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Geometric Linearization for Constraint Hamiltonian Systems

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Abstract: This study investigates the geometric linearization of constraint Hamiltonian systems using the Jacobi metric and the Eisenhart lift. We establish a connection between linearization and maximally symmetric spacetimes, focusing on the Noether symmetries admitted by the constraint Hamiltonian systems. Specifically, for systems derived from the singular Lagrangian $L\left(N, q^k, \dot{q}^k\right) = \frac{1}{2N} g_{ij}\dot{q}^i\dot{q}^j - NV(q^k)$, where N and q^i are dependent variables and $\dim g_{ij} = n$, the existence of $\frac{n(n+1)}{2}$ Noether symmetries is shown to be equivalent to the linearization of the equations of motion. The application of these results is demonstrated through various examples of special interest. This approach opens new directions in the study of differential equation linearization.

Keywords: constraint Hamiltonian systems; linearization; exact solutions; Noether symmetries

1. Introduction

The theory of invariant infinitesimal transformations, known as Lie symmetry analysis [1–3], is a powerful tool for the analytic treatment of nonlinear differential equations. The concept of symmetry is based on the existence of invariant functions in the solution space of the differential equations [4]. These invariant functions are mainly used to simplify differential equations and to derive solutions [5–7]. Lie symmetry analysis systematically treats dynamical systems and has been widely explored in various areas of applied mathematics; see, for instance, Refs. [8–19] and the references therein.

Symmetry in physics is crucial because it is related to conservation laws [20–22]. The conservation laws of momentum for free particles, energy for conservative systems, Kepler's laws of motion, and many others are connected to the infinitesimal transformations that keep the differential equations invariant [5]. The existence of a sufficiently large number of Lie symmetries for a given differential equation allows for the solution of the equation by repeated-order reduction using similarity transformations or by determining a sufficient number of first integrals [6]. Additionally, Lie symmetries can be used to classify dynamical systems through their admitted invariant functions [7].

This latter characteristic is important for the concept of the global linearization of differential equations via point transformations. Specifically, nonlinear differential equations can be linearized if they admit a specific number of Lie symmetries [4], as will be detailed in the following sections. In this study, we focus on the global linearization of a family of constraint Hamiltonian systems. We derive a set of geometric criteria that allow us, for the first time, to express nonlinear dynamical systems with fewer symmetries in a linear form.

We employ two different methods for the geometrization of non-relativistic dynamical systems with conservative forces: the Einsenhart–Duval lift and the Jacobi metric. In the Jacobi metric approach, the conservative force is absorbed in the kinetic terms, such that the dynamical effects of the force are part of the new geometry that describes the dynamical system. This is achieved by the reparametrization of the independent variables. There are

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various applications of the Jacobi metric for the study of dynamical systems [23–25], while the method has been generalized for the description of relativistic systems [26,27]. For a more general discussion, we refer the reader to [28].

On the other hand, in the Eisenhart–Duval method, the conservative forces are embedded into an extended geometry by introducing new independent variables and degrees of freedom. This method was introduced by Eisenhart in the early twentieth century [29] and it was rediscovered later by Duval et al. in [30]. The Eisenhart–Duval method is the nonrelativistic case of the Kaluza–Klein framework [31,32]. Due to the simplicity of the method, it has been widely applied for the study of nonlinear dynamical systems. The Eisenhart lift of two-dimensional mechanical systems with or without varying mass terms is discussed in [33]. The superintegrability property of three-dimensional Newtonian systems was investigated by using the Eisenhart lift in [34]. In [35], the Eisenhart lift is applied for the study of the Toda chain in nearest-neighbor interacting particles on a line. The geodesic description of the two fixed centers problem using the Eisenhart lift was studied in [36].

Nevertheless, the Eisenhart lift has been applied not only in classical mechanics but also for the analysis of the Schrödinger [37,38] and the Dirac [39] equations of quantum mechanics. There is a plethora of applications of the Eisenhart lift in relativistic physics and cosmology; see, for instance, Refs. [40–42] and the references therein. The extended minisuperspace via the Eisenhart lift for the study of quantum cosmology was introduced in [43]. Recently, in [44], the Eisenhart lift was employed for the derivation of the analytic solutions of the field equations in scalar field cosmology. It was found that the field equations can be linearized in the framework of minimally coupled scalar field cosmological theory for a Friedmann–Lemaître–Robertson–Walker background geometry and for an exponential potential for the description of the mass of the scalar field. Moreover, in [45], it was found that the cosmological constant in the Friedmann–Lemaître–Robertson–Walker geometry can be recovered by means of the application of the hidden symmetries for the extended minisuperspace in quantum cosmology.

Furthermore, in [46], the Eisenhart lift was employed to write the equation of motion for the oscillator in terms of a free particle. This is a well-known result provided by symmetry analysis, which relates the two dynamical systems by sharing a common symmetry group [47]. We remark that by geometrizing a given dynamical system, it is feasible to employ important results from the differential geometry in a systematic way. This property is considered in the following in order to perform geometric linearization for a class of constraint Hamiltonian systems. We shall see that there exists a relation between the geometric properties of the extended geometry in the Eisenhart lift and the admitted symmetries of the original system. The structure of the paper is as follows.

In Section 2, we discuss the mathematical tools primarily applied in this study. In particular, we discuss the Lie symmetry analysis for differential equations and we focus on the case of a second-order differential. We also present Noether's theorems, which play an important role in the main analysis of this study. Furthermore, in Section 3, we provide a literature review on the global linearization of differential equations through Lie symmetry analysis.

The family of dynamical systems considered in this work is introduced in Section 4. Specifically, we examine a family of dynamical systems described by a singular Lagrangian function, leading to equations of motion where the Hamiltonian function is constrained. Additionally, we discuss in detail two geometrization approaches for dynamical systems within this family. Section 5 includes the main results of this study, where we derive a new set of geometric conditions and criteria under which constraint Hamiltonian systems can be transformed into an equivalent system of a free particles in a flat space, thereby becoming linearizable.

We demonstrate the application of this geometric approach in Section 6, where we explore the linearization of nonlinear dynamical systems of special interest in gravitational physics. The geometric linearization of the Szekeres system with or without the cosmological constant is discussed. Furthermore, we investigate the linearization of the minisuperspace for the gravitational model in a static spherically symmetric spacetime with a charge, i.e., the exact solution for the Reissner–Nordström black hole metric can be constructed via the solution of the free particle. Lastly, dynamical systems of interest in Newtonian mechanics are discussed. Finally, in Section 7, we summarize our results.

2. Preliminaries

In this section, we briefly discuss the basic mathematical elements necessary for this study.

2.1. Lie Symmetries of Differential Equations

In the following lines, we review the basic definitions for the Lie symmetry analysis of ordinary differential equations.

Let us assume the n-dimensional dynamical system of μth-order differential equations of the form

$$q^{(\mu)i} = \omega^i\left(t, q^k, \dot{q}^k, \ddot{q}^k, \dots, q^{(\mu-1)i}\right) \tag{1}$$

in which t is the independent variable and q^i denotes the dependent variables, $q^i = (q^1, q^2, \dots, q^N)$. Moreover, a dot represents the total derivative with respect to the independent variable t, i.e.,

$$\dot{q}^i = \frac{dq^i}{dt},\ \ddot{q}^i = \frac{d^2q^i}{dt^2},\ \dots,\ q^{(\mu)} = \frac{d^\mu q}{dt^\mu}. \tag{2}$$

Consider the infinitesimal transformation

$$\bar{t} = t + \varepsilon\xi\left(t, q^k\right) \tag{3}$$

$$\bar{q}^i = q^i + \varepsilon\eta^i\left(t, q^k\right) \tag{4}$$

with the generator vector field $X = \xi\left(t, q^k\right)\partial_t + \eta^i\left(t, q^k\right)\partial_i$.

The vector field $q^{[\mu]}$ is the μth-prolongation of q in the jet space $J_V = \left\{t, q^i, \dot{q}^i, \ddot{q}^i, \dots, q^{(\mu)}\right\}$ defined as

$$X^{[\mu]} = X + \eta^i_{[1]}\partial_{\dot{q}^i} + \eta^i_{[2]}\partial_{\ddot{q}^i} + \dots + \eta^i_{[\mu]}\partial_{q^{(\mu)i}}, \tag{5}$$

in which $\eta^i_{[1]},\ \eta^i_{[2]}, \dots, \eta^i_{[\mu]}$ are given by the following expressions:

$$\eta^i_{[1]} = \dot{\eta}^i - \dot{q}^i\dot{\xi}, \tag{6}$$

$$\eta^i_{[2]} = \dot{\eta}^i_{[2]} - \ddot{q}^i\dot{\xi}, \tag{7}$$

$$\dots \tag{8}$$

$$\eta^i_{[\mu]} = \dot{\eta}^i_{[\mu-1]} - q^{(\mu)i}\dot{\xi}. \tag{9}$$

Then, we say that the system of differential equations (1) will be invariant under the application of the infinitesimal transformations (3), (4) if and only if there exists a function λ such that the following condition holds [4]:

$$\left[X^{[\mu]}, A\right] = \lambda A, \tag{10}$$

where

$$\left[X^{[\mu]}, A\right] = X^{[\mu]}A - AX^{[\mu]}, \tag{11}$$

and operator A is Hamilton's vector

$$A = \frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + \ddot{q}^i\frac{\partial}{\partial \dot{q}^i} + \dots + \omega^i\left(t, q^k, \dot{q}^k, \ddot{q}^k, \dots, q^{(\mu-1)i}\right)\frac{\partial}{\partial q^{(\mu)i}}. \tag{12}$$

Condition (10) is known as the Lie symmetry condition, and X is a Lie point symmetry for the dynamical system (1).

Assume that f^i is a solution vector for the dynamical system (1), which is $Af^i = 0$; then, the Lie symmetry condition (10) becomes $\left[X^{[\mu]}, A\right] f = \lambda A f$, i.e., [4]

$$X\left(Af^i\right) = 0, \tag{13}$$

or equivalently

$$\eta^i_{[\mu]} = X^{[\mu-1]}\omega^i\left(t, q^k, \dot{q}^k, \ddot{q}^k, \dots, q^{(\mu-1)i}\right). \tag{14}$$

The solution of the latter linear system defines the functional form of the generator vector X for the infinitesimal transformation (3) and (4).

2.2. Second-Order Differential Equations

In this study, we focus on second-order differential equations of the form [5]

$$\ddot{q}^i = \omega^i\left(t, q^k, \dot{q}^k\right). \tag{15}$$

Therefore, the components $\eta^{[1]}, \eta^{[2]}$ of the second prolongation read [5]

$$\eta^{[1]} = \eta^i_{,t} + q^i\left(q^k\eta^i_{,k} - \xi_{,t}\right) - \dot{q}^i\dot{q}^k\xi_{,k}, \tag{16}$$

$$\eta^i_{[2]} = \eta^i_{,tt} + 2\left(\eta^i_{,tk} - \xi_{,tt}\right)\dot{q}^k + \left(\eta^i_{,kr} - 2\xi_{,tq}\right)\dot{q}^k\dot{q}^r - \dot{q}^i\dot{q}^k\dot{q}^r\xi_{,kr} + \left(\eta^i_{,k} - 2\xi_{,t} - 3\xi_{,kr}\dot{q}^r\right)\ddot{q}^k. \tag{17}$$

Thus, the symmetry conditions (14) become [5]

$$\begin{aligned} 0 \quad = \quad & \eta^i_{,tt} + 2\left(\eta^i_{,tk} - \xi_{,tt}\right)\dot{q}^k + \left(\eta^i_{,kr} - 2\xi_{,tq}\right)\dot{q}^k\dot{q}^r \\ & -\dot{q}^i\dot{q}^k\dot{q}^r\xi_{,kr} + \left(\eta^i_{,k} - 2\xi_{,t} - 3\xi_{,kr}\dot{q}^r\right)\omega^i\left(t, q^k, \dot{q}^k\right) \\ & -\omega^i_{,t} - \omega^i_{,k}\eta^k - \omega^i_{,\dot{q}^j}\left(\eta^j_{,t} + q^j\left(q^k\eta^j_{,k} - \xi_{,t}\right) - \dot{q}^j\dot{q}^k\xi_{,k}\right). \end{aligned} \tag{18}$$

Lie symmetries have numerous applications. They are used to derive invariant functions and conservation laws. Furthermore, they are applied to categorize differential equations and establish criteria for when differential equations are equivalent to a linear differential equation under a point transformation. In the following lines, we discuss Noether's theorems for the construction of conservation laws and geometric linearization through symmetries.

2.3. Noether's Theorems

Let us turn our attention to the case where the second-order dynamical system (15) arises from the variation of the action integral

$$S = \int L(t, q, \dot{q})dt, \tag{19}$$

where $L(t, q, \dot{q})$ is the so-called Lagrangian function.

In 1918 [48], Emmy Noether published two groundbreaking theorems that relate the symmetries of the variation principle to conservation laws in dynamical systems.

The first theorem states that if, under the application of the infinitesimal transformations (3), (4), there exists a function f such that the following condition holds

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{df}{dt}, \tag{20}$$

then X is called a Noether symmetry. It is evident that Noether symmetries are Lie symmetries for the dynamical system (15), but the converse is not necessarily true; in other words, Noether symmetries form a subalgebra of Lie symmetries for the dynamical system. The function f in (20) represents a boundary term introduced to account for infinitesimal changes in the action integral due to infinitesimal changes in the boundary of the domain, caused by infinitesimal transformations of the variables in the action integral.

Noether's second theorem provides a systematic way to derive conservation laws. In particular, if $X = \xi\partial_t + \eta\partial_i$ is a Noether symmetry for the dynamical system (15), then the function

$$\Phi\left(t, q^k, \dot{q}^k\right) = \xi\left(\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L\right) - \eta^i \frac{\partial L}{\partial q^i} + f, \tag{21}$$

is a conservation law, i.e., $A(\Phi) = 0$; consequently, $X(\Phi) = 0$.

For a recent extended discussion of Noether's theorems, we refer the reader to [49].

3. Linearization through Symmetries

One of the significant findings of Lie symmetry analysis is the linearization criterion for a second-order ordinary differential equation. The linearization process is crucial because it provides a method to express the analytic solution in terms of simple functions, thereby offering a better understanding of the dynamics of the dynamical system.

Sophus Lie's theorem states that if a second-order ordinary differential equation $\ddot{q} = \omega(t, q, \dot{q})$ admits eight Lie symmetries forming the $sl(3, R)$ Lie algebra, then there exists a transformation that can bring the equation to the form of the free particle equation $\bar{q}'' = 0$. For further discussion, see [50] and the references therein.

For third-order ordinary differential equations, various approaches have been developed to address their linearization. Criteria for linearization through the point and contact symmetries of third-order differential equations have been established in [51,52]. Additionally, the Sundman transformation as a method for linearization has been studied in [53,54]. More recently, the Cartan equivalency method was considered in [55], where a straightforward procedure was established for the linearization of third-order differential equations using a four-dimensional Lie algebra [55].

The linearization of higher-order ordinary differential equations through Lie symmetries has been extensively discussed in [47,56,57] and related references. However, the linearization of partial differential equations holds special interest and has yielded many important results, as seen in [58].

In the context of systems of nonlinear differential equations, the existence of transformations that linearize these equations is highly significant, as it offers an alternative approach to studying the integrability of dynamical systems. Due to the complexity of the problem, various criteria have been proposed in the literature for the linearization of higher-dimensional dynamical systems. For example, in [59], it was demonstrated that a system of second-order differential equations admitting four Lie point symmetries forming the $A_{4,1}$ or $A_{4,2}$ Lie algebra [60] can be transformed into a linear form through a point transformation. Furthermore, the introduction of complex Lie symmetries has led to new linearization criteria for systems of second-order ordinary differential equations, as discussed in the series of studies [61–63].

Nevertheless, the linearization process is inherently a geometric approach. Therefore, studies in the literature have shown that a system of differential equations is linearizable based on specific geometric properties [64]. For systems of second-order ordinary differential equations of the form

$$\ddot{q}^i + \alpha^i_{srj}\left(t, q^k\right)\dot{q}^s\dot{q}^r\dot{q}^j + \beta^i_{rj}\left(t, q^k\right)\dot{q}^r\dot{q}^j + \gamma^i_r\left(t, q^k\right)\dot{q}^r + \delta^i\left(t, q^k\right) = 0,$$

the coefficients α^i_{srj}, β^i_{rj}, γ^i_r, δ^i can be related to the connection coefficient of an extended manifold [13], where, if the connection has zero curvature, i.e., the geometry is flat, then there exists a point transformation that linearizes the latter system. For some applications of this method, we refer the reader to [65].

Recently, in [66], a novel approach to geometric linearization for a family of second-order differential equations was discovered. It was found that this new family of dynamical systems can be linearized by introducing new dependent variables. The solution of the extended system can then be expressed in terms of solutions to linear equations. In a similar vein, it was discovered that solutions to Einstein's field equations for certain gravitational models can also be represented in terms of linear equations [67].

In the following section, we establish a new geometric criterion for the global linearization of a family of constraint Hamiltonian systems.

4. Constraint Hamiltonian Systems

We introduce the Lagrangian function

$$L\left(N,q^k,\dot{q}^k\right) = \frac{1}{2N}g_{ij}\dot{q}^i\dot{q}^j - NV(q^k), \tag{22}$$

where $g_{ij}\left(q^k\right)$ is a second-rank tensor with inverse g^{ij}, and $N(t)$, $q^i(t)$ are the $(1+n)$ degrees of freedom.

For the Lagrangian (22), it follows $\det\left|\frac{\partial^2 L}{\partial Q^A \partial Q^B}\right| = 0$, where $Q^A = \left(N(t), q^i(t)\right)$. Hence, the dynamical system described by the Lagrangian (22) is singular.

Variation with respect to the variable $N(t)$ leads to the constraint equation $\frac{\partial L}{\partial N} = 0$, i.e.,

$$\frac{1}{2N^2}g_{ij}\dot{q}^i\dot{q}^j + V(q^k) = 0. \tag{23}$$

We introduce the momentum $p_i = \frac{1}{N}g_{ij}\dot{q}^j$; thus, the latter constraint reads

$$H(q,p) \equiv \frac{1}{2}g^{ij}p_i p_j + V(q^k) = 0. \tag{24}$$

Function $H(q,p)$ represents the Hamiltonian of (22), which is a constraint due to the existence of expression (24). Dynamical systems described by Lagrangians of the form (22) are of special interest in gravitational physics [68] and in other physical theories (see the discussion in [69]). Nevertheless, any regular Hamiltonian system of the form

$$H_R(q,p,h) \equiv \frac{1}{2}g^{ij}p_i p_j + U(q^k) = h, \tag{25}$$

can be written in the singular form by absorbing the integration constant h inside the potential, i.e., $U\left(q^k\right) = V(q^k) - h$.

An important characteristic of this family of constrained dynamical systems is their invariance under time reparametrization. Below, we demonstrate this property.

The action integral related to the Lagrangian function (22) is

$$S = \int\left(\frac{1}{2N}g_{ij}\dot{q}^i\dot{q}^j - NV(q^k)\right)dt, \tag{26}$$

and the corresponding equations of motion are

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k + V^{,i} - 2(\ln N)^{,i}\left(\frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j + V(q^k)\right) = 0, \tag{27}$$

$$\frac{1}{2N^2}g_{ij}\dot{q}^i\dot{q}^j + V(q^k) = 0, \tag{28}$$

or, due to the constraint equation, the second-order differential equation reads

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k + V^{,i} = 0, \tag{29}$$

where Γ^i_{jk} is the Levi–Civita connection related to the tensor g_{ij}.

Under the change in the independent variable $M(\tau)d\tau = dt$, it follows that

$$S = \int \tilde{L}\left(\tilde{N}, q^k, \frac{dq^k}{d\tau}\right) d\tau \;,\; \tilde{N} = N(\tau)M(\tau), \tag{30}$$

where

$$\tilde{L}\left(\tilde{N}, q^k, \frac{dq^k}{d\tau}\right) = \frac{1}{2\tilde{N}} g_{ij} \frac{dq^i}{d\tau}\frac{dq^j}{d\tau} - \tilde{N}V\left(q^k\right), \tag{31}$$

is the conformally related Lagrangian. The corresponding equations of motion are derived as

$$\frac{d^2q^i}{d\tau^2} + \Gamma^i_{jk}\frac{dq^j}{d\tau}\frac{dq^k}{d\tau} + V^{,i} = 0. \tag{32}$$

$$\frac{1}{2N^2} g_{ij}\frac{dq^i}{d\tau}\frac{dq^j}{d\tau} + V\left(q^k\right) = 0 \tag{33}$$

It is evident that the equations of motion remain invariant under a time reparametrization where the parameter $N(t)$ does not affect the dynamical behavior of the constrained system. Specifically, the conformally related Lagrangians $L\left(N, q^k, \dot{q}^k\right)$, $\tilde{L}\left(\tilde{N}, q^k, \frac{dq^k}{d\tau}\right)$ have common equations of motion.

4.1. Symmetry Analysis

The Lie and Noether symmetries for the constraint Hamiltonian systems with Lagrangians of the form (22) have been investigated in detail before in [70]. Owing to the constraint (23), the symmetry analysis differs from that of regular systems; see, for example, [71]. Indeed, for regular dynamical systems and for the equations of motion of the form (29), the Lie symmetries are constructed by the elements of the projective algebra of the connection Γ^i_{jk}. Additionally, Noether symmetries are linked to homothetic symmetries of the metric tensor g_{ij} [71].

Indeed, if $\Phi\left(t, q^k, p^k\right)$ is a conservation law for the dynamical system with Hamiltonian (24), it holds that

$$\frac{d\Phi}{dt} \equiv \frac{\partial\Phi}{\partial t} + \{\Phi, H\} = 0, \tag{34}$$

where $\{,\}$ is the Poisson bracket.

Nevertheless in order to make use of the Hamiltonian constraint, the latter condition can be relaxed such that

$$\frac{d\Phi}{dt} \equiv \frac{\partial\Phi}{\partial t} + \{\Phi, H\} = \chi H \approx 0, \tag{35}$$

in which χ is a conformal factor.

Thus, from (35), it follows that the symmetries of constraint Hamiltonian systems are determined by the conformal symmetries of the metric tensor g_{ij} [70].

4.2. Geodesic Description

The process of geometrizing dynamical systems of the form (22) involves formulating the dynamical system as a set of geodesic equations, where the potential term can be interpreted as part of the geometry. Two different approaches for this are the Jacobi metric and the Eisenhart lift, both of which incorporate the potential, i.e., conservative forces, into the geometry.

In the Jacobi metric approach (for a recent discussion, see [28]), we introduce $\tilde{N} = \frac{1}{V}$, such that the Lagrangian is

$$\bar{L}\left(q^k, \dot{q}^k\right) = \frac{1}{2}\bar{g}_{ij}\dot{q}^i\dot{q}^j - 1\ ,\ \bar{g}_{ij} = V(q)g_{ij}. \tag{36}$$

Thus, the constraint Equation (23) becomes

$$\frac{1}{2}\bar{g}_{ij}\dot{q}^i\dot{q}^j + 1 = 0 \tag{37}$$

which means that Lagrangian (36) describes the time/space-like geodesic equations for the conformal metric $\bar{g}_{ij}$. It is well known that the geodesic equations of the metric $\bar{g}_{ij}$, which can be linearized, are those of the flat space, i.e., $\bar{g}_{ij}$ should be flat.

The Jacobi metric is obtained through a conformal transformation where $\bar{g}_{ij}$ is conformally related to the metric tensor g_{ij}. Conformally related spacetimes share the same conformal symmetries, implying that the symmetries and conservation laws of the original dynamical system also exist for the Jacobi metric. Notably, conformal symmetries in this context transform into isometries. On the other hand, the Eisenhart lift approach involves augmenting the dimensionality of the system [29,72].

Geometrization is achieved by introducing additional dimensions through new dependent variables. The potential term becomes integrated into the newly extended metric tensor. These extended spaces possess isometries that correspond to Noetherian conservation laws for the geodesic equations. When these conservation laws are applied within the extended system, the original dynamics are recovered.

In the Eisenhart approach, we write the extended Lagrangian function

$$L_{lift}\left(N, q^k, \dot{q}^k, z, \dot{z}\right) = \frac{1}{2N}\left(g_{ij}\dot{q}^i\dot{q}^j + \frac{1}{V(q^k)}\dot{z}^2\right), \tag{38}$$

with the constraint equation and conservation law

$$g_{ij}\dot{q}^i\dot{q}^j + \frac{1}{V(q^k)}\dot{z}^2 = 0\ ,\ \frac{1}{V(q^k)}\dot{z} = I_0. \tag{39}$$

Thus, by replacing the $\dot{z} = I_0 V\left(q^k\right)$ in the equations of motion and the constraint equation, the original system is recovered when $(I_0)^2 = 1$.

The Lagrangian functions (38) describe a set of null geodesic equations in the extended manifold with metric

$$\hat{g}_{AB}\left(q^k, z\right) = g_{ij}\left(q^k\right) + \frac{1}{V(q^k)}\delta^z_A\delta^z_B, \tag{40}$$

where $A, B = i, j \ldots, z$, are the indices of the extended geometry.

5. Geometric Linearization

In this section, we will establish new geometric criteria for the global linearization of the equations of motion described by the Lagrangian (22).

We will utilize the two geometrization approaches described previously. Specifically, for the equivalent Lagrangian (36) obtained through the Jacobi metric approach, we will employ the Eisenhart lift. This allows us to describe the dynamical system using the equivalent/extended Lagrangian function

$$\bar{L}_{lift} = \frac{1}{2}\bar{g}_{ij}\dot{q}^i\dot{q}^j + \frac{1}{2}\dot{z}^2 \tag{41}$$

with constraints

$$\bar{g}_{ij}\dot{q}^i\dot{q}^j + \dot{z}^2 = 0\ ,\ \dot{z} = \pm 1. \tag{42}$$

The Lagrangian (41) describes the equations of motion for the null geodesics of the extended metric tensor $\tilde{g}_{AB}\left(q^k,z\right) = \bar{g}_{ij}\left(q^k\right) + \delta^z_A\delta^z_B$.

Consequently, if the metric tensor $\tilde{g}_{AB}$ is conformally flat, then there exists a point transformation $\left(q^A,z\right) = Q^A$ such that $\tilde{g}_{AB} = e^{2U}\eta_{AB}$, where η_{AB} is the flat space in diagonal coordinates. Thus, the Lagrangian (41) and the constraint (42) become

$$\bar{L}_{lift} = e^{2U}\eta_{AB}\dot{Q}^A\dot{Q}^B, \tag{43}$$

$$\eta_{AB}\dot{Q}^A\dot{Q}^B = 0. \tag{44}$$

Therefore, after the change in the independent variable $d\tau = e^{-2U}dt$, we obtain the equivalent geodesic equations

$$\ddot{Q}^A = 0\,, \tag{45}$$

which are the equations of motion for the free particle in the flat space.

Thus, regarding the linearization of a dynamical system described by the singular Lagrangian function (22), the following theorem can be stated.

Theorem 1. *The n-dimensional, with $n \geq 2$, constraint Hamiltonian system described by the Lagrangian function (22) is globally linearizable if one of the following equivalent statements is true.*

(A) The admitted (non-trivial) Noether symmetries of the constraint Hamiltonian system are $\frac{n(n+1)}{2}$.

(B) The Jacobi metric $\bar{g}_{ij}$ is maximally symmetric.

(C) The extended $1+n$ decomposable space with metric $\tilde{g}_{AB}$ is conformally flat.

Proof. All statements of Theorem 1 are equivalent. As discussed before, if the extended $1+n$ decomposable space with metric $\tilde{g}_{AB}$ is conformally flat, then the equations of motion are linearizable through point transformations in the space $\{q^A,z\}$.

Nevertheless, $\tilde{g}_{AB}$ is conformally flat if and only if the n-dimensional space with metric $\bar{g}_{ij}$ is maximally symmetric (see Proposition 2 in [73]).

However, when $\bar{g}_{ij}$ is maximally symmetric, it means that $\bar{g}_{ij}$ admits $\frac{n(n+1)}{2}$ isometries, as many as the Noether symmetries for the original Lagrangian (22).

The inverse proof is straightforward. □

It is important to note that two-dimensional spaces are maximally symmetric when they have a constant curvature. Conversely, three-dimensional spaces are conformally flat when the Cotton–York tensor vanishes. Therefore, for the case of two-dimensional systems, from Theorem 1, we derive the following corollary.

Corollary 1. *The two-dimensional constraint Hamiltonian systems described by the Lagrangian function (22) are globally linearizable if one of the following equivalent statements is true.*

(A) The admitted Noether symmetries of the constraint Hamiltonian system total three.

(B) There exists a point transformation where the two-dimensional Jacobi metric can be written as $\bar{g}_{ij} = diag\left(e^{2U(x,y)}, e^{2U(x,y)}\right)$, and $U(x,y)$ is a solution of the equation $U_{,xx} + U_{,yy} + 2\kappa e^{2U} = 0$, in which κ is the curvature of the two-dimensional space and $V(x,y) = e^{-2U(x,y)}$.

(C) The Cotton–York tensor $C_{\mu\nu\kappa} = R_{\mu\nu;\kappa} - R_{\kappa\nu;\mu} + \frac{1}{4}\left(R_{;\nu}g_{\mu\kappa} - R_{;\kappa}g_{\mu\nu}\right)$ for the three-dimensional extended metric $\tilde{g}_{AB}$ has zero components.

Corollary 2. *The one-dimensional constraint Hamiltonian system described by the Lagrangian function (22) is always globally linearizable for an arbitrary potential function.*

Proof. Consider the one-dimensional constraint Hamiltonian system with Lagrangian function $L(N,q,\dot{q}) = \frac{1}{2N}\dot{q}^2 - NV(q)$. After the application of Jacobi and Eisenhart's methods, we find the equivalent dynamical system of null geodesics for the two-dimensional space. All two-dimensional spaces are conformally flat, which means that the null geodesics can be globally linearizable. In particular, the extended two-dimensional space is described by the decomposable line element $ds^2 = \frac{1}{V(q)}dq^2 + dz^2$. Thus, under the change in variable $\sqrt{\frac{1}{V(q)}}dq = dY$, it follows that $ds^2 = dY^2 + dz^2$, which is a flat space.

On the other hand, the Jacobi metric approach leads to the equivalent system of time/space-like geodesic equations for the one-dimensional line element $d\bar{s}^2 = \frac{1}{V(q)}dq^2$. Because the one-dimensional space is the flat space, the point transformation $\sqrt{\frac{1}{V(q)}}dq = dY$ leads to the linearization of the dynamical system. □

6. Applications

In the following lines, we demonstrate the application of Theorem 1 for the geometric linearization of Hamiltonian systems of special interest.

6.1. Exponential Interaction

Consider two particles that interact with an exponential law, similarly to that of the Toda lattice. The Lagrangian that describes the dynamical system is

$$L(q_1, \dot{q}_1, q_2, \dot{q}_2) = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - V_0 e^{2\gamma(q_1 - q_2)}. \tag{46}$$

where the Hamiltonian function is $H = h$.

We introduce the equivalent singular Lagrangian

$$\bar{L}(N, q_1, \dot{q}_1, q_2, \dot{q}_2) = \frac{1}{2N}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - N\left(V_0 e^{q_1 - q_2} - h\right), \tag{47}$$

with constraint equation

$$\bar{H} \equiv \frac{1}{2}\left(p_1^2 + p_2^2\right) + V_0 e^{q_1 - q_2} - h = 0, \tag{48}$$

in which $p_1 = \dot{q}_1$, $p_2 = \dot{q}_2$.

We employ the Eisenhart lift and we write the equivalent Lagrangian of the form (38), i.e.,

$$\bar{L}_{lift}(N, q_1, \dot{q}_1, q_2, \dot{q}_2) = \frac{1}{2N}\left(\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{(V_0 e^{q_1 - q_2} - h)}\dot{z}^2\right), \tag{49}$$

where the extended three-dimensional metric reads

$$ds_{lift}^2 = dq_1^2 + dq_2^2 + \frac{1}{(V_0 e^{q_1 - q_2} - h)}dz^2. \tag{50}$$

We calculate the Cotton–York tensor and we find that it is zero when $h = 0$. Thus, from Corollary 1, it follows that the dynamical system can be written in the equivalent form of the three-dimensional free particle of the flat geometry.

Under the conformal transformation $d\bar{s}_{lift}^2 = \frac{1}{XY}ds_{lift}^2$ and the change in variables

$$ix = -\frac{1}{4}(1+i)\ln Y + \frac{1}{4}(1-i)\ln X, \tag{51}$$

$$iy = \frac{1}{4}(1-i)(i\ln X - \ln Y), \tag{52}$$

the extended space $d\bar{s}_{lift}^2$ becomes

$$ds_{lift}^2 = \frac{1}{2}dXdY + dz^2. \tag{53}$$

where the null geodesics read

$$\ddot{X}=0\,,\ddot{Y}=0\,,\ \ddot{z}=0, \tag{54}$$

and the constraint equation is

$$\frac{1}{2}\dot{X}\dot{Y}+\dot{z}^2=0. \tag{55}$$

6.2. Two-Dimensional Oscillator with Corrections

We assume the following singular Lagrangian, which describes a two-dimensional oscillator with correction terms:

$$L(N,x,\dot{x},y,\dot{y})=\frac{1}{2N}\left(\dot{x}^2+\dot{y}^2\right)+N\left(1+\frac{\kappa}{4}\left(x^2+y^2\right)\right)^2. \tag{56}$$

We calculate the Jacobi metric, i.e., the corresponding line element is of the form

$$ds^2_{Jacobi}=\frac{1}{\left(1+\frac{\kappa}{4}(x^2+y^2)\right)^2}\left(dx^2+dy^2\right) \tag{57}$$

from which we observe that it is a space of constant κ non-zero curvature, i.e., it is a maximally symmetric space.

We employ Eisenhart's lift and we write the equivalent extended Lagrangian

$$L_{lift}(N,x,\dot{x},y,\dot{y})=\frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{\left(1+\frac{\kappa}{4}(x^2+y^2)\right)^2}+\frac{1}{2}\dot{z}^2, \tag{58}$$

with constraint

$$\frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{\left(1+\frac{\kappa}{4}(x^2+y^2)\right)^2}+\frac{1}{2}\dot{z}^2=0. \tag{59}$$

For the three-dimensional extended space with line element

$$ds^2_{Lift}=ds^2_{Jacobi}+dz^2 \tag{60}$$

we calculate the Cotton–York tensor components, which are zero. Hence, the line element (60) is conformally flat.

Under the change in variables

$$x=\frac{X\left(Z+\sqrt{Z^2+16\kappa(X^2+Y^2)}\right)}{2\kappa(X^2+Y^2)}\,,\ y=\frac{Y\left(Z+\sqrt{Z^2+16\kappa(X^2+Y^2)}\right)}{2\kappa(X^2+Y^2)}, \tag{61}$$

$$e^{\sqrt{K}z}=2\kappa\frac{\left(X^2+Y^2\right)}{Z+\sqrt{Z^2+16\kappa(X^2+Y^2)}}\left(\frac{8\left(X^2+Y^2\right)}{\left(Z+\sqrt{Z^2+16\kappa(X^2+Y^2)}\right)}+Z\right), \tag{62}$$

the three-dimensional space (60) reads

$$ds^2_{Lift}=e^{\sqrt{K}z(X,Y,Z)}\left(dX^2+dY^2+dZ^2\right), \tag{63}$$

with null geodesic equations

$$\ddot{X}=0\,,\ \ddot{Y}=0\,,\ \ddot{Z}=0, \tag{64}$$

and constraint $\dot{X}^2+\dot{Y}^2+\dot{Z}^2=0$.

This two-dimensional linearizable example can be generalized to an n-dimensional system in a similar way.

Corollary 3. *The n-dimensional dynamical system with Lagrangian*

$$L\left(N,q^k,\dot{q}^k\right)=\frac{1}{2N}\left(\eta_{ij}\dot{q}^i\dot{q}^j\right)+N\left(1+\frac{\kappa}{4}\left(\eta_{ij}q^iq^j\right)\right)^2, \tag{65}$$

can be linearized though the Jacobi metric and the Eisenhart lift.

6.3. The Szekeres System

Consider the two-dimensional dynamical system described by the regular Lagrangian function [74,75]

$$L_R(u,\dot{u},v,\dot{v})=\dot{u}\dot{v}-\frac{v}{u^2}, \tag{66}$$

and equations of motion

$$\ddot{u}+u^{-2}=0\,,\;\ddot{v}-2vu^{-3}=0. \tag{67}$$

Furthermore, the Hamiltonian function is

$$H\equiv\dot{u}\dot{v}+\frac{v}{u^2}=h \tag{68}$$

where h is a constant.

The system described above corresponds to the dynamics governed by the Einstein field equations for Szekeres spacetimes, commonly referred to as the Szekeres system [76]. Szekeres spacetimes represent exact inhomogeneous solutions with a dust fluid and find various applications in gravitational physics. For more detailed information, the interested reader is referred to [77].

We introduce the equivalent singular Lagrangian function

$$L(N,u,\dot{u},v,\dot{v})=\frac{\dot{u}\dot{v}}{N}-N\left(\frac{v}{u^2}-h\right), \tag{69}$$

such that we write the original system in the form of a constraint Hamiltonian dynamical system.

The latter singular Lagrangian for arbitrary parameter h has no Noetherian symmetry. In particular, it admits a hidden symmetry related to a quadratic conservation law [74]. However, for $h=0$, Lagrangian (69) admits three Noether symmetries:

$$X_1=\frac{1}{v}\partial_v\,,\;X_2=u^2\partial_u\,,\;X_3=v\partial_v+2u\partial_u. \tag{70}$$

Thus, according to the first statement of Theorem 1, on the surface with $h=0$, the equations of motion can be linearized. Moreover, the solution space with initial conditions $h=0$ is equivalent to the equations of motion for the two-dimensional flat space.

Indeed, we employ Jacobi's approach and, from (69), we define the equivalent geodesic Lagrangian

$$\bar{L}(u,\dot{u},v,\dot{v})=\frac{v}{u^2}\dot{u}\dot{v}-1. \tag{71}$$

Hence, under the change in variables

$$\frac{du}{u^2}=dU,\;vdv=dV, \tag{72}$$

it follows that

$$L\left(U,\dot{U},V,\dot{V}\right)=\dot{U}\dot{V}-1, \tag{73}$$

with the equations of motion

$$\ddot{U}=0,\;\ddot{V}=0 \tag{74}$$

and the constraint equation

$$\dot{U}\dot{V}+1=0. \tag{75}$$

6.4. The Cosmological Constant in the Szekeres Model

The introduction of the cosmological constant in the Szekeres model [78] leads to the modification of the regular Lagrangian (66).

Specifically, the new dynamics follows from the Lagrangian function [79]

$$L_R^{\Lambda}(u,\dot{u},v,\dot{v}) = \dot{u}\dot{v} - \left(\frac{v}{u^2} - \Lambda uv\right), \tag{76}$$

in which Λ is the cosmological constant term.

In a similar approach as before, we introduce the singular Lagrangian function

$$L^{\Lambda}(N,u,\dot{u},v,\dot{v}) = \dot{u}\dot{v} - \left(\frac{v}{u^2} - \Lambda uv - h\right) \tag{77}$$

Thus, the Jacobi metric is defined as

$$ds^2_{Jacobi} = \frac{1}{\left(\frac{v}{u^2} - \Lambda uv - h\right)} dudv. \tag{78}$$

We calculate the Ricci scalar for the two-dimensional space, and it is

$$R_{Jacobi} = -\frac{4\left(2+\Lambda u^3\right)h}{u\left(v\left(\Lambda u^3 - 1\right) + hu^2\right)}. \tag{79}$$

Hence, for $h = 0$, the two-dimensional Jacobi metric (78) describes the flat space, i.e., the equations of motion can be linearized though a point transformation.

Indeed, under the point transformation

$$\frac{du}{\left(\frac{1}{u^2} - \Lambda u\right)} = dU\ ,\ \frac{dv}{v} = dV, \tag{80}$$

the Jacobi metric reads

$$ds^2_{Jacobi} = dUdV, \tag{81}$$

which leads to the geodesic equations of the flat space, i.e.,

$$\ddot{U} = 0\ ,\ \ddot{V} = 0. \tag{82}$$

6.5. Static Spherical Symmetric Spacetime with Charge

Einstein's gravitational field equations for a static spherical symmetric spacetime with a charge are described by the variation of the singular Lagrangian [80]

$$L^{RN}\left(N,a,a',b,b',\zeta,\zeta'\right) = \frac{1}{2N}\left(8ba'b' + 4ab'^2 + 4\frac{b^2}{a}\zeta'^2\right) + 2Na, \tag{83}$$

where N, a, b are the scale factors of the background geometry with line element

$$ds^2 = -a(r)^2dt^2 + N(r)^2dr^2 + b^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{84}$$

and a prime means a total derivative with respect to the independent parameter r, i.e., $a' = \frac{da}{dr}$. Function ζ is related to the charge. The solution of the field equations is known as the Reissner–Nordström black hole [81,82]

The kinetic term of the singular Lagrangian (83) is defined by a three-dimensional space. The admitted Noether symmetries of Lagrangian (83) are calculated to be six [80]; they are [67]

$$X^1 = \frac{1}{ab}\partial_a\ ,\ X^2 = -a\partial_a + b\partial_b - z\partial_\zeta\ ,$$

$$X^3 = -\left(\frac{a}{2b} + \frac{z^2}{ab}\right)\partial_a + \partial_b - \frac{\zeta}{b}\partial_\zeta\ ,$$

$$X^4 = -a\zeta\partial_a + b\zeta\partial_b + \left(\frac{a^2}{4} - \frac{z^2}{2}\right)\partial_\zeta,$$

$$X^5 = \frac{2\zeta}{ab}\partial_a + \frac{1}{b}\partial_\zeta\ ,\ X^6 = \partial_\zeta.$$

Therefore, case A of Theorem 1 states that the field equations can be written in the equivalent form of the free particle. This transformation is derived before in [67], where the common solution space for a large family of gravitational models is investigated.

We employ the Eisenhart lift and we write the extended line element

$$ds^{2\ m}_{lift} = \frac{1}{N}\left(8b\ da\ db + 4a\ db^2 + 4\frac{b^2}{a}d\zeta^2 - \frac{d\psi^2}{2a}\right) \tag{85}$$

and, under the point transformation

$$a = \sqrt{\frac{X+Y}{X-Y} + \frac{z^2}{(X-Y)^2}},\ \zeta = \frac{z}{(X-Y)^2}, \tag{86}$$

the latter line element becomes

$$ds^{RN\ 2} = \frac{1}{n}\frac{X-Y}{\sqrt{X^2-Y^2+z^2}}\left(4dX^2 - 4dY^2 + 4dz^2 - d\psi^2\right) \tag{87}$$

where the corresponding null geodesics are written in the linear form

$$X'' = 0\ ,\ Y'' = 0,\ z'' = 0\ ,\ \psi'' = 0. \tag{88}$$

and the Hamiltonian constraint is

$$\dot{X}^2 - \dot{Y}^2 + \dot{z}^2 - \frac{1}{4}\dot{\psi}^2 = 0. \tag{89}$$

7. Conclusions

In this study, by using the Jacobi metric and the Eisenhart lift, we establish a new criterion for the global linearization of constrained Hamiltonian systems. The requirements for the dynamical system are to be in the form of (22) with the constraint expression (23). The n-dimensional dynamical system must admit $\frac{n(n+1)}{2}$ Noether point symmetries, which correspond to a number of $\frac{n(n+1)}{2}$ independent conservation laws. This property indicates that the given dynamical system, constrained by Equation (23), posseses the property of superintegrability.

The linearization of the dynamical system is achieved through geometry and is based on the linearization of the equivalent system that describes the geodesic equations for an extended geometry. The main result of this analysis is summarized in Theorem 1, where three equivalent statements describe the number of admitted symmetries and the geometric characteristics of the Jacobi metric and of the extended Eisenhart metric.

Two immediate results are given in Corollaries 1 and 2. Corollary 1 specifies the statements of Theorem 1 for the case of two-dimensional dynamical systems, while Corollary 2 states that all one-dimensional constraint Hamiltonian systems can be globally linearized. This geometric linearization for one-dimensional systems is possible because the two-dimensional extended Eisenhart space is always conformally flat.

To demonstrate the application of this new geometric approach, we present a series of applications of special interest. We focus on two dynamical systems of analytic mechanics and on some gravitational models. Specifically, we consider the Szekeres system, which describes inhomogeneous cosmological models with or without the cosmological constant term, and the Reissner–Nordström black hole.

A criticism that can be made is that this specific algorithm fails in the case of the simplest maximally symmetric system, which is that of the n-dimensional harmonic oscillator with Lagrangian

$$L\left(N,q^k,\dot{q}^k\right) = \frac{1}{2}\eta_{ij}\dot{q}^i\dot{q}^j - \frac{\omega^2}{2}\eta_{ij}q^iq^j. \quad (90)$$

We should clarify that the geometric linearization discussed in this study is based on the existence of symmetries generated by conformal symmetries of the metric tensor g_{ij}, which defines the kinetic term in (22). As far as the symmetries of the oscillator (90) are concerned, they are related to the projective algebra [83] and not to the conformal algebra. Nevertheless, only the isometries are the common subalgebra for the projective and conformal algebras. An alternative Eisenhart lift has been proposed in [66], where the oscillator is reduced to the free particle. This is possible with the introduction of an extended space that belongs to the pp-wave geometries. In this case, the dimension of the extended space is increased by two and the resulting Eisenhart metric is conformally flat.

Thus, any dynamical system of the form (22), where an extended Eisenhart metric can be constructed to be conformally flat, can be linearized. This means that Theorem 1 can be generalized for other families of Eisenhart lifts.

This work opens new directions in the study of nonlinear differential equations and demonstrates that geometry is a powerful tool for the study of dynamical systems.

A natural extension of this geometric consideration is to the study of the Klein–Gordon equation, which describes the quantum limit of the constraint Hamiltonian system with Lagrangian (22).

Specifically, the quantization of the constraint Hamiltonian (24) leads to the Yamabe equation [84]

$$\hat{\Delta}\Psi + V\left(q^k\right)\Psi = 0, \quad (91)$$

where

$$\hat{\Delta} = \Delta + \frac{n-2}{4(n-1)}R \quad (92)$$

and $\Delta = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial q^i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x^j}\right)$ is the usual Laplace operator, and R is the Ricci scalar for the metric tensor g_{ij} that defines the kinetic term and $n = \dim g_{ij}$. The introduction of the second term in (92) is necessary in order for the equation to be invariant under conformal transformations.

We demonstrate the application of the geometric approach in Equation (91). Let us study the Yamabe equation for the Szekeres system (69) with $h = 0$. The equation that describes the quantum system is

$$\frac{u^2}{v}\left(\Psi_{,uv} - \frac{v}{u^2}\Psi\right) = 0,\ \Psi = \Phi(u,v) \quad (93)$$

Assume now the following equation:

$$\frac{u^2}{v}\Phi_{,uv} + \Phi_{,zz} = 0\ ,\ \Phi = \Phi(u,v,z). \quad (94)$$

The vector field $\partial_z - i\Phi\partial_\Phi$, is a Lie symmetry for Equation (94). The corresponding invariants are $\{u,v,\Phi = \Psi e^{iz}\}$. Thus, by replacing (94), we end with Equation (93).

Under the change in variables (72), and $U = X + Y$, $V = X - Y$, Equation (94) reads

$$\Phi_{,XX} - \Phi_{,YY} + \Phi_{,zz} = 0, \quad (95)$$

which is the wave equation for the three-dimensional flat space. Equation (95) is maximally symmetric and admits ten Lie point symmetries plus the infinity symmetries related to the infinite number of solutions of the linear equation (for more details, we refer to [85]). On the other hand, Equation (93) admits only three Lie point symmetries (plus the infinity symmetries). Therefore, the symmetries of the maximally symmetric Equation (94), which does not survive under the reduction with the invariants $\{u, v, \Phi = \Psi e^{iz}\}$, become nonlocal symmetries, which can be used for the construction of new solutions for the inhomogeneous Equation (93) related to solutions for the homogeneous Equation (95).

In a future work, we plan to investigate further this geometric consideration for the analysis of other dynamical systems and of partial differential equations.

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References

1. Lie, S. *Theorie der Transformationsgrupprn: Vol I*; Chelsea: New York, NY, USA, 1970.
2. Lie, S. *Theorie der Transformationsgrupprn: Vol II*; Chelsea: New York, NY, USA, 1970.
3. Lie, S. *Theorie der Transformationsgrupprn: Vol III*; Chelsea: New York, NY, USA, 1970.
4. Ibragimov, N.H. *CRC Handbook of Lie Group Analysis of Differential Equations, Volume I: Symmetries, Exact Solutions, and Conservation Laws*; CRS Press LLC: Boca Raton, FL, USA, 2000.
5. Stephani, H. *Differential Equations: Their Solutions Using Symmetry*; Cambridge University Press: New York, NY, USA, 1989.
6. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1993.
7. Bluman, G.W.; Kumei, S. *Symmetries of Differential Equations*; Springer: New York, NY, USA, 1989.
8. Meleshko, S.; Samatova, N. Group classification of the two-dimensional shallow water equations with the beta-plane approximation of coriolis parameter in Lagrangian coordinates. *Comm. Nonl. Sci. Num. Sim.* **2020**, *90*, 105337. [CrossRef]
9. Chesnokov, A.A. Symmetries and exact solutions of the rotating shallow-water equations. *J. Appl. Mech. Techn. Phys.* **2008**, *49*, 737. [CrossRef]
10. Torrisi, M.; Tracina, R. Symmetries and Conservation Laws for a Class of Fourth-Order Reaction–Diffusion–Advection Equations. *Symmetry* **2023**, *15*, 1936. [CrossRef]
11. Diaz, L.B.; Sardon, C.; Alburquerque, F.J.; de Lucas, J. Geometric Numerical Methods for Lie Systems and Their Application in Optimal Control. *Symmetry* **2023**, *15*, 1285. [CrossRef]
12. Dorodnitsyn, V.A.; Kaptsov, E.I. Shallow water equations in Lagrangian coordinates: Symmetries, conservation laws and its preservation in difference models. *Comm. Nonl. Sci. Num. Sim.* **2020**, *89*, 105343. [CrossRef]
13. Aminova, A.V. Projective transformations and symmetries of differential equations. *Sb. Math.* **1995**, *186*, 1711. [CrossRef]
14. Liu, H.; Yun, Y. Lie Symmetry Analysis and Conservation Laws of Fractional Benjamin–Ono Equation. *Symmetry* **2024**, *16*, 473. [CrossRef]
15. Bokhari, A.H.; Kara, A.H.; Kashif, A.R.; Zaman, F.D. Noether Symmetries Versus Killing Vectors and Isometries of Spacetimes. *Int. J. Theor. Phys.* **2006**, *45*, 1029. [CrossRef]
16. Bokhari, A.H.; Johnpillai, A.G.; Kara, A.H.; Mahomed, F.M.; Zaman, F.D. Classification of Static Spherically Symmetric Spacetimes by Noether Symmetries. *Int. J. Theor. Phys.* **2013**, *52*, 3534. [CrossRef]
17. Oliveri, F. Lie Symmetries of Differential Equations: Classical Results and Recent Contributions. *Symmetry* **2010**, *2*, 658. [CrossRef]
18. Belmonte-Beitia, J.; Pérez-García, V.M.; Vekslerchik, V.; Torres, P.J. Lie Symmetries and Solitons in Nonlinear Systems with Spatially Inhomogeneous Nonlinearities. *Phys. Rev. Lett.* **2007**, *98*, 064102. [CrossRef]
19. Jamal, S. Quadratic integrals of a multi-scalar cosmological model. *Mod. Phys. Lett. A* **2020**, *35*, 2050068. [CrossRef]
20. Hojman, S.A. A new conservation law constructed without using either Lagrangians or Hamiltonians. *J. Phys. A Math. Gen.* **1992**, *25*, L291. [CrossRef]
21. Pillay, T.; Leach, P.G.L. Comment on a theorem of Hojman and its generalizations. *J. Phys. A Math. Gen.* **1996**, *29*, 6999. [CrossRef]
22. Zhang, H.-B.; Chen, L.-Q. The Unified Form of Hojman's Conservation Law and Lutzky's Conservation Law. *J. Phys. Soc. Jpn.* **2005**, *74*, 905. [CrossRef]
23. Chanda, S.; Gibbons, G.W.; Guha, P. Jacobi–Maupertuis metric and Kepler equation. *Int. J. Geom. Meth. Mod. Phys.* **2017**, *14*, 1730002. [CrossRef]
24. Szydłowski, M. Desingularization of Jacobi metrics and chaos in general relativity. *J. Math. Phys.* **1999**, *40*, 3519. [CrossRef]
25. Duenas-Vidal, A.; Andino, O.L. The Jacobi metric approach for dynamical wormholes. *Gen. Relat. Grav.* **2023**, *55*, 9. [CrossRef]

26. Szydlowski, M.; Heller, M.; Sasin, W. Geometry of spaces with the Jacobi metric. *J. Math. Phys.* **1996**, *37*, 346. [CrossRef]
27. Gibbons, G.W. The Jacobi metric for timelike geodesics in static spacetimes. *Class. Quantum Grav.* **2016**, *33*, 025004. [CrossRef]
28. Maraner, P. On the Jacobi metric for a general Lagrangian system. *J. Math. Phys.* **2019**, *60*, 112901. [CrossRef]
29. Eisenhart, L.P. Dynamical Trajectories and Geodesics. *Ann. Math.* **1928**, *30*, 591. [CrossRef]
30. Duval, C.; Burdet, G.; Kunzle, H.P.; Perrin, M. Bargmann structures and Newton-Cartan theory. *Phys. Rev. D* **1985**, *1841*, 31. [CrossRef]
31. Kaluza, T. *Zum unitätsproblem der physik Itzungsber*; Sitzber Kgl Preuss Akad Wiss: Berlin, Germany, 1921; p. 966.
32. Klein, O. The atomicity of electricity as a quantum theory law. *Nature* **1926**, *118*, 516. [CrossRef]
33. Fordy, A.P.; Galajinsky, A. Eisenhart lift of 2-dimensional mechanics. *Eur. Phys. J. C* **2019**, *79*, 301. [CrossRef]
34. Cariñena, M.; Jose, F.; Herranz, F.J.; Fernández-Rañada Menéndez De Luarca, M. Superintegrable systems on 3-dimensional curved spaces: Eisenhart formalism and separability. *J. Math. Phys.* **2017**, *58*, 022701. [CrossRef]
35. Cariglia, M.; Gibbons, G.W. Generalised Eisenhart lift of the Toda chain. *J. Math. Phys.* **2014**, *55*, 022701. [CrossRef]
36. Zhang, H.; Hao, Q.-Y. Eisenhart lift for Euler's problem of two fixed centers. *Appl. Math. Compt.* **2019**, *350*, 305. [CrossRef]
37. Fin, K.; Karamitsos, S.; Pilaftsis, A. Quantizing the Eisenhart lift. *Phys. Rev. D* **2021**, *103*, 065004. [CrossRef]
38. Fin, K. The Eisenhart Lift. In *Geometric Approaches to Quantum Field Theory*; Springer Theses; Springer: Cham, Switzerland, 2021.
39. Cariglia, M. Hidden symmetries of Eisenhart-Duval lift metrics and the Dirac equation with flux. *Phys. Rev. D* **2012**, *86*, 084050. [CrossRef]
40. Cariglia, M.; Galajinsky, A.; Gibbons, G.W.; Horvathy, P.A. Cosmological aspects of the Eisenhart–Duval lift. *Eur. Phys. J. C* **2018**, *78*, 314. [CrossRef]
41. Balcerzak, A.; Lisaj, M. Spinor wave function of the Universe in non-minimally coupled varying constants cosmologies. *Eur. Phys. J. C* **2023**, *83*, 401. [CrossRef]
42. Duval, C.; Gibbons, G.W.; Horvathy, P.A. Conformal and projective symmetries in Newtonian cosmology. *J. Geom. Phys.* **2017**, *112*, 197. [CrossRef]
43. Kan, N.; Aoyama, T.; Hasegawa, T.; Shiraishi, K. Eisenhart-Duval lift for minisuperspace quantum cosmology. *Phys. Rev. D* **2021**, *104*, 086001. [CrossRef]
44. Paliathanasis, A. Classical and quantum solutions in scalar field cosmology via the Eisenhart lift and linearization. *Phys. Dark Univ.* **2024**, *44*, 101466. [CrossRef]
45. Paliathanasis, A. Cosmological Constant from Equivalent Transformation in Quantum Cosmology. *arXiv* **2024**, arXiv:2405.20683.
46. Dhasmana, S.; Sen, A.; Silagadze, Z.K. Equivalence of a harmonic oscillator to a free particle and Eisenhart lift. *Ann. Phys.* **2021**, *434*, 168623. [CrossRef]
47. Mahomed, F.M.; Leach, P.G.L. Symmetry Lie Algebras of nth Order Ordinary Differential Equations. *J. Math. Anal. Appl.* **1990**, *151*, 80. [CrossRef]
48. Noether, E. Invariante Variationsprobleme, Koniglich Gesellschaft der Wissenschaften Gottingen Nachrichten. In *Mathematik-Physik Klasse*; 1918; Volume 2, pp. 235–267. Available online: https://books.google.co.in/books/about/Mathematisch_Physikalische_Klasse.html?id=qPgMzQEACAAJ&redir_esc=y (accessed on 1 July 2024).
49. Halder, A.; Paliathanasis, A.; Leach, P.G.L. Noether's Theorem and Symmetry. *Symmetry* **2018**, *10*, 744. [CrossRef]
50. Sarlet, W.; Mahomed, F.M.; Leach, P.G.L. Symmetries of non-linear differential equations and linearisation. *J. Phys A Math. Gen.* **1987**, *20*, 277. [CrossRef]
51. Ibragimov, N.H.; Meleshko, S.V. Linearization of third-order ordinary differential equations by point and contact transformations. *J. Math. Anal. Appl.* **2005**, *308*, 266. [CrossRef]
52. Meleshko, S.V. On linearization of third-order ordinary differential equations. *J. Phys. A Math. Gen.* **2006**, *39*, 15135. [CrossRef]
53. Berkovich, L.M. Method of Factorization of ordinary differential operators and some of its applications. *Appl. Anal. Disc. Math.* **2007**, *1*, 122.
54. Nakpim, W.; Meleshko, S.V. Linearization of third-order ordinary differential equations by generalized Sundman transformations: $X''' + \alpha X = 0$. *Commun. Nonlinear Sci. Numer. Simulat.* **2010**, *15*, 1717. [CrossRef]
55. Al-Dweik, A.Y.; Mustafa, M.T.; Mahomed, F.M.; Alassar, R.S. Linearization of third-order ordinary differential equations $u''' = f(x, u, u', u')$ via point transformations. *Math. Meth. Appl. Sci.* **2018**, *41*, 6955. [CrossRef]
56. Berkovich, L.M. *Factorization and Transformations of Ordinary Differential Equations*; Saratov Univiversity Publications: Saratov, Russia, 1989.
57. Campoamor-Stursberg, R.; Guerón, J. Linearizing Systems of Second-Order ODEs via Symmetry Generators Spanning a Simple Subalgebra. *Acta Appl. Math.* **2013**, *127*, 105. [CrossRef]
58. Bluman, G.W. Potential Symmetries and Linearization. In *Applications of Analytic and Geometric Methods to Nonlinear Differential Equations*; Clarkson, P.A., Ed.; NATO ASI Series; Springer: Dordrecht, The Netherlands, 1993; Volume 413.
59. Soh, C.W.; Mahomed, F.M. Linearization criteria for a system of second-order ordinary differential equations. *Int. J. Non-Linear Mech.* **2001**, *36*, 671. [CrossRef]
60. Patera, J.; Sharp, R.T.; Winternitz, P.; Zassenhaus, H. Invariants of real low dimension Lie algebras. *J. Math. Phys.* **1976**, *17*, 986. [CrossRef]
61. Ali, S.; Mahomed, F.M.; Qadir, A. Linearizability criteria for systems of two second-order differential equations by complex methods. *Nonlinear Dyn.* **2011**, *66*, 77. [CrossRef]

62. Ali, S.; Safdar, M.; Qadir, A. Linearization from complex Lie point transformations. *J. Appl. Math.* **2014**, *2014*, 793247. [CrossRef]
63. Safdar, M.; Ali, S.; Mahomed, F.M. Linearization of systems of four second-order ordinary differential equations. *Pranama-J. Phys.* **2011**, *77*, 581. [CrossRef]
64. Qadir, A. Geometric Linearization of Ordinary Differential Equations. *Sigma* **2007**, *3*, 1. [CrossRef]
65. Tsamparlis, M. Linearization of Second-Order Non-Linear Ordinary Differential Equations: A Geometric Approach. *Symmetry* **2023**, *15*, 2082. [CrossRef]
66. Paliathanasis, A. Solving Nonlinear Second-Order ODEs via the Eisenhart Lift and Linearization. *Axioms* **2024**, *13*, 331. [CrossRef]
67. Paliathanasis, A. The Common Solution Space of General Relativity. *arXiv* **2024**, arXiv:2406.01998.
68. Ryan, M.P.; Shepley, L.C. *Homogeneous Relativistic Cosmologies*; Princeton University Press: Princeton, NJ, USA, 1975.
69. Terzis, P.A.; Dimakis, N.; Christodoulakis, T.; Paliathanasis, A.; Tsamparlis, M. Variational contact symmetries of constrained Lagrangians. *J. Geom. Phys.* **2016**, *101*, 52. [CrossRef]
70. Christodoulakis, T.; Dimakis, N.; Terzis, P.A. Lie point and variational symmetries in minisuperspace Einstein gravity. *J. Phys. A Math. Theor.* **2014**, *47*, 095202. [CrossRef]
71. Tsamparlis, M.; Paliathanasis, A. Two-dimensional dynamical systems which admit Lie and Noether symmetries. *J. Phys. A Math. Theor.* **2011**, *44*, 175202. [CrossRef]
72. Galajinsky, A.; Masterov, I. Eisenhart lift for higher derivative systems. *Phys. Lett. B* **2017**, *765*, 86. [CrossRef]
73. Tsamparlis, M.; Paliathanasis, A.; Karpathopoulos, L. Exact solutions of Bianchi I spacetimes which admit Conformal Killing vectors. *Gen. Rel. Grav.* **2015**, *47*, 15. [CrossRef]
74. Paliathanasis, A.; Leach, P.G.L. Symmetries and singularities of the Szekeres system. *Phys. Lett. A* **2017**, *381*, 1277. [CrossRef]
75. Paliathanasis, A.; Zampeli, A.; Christodoulakis, T.; Mustafa, M.T. Quantization of the Szekeres system. *Class. Quantum Grav.* **2018**, *35*, 125005. [CrossRef]
76. Szekeres, P. A class of inhomogeneous cosmological models. *Commun. Math. Phys.* **1975**, *41*, 55. [CrossRef]
77. Krasinski, A. *Inhomogeneous Cosmological Models*; Cambridge University Press: Cambridge, UK, 2011.
78. Barrow, J.D.; Stein-Schabes, J. Inhomogeneous cosmologies with cosmological constant. *Phys. Lett. A* **1984**, *103*, 315. [CrossRef]
79. Zampeli, A.; Paliathanasis, A. Quantization of inhomogeneous spacetimes with cosmological constant term. *Class. Quantum Grav.* **2021**, *38*, 165012. [CrossRef]
80. Christodoulakis, T.; Dimakis, N.; Terzis, P.A.; Vakili, B.; Melas, E.; Grammenos, T. Minisuperspace canonical quantization of the Reissner-Nordström black hole via conditional symmetries. *Phys. Rev. D* **2014**, *89*, 044031. [CrossRef]
81. Reissner, H. Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie. *Ann. Phys.* **1916**, *50*, 106. [CrossRef]
82. Nordström, G. On the Energy of the Gravitational Field in Einstein's Theory. *Proc. Ser. Phys. Sci.* **1918**, *26*, 1201.
83. Paliathanasis, A. Projective Collineations of Decomposable Spacetimes Generated by the Lie Point Symmetries of Geodesic Equations. *Symmetry* **2021**, *13*, 1018. [CrossRef]
84. Lee, J.M.; Parker, T.H. The Yamabe problem. *Bull. Am. Math. Soc.* **1987**, *17*, 37. [CrossRef]
85. Abraham-Shrauner, B.; Govinder, K.S.; Arrigo, D.J. Type-II hidden symmetries of the linear 2D and 3D wave equations. *J. Phys. A Math. Gen.* **2006**, *39*, 5739. [CrossRef]