual, $\star_5$. In this Hodge dual, 4-dimensional indices are raised with flat Minkowski metric, while the r-index is raised with $h^{-1}(r)$. Thus for example

$$\star_5 dx_\mu = \eta_{\mu\mu'}\epsilon^{\mu'}_{\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dr, \quad \star_5 dr = h^{-1}(r)\epsilon_{\mu\nu\rho\sigma}dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \quad (4.19)$$

etc. With this notation, we find that the self-duality condition (38) reduces to the following two equations:

$$K^3 = dK^2 + \frac{3}{r} \star_5 dK^1, \quad (4.20)$$

$$dK^3 + \frac{12}{hr^3} \star_5 (dK^0 + 2K^1) + \frac{8(R^4 + 2L^4 \log \frac{r}{r_0})}{9hr^3} \star_5 W = 0. \quad (4.21)$$

Without loss of generality, we can set $K^0 = 0$ and $K^2 = 0$. Then the above reduces to

$$K^3 = \frac{3}{r} \star_5 dK, \quad (4.22)$$

$$hr\partial_r \frac{1}{hr}(\partial_r K_\mu - \partial_\mu K_r) + h\partial_\nu(\partial_\nu K_\mu - \partial_\mu K_\nu) - \frac{8}{r^2}K_\mu - \frac{8(R^4 + 2L^4 \log(r/r_0))}{27r^2}W_\mu = 0, \quad (4.23)$$

$$h(\square K_r - \partial_r(\partial \cdot K)) - \frac{8}{r^2}K_r - \frac{8(R^4 + 2L^4 \log(r/r_0))}{27r^2}W_r = 0, \quad (4.24)$$

where we now denote the vector K^1 simply by K. We have separated the x^μ and r components of the equations of motion for K; $\partial \cdot K$ denotes the 4-dimensional divergence.

We now turn to the Einstein equations (4.12). The variation of the Ricci tensor in terms of the metric variation is

$$\delta R_{MN} = \Box_{10} h_{MN} + D_M D_N h_P^P - D_M D^P h_{PN} - D_N D^P h_{MP} - 2R_{MPSN} h^{PS} + R_M^P h_{PN} + R_N^P h_{MP}.$$

$$(4.25)$$

where $\Box_{10}$ is the 10-dimensional Laplace operator, $h_{MN} = \delta g_{MN}$, and all covariant derivatives, as well as raised indices, are taken with respect to the background metric. Plugging this into eq. (4.12) and using equations (4.5,4.8) as well as the equations of motion (4.22,4.23,4.24) for K^1 and K^3, this becomes

$$\frac{1}{hr^7}\partial_r hr^7 (\partial_r W_\mu - \partial_\mu W_r) + h\partial_\nu(\partial_\nu W_\mu - \partial_\mu W_\nu) - \frac{8L^4}{hr^6}W_\mu -$$

$$-16\frac{(R^4 + 2L^4 \log(r/r_0))^2}{h^2 r^{10}}(W_\mu + \frac{27}{R^4 + 2L^4 \log(r/r_0)}K_\mu) = 0, \qquad (4.26)$$

$$h(\Box W_r - \partial_r(\partial \cdot W)) - \frac{8L^4}{hr^6}(W_r - \frac{3C}{2r}) -$$

$$-16\frac{(R^4 + 2L^4 \log(r/r_0))^2}{h^2 r^{10}}(W_r + \frac{27}{R^4 + 2L^4 \log(r/r_0)}K_r) = 0, \qquad (4.27)$$

where again we have separated the x^μ and r components.

In general, whenever we have Lorentz-invariant equations of motion involving a vector mode A_μ[2], they can be separated into transverse and longitudinal components $\tilde{A}_\mu$ and $\partial \cdot A$ by setting

$$A_\mu = \tilde{A}_\mu + \frac{\partial_\mu(\partial \cdot A)}{\Box}. \qquad (4.28)$$

The transverse mode $\tilde{A}_\mu$ that satisfies $\partial \cdot \tilde{A} = 0$ then decouples from all scalar fluctuations and can only couple to other transverse vectors.

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[2] By "vectors" we mean vectors with respect to the 4-dimensional Lorentz group; the r-components A_r are scalars with respect to that group.

In the present case we have two vector fields K_μ, W_μ. Defining their transverse parts as above, we find that they only couple to each other, and satisfy the equations:

$$(\frac{1}{hr^7}\partial_r hr^7 \partial_r + h\square)\tilde{W}_\mu - \frac{8L^4}{hr^6}\tilde{W}_\mu -$$

$$-16\frac{(R^4 + 2L^4\log(r/r_0))^2}{h^2 r^{10}}(\tilde{W}_\mu + \frac{27}{R^4 + 2L^4\log(r/r_0)}\tilde{K}_\mu) = 0, \qquad (4.29)$$

$$(hr\partial_r \frac{1}{hr}\partial_r + h\square)\tilde{K}_\mu - \frac{8}{r^2}\tilde{K}_\mu - \frac{8(R^4 + 2L^4\log(r/r_0))}{27r^2}\tilde{W}_\mu = 0. \qquad (4.30)$$

These coupled equations can be diagonalized by taking the following linear combinations:

$$W^1 = W - \frac{54}{hr^4}K, \quad W^2 = W + \frac{27}{hr^4}K. \qquad (4.31)$$

Then the transverse components $\tilde{W}^1$ and $\tilde{W}^2$ satisfy the equations:

$$(\frac{1}{hr^7}\partial_r hr^7 \partial_r + h\square - \frac{4L^8}{r^2(R^4 + 2L^4(\log(r/r_0) + \frac{1}{4}))^2})\tilde{W}^1_\mu = 0, \qquad (4.32)$$

$$(\frac{1}{hr^7}\partial_r hr^7 \partial_r + h\square -$$

$$-4\frac{6R^8 + 3R^4 L^4 + (24R^4 L^4 + 6L^8)\log(r/r_0) + 24L^8 \log^2(r/r_0)}{r^2(R^4 + 2L^4(\log(r/r_0) + \frac{1}{4}))^2})\tilde{W}^2_\mu = 0. \qquad (4.33)$$

By inspection we see that W^2 is massive in the $AdS_5 \times T^{1,1}$ limit where we take $L = 0$; the mode we're interested in is W^1. This is the Goldstone vector that acquires a mass, corresponding to the spontaneous breaking of R-symmetry.

The authors of ref. [57] made a general prediction for the mass of a vector associated with such symmetry breaking. We can now compare that prediction to our result. Eq. (193) of [57] reads

$$(e^{-2T}\partial_q e^{2T}\partial_q + e^{-2T}\square + 2\frac{\partial^2 T}{\partial q^2})\tilde{V}_\mu = 0, \qquad (4.34)$$

where V_μ is related to W^1_μ by a rescaling and q, T are such that the reduced metric in 5 dimensions is

$$ds_5^2 = dq^2 + e^{2T}dx_\mu dx^\mu. \qquad (4.35)$$

In terms of the r coordinate, the reduced KT metric (6) in 5 dimensions is

$$ds_5^2 = (h(r)r^4/R_0^4)^{5/6}(h^{1/2}(r)dr^2 + h^{-1/2}(r)dx_\mu dx^\mu), \qquad (4.36)$$

where R_0 is some reduction scale. Comparing this to (4.35) we get

$$dq = \frac{dr}{rR_0^{5/3}}(R^4 + 2L^4(\log(r/r_0) + 1/4))^{2/3}, \qquad (4.37)$$

$$e^{2T} = r^2 R_0^{-10/3}(R^4 + 2L^4(\log(r/r_0) + 1/4))^{1/3}. \qquad (4.38)$$

We now transform (4.34) to the r coordinate. To obtain agreement between the kinetic terms we also need to rescale:

$$V_\mu = (h(r)r^4/R_0^4)^{2/3}W_\mu^1. \qquad (4.39)$$

Plugging the above expressions in, we get

$$(\frac{1}{hr^7}\partial_r hr^7 \partial_r + h\square - \frac{4L^8}{r^2(R^4 + 2L^4(\log(r/r_0) + \frac{1}{4}))^2})\tilde{W}_\mu^1 = 0, \qquad (4.40)$$

which is precisely our eq.(4.32). In terms of the gauged supergravity conventions of ref. [57], the vector W^1 has picked up a mass

$$m^2 = -2\frac{d^2T}{dq^2} = \frac{4}{\alpha'(3\pi)^{3/2}}\frac{(g_s M)^2}{(g_s N)^{3/2}}, \qquad (4.41)$$

where we have used the relations (3.14).

Before proceeding to derive the remaining equations of motion for the scalar sector, let us first calculate the leading order 2-point functions $\langle \tilde{J}_\mu^{1,2}(x)\tilde{J}_\nu^{1,2}(x')\rangle$ for the transverse components of the gauge theory currents $J^{1,2}$ dual to the supergravity modes $W^{1,2}$ we have found. We follow the method outlined in the previous chapter. Using the change of variables (18,19), eqs. (52,53) in momentum space become

$$(\frac{y}{\log(Y/y)}\partial_y y^{-1} \log(Y/y)\partial_y - \frac{1}{y^2 \log^2(Y/y)} - \log(Y/y))\tilde{W}^1 = 0, \qquad (4.42)$$

$$(\frac{y}{\log(Y/y)}\partial_y y^{-1}\log(Y/y)\partial_y - \frac{24\log^2(Y/y) - 6\log(Y/y) + 1}{y^2\log^2(Y/y)} - \log(Y/y))\tilde{W}^2 = 0.$$
(4.43)

In the IR region $1/Y \ll y \ll Y$, these reduce to Bessel equations:

$$(y\partial_y y^{-1}\partial_y - \log Y)\tilde{W}_{IR}^1 = 0,$$
(4.44)

$$(y\partial_y y^{-1}\partial_y - \frac{24}{y^2} - \log Y)\tilde{W}_{IR}^2 = 0,$$
(4.45)

where we have also used $Y \gg 1$. Note that in going from eq. (4.42) to (4.44), the mass term $1/(y^2\log^2(Y/y))$ is left out since it becomes suppressed by a factor of $1/(\log Y)^2$. This is the term responsible for the anomalous dimension of the R-current, so we see that this anomalous dimension will not show up in our calculations, which are at leading order in high energy. If this term were included, it would modify the order of the Bessel function, and thus the power of k in the correlator. The solutions of these Bessel equations (4.44,4.45) that remain regular at large y are, up to a multiplicative constant

$$\tilde{W}_{IR}^1 \sim 1 + B_1 y^2 \log Y \log\log Y + \ldots$$
(4.46)

$$\tilde{W}_{IR}^2 \sim \frac{1}{y^4 \log Y} + \ldots + B_2 y^6 \log^4(Y) \log\log Y + \ldots$$
(4.47)

where we have only included the terms relevant to our matching. B_1, B_2 are constants whose exact value will not matter to us. In the UV region $y \ll 1$, we perform an iterative expansion similar to eqs. (23,24). We find

$$\tilde{W}_{UV}^1 \sim \frac{1}{2}\frac{2\log(Y/y) - 1}{\log(Y/y)} - \frac{1}{6}y^2 \log^2(Y/y) + C_1\frac{y^2}{\log(Y/y)} + \ldots$$
(4.48)

$$\tilde{W}_{UV}^2 \sim \frac{1}{y^4\log(Y/y)} - \ldots - \frac{1}{7200}y^6 \log^5(Y/y) + C_2 y^6 + \ldots$$
(4.49)

where again we omitted terms not relevant to the matching. Performing the matching, we find that, just as in the case of the minimal scalar, the UV expansions $\tilde{W}_{UV}^{1,2}$ have a higher power of $\log Y$ in the coefficient of the critical power of y than the IR functions that remain regular at large y. We must use the arbitrary constants C_1,

C_2 to cancel this leading log; these constants then encode the leading-order 2-point functions. Thus we have

$$C_1 = \frac{1}{6} \log^3 Y + \dots, \quad C_2 = \frac{1}{230400} \log^5 Y + \dots \tag{4.50}$$

Restoring the k-dependence of y and Y, we obtain

$$\tilde{W}_{UV}^1 \sim 1 + \frac{L^4 k^2}{3 r^2 \log(r/r_s)} \log^3(k/\Lambda) + \dots, \tag{4.51}$$

$$\tilde{W}_{UV}^2 \sim \frac{r^4}{\log(r/r_s)} + \dots + \frac{L^{20} k^{10} \log^5(k/\Lambda)}{7200 r^6} + \dots \tag{4.52}$$

where Λ is given by $\Lambda \sim r_s/(M\alpha')$. As usual, the momentum space 2-point functions are proportional to the lowest-order nonanalytic terms in k, so we have

$$\langle \tilde{J}_\mu^R(k) \tilde{J}_\nu^R(-k) \rangle \sim g_s^2 M^4 \pi_{\mu\nu}(k) k^2 \log^3(k/\Lambda), \tag{4.53}$$

$$\langle \tilde{J}_\mu^2(K) \tilde{J}_\nu^2(-k) \rangle \sim g_s^{10} M^{12} (\alpha')^8 \pi_{\mu\nu}(k) k^{10} \log^5(k/\Lambda), \tag{4.54}$$

where we have renamed $J^1 \equiv J^R$ since it is in fact the R-current. The transverse projector $\pi_{\mu\nu}$ is defined in (4.4). These translate into position space 2-point functions

$$\langle \tilde{J}_\mu^R(x) \tilde{J}_\nu^R(x') \rangle \sim g_s^2 M^4 (\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}) \frac{\log^2(\Lambda|x - x'|)}{|x - x'|^6}, \tag{4.55}$$

$$\langle \tilde{J}_\mu^2(x) \tilde{J}_\nu^2(x') \rangle \sim g_s^{10} M^{12} (\alpha')^8 (\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}) \frac{\log^4(\Lambda|x - x'|)}{|x - x'|^{14}}. \tag{4.56}$$

We now make a brief digression to compute yet another transverse current-current correlator; the purpose is to once again demontrate the general pattern of these calculations. First, we note[3] that if we vary the NS-NS 2-form as

$$B_2 \to B_2 + A_i dx^i \wedge g^5, \tag{4.57}$$

while leaving all other fields in the KT solution constant, the transverse components of the vector field A_i decouple from all other modes. Inserting the variation (4.57)

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[3]This observation is due to Igor Klebanov.

into the equation of motion (2) for B_2, and, as usual, expanding to linear order in A_i, we find that these transverse components $\tilde{A}_\mu$ obey the equations of motion

$$(r^{-3}\partial_r r^3 \partial_r + h\Box - \frac{8}{r^2})\tilde{A}_\mu = 0. \tag{4.58}$$

Performing once again the change of variables (18,19), this becomes

$$(y\partial_y y^{-1}\partial_y - \frac{8}{y^2} - \log(Y/y))\tilde{A}_\mu = 0. \tag{4.59}$$

The IR and UV expansions are obtained in the usual way:

$$\tilde{A}_{IR} \sim \frac{1}{y^2} + \ldots + By^4 \log^3 Y \log\log Y + \ldots \tag{4.60}$$

$$\tilde{A}_{UV} \sim \frac{1}{y^2} - \ldots - \frac{1}{192} y^4 \log^4(Y/y) + Cy^4 + \ldots \tag{4.61}$$

Again we see that in the matching region the UV expansion has a higher power of $\log Y$ in the critical term coefficient, and we use the constant C to cancel it. Thus

$$C = \frac{\log^4 Y}{192} + \ldots \tag{4.62}$$

With the usual transformations, we find a corresponding position space 2-point function

$$\langle \tilde{J}_\mu^A(x)\tilde{J}_\nu^A(x')\rangle \sim (\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box})\frac{\log^3(\Lambda|x-x'|)}{|x-x'|^{10}}. \tag{4.63}$$

The lesson from the above calculations is that, to leading order, the high energy behavior of the 2-point functions can be extracted from the UV iterative expansion alone; the matching with the IR solution always has the consequence that we must choose the undetermined constant C_k in the UV expansion in such a way as to cancel the leading log coefficient of the critical power of y in the IR limit. We will now use this shortcut to compute the longitudinal part of the R-current correlator $\langle J_\mu^R J_\nu^R\rangle$. First, we need to derive the remaining scalar equations of motion. Eqs. (4.13,4.14) yield

$$(r^{-5}\partial_r r^5 \partial_r + h\Box)C + \frac{16L^4}{3hr^5}(W_r - \frac{3C}{2r}) = 0, \tag{4.64}$$

$$(\partial \cdot W) = -\frac{108}{r^5 h^2}(K_r + \frac{R^4 + 2L^4 \log(r/r_0)}{27} W_r) - \frac{1}{2r}\partial_r(\frac{r}{h}(W_r - \frac{3C}{2r})). \quad (4.65)$$

Also, by taking divergences of eqs. (4.23,4.26) we find

$$hr\partial_r\frac{1}{hr}\partial_r(\partial \cdot K) - \frac{8}{r^2}(\partial \cdot K) -$$

$$-\frac{8(R^4 + 2L^4 \log(r/r_0))}{27r^2}(\partial \cdot W) - hr\partial_r\frac{1}{hr}\Box K_r = 0, \quad (4.66)$$

$$\frac{1}{hr^7}\partial_r hr^7\partial_r(\partial \cdot W) - \frac{8L^4}{hr^6}\partial \cdot W -$$

$$-16\frac{(R^4 + 2L^4 \log(r/r_0))^2}{h^2 r^{10}}(\partial \cdot W + \frac{27}{R^4 + 2L^4 \log(r/r_0)}\partial \cdot K) -$$

$$-\frac{1}{hr^7}\partial_r hr^7\Box W_r = 0. \quad (4.67)$$

Equations (4.64-4.67), along with eqs. (4.24,4.27) are the equations of motion in the scalar sector. The independent fields may be taken to be the scalars C, W_r, K_r; it is possible to check, as must of course be the case, that these six equations for three independent fields are consistent.

We now want to extract the longitudinal part of the $\langle J_\mu^R J_\nu^R \rangle$ correlator. As discussed above, we only need to solve the equations to the critical order in the UV expansion. Recall once again that in practice, this means that in the 0-th approximation, we drop all the '$\Box$' terms, since they scale as r^{-4}, whereas all other terms scale as r^{-2}. The '$\Box$' terms operating on the 0-th order solution are then included in the equations for the 1-st order solutions, etc. Since the vector dual to the R-current operator is W^1 as defined in eq. (4.31), we impose the boundary conditions

$$\partial \cdot W^1(r, x)(r \to \infty) \to \partial \cdot W^1(x), \quad \partial \cdot W^2(r, x)(r \to \infty) \to 0,$$

$$C(r, x)(r \to \infty) \to 0, \quad W_r(r, x)(r \to \infty) \to 0, \quad K_r(r, x)(r \to \infty) \to 0. \quad (4.68)$$

As usual, we transform to momentum space, and seek normalized solutions of the form

$$\partial \cdot W^1(r, k)(r \to \infty) \to 1, \quad \partial \cdot W^2(r, k)(r \to \infty) \to 0,$$

$$C(r,k)(r \to \infty) \to 0, \quad W_r(r,k)(r \to \infty) \to 0, \quad K_r(r,k)(r \to \infty) \to 0. \quad (4.69)$$

We are indeed able to find a solution with these boundary conditions. To first order in the UV expansion (i.e. iterating once), the solution is:

$$\partial \cdot W^1(r) = 1 - \frac{1}{3}\frac{k^2 L^4}{r^2}\log(r/r_s) + C_1\frac{1}{r^2\log(r/r_s)}, \quad \partial \cdot W^2(r) = -\frac{1}{72}\frac{k^2 L^4}{r^2}.$$

$$C(r) = -\frac{2}{9}\frac{L^4}{r^2} + \frac{1}{27}\frac{k^2 L^8}{r^4}\log^3(r/r_s) - \frac{2}{3}C_1\frac{L^4}{r^4}\log(r/r_s),$$

$$W_r(r) = -\frac{1}{3}\frac{L^4}{r^3}\log(r/r_s) + \frac{2}{9}\frac{k^2 L^8}{r^5}\log^3(r/r_s) + 2C_1\frac{L^4}{r^5}\log(r/r_s),$$

$$K_r(r) = \frac{2}{81}\frac{L^8}{r^3}\log^2(r/r_s) - \frac{4}{243}\frac{k^2 L^{12}}{r^5}\log^4(r/r_s) - \frac{4}{27}C_1\frac{L^8}{r^5}\log^2(r/r_s), (4.70)$$

where we have kept only the leading log terms at each power of r, and C_1 is the undetermined constant that contains the information we need. Looking at the solution (4.70), we see that in going to the IR limit $1/Y \ll y \ll Y$ with y, Y defined as in (3.36,3.37), to cancel the leading log in the critical term, we have to choose the undetermined constant

$$C_1 \sim k^2 L^4 \log^2(k/\Lambda), \quad (4.71)$$

which leads to the longitudinal momentum space R-current 2-point function

$$\langle J_\mu^\parallel(k) J_\nu^\parallel(-k)\rangle \sim g_s^2 M^4 k_\mu k_\nu \log^2(k/\Lambda). \quad (4.72)$$

Mulitplying the above by $k_\mu k_\nu$, we also get

$$\langle \partial \cdot J^R(k)\partial \cdot J^R(-k)\rangle \sim g_s^2 M^4 k^4 \log^2(k/\Lambda). \quad (4.73)$$

Thus, the 2-point function of the R-anomaly scalar $\partial \cdot J^R$ has a different leading-order logarithmic behavior from that of the minimal scalar (3.45) found in the previous section.

Unfortunately, the above analysis does not in itself allow us to determine the numerical value of the constant C_1; to do that, we would need to diagonalize the

fluctuation equations in the IR and match them in detail to the UV solution (4.70), which we are unable to do at present. Thus, by combining eqs. (4.53,4.72) we can write the total R-current 2-point function as

$$\langle J_\mu^R(k) J_\nu^R(-k)\rangle = g_s^2 M^4 (C_0 \pi_{\mu\nu}(k) k^2 \log^3(k/\Lambda) + C_1 k_\mu k_\nu \log^2(k/\Lambda)), \quad (4.74)$$

where we could in principle compute the prefactor C_0 exactly (though we do not bother do that here), but we have not been able to determine the prefactor C_1.

4.2   The EM tensor and the graviton

In this section we would like to compute the short-distance behavior of the field theory energy-momentum tensor 2-point function $\langle T_{\mu\nu}(x) T_{\rho\sigma}(x')\rangle$. This is an object of interest because in a conformal theory, the structure of this correlator is completely determined by conformal symmetry. Thus, any deviation from the CFT result will exhibit some of the structure of the breaking of conformal symmetry, and yield information about the flow of the beta function. More specifically, the most general form of the $\langle TT\rangle$ correlator allowed by translation invariance is

$$\langle T_{\mu\nu}(k) T_{\rho\sigma}(-k)\rangle = C(k^2) \pi_{\mu\nu\rho\sigma}(k) + D(k^2) \pi_{\mu\nu}(k) \pi_{\rho\sigma}(k), \quad (4.75)$$

where $\pi_{\mu\nu}$ is the transverse projector defined in eq. (4.4), and

$$\pi_{\mu\nu\rho\sigma} = \frac{1}{2}(\pi_{\mu\sigma}\pi_{\nu\rho} + \pi_{\mu\rho}\pi_{\nu\sigma}) - \frac{1}{3}\pi_{\mu\nu}\pi_{\rho\sigma} \quad (4.76)$$

is the transverse traceless projector. Our purpose is to compute the form factors C and D. Note that in a scale invariant theory, $T_\mu^\mu = 0$, and therefore $D = 0$; thus a nonzero D is an indication of the trace anomaly. Moreover, conformal symmetry dictates $C(k) \sim k^4$, so a nontrivial C also manifests the breaking of conformal symmetry.

The supergravity field dual to the EM-tensor operator $T_{\mu\nu}$ is the graviton γ_ν^μ along the brane directions, normalized with respect to the background metric (see

ref.[68]). In other words, we vary the 4-dimensional part of the metric as follows:

$$g_{\mu\nu} \to g_{\mu\nu} + g_{\mu\rho}\gamma^\rho_\nu. \tag{4.77}$$

The 2-point function is then

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(x')\rangle = \frac{\delta^2 S(\gamma, \phi_i)}{\delta\hat{\gamma}_{\mu\nu}(x)\delta\hat{\gamma}_{\rho\sigma}(x')}, \tag{4.78}$$

where γ denotes the graviton along the branes and ϕ_i denotes collectively all other supergravity fields, and the action S is evaluated at the solution to the linearized SUGRA equations of motion with the boundary conditions

$$\gamma^\mu_\nu(x, r \to \infty) \to \hat{\gamma}_{\mu\nu}(x), \quad \phi_i(x, r \to \infty) \to 0. \tag{4.79}$$

In what follows, we will not be careful about distinguishing upper and lower indices: rather, we will assume that all indices are raised and lowered with flat metric, and tacitly insert appropriate factors of $h(r)$ as needed.

Our task is to solve the linearized SUGRA equations of motion around the KT background with boundary conditions (4.79). The graviton $\gamma_{\mu\nu}$ couples to other fields, so we must include their fluctuations as well. To simplify the calculations somewhat, we will set $R = 0$ in the definition of $h(r)$ (see eq. 3.13). A self-consistent ansatz is:

$$ds^2 = h^{-1/2}(r)(\eta_{\mu\nu} + \gamma_{\mu\nu}(x, r))dx^\mu dx^\nu +$$
$$+h^{1/2}(r)(\gamma_{rr}(x, r)dr^2 + r^2[\frac{1}{6}(1 + \frac{1}{4}s_1(x, r))\sum_{i=1}^{4}(g^i)^2 + \frac{1}{9}(1 + s_2(x, r))(g^5)^2]),$$
$$B_2 = \frac{2L^2}{3}\log(r/r_0)(1 + \delta B(x, r))\omega_2,$$
$$C_4 = (\partial_r h^{-1})(1 + \delta C(x, r))d^4x, \quad \Phi = \delta\Phi(x, r) \tag{4.80}$$

with all other fields equal to their background values. The self-duality condition (1.6) for $\tilde{F}_5$ allows us immediately to solve for the field δC:

$$\delta C = \delta B + \frac{1}{2}(\gamma + \gamma_{rr} - s), \tag{4.81}$$

where we have defined the traces

$$\gamma = \delta^{\mu\nu}\gamma_{\mu\nu}, \quad s = s_1 + s_2. \tag{4.82}$$

We now turn our attention to the Einstein equations (1.3). Using the expansion (4.25) and looking at the '$\mu\nu$' and 'μr' components, we obtain

$$(h\Box + \partial_r^2 + \frac{5}{r}\partial_r)\gamma_{\mu\nu} + h\partial_\mu\partial_\nu(\gamma + \gamma_{rr} + s) - h\partial_\mu\partial_\rho\gamma_{\rho\nu} - h\partial_\nu\partial_\rho\gamma_{\mu\rho} +$$
$$+\delta_{\mu\nu}[\frac{1}{r}\frac{\log(r/r_0)}{\log(r/r_0) + 1/4}\partial_r(\gamma + s - \gamma_{rr} - 2\delta B) -$$
$$-4\frac{32\log^2(r/r_0) + 4\log(r/r_0) + 1}{r^2(4\log(r/r_0) + 1)^2}(\gamma_{rr} - s + 2\delta B)] = 0 \tag{4.83}$$

and

$$\partial_r\partial_\mu\gamma_{\mu\nu} + (\frac{5}{r} + \frac{h'}{2h})\partial_\mu\gamma_{rr} - \partial_\mu\partial_r(\gamma + s) - (\frac{1}{r} + \frac{h'}{2h})\partial_\mu s - \frac{4}{r}\frac{\log(r/r_0)}{\log(r/r_0) + 1/4}\partial_\mu\delta B = 0. \tag{4.84}$$

where we have used the relation (4.81). Following ref. [69], we define the transverse traceless part of the graviton $\bar{\gamma}_{\mu\nu}$ as

$$\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{\Box}\partial_\mu\partial_\rho\gamma_{\rho\nu} - \frac{1}{\Box}\partial_\nu\partial_\rho\gamma_{\mu\rho} + \frac{\partial_\mu\partial_\nu}{\Box^2}\partial_\rho\partial_\sigma\gamma_{\rho\sigma} +$$
$$+\frac{1}{3}(\frac{\partial_\mu\partial_\nu}{\Box} - \delta_{\mu\nu})(\gamma - \frac{1}{\Box}\partial_\rho\partial_\sigma\gamma_{\rho\sigma}). \tag{4.85}$$

This tensor satisfies:

$$\delta^{\mu\nu}\bar{\gamma}_{\mu\nu} = 0, \quad \partial_\mu\bar{\gamma}_{\mu\nu} = 0. \tag{4.86}$$

From the index structure of eq. (4.83), it is clear that $\bar{\gamma}_{\mu\nu}$ decouples from all other fields and satisfies the equation:

$$(h\Box + r^{-5}\partial_r r^5\partial_r)\bar{\gamma}_{\mu\nu} = 0. \tag{4.87}$$

This is precisely the equation for the minimal massless scalar, so following the steps outlined in section 3, we obtain the transverse traceless (TT) part of the energy-momentum tensor 2-point function:

$$\langle T_{\mu\nu}(k)T_{\rho\sigma}(-k)\rangle_{TT} \sim g_s^2 M^4 \pi_{\mu\nu\rho\sigma}(k)k^4\log^3(k/\Lambda), \tag{4.88}$$

where the transverse traceless projector $\pi_{\mu\nu\rho\sigma}$ was defined in eq. (4.76). In terms of the form factors C, D defined in (4.75), we find $C(k) \sim k^4 \log^3 k$. This is different from the k^4 behavior required by conformal symmetry.

Of course, eq. (4.87) does not exhaust the information contained in eq. (4.83). By substituting (4.85) into (4.83) and taking the trace or multiplying it by ∂_μ, we obtain

$$r^{-5}\partial_r r^5 \partial_r \gamma + h(-2\partial_\mu \partial_\nu \gamma_{\mu\nu} + 2\Box\,\gamma + \Box\,(\gamma_{rr} + s)) + 4V = 0, \qquad (4.89)$$

$$r^{-5}\partial_r r^5 \partial_r (\partial_\nu \gamma_{\mu\nu}) + h((-\partial_\mu \partial_\rho \partial_\sigma \gamma_{\rho\sigma} + \partial_\mu \Box\,(\gamma + \gamma_{rr} + s)) + \partial_\mu V = 0. \qquad (4.90)$$

where we have defined

$$V = \frac{1}{r} \frac{\log(r/r_0)}{\log(r/r_0) + 1/4} \partial_r (\gamma + s - \gamma_{rr} - 2\delta B) -$$

$$-4 \frac{32 \log^2(r/r_0) + 4\log(r/r_0) + 1}{r^2(4\log(r/r_0) + 1)^2} (\gamma_{rr} - s + 2\delta B). \qquad (4.91)$$

Next, we define a new scalar field φ by

$$\partial_\nu \gamma_{\mu\nu} = \partial_\mu \varphi + C_\mu(x). \qquad (4.92)$$

φ is well-defined because eq. (4.84) shows that the r-derivative of the vector $\partial_\nu \gamma_{\mu\nu}$ is a 4-dimensional gradient. Hence it is itself a 4-dimensional gradient up to an r-independent vector $C_\mu(x)$. With the definition (4.92), we have

$$r^{-5}\partial_r r^5 \partial_r \gamma + h(\Box\,(2\gamma - 2\varphi + \gamma_{rr} + s) - 2\partial_\mu C_\mu) + V = 0, \qquad (4.93)$$

$$\partial_r \varphi + X = 0, \qquad (4.94)$$

where we have also defined

$$X = (\frac{5}{r} + \frac{h'}{2h})\gamma_{rr} - \partial_r(\gamma + s) - (\frac{1}{r} + \frac{h'}{2h})s - \frac{4}{r} \frac{\log(r/r_0)}{\log(r/r_0) + 1/4}\delta B. \qquad (4.95)$$

Let us now write out the equations of motion for the other scalar fields $s_1, s_2, \gamma_{rr}, \delta B, \delta\Phi$. Expanding eqs. (1.3-1.5) linearly in these fields with the ansatz (4.80), we get

$$(h\Box + r^{-5}\partial_r r^5 \partial_r)\delta\Phi - \frac{8L^4}{r^6 h}(r\partial_r[\log(r/r_0)\delta B] + \frac{1}{2}(s_2 - \gamma_{rr} - 2\delta\Phi)) = 0, \qquad (4.96)$$

$$(h\Box + \frac{h}{r\log^2(r/r_0)}\partial_r \frac{r\log^2(r/r_0)}{h}\partial_r)\delta B +$$

$$+\frac{1}{2r\log(r/r_0)}\partial_r(\gamma - \gamma_{rr} + s_2 - 2\delta\Phi) +$$

$$+\frac{h(\partial_r h^{-1})}{2r\log(r/r_0)}(-2\gamma_{rr} + 2s_2 + s_1 - 2\delta\Phi) = 0, \qquad (4.97)$$

$$h\Box\,\gamma_{rr} + \partial_r^2(s + \gamma) + 6\frac{\log(r/r_0)}{r(\log(r/r_0) + 1/4)}\partial_r\delta B + 3\frac{\log(r/r_0)}{r(\log(r/r_0) + 1/4)}\partial_r\gamma -$$

$$-\frac{64\log^2(r/r_0) + 36\log(r/r_0) + 5}{r(4\log(r/r_0) + 1)^2}\partial_r\gamma_{rr} + \frac{2(8\log^2(r/r_0) + 6\log(r/r_0) + 1)}{r(4\log(r/r_0) + 1)^2}\partial_r s -$$

$$-\frac{4(32\log^2(r/r_0) + 4\log(r/r_0) + 1)}{r^2(4\log(r/r_0) + 1)^2}(\gamma_{rr} - s) - \frac{2}{r^2(\log(r/r_0) + 1/4)}s_1 -$$

$$-\frac{4}{r^2(\log(r/r_0) + 1/4)^2}\delta\Phi - \frac{8(32\log^2(r/r_0) - 12\log(r/r_0) - 3)}{r^2(4\log(r/r_0) + 1)^2}\delta B = 0, \quad (4.98)$$

$$(h\Box + r^{-5}\partial_r r^5\partial_r)s_1 + \frac{1}{r(\log(r/r_0) + 1/4)}\partial_r(\gamma + s - \gamma_{rr}) + \frac{32\log(r/r_0)}{r(4\log(r/r_0) + 1)}\partial_r\delta B -$$

$$-\frac{4(112\log^2(r/r_0) + 8\log(r/r_0) + 3)}{r^2(4\log(r/r_0) + 1)^2}s_1 - \frac{64\log(r/r_0)(4\log(r/r_0) - 1)}{r^2(4\log(r/r_0) + 1)^2}s_2 -$$

$$-\frac{16(12\log(r/r_0) + 1)}{r^2(4\log(r/r_0) + 1)^2}\gamma_{rr} + \frac{32(32\log^2(r/r_0) + 4\log(r/r_0) + 1)}{r^2(4\log(r/r_0) + 1)^2}\delta B = 0, \quad (4.99)$$

$$(h\Box + r^{-5}\partial_r r^5\partial_r)s_2 + \frac{1}{4r(\log(r/r_0) + 1/4)}\partial_r(\gamma + s - \gamma_{rr}) -$$

$$-\frac{8\log(r/r_0)}{r(4\log(r/r_0) + 1)}\partial_r\delta B - \frac{16(4\log^2(r/r_0) - \log(r/r_0))}{r^2(4\log(r/r_0) + 1)^2}s_1 -$$

$$-\frac{4(64\log^2(r/r_0) + 28\log(r/r_0) + 5)}{r^2(4\log(r/r_0) + 1)^2}s_2 - \frac{4(4\log(r/r_0) - 1)}{r^2(4\log(r/r_0) + 1)^2}\gamma_{rr} +$$

$$+\frac{8(32\log^2(r/r_0) - 4\log(r/r_0) - 1)}{r^2(4\log(r/r_0) + 1)^2}\delta B + \frac{16\delta\Phi}{r^2(4\log(r/r_0) + 1)} = 0. \quad (4.100)$$

The 5-dimensional graviton γ_{ij} includes 5 unphysical degrees of freedom associated with the (linearized) gauge transformations

$$\gamma_{ij} \to \gamma_{ij} + D_i\xi_j + D_j\xi_i, \qquad (4.101)$$

where ξ_i is an arbitrary 5-dimensional vector. In choosing $\gamma_{\mu r} = 0$ in the ansatz (104), we used 4 of these 5 degrees of freedom, so we still have to make one more gauge choice before solving the equations of motion. We find it convenient to choose the gauge

$$\partial_r \gamma = 0, \tag{4.102}$$

so that the trace γ of the 4-dimensional graviton is r-independent and completely determined by its boundary value $\hat{\gamma}$. With this gauge choice, the graviton $\gamma_{\mu\nu}$ decouples from the scalar equations (4.96-4.100). The solutions of these equations then enter into the graviton equations (4.93,4.94) through the quantities V, X defined in (4.91,4.95). Using eqs. (4.96-4.100), we find that V, X satisfy the equations

$$\partial_r \Big(\frac{r^4}{\log(r/r_0) + 1/4} V \Big) = 2L^4 \Box \Big(\frac{1}{4} \partial_r (s - \gamma_{rr}) +$$
$$+ \frac{2}{r} s + \frac{2}{r} \frac{\log(r/r_0)}{\log(r/r_0) + 1/4} \delta B + \frac{1}{8r} \frac{12 \log(r/r_0) + 5}{\log(r/r_0) + 1/4} (s - \gamma_{rr}) \Big), \tag{4.103}$$

$$\Big(\partial_r^2 + \Big(\frac{9}{r} - \frac{1}{r(\log(r/r_0) + 1/4)} \Big) \partial_r + \Big(\frac{15}{r^2} - \frac{5}{r^2(\log(r/r_0) + 1/4)} \Big) \Big) X =$$
$$= h \Box \Big(-\frac{3}{4} \partial_r \gamma_{rr} - \frac{1}{4} \partial_r s -$$
$$- \frac{1}{2r} \frac{12 \log(r/r_0) + 5}{4 \log(r/r_0) + 1} \gamma_{rr} - \frac{1}{2r} \frac{4 \log(r/r_0) - 1}{4 \log(r/r_0) + 1} s + \frac{8 \log(r/r_0)}{r(4 \log(r/r_0) + 1)} \delta B \Big). \tag{4.104}$$

Looking ahead, we see that in the UV expansion, the right-hand sides of the above equations are treated as perturbations, so to 0th order we have

$$V_0 = A_0 \frac{\log(r/r_0) + 1/4}{r^4}, \quad X_0 = A_1 \frac{\log(r/r_0) - 1/4}{r^3} + A_2 \frac{1}{r^5}. \tag{4.105}$$

Substituting the above into eq. (4.93), and using again the gauge choice (4.102), we find

$$A_0 = L^4 (k^2 \hat{\gamma} - k_\mu k_\nu \hat{\gamma}_{\mu\nu}), \tag{4.106}$$

so A_0 is completely determined by the boundary data. We see that to find the leading order 2-point function, we need to solve the eqs. (4.96-4.100) to first order

in the UV expansion. The first order solutions will have 2 arbitrary constants which we can express in terms of A_0 and A_2. Then plugging the 0th order part of the first order solutions into the right-hand-side of eq. (4.104), we will solve for X to 1st order, and choose A_2 so as to eliminate the leading order *log* term in the IR limit. In solving eqs. (4.96-4.100), the boundary conditions (4.79) mean that all scalar fields should approach 0 at $r \rightarrow \infty$. The first order solutions with these boundary conditions, and with (4.105,4.106) are:

$$\gamma_{rr} = -\frac{A_0 \log(r/r_0)}{6r^2} - \frac{A_2}{4r^4}, \quad \delta B = \frac{A_0}{8r^2} + \frac{A_2}{2r^4},$$

$$s_1 = \frac{7A_0}{36r^2} + \frac{A_2}{3r^4}, \quad s_2 = \frac{13A_0}{144r^2} + \frac{5A_2}{12r^4}, \quad \delta\Phi = -\frac{A_0}{6r^2} + \frac{13A_2}{12r^4}, \tag{4.107}$$

where we have only kept the leading *log* terms. Plugging these into (4.95), we find

$$A_1 = -\frac{1}{2}A_0. \tag{4.108}$$

Substituting the solutions (4.107) into the right hand side of (4.104) and going to next order, we obtain

$$X = -A_0 \frac{\log(r/r_0) - 1/4}{2r^3} - \frac{5}{24} A_0 L^4 k^2 \frac{\log^2(r/r_0)}{r^5} + \frac{A_2}{r^5}, \tag{4.109}$$

where we have again kept only leading logs. From this we see that in taking the usual IR limit, we will need to choose

$$A_2 \sim A_0 L^4 k^2 \log^2(k/\Lambda) = L^8 k^4 \log^2(k/\Lambda)(\hat{\gamma} - \frac{k_\mu k_\nu \hat{\gamma}_{\mu\nu}}{k^2}). \tag{4.110}$$

As in the previous section, the above considerations only allow us to determine the coefficient A_2 up to a constant factor.

To complete the calculation of the EM tensor 2-point function, we now have to substitute the above solutions into the SUGRA action. We only need to be concerned with the gravitational part of the action; this is because our boundary conditions stipulate that all scalar fields (except γ and φ) approach 0 at $r \rightarrow \infty$, so

all contributions to the 2-point function from nongravitational parts of the action will vanish. The quadratic gravitational action is (see e.g. [69]):

$$S \sim \frac{1}{\kappa^2} \int d^4x dr \sqrt{g} (D_K h_{MN} D^K h^{MN} - 2 D_M h_{KN} D^K h^{MN} +$$
$$+ 2 D_N h_K^K - D_M h_K^K D^M h_N^N) \qquad (4.111)$$

Integrating by parts in the usual way, and using eqs. (4.80,4.85,4.92), this becomes

$$S \sim \frac{1}{\kappa^2} \int_{r=\infty} d^4x \, r^5 (\bar{\gamma}_{\mu\nu} \partial_r \bar{\gamma}_{\mu\nu} + (\frac{4}{3} \frac{\partial_\rho \partial_\sigma \hat{\gamma}_{\rho\sigma}}{\Box} - \frac{1}{3} \hat{\gamma}) \partial_r \varphi - \hat{\gamma} \partial_r (2\gamma_{rr} + s)) \quad (4.112)$$

Substituting eqs. (4.94,4.107,4.109,4.110) the terms conspire to add to a transverse momentum space 2-point function

$$\langle T_{\mu\nu}(k) T_{\rho\sigma}(-k) \rangle = \frac{\delta^2 S}{\delta \hat{\gamma}_{\mu\nu} \delta \hat{\gamma}_{\rho\sigma}} \sim$$
$$\sim g_s^2 M^4 k^4 (\pi_{\mu\nu\rho\sigma}(k) \log^3(k/\Lambda) + C \pi_{\mu\nu}(k) \pi_{\rho\sigma}(k) \log^2(k/\Lambda)), \qquad (4.113)$$

where the projectors $\pi_{\mu\nu}$ and $\pi_{\mu\nu\rho\sigma}$ have been defined in eqs. (4.4,4.76) and we are unable to determine the numerical value of the constant C. Note that by multiplying eq. (4.113) by $\delta_{\mu\nu} \delta_{\rho\sigma}$, we find

$$\langle T_\mu^\mu(k) T_\mu^\mu(-k) \rangle \sim g_s^2 M^4 k^4 \log^2(k/\Lambda). \qquad (4.114)$$

Thus the leading order logarithmic behavior of the 2-point function of the trace anomaly scalar T_μ^μ is different from that of the minimal massless scalar discussed in section 3, but the same as that of the R-anomaly scalar $\partial \cdot J^R$ discussed in the previous section (see eq. (4.73)).

Chapter 5

Is there a baryonic symmetry?

Consider the following transformation of the chiral fields A_i, B_i of the superconformal $SU(N) \times SU(N)$ gauge theory corresponding to the $AdS_5 \times T^{1,1}$ background (see section 3.1 for a review):

$$A_i \to e^{i\theta} A_i, \quad B_i \to e^{-i\theta} B_i. \tag{5.1}$$

This transformation leaves the superpotential (3.10) invariant, and is therefore a $U(1)$ symmetry of the theory. Moreover, it is a nontrivial symmetry: there are gauge-invariant operators that are charged under it. The simplest (having the lowest dimension) such operators are the baryon-like color singlets

$$\mathcal{B}_i^+ = det(A_i), \quad \mathcal{B}_i^- = det(B_i), \tag{5.2}$$

so that we can think of the transformation (5.1) as a baryonic symmetry. The SUGRA picture of this symmetry, which we will review below, is well understood. The string modes corresponding to the baryonic operators are D3-branes wrapped around the 3-cycle of $T^{1,1}$ [5]. The SUGRA spectrum contains a massless vector arising from a topological mode of the RR 4-form that corresponds to the conserved baryonic symmetry current [6, 49].

In this chapter we would like to address the question: does a nontrivial baryonic symmetry akin to (5.1) exist in the cascading $SU(N + M) \times SU(N)$ gauge theory?

The problem is that while the transformation (5.1) can still be defined and appears to leave the action invariant, the running of N makes it difficult to construct globally well-defined charged operators like (5.2). This question was discussed qualitatively in ref. [71], where it was argued that for $M > 0$, the symmetry is broken and there is no nontrivial conserved baryonic current. We will use the SUGRA dual of the cascading theory to try to obtain a better understanding of whether, and how, the symmetry gets broken. We will argue that in the full warped deformed conifold (KS) background, there can indeed be no well-defined baryon number. However, in the UV limit of that background, the KT background, where the conifold is warped but not deformed, the situation may be more subtle. We will see that there is no way to define a bulk massless vector dual to a conserved baryon current, because, in a novel mechanism resembling but not identical to Higgsing, the vector degrees of freedom are dualized into 2-form degrees of freedom from which one cannot recover a vector in the bulk. On the other hand we will show that the dimension of this 2-form receives no corrections, so it might still be possible to dualize it into a nontrivial conserved current at the boundary. There seems to be no spontaneous symmetry breaking and we find no Goldstone mode. Thus, although there is no baryonic $U(1)$ symmetry in the full field theory, it is possible that there is an effective (nontrivial) symmetry at high energies.

Let us first review the $M = 0$ case in more detail. As shown by the authors of [5], a D3-brane wrapped around the 3-cycle of $T^{1,1}$ should be identified with a baryonic operator $\mathcal{B}_\pm$ as defined in (5.2). As evidence for this, it can be noted that since the A_i, B_i have dimension 3/4, the operators $\mathcal{B}$ have dimension $3N/4$; a D3-brane wrapped around the S^3 of $T^{1,1}$ has a mass equivalent to exactly that dimension [5]. The two types of baryonic operators defined in (5.1) correspond to D3-branes localized at either of the two S^2s of $T^{1,1}$, i.e. either at (θ_1, ϕ_1) or at (θ_2, ϕ_2). Moreover, by forming appropriate linear combinations with Clebsh-Gordan coefficients of the operators $\mathcal{B}_\pm$, they are seen to fall into $(N + 1, 1)$ and $(1, N + 1)$

representations of the $SU(2) \times SU(2)$ global symmetry of the field theory; these are precisely the $SU(2) \times SU(2)$ quantum numbers of a D3-brane wrapped around the 3-cycle of $T^{1,1}$ and localized at one of the S^2s.

Since the baryonic charge in the SUGRA picture is carried by D3-branes wrapped around the 3-cycle, and since D3-branes carry RR 4-form charge, it follows that the bulk SUGRA mode that we expect to be dual to the baryonic current should be an excitation of the RR 4-form with 3 indices along the ω_3 of $T^{1,1}$. This is indeed the case. Consider the following fluctuation of the RR 4-form:

$$\delta C_4 = B \wedge \omega_3 + K_2 \wedge \omega_2, \tag{5.3}$$

and the resulting RR 5-form field strength

$$\delta \tilde{F}_5 = dB \wedge \omega_3 + dK_2 \wedge \omega_2. \tag{5.4}$$

The fields B, K_2 decouple from all other SUGRA fluctuations. The self duality equation, $\tilde{F}_5 = \star \tilde{F}_5$ implies

$$dK_2 = \frac{3}{r} \star_5 dB, \tag{5.5}$$

where the 5-dimensional Hodge dual $\star_5$ is defined as in (4.19). From this we obtain

$$d\frac{1}{r} \star_5 dB = 0, \tag{5.6}$$

so that B is a massless vector, dual to a conserved current of dimension 3 – the baryonic current [49]. Choosing the gauge $B_r = 0$, we find $\partial_r(\partial \cdot B) = 0$, so that $(\partial \cdot B)$ does not propagate. Thus B contains 3 propagating degrees of freedom, corresponding (in this gauge choice) to the transverse components of B_μ. The 2-form K_2 is not an independent field but is expressed through B by eq. (5.5), up to another gauge choice. At this point it is important to note that we could just as well have chosen our degrees of freedom to reside in the 2-form K_2. From eq. (5.5) we can also obtain

$$d\, rh \star_5 dK_2 = 0. \tag{5.7}$$

This is the equation for a massless 2-form field, which again has 3 propagating degrees of freedom; they can be chosen as the transverse components of $K_{\mu\nu}$. It is dual to a field theory 2-form operator of dimension 2. What is the meaning of that operator? For any 4-dimensional conserved current J, the continuity equation can be written as $d \star_4 J = 0$, where $\star_4$ is the ordinary flat space Hodge dual. Thus, the 3-form $\star_4 J$ is closed, from which it follows (in flat space) that it is exact. So there is a 2-form S_2 such that

$$\star_4 J = dS_2. \tag{5.8}$$

This relation also shows that S_2 is 1 dimension lower than J, so if J has dimension 3, S_2 has dimension 2. Thus either S_2 or equivalently $\star_4 S_2$ must be the 2-form operator of dimension 2 dual to the SUGRA field K_2. In ordinary electrodynamics, if J is the usual electric current, S_2 is simply the dual of the EM field strengh $\star_4 F_{\mu\nu}$.

Note that conversely, given any 2-form operator S_2 we can in principle use eq. (5.8) to define an associated conserved current J. However, for the associated charge $Q = \int d^3x J_0 = lim(|x| \to \infty) \int_{S^2} \star_4 S_2$ to be nontrivial, S_2 must scale as $1/|x|^2$, i.e. it must have dimension exactly 2.

The conserved baryonic current J is a component of a conserved superfield $\mathcal{J}$, whose other components are of course related to it by supersymmetry. This superfield can be written as $\mathcal{J} = Tr(Ae^V \bar{A}e^{-V} - Be^V \bar{B}e^{-V})$ [49, 6]. This superfield includes the dimension 2 scalar $\mathcal{U} = Tr(A\bar{A} - B\bar{B})$. Since each pair (A_i, B_i) of chiral fields is associated with an S^2 of $T^{1,1}$, we might guess that the SUGRA mode dual to $\mathcal{U}$ corresponds to turning on the difference between the volumes of the two S^2s. This is known as resolving the conifold, and the related scalar s as the Kähler modulus. We take a graviton fluctuation of the form

$$\delta(ds^2) = 2r^2 h^{1/2} s(x,r)(g^1 g^3 + g^2 g^4), \tag{5.9}$$

which when written out in coordinates is proportional to $\delta(ds^2) \sim d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 - d\theta_2^2 - \sin^2\theta_2 d\phi_2^2$, i.e. the difference between the volumes of the S^2s. The scalar s

decouples from all other SUGRA fluctuations and satisfies the equation

$$(h\Box + r^{-5}\partial_r r^5 \partial_r + \frac{4}{r^2})s = 0. \tag{5.10}$$

In the UV, this has the solutions $s(r) = (A + B\log(r))/r^2$, which shows that the operator $\mathcal{U}$ dual to s indeed has dimension 2.

Before moving on to $M > 0$, we will consider some other SUGRA fluctuations at $M = 0$. The reason is that for $M > 0$ these modes will couple to modes in the baryonic multiplet, so it is important to establish their behavior for $M = 0$ and count the associated degrees of freedom. Consider first varying the RR and NS-NS 2-forms:

$$\delta B_2 = L_2, \quad \delta C_2 = M_2. \tag{5.11}$$

The fields L_2, M_2 couple to each other and decouple from all other SUGRA fields. They satisfy the equations of motion:

$$d \star dM_2 = \tilde{F}_5 \wedge dL_2, \quad d \star dL_2 = -g_s^2 \tilde{F}_5 \wedge dM_2. \tag{5.12}$$

Using the L_2 equation of motion and substituting the background value of $\tilde{F}_5$, we can express L_2 through M_2 as

$$dL_2 = -\frac{27\pi(\alpha')^2 N g_s^2}{r^5 h^2} \star_5 M_2,$$

and plugging this into the M_2 equation of motion, we obtain

$$(r^{-5} \star_5 dr^5 h^2 \star_5 d - \frac{16}{r^2})M_2 = 0. \tag{5.13}$$

M_2 is thus a massive 2-form field. We now separate the equations (5.13) into equations for transverse 2-forms, transverse vectors, and scalars. For any 2-form $G_{\mu\nu}$, we can define its transverse part by

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{\partial_\mu \partial_\rho G_{\rho\nu}}{\Box} - \frac{\partial_\nu \partial_\rho G_{\mu\rho}}{\Box}. \tag{5.14}$$

This 2-form satisfies

$$\partial_\mu \tilde{G}_{\mu\nu} = 0.$$

Transverse 2-form can only couple to other transverse 2-forms. In this case we have only one transverse 2-form $\tilde{M}_{\mu\nu}$ that satisfies the equation

$$(h\Box + r^{-1}\partial_r r \partial_r - \frac{16}{r^2})\tilde{M}_{\mu\nu} = 0. \tag{5.15}$$

In the UV this has the solutions $\tilde{M}(r) = Ar^4 + Br^{-4}$, so the operator that couples to $\tilde{M}$ has dimension 6. There are three degrees of freedom associated with this transverse 2-form. From eq. (5.13) we also derive the conservation law:

$$r\partial_r \frac{1}{rh} M_{r\mu} - \partial_\nu M_{\mu\nu} = 0.$$

Thus there is one independent transverse vector, and no propagating scalars (since $\partial_\nu M_{\mu\nu}$ is purely transverse). The transverse vector $M_{r\mu}$ satisfies the equation

$$(h\Box + r^{-7}\partial_r r^7 \partial_r - \frac{7}{r^2})M_{r\mu} = 0. \tag{5.16}$$

In the UV this has the solutions $M_{r\mu} = Ar + Br^{-7}$, so the operator that couples to $M_{r\mu}$ also has dimension 6. There are three degrees of freedom associated with this transverse vector.

Finally, we would like to consider fluctuations of the form

$$\delta C_2 = A^1 \wedge g^5, \quad \delta B_2 = A^2 \wedge g^5. \tag{5.17}$$

The vectors A^1, A^2 decouple from all other fields (and from each other). They obey identical equations of motion

$$(rh^2 \star_5 dr^{-1} \star_5 d - \frac{8}{r^2})A^{1,2} = 0. \tag{5.18}$$

As usual, the transverse components decouple and satisfy the equation

$$(h\Box + r^{-3}\partial_r r^3 \partial_r - \frac{8}{r^2})tA^{1,2} = 0. \tag{5.19}$$

In the UV, this has solutions $\tilde{A}^{1,2}(r) = Ar^2 + Br^{-4}$, so $\tilde{A}^{1,2}$ are transverse massive vectors that couple to operators of dimension 5. Each carries three degrees of freedom. From (5.18), the scalars satisfy the conservation law

$$\frac{1}{r}\partial_r(rh^{-1}A_r^{1,2}) + \partial_\mu A_\mu^{1,2} = 0.$$

Thus in each case there is one independent scalar. These scalars have the equation of motion

$$(h\square + r^{-9}\partial_r r^9 \partial_r + \frac{7}{r^2})A_r^{1,2} = 0. \tag{5.20}$$

In the UV this has the solutions $A_r^{1,2}(r) = Ar^{-1} + Br^{-7}$, so these are massive scalars that couple to operators of dimension 5. Each carries one degree of freedom.

Let us now investigate the situation for $M > 0$. Our main interest lies in the mode coupling to the baryon current. However, now B_2 and F_3 have background values, and fluctuations of these modes will couple to the mode we're interested in. We are led to consider the following ansatz for fluctuations of the fields:

$$\delta C_4 = \frac{1}{2}B \wedge \omega_3 + K_2 \wedge \omega_2,$$

$$\delta B_2 = \frac{1}{M\alpha'}L_2,$$

$$\delta C_2 = \frac{2}{3g_s M\alpha'}M_2 + \frac{1}{3g_s M\alpha'}A^1 \wedge g^5.$$

The vector field B would be the natural candidate for the mode dual to the baryon number current. The fluctuations are normalized for future convenience. The variation of the self-dual RR 5-form field strength is then

$$\delta F_5 = \frac{1}{2}(L_2 + dB + \log(r/r_0)dA) \wedge \omega_3 + (dK_2 + \log(r/r_0)dM_2) \wedge \omega_2.$$

We see that B will only appear in the equations of motion in the combination $L_2 + dB$. Since L_2 is a gauge 2-form, all physical fields are invariant under the gauge transformations

$$B \to B - \Lambda, \quad L_2 \to L_2 - d\Lambda,$$

where Λ is an arbitrary vector field. From this we see that the field B can be gauged away – it is swallowed by the 2-form L_2. Henceforth we set $B = 0$. So far this looks like a Higgs mechanism: a field which is a physical field at $M = 0$ (the vector B) is now swallowed by gauge invariance of a higher index field (the 2-form L_2).

From the self-duality condition

$$\star \delta F_5 = \delta F_5,$$

we find

$$dK_2 + \log(r/r_0)dM_2 = \frac{3}{2r} \star_5 (L_2 + \log(r/r_0)dA^1), \qquad (5.21)$$

Eq. (5.21) allows us to express the 2-form L_2 through the 2-forms K_2 and M_2 and the vector A^1:

$$L_2 + \log(r/r_0)dA^1 = -\frac{2rh}{3}(dK_2 + \log(r/r_0)dM_2). \qquad (5.22)$$

Next, look at the equation of motion for the NS-NS 2-form,

$$d \star H_3 = -g_s^2 F_5 \wedge F_3.$$

Taking the variation, this becomes

$$d \star dL_2 = -\frac{1}{2}(g_s M \alpha')^2 dK_2 \wedge \omega_2 \wedge \omega_3.$$

This is solved (up to a gauge choice) to yield

$$\star dL_2 = -\frac{1}{2}(g_s M \alpha')^2 K_2 \wedge \omega_2 \wedge \omega_3,$$

or

$$dL_2 = \frac{27(g_s M \alpha')^2}{r^5 h} \star_5 K_2. \qquad (5.23)$$

Eq. (5.23) immediately results in the conservation law

$$d\frac{1}{r^5 h} \star_5 K_2 = 0.$$

Explicitly, this is

$$\partial_\nu K_{\mu\nu} = r^5 h \partial_r \frac{1}{r^5 h^2} K_{r\mu},$$ (5.24)

so we can express the vector $\partial_\nu K_{\mu\nu}$ through the vector $K_{r\mu}$. Furthermore, by substituting L_2 from eq. (5.21) into eq. (5.23) we obtain the equation of motion

$$r^{-1} h \star_5 d\, r h \star_5 (dK_2 + \log(r/r_0)dM_2) + \frac{3}{2r^2} \star_4 dA^1 - \frac{81}{2}(g_s M\alpha')^2 \frac{K_2}{r^6 h} = 0,$$ (5.25)

where all indices in $\star_4$ are raised with flat metric.

Next, consider the equation of motion for the RR 2-form,

$$d \star F_3 = F_5 \wedge H_3.$$

Taking the variation, this becomes

$$d\star(dM_2 + \frac{1}{2}dA\wedge g^5 - \frac{1}{2}A\wedge dg^5) = -\frac{9}{8}(g_s M\alpha')^2(dL_2 + (L_2 + \log(r/r_0)dA)\wedge \frac{dr}{r})\wedge\omega_2\wedge\omega_3 +$$

$$+\frac{9}{4}(g_s M\alpha')^2(dK + \log(r/r_0)dM) \wedge \frac{dr}{r} \wedge \omega_2 \wedge \omega_2.$$

This equation has three components. The first gives the conservation law

$$d(r \star_5 A^1) = 0,$$

or explicitly,

$$\partial_\mu A_\mu^1 = -\frac{1}{r}\partial_r(rh^{-1}A_r^1).$$ (5.26)

Thus we have expressed the scalar $\partial_\mu A_\mu^1$ through the scalar A_r^1. The second component gives the A^1 equation of motion:

$$r^{-3} h \star_5 d\, r^3 h \star_5 dA^1 - \frac{8}{r^2}A = \frac{27(g_s M\alpha')^2}{r^4} \star_4 (dK_2 + \log(r/r_0)dM_2).$$ (5.27)

The third component gives the equation of motion for M_2:

$$r^{-5} \star_5 d\, r^5 h^2 \star_5 dM_2 + \frac{81(g_s M\alpha')^2 h}{2r^6} \star_5 (\star_5(dK_2 + \log(r/r_0)dM_2) \wedge dr) +$$

$$+\frac{16\log(r/r_0)}{r^2(\log(r/r_0) + 1/4)^2}K_2 = 0.$$ (5.28)

We now separate the above equations into equations for transverse 2-forms, transverse vectors, and scalars. As usual, transverse 2-form can only couple to other transverse 2-forms. In our case we have the three transverse 2-forms $\tilde{M}_{\mu\nu}$, $\tilde{K}_{\mu\nu}$ and $F_{\mu\nu}$, where F is related to the vector A^1 by

$$F = \star_4 dA^1.$$

Note that F is automatically transverse. The equations (5.25,5.28,5.27) translate into the following equations of motion for the transverse 2-forms $\tilde{K}, \tilde{M}, F$, where we have dropped the indices:

$$(h\Box + r^{-1}\partial_r r \partial_r - \frac{4}{r^2(\log(r/r_0) + 1/4)})\tilde{K} +$$
$$+(h\Box \log(r/r_0) + r^{-1}\partial_r r \log(r/r_0)\partial_r)\tilde{M} + \frac{3}{2r^2}F = 0 \qquad (5.29)$$

$$(h\Box + r^{-5}\partial_r r^5 \partial_r)\tilde{M} + (\frac{4}{r(\log(r/r_0) + 1/4)}\partial_r + \frac{16\log(r/r_0)}{r^2(\log(r/r_0) + 1/4)^2})\tilde{K} = 0 \quad (5.30)$$

$$(h\Box + r^{-3}\partial_r r^3 \partial_r - \frac{8}{r^2})F - \frac{27(g_s M\alpha')^2}{r^4}\Box(\tilde{K} + \log(r/r_0)\tilde{M}) = 0. \qquad (5.31)$$

In the UV (i.e. for large r), we can solve these equations by retaining all terms of order $1/r^2$, and dropping all the $\Box$ terms that go as $1/r^4$. The solution is

$$\tilde{M}(r,k) = C_1 + C_2\frac{1}{\log(r/r_0) + 1/4} - C_3\frac{r^4}{(\log(r/r_0) + 1/4)} -$$

$$-2C_4\frac{\log(r/r_0) + 1/8}{r^4(\log(r/r_0) + 1/4)} - C_5\frac{3}{64r^4(\log(r/r_0) + 1/4)} - C_6\frac{3r^2}{4(\log(r/r_0) + 1/4)}(5.32)$$

$$\tilde{K}(r,k) = C_2\frac{1}{4(\log(r/r_0) + 1/4)} + C_3\frac{r^4\log(r/r_0)}{\log(r/r_0) + 1/4} +$$
$$+C_4\frac{2\log^2(r/r_0) + (5/4)\log(r/r_0) + (1/4)}{r^4(\log(r/r_0) + 1/4)} -$$

$$-C_5 \frac{6 \log(r/r_0) + 3}{128(\log(r/r_0) + 1/4)} + C_6 \frac{3r^2 \log(r/r_0)}{8(\log(r/r_0) + 1/4)}, \tag{5.33}$$

$$F(r, k) = C_5 \frac{1}{r^4} + C_6 r^2. \tag{5.34}$$

From these solutions we see that the fields $\tilde{K}$, $\tilde{M}$ and F contain a mixture corresponding to an operator of dimension 6, a mixture corresponding to an operator of dimension 5, and a mixture corresponding to an operator of dimension 2. Before discussing them, let us see what happens to the remaining degrees of freedom. The vector A^1 contains, in addition to the transverse vector $\tilde{A}^1$ which is equivalent to the transverse 2-form F, also the scalars A_r^1 and $\partial_\mu A_\mu^1$. Eq. (5.26) shows that only one of them is independent. Let us take that to be A_r^1. Using equations (5.26, 5.27), we find that A_r^1 obeys the equation:

$$(h\Box + r^{-1}h^2\partial_r r h^{-2}\partial_r + \frac{112 \log^2(r/r_0) - 72 \log(r/r_0) + 7}{r^2(4\log(r/r_0) + 1)^2})A_r^1 = 0. \tag{5.35}$$

In the UV, this has the solution

$$A_r = C_1 \frac{\log(r/r_0) + 1/4}{r} + C_2 \frac{\log(r/r_0) + 1/4}{r^7}, \tag{5.36}$$

so A_r^1 corresponds to a scalar operator of dimension 5. Next, the 2-forms K_2, M_2 contain also the vectors $K_{r\mu}$, $\partial_\nu K_{\mu\nu}$, $M_{r\mu}$, $\partial_\nu M_{\mu\nu}$. From eq. (5.24) we can eliminate $\partial_\nu K_{\mu\nu}$. Using equations (5.25, 5.28), we find that the vector $K_{r\mu}$ satisfies the equation

$$(h\Box + r^5 h^3 \partial_r r^{-5} h^{-3} \partial_r - \frac{112 \log^2(r/r_0) + 136 \log(r/r_0) - 21}{r^2(4\log(r/r_0) + 1)^2})K_{r\mu} = 0. \tag{5.37}$$

In the UV, this has the solution

$$K_{r\mu} = C_1 r(\log(r/r_0) + 1/4) + C_2 \frac{32 \log^2(r/r_0) + 20 \log(r/r_0) + 3}{r^7} \tag{5.38}$$

so $K_{r\mu}$ corresponds to a transverse vector of dimension 6. Note that because of eq. (5.24), $\partial_\mu K_{r\mu} = 0$. Finally, the vectors $M_{r\mu}$ and $\partial_\nu M_{\mu\nu}$ only appear in the equations

of motion in the gauge invariant coombination $\partial_\nu (dM)_{r\mu\nu}$ and, using equations (5.25, 5.28) this can be expressed through $K_{r\mu}$ so it is not an independent field.

Let us now address our main question: what happened to the baryonic current? Surveying the above modes, we see that there is no vector of dimension 3: the vector B was swallowed by the 2-form L_2 and gauged away. However, we still have a 2-form of dimension 2. Looking at the equations (5.29,5.30), we identify, to leading order in the UV, the mixture $\tilde{K}_2 + \log(r/r_0)\tilde{M}_2$ as the dimension 2 2-form. Indeed, from equations (5.32,5.33), we find that this mixture has the leading UV behavior

$$K_2 + \log(r/r_0)\tilde{M}_2 = C_1 + C_2 \log(r/r_0),$$

which is the behavior expected of a 2-form of dimension exactly 2 (with no corrections). Note that this precisely the mixture related by eq. (5.21) to the 2-form L_2 that has swallowed the vector B. Comparing eq. (5.21) to its $M = 0$ counterpart, eq. (5.5), we see that, up to trivial normalization factors, the crucial difference is that in the $M > 0$ case we can no longer recover a bulk vector from the bulk 2-form: the right hand side of eq. (5.22) is no longer exact. Thus, while for $M = 0$ we could choose the relevant bulk degrees of freedom to be either a dimension 3 vector, or equivalently, a dimension 2 2-form, for $M > 0$ only the latter choice is possible. This is a different phenomenon from Higgsing: the vector B has not been swallowed by a massless 2-form that became massive, rather it has been dualized into a massless 2-form with a source from which one can no longer recover a vector in the bulk. Indeed, counting the degrees of freedom, we see that the 2-forms K_2 and M_2 and the vector A^1 still contribute three degrees of freedom from a transverse 2-form of dimension 2, three from a transverse 2-form of dimension 6, three from a transverse vector of dimension 6, three from a transverse vector (or equivalently, transverse 2-form) of dimension 5, and one from a scalar of dimension 5, just as for $M = 0$. There is no exchange of degrees of freedom between different modes that is characteristic of Higgsing, and we do not find a Goldstone mode associated with

symmetry breaking.

To shed more light on the situation, let us see what happens to the Kähler modulus s which resides in the same supermultiplet as the dimension 2 2-form. We have to consider a fluctuation of the form

$$\delta ds^2 = 2r^2 h^{1/2} s(g^1 g^3 + g^2 g^4), \quad \delta B_2 = A^2 \wedge g^5. \tag{5.39}$$

The transverse vector $\bar{A}^2$ decouples from all other modes and satisfies the same equation of motion as $\bar{A}^1$, so this remains a transverse vector of dimension 5. The scalars s and A_r^2 couple to each other. Their equations of motion are:

$$(h\Box + r^{-5}\partial_r r^5 \partial_r + \frac{4}{r^2})s = \frac{2}{r^2(\log(r/r_0) + 1/4)}(s - \frac{2}{9}r A_r^2), \tag{5.40}$$

$$h\Box A_r^2 + (r^{-1}h^2 \partial_r r h^{-2}\partial_r + $$
$$+ \frac{112\log^2(r/r_0) - 72\log(r/r_0) + 7}{r^2(4\log(r/r_0) + 1)^2})(A_r^2 - \frac{9}{2r}s) = 0. \tag{5.41}$$

We see that, to leading order in the UV, the linear combination $A_r^2 - 9s/2r$ decouples and is a scalar of dimension 5. Setting this combination to 0, we find that to this leading order, s satisfies the equation for a scalar of dimension exactly 2:

$$(r^{-5}\partial_r r^5 \partial_r + \frac{4}{r^2})s = 0.$$

Since s is related by supersymmetry to the 2-form of dimension 2 $K_2 + \log(r/r_0)M_2$, the equation of motion for s provides further evidence that in the UV, there is no correction to the dimension – not even a small correction that vanishes as $1/\log(r)$. Once again, there is no Higgsing: we still have three degrees of freedom from a transverse vector of dimension 5, one from a scalar of dimension 5, and one from a scalar of dimension 2.

So far we have been considering the UV limit of the SUGRA solution. What happens in the full (deformed conifold) KS solution? It is a well known fact about conifolds that they cannot be deformed and resolved at the same time – we have

to choose between complex structure and Kähler structure. In our case, this means that in the full deformed conifold background, the Kähler modulus s can no longer have zero modes that behave as $1/r^2$. Thus, in the IR, the dimension of s must be modified, and by SUSY, the same is true for the dimension of $K_2 + \log(r/r_0)M_2$. Since this form has IR dimension different from 2, there can be no nontrivial baryonic current in the IR (see the discussion after eq. (5.8)). Another argument for the same conclusion is that in the IR, the gauge group becomes simply $SU(M)$, and there is no way to define nontrivial baryonic operators like (5.2).

The bottom line seems to be that while there is certainly no baryonic symmetry and no charges at low energies, the situation at high energies remains somewhat mysterious. There is no bulk dimension 3 vector dual to a baryonic current, but there is a 2-form of UV dimension exactly 2. Thus, at high enough energies, there may still be an effective baryonic symmetry. This correponds to the fact that for a given $N > 0$, the $SU(N+M) \times SU(N)$ theory contains baryonic operators like (5.2), though it is not clear what happens to them as N cascades. It would be interesting to obtain a better understanding of this.

Chapter 6

Glueballs in KS background

In this chapter, based on parts of the paper [53], we demonstrate explicitly the existence of a mass gap in the KS solution given by the metric (3.22) by computing the spectrum of low-lying glueball modes. This calculation proceeds as follows[64, 72, 73]: we need to go from Euclidean to Minkowski space, and find solutions to eq. (3.27) that are normalizable both in the UV and the IR. We will see that such solutions exist only for a discrete set of k^2 which give the glueball masses. The IR can no longer be ignored, so we need to use the full KS background with the metric of (3.22). The equation of motion for a massless minimal scalar in this background is

$$[(\sinh 2\tau - 2\tau)^{-2/3}\partial_\tau(\sinh 2\tau - 2\tau)^{2/3}\partial_\tau + \frac{(m\tilde{R})^2 \sinh^2 \tau}{(\sinh 2\tau - 2\tau)^{2/3}}h(\tau)]\phi = 0. \quad (6.1)$$

Here $m^2 = k^2$ is the mass squared of the 4-dimensional mode. Our goal is to find values of m^2 for which eq. (6.1) has solutions that are normalizable at both the UV and the IR. This means that the flux factor

$$\mathcal{F} = \phi(\tau)(\sinh 2\tau - 2\tau)^{2/3}\partial_\tau\phi(\tau) \quad (6.2)$$

must remain finite at both $\tau = \infty$ and $\tau = 0$ (in fact for the normalizable solutions it will vanish at both ends). Let $f(\tau) = (\sinh 2\tau - 2\tau)^{2/3}$, and define the new field $\psi(\tau)$ by $\phi(\tau) = f^{-1/2}(\tau)\psi(\tau)$. Then the equation (6.1), written in terms of ψ, reduces to

the more familiar "Schroedinger" form

$$[\partial_\tau^2 - \bar{k}^2(\tau)]\psi = 0, \tag{6.3}$$

where

$$\bar{k}^2(\tau) = \frac{4}{3}\frac{\sinh 2\tau}{\sinh 2\tau - 2\tau} - \frac{8}{9}\frac{(\cosh 2\tau - 1)^2}{(\sinh 2\tau - 2\tau)^2} - \frac{(m\bar{R})^2 \sinh^2 \tau}{(\sinh 2\tau - 2\tau)^{2/3}}h(\tau) \tag{6.4}$$

with the flux (6.2) now expressed as

$$\mathcal{F} = \psi\partial_\tau\psi - \frac{1}{2}(\partial_\tau \log f)\psi^2. \tag{6.5}$$

At $\tau \to \infty$, $\bar{k}^2 \to 4/9$ and we have the solutions $\psi_\pm \sim e^{\pm 2\tau/3}$; only ψ_- is normalizable. At $\tau \to 0$, the normalizable solution behaves as $\psi \sim \sinh(\bar{k}(0)\tau)$, where

$$\bar{k}^2(0) = \frac{2}{5} - m^2(\bar{A}\bar{R})^2 \int_0^\infty dx \frac{x\coth x - 1}{\sinh^2 x}(\sinh(2x) - 2x)^{1/3}. \tag{6.6}$$

We now find the eigenvalues of eq. (6.3) using the WKB approximation (see e.g. [74]), which gives a sensible estimate for a smooth potential like $\bar{k}^2$ and is increasingly accurate for more excited states. In this approximation, (6.3) has the solutions

$$\psi_\pm(\tau) \sim \bar{k}^{-1/2}(\tau)e^{\pm \int^\tau \bar{k}(x)dx}$$

which are valid away from the turning points $\bar{k} = 0$. At a turning point τ_0, the exponentially decreasing solution on one side is matched to an oscillatory solution on the other through

$$\bar{k}^{-1/2}e^{-\int^\tau \bar{k}(x)dx} \to k^{-1/2}\cos(\zeta(\tau) - \pi/4),$$

where $k^2 = -\bar{k}^2$ when $\bar{k}^2 < 0$, and

$$\zeta(\tau) = \int_\tau^{\tau_0} k(x)dx.$$

In our case, it's not hard to see that the function $\bar{k}^2(\tau)$ increases monotonically with τ from its value $\bar{k}^2(0) < 2/5$ to $\bar{k}^2(\infty) = 4/9$ (see fig. 6.1). When m^2 is small
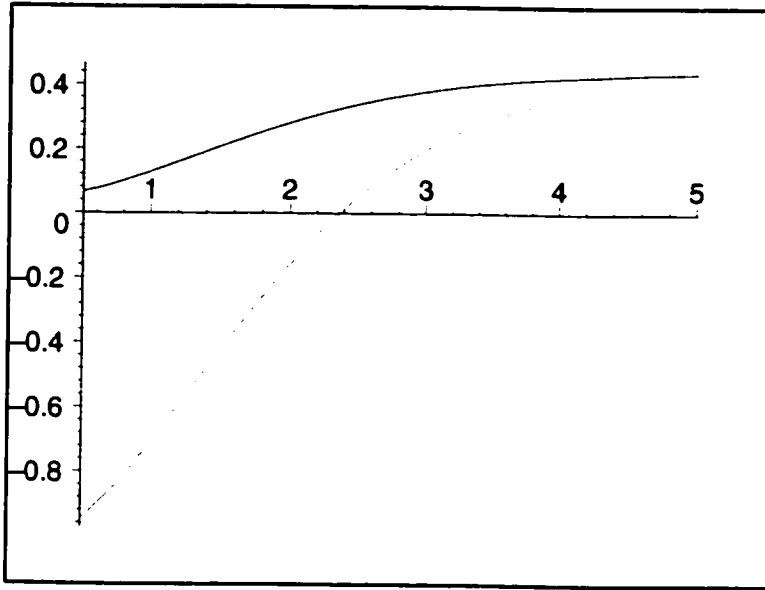
Figure 6.1: $\bar{k}^2(\tau)$ vs. τ for $m^2 = 0.5$ (solid line) and $m^2 = 2$ (dashed line)

enough, $\bar{k}^2(0) > 0$ and there are no turning points at all. Then the exponentially decreasing solution, which is not normalizable at the IR, is valid for all τ, so there are no normalizable modes; in other words, there is a mass gap, as advertised. For sufficiently large m^2 there are normalizable modes determined by the transcendental equation

$$\int_0^{\tau_0(m^2)} k(x)dx = \frac{3\pi}{4} + (n-1)\pi \tag{6.7}$$

where $n = 1, 2, 3...$ and τ_0 is given by $\bar{k}^2(\tau_0) = 0$. The phase in (6.7) must be such that ψ behaves as a pure sine wave near $\tau = 0$. Solving this equation numerically we obtain the glueball modes:

$$m_n^2 = c_n(\tilde{A}\tilde{R})^{-2}$$

with the first few coefficients being

$$c_1 = 1.79, c_2 = 4.03, c_3 = 7.16, c_4 = 11.2, c_5 = 16.2, c_6 = 22.0, c_7 = 28.8$$

and so on. The overall normalization of these coefficients is of course arbitrary but their ratios are dimensionless numbers that constitute a prediction for the gauge

theory. As expected, we have $m^2 \sim \Lambda_s^2/(g_s M)^2$, where $\Lambda_s \sim 1/R$ is the string tension scale. This demonstrates dimensional transmutation.

Chapter 7

Discussion

Let us restate our main results. To leading order at high energies, we have found the following momentum space 2-point functions for the R-current and the energy-momentum tensor:

$$\langle J_\mu^R(k) J_\nu^R(-k) \rangle = g_s^2 M^4 k^2 (A_0 \pi_{\mu\nu}(k) \log^3(k/\Lambda) + B_0 \frac{k_\mu k_\nu}{k^2} \log^2(k/\Lambda)), \qquad (7.1)$$

$$\langle T_{\mu\nu}(k) T_{\rho\sigma}(-k) \rangle = g_s^2 M^4 k^4 (C_0 \pi_{\mu\nu\rho\sigma}(k) \log^3(k/\Lambda) + D_0 \pi_{\mu\nu}(k) \pi_{\rho\sigma}(k) \log^2(k/\Lambda)) \qquad (7.2)$$

where A_0, B_0, C_0, D_0 are k-independent constants. The most general structure of the correlators allowed by the symmetries is

$$\langle J_\mu^R(k) J_\nu^R(-k) \rangle = A(k^2) \pi_{\mu\nu}(k) + B(k^2) \frac{k_\mu k_\nu}{k^2}, \qquad (7.3)$$

$$\langle T_{\mu\nu}(k) T_{\rho\sigma}(-k) \rangle = C(k^2) \pi_{\mu\nu\rho\sigma}(k) + D(k^2) \pi_{\mu\nu}(k) \pi_{\rho\sigma}(k), \qquad (7.4)$$

where A, B, C, D are k-dependent form factors. The presence of nonzero form factors B and D indicates, respectively, the anomalous breaking of R and conformal symmetries. Indeed, the longitudinal part of the $\langle JJ \rangle$ correlator should be proportional to the R-symmetry anomaly $\partial_\mu J_\mu$, and the trace part of the $\langle TT \rangle$ correlator should be proportional to the trace anomaly $T_\mu{}^\mu$. Moreover, supersymmetric Ward identities are expected to relate A to C and B to D [61]. From the functional form of

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the form factors in eqs.(7.1,7.2) (the identical powers of the leading order logarithm in A and C and in B and D), it seems plausible that these identities are indeed satisfied. This provides a qualitative check on our results. Note that although the $\langle JJ \rangle$ and $\langle TT \rangle$ correlators are related to each other by Ward identities, the sectors of SUGRA fluctuations that are dual to them are completely decoupled from each other; there is no interaction between the modes considered in section 4 and those in section 5. In this, the KT background is similar to AdS, and markedly different from the RG flows studied in refs. [58, 60, 61]. The reason for this is unbroken chiral symmetry. In the above papers, it is the breaking of chiral symmetry, which is an IR phenomenon, that mixes the T and J modes. If we were interested in the IR behavior of the same correlators in the full KS background, and not only in its UV limit – the KT background – we would encounter the same kind of mixing. Unlike the RG flow backgrounds, we see little hope of obtaining analytic results in the KS background.

It is interesting to contrast our results with the correlators of irrelevant operators responsible for the breaking of scale symmetry in the full DBI action on 3-branes that we computed in chapter 2. In that case, we found that the trace of the energy-momentum tensor had twice the dimension of the traceless part. In the case of KT solution, the R and dilatation symmetries are broken not by explicitly adding irrelevant operators to the action, but via quantum anomalies. This results in correlators whose behavior differs only logarithmically from the behavior of correlators present in the conformal theory.

In all the 2-point functions we have computed, we encounter at leading order logarithmic factors of the form $\log^n(k)$, where n is a positive integer (an exponent of n in momentum space translates to an exponent of $n - 1$ in position space). From the way these logarithms arise in the UV solutions to the fluctuation equations, it is easy to see that in general, the larger the dimension of the field theory operator $\mathcal{O}$ whose 2-point function $\langle \mathcal{O}\mathcal{O} \rangle$ we are calculating, the more times we need to iterate

the UV expansion, and since in each iteration we effectively pick up a factor of $g_s M \log(k)$, the higher the power of the logarithm that will appear in the 2-point function. The interpretation of these logarithms from the field theory point of view is somewhat mysterious. On the one hand, the $SU(N+M) \times SU(M)$ gauge theory has a nontrivial beta function [75], with the relative coupling flowing as

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim M \log(\mu), \tag{7.5}$$

where μ is the energy scale; the Yang-Mills coupling $1/g_s \sim 1/g_1^2 + 1/g_2^2$ remains constant in the supergravity approximation [9]. Thus a perturbative expansion in the coupling would be an expansion in inverse powers of the logarithm, but it is difficult to see how it could give rise to the large positive powers expected to appear for operators of large dimension. It is tempting to speculate that the logarithmic growth of the correlators is a manifestation of a logarithmic growth with scale of the effective number of degrees of freedom in the theory. Indeed, as noted above, the logarithms always appear in the combination $M \log(k)$ which perhaps somehow represents the "effective number of colors" in the UV. Some support for this is provided by the fact that when the finite temperature theory is considered, one finds an entropy that grows logarithmically with scale [70]. It would be interesting to better understand the field theory origin of these logarithmic factors.

Finally let us say that a lot remains to be done. Even in the context of the Klebanov-Strassler solution, there are many outstanding problems involving both the IR limit (there is still no systematic description of all glueball states) and the UV limit – how can one tighten the cascade interpretation of the theory in the UV and really count the number of degrees of freedom as a function of the scale? Beyond this, of course, lies the continuation of the search for supergravity duals of yet more realistic gauge theories, in particular, theories that are not only confining in the IR, but also asymptotically free in the UV; and in the end, we will have to go beyond the supergravity approximation and describe sigma models of strings moving on

Ramond-Ramond backgrounds. All this is surely in the future.

Bibliography

[1] J. Maldacena, "The Large N limit of superconformal field theories and super-gravity," *Adv. Theor. Math. Phys.* **2** (1998) 231, hep-th/9711200.

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys. Lett.* **B428** (1998) 105, hep-th/9802109.

[3] E. Witten, "Anti de-Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253, hep-th/9802150.

[4] I.R. Klebanov and E. Witten, "Superconformal field theory on threebranes at a Calabi-Yau singularity," *Nucl. Phys.* **B536** (1998) 199, hep-th/9807080.

[5] S.S. Gubser and I.R. Klebanov, "Baryons and Domain Walls in an N=1 Super-conformal Gauge Theory," *Phys. Rev.* **D58** (1998), hep-th/9808075.

[6] I.R. Klebanov and E. Witten, "AdS/CFT correspondence and symmetry breaking," *Nucl. Phys.* **B556** (1999) 89, hep-th/9905104.

[7] I.R. Klebanov and N. Nekrasov, "Gravity Duals of Fractional Branes and Logarithmic RG Flow," *Nucl. Phys.* **B574** (2000) 263, hep-th/9911096.

[8] I.R. Klebanov and A. Tseytlin, "Gravity Duals of Supersymmetric $SU(N)XSU(N + M)$ Gauge Theories," *Nucl. Phys.* **B578** (2000) 123, hep-th/0002159.

[9] I. Klebanov and M. Strassler, "Supergravity and a Confining Gauge Theory: Duality Cascades and χSB-Resolution of Naked Singularities," *JHEP* **0008** (2000) 052, hep-th/0007191.

[10] J. Maldacena and C. Nunez, "Towards the large N limit of pure $\mathcal{N} = 1$ Super Yang Mills," *Phys. Rev. Lett.* **86** (2001) 588, hep-th/0008001.

[11] C. Vafa, "Superstrings and Topological Strings at Large N," J. Math. Phys. **42** (2001) 2798, hep-th/0008142.

[12] I. Klebanov, "TASI lectures: Introduction to the AdS/CFT correspondence," hep-th/0009139.

[13] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183, hep-th/9905111.

[14] E. D'Hoker and D. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," hep-th/0201253.

[15] Y. Nambu, "Quark model and the factorization of the Veneziano amplitude," in it Symmetries and Quark Models, ed. Chand, Gordon and Breach (1970); H.B. Nielsen, "An almost physical interpretation of the integrand of the n-point Veneziano amplitude," submitted to the 15th International Conference on High Energy Physics, Kiev (1970); L. Susskind, "Dual-symmetric theory of hadrons," *Nuovo Cim.* **69A** (1970) 457.

[16] J. Scherk and J. Schwarz, "Dual models for non-hadrons," *Nucl. Phys.* **B81** (1974) 118.

[17] K. Wilson, "Confinement of quarks," *Phys. Rev.* **D10** (1974) 2445.

[18] G. 't Hooft, "A planar diagram theory for strong interactions," *Nucl. Phys.* **B72** (1974) 461.

[19] A. Polyakov, "Quantum geometry of bosonic strings," *Phys. Lett.* **B103** (1981) 207.

[20] A. Polyakov, "String theory and quark confinement," *Nucl. Phys. B (Proc. Suppl.)* **68** (1998) 1, hep-th/9711002; "The wall of the cave," *Int. J. Mod. Phys.* **A14** (1999) 645, hep-th/9809057.

[21] J. Schwarz, "Covariant field equations of chiral $N = 2$, $d = 10$ supergravity," *Nucl. Phys.* **B226** (1983) 269.

[22] G.T. Horowitz and A. Strominger, "Black strings and p-branes," *Nucl. Phys.* **B360** (1991) 197.

[23] M. Duff and J. Lu, "The selfdual type IIB superthreebrane," *Phys. Lett.* **B273** (1991) 409; M. Duff, J. Khuri and J. Lu, "String solitons," *Phys. Rept.* **259** (1995) 213, hep-th/9412184.

[24] J. Polchinski, "Dirichlet branes and Ramond-Ramond charges," *Phys. Rev. Lett.* **75** (1995) 4724, hep-th/9510017.

[25] C. Hull and P. Townsend, "Unity of superstring dualities," *Nucl. Phys.* **B438** (1995) 109, hep-th/9410167; P. Townsend, "The eleven-dimensional supermembrane revisited," *Phys. Lett.* **B350** (1995) 184, hep-th/9501068; E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.* **B443** (1995) 85, hep-th/9503124.

[26] E. Witten, "Bound states of strings and p-branes," *Nucl. Phys.* **B460** (1996) 335, hep-th/9510135.

[27] S. S. Gubser, I. R. Klebanov and A. W. Peet, "Entropy and temperature of black 3-branes," *Phys. Rev.* **D54** (1996) 3915, hep-th/9602135.

[28] I.R. Klebanov, "World volume approach to absorption by nondilatonic branes," *Nucl. Phys.* **B496** (1997) 231, hep-th/9702076.

[29] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, "String theory and classical absorption by three-branes," *Nucl. Phys.* **B499** (1997) 217, hep-th/9703040.

[30] S.S. Gubser and I.R. Klebanov, "Absorption by branes and Schwinger terms in the world volume theory," *Phys. Lett.* **B413** (1997) 41, hep-th/9708005.

[31] I. Klebanov, W. Taylor and M. Van Raamsdonk, "Absorption of dilaton partial waves by D3-branes," *Nucl. Phys.* **B560** (1999) 207, hep-th/9905174.

[32] B. Kol and A. Rajaraman, "Fixed scalars and suppression of Hawking evaporation," *Phys. Rev.* **D56** (1997) 983, hep-th/9608126.

[33] C.G. Callan, S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, "Absorption of fixed scalars and the D-brane approach to black holes," *Nucl. Phys.* **B489** (1997) 65, hep-th/9610172.

[34] I.R. Klebanov and M. Krasnitz, "Fixed scalar gray body factors in five-dimensions and four-dimensions," *Phys. Rev.* **D55** (1997) 3250, hep-th/9612051.

[35] M. Krasnitz and I.R. Klebanov, "Testing effective string models of black holes with fixed scalars," *Phys. Rev.* **D56** (1997) 2173, hep-th/9703216.

[36] S. Gubser, A. Hashimoto, I. Klebanov and M. Krasnitz, "Scalar absorption and the breaking of the world volume conformal invariance," *Nucl. Phys.* **B526** (1998) 393, hep-th/9803023.

[37] A.A. Tseytlin, "Vector field effective action in the open superstring theory," *Nucl. Phys.* **B276** (1986) 391.

[38] D.J. Gross and E. Witten, "Superstring modifications of Einstein's equations," *Nucl. Phys.* **B277** (1986) 1.

[39] A.A. Tseytlin, "On non-Abelian generalization of Born-Infeld action in string theory," *Nucl. Phys.* **B501** (1997) 41, hep-th/9701125.

[40] G.T. Horowitz and H. Ooguri, "Spectrum of large N gauge theory from supergravity," *Phys. Rev. Lett.* **80** (1998) 4116, hep-th/9802116.

[41] M. Dine and N. Seiberg, "Comments on higher derivative operators in some SUSY field theories," *Phys. Lett* **B409** (1997) 239, hep-th/9705057.

[42] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen, "The mass spectrum of chiral $N = 2$ $d = 10$ supergravity on S^5," *Phys. Rev.* **D32** (1985) 389.

[43] S. Kachru and E. Silverstein, "4d comformal field theories ans strings on orbifolds," *Phys. Rev. Lett.* **80** (1998) 4855, hep-th/9802183; A. Lawrence, N. Nekrasov and C. Vafa, "On conformal field theories in four dimensions," *Nucl. Phys.* **B533** (1998) 199, hep-th/9803015.

[44] A. Kehagias, "New type IIB vacua and their F-theory interpretation," *Phys. Lett.* **B435** (1998) 337, hep-th/9805131.

[45] D. Morrison and R. Plesser, "Non-spherical horizons, I," *Adv. Theor. Math. Phys.* **3** 1999 1, hep-th/9810201.

[46] C. Herzog, I. Klebanov and P. Ouyang, "D-branes on the conifold and $\mathcal{N} = 1$ gauge/gravity dualities," hep-th/0205100.

[47] L. Romans, "New compactifications of chiral $N = 2$, $d = 10$ supergravity," *Phys. Lett.* **B153** (1985) 392.

[48] P. Candelas and X. de la Ossa, "Comments on conifolds," *Nucl. Phys.* **B342** (1990) 246.

[49] A. Ceresole, G. Dall'Agata, R. D'Auria and S. Ferrara, "Spectrum of type IIB supergravity on $AdS_5 \times T^{1,1}$: predictions on N=1 SCFT's," *Phys. Rev.* **D61** (2000) 066001, hep-th/9905226.

[50] N. Seiberg, "Electric-magnetic duality in supersymmetric nonAbelian gauge theories," *Nucl. Phys.* **B435** (1995) 129, hep-th/9411149.

[51] R. Minasian and D. Tsimpis, "On the geometry of non-trivially embedded branes," *Nucl. Phys.* **B572** (2000) 499, hep-th/9911042.

[52] K. Ohta and T. Yokono, "Deformation of conifold and intersecting branes," *JHEP* **0002** (2000) 023, hep-th/9912266.

[53] M. Krasnitz, "A two point function in a cascading $\mathcal{N} = 1$ gauge theory from supergravity," hep-th/0011179.

[54] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," *JHEP* **9807** (1998) 023, hep-th/9806087.

[55] S. de Haro, K. Skenderis and S. Solodukhin, "Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence," *Commun. Math. Phys.* **217** (2001) 595, hep-th/0002230.

[56] K. Skenderis, "Lecture notes on holographic renormalization," hep-th/0209067.

[57] M. Bianchi, O. DeWolfe, D. Z. Freedman, K. Pilch, "Anatomy of two holographic renormalization group flows," *JHEP* **0101** (2001) 021, hep-th/0009156.

[58] O. DeWolfe and D. Z. Freedman, "Notes on fluctuations and correlation functions in holographic renormalization group flows," hep-th/0002226.

[59] G. Arutyunov, S. Frolov and S. Theisen, "A note on gravity-scalar fluctuations in holographic RG flow geometries," *Phys. Lett.* **B484** (2000) 295. hep-th/0003116.

[60] M. Bianchi, D. Z. Freedman and K. Skenderis, "How to go with an RG flow," *JHEP* **0108** (2001) 041, hep-th/0105276.

[61] M. Bianchi, D. Z. Freedman and K. Skenderis, "Holographic renormalization," *Nucl. Phys.* **B631** (2002) 159, hep-th/0112119.

[62] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, "The supergravity dual of N=1 Super Yang-Mills theory," *Nucl. Phys.* **B569** (200-) 451. hep-th/9909047.

[63] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, "Renormalization group flows from holography-supersymmetry and a c-theorem," *Adv. Theor. Math. Phys.* **3** (1999) 363, hep-th/9904017.

[64] E. Witten, "Anti de-Sitter space, thermal phase transition, and confinement in gauge theories," *Adv. Math. Theor. Phys.* **2** (1998) 505, hep-th/9803131.

[65] M. Krasnitz, "Correlation functions in a cascading $\mathcal{N} = 1$ gauge theory from supergravity," hep-th/0209163.

[66] A. Brandhuber and K. Sfetsos, "Current correlators in the Coulomb branch of N=4 SYM," *JHEP* **0012** (2000) 014, hep-th/0010048.

[67] I.R. Klebanov, P. Ouyang and E. Witten, "A gravity dual of the chiral anomaly," *Phys. Rev.* **D65** (2002) 105007, hep-th/0202056.

[68] H. Liu and A.A. Tseytlin, "D=4 Super Yang Mills, D=5 gauged supergravity and D=4 conformal supergravity," *Nucl. Phys.* **B533** (1998) 88, hep-th/9804083.

[69] G.E. Arutyunov and S.A. Frolov, "On the origin of supergravity boundary terms in the AdS/CFT correspondence," *Nucl. Phys.* **B544** (1999) 576, hep-th/9806216.

[70] A. Buchel, "Finite temperature resolution of the Klebanov-Tseytlin singularity," *Nucl. Phys.* **B600** (2001) 219, hep-th/0011146.

[71] O. Aharony, "A note on the holographic interpretation of string theory backgrounds with varying flux," *JHEP* **0103** (2001) 012, hep-th/0101013.

[72] C. Csaki, H. Ooguri, Y. Oz and J. Terning, "Glueball mass spectrum from supergravity," *JHEP* **9901** (1999) 017, hep-th/9806021.

[73] R. de Mello Koch, A. Jevicki, M. Mihailescu and J. Nunes, "Evaluation of glueball masses from supergravity," *Phys. Rev.* **D58** (1998) 105009, hep-th/9806125.

[74] L. Schiff, *Quantum Mechanics* 3rd ed. McGraw-Hill 1968.

[75] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, "Exact Gell-Mann-Low function of supersymmetric Yang-Mills theories from instanton calculus," *Nucl. Phys.* **B229** (1983) 381.