Classification of Future Phantom-to-Normal Oscillations in $f(R)$ Gravity

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Abstract

We consider oscillations of the dark energy effective equation of state $w_{\rm DE}$ around the phantom divide line $w_{\rm DE} = -1$ in the future evolution of viable cosmological models in $f(R)$ gravity. We present an analytical condition for the existence of an infinite number of such oscillations and numerically determine the region of model parameters where it is satisfied. It is shown that the amplitude of the oscillations decreases very fast with the increase of the present mass of scalaron, which is the scalar particle appearing in $f(R)$ gravity. As a result, the effect quickly becomes very small and its beginning is shifted to the remote future.

1 Introduction

$f(R)$ gravity is a natural generalization of the Einstein gravity which can provide a self-consistent and nontrivial alternative to the ΛCDM model of the present Universe [1–3], as well as a viable model of inflation in the early Universe [4]. This theory adopts a new phenomenological function of the Ricci scalar R, $f(R)$. As compared to the Einstein gravity, it contains a new scalar degree of freedom, in quantum language – a new scalar particle dubbed "scalaron" in [4]. Thus, this generalization is *nonperturbative*. Scalaron is a massive particle which mass depends on R.

To distinguish $f(R)$ gravity as the model for "present Dark Energy (DE)" which is responsible for the current cosmic acceleration (as opposed to "primordial DE" which drove inflation in the early Universe) from the standard ΛCDM model, it is useful to focus on two parameters, the effective equation-of-state (EoS) parameter for dark energy $w_{\rm DE}$ and the gravitational growth index γ. The latter is defined as $d\ln\delta/d\ln a \equiv \Omega_m(z)^{\gamma(z)}$ where $\delta \equiv \delta\rho_m/\rho_m$ and $\Omega_m \equiv 8\pi G\rho_m/3H^2$ are matter density fluctuation and density parameter for matter, respectively. In $f(R)$ gravity, $w_{\rm DE}$ is time dependent and γ is time and scale dependent, whilst $w_{\rm DE} \equiv -1$ and $\gamma \simeq 6/11$ in the ΛCDM model. Viable $f(R)$ models generically exhibit crossing of the phantom divide $w_{\rm DE} = -1$, similar to a more general case of scalar-tensor gravity. Time and scale dependency of γ generates an additional transfer function for matter density fluctuations that constraints the region of viable model parameters [5–7].

It was noted recently that the EoS parameter $w_{\rm DE}$ oscillates around the de Sitter solution in the future evolution of viable $f(R)$ models of dark energy[8]. However, it has not been clarified yet whether the phantom crossing occurs infinitely many times or not, and under which condition. Although this property is not observable since it refers to the remote future, it is very interesting from the theoretical point of view. Here we derive this conditions for a general $f(R)$ gravity, and present results of numerical calculations for a specific viable model.

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2 Conditions

$f(R)$ gravity is defined by the following action:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_m, \tag{1}$$

where $f(R)$ is a function of Ricci scalar and S_m denotes the matter action. Field equations are derived as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} + T^{\rm DE}_{\mu\nu}), \tag{2}$$

$$8\pi G T^{\rm DE}_{\mu\nu} = (1-F)R_{\mu\nu} - \frac{1}{2}(R-f)g_{\mu\nu} + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F, \tag{3}$$

where $F = df/dR$ and $T^{\rm DE}_{\mu\nu}$ is the energy-momentum tensor for effective DE. The trace equation is

$$RF - 2f + 3\Box F = 8\pi G T. \tag{4}$$

In de Sitter regime, matter density decreases rapidly as $\rho \propto e^{-3H_1 t}$. It follows from Eq. (4), that a constant value of the Ricci scalar $R = R_1 =$ const. characterizing the de Sitter regime should be a root of the algebraic equation

$$2f_1 = R_1F_1, \tag{5}$$

where $f_1 \equiv f(R_1)$ and $F_1 \equiv F(R_1)$. Effective dark energy at de Sitter regime is characterized by $8\pi G\rho_{\rm DE,1} = -8\pi G P_{\rm DE,1} = \frac{R_1}{4}$, thus $w_{\rm DE,1} = -1$.

To investigate stability of the de Sitter solution and to find the condition for the existence of oscillations around it, we use the first order of perturbation theory with respect to $\delta R \equiv R - R_1$. The evolution equation for δR is derived from Eq. (4),

$$\delta R'' + 3\delta R' + \frac{1}{3H_1^2}\left(\frac{F_1}{F_{R1}} - R_1\right)\delta R = \frac{8\pi G\rho_m}{3F_{R1}H_1^2}. \tag{6}$$

where prime denotes the derivative with respect to number of e-folding $N \equiv \ln a = -\ln(1+z)$ and $F_{R1} \equiv F_R(R_1) \equiv dF(R_1)/dR$. We include the matter density term $\rho_m = \rho_{m0}e^{-3N}$ into the right-hand side since $\delta R_{\rm dec}$ is much smaller than background quantities at the de Sitter stage.

The solution for Eq. (6) takes the form $\delta R = \delta R_{\rm dec} + \delta R_{\rm osc}$, where $\delta R_{\rm osc}$ is the homogeneous solution with an integration constant and $\delta R_{\rm dec} = \frac{8\pi G\rho_{m0}}{F_1 - R_1F_{R1}}e^{-3N}$ is the special solution for the full equation. Whilst $\delta R_{\rm dec}$ is a monotonically decaying mode, $\delta R_{\rm osc}$ may have oscillatory behaviour. The de Sitter solution is future stable, $dR_{\rm osc} \to 0$ for $t \to \infty$, if the following stability condition is satisfied:

$$\frac{F_1}{F_{R1}} > R_1. \tag{7}$$

Further, the criterion for the existence of an infinite number of oscillations around the de Sitter asymptote for $t \to \infty$ is obtained by setting negative the discriminant of the second order algebraic equation for characteristic exponents of homogeneous solutions of Eq. (6):

$$\frac{F_1}{F_{R1}} > \frac{25}{16}R_1. \tag{8}$$

If this condition is satisfied, $\delta R_{\rm osc} = Ae^{-3N/2}\sin(\omega N + \phi)$, where $\omega \equiv 2\sqrt{\frac{F_1}{R_1F_{R1}} - \frac{25}{16}}$, and A and ϕ are integration constants.

The perturbation of EoS parameter $\delta w_{\rm DE} = (\delta P_{\rm DE} + \delta\rho_{\rm DE})/\rho_{\rm DE,1}$ is calculated from $8\pi G(\rho_{\rm DE} + P_{\rm DE}) = -2\dot{H} - 8\pi G\rho_m$. We decompose $\delta w_{\rm DE} \equiv \delta w_{\rm dec} + \delta w_{\rm osc}$ as

$$\delta w_{\rm dec} = \frac{4}{R_1}\left(\frac{1}{F_1 - R_1F_{R1}} - 1\right)8\pi G\rho_{m0}(1+z)^3 \tag{9}$$

$$\delta w_{\rm osc} = A(1+z)^{3/2}\frac{4}{R_1}\left[-\frac{R_1F_{R1}}{3F_1}\omega\cos(\omega N + \phi) + \frac{1}{3}\left(\frac{5R_1F_{R1}}{2F_1} - 1\right)\sin(\omega N + \phi)\right]. \tag{10}$$

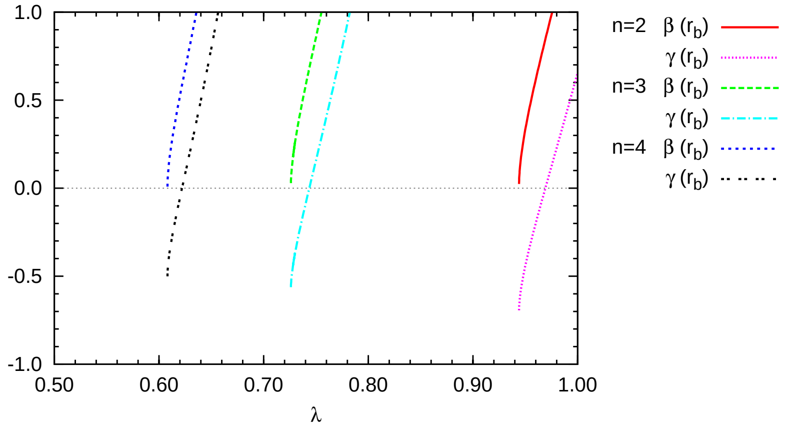

Figure 1: The parameter region $\gamma(r_b) < 0 < \beta(r_b)$ corresponds to stable de Sitter solution without the oscillatory behaviour.

3 The specific model

Hereafter, we consider the following viable cosmological model of present DE in $f(R)$ gravity [3]:

$$f(R) = R + \lambda R_s \left[\left(1 + \frac{R^2}{R_s^2} \right)^{-n} - 1 \right], \tag{11}$$

where n and λ are model parameters, and R_s is determined by the present observational data, namely, the ratio R_s/H_0^2 is well fit by a simple power-law $R_s/H_0^2 = c_n \lambda^{-p_n}$ with $(n, c_n, p_n) = (2, 4.16, 0.953)$, $(3, 4.12, 0.837)$, and $(4, 4.74, 0.702)$, respectively[5].

From Eq. (5), the de Sitter curvature is given by

$$\alpha(r) \equiv r + 2\lambda \left[\frac{1 + (n+1)r^2}{(1+r^2)^{n+1}} - 1 \right] = 0, \tag{12}$$

where $r \equiv R_1/R_s$. It is obvious that the Minkowski space, $r = 0$, is one of the solutions. We denote the other positive solutions for $\alpha(r) = 0$ as $r_a \equiv R_{1a}/R_s$ and $r_b \equiv R_{1b}/R_s$. We can estimate r_a and r_b by considering limiting cases. For $r \ll 1$, $\alpha(r) \simeq r[1 - 2\lambda(n+1)^2 r^3]$, and for $r \gg 1$, $\alpha(r) \simeq r - 2\lambda$. Therefore, for large n and λ the de Sitter solutions are given by $r = r_a \simeq [2\lambda(n+1)^2]^{-1/3}$ and $r = r_b \simeq 2\lambda$. Numerical analysis shows that this approximation is enough close to the exact answer even for $n = 2$ and $\lambda = 3$.

Once one obtained the de Sitter solutions, one can check their stability and oscillatory behaviour around them by using the stability condition and the oscillation condition derived in Eq. (7) and (8),

$$\beta(r) \equiv \frac{(1+r^2)[(1+r^2)^{n+1} - 2n\lambda r]}{2n\lambda[(2n+1)r^2 - 1]} - r > 0, \tag{13}$$

$$\gamma(r) \equiv \frac{(1+r^2)[(1+r^2)^{n+1} - 2n\lambda r]}{2n\lambda[(2n+1)r^2 - 1]} - \frac{25}{16} r > 0. \tag{14}$$

Since $\gamma(r) = \beta(r) - 9r/16$, there is no oscillation for the unstable de Sitter state, as it should be. From these criteria, we note that $r = r_a$ and r_b are unstable and stable, respectively.

For fixed n and various values of λ, we obtain λ_β and λ_γ as roots of $\beta(r_b) = 0$ and $\gamma(r_b) = 0$ respectively. Now the whole range of λ can be divided into 3 regions $\lambda < \lambda_\beta$, $\lambda_\beta < \lambda < \lambda_\gamma$, and $\lambda > \lambda_\gamma$, in which the de Sitter solution $r = r_b$ is stable with oscillations, stable without oscillations, and unstable correspondingly. Although for the most of the parameters values the stable de Sitter solution with oscillations is realized, there exists a parameter region corresponding to the stable de Sitter solution

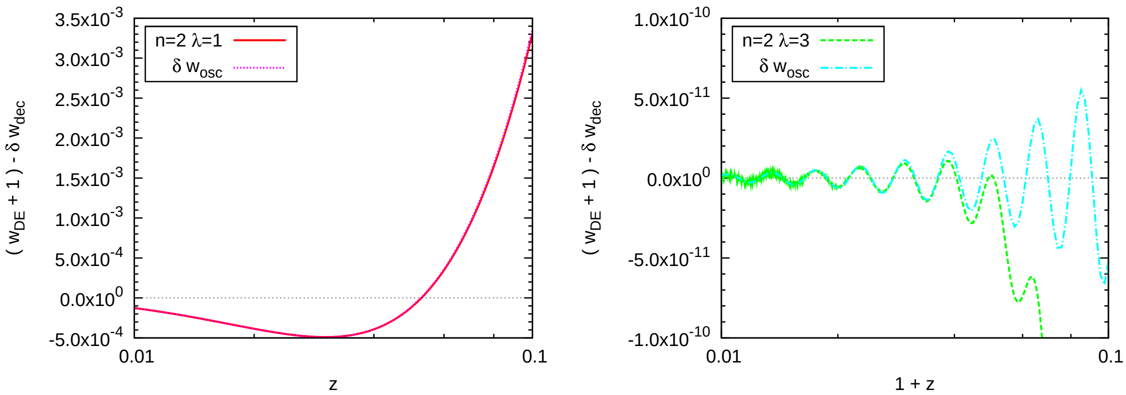

Figure 2: Numerical results for $(1+w) - \delta w_{\rm dec}$ using the analytic solution for $\delta w_{\rm osc}$.

without oscillations. Fig. 1 suggests that such parameter regions are $0.944 < \lambda < 0.970$, $0.726 < \lambda < 0.744$ and $0.608 < \lambda < 0.622$ for $n = 2,\ 3$ and 4 respectively.

We integrate the evolution equation numerically. The initial condition is taken to be the same as in the ΛCDM model at $z = 10$. The present time is identified as the moment when $\Omega_m = 0.27$. Fig. 2 depicts oscillations of the EoS parameter for $n = 2$ and $\lambda = 1,\ 3$. We subtract $\delta w_{\rm dec}$ and present $\delta w_{\rm osc}$ using the analytic solution for it.

4 Conclusion

We have considered future oscillations of the effective EoS parameter $w_{\rm DE}$ for dark energy in $f(R)$ gravity around the phantom divide $w_{DE} = -1$. They occur due to scalaron oscillations around the future stable de Sitter solution in the first order of perturbations theory. We have derived the analytical expression, Eq. (8), for the existence of an infinite number of such oscillations. There are two types of models which correspond to stable de Sitter solutions with and without oscillations. An analytic solution for the EoS perturbation $\delta w_{\rm DE}$ is obtained which contains a monotonically decaying part $\delta w_{\rm dec}$ and a damped oscillatory part $\delta w_{\rm osc}$. This is confirmed by numerical calculations for a specific viable cosmological $f(R)$ model.

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