Geometry and entanglement in AdS/CFT and beyond

by

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Submitted to the Department of Physics
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Abstract

Holographic duality is the most well-understood non-perturbative framework of quantum gravity that we have. In particular, it has revealed a deep connection between quantum entanglement and the dual gravitational geometry. A semiclassical analysis on geometries has shed insights on the non-perturbative aspects of quantum field theories, some of which does not seem to require an asymptotic anti-de Sitter boundary. This raises the question, *"What does the semiclassical gravity actually know, and how?"* This thesis aims to approach a very narrow aspect of this question by, first, summarizing developments in the leading order perturbation theory. Then, we make some small advances on generalizing the perturbative framework to geometries *beyond* AdS/CFT. On the field theory we side, we discuss perturbation theory with irrelevant deformations; on the gravity side, we study cutoff AdS and D3 brane geometry. The result is not conclusive, but we believe that the framework we set up, along with knowledge in the first few chapters, will lead us to a better understanding of what the perturbation theory of semiclassical gravity is actually capable of.

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Chapter 1

Introduction

The AdS/CFT correspondence [1] is arguably the most well-established non-perturbative quantum gravity we have. The utility of such duality goes in two ways. On one hand, we can apply our knowledge in quantum field theories to understand aspects of quantum gravity, which is the biggest unknown. On the other hand, a difficult question about strongly interacting theories may be dual to a simpler question about semiclassical gravity. In particular, the "difficult questions" of our interest are *entanglement* and *renormalization group (RG) flow*.

Why do we care about entanglement and RG flow? Wilsonian RG flow[2, 3] was a milestone in theoretical physics that says that physics are organized by energy/length scales, and that to obtain an "effective description," the high energy modes are irrelevant. In other words, to compute macroscopic observables, we should *coarse-grain* the fundamental degrees of freedom and identify the relevant modes at the scale of interest. This brings up the natural question, does there exist a measure of effective degrees of freedom?

This has been a very difficult question in the study of quantum field theories. Nevertheless, thanks to the modern development in quantum information science, it seems that quantum entanglement may shed insight on this question, such as the entropic functions constructed from the entanglement entropy. The connection is not very direction though; for example, we do not know how to spell out entanglement between quarks or gluons when flowing to the IR. It turns out that for certain quantum field theories, the *holographic* theories, these quantities can be obtained from simple geometric quantity on the gravity side.

Despite some difficulties at the formal level, entanglement has successfully played an

essential role in understanding quantum many-body system[4, 5, 6], especially topological phases[7, 8, 9], and holography[10, 11]. One should notice that entanglement is interesting on its own right in quantum information science.

For holographic theories particularly, the entanglement entropy, which we will define later, has been proposed to be quantified by certain geometrical quantities in the semiclassical gravity dual, *e.g.,* the area of a minimal surface. This is reminiscent of black hole entropy and universal UV divergence in quantum field theories:

- Bekenstein and Hawking proposed that the black hole entropy is proportional to the area of its even horizon X,

$$S_{BH} = \frac{\text{Area}[X]}{4G\hbar}. \tag{1.1}$$

- The vacuum of free theories (or renormalizable theories with CFT UV fixed point) has an *area law of entanglement,*

$$S_A = \frac{\text{Area}[\partial A]}{\epsilon_{UV}} + ..., \tag{1.2}$$

  where ∂A is the boundary of subregion, or the *entangling surface.*

The program has been extended to many other fields in physics and gained much success.

In fact, the program of holographic entanglement may have been "too successful." For one thing, it seems that the perturbative results from classical gravitation could capture non-perturbative behaviors of strongly-interacting quantum theories, which are rarely tractable. Moreover, it was recently found [12, 13] that a semiclassical analysis on the entropy of hawking radiation may produce an unitary Page curve, *i.e.,* partially resolving the black hole information paradox[1]. None of these advances seem plausible before the appearance of holographic duality. Therefore, a central question is, *"What does semiclaasical gravity actually know, and how?"*

Of course, this is a very broad question. In this thesis, we are interested in an even narrower question, "*What does the leading order perturbation theory in semiclassical gravity know?*" Also, we would like to explore the possibility of going beyond asymptotic anti-de Sitter space. There are many prerequisite knowledge beyond usual undergraduate-level

[1] Also the works cited in [14, 15, 16].

courses, for which we tried to present a review that are complementary to the other known reviews, particularly [17, 18]. The outline of the paper is the following.

In Chapter 2, we present a review on AdS/CFT. Notice that this is not a self-contained review, but the references are cited. The techniques that will be useful for later chapters are highlighted, especially the embedding space formalism. Occasionally, concepts that had confused the author for a long time will be emphasized as well, even though they may not be essential for the rest of this thesis.

In Chapter 3, both entanglement and holographic entanglement are defined. This is a rather self-contained section, but additional references are cited as well. The most important thing is the RT formula. The comments on corrections will come back at the end of the thesis too.

In Chapter 4, we first carefully examine the paper [19] and reproduce the main result: under an relevant perturbation,

$$I_E = I_0 + \lambda \int \mathcal{O}, \tag{1.3}$$

the entanglement entropy of a spherical region A becomes

$$S_A = S_0 + \lambda \cdot s_1 + \lambda^2 \cdot s_2 + \mathcal{O}(\lambda^3), \tag{1.4}$$

where

$$s_2 = -R^{2(d-\Delta)} \frac{\pi^{\frac{d+1}{2}} (d-\Delta)\Gamma(\frac{d+2}{2} - \Delta)}{2\Gamma(\frac{d+3}{2} - \Delta)}. \tag{1.5}$$

Along the way, we explicitly see the bulk emergence of a scalar field. The paper contains many technical subtleties and is not reviewed anywhere else as far as the author knows. However, we believe that ingredients of this calculation will be useful for understanding our central question about semiclassical perturbation theory. We also discuss possible generalization to the irrelevant deformations. After this, we discuss more aspects on RG flow and anomalies.

In Chapter 5, we make efforts to go beyond gravity duals with asymptotic AdS boundary. We first review the so-called $T\overline{T}$-deformed theories and their gravity dual, namely the cutoff AdS. Then we tried to do some calculation on entanglement entropy in cutoff AdS and D3

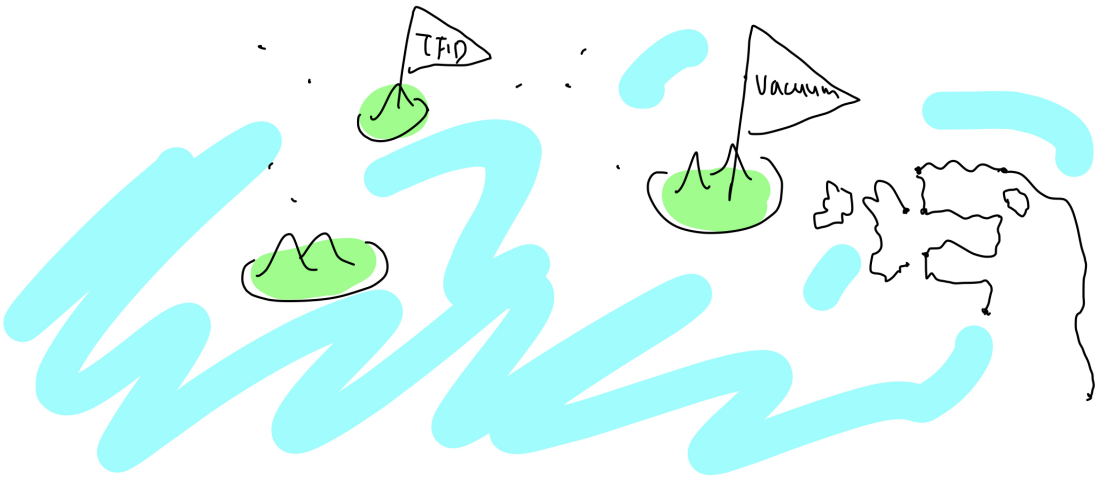


Figure 1-1: A picture of archipelago of code subspaces. Each island represents a code subspace, associated with a smooth geometry dual. Outside of code subspaces, we have stringy solutions that are not dual to smooth geometries. A naive perturbation theory explored in Chapter 4 seems to only tour around the island that contains the vacuum state, *i.e.,* pure AdS, and cannot transition to the other code subspaces.

brane geometry. The latter may be dual to a variation of $T\overline{T}$.

In Chapter 6, we conclude by piecing the loose ends of this thesis together. Roughly speaking, it seems that an naive perturbation theory only works within a *code subspace, i.e.,* perturbations around a fixed spacetime; see Fig. 1-1. However, we discuss some possible way to go beyond such picture. Hopefully some interesting, tractable research questions will grow out of this.

Chapter 2

AdS/CFT Primer

This section aims to be complementary to some modern reviews on holographic entanglement entropy [17, 18], providing more field-theoretic details. In Section 2.1, our main goal is to introduce the *embedding space formalism*. We will also introduce some other more formal aspects of CFT kinematics that are not very relevant for us. However, they are included because of their relevance to plausible generalization discussed in Chapter 6. The application of the AdS section 2.2 is more immediate in later sections.

Kinematics-wise, a d-dimensional CFT shares the same symmetry group with a generic quantum field theory in $d+1$-dimensional AdS spacetime. Naturally, the operators and states on both sides can be identified for each representation of the symmetry group. As an explicit example, the *generalized free field theory* (the simplest possible CFTs) is dual to the free scalar theory in the fixed AdS spacetime. Then, we will add a small perturbation on the AdS side and see, at the perturbative level, its correspondence to another CFT, whose anomalous dimension is related to the enegy shift in the interacting AdS theory. Notice that although we have assumed semiclassical gravity, *i.e.,* fixed AdS spacetime, a dynamical metric in the bulk is necessary for having a stress energy tensor on the CFT side. For a complete pedagogical references on these, we recommend [20, 21].

2.1 Conformal field theories

For pedagogical references on conformal bootstrap, we recommend [22, 23, 24]. For more advanced topics, one may start from [25] and references therein. [20, 21] discuss application of embedding space formalism in AdS/CFT. We will not discuss analytic bootstrap.

2.1.1 Basic kinematics

A conformal transformation transforms a theory such that the metric satisfies

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x), \tag{2.1}$$

where $\Omega(x)$ will be called the *conformal factor*. Notice that this is a weird relation since the RHS and LHS are evaluated at *different* points. Some may find it helpful to think of the conformal transformation on the metric as a two-step process: physically rescale the distance of the theory according to the conformal factor, followed by a trivial relabeling of coordinates, *i.e.,* the passive realization of diffeomorphism[1].

The conformal transformations include translations, rotation, dilation, and special conformal transformations (SCT), which schematically are SCT = Trans. ∘ Inversion ∘ Trans..

Conformal algebra in $\mathbb{R}^d$:

- translations: $p_\mu = \partial_\mu$
- rotations: $m_{\mu\nu} = x_\nu \partial_\mu - x_\nu \partial_\mu$
- dilations: $d = x^\mu \partial_\mu$
- SCT: $k_\mu = 2x_\mu (x \cdot \mu) - x^2 \partial_\mu$[a]

In addition to the algebra of $SO(d)$, these generators satisfy

$$[D, P_\mu] = P_\mu, \tag{2.2}$$

$$[D, K_\mu] = -K_\mu, \tag{2.3}$$

$$[K_\mu, P_\nu] = 2\delta_{\mu\nu} D - 2M_{\mu\nu}, \tag{2.4}$$

where the capital letters stand for the symmetry charges that generate the associated transformation. The conformal algebra in $\mathbb{R}^d$ is isomorphic to the algebra of $SO(1, D+1)$.

[a] Let I be the inversion: $x^\mu \to \frac{x^\mu}{x^2}$. Then $k_\mu = -I P_\mu I$ that sends $x^\mu \to \frac{x^\mu - a^\mu x^2}{1 - 2(a\cdot x) + a^2 x^2}$.

[1] If the scaling is uniform, then this step is the RG flow.

Under the uniform rescaling, a local scalar operator transforms as

$$x \to \lambda x, \quad \phi(x) \to \phi'(x') = \lambda^{-\Delta}\phi(x), \tag{2.5}$$

where Δ is the scaling dimension of the field ϕ. The field ϕ' may be understood as the *same* field as ϕ but in a *transformed* theory. A theory is said to be *scale-invariant* if the correlation functions of ϕ' and ϕ are identified. A conformal theory is always scale-invariant by definition, so the statement holds, even though it is not clear whether the converse is true[2].

The Jocobian of a $d-$dimensional conformal transformation reads

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Omega(x) M^{\mu}_{\nu}(x), \tag{2.6}$$

where M is an orthogonal matrix in $SO(d)$. An operator is called a *primary operator* if it transforms as

$$\phi(x) = \Omega(x)^{-\Delta} R[M^{\mu}_{\nu}]\phi(x), \tag{2.7}$$

where R is the appropriate representation of $SO(d)$ for ϕ. For example,

$$\begin{aligned} R[M] &= 1 && \text{scalar representation,} \\ R[M]^{\mu}_{\nu} &= M^{\mu}_{\nu} && \text{vector representation,} \end{aligned}$$

...etc. The derivatives of primaries are called the *descendants*. It turns out that all the operators in a CFT is either primary or descendants.

The M^{μ}_{ν} associated with the inversion has a conventional notation: I^{μ}_{ν} the inversion tensor.

Some properties of $I_{\mu\nu}$:

- The inversion tensor takes the form

$$I^{\mu}_{\nu} = \delta^{\mu}_{\nu} - \frac{2x^{\mu}x_{\nu}}{x^2}. \tag{2.8}$$

[2]See references cited at the end of Lecture 1 of [23].

- The inversion tensor determines parallel transport under a conformal transformation,

$$I^{\mu\nu} = M^{\mu}_{\rho}(x_i) M^{\nu}_{\lambda}(x_j) I^{\rho\lambda}(x_{ij}), \tag{2.9}$$

where $x_{ij} \equiv x_i - x_j$. As a consequence, the distance is scaled in a simple way,

$$x'^{\,2}_{ij} = \Omega(x_i)\Omega(x_j) x^2_{ij}. \tag{2.10}$$

- The two-point function on $\mathbb{R}^d$ of any primaries with its conjugate is determined by the conformal symmetry,

$$\langle \mathcal{O}^a(x_1) \overline{\mathcal{O}}_b(x_2) \rangle_{\mathbb{R}^d} = \frac{c_{\mathcal{O}} R\,[I(x_{12})]^a_b}{x_{12}^{2\Delta}}, \tag{2.11}$$

where $c_{\mathcal{O}}$ is a normalization constant and R is the appropriate representation for $\mathcal{O}^a$.

- For example, for conserved currents j_μ of dimension $\Delta = d-1$ and the stress energy tensor $T_{\mu\nu}$ of dimension d, the two-point functions are

$$\langle j_\mu(x_1) j_\nu(x_2) \rangle = C_J \frac{I_{\mu\nu}(x_{12})}{x_{12}^{2(d-1)}}, \tag{2.12}$$

$$\langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2) \rangle = C_T \frac{1}{x_{12}^{2(d)}} \left[\frac{1}{2} \left(I_{\mu\rho}(x_{12}) I_{\nu\sigma}(x_{12}) + I_{\mu\sigma}(x_{12}) I_{\nu\rho}(x_{12}) \right) - \frac{1}{d} \delta_{\mu\nu}\delta_{\rho\sigma} \right]. \tag{2.13}$$

From the algebra, it is clear that the translation and SCT acts as the raising and lowering operators of dilation respectively. Therefore it is natural to define a basis of operators that diagonalizes the dilation operator, *i.e.,*

$$[D, \phi(0)] = \Delta \phi(0), \tag{2.14}$$

and that the lowest dimension operators are the primaries, *i.e.,*

$$[K_\mu, \mathcal{O}(0)] = 0. \tag{2.15}$$

We will try to reserve the letter $\mathcal{O}$ for primaries. The descendants are constructed by repeatedly acting translation generators on a primary.

2.1.2 Embedding space formalism

We saw that the conformal algebra is isomorphic to the Lorentz group in two higher dimensions. This implies that the conformal transformation may be linearized on $\mathbb{R}^{1,1+d}$. Indeed, define the coordinate $X^{-1}, X^0, X^1, \cdots, X^d$, whose metric takes the form

$$ds^2_{\mathbb{R}^{1,1+d}} = -dX^+ dX^- + \sum_{i=1}^{d} dX_i^2, \tag{2.16}$$

where the light-cone coordinate is

$$X^+ = X^{-1} + X^0, \qquad X^- = X^{-1} - X^0. \tag{2.17}$$

Then the Lorentzian generators J_{MN} can be identified with conformal generators easily:

$$J_{\mu\nu} = M_{\mu\nu}, \tag{2.18}$$

$$J_{\mu+} = P_\mu, \tag{2.19}$$

$$J_{\mu-} = K_\mu, \tag{2.20}$$

$$J_{+-} = D, \tag{2.21}$$

and that

$$J_{MN} \equiv X_M \frac{\partial}{\partial X_N} - X_N \frac{\partial}{\partial X_M}. \tag{2.22}$$

We argue that in the embedding space, a conformal transformation can be realized as a Lorentz transformation on the null cone, followed by a radial projection onto a section on the null cone. A null cone is a codimension-1 hypersurface that satisfies

$$X^2 = 0. \tag{2.23}$$

A section on the null cone can be specified by

$$X^+ = f\left(X^\mu\right). \tag{2.24}$$

Heuristically, these two constraints eliminate the two extra degrees of freedom from embedding. Schematically, the procedure is:

Embedding space formalism:

$$x^\mu \xrightarrow{\qquad\qquad\text{Conformal transf.}\qquad\qquad} x'^\mu \tag{2.25}$$

$$x^\mu \longrightarrow X^M \Leftrightarrow \text{light ray} \xrightarrow{\text{Lorentz transf.}} \text{light ray'} \xRightarrow{\text{Projection}} X'^M \longrightarrow x'^\mu \tag{2.26}$$

A point X^M on the null cone is naturally identified with a light ray, which is the line connecting X^M and the origin. Conversely, every light ray intersects a section exactly once, so we can project light rays to points on a section.

Notice that if we rescale the coordinate $X' = \lambda(X)X$, then an infinitesimal distance becomes

$$(dX'^2) = \lambda(X)^2 dX^2, \tag{2.27}$$

where we need to use $X^2 = 0$. In the original CFT on flat space, this translates into

$$ds'^2 = \lambda(x)^2 ds^2, \tag{2.28}$$

where we choose the section to be

$$X^M = \left(X^+, X^-, X^\mu\right) = \left(1, x^2, x^\mu\right). \tag{2.29}$$

By rescaling on $X^+ = f\left(X^\mu\right)$, we can arrive at any other conformally flat space. For example, the one we will use later is the hyperbolic space:

$$\left(X^{-1}, X^0, X^i, X^d\right) = \left(Y^{-1}, \cos\tau, Y^i, \sin\tau\right), \qquad \left(Y^{-1}, Y^i\right) \in \mathbb{H}_{d-1}, \tag{2.30}$$

i.e., $-(Y^{-1})^2 + (Y^i)^2 = -1$ and $Y^{-1} > 0$.

To produce the scaling behaviors of operators in CFT, we can simply demand that operators transform homogeneously in the embedding space,

$$\phi(\lambda X) = \lambda^{-\Delta}\phi(X) \tag{2.31}$$

and then identify

$$\phi(X)|_{\text{section}} = \phi(x). \tag{2.32}$$

This way we can easily fix the structure of correlations functions; for example,

Some correlation functions in the embedding space:

Here we show how to obtain scalar correlation functions from embedding space.

- Two-point function:

$$\langle\phi(X)\phi(Y)\rangle = \frac{c}{(2X\cdot Y)^{\Delta}} \Rightarrow \langle\phi(x)\phi(y)\rangle = \frac{c}{(x-y)^{2\Delta}} \tag{2.33}$$

- Three-point function:

$$\langle\phi_1(X_1)\phi_2(x_2)\phi_3(X_3)\rangle = \frac{C_{123}}{(2X_1\cdot X_2)^{\alpha_{123}}(2X_1\cdot X_3)^{\alpha_{132}}(2X_2\cdot X_3)^{\alpha_{231}}} \tag{2.34}$$

$$\Rightarrow\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{2\alpha_{123}}x_{13}^{2\alpha_{132}}x_{23}^{2\alpha_{231}}} \tag{2.35}$$

The whole power of embedding space formalism is actually to compute the correlation functions with spins, but that won't be useful for us.

2.1.3 Conformal blocks and shadows

[Incomplete, But this won't be useful for us]The quadratic Casimir of the conformal group takes the form

$$C \equiv D^2 + \frac{1}{2}(K\cdot P + P\cdot K) + M_{\mu\nu}M^{\mu\nu}, \tag{2.36}$$

with eigenvalues

$$m^2 \equiv \Delta(\Delta - d) + l(l + d - 2). \tag{2.37}$$

2.2 Anti-de Sitter spacetime

Lorentzian AdS_{d+1} is a spacetime of constant negative curvature, satisfying the Einstein equation:

$$R_{AB} + \frac{d}{\ell_{AdS}^2} g_{AB} = 0, \tag{2.38}$$

where ℓ_{AdS} sets the curvature scale. To derive this, recall that for AdS_{d+1}, the Ricci scalar $R = -\frac{d(d+1)}{\ell_{AdS}^2}$ and the cosmological constant $\Lambda = -\frac{d(d-1)}{2\ell_{AdS}^2}$. The solution is a hyperboloid embedded in $\mathbb{R}^{2,d}$,

$$-X_{-1}^2 - X_0^2 + \Sigma_{i=1}^{d} X_d^2 = -\ell_{AdS}. \tag{2.39}$$

Notice that the hyperboloid admits $SO(d, 2)$ isometries. In order to write the metric for AdS, we write the metric for $\mathbb{R}^{2,d}$

$$ds^2_{\mathbb{R}^{2,d}} = -dX_{-1}^2 - dX_0^2 + \sum_{i=1}^{d} dX_d^2 \tag{2.40}$$

subject to the constraint 2.39. Now we will exploit different parametrizations of the hyperboloid.

2.2.1 Global, Poincare, and hyperbolic coordinate

To avoid cumbersome notations, we will let $R \equiv \ell_{AdS}$ from now on.

Global coordinate:

Consider the parametrization,

$$\begin{aligned} X_{-1} &= R\cosh\rho\cos t, \\ X_0 &= R\cosh\rho\sin t, \\ X_i &= R\sinh\rho\Omega_{d-1,i}, \end{aligned} \tag{2.41}$$

where $\Omega_{d-1,i}$ are unit vectors on S^{d-1}. Plugging into the metric,

$$ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2) \tag{2.42}$$

$$= -\left(\frac{r^2}{R^2}+1\right)dt^2 + \frac{dr^2}{\frac{r^2}{R^2}+1} + r^2 d\Omega_{d-1}^2 \tag{2.43}$$

$$= \frac{1}{\sin\mu}(-dt^2 + d\mu^2 + \cos^2\mu d\Omega_{d-1}^2) \tag{2.44}$$

$$\tag{2.45}$$

where $\cosh\rho = \sqrt{R^2+r^2}$ in the second line and $r = \cot\mu$ in the third line. The boundary is sitting at $r\to\infty$, or $\rho\to\infty$, or $\mu\to 0$[3]. The boundary metric is conformally equivalent to the Einstein static universe, *i.e.,* $\mathbb{R}\times S^{d-1}$.

Poincare coordinate:

Consider the parametrization,

$$\begin{aligned} X_{-1} &= \frac{R}{z}t, \\ X_0 &= \frac{R^2+z^2+(x_i)^2-t^2}{2z}, \\ X_i &= \frac{R}{z}x_i, \\ X_d &= \frac{z^2+(x_i)^2-R^2-t^2}{2z}, \end{aligned} \tag{2.46}$$

where $i = 1.2.\cdots, d-1$. The metric reads

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + (dx_i)^2 + dz^2\right) \tag{2.47}$$

$$= R^2\left(-u^2dt^2 + u^2(dx_i)^2 + \frac{du^2}{u^2}\right), \quad u\equiv\frac{1}{z} \tag{2.48}$$

The boundary is sitting at $z\to 0$ or $u\to\infty$, conformally equivalent to the Minkowski space, $\mathbb{R}^{1,d-1}$.

[3]The origin is at $\mu\to\frac{\pi}{2}$. One can define $\theta = \frac{\pi}{2}-\mu$, so that the coordinates for origin and boundary swap.

Hyperbolic black hole: Consider the parametrization:

$$\begin{aligned} X_{-1} &= r\cosh u, \\ X_0 &= \sqrt{r^2 - R^2}\sinh t, \\ X_i &= r\sinh u \Omega_{d-1,i}, \\ X_d &= \sqrt{r^2 - R^2}\cosh t, \end{aligned} \tag{2.49}$$

so that the metric reads

$$ds^2 = R^2\left[-\left(\frac{r^2}{R^2} - 1\right)dt^2 + \frac{dr^2}{\frac{r^2}{R^2} - 1} + r^2\left(du^2 + \sinh^2 u d\Omega_{d-2}^2\right)\right]. \tag{2.50}$$

Notice that this describes a black hole with the horizon radius equals the AdS radius, $r_H = R$. The boundary is at $r \to \infty$, conformally equivalent to $\mathbb{R} \times \mathbb{H}_{d-1}$. In Euclidean signature the Euclidean time $\tau \equiv -it$ has to be periodically identified to avoid singularity at the horizon, as any black holes.

Common black hole metrics:

We collect the metrics that appear quite often in the literature. There are two length scales: the AdS radius ℓ_{AdS} and the size of black hole r_H or z_H, where the subscript "H" stands for "event horizon"

- The **global Schwarzschild-AdS$_{d+1}$** black hole:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \tag{2.51}$$

with

$$f(r) \equiv 1 + \frac{r^2}{\ell_{AdS}^2} - \frac{r_H^{d-2}}{r^{d-2}}\left(1 + \frac{r_H^2}{\ell_{AdS}^2}\right), \quad r_H \in \mathbb{R}. \tag{2.52}$$

The associated temperature is

$$T_{global} = \frac{1}{4\pi\ell_{AdS}}\left(d\frac{r_H}{\ell_{AdS}} + (d-2)\frac{\ell_{AdS}}{r_H}\right). \tag{2.53}$$

- The **planar Schwarzschild-AdS$_{d+1}$** black hole:

$$ds^2 = \frac{\ell_{AdS}^2}{z^2}\left(-f(z)dt^2 + dx_i^2 + \frac{dz^2}{f(z)}\right), \tag{2.54}$$

with

$$f(z) \equiv 1 - \frac{z^d}{z_H^d}, \quad z_H \in \mathbb{R}. \tag{2.55}$$

The associated temperature is

$$T_{Poincare} = \frac{d}{4\pi z_H}. \tag{2.56}$$

- The $d+1$-dimensional **hyperbolic** black holes:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(du^2 + \sinh^2 u d\Omega_{d-2}^2\right), \tag{2.57}$$

with

$$f(r) \equiv \frac{r^2}{\ell_{AdS}^2} - 1 - \frac{r_H^{d-2}}{r^{d-2}}\left(\frac{r_H^2}{\ell_{AdS}^2 - 1}\right). \tag{2.58}$$

The associated temperature is

$$T_{hperb.} = \frac{1}{4\pi\ell_{AdS}}\left(d\frac{r_H}{\ell_{AdS}} - (d-2)\frac{\ell_{AdS}}{r_H}\right). \tag{2.59}$$

- The **global BTZ** black hole in $2+1$ dimensions:

$$ds^2 = -\frac{r^2 - r_H^2}{\ell_{AdS}^2}dt^2 + \frac{\ell_{AdS}^2}{r^2 - r_H^2}dr^2 + r^2 d\phi^2, \quad r_H \in \mathbb{R}, \phi \in [0, 2\pi] \tag{2.60}$$

Notice that for global black holes, there exists a minimum temperature,

$$T_{global} \geq T_{min} \equiv \frac{\sqrt{d(d-2)}}{2\pi\ell_{AdS}}. \tag{2.61}$$

The radius associated with the minimum temperature, $r^* = \sqrt{\frac{d-2}{d}}\ell_{AdS}$, can be used to categorize global black holes:

Two types of global BH given T:

The separating radius is $r^* \equiv \sqrt{\frac{d-2}{d}}\ell_{AdS}$.

- Small black hole: $r < r^*$.
- Large black hole: $r > r^*$.

The large black holes are always more dominant, which then compete with the thermal AdS, *i.e.,* the bulk solution without black holes. The transition of dominance between large black holes and thermal AdS is known as the *Hawking-Page transition.*

2.2.2 Free scalar field in AdS

Let's work out the simplest QFT in AdS: a free scalar theory. We will add a small $\lambda\phi^4$ deformation near the end. The scalar field ϕ with mass m is minimally coupled to the metric and the backreaction is ignored:

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right), \tag{2.62}$$

where $g = g_{AdS}$, the AdS_{d+1} metric. Extremizing the action, the equation of motion is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial\phi\right) - m^2\phi = 0. \tag{2.63}$$

Use the ansatz $\phi(z) \sim z^\Delta$, the equation becomes

$$\begin{aligned} &\Delta(\Delta-1) + (1-d)\Delta - m^2R^2 = 0 \\ \implies\ &\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2R^2}. \end{aligned} \tag{2.64}$$

It will turn out that the dimension Δ is the scaling dimension of the dual operator in CFT. This is the famous *mass-dimension relation*: the mass in the bulk is dual to the scaling dimension on the boundary.

We can solve the Klein-Gordon equation in Euclidean AdS to obtain the two-point correlation function $G_\Delta(X,Y) \equiv \langle\phi(X)\phi(Y)\rangle$,

$$(\nabla^2 + m^2)G_\Delta(X,Y) = (2\pi)^{d+1}\delta^{d+1}_{AdS}(X-Y), \tag{2.65}$$

which gives

$$G_\Delta(X,Y) = \frac{C_\Delta}{\xi^\Delta} {}_2F_1(\Delta, \Delta - \frac{d}{2} + \frac{1}{2}, 2\Delta - 2h + 1, -\frac{4}{\xi}), \tag{2.66}$$

$$C_\Delta = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2} + 1)}. \tag{2.67}$$

This is called the *bulk-to-bulk* propagator, in contrast to the *boundary-to-bulk* propagator $G_\Delta(X,P)$[4],

$$G_\Delta(X,P) \equiv \lim_{\lambda\to\infty} \lambda^\Delta G_\Delta(X, Y = \lambda P + \cdots) \tag{2.68}$$

$$= \frac{C_\Delta}{(2P\cdot X)^\Delta}, \tag{2.69}$$

where the dots "$\cdots$" in the definition are terms that does not scale with λ[5].

It was found that the boundary-to-bulk propagator has a more computationally-convenient expression,

$$G_\Delta(X,P) = \frac{1}{2\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2} + 1)} \int_0^\infty \frac{dt}{t} t^\Delta e^{-2tP\cdot X}. \tag{2.70}$$

When interaction, say $\lambda\phi^4$, is added, the boundary four-point function is computed as

$$\begin{aligned}\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\mathcal{O}(P_4)\rangle =& \frac{\lambda}{(2\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2} + 1))^4} \int_{AdS} d^{d+1}Y \prod_{i=1}^4 \int_0^\infty \frac{dt_i}{t_i} t_i^\Delta e^{-2t_iP_i\cdot Y} \\ =& \frac{\lambda}{(2\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2} + 1))^4} \prod_{i=1}^4 \int_0^\infty \frac{dt_i}{t_i} t_i^\Delta \int_{AdS} d^{d+1}Y e^{-2(\sum_i t_iP_i)\cdot Y}.\end{aligned} \tag{2.71}$$

Since the integral is over the whole AdS, we can rotate $\sum t_iP_i$ such that it lies in the 0-th direction,

$$\int_{AdS} e^{2(\sum t_iP_i)\cdot Y} = \int_0^\infty \frac{dz}{z} z^{-d} \int [d^dr] e^{-\frac{1+z^2+r^2}{z}|\sum t_iP_i|^2}. \tag{2.72}$$

[4] Although we use the same notation G_Δ for both propagators, the distinction should be clear from the argument.

[5] For minor subtlety on stripping off such terms, see [20].

The integral over r is an Gaussian integral, so the four point function reads

$$\begin{aligned}\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\mathcal{O}(P_4)\rangle =& \frac{\lambda}{(2\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2}+1))^4}\prod_{i=1}^{4}\int_0^\infty \frac{dt_i}{t_i}t_i^\Delta \int_0^\infty \frac{dz}{z}z^{-h}e^{-z+\frac{1}{z}\sum_{i,j}t_it_jP_i\cdot P_j}\\ \overset{t_i\to t_i\sqrt{z}}{=}& \frac{\lambda}{(2\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2}+1))^4}\prod_{i=1}^{4}\frac{dt_i}{t_i}t_i^\Delta e^{-\sum t_it_jP_i\cdot P_j}. \end{aligned} \tag{2.73}$$

In the last equality, we rescale t_i to factor out the dependence on z[6].

To summarize,

Correlators in AdS:

Let $C_\Delta = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2}+1)}$,

- bulk-to-bulk:

$$G_\Delta(X,Y) = \frac{C_\Delta}{\xi^\Delta}{}_2F_1(\Delta,\Delta-\frac{d}{2}+\frac{1}{2},2\Delta-d+1,-\frac{4}{\xi}). \tag{2.74}$$

- boundary-to-bulk:

$$G_\Delta(X,P) = \frac{C_\Delta}{(2P\cdot X)^\Delta} \textbf{ OR } \frac{1}{2\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2}+1)}\int_0^\infty \frac{dt}{t}t^\Delta e^{-2tP\cdot X} \tag{2.75}$$

- In the presence of $\lambda\phi^4$ interaction, the boundary four-point function

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\mathcal{O}(P_4)\rangle = \frac{\lambda}{(2\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2}+1))^4}\prod_{i=1}^{4}\frac{dt_i}{t_i}t_i^\Delta e^{-\sum t_it_jP_i\cdot P_j}. \tag{2.76}$$

2.2.3 Asymptotic AdS and stress energy tensor

It is conventional to write an asymptotic AdS metric in the *Fefferman-Graham gauge*[26],

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = \frac{dz}{z^2} + \frac{1}{z^2}g_{ij}(x,z)dx^idx^j. \tag{2.77}$$

[6]Take my prefactor with a grain of salt. One may check whether it is consistent with, e.g., Eq.180 in [21]. It is quite plausible there are typos in my calculation.

Assuming that there exists a CFT living on the boundary, we can then Taylor expand $g_{ij}(x,z)$ in z near the boundary, disregarding convergence issues[27][7]:

$$\begin{aligned} g_{ij} =& h_{ij}(x) + z^2 h^{(2)}_{ij}(x) + z^4 h^4_{ij}(x) + \cdots + z^{2k} h^{(2k)}_{ij}(x) + \cdots \\ &+ z^d \log h^{(d)}(x) + z^d T_{ij}(x) + \mathcal{O}(z^{d+1}), \end{aligned} \tag{2.78}$$

where $k = 1, 2, \cdots, \lfloor \frac{d}{2} \rfloor$ and T_{ij} is the expectation value of stress energy tensor on the boundary. The log term only appears in even dimensions since it comes from a conformal anomaly.

We put an upper bound on k because that is the range where h_{ij}, its derivatives, and T_{ij} determine everything. For example,

$$h^{(2)}_{ij} = \frac{1}{d-2}\left(R_{ij} - \frac{1}{2(d-1)} R h_{ij}\right), \tag{2.79}$$

where the Ricci tensor and scalar are associated with the zero-th order piece, namely h_{ij}. The determination of T_{ij} is a much harder question. We may have the formula [28]

$$T_{ij} = \lim_{\Lambda_c \to \infty} 4\pi \Lambda_c^{d-2} \left[K_{ij} - K h_{ij} - (d-1) h_{ij} - \frac{1}{d-2}\left(R_{ij} - \frac{1}{2} R h_{ij}\right)\right], \tag{2.80}$$

where K_{ij} is the extrinsic curvature (see Section 3.5.1) and Λ_c is a cut-off in the global coordinate.

2.3 The dictionary

A generic QFT has two basic parameters

- the coupling coefficient: λ
- the number of degrees of freedom: N or c_{eff}.

Of course there can be more than one couplings, but the generalization is straightforward. In the dual gravity theory, the natural parameters are

- the AdS radius (which also sets the curvature scale): ℓ_{AdS}
- the string scale: l_s.

[7] Here we follow the presentation of [17]

. The dictionary between the parameters are

Matching the parameters:

$$c_{eff} = \frac{\ell_{AdS}^{d-1}}{16\pi G_N^{(d+1)}}, \quad \lambda = (\frac{\ell_{AdS}}{l_s})^{\gamma}, \tag{2.81}$$

where γ is positive and theory-dependent. For example, for $\mathcal{N} = 4$ super-Yang-Mills, γ equals to $\frac{1}{4}$.

Notice that the $d+1$ dimensional Newton constant scales with the Planck length l_p as $G_N^{(d+1)} \sim (l_p)^{d-1}$. From now on, we will just write G_N, and it should be clear from the context the dimension of the bulk spacetime.

The local operators[8] of the field theory are mapped to the local operators in the bulk. More concretely, consider the free scalar theory of ϕ as before in the bulk. The asymptotics of normalizable mode are non-normalizable mode are associated with the expectation value of boundary operator $\mathcal{O}$ and the classical source $\mathcal{J}$ respectively:

Matching operators:

$$\phi(z, t, \mathbf{x}) \sim \mathcal{J}(t, \mathbf{x}) z^{d-\Delta} + \langle O(t, \mathbf{x}) \rangle z^{\Delta}. \tag{2.82}$$

In fact, in the literature, there are *two* dictionaries for operators, which had caused great confusion to the author.

Two operator dictionaries:

- GPK-W dictionary:
  - a dictionary between generating functionals
  - practically useful for computing correlation functions on either side.
- Extrapolate dictionary:
  - a dictionary between Hilbert spaces directly
  - perhaps more intuitive
  - useful for recent works on bulk reconstruction.

[8]The operators has to be well defined. This might be a bit tricky in gauge theories, but, at the very least, they have to be gauge-invariant.

These two dictionaries turn out to be equivalent in AdS/CFT but not in dS/CFT [29].

Chapter 3

Holographic Entanglement Entropy

We will review the holographic principle and, in particular, the development of black hole entropy, following [30, 31]. Then we will define slightly more formally the entanglement in quantum field theories, following [17, 18] and lecture notes from the class at MIT 8.871 (Fall 2020) by Hong Liu. An even more formal discussion can be found in, for example, [32, 33] and a very introductory nice note by Witten [34].

Then, we introduce the propsals for the *holographic entanglement entropy,* which are endeavors to connect entanglement with geometric quantities in semiclassical gravity, following [17, 18] and references therein. The TASI lectures by Netta Engelhardt is also helpful. We remark that it is very surprising that a classical quantity in gravity, *e.g.,* the area of minimal surface, can tell us something quantum mechanical, namely the entanglement entropy. This is our first hint that the General Relativity knows more than we think, more than we should expect from a *a priori* classical theory. This, however, makes the inclusion of quantum corrections much more tricky and confusing, which we will see later in this chapter.

3.1 Black hole entropy and information paradox

The first time humans envisioned the principle of holography was probably when the black hole entropy was discovered by Bekenstein and Hawking,

$$S_{BH} = \frac{A}{4G\hbar}, \tag{3.1}$$

where A is the area of event horizon. This result is based on the observation that the area of event horizon never decreases in GR, known as the *area theorem,*

$$dA \geq 0. \tag{3.2}$$

This is reminiscent of the second law of thermodynamics, which states that the entropy of the whole system is never decreasing. Notice that this entropy formula S_{BH} is intrinsically quantum mechanical, *i.e.,* $S_{BH} \sim \frac{1}{\hbar}$.

This entropy formula of black holes thus proposes a potential resolution to the paradox that the information falling into black holes seems to be erased. The paradox may be resolved if the black hole itself has entropy. More specifically, Bekenstein proposed that, analogous to the second law of thermodynamics, the *total entropy*, which is the sum of S_{BH} and the ordinary entropy of matters outside black holes, is never decreasing,

$$dS_{tot} \geq 0, \quad S_{tot} \equiv S_{BH} + S_{matter}. \tag{3.3}$$

This is known as the *generalized second law.*

In words, the generalized second law says that the information of matters falling into black holes will be stored as the horizon entropy. Conversely, when black hole evaporates, the horizon entropy turns into the entropy of radiation, *i.e.,* the information, as far as the entropy is concerned, may come out. This, however, is a too naive argument, since it does not tell us the precise content of states. Indeed, as Hawking showed, a standard, field-theoretic calculation, assuming locality and effective field theories, does not produce the correct outgoing states: the state of black hole radiation looks almost thermal.

Heuristically, this is because the black hole radiation originates from the short-distance modes near the event horizon and then gets red-shifted as propagating away to the infinity. In the framework of effective field theories, the short-distance physics of the vacuum in any UV-complete QFTs are the same. In other words, if someone only have partial access to the black hole radiation, *e.g.,* starting to receive the radiation at sufficiently late time, then they effectively only have access to the modes coming from very short distance, and thus almost identical. The statement is that the time threshold for the information being washed away is exponentially fast.

The resulting thermal state from the vacuum near event horizon can be made more

precise in the following way. In Hawking's paper, he adopted a principle he called "the principle of ignorance," which means that the field configurations inside the black hole can be integrated out from an outside observer perspective. This motivates the definition of the *Hartle-Hawking* state ρ_{HH} for a static black hole at thermal equilibrium, which can be prepared via the Euclidean path integral. The Hartle-Hawking state looks thermal from the outside observer due to the analyticity of Eclidean geometry, while it is also the ground state of null translation at the bifurcation surface.

This is a version of the *black hole information paradox.* One can show that the issue cannot be fixed by including subleading quantum corrections in Hawking's calculation. This is known as the *small correction problem.* Therefore, it had been believed for a long time that the paradox cannot be resolved by semiclassical approaches. Nevertheless, very recently, a promising proposal was given, which includes *non-perturbative* corrections that become *leading at late time* of black hole evaporation. The calculation seems to only require a careful saddle-point analysis of the Euclidean path integral of semiclassical gravity but nothing like, for example, string theory. In other words, at late time, the most dominant saddle is no longer the Hartle-Hawking state, but a non-perturbative contribution, called the *replica wormholes.* This is the progress that motivates this thesis: how can semiclassical gravity know so much?

Getting ahead of ourselves, we want to remark that we will see soon the total entropy in the generalized second law has been generalized to include quantum corrections and claimed to play an essential role in the holographic picture,

$$S_{gen} = \frac{\langle A \rangle}{4G\hbar} + S_{out}, \tag{3.4}$$

where $\langle \cdot \rangle$ denote the expectation value when the back-reaction from quantum fields is included perturbatively and $S_{out} = -\mathrm{Tr}\rho_{out} \log \rho_{out}$ is the entanglement entropy of the matters outside black holes. For an evaporating black hole, the first term is roughly the black hole entropy, and the second term is the entropy of black hole radiation. An amazing thing is that their UV divergences "happen" to cancel out, This may be a hint/evidence that we should take this quantity seriously as a fundamental measure of entanglement in quantum gravity.

Also, an interesting direction that we won't explore here is the energy conditions in

quantum field theories and quantum gravity. Quite remarkably, the entropy inequalities that we will introduce momentarily can be used to prove, for example[1],

- the generalized second law 3.4, where the essential ingredient is

$$D(\rho||\rho_{HH}) = -S_{gen}, \tag{3.5}$$

  the monotonicity of relative entropy under reduced trace,

$$D(\rho||\sigma) \geq D(\rho_A||\sigma_A), \tag{3.6}$$

  where A will be the region outside the black hole, and the Raychaudhuri equation, which describes the rate of change of expansion and thus the change of $\langle A \rangle$.

- the *averaged null energy condition*, which serves to rule out geometry that violates causality in GR:

$$\int_{-\infty}^{+\infty} T_{ab}\nu^a\nu^b d\nu \geq 0, \tag{3.7}$$

  where T is the stress energy tensor and ν is an affine, null coordinate. This is proven in local quantum field theories in general [35][2].

- The quantum null energy condition,

$$\langle T_{\mu\nu} \rangle \geq \frac{\hbar}{2\pi} S'', \tag{3.8}$$

  where S'' is the second derivative of the entanglement entropy with respect to the null coordinate ν, evaluated on the null surface.

While not having space to discuss these beautiful results, we will discuss the use of entanglement inequalities in RG flow soon. We remark again that the fact that the entanglement-based perspective is so useful is totally unexpected and perhaps telling us that this is a more universal and fundamental perspective.

[1] See [30] and references therein.
[2] Also see [36] for a different, field-theoretic proof from analytic bootstrap.

3.2 Entanglement entropy

Here we will define entanglement entropy and discuss its computation in relativistic quantum field theories, namely the replica trick. The goal is to prepare readers for the holography-related calculations in later sections. Therefore, we want to warn the readers that we will always assume the *tensor product structure* of quantum field theories after UV regularization, even though there are good reasons to believe that we may lost essential physics in doing so. For example, as emphasized in [34], the universal UV divergence in QFTs is hinting to us that the entanglement is not merely a property of the states but of the algebras of observables. This subtlety may have essential implications on quantum gravity and the emergent time, *e.g.,* [37].

3.2.1 Some definitions

Since we will assume the tensor product structure of QFTs, let's first discuss entanglement in quantum mechanics, *i.e,* on the lattice. On each site α of the (spatial) lattice, we assign a local degree of freedom, *e.g.,* a spin, associated with a Hilbert space $\mathcal{H}_\alpha$. Consider a bipartition of a lattice Hilbert space,

$$\bigotimes_\alpha \mathcal{H}_\alpha \cong \mathcal{H}_A \otimes \mathcal{H}_{A^c}, \tag{3.9}$$

where A is a subsystem and A^c is its complement. We call the codimension-1 interface ∂A the *entangling surface.*

Now consider a pure state $|\Psi\rangle \in \bigotimes_\alpha \mathcal{H}_\alpha$ on the full system and its *reduced* state on the subsystem A. Having a state be pure means that we have "full knowledge" of the state in the sense of a single vector on the Bloch sphere. If there exist nonlocal correlations (namely, the quantum entanglement) across the entangling surface, then the reduced state on A would not be pure, *i.e.,* not a vector on the Bloch sphere, anymore due to the ignorance of the correlation from A^c; instead the state may be described by the *reduced density matrix,*

$$\rho_A \equiv \mathrm{Tr}_{A^c} |\Psi\rangle \langle\Psi| , \tag{3.10}$$

where ρ_A has more than 1 eigenvector and, thereby, resembles a classical ensemble of states.

Inspired by the Shannon entropy, it is natural to quantify the amount of entanglement

across ∂A as EPR pairs needed via the von Neumann entropy

$$S_A \equiv -\mathrm{Tr}_A \rho_A \log \rho_A = -\sum_i \lambda_i \log \lambda_i, \tag{3.11}$$

where $\{\lambda_i\}$ is the set of eigenvalues of the reduced density matrix and is referred to as the *entanglement spectrum.* Clearly, the von Neumann entropy equals the Shannon entropy of the entanglement spectrum.

However, computing $\log \rho_A$ is tricky in the continuum limit, both numerically and analytically. Therefore we introduce the *$q-$th Renyi entropies*

$$S_A^{(q)} \equiv \frac{1}{1-q} \log \mathrm{Tr}_A (\rho_A)^q = \frac{1}{1-q} \log \sum_i (\lambda_i)^q, \quad q \in \mathbb{Z}_+. \tag{3.12}$$

In quantum field theories, the trace over ρ_A^q can be computed via the Euclidean path integral on a $q-$cover of the original manifold. Then the trick is to analytically continue q to the whole positive real line $\mathbb{R}_+$ and then take the $q \to 1$ limit,

Replica trick:

$$S_A = \lim_{q \to 1} S_A^{(q)}. \tag{3.13}$$

This is the *replica trick,* which we will give more details momentarily.

Notice that the Renyi entropies look analogous to, instead of entropies, thermodynamical free energies. More specifically, it looks like a free energy at temperature $\frac{1}{q}$ with respect to the *modular Hamiltonian* (or *entanglement* Hamiltonian)

$$K_A \equiv -\log \rho_A. \tag{3.14}$$

Plugging into the definition of the Renyi entropies, $S_A^{(q)}$ looks like a "modular free energy"

$$S_A^{(q)} = \frac{1}{1-q} \log \mathrm{Tr} e^{-qK_A}, \tag{3.15}$$

where the usual free energy is $F = -\frac{1}{\beta} \log \mathrm{Tr}(e^{-\beta H})$ and β is the inverse temperature. Therefore, one may try to write another quantity that resembles the usual thermodynamical entropy $S = -\beta^2 \partial_\beta (\frac{1}{\beta} \log \mathrm{Tr} e^{-\beta H})$ with respect to the modular Hamiltonian. Indeed, we

define the *modular entropy*

$$\tilde{S}_A^{(q)} \equiv \frac{1}{q^2}\partial_q\left(\frac{q-1}{q}S_A^{(q)}\right). \tag{3.16}$$

Remarkably the modular entropy seems to have a better holographic interpretation than the Renyi entropies, involving the insertion of *cosmic branes* in the gravitational path integral. A better understanding for this fact seems crucial for understanding why the tensor network and quantum error correction picture works well for semiclassical gravity and where the limitations are.

Finally, all the entropies above suffer from the UV divergences and therefore are not (yet) mathematically well-defined. A more well-defined quantity is the quantum relative entropy,

$$D(\rho||\sigma) \equiv \mathrm{Tr}\rho\log\rho - \rho\log\sigma, \tag{3.17}$$

which measures the distinguishability between two states ρ and σ. Heuristically, this is because the UV divergence in QFTs (at least for renormalizable theories) is universal and originate from the vacuum state.

The relative entropy is also like the "mother" of entanglement measures, since they can often be understood via relative entropy; for example, the entanglement entropy can be understood as how different a state is from the maximally mixed state,

$$S_A = \log\left[d_A\right] - D\left(\rho_A||\frac{\mathbb{1}}{d_A}\right). \tag{3.18}$$

Another similar identity is

<u>Connection to modular Hamiltonian</u>:

$$\begin{aligned} D(\rho||\sigma) &= \mathrm{Tr}\rho\log\rho - \mathrm{Tr}\sigma\log\sigma + \mathrm{Tr}\sigma\log\sigma - \mathrm{Tr}\rho\log\sigma \\ &= \Delta\langle K_\sigma\rangle - \Delta\langle S\rangle, \end{aligned} \tag{3.19}$$

where $\Delta\langle\cdot\rangle \equiv \langle\cdot\rangle_\rho - \langle\cdot\rangle_\sigma$ is the difference of expectation values between states ρ and σ.

From this expression, the cancellation between entanglement entropies is more transparent: the universal UV divergence will be canceled out. This formula also plays an important role in Casini's proof of Bekenstein's entropy bound.

Relative entropy also satisfies the non-negativity

$$D(\rho||\sigma) \geq 0 \tag{3.20}$$

with the equality holds if and only if $\rho = \sigma$ and the monotonicity

$$D(\rho||\sigma) \geq D(\mathcal{N}[\rho]||\mathcal{N}[\sigma]), \tag{3.21}$$

where $\mathcal{N}$ is any quantum channel; in particular, $\mathcal{N}$ can be taken to be the partial trace, *i.e.,* two states are less distinguishable when less information are accessible to the observer,

$$D(\rho||\sigma) \geq D(\rho_A||\sigma_A). \tag{3.22}$$

In holography, it can be derived, under suitable assumptions, that the bulk relative entropy equals the boundary's [38],

Relative entropy equals bulk relative entropy:

$$D_{bdry}(\rho||\sigma) = D_{bulk}(\rho||\sigma)[X], \tag{3.23}$$

where X is the *entanglement wedge* dual to a boundary subregion.

We will define these things properly after we have more knowledge on holographic entanglement entropies. For now, the main point is that such equality can be used for proving the *entanglement wedge reconstruction* (EWR): the data inside the entanglement wedge can be reconstructed from the boundary via the modular flow.

All in all, we summarize different entanglement measures in Table 3.1.

3.2.2 Inequalities

We won't prove the following inequalities, but their proofs are well-known. We mention them because they will be useful for our later discussions. For the most parts, we consider a bipartite system with subsystems A and B, with the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

First, we have the so called subadditivity condition, which translates into the condition

Entanglement measure	Formula	Classical analogue	Significance
von Nuemann entropy	$S = -\mathrm{Tr}\rho\log\rho$	Shannon entropy	Quantify entanglement as EPR pairs needed.
Renyi entropy	$S^{(q)} = -\frac{1}{1-q}\log\mathrm{Tr}\rho^q$	Thermo. free energy	Can be computed analytically via the replica trick.
Modular entropy	$\tilde{S}^{(q)} = \frac{1}{q^2}\partial_q(\frac{q-1}{q}S^{(q)})$	Thermo. entropy	Has a better holographic interpretation.
Relative entropy	$D(\rho\|\|\sigma) = \mathrm{Tr}\rho\log\rho - \rho\log\sigma$	Relative entropy	Fundamental; EWR.
Mutual information	$I(A,B) = S_A + S_B - S_{A\cup B}$	Mutual information	Bounding 2-point correlation functions.

Table 3.1: The comparison between entanglement measures and their significance.

that the mutual information is non-neqative:

$$I(A,B) = S_A + S_B - S_{A\cup B} \geq 0. \tag{3.24}$$

We also remark that the mutual information can be understood as how distinguishable the state is from the product of its reduced states:

$$I(A,B) = D(\rho_{A\cup B}||\rho_A \otimes \rho_B). \tag{3.25}$$

With the Schwarz inequality,

$$||X|| \geq \frac{\mathrm{Tr}(XY)}{||Y||}, \quad ||X|| \equiv \mathrm{Tr}\sqrt{X^\dagger X}, \tag{3.26}$$

where Y is any bounded operator, one can show that the mutual information bounds the connected correlation functions between any bounded operators $\mathcal{O}_A$ and $\mathcal{O}_B$ from each subsystem,

Bounding 2-point correlators:

$$I(A,B) \geq \frac{1}{2}\frac{(\langle\mathcal{O}_A\mathcal{O}_B\rangle - \langle\mathcal{O}_A\rangle\langle\mathcal{O}_B\rangle)^2}{||\mathcal{O}_A||^2||\mathcal{O}_B||^2}. \tag{3.27}$$

Second, the entanglement entropy satisfies a “triangle inequality", known as the Araki-Lieb ineqaulity,

$$|S_A - S_B| \leq S_{A\cup B}. \tag{3.28}$$

[3]

Finally, we have the the strong subadditivity condition, which is the most fundamental one:

$$\begin{aligned} S_{A\cup B\cup C} + S_B &\leq S_{A\cup B} + S_{B\cup C} \\ S_A + S_C &\leq S_{A\cup B} + S_{B\cup C}. \end{aligned} \tag{3.29}$$

We can derive the subadditivity and Araki-Lieb ineqaulity from the strong subadditivity. In fact, the only known proof of Araki-Lieb ineqaulity in QFTs is via the purification of the strong subadditivity. The strong subadditivity itself can be proved in QFTs via the relative entropy and Eq.

3.2.3 Causality constraints

Noe, let's move on to the continuum, relativistic quantum field theories. Again, since we assume the tensor product structure of the Hilbert space, the continuum limit does not introduce much subtleties for us. The only subtlety is that the degrees of freedom are the fields $\phi(x)$, which are functions of space. Thus, we need to define the reduced density matrix more carefully in QFTs. The wave function, instead of a function of space $\Psi[x]$, is a functional of fields $\Psi[\Phi(x)]$. Notice that in QFTs, $\phi(x)$ replaces the role of x, and the Hilbert space of a QFT is spanned by $|\phi(x)\rangle$, not $|x\rangle$.

In other words, the wave function describes the probability of a certain field configuration on the whole space, instead of the probability of a particle being at position x. In this sense, quantum mechanics is rather like a $0+1$ dimensional quantum field theory. Then the reduced density matrix ρ_A is then defined by integrating out field configurations in the complement domain (*i.e.,* A^c). Similarly, in the definition of entanglement entropy, we understand the trace as path integral over field configurations.

The relativistic limit, on the other hand, introduces some causality constraints. Let the full spacetime be a d-dimensional Lorentzian manifold $\mathcal{M}_d$. In order to define the Hilbert space, we need to find a Cauchy slice Σ_{d-1}[4] on which the Hilbert space is defined. Notice

[3] In [17], it is claimed that, combined with the subadditivity 3.24, we obtain the monotonicity of entanglement entropy $S_{A\cup B} \geq \max\{S_A, S_B\}$. I think this is clearly **incorrect**.

[4] We always assume the global hyperbolicity, or that some specific treatments can be done to define a Cauchy slice, *e.g.,* the anti-de Sitter space.

that a Cauchy slice is achronal and codimension-1 in $\mathcal{M}_d$. We then define the subsystem A on the Cauchy slice, $A \subset \Sigma_{d-1}$. Its complement A^c and the entangling surface ∂A are defined as before. Notice that the entangling surface is always spacetime codimension-2, and that it can be drawn as a dot on a two-dimensional plane, associated with two normal directions.

An important observation is that the domains of dependence of A and A^c do not make up for the whole spacetime, *i.e.,* $D[A] \cup D[A^c] \neq \mathcal{M}_d$. The other parts of the spacetime is covered by the causal future J^+ and past J^- of the entangling surface,

Causal partitions of spacetime:

$$\mathcal{M}_d = D[A] \cup D[A^c] \cup J^+[\partial A] \cup J^-[\partial A]. \tag{3.30}$$

This is hinting that the UV physics near the entangling surface plays a significant role.

Based on the above definitions, we see that the entanglement spectrum is not just a property of A, but of the domain of dependence $D[A]$. This is because that we can evolve a state on A to another state on any Cauchy slice of $D[A]$ via unitary operators supported only in $D[A]$[5]. Then, by the cyclicity of trace, we see that the entropy of the reduced density matrix $S_A = -\mathrm{Tr}_A \rho_A \log \rho_A$ is invariant under such conjugation of unitaries.

3.2.4 Replica trick and twist operator

The standard approach to compute EE in genric quantum field theories is the so-called *replica trick,* which evaluates the partition function on a $n-$folded cover of the original manifold and then take the limit:

$$S_A = -\lim_{n\to 1} \frac{\log \mathrm{Tr}(\rho_A)^n}{n-1} = -\partial_n \log \mathrm{Tr}(\rho_A)^n|_{n=1^+}. \tag{3.31}$$

The partition function on the replicas will be evaluated by the Euclidean path integral. This, however, has only been done explicitly in few examples, where the CFTs at the fixed point are mostly free theories on flat space $\mathbb{R}^d$ or the sphere S^d. There exist a real-time version, where one needs to adopt so-called *Schwinger-Keldysh* formalism. There exist many reviews

[5]We always assume the existence of stress energy tensor $T_{\mu\nu}$. On a spacial slice, the time component "T_{00}" acts as the Hamiltonian locally and generates the desired time evolution.

on the Euclidean path integrals and application to replica trick, *e.g.,* [17, 18], supplemented by delicate figures, so we will direct the readers to those references.

Here, we only remark that in 2D a simplification occurs due to the simple form of correlation function of *twist operators*, $\tau_n(x,t)$. Twist operators appear in the orbifold theory. When implementing replica trick, one can either think of the original theory on a $n-$folded manifold, or an orbifold theory on the original manifold with twist operators sitting at the entangling surface. They serve the role of conical singularity when the $\mathbb{Z}_n$ gauge replica symmetry is quotioned out.

In 2D, the partition function on the $n-$folded manifold can be evaluated as

$$Z_n = \langle \tau_n(-a/2,0)\tau_n(a/2,0)\rangle. \tag{3.32}$$

The scaling dimension of twist operators can be obtained by mapping from flat space to an infinite cylinder and compare the Casimir energy. The result is

$$h_n = \overline{h}_n = \frac{c}{24}\left(n - \frac{1}{n}\right). \tag{3.33}$$

Therefore, the partition function equals

$$Z_n = \left(\frac{a}{\epsilon_{UV}}\right)^{-\frac{c}{6}(n-\frac{1}{n})}, \tag{3.34}$$

and there the entanglement entropy is

$$S_A = \frac{c}{3}\log\frac{a}{\epsilon_{UV}}. \tag{3.35}$$

We will derive the same result from trace anomaly in Section 4.2.

Entanglement entropy in 2D CFT:

- Flat space $\mathbb{R}^2$:

$$S_A = \frac{c}{3}\log\frac{a}{\epsilon_{UV}}. \tag{3.36}$$

- Cylinder $\mathbb{R} \times S^1$:

$$S_A = \frac{c}{3} \log \left(\frac{R}{\pi \epsilon_{UV}} \sin \left(\frac{a}{R} \right) \right), \tag{3.37}$$

  where R is the circumference of S^1.

- Thermal entropy on flat space:

$$S_{thermo.,A} = \frac{c}{3} \log \left(\frac{\beta}{\pi \epsilon_{UV}} \sinh \left(\frac{\pi a}{\beta} \right) \right), \tag{3.38}$$

3.2.5 UV divergences

UV divergences is a bizarrely fruitful subjects in quantum field theories. On one hand, we do not have a complete, mathematically rigorous understanding of them; on the other hand, they have already produced many interesting results that have no analogue in finite-dimensional quantum mechanics.

For example, when going from lattice to continuum limit, one may imagine adding more and more lattice points and modify the theory accordingly. Since the addition of lattice points is arbitrary, it may not be surprising if the outcome turns out to be non-unique. This is clearly not ideal! However, working within such non-ideal framework, people still have managed to extract some *universal* pieces, which then turn out to be one of the rare tool we have to study non-perturbative aspects of QFTs. This is what the author meant by "bizarrely fruitful".

See [18] for technical details, especially the connection to anomalies.

3.3 Entropy bounds

3.4 Ryu-Takayanagi formula

Ryu and Takayanagi gave the following proposal for holographic entanglement entropy at the moment of time reflection symmetry:

RT formula:

In a classical asymptotic AdS spacetime, the von Neumann entropy of a subsystem A

on a stationary Cauchy slice is given by

$$S_A = \min_{m_A} \frac{\text{Area}[m_A]}{4G\hbar} + \cdots , \tag{3.39}$$

where m_A are the minimal surfaces anchored on ∂A and homologous to A.

Since we are going to describe in details the covariant version, which is more general, we won't talk too much here. We simply remark that the minimality condition implies the vanishing of the spatial extrinsic curvature.

3.5 A covariant formula

When the bulk geometry lacks the moment of time reflection symmetry, the variation of surfacess in time direction has to be considered too. Then, it is clear that an alternative proposal for holographic entanglement entropy is needed, since, otherwise, minimizing the area will take us to a null surface, where the associated entropy is simply zero. A covariant proposal is given by Hubney, Rangamani, and Takayanagi. Before explaining the proposal, we will equip the readers with some basics of hypersurfaces. They are not essential for understanding the proposal, but absolutely necessary if one wants to actually do calculations.

3.5.1 Basics of hypersurfaces

We briefly review the standard technique to the hypersurfaces in general relativity. For simplicity, we consider the codimension-1 hypersurface, but the generalization to codimension-2 is obvious. The readers can find the details in many GR textbooks. Here we follow Chapter 3 of [39]. Notice that the formulation we give involves defining a coordinate on the hypersurface, so that Eq.3.53 would hopefully make more sense.

Consider a manifold with the metric $g_{\mu\nu}$ and a codimension-1 hypersurface Σ subject to the constraint

$$\Phi(x^\mu) = 0. \tag{3.40}$$

If the hypersurface is time- or space-like, we define the normal vector as

$$n_\mu \equiv \frac{\epsilon \cdot \partial_\mu \Phi}{|g^{\mu\nu}\partial_\mu\partial_\nu|^{1/2}}, \tag{3.41}$$

where ϵ equals to 1 is Σ is timelike and -1 is spacelike. Such normal vector satisfies

$$n^\mu n_\mu = \epsilon. \tag{3.42}$$

If we pick a coordinate for the hypersurface, say y^a, then the induced metric h_{ab} is defined as

$$ds^2_\Sigma = g_{\mu\nu}x^\mu x^\nu|_\Sigma \tag{3.43}$$

$$= h_{ab}dy^a, dy^b, \quad h_{ab} \equiv g_{\mu\nu}\frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b}. \tag{3.44}$$

This way, we can decompose the metric into

$$g^{\mu\nu} = \epsilon \cdot n^\mu n^\nu + h_{ab} \cdot \frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b}. \tag{3.45}$$

Notice that there is an alternative definition of induced metric which does not involve defining a coordinate on the hypersurface:

Induced metric:

$$h_{\mu\nu} = g_{\mu\nu} - \epsilon \cdot n_\mu n_\nu. \tag{3.46}$$

In this formulation, the induced metric is like the projection onto the hypersurface.

When Σ is null, the definition 3.41 does not make sense, since the denominator vanishes. Instead, we define the normal vector without normalization. *i.e.,*

$$k_\mu \equiv \partial_\mu \Phi, \tag{3.47}$$

and an auxiliary null vector field N^μ such that

$$N^\mu k_\mu = -1, \quad N_\mu \frac{\partial x^\mu}{\partial y^a} = 0. \tag{3.48}$$

With two normal vectors, the induced metric h_{AB} has one less rank. In other words, if $\mu = 1, ..., d$ and $a = 1, ..., d-1$, then $A = 1, ..., d-2$. Since the null hypersurface consists of null geodesics, where k^μ is tangent to the generators. Let λ be the parameter associated

with k^μ, then we can choose a coordinate

$$y^a = (\lambda, \theta^A), \tag{3.49}$$

where θ^A is constant on a displacement along a single generator. This way, the metric decomposes into

$$g^{\mu\nu} = -k^\mu N^\nu - k^\nu N^\mu + h_{AB} \cdot \frac{\partial x^\mu}{\partial \theta^A} \frac{\partial x^\nu}{\partial \theta^B}. \tag{3.50}$$

When Σ is timelike or spacelike, we can also define the extrinsic curvature

$$K_{ab} \equiv \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} \nabla_\mu n_\nu. \tag{3.51}$$

It has the interesting property that the "trace" measures the expansion of congruence of geodesics that intersect Σ orthogonally,

$$K \equiv h^{ab} K_{ab} = \nabla_\mu n^\mu. \tag{3.52}$$

In Riemann geometry, the induced metric h_{ab} is called the *first fundamental form,* while the extrinsic curvature K_{ab} is called the *second fundamental form.* All together, they completely characterize the hypersurfaces. For null hypersurfaces, however, the situation is trickier. We refer the readers to, for example, Section 3.11 of [39].

Finally, when the spacetime has a timelike and a spacelike normal direction, *i.e.,* locally isomorphic to $\mathbb{R}^{1,1} \times \Sigma$, the metric (for simplicity, consider the Euclidean version) takes a succinct form[6],

Metric near a codimension-2 surface:

$$ds_E^2 = dx^2 + dt_E^2 + (h_{ij} + 2K^x_{ij} x + 2K^t_{ij} t_E) dy^i dy^j + \mathcal{O}(x^2 + t_E^2), \tag{3.53}$$

or in the polar coordinate ($x = r\cos\tau, t_E = r\sin\tau$),

$$ds_E^2 = dr^2 + r^2 d\tau^2 + (h_{ij} + 2r\cos\tau K^x_{ij} + 2r\sin\tau K^t_{ij}) dy^i dy^j + \mathcal{O}(r^2). \tag{3.54}$$

[6]Here, we expand the metric around $x = t_E = 0$ in the *Riemann normal coordinate*

3.5.2 Extremal surface

The extremal surface $\mathcal{E}_A$ of our interest is a spacetime codimension-2 hypersurface that has a timelike and a spacelike normal vector, say t^μ and s^μ respectively. Alternatively, we can also define two null normal vectors N^μ and k^μ such that $N^\mu k_\mu = -1$ and the induced metric

$$\begin{aligned} h_{\mu\nu} &= g_{\mu\nu} + t^\mu t^\nu - s^\mu s^\nu \\ &= g_{\mu\nu} + N^\mu k^\nu + N^\nu k^\mu. \end{aligned} \tag{3.55}$$

The extremality of surface area implies that the

$$\frac{\delta\sqrt{h}}{\delta k^\mu} = 0, \tag{3.56}$$

which is equivalent to

$$h^{\mu\nu}\nabla_\nu h^\lambda_\mu = 0. \tag{3.57}$$

When the bulk Cauchy slice admits the moment of time reflection symmetry, Eq. 3.57 reduces to that the extrinsic curvature in the spatial direction, *i.e.*, the trace of K^x_{ij}, vanishes:

$$K^x = 0. \tag{3.58}$$

This then implies the minimality condition on the Cauchy slice. In other words, the extremal surface reduces to the minimal surface in the RT formula.

3.5.3 Hubney-Rangamani-Takayanagi formula and the maxmin version

Hubney, Rangamani, and Takayanagi gave a proposal for covariant holographic entanglement entropy:

HRT formula:

In a classical asymptotic AdS spacetime satisfying the *Null Energy Condition*, $R_{ab}k^a k^b \geq 0$, the von Nuemann entropy of a subsystem A is given by

$$S_A = \min_{\mathcal{E}_A} \frac{\text{Area}[\mathcal{E}_A]}{4G\hbar} + \cdots, \tag{3.59}$$

where $\mathcal{E}_A$ are the extremal surfaces anchored on ∂A and homologous to A.

Basically, this is just replacing the word "minimal" in the RT formula with the word "extremal". In practice, however, an alternative, equivalent version of the HRT formula is more convenient, called the *maxmin* construction. The construction is the following:

Maxmin construction:

1. Given a boundary Cauchy slice $\Sigma = A \cup A^c$, where A is the subsystem of interest, pick an arbitrary bulk Cauchy slice $\tilde{\Sigma}$ anchored on Σ, *i.e.,* $\partial\tilde{\Sigma} = \Sigma$.
2. Find the minimal surface on $\tilde{\Sigma}$ anchored on ∂A.
3. Enumerate over all $\tilde{\Sigma}$, and find the maximal area surface over the family of minimal surfaces.

Since the bulk Cauchy slice $\tilde{\Sigma}$ has to be spacelike, they all sit in the so-called *FRW* wedge, which is basically the domain of dependence of $\tilde{\Sigma}$.

As mentioned in Englehardt's lectures, the maxmin construction is more useful because it guarantees the existence of a Cauchy slice on which the extremal surface sits. Also, an similar subtlety about entanglement in the time-dependent geometry is that for different subregions on the boundary Cauchy slice, the bulk Cauchy slice that contains the extremal surface may be different. To compare their surface areas, one may employ the so-called *representative trick,* which states that the surface area of $\mathcal{E}_A$ decreases along the null direction. With this, it is easy to prove a covariant proof of strong subadditivity. See Englehardt's lectures for more details.

3.6 Derivation of holographic entanglement

3.6.1 Casini-Huerta-Myers construction

The main result of the CHM construction [40] is that the entanglement entropy of a spherical subsystem equals to the area of the minimal surface prescribed by the RT formula, assuming a thermal CFT state is dual to a black hole in AdS.

The main idea is the following. Consider a $d-$dimensional CFT on the Minkowski space $\mathbb{R}^{1,d-1}$ and spherical subsystem A, *i.e.,* $\partial A = S^{d-2}$. They found a conformal transformation that maps the domain of dependence of A, namely $D[A]$, to the hyperbolic space $H =$

$\mathbb{R} \times \mathbb{H}^{d-1}$, such that the vacuum correlation functions on $D[A]$ get mapped to thermal correlators on H. The entanglement entropy of $D[A]$ is also mapped to the thermal entropy of H. For holographic CFTs, the latter can be computed as the area of even horizon of a black hole in one higher dimension. Reversing the conformal transformation, we will find that the horizon is mapped to the desired RT surface of A.

This fact that the vacuum state ρ on a spherical subsystem takes the form of Gibbs state under certain conformal transformation implies that the state takes the form

$$\rho \sim e^{-K}, \tag{3.60}$$

where K is an integral of local functions. Indeed, K can be obtained from the modular Hamiltonian of Rindler wedge.

Modular Hamiltonian of Rindler wedge:

Conformal maps:

3.6.2 Lewkowycz-Maldecena construction

The CHM constructions derive the RT formula for states equivalent to thermal states under unitary transformations. The LM construction [41] generalizes this to any static states, *i.e.,* proving the RT formula. Although some assumptions were made, the construction sets up the framework that the saddles of Euclidean path integral of semiclassical gravity computes the Renyi entropies in the replica trick. These saddles are simply Hilbert-Einstein action, evaluated at geometries satisfying the Einstein equations and the desired boundary condition. In other words, all the calculations are done *on-shell* and classically. More precisely,

$$\begin{aligned} Z_{QG}[\mathcal{R}_q] &\overset{c_{eff}\to\infty}{\approx} \int_{\mathcal{R}_q} D[g] e^{-I_{EH}}, \quad I_{EH} = \frac{1}{16\pi G_N} \int (R + \frac{d(d-1)}{l_{AdS}^2}) \\ &\overset{\text{saddles}}{\approx} \sum_{g_{on-shell}} e^{-I_{EH}[g_{on-shell}]}, \end{aligned} \tag{3.61}$$

where $g_{on-shell}$ satisfies the Einstein equation for AdS[7],

$$R_{ab} + \frac{d}{l_{AdS}^2} g_{ab} = T_{ab}. \tag{3.62}$$

We strongly recommend the interested reader to read the original paper [41] and work through the simple examples in it. There examples are instructive but are not included in most review papers. Nevertheless, Section V of [18] on "Heat kernel expansion" contains calculations that have the same taste but in the field-theoretic context. After learning the LM construction, the readers may read [42] for a more modern interpretation for the intermediate steps in the LM construction, involving the holographic dual of the modular entropy $\tilde{S}_q$ as defined in Eq. 3.16.

3.7 Comments on quantum corrections and QES formula

Higher-derivative corrections can be included in addition to General Relativity,

$$I_{bulk} = I_{EH} + I_{matter} + \frac{1}{16\pi G_N} \int \sum_k [\alpha_k \nabla^{2k} R], \tag{3.63}$$

where the bracket denotes schematically higher-derivative terms at order $2k$. We won't discuss anything beyond General Relativity in this thesis, so these corrections are not in urgent need for us. One can find an explicit formula for entanglement entropy in [43].

$$S_A = \frac{1}{8G_N} \int_X \mathrm{D}_{\mathcal{L}}, \tag{3.64}$$

where $\mathrm{D}_{\mathcal{L}}$ is a complicated functional. The derivation is similar to the LM construction, but with a different equation of motion.

Since the minimality condition in the LM construction comes from the GR equation of motion, the surface X in Eq. 3.64, unfortunately, is not an extremal surface in general; instead, it is subject to a rather contrived constraint. It is an open question whether there is a nicer interpretation for such surfaces.

Besides these higher derivatives terms, we also remark that one may include a Chern-Simons term in the bulk, which will result in gravitational anomalies on the boundary,

[7] Remember that, for AdS_{d+1}, the Ricci scalar $R = -\frac{d(d+1)}{l_{AdS}^2}$ and the cosmological constant $\Lambda = -\frac{d(d-1)}{2l_{AdS}^2}$.

i.e., the stress energy tensor is not conserved. Such theories are sensitive to boosts. The entanglement entropy in the presence of anomalies has been discussed in, *e.g.,* [44, 45].

As mentioned several times, it is bizarre that the RT formula only utilizes classical gravity to extract information about entanglement, which is something intrinsically quantum-mechanical. But, at the same time, the RT formula only exact *partial* information about entanglement, namely the leading order contribution *i.e.,* at the order of $c_{eff} \sim \frac{1}{G_N} \sim N^2$ for $SU(N)$ gauge theories.

The higher-order quantum corrections of the RT formula has been discussed by Faulkner, Lewkowycz, and Maldacena (FLM) [46]. The main statement is that the leading order quantum correction is the entanglement entropy of the entanglement wedge, *i.e.,* the amount of EPR pairs across the extremal surface.

More precisely, FLM proposed that

$$S_A = \frac{\text{Area}[\mathcal{E}_A]}{4G_N} + S_{\mathcal{W}_A} + \mathcal{O}(G_N), \quad S_{\mathcal{W}_A} \equiv S_{area}(A) + S_{bulk}(A), \tag{3.65}$$

where $S_{\mathcal{W}_A}$ is the entanglement entropy of entanglement wedge, bipartitioning into two pieces: $S_{area}(A)$ is the universal, UV divergent, area piece, and $S_{bulk}(A)$ depends on the bulk matter content in $\mathcal{W}_A$. Then one define an *area operator*

$$\widehat{\text{Area}}[\mathcal{E}_A] \equiv \text{Area}[\mathcal{E}_A] + 4G_N S_{area}. \tag{3.66}$$

This may alternatively be realized as the one-loop renormalization of the Newton constant G_N due to the matter fields. Thus a modified formula for holographic entropy, including the quantum correction, is

$$S_A = \frac{\widehat{\text{Area}}[\mathcal{E}_A]}{4G_N} + S_{bulk}(A) + \mathcal{O}(G_N). \tag{3.67}$$

Quantum corrections to RT:

- On the field theory side,

$$S_A = c_{eff} S_{saddle} + S_{1-loop} + \mathcal{O}(c_{eff}^{-1}), \tag{3.68}$$

where S_{saddle} is the contribution from the saddle point of Euclidean path integral, *i.e.,* classical. Thus, the natural next-order QFT correction comes from one-loop diagrams.

- On the gravity side, FLM proposed

$$S_A = \frac{\widehat{\text{Area}}[\mathcal{E}_A]}{4G_N} + S_{bulk}(A) + \mathcal{O}(G_N). \tag{3.69}$$

The equivalence has been demonstrated for some cases in 2D CFT [47]. Nowadays, the formula may be realized by *operator algebra quantum error correcting code*; see, *e.g.,* [48].

Recently, it was found that [49] at the point of Page transition, a new subleading term appears at the order of $\frac{1}{\sqrt{G_N}}$. Mathematically, this comes from the fact when there are two equally-dominant saddles, the saddle point analysis has to be done more carefully: summing over saddle contributions will lead to significant double counting. It is interesting to investigate the origin of such term and its implication on the bulk and boundary dynamics.

Finally, we want to bring up the most modern formula for holographic entanglement entropy, known as the *quantum extremal surface formula*[50],

$$S_A = \min \operatorname*{ext}_{\gamma} \left\{ \frac{\widehat{\text{Area}}[\gamma]}{4G_N} + S_{bulk}(A) \right\}. \tag{3.70}$$

Notice that the extremization is taken *after* including the quantum correction from bulk matter content. This formula has successfully produced the unitary Page curve for black hole evaporation[12, 13]. Nevertheless, it was recently found[51] that the QES formula may suffer large corrections, and that a more proper measure is probably the *one-shot entropy.*

Chapter 4

Perturbation theory and RG Flow

4.1 From entanglement to RG flow

4.1.1 The correct formula for δS

Consider a perturbed state

$$\rho = \rho_0 + \delta\rho, \tag{4.1}$$

where ρ_0 is the unperturbed, normalized state. Notice that without loss of generality, we can assume that ρ is also normalized, *i.e.,* $\mathrm{Tr}\rho = 1$. This imposes that

$$\mathrm{Tr}\delta\rho = 0. \tag{4.2}$$

Now we want to express the perturbed entanglement entropy S in terms of the unperturbed entropy S_0, modular Hamiltonian K_0, and the perturbation $\delta\rho$. In order to do this ,we need to expand the non-linear function $\log\rho = \log(\rho_0 + \delta\rho)$.

A naive procedure, which appears quite often in the literature, is to expand it as a scalar function,

$$\log(\rho_0 + \delta\rho) = \log\rho_0 + \rho_0^{-1}\delta\rho - \frac{1}{2}\rho_0^{-1}\delta\rho\rho_0^{-1}\delta\rho + \mathcal{O}(\delta\rho^3). \tag{4.3}$$

With this ,the perturbed S is

$$S = S_0 + \mathrm{Tr}(\delta\rho K_0) - \frac{1}{2}\mathrm{Tr}(\delta\rho_0^{-1}\delta\rho) + \mathcal{O}(\delta\rho^3), \tag{4.4}$$

where, as a reminder, the unperturbed quantities are defined as

$$\begin{aligned} S_0 &= -\mathrm{Tr}\rho_0 \log\rho_0 \\ K_0 &= -\log\rho_0. \end{aligned} \tag{4.5}$$

Notice that Eq.4.2 has been used to obtain Eq.4.4.

This, however, is wrong when ρ_0 and $\delta\rho$ do not commute. The correct formula for expanding the log of a matrix is,

$$\log(A+tB) = \log A + t\int_0^\infty dz \frac{1}{A+z}B\frac{1}{A+z} - t^2\int_0^\infty dz \frac{1}{A+z}B\frac{1}{A+z}B\frac{1}{A+z} + \mathcal{O}(t^3). \tag{4.6}$$

This can be derived from the following two formulas,

$$\frac{1}{A+tB} = \frac{1}{A} - t\frac{1}{A}B\frac{1}{A} + t^2\frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \mathcal{O}(t^3) \tag{4.7}$$

$$\int_0^u dz \frac{1}{A+z} = \log(A+u) - \log A. \tag{4.8}$$

Then, the correct formula for the entropy of a perturbed state should be

$$\delta S \equiv S - S_0 = \mathrm{Tr}(\delta\rho K_0) - \mathrm{Tr}\int_0^\infty dz z\frac{1}{(\rho+z)^2}\delta\rho\frac{1}{\rho+z}\delta\rho + \mathcal{O}(\delta\rho^3). \tag{4.9}$$

Notice that the first order term is the same and can be written as

$$\delta S = \delta\langle K_0\rangle + \mathcal{O}(\delta\rho^2). \tag{4.10}$$

This is known as the *first law of the entanglement.* This can also be derived from Eq.3.19 and noting that the first order term in the relative entropy vanishes, *i.e.,* $D(\rho||\rho_0) = \mathcal{O}(\delta\rho^2)$. Combined with an argument similar to the CHM construction, one can derive the *linearized Einstein equation* in the hyperbolic black hole geometry.

In the next subsection, we will compute δS under relevant perturbations for spherical

entangling surface, where the first order term vanishes. Therefore, the use of correct formula is absolutely essential, despite the potential difficulty to compute exactly. We will see that this is a place where holography may shed insight on.

4.1.2 Entanglement flow under relevant perturbations

Here we reproduce the result in [19]. This is a generalization of CHM construction that includes relevant perturbation.

> [19] generalizes the CHM construction: they consider entanglement entropy of spherical system under relevant deformation, $\lambda \int \mathcal{O}$. They show the emergence of the bulk non-renormalizable mode ϕ associated with the source of CFT's deformation, *i.e.,* $\phi \sim \lambda z^{d-\Delta}$, as predicted by the AdS/CFT dictionary.

Consider a relevant deformation $\lambda \int \mathcal{O}$, the partition function (or Euclidean path integral) reads

$$\begin{aligned} Z &= \int_{\mathcal{M}} D[\phi] e^{-I_E[\phi] - \int \lambda \mathcal{O}} \\ &= \int_{\mathcal{M}} D[\phi] e^{-I_E[\phi]} (1 - \lambda \int \mathcal{O}(a) + \frac{1}{2}\lambda^2 \int \int \mathcal{O}(a)\mathcal{O}(b) + \cdots). \end{aligned} \tag{4.11}$$

As we mentioned in the previous section, the first order term is well-known, and it is the second order term that is subtle. For flat and spherical entangling surface, the first order term simply vanish. It is quite amazing that they show that the second order correction to the holographic entanglement entropy is actually universal, *i.e.,* true for non-holographic theories as well. The result may seem unsurprising if one trusts the principle that *all CFTs near vacuum looks like holographic theories near the asymptotic boundary.*

Albeit the elegance of the result, the calculation in [19] seems obscure and subtle. This may be the reason why it is not reviewed in any of the review papers. We, however, believe that there may be something to learn out of this calculation, so we would like to spell out some of the most confusing parts of the calculation. That's being said, we focus on motivating the steps related to the *emergence of geometry* and leave out the subtleties on analytic continuation.

Let's begin. Instructed by CHM, we know the vacuum of a spherical system is unitarily equivalent to the thermal state on the hyperbolic space $S^1 \times \mathbb{H}_{d-1}$. In other words, in the

hyperbolic space, the theory has a Killing symmetry associated with translation in S^1, which is also the modular Hamiltonian. The relation between correlation functions in flat space and that in hyperbolic space is

$$\langle \mathcal{O}(a)\mathcal{O}(b)\rangle = \Omega^{\Delta}(a)\Omega^{\Delta}(b)G(\tau_{ba}, Y_a \cdot Y_b), \tag{4.12}$$

where τ parametrizes S^1 (and $\tau_{ba} \equiv \tau_b - \tau_a$) and Y parametrizes $\mathbb{H}_{d-1}$. The argument of the hyperbolic space correlation function is simply the invariant distant between points a and b[1].

We now invoke the formula for δS above, Eq. 4.9, and plug in the expression for $\delta\rho$ that can be obtained from path integral,

$$\delta\rho = \lambda[\int_{\mathbb{H}_{d-1}} dY \int_0^{2\pi} d\tau \Omega^{\Delta-d}(\tau, Y)] \cdot \rho_0 U(\tau)\mathcal{O}(0, Y)U(-\tau) + \mathcal{O}(\lambda^2), \tag{4.13}$$

where $U(\tau)$ generates Euclidean time translation (and modular flow); we write it as $U(\tau) = \rho^{\tau/2\pi}$[2]. Very importantly, the $\mathcal{O}(\lambda^2)$ term will of course contribute at the leading order from $\delta\langle K_0\rangle$. However, this term is less subtle in the sense that it has been discussed in [52, 53]. For convenience, we will call denote these two $\mathcal{O}(\lambda^2)$ terms differently,

Two $\mathcal{O}(\lambda^2)$ terms:

- $\mathcal{O}(\delta\rho)$ with $\delta\rho = \mathcal{O}(\lambda^2)$: called $\delta S^{(1)}$ ($\sim \langle K\mathcal{O}\mathcal{O}\rangle$ in [52, 53]).
- $\mathcal{O}(\delta\rho^2)$ with $\delta\rho = \mathcal{O}(\lambda)$: called $\delta S^{(2)}$($\sim \langle \mathcal{O}\mathcal{O}\rangle$ in [52, 53]).

Despite the similarity with [52, 53], [19] found that $\delta S^{(2)}$ has a prefactor involving $\sinh^2()$, c.f. Eq. 4.14, that in some sense encodes the emergent radial direction.

The trace in Eq. 4.9 can be worked out by going to the spectral representation and integrate in new variables; see Appendix C of [19]. The analytic continuation is subtle but we will trust it. The final result is that

$$\delta S^{(2)} = \int d\mu \int_{C(\tau_{ba})} \frac{ds}{4\sinh^2(s/2)} G((\tau_b + s) - \tau_a), \tag{4.14}$$

[1] As a reminder, $Y_a \cdot Y_b$ can be the natural invariant distance since $Y^2 = -1$. The conformal factor is $\Omega = R^{-1}(y^1 + \cos\tau)$.

[2] We assume the period of time is 2π.

where $\int du$ is the integral measure over the hyperbolic space along with the conformal factors and λ dependence; more precisely,

$$\int d\mu = \frac{1}{2}\lambda^2 \int_0^{2\pi} d\tau_a \int_0^{2\pi} d\tau_b \int_{\mathbb{H}_{d-1}} dY_a \int_{\mathbb{H}_{d-1}} dY_b \Omega^{\Delta-d}(a)\Omega^{\Delta-d}(b). \tag{4.15}$$

The integral contour C depends on the time ordering of a and b, $C = s + i\epsilon \operatorname{sgn} \tau_{ab}$. If we trust it, then the integral contour can be deformed so that

$$\delta S^{(2)} == \int d\mu \int_{-\infty}^{\infty} \frac{ds}{4\sinh^2((s - i\tau_{ba})/2)} G((s - i\epsilon \operatorname{sgn}(\tau_{ba})), \tag{4.16}$$

which may be more tractable. An interesting/annoying/confusing feature of this expression is that s serves as a real time. This feature appears quite often through out the derivation. It seems to do with the fact that CHM construction maps the domain of dependence of some subsystem A to the hyperbolic space, where the former physically only makes sense in Lorentzian signature.

Assuming the deformation operator is primary, we can then plug in formula for two-point function,

$$G(\tau, Y_a \cdot Y_b) = \frac{c_\Delta}{(-2Y_a \cdot Y_b - 2\cos(\tau))^\Delta}, \quad c_\Delta = \frac{2(\Delta - \frac{d}{2})\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2})}, \tag{4.17}$$

where the normalization *a priori* is arbitrary, but here we use the same one as the AdS propagators. Plugging in the explicit form of conformal factor and integral measure, we write out the integral explicitly,

Explicit integral for $\delta S^{(2)}$:

$$\delta S^{(2)} = \frac{1}{2}\lambda^2 c_\Delta R^{2(d-\Delta)} \int_C d\tau_a d\tau_b dY_a dY_b \int_{-\infty}^{\infty} ds \frac{(Y_a^1 + \cos\tau_a)^{\Delta-d}(Y_b^1 + \cos\tau_b)^{\Delta-d}}{(4\sinh^2(\frac{s - i\tau_{ba}}{2})(-2Y_a \cdot Y_b - 2\cosh(s - i\epsilon \operatorname{sgn}(\tau_{ba}))))^\Delta}. \tag{4.18}$$

Comments:

- One may **NOT** take the prefactor $R^{2(d-\Delta)}$ too serious yet, since the integral may diverge and require a regularization that spoils this scaling. In fact, this scaling

simply comes from the conformal factor, *i.e.,* purely kinematic. One may expect RG flow to generate something more interesting. This, however, does turn out to be the correct scaling for the physical part.

- The integral looks scary, but it is still a two-point function with a prefactor involving $\sinh^2()$. The numerator simply comes from the conformal factor. Again, the appearance of $\sinh^2()$ and s as real time is a bit mysterious to the author, but seems essential for bulk emergence.

Focusing on the integral over s,

$$I \equiv \int_{-\infty}^{\infty} -\frac{1}{(4\sinh^2(\frac{s-i\tau_{ba}}{2})(-2Y_a\cdot Y_b - 2\cosh(s-i\epsilon\,\mathrm{sgn}(\tau_{ba}))))^{\Delta}}, \tag{4.19}$$

we may perform a slight change of integral contour[3] so that integral simplifies

$$I = -\int_{-\infty}^{\infty} ds \frac{1}{(4\cosh^2(\frac{s-i\tau_{ba}}{2})(-2Y_a\cdot Y_b - 2\cosh(s)))^{\Delta}}. \tag{4.20}$$

In order to do the integral, we will start introducing auxiliary coordinates. The procedure goes like the reverse of what we did in Section. 2.2.2. We first introduce a Schwinger parameter to the two-point function as in Eq. 2.70 and let $\beta \equiv e^s$,

$$I = -\frac{e^{i\tau_{ab}}}{\Gamma(\Delta)}\int_0^{\infty}\frac{d\beta}{\beta}\int_0^{\infty}\frac{dt}{t}\frac{\beta t^{\Delta}}{(\beta+e^{i\tau_{ba}})^2}e^{t(2Y_a\cdot Y_b-\beta-\beta^{-1})}, \tag{4.21}$$

where we intentionally write the integral in a way that variables can be rescaled easily. Indeed, we change variables from (β, t) to $(t_a, t_b) = (\sqrt{t/\beta}, \sqrt{t\beta})$, so that

$$t(\beta+\beta^{-1}-2Y_a\cdot Y_b) = -(t_aY_a + t_bY_b)^2 \tag{4.22}$$

and that

$$I = -\frac{2e^{-i\tau_a-i\tau_b}}{\Gamma(\Delta)}\int_0^{\infty}\frac{dt_a}{t_a}\int_0^{\infty}\frac{dt_b}{t_b}(t_at_b)^{\Delta+1}\frac{\exp\{(t_aY_a+t_bY_b)^2\}}{(t_ae^{-i\tau_a}+t_be^{-i\tau_b})^2}. \tag{4.23}$$

Looking at the integrand, we see the familiar terms in the numerator and an unfamiliar term in the denominator that comes from $\sinh^2()$. Now, we will introduce arguably the

[3] See [19] for details. Here we take it as granted.

most essential auxiliary variable, which turns out to be the "time" that parametrizes the evolution of event horizon of hyperbolic black hole. We introduce a Schwinger parameter ℓ_B to the denominator,

$$\frac{1}{(t_a e^{-i\tau_a}+t_b e^{-i\tau_b})^2}=\int_0^\infty d\ell_B \ell_B \exp\{\ell_B(t_a e^{-i\tau_a}+t_b e^{-i\tau_b})\}. \tag{4.24}$$

For the familiar numerator, we reverse the procedure in Eq. 2.72 by introducing an auxiliary coordinate Y_B for $\mathbb{H}_{d-1}$,

$$\exp\{(t_a Y_a+t_b Y_b)^2\}\to \frac{\pi c_\Delta}{2(\Delta-\frac{d}{2})^2\Gamma(\Delta)}\int_{\mathbb{H}_{d-1}} dY_B \exp\{2Y_B\cdot(t_a Y_a+t_b Y_b)\}. \tag{4.25}$$

Notice that now the integrals over t_a and t_B become very easy. We thus obtain

$$I=-\frac{\pi c_\Delta}{(\Delta-\frac{d}{2})^2}\int_0^\infty d\ell_B\ell_B\int_{\mathcal{H}_{d-1}} dY_B\left[\partial_{\ell_B}\frac{1}{(-2Y_a\cdot Y_B-\ell_B e^{-i\tau_a})^\Delta}\right]\left[\partial_{\ell_B}\frac{1}{(-2Y_b\cdot Y_B-\ell_B e^{-i\tau_b})^\Delta}\right] \tag{4.26}$$

Notice that we intentionally write the integrand of Y_B's integral as the boundary-to-bulk propagator.The appearance of the square of $\partial_{\ell_B}(\cdots)$ indicates the analogue to stress energy tensor in the null direction parametrized by ℓ_B. ℓ_B and Y_B altogether parametrize the codimension-1 event horizon in hyperbolic black hole geometry, which is the Rindler-like future horizon, named $\mathcal{H}^+$, in Poincare coordinate. ℓ_B parametrizes the null direction along the null horizon $\mathcal{H}^+$; Y_B parametrizes the transverse directions.

To summary,

Bulk formula for $\delta S^{(2)}$:

We obtain a "bulk" formula for $\delta S^{(2)}$,

$$\delta S^{(2)}=-2\pi\int s\ell_B\ell_B\int dY_B(\partial_{\ell_B}\phi)^2, \tag{4.27}$$

where ϕ is an emergent scalar bulk field

$$\phi(X_B)\equiv\lambda\frac{c_\Delta}{2(\Delta-\frac{d}{2})}\int d\tau dY\frac{(Y^1+\cos\tau)^{\Delta-d}}{(-2Y\cdot Y_B-\ell_B e^{-i\tau})^\Delta}. \tag{4.28}$$

One may be more familiar with the expression in Poincare coordinate[a],

$$\phi(X_B) = \lambda \frac{c_\Delta}{2(\Delta - \frac{d}{2})} \int d^{d-1}\vec{x} \int dx^0 \frac{z^\Delta}{(z^2 + (x^0 - it_B)^2 + (\vec{x} - \vec{x_B})^2)^\Delta}, \tag{4.29}$$

where $t_B \equiv -ix_B^0$ is the real time for labeling points on the future horizon $\mathcal{H}^+$.

[a]For a standard AdS/dictionary for the source term, see, *e.g.,* Sec. 3.2.2. of [21] or Sec. 12 of [20].

It is interesting that such second order correction to the entanglement entropy is actually proportional to a conserved charge. More precisely, one can show that Eq. 4.27 is equivalent to

$$\delta S^{(2)} = -2\pi \int_{\mathcal{R}} d\Sigma^a \xi^b T_{ab}, \tag{4.30}$$

where $\mathcal{R}$ is some bulk hypersurface homologous to the future horizon $\mathcal{H}^+$, ξ is the conformal Killing vector, and T is the stress energy tensor. The formula reduces to the previous one if we take $\mathcal{R} = \mathcal{H}^+$. One may check this by plugging in

$$T_{B,ab} = \partial_a\phi\partial_b\phi - \frac{1}{2}(m^2\phi^2 + (\partial\phi)^2)g_{AdS,ab}, \quad m^2 \equiv \Delta(\Delta - d) \tag{4.31}$$

and

$$\xi_B = \frac{R^2 - \vec{x}^2 - t^2 - z^2}{2R}\partial_t - \frac{t}{R}(\vec{x}\cdot\partial + z\partial_z). \tag{4.32}$$

Notice that Eq. 4.32 is quoted from [19], but we strongly suspect that the minus sign in "$\ldots - t^2 \ldots$" should be a plus. The subscript B is just there to remind us that these quantities are evaluated at the place where we previously defined Y_B and ℓ_B, namely the Rindler-like codimension-1 event horizon of hyperbolic black hole or the future horizon in Poincare coordinate.

In light of the covariance of the formula, we may choose a special $\mathcal{R}$ that simplifies the calculation. Indeed, we may choose $\mathcal{R} = A_B \cup D^+[A]$, where A_B sits on a spatial slice that connects the bifurcation surface to the subsystem A on boundary, stopped at the cutoff z_Σ. See Fig. 4-1 for a schematic expression. Clearly, the contribution from the integral along the boundary, *i.e.,* the piece associated with $D^+[A]$, is UV divergent in z_Σ. Therefore, [19]

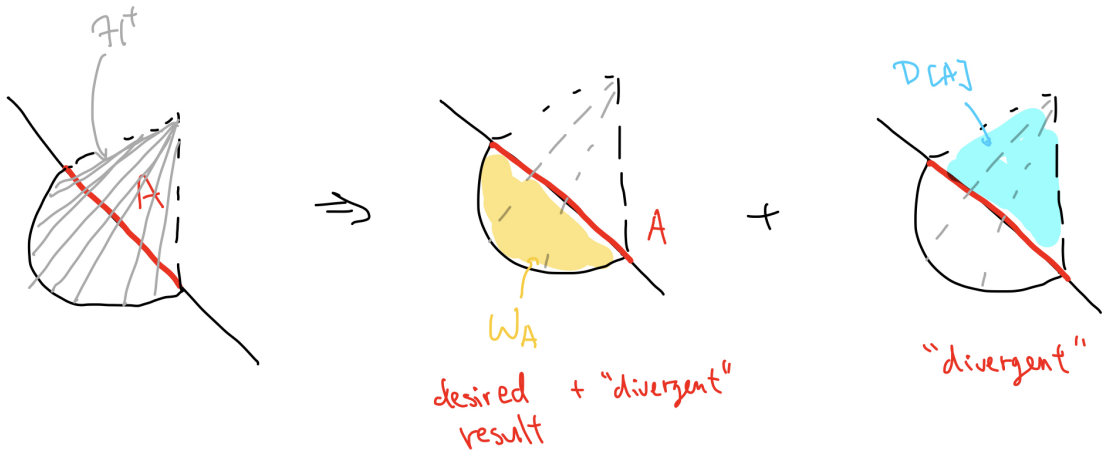


Figure 4-1: $\delta S^{(2)}$ is found to be a conserved charge, integrated over the null horizon $\mathcal{H}^+$, namely the figure on the farthest left. However, we deform the integration surface to be the union of entanglement wedge and domain of dependence of A, namely the two figures on the right. Here we ignore the subtlety about the cutoff z_Σ

treats this term as a counter term,

$$\delta S^{(2)} = -2\pi \int_{A_B} d\Sigma^a \xi^b T_{ab} + S_{ct}, \quad S_{ct} \equiv -2\pi \int_{D^+[A]} d\Sigma^a \xi^b T_{ab}. \tag{4.33}$$

The analysis on S_{ct} is very subtle, so we won't get into it. Yet, we would like to present the result and, later, discuss the possibility of where this analysis may go wrong/be modified for irrelevant deformations.

[19] shows that the boundary piece is

$$S_{ct} = 2\pi \int_A d^{d-1}\vec{x}\xi^t \left(-\frac{d-\Delta}{2}\lambda^2 z_\Sigma^{d-2\Delta} - c'\lambda^2 (2\delta)^{d-2\Delta} \right), \tag{4.34}$$

where δ is a UV cutoff for the collision of operators. It is used in $\delta S^{(1)}$ to avoid the collision of three operators in $\langle T\mathcal{O}\mathcal{O}\rangle$. Thus, the second term will be used for canceling out such UV divergence in $\delta S^{(1)}$, and the first term will be used to cancel the UV divergence that comes from the near-boundary region of A_B.

In other words, $\delta S^{(2)}$ comes entirely from the finite term of the integral over A_B.

Explicitly,

$$\delta S^{(1)} = (\text{divergences in } \delta),$$
$$\delta S^{(2)} = -\lambda^2 R^{2(d-\Delta)} \frac{\pi^{\frac{d+1}{2}}(d-\Delta)\Gamma(\frac{d+2}{2}-\Delta)}{2\Gamma(\frac{d+3}{2}-\Delta)} + (\text{divergences in } z_\Sigma, \delta). \tag{4.35}$$

[19] provides an argument that in the total contribution to δS, the divergences should cancel out and thus

Correction to entanglement entropy under relevant perturbations:

$$\begin{aligned}\delta S &= \delta S^{(1)} + \delta S^{(2)} \\ &= -\lambda^2 R^{2(d-\Delta)} \frac{\pi^{\frac{d+1}{2}}(d-\Delta)\Gamma(\frac{d+2}{2}-\Delta)}{2\Gamma(\frac{d+3}{2}-\Delta)}.\end{aligned} \tag{4.36}$$

Notice that the first order term in λ vanishes for flat and spherical entangling surface.

We remark that

- The scaling behavior is $R^{2(d-\Delta)}$, *i.e.,* the cutoff scales do not spoil a naive dimensional analysis. This may have to do with the fact that we are not *flowing* anywhere but just doing naive perturbation theory, *i.e.,* no resummation or RG-improvement. This could be the limitation of this analysis

- $\delta S^{(2)}$ equals the modular Hamiltonian of bulk field integrated over the entanglement wedge, which equals the causal wedge in this case. This may be reminiscent of the first law of entanglement, *i.e.,* $\delta S = \delta\langle K_0\rangle$. However, what we have here is more like

$$\delta S = \langle \delta K\rangle_0, \tag{4.37}$$

  where δK is due to the presence of bulk field ϕ and will be computed from QFT in curved spacetime. We do think this is reminiscent of FLM 3.7(or actually the JLMS version[38]):

$$K_A = c_{eff}\langle \widehat{\text{Area}}(\mathcal{E}_A)\rangle + K^{bulk}_{\mathcal{W}_A} + \mathcal{O}(c_{eff}^{-1}). \tag{4.38}$$

The paper [19] then discuss the case of non-uniform coupling $\lambda(x)$, following a similar

analysis from [54], which use the machinery set up in [55]. The idea is that in this non-uniform-coupling case, the emergent bulk field obtains an additional term, namely the normalizable mode,

$$\phi(X_B) \sim \beta(\vec{x_B}, x_B^0) z^{\Delta} + \lambda(\vec{x_B}, x_B^0) z^{d-\Delta} + \cdots , \tag{4.39}$$

since the stress energy tensor is not traceless, $\langle T^\mu_\mu \rangle^{\rm ren}_\lambda = -\Delta(2\Delta - d)\beta\lambda$. In Appendix E of [19], the UV divergences and analytic properties of S_{ct} and $\delta S^{(1)}$ are discussed, and the conclusion is

$$S_{ct} + \delta S^{(1)} = 2\pi \int d^{d-1}\vec{x_B} \xi^t \left(-\Delta\beta\lambda - \frac{d-\Delta}{2} g^2 z_\Lambda^{d-2\Delta} + \langle T_{tt} \rangle^{\rm ren}_\lambda \right), \tag{4.40}$$

where the renormalized T is

$$\langle T_{\mu\nu} \rangle^{\rm ren}_\lambda \equiv \frac{1}{2}\lambda^2 \int_a \int_b \langle T_{\mu\nu} \mathcal{O}(a) \mathcal{O}(b) \rangle - c'(2\delta)^{d-2\Delta}\lambda^2. \tag{4.41}$$

This is the same as before, where δ is to regularize the composite operator: $x_{ab} > \delta$.

Now, we can use these to constrain the linearized metric $h_{\mu\nu}$ so that it satisfies the RT formula, *i.e.,*

$$\delta S = \delta S^{(2)} + \delta S^{(1)} = \frac{1}{4G_N} \delta A(h), \quad \delta A(h) = \frac{1}{2R} \int d^{d-1}\vec{x_B} z^{2-d} \left(\delta^{ij} R^2 - x_B^i x_B^j \right) h_{ij}. \tag{4.42}$$

The rest follows a similar analysis from [54], and then one may find a solution

$$h_{\mu\nu} = \frac{16\pi G_N}{d} \left(\langle T_{\mu\nu} \rangle^{\rm ren}_\lambda - \frac{\eta_{\mu\nu}}{d-1} \left(\langle T^\mu_\mu \rangle^{\rm ren}_\lambda + \Delta\beta\lambda \right) \right) z^{d-2} - \frac{4\pi G_N}{(d-1)} \eta_{\mu\nu} \lambda^2 z^{2(d-\Delta-1)} + \cdots . \tag{4.43}$$

Combined with imposing the Einstein equation, this solution is compatible with RT formula in AAdS. Notice that we do not know if Eq. 4.43 is the *unique* solution.

4.1.3 Comments on irrelevant perturbations

What happens if we have irrelevant deformations, *i.e.,* $d > \Delta$?

We believe that at this level of perturbation theory, the correction to the entanglement

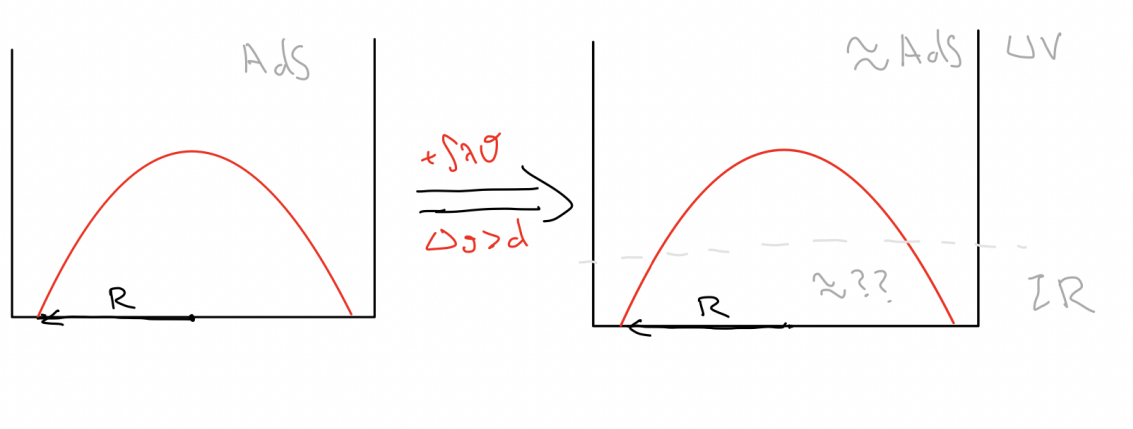


Figure 4-2: (Left) The minimal surface in AdS, anchored on the asymptotic boundary. (Right) After introducing a small irrelevant deformation, the metric in the UV may go wild, but the shape of minimal surface won't change at the leading order of perturbation theory.

entropy still contains the same term, *i.e.,*

$$\delta S \supset -\lambda^2 \left(\frac{1}{R}\right)^{2(\Delta-d)} \frac{\pi^{\frac{d+1}{2}}(d-\Delta)\Gamma(\frac{d+2}{2}-\Delta)}{2\Gamma(\frac{d+3}{2}-\Delta)}. \tag{4.44}$$

This is because that this term has a naive/classical scaling in system size. As remarked in the box above, this may mean that the leading order piece is protected against quantum fluctuations from RG. We may, however, no longer trust the analysis on UV divergences in [19], which means there could be new, bizarre terms., but we still believe that the first order piece has to vanish and the term above is the leading one (in terms of $\frac{1}{R}$ exponent) at second order in λ.

Another way to state this is that we are performing the perturbation theory *within* the code subspace that contains the vacuum state. The minimal surfaces still have the same shape, but we evaluate the surface area with a different metric. Therefore, at large R limit, even though the metric near the boundary can go wild, its contribution to the surface area is dominated over by the piece near the core, where the metric is close to AdS. See Fig. 4-2.

One may still try to find a constraint on the metric similar to the analysis above. The difficulty would be that we do not know the boundary condition in IR, *e.g.,* whether or not we should require the modes to be normalizable near the boundary. However, in the next chapter, we will introduce a certain class of theories, called $T\overline{T}$-deformed theories. It is proposed that the pure gravity dual is to place a Dirichlet boundary condition in the bulk. A plausible research is to see whether or not that arise naturally from an analysis on the

constraints of metric, assuming RT formula.

4.2 A proof of the c theorem

As said in the introduction, entanglement characterizes the renormalization group flow in a non-trivial and interesting way. We may explore such connection via the *conformal anomalies,* whose connection to RG is well-known and very elegant.

What are conformal anomalies? Due to the conformal invariance, one may expect that the trace of the stress energy tensor, T^μ_μ, vanishes. However, in even dimensions, the trace is non-vanishing on curved manifold; more precisely,

$$\langle T^\mu_\mu \rangle = \sum_i c_i I_i - (-1)^{\frac{d}{2}} a_d E_d, \tag{4.45}$$

where I_i's are invariants built from the Weyl tensor $C_{\mu\nu\rho\sigma}$ and E_d is the topological Euler density,

$$E_{d=2n} = \frac{1}{2^n} \epsilon_{\mu_1\nu_1\mu_2\nu_2\ldots\mu_n\nu_n} R^{\mu_1\nu_1} \wedge R^{\mu_2\nu_2} \wedge \cdots R^{\mu_n\nu_n}. \tag{4.46}$$

The conformal anomalies are then defined as the coefficients

Trace anomalies:

- Type-a: a_d in Eq. 4.45
- Type-c: c_i in Eq. 4.45

Warning: It is unfortunate that the central charge c in 2D CFT is actually a Type-a anomaly, since

$$\langle T^\mu_\mu \rangle_{2D} = -\frac{c}{12} R. \tag{4.47}$$

Now we present a derivation of entanglement entropy of single interval in 2D CFT. Consider the theory on $n-$folded cover, $\mathcal{M}_n$, glued together at the interval, which has length ℓ. For a theory with only one length scale ℓ, one can write, modulo subtleties about

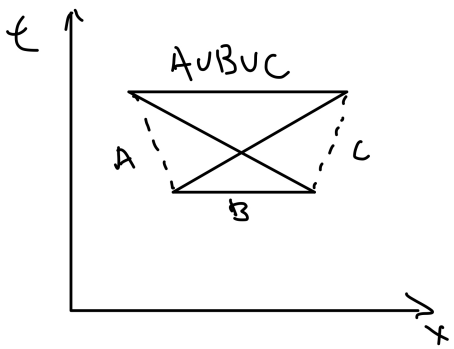


Figure 4-3: The proof of $c-$ theorem. Dashed lines are supposed to be the light rays.

dimensional transmutation,

$$\ell \frac{d}{d\ell} \log Z_n = -\frac{1}{2\pi} \int_{\mathcal{M}_n} d^d x \sqrt{g} \langle T^\mu_\mu \rangle, \tag{4.48}$$

where Z_n is the partition function on the $n-$folded cover. With the trace anomaly above and standard replica trick 3.2.4,

$$\ell \frac{d}{d\ell} S_A = \frac{c}{6} \int_\Sigma 1 = \frac{c}{3}, \tag{4.49}$$

where in the last step, Σ is the entangling surface, namely two points on a line. Thus 2D CFT has the following entanglement entropy,

$$S_A = \frac{c}{3} \log \frac{\ell}{\epsilon_{UV}} + \cdots, \tag{4.50}$$

where $\cdots$ are finite terms for excited states.

Heuristically, the central charge c characterizes the degrees of freedom of 2D CFT, *i.e.,* fixed points of RG flow. Can we extend such measure of d.o.f. to the whole RG trajectory? In other words, can we find a function $C(\ell)$ such that C is monotonic and satisfies

$$C_{UV} \geq C_{IR}. \tag{4.51}$$

Remarkably the answer is yes. This is known as the *c-theorem*. This is first proved in QFT by Zamolodchikov [56]. Here we give the proof based on strong subadditivity of S_A, following [57]

See Fig. 4-3. Define the entanglement entropy of an interval with length ℓ to be $S(\ell)$. The regions $A \cup B \cup C$ and B have length R and r respectively, *i.e.,*

$$S_{A\cup B\cup C} = S(R), \qquad S_B = S(r). \tag{4.52}$$

The diagonals, $A \cup B$ and $B \cup C$, both have length $\sqrt{rR}$, *i.e.,*

$$S_{A\cup B} = S_{B\cup C} = S(r\sqrt{R}). \tag{4.53}$$

By the causality constraint 3.2.3 and the strong subadditivity, one finds the inequality,

$$S(R) + S(r) \leq 2S(\sqrt{rR}), \qquad \forall r, R. \tag{4.54}$$

If we define the *entropic c−function* to be

$$C(\ell) \equiv 3\ell \frac{d}{d\ell} S(\ell), \tag{4.55}$$

then from the inequality above, we have

$$C'(\ell) = 3S'(\ell) + 3\ell S''(\ell) \leq 0, \tag{4.56}$$

i.e., the entropic function is monotonic, as desired. Notice that in the definition, the prefactor has been chosen such that

$$C(\ell)|_{\text{CFT}} = c, \tag{4.57}$$

the entropic function equals the central charge of CFT at the fixed points.

The higher dimensional generalization has been proposed. As in 2D, people believe that the type-a anomaly may characterize the RG flow in 2D. This is known as the *a-theorem*, coined by Cardy[58]. The $a-$theorem is beautifully proven in [59] by Komargodski and Schwimmer. They perform a "spurion analysis" by introducing dilatons to restore conformal symmetry along the RG flow. This enables the t' Hooft anomaly matching, and thus proves the $a-$theorem. A more recent proof based on entanglement is given in [60].

In three dimensions, there are no trace anomaly, so there does not seem to be a natural

candidate for the entropic function. The closest analogue is the sphere partition function, which does pick up type-a anomaly in even dimensions. This eventually leads to [61, 62],

$$\tilde{F}_{(d)} \equiv -\sin\left(\frac{\pi d}{2}\right)\log Z_{S^d} = \begin{cases} (-1)^{\frac{d-1}{2}}\log Z_{S^d} & ,d \text{ is odd} \\ (-1)^{\frac{d}{2}}\, a_d & ,d \text{ is even.} \end{cases} \tag{4.58}$$

More recently, an entanglement-based proof, analogous to our proof of $c-$theorem, was given [63, 64].

Motivated by these examples, a more general formula, called the *renormalized entropy*, was proposed [65] that systematically picks up universal coefficients of UV and IR divergences.

Entropic functions: The monotonic entropic functions of spherical subsystem, characterized by length scale ℓ are

- In 2D,

$$C(\ell) = \ell\frac{\partial}{\partial\ell}S(\ell) \tag{4.59}$$

- In 3D,

$$C(\ell) = \left(\ell\frac{\partial}{\partial\ell} - 1\right)S(\ell) \tag{4.60}$$

- In higher dimensions, the renormalized entropy is proposed:

$$\mathcal{S}_{(d)}(\ell) = \begin{cases} \frac{1}{(d-2)!!}\left(\ell\frac{d}{d\ell}-1\right)\left(\ell\frac{d}{d\ell}-3\right)\cdots\left(\ell\frac{d}{d\ell}-(d-2)\right)S(\ell) & ,\text{d is odd} \\ \frac{1}{(d-2)!!}\left(\ell\frac{d}{d\ell}-2\right)\left(\ell\frac{d}{d\ell}-4\right)\cdots\left(\ell\frac{d}{d\ell}-(d-2)\right)S(\ell) & ,\text{d is even.} \end{cases} \tag{4.61}$$

  For CFTs, the renormalized entropy $\mathcal{S}_{(d)}(\ell)$ does pick up central charges in all dimensions; for general QFTs, it seems to interpolate between c_{UV} and c_{IR}. However, in 4D, there exists examples where $\mathcal{S}_{(d)}(\ell)$ is neither monotonic or non-negative.

4.3 Examples of entropic functions

In 2D,

$$c = \frac{3\ell}{2G_N} \tag{4.62}$$

4.3.1 $\mathbf{CFT}_2$ on $\mathbb{R}^{1,1}$

4.3.2 $\mathbf{CFT}_3$ on $\mathbb{R}^{1,2}$

4.3.3 $\mathbf{CFT}_2$ on $S^1 \times \mathbb{R}$

Chapter 5

Beyond Asymptotic AdS?

In the usual renormalization theory in QFT, the irrelevant perturbations require an infinite number of counter terms and thus are often un-controllable. Nevertheless, it was recently discovered that a certain class of irrelevant deformations in 2D is exactly solvable, known as the $T\overline{T}$ *deformation*[66, 67]. This includes, for example, knowing the finite volume spectrum[66, 67] on an infinite cylinder and the torus partition function[68, 69, 70]. When there exists a conserved holomorphic $U(1)$ current, an analogous "$J\overline{T}$ deformation" can be defined[71]. For integrable theories with higher conserved charges, these exists a family of deformations from conserved currents[72].

It is still not completely clear why $T\overline{T}$ deformation is exactly solvable. One hint is that conserved charges remain conserved under $T\overline{T}$ flow[66], *i.e.,* integrable theories remain integrable. For the other QFTs, [68] proposed a random geometry picture that relates this question to the solvability of coupled "gravity sector". Another perhaps more direct connection to gravity is [73] that $T\overline{T}$ may be equivalent to turning on Jackiw-Teitelbohm gravity , and that the $S-$matrix becomes gravitationally dressed accordingly.

We will briefy introduce features of $T\overline{T}$ in Section 5.1. For more details, see [67, 74]. Then we move on to discuss proposals on its holographic dual in Section 5.2. Finally in Section 5.3, we will show some calculation on entanglement entropy explored by the author.

5.1 $T\overline{T}$ in $2D$ CFT

5.1.1 Definition

The trajectory of theories is described by a differential equation

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\frac{1}{\pi^2}\,\mathrm{T}\overline{\mathrm{T}}(\lambda), \quad \mathrm{T}\overline{\mathrm{T}}(\lambda) \equiv -\pi^2 \det T_{\mu\nu}(\lambda). \tag{5.1}$$

Even more explicitly, in 2D, the stress energy tensor is 2×2, and it is conventional to define

$$T \equiv -2\pi T_{zz}, \quad \overline{T} \equiv -2\pi T_{\overline{z}\overline{z}}, \quad \Theta \equiv= 2\pi T_{z\bar{z}} \tag{5.2}$$

and therefore

$$\mathrm{T}\overline{\mathrm{T}} = T\overline{T} - \Theta^2. \tag{5.3}$$

Notice that in 2D CFT, $\Theta = 0$ classically. Since a theory with a single mass scale μ[1] satisfies

$$\mu \frac{\partial \mathcal{L}}{\partial \mu} = \frac{1}{2\pi} T_i^i, \tag{5.4}$$

we have an alternative and equivalent definition of $T\overline{T}$-deformed theory, the *trace relation:*

Definition of a $T\overline{T}$-deformed theory:

$$T_i^i = \frac{4}{\pi}\,\mathrm{T}\overline{\mathrm{T}} \quad \Longleftrightarrow \quad \frac{\partial \mathcal{L}}{\partial \lambda} = -\frac{1}{\pi^2}\,\mathrm{T}\overline{\mathrm{T}} \tag{5.5}$$

Consider the simplest example: the free scalar theory,

$$\mathcal{L}(\lambda = 0) = \partial\phi\overline{\partial}\phi. \tag{5.6}$$

It is a simple differential equation to solve for the deformed Lagrangian, which turns out to be

$$\mathcal{L}(\lambda) = \frac{1}{2\lambda}\left(\sqrt{4\lambda\partial\phi\overline{\partial}\phi + 1} - 1\right) \tag{5.7}$$

$$= -\frac{1}{2\lambda} + \mathcal{L}_{NG}, \quad \mathcal{L}_{NG} \equiv \frac{1}{2\lambda}\sqrt{\det\left(\partial_\mu X \cdot \partial_\nu X\right)}, \tag{5.8}$$

[1] In our case, $\lambda = \frac{1}{\mu^2}$. We ignore subtleties on dimensional transmutation from RG flow.

where $\mathcal{L}_{NG}$ is the Nambu-Goto Lagrangian in the static gauge. The Nambu-Goto Lagrangian is known to describe free bosonic strings, where $\frac{1}{\lambda}$ plays the role of string tension. Therefore, we see that a local field theory flows to a non-local theory with extended object, namely strings.

This may be interesting from the RG point of view, since it seems like we are "flowing up" the RG flow, while, strictly speaking, all we did is to solve a different equation for classical Lagrangian, *i.e.,* no quantization yet. In fact, it is a well-known difficult problem in string theory to quantize the Nambu-Goto action in arbitrary dimensions. Nevertheless, we will see in the next subsection that the spectrum of a $T\overline{T}$-deformed quantum field theory on an infinite cylinder is exactly solvable given the un-deformed spectrum.

$T\overline{T}$ flow:

$\mathcal{L}(0)$	$\xrightarrow{\mathrm{T\overline{T}}(\lambda)}$	$\mathcal{L}(\lambda)$
IR	$\longrightarrow$	UV
local	$\longrightarrow$	non-local
free scalars	$\longrightarrow$	free bosonic strings

5.1.2 Factorization formula and spectrum on cylinder

Before stating the formula, we want to list the assumptions needed [75]:

1. Local translational and rotational symmetry: this guarantees the existence of a symmetric, conserved stress energy tensor.

2. Global translational symmetry: For any local field $\mathcal{O}_i(z)$, the expectation value $\langle\mathcal{O}_i(z)\rangle$ is constant.

3. Infinite separation: there exist an uncompact direction where cluster decomposition theorem holds.

4. CFT limit at short distances: certain no-resonance condition is satisfied.

Notice that assumption 2 and 3 constraint the geometry of 2D space to be either flat space or infinite cylinder.

The spectrum of a $T\overline{T}$-deformed QFT on an infinite cylinder is exactly solvable thanks to the Zamolodchikov's factorization formula[75],

Zamolodchikov's factorization formula:

$$\langle n|\,\mathrm{T}\overline{\mathrm{T}}\,|n\rangle = \langle n|\,T\,|n\rangle\,\langle n|\,\overline{T}\,|n\rangle - \langle n|\,\Theta\,|n\rangle\,\langle n|\,\Theta\,|n\rangle\,, \tag{5.9}$$

where $|n\rangle$ are non-degenerate energy and momentum eigenstates, *i.e.*,

$$H\,|n\rangle = E_n\,|n\rangle\,, \quad P\,|n\rangle = P_n\,|n\rangle\,. \tag{5.10}$$

Proof sketch:

- Show that

$$C(z,w) \equiv \langle n|\,T(z)\overline{T}(w)\,|n\rangle - \langle n|\,\Theta(z)\Theta(w)\,|n\rangle \tag{5.11}$$

  is a constant, *i.e.*, $C(z,w) = C$.

- Show that $\mathrm{T}\overline{\mathrm{T}}$ is a well-defined composite operator; in other words,

$$\mathrm{T}\overline{\mathrm{T}} = \lim_{z\to w}\left(T(z)T(w) - \Theta(z)\Theta(w)\right) + (\text{total derivatives}) \tag{5.12}$$

  is independent of the path taken from z to w. The total derivatives terms vanish on-shell due to translational symmetry.

- Look at the spectral decomposition[a]

$$\langle n|\,T(z)T(w)\,|n\rangle = \sum_m \langle n|\,T(z)\,|m\rangle\,\langle m|\,\overline{T}(z)\,|n\rangle \times e^{(E_n-E_m)|y-y'|+i(P_n-P_m)|x-x'|}. \tag{5.13}$$

  Similarly for $\langle n|\,\Theta(z)\Theta(w)\,|n\rangle$.

- Since $C(z,w) = C$, the non-diagonal terms between two two-point functions must

cancel out. Hence

$$C = \langle n|\, T(z)\, |n\rangle \, \langle n|\, \overline{T}(w)\, |n\rangle - \langle n|\, \Theta(z)\, |n\rangle \, \langle n|\, \Theta(w)\, |n\rangle \,. \tag{5.14}$$

The desired result follows from taking the $z \to w$ limit.

Notice that the factorization formula holds for the whole $T\overline{T}$ flow and implies that $\mathrm{T\overline{T}}$ has exact scaling dimension 4, *i.e.,* no anomalous dimension.

[a] There is a typo in (3.21) of [74].

To prove $C(z, w) = C$, one need to show that $\partial_z C(z, w) = 0$, which follows from the assumption 1. The second step is harder, so we leave out the details. We will just comment that one special thing about 2D is that the correlation function

$$\langle T_{\mu\nu} T^{\mu\nu} \rangle = \frac{d-2}{x^{2d}} \tag{5.15}$$

vanishes.

Consider QFT on an infinite cylinder with radius R. By the definition of stress energy tensor,

$$E_n(R, \lambda) = -R \langle n|\, T_{yy}\, |n\rangle \,, \partial_R E_n(R, \lambda) = -\langle n|\, T_{xx}\, |n\rangle \,, P_n = -i \langle n|\, T_{xy}\, |n\rangle \,, \tag{5.16}$$

where λ is just used to label the theory that the energy is measured in. By the definition of $T\overline{T}$ flow,

$$\partial_\lambda E_n(R, \lambda) = -R \langle n|\, \det T_{\mu\nu}\, |n\rangle \,. \tag{5.17}$$

Combining these equations, we can obtain the differential equation for the spectrum and its solution:

Spectrum of $T\overline{T}$ on infinite cylinder:

The solutions to the differential equation

$$\partial_\lambda E_n(R, \lambda) = E_n(R, \lambda) \partial_R E_n(R, \lambda) + \frac{1}{R} P_n^2(R, \lambda) \tag{5.18}$$

are

$$E_n(R,\lambda) = \frac{R}{2\lambda}\left(\sqrt{1+\frac{4\lambda E_n}{R}+\frac{4\lambda^2 P_n^2}{R^2}}-1\right), \tag{5.19}$$

where

$$E_n \equiv \frac{1}{R}\left(n+\overline{n}-\frac{c}{12}\right), P_n(R) \equiv \frac{1}{R}\left(n-\overline{n}\right) \tag{5.20}$$

is the initial condition by the assumption 4. n and $\overline{n}$ are the number of left and right movers.

When $n = \overline{n} = 0$, we obtain the ground state energy

$$E_0 = \frac{R}{2\lambda}\left(\sqrt{1-\frac{\lambda c}{3R^2}}-1\right). \tag{5.21}$$

Notice that the UV behaviors of two signs of λ are drastically different. In particular, when λ is positive, there exists a singularity when $\lambda > \frac{3R^2}{c}$. This corresponds to a critical temperature

$$T_c \equiv \frac{1}{R} = \sqrt{\frac{3}{\lambda c}}, \tag{5.22}$$

and the associated phase transition is known as the *Hagedorn transition.* This sign, $\lambda > 0$, is thus called the "bad" sign of $T\overline{T}$. We will discuss its holographic dual in the following section.

5.2 $T\overline{T}$'s holographic dual

As we just saw, there are two signs of $T\overline{T}$: the good sign and the bad sign. The bad sign without matter content is proposed to be dual to the *cutoff AdS*, *i.e.,* putting a Dirichlet boundary condition in the bulk[76, 77, 78]. We will work in the Euclidean signature from now on and set the AdS radius $\ell_{AdS} = 1$. The boundary is at finite radius r_c.

There are many evidences for the duality, but here we will just reproduce the one for

energy spectrum, namely reproducing Eq. 5.19 from gravity. The action of cutoff AdS is

$$S = -\frac{1}{16\pi G_N}\int_{\mathcal{M}}(R+2) - \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}}(K-1), \tag{5.23}$$

where $\mathcal{M}$ is a finite-radius AdS_3 with boundary. The second term above is the standard Gibbons-Hawking term. Let's first consider the vacuum. We fix a gauge such that the metric takes the form

$$ds^2 = d\rho^2 + g_{ij}(x,\rho)dx^i dx^j, \tag{5.24}$$

and the extrinsic curvature 3.5.1 is

$$K_{ij} = \frac{1}{2}\partial_\rho g_{ij}. \tag{5.25}$$

The stress energy tensor can be computed[2],

$$T^{ij} = \frac{2}{\sqrt{h}}\frac{\delta S}{\delta h_{ij}} \sim \frac{1}{G_N}\left(K^{ij} - Kg^{ij} + g^{ij}\right), \tag{5.26}$$

where we ignore the prefactor. By virtue of the Einstein equations, we obtain constraints on the extrinsic curvature, which then can be translated into constraints on the stress energy tensor; for example,

$$\nabla^i T_{ij} = 0, \tag{5.27}$$

$$T^i_i \sim G_N\,\mathrm{T\overline{T}}\,. \tag{5.28}$$

This is analogous to the $T\overline{T}$-deformed theory, if one makes the identification

$$\lambda = \#G_N, \tag{5.29}$$

where the prefactor can be worked out if one keeps track of it. This is our first hint of the duality.

Let's move on to the excited state of $T\overline{T}$, which may dual to the black hole geometry in

[2]The bulk is empty, so it suffices to take variations with respect to the boundary metric.

cutoff AdS_3, which is the rotating BTZ black hole 2.60,

$$ds^2 = \frac{dt^2}{f(r)} + f(r)dt^2 + r^2(d\phi - i\omega dt)^2 \tag{5.30}$$

with

$$f(r) = r^2 - 8G_N M + \frac{16G_N^2 J^2}{r^2}, \qquad \omega(r) = \frac{4G_N J}{r^2}. \tag{5.31}$$

The energy of such black hole is well known [79],

$$E = \frac{R}{8\pi G_N}\left(1 - \sqrt{1 - \frac{16\pi G_N}{R}M + (\frac{8\pi G_N}{R}J)^2}\right), \tag{5.32}$$

where R is the proper length of the boundary, $\int_{r=r_c} d\phi\sqrt{g_{\phi\phi}}$. Clearly this is consistent with the energy spectrum of a $T\overline{T}$-deformed theory.

Besides cutoff AdS, there is another proposal [80] about a variation of $T\overline{T}$, which may be dual to the *little string theory*, which are asymptotically flat. Therefore, in the following section, we also explore the entanglement entropy in D3 brane geometry, which is also asymptotically flat.

5.3 Entanglement entropy in cutoff AdS and $D3$ brane holography

The entanglement entropy of cutoff AdS is also studied by [81, 82, 83]. We study the entanglement entropy between two codimension-1 hemispheres in AdS_{d+1}. The global AdS metric is

$$ds^2 = \ell_{AdS}^2\left(d\rho^2 + \sinh^2\rho\left(d\theta^2 + \sum_{j=2}^{d}\prod_{i=1}^{j-1}\sin^2\theta_i d\theta_j^2\right)\right). \tag{5.33}$$

With the entangling surface S^{d-1} sitting on a cutoff with radius R, the RT formula simply gives

$$S_A = \frac{2\pi\ell_{AdS}^{d-1}\omega_{d-2}}{\ell_p^{d-1}}\int_0^{\rho_c}\sinh^{d-2}(\rho)d\rho, \tag{5.34}$$

where ω_{d-2} is the area of a unit $(d-2)$-sphere and $\sinh\rho_c = \frac{R}{\ell_{AdS}}$.

The integral can be performed explicitly:

Entanglement entropy of halved-S^d in cutoff AdS$_{d+1}$:

$$S_A = \frac{2\pi\omega_{d-2}}{(d-1)}\left(\frac{R}{\ell_p}\right)^{d-1} {}_2F_1\left[\frac{1}{2}, \frac{d-1}{2}, \frac{d+1}{2}, -\frac{R^2}{\ell_{AdS}^2}\right] \tag{5.35}$$

- $d=2$, $\quad S_A = \frac{4\pi\ell_{AdS}\sinh^{-1}(\frac{R}{\ell_{AdS}})}{\ell_p}$
- $d=3$, $\quad S_A = \frac{4\pi^2\ell_{AdS}\left(\sqrt{R^2+\ell_{AdS}^2}-\ell_{AdS}\right)}{\ell_p}$

  ...

At the leading order, they agree with the field theory results [83].

The entanglement entropy of such subsystem can be computed from the field theory by sphere partition functions, which has been vastly calculated. In addition, one may compute the entanglement entropy of a spherical region on flat space. The computation from gravity side is very straightforward; in fact, it is almost trivial if one just replace the UV cutoff in the standard formulas by the radius of Dirichlet boundary condition. The author is curious whether the derivation from field theory has been attempted. If one follows the CHM construction, this requires knowledge of modular Hamiltonian of a $T\overline{T}$-deformed theory, which seems like a non-trivial task.

Now, we try to compute the holographic entanglement entropy of spherical region in (a variation of) D3 brane geometry,

The metric:

$$ds^2 = \frac{1}{\sqrt{f_3(r)}}\left(-dt^2+\sum_{i=1}^{3}dx_i^2\right) + \sqrt{f_3(r)}\left(dr^2+r^2d\Omega_5^2\right), \quad f_3(r) = h + \frac{r_3^4}{r^4}. \tag{5.36}$$

- When $h=1$, D3 brane geometry.
- When $h=0$, AdS$_5\times S^5$

Here, we will work in the small h regime, *i.e.,* close to AdS. In this limit, we can compute the entanglement entropy by evaluating the area of the old minimal surface with the new metric.

Let's write the metric in a slightly different form,

$$ds^2 = \frac{1}{\sqrt{f_3(r)}}\left(-dt^2 + d\rho^2 + \rho^2 d\Omega_2^2\right) + \sqrt{f_3(r)}\left(\frac{dz^2}{z^4} + \frac{1}{z^2}d\Omega_5^2\right), \quad f_3(r) = h + \frac{z^4}{z_3^4}, \tag{5.37}$$

where $z \equiv \frac{1}{r}$ and $z_3 \equiv \frac{1}{r_3}$. As a practice, let $h = 0$ and write down the area functional

$$A = \frac{\omega_2\omega_5}{z_3^2}\int \frac{\rho^2}{z^3}\sqrt{\dot{\rho}^2 + \frac{1}{z_3^4}}dz \equiv \frac{\omega_2\omega_5}{z_3^2}\int \mathcal{L}(\rho, \dot{\rho}, z)dz, \tag{5.38}$$

where ω_n is the volume of a unit n-sphere and $\dot{\rho} \equiv \frac{d\rho}{dz}$. To obtain the minimal surface, we need to solve for the Euler-Lagrangian equation,

$$\frac{\partial\mathcal{L}}{\partial\rho} - \frac{d}{dz}\left(\frac{\partial\mathcal{L}}{\partial\dot{\rho}}\right) = 0 \tag{5.39}$$

$$\Rightarrow \quad \frac{2\rho}{z^3}\sqrt{\dot{\rho}^2 + \frac{1}{z_3^4}} - \frac{d}{dz}\left(\frac{\rho^2\dot{\rho}}{z^3\sqrt{\dot{\rho}^2 + \frac{1}{z_3^4}}}\right) = 0, \quad \rho(0) = R. \tag{5.40}$$

Since we already know that the minimal surface of spherical region on the boundary of AdS is an hemisphere, we just try the solution,

$$\rho = \sqrt{R^2 - \frac{z^2}{z_3^4}}, \tag{5.41}$$

and, indeed, this would work.

We can try to repeat the same procedure and compute the area functional when $h = 1$,

$$A = \frac{\omega_2\omega_5}{z_3^3}\int dz\rho^2\left(1 - \frac{z_3^4}{2z^4}\right)\left[\left(1 + \frac{z_3^4}{4z^4}\right)\frac{1}{z_3}\right]^5\sqrt{\left(1 - \frac{z_3^4}{2z^4}\right)(\dot{\rho}^2 + \frac{1}{z_3^4})} \tag{5.42}$$

This area may be extremized numerically. We decided, however, to analytically explore the small h limit. To the leading order in h, the metric reads

$$ds^2 = \frac{z_3^2}{z^2}\left(1 - \frac{hz_3^4}{2z^4}\right)(-dt^2 + d\rho^2 + \rho^2 d\Omega_2^2) + \frac{z^2}{z_3^2}\left(1 + \frac{hz_3^4}{2z^4}\right)\left(\frac{dz^2}{z^4} + \frac{1}{z^2}d\Omega_5^2\right) + \mathcal{O}(h^2) \tag{5.43}$$

and the area functional becomes

$$A = \frac{2\omega_2\omega_5}{z_3^8}\int_k^0 \left[1 + \frac{h}{z_3^4 R'^4}\left(\frac{1}{(1-u^2)^2} - \frac{1}{2(1-u^2)}\right)\right]\frac{u^2 du}{(1-u^2)^2}, \quad u \equiv \sin\theta. \tag{5.44}$$

The final result is

Entanglement entropy at leading order in h:

$$A = \frac{3k + \left(\frac{h}{z_3^4 R'^4} - 6\right)k^3 + 3k^5 + 3(k^2-1)^3\,\mathrm{ArcTanh}[k]}{6(k^2-1)^3} + \mathcal{O}(h^2) \tag{5.45}$$

where

$$R' = \sqrt{\frac{z_c^2}{z_3^2}\left(1 + \frac{h z_3^4}{2 z_c^4}\right)R^2 + \frac{z_c^2}{z_3^4}}, \tag{5.46}$$

$$k = \frac{\left(1 + \frac{h z_3^4}{4 z_c^4}\right)R}{\sqrt{\left(1 + \frac{h z_3^4}{2 z_c^4}\right)R^2 + \frac{z_c^2}{z_3^4}}}, \tag{5.47}$$

and R is the radius of the spherical system on a cut-off surface at $z = z_c$.

From here, one can then compute the entropic functions in specific dimensions. We observe that the entropic function is monotonically decreasing from IR to UV, which is unusual for local quantum field theories but may be a common feature of irrelevantly-deformed theories.

5.4 Holography with branes?

As introduced in Section 3.7, the information paradox may be resolved given the QES formula. However, this is in general hard to do, except in 2D. The main difficulty boils down to the replica trick (Section 3.2.4), which computes the bulk entanglement entropy of matter fields. We do not have a simple way to compute correlation functions of twist fields in $D > 2$. One possible simplification occurs when the bulk fields are themselves *holographic*, and thus the leading order entanglement entropy can be computed from the classical gravity in even one-higher dimensions, *i.e.,* two dimensions higher than the boundary CFT. Such theories are called *double-holographic.*[84, 85, 86].

For example, [85] studied the entropy of Hawking radiation in $3+1D$ via holography in AdS_5. The double-holographic models naturally arise from the Karch-Randall braneworld[87,

88, 89, 90], where [90] studied the so-called *AdS/BCFT* duality. A known issue about such models is that the physical graviton modes are necessarily massive, since the massless ones are non-renormalizable[91][3]. A series of reviews in this direction appeared very recently [93, 94, 95].

[3] An attempt to address thjs, for example, is [92].

Chapter 6

Prospects

In this thesis, we first summarized previous knowledge on geometry from entanglement in the leading order perturbation theory. The summary is of course in-complete, but we went into specifically-chosen technical details, with a goal of approaching the question,

"*What does the leading order perturbation theory in gravity know?*"

Then we made some small advance on how the perturbation theory may be realized when going beyond asymptotic anti-de Sitter (AAdS) spacetime. Perhaps unsurprisingly, we are only able to specify leading order boundary condition for bulk fields near UV. Yet, in the framework we set up, the bulk metric are constrained by the holographic entanglement. Thus, we hope that the entanglement of certain non-AAdS geometries will turn out to instruct us how to regularize conformal perturbation theories with irrelevant perturbations via holographic duality. We have explored cutoff AdS and D3 brane holography, although neither of which have provided a clear connection yet.

There are several loose ends in the thesis. Although it seems like the leading order perturbation theory does not take us away frow the vacuum code susspace, there may be some plausible ways to make generalization[1]. We discuss some of them in the following (in the order of decreasing concreteness):

- **Subtleties on corrections to entanglement:**While the corrections to entanglement entropy can naively be written as the integral of local correlation functions[52, 53], the subtleties are essential for bulk emergence, such as the $\sinh^2()$ factor we saw in Section 3.7. [52, 53] mentioned that the subtleties are attributed to the contact terms. What

[1]The author is actually not sure whether these questions have been addressed in the literature. If they are, then the task is to collect the results and reflect on their implications on our central question.

exactly is the role of such contact terms, and how do they relate to our calculation? Also, in standard QFT, one figures out the structure of Feynman diagrams, namely the *one-particle-irreducible* diagrams, and thus is able to perform resummation. Do we have a similar story for the entanglement-based perturbation theory, including subtleties of contact terms?[2]

- **Constraints on emergent bulk metric for $T\overline{T}$:** It is important that in $T\overline{T}$-deformed theories, the deformation differentially evolves along the RG flow, instead of being fixed and inserted at the beginning. Then a potential research question is how the constraint on metric sketched in Section 4.1.3 may change. Would it also take a form of differential equation? This may be related to the question studied in [98, 99, 100], while here we want to develop an entanglement-based approach.

- **Conserved charges and corrections to entanglement:** It is a bit surprising, at least to the author, that the second order correction we obtained in Section 3.7 is a conserved charge in semiclassical gravity. Under what condition is this a true statement, *e.g.,* true in general, true for a particular class of entangling surface's geometry, or only true at the first couple orders of perturbation theory? These are directly related to our central question, "*What does the semiclassical gravity actually know?*". A starting point may be to learn in details the formalism set up in [55, 101] and then to think more carefully about arguments in the FLM proposal and [54].

- **Entanglement island, double holography, and dS/dS correspondence:** It was a huge surprise [12, 13] that it seems possible to produce an unitary Page curve within the leading order perturbation theory. As already mentioned in Section 5.4, this is vastly being studied in the double-holographic context [84, 85], with which one can even study information paradox in dS/dS correspondence [102, 103, 86]. The staring point is the Karch-Randall braneworld[87, 88, 89, 90].This seems to be a more difficult but popular/important research direction.

Finally, gauge/gravity duality and entanglement are very broad. There are many interesting topics that we did not discuss at all in this thesis. Here we bring up some of them that the author wish to learn in the future:

[2]The connection between CFT correlation functions and Feynman diagrams in AdS has been vastly studied, *e.g.,*[96, 97].

- **Stringy models:** Despite the success of the holographic principle or a general notion of gauge/gravity duality, string theory is still the best platform for generating explicit models of quantum gravity and testing the holographic correspondence. Some examples include

  - The Maldacena's original duality: $\mathcal{N} = 4$ super-Yang-Mills dual to Type IIB superstrings in $AdS_5 \times S^5$[1].
  - 2D conformal field theories with large central charge dual to classical gravitation theories in AdS_3, *e.g.,* [104, 105, 106, 107].
  - The ABJM theory dual to the classical gravitational theories on $AdS_4 \times S^7$ [108].
  - The large N limit of a 6D $(2, 0)$ superconformal field theories dual to classical gravity on $AdS_7 \times S^4$ [109].
  - Stringy ER=EPR [110].

- **Tensor network:** A tensor network realization of entanglement renormalization[111] may be applicable to holography[112, 113]. In particular, given the insight of holographic quantum error-correcting code[114], the network of perfect tensors[115] is constructed as well as the entanglement wedge reconstruction[116]. Later on, the random tensor network version[117, 118] was developed to address issues for multi-interval subsytems. However, all these models suffer from the flat entanglement spectrum, *i.e.,* the $n-$th Renyi entropy is independent of n, and hence modifications are necessary, *e.g.,* [119], to go beyond the *fixed area states*[120, 121].

- **Application to many-body systems and hydrodynamics:** Entanglement has been a more and more popular phrase used in condensed matter physics. Some techniques of holographic entanglement may be be applicable to strongly correlated phases, *e.g.,* strange metals, very gapless phases. *e.g.,* Fermi liquid, or hydrodynamics[122, 123]. For references, see, for example, Section 9 of [17] or [124, 125].

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